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Global dynamics and asymptotic spreading of a diffusive age-structured model in spatially periodic media

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Abstract

The paper is concerned with the persistence and spatial propagation of populations with age structure in spatially periodic media. We first provide a complete characterization of the global dynamics for the problem via investigating the existence, uniqueness and global stability of the nontrivial equilibrium. This leads to a necessary and sufficient condition for populations to survive, in terms of the principal eigenvalue of the associated linearized problem with periodic boundary conditions. We next establish the spatial propagation dynamics for the problem and derive the formula for the asymptotic spreading speed. The result suggests that the propagation fronts of populations are uniform for all age groups with a common spreading speed. Our approach involves developing the theory of generalized principal eigenvalues and the homogenization method to address novel challenges arising from the nonlocal age boundary condition.

Mathematics Subject Classification 35B40 · 35K57 · 35B27 · 92D25

1 Introduction

In this paper, we are concerned with an age-dependent population dynamics model with spatial diffusion in the spatially periodic media:

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$$\begin{cases} \partial_t u + \partial_a u = D\partial_{xx} u - \mu(a,x)u, & (a,x) \in (0,a_m) \times \mathbb{R},\ t > 0, \\ u(t,0,x) = f\left(x, \int_0^{a_m} \beta(a,x)u(t,a,x)da\right), & x \in \mathbb{R},\ t > 0, \\ u(0,a,x) = u_0(a,x), & (a,x) \in (0,a_m) \times \mathbb{R}, \end{cases} \tag{1.1}$$

where $u(t, a, x)$ represents the density of the population with age a at location x and time t, and the maximal age is parameterized by $a_m \in (0,+\infty]$. Function $\mu \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R} \times [0,a_m))$ with $\alpha \in (0,1)$ represents the age-specific death rate of individuals, which is assumed to be nonnegative and periodic in x with period $L > 0$. Let nonnegative function $\beta \in C^{2+\alpha}(\mathbb{R}, L^\infty(0,a_m))$ with $\alpha \in (0,1)$ denote the age-specific birth rate of individuals, which is L-periodic in x. The total birth rate at location x and time t is given by the nonlocal term

$$\int_0^{a_m} \beta(a,x)u(t,a,x)da.$$

The age boundary condition is given by the nonlinear function $f \in C^{2+\alpha,1}(\mathbb{R} \times [0,\infty))$ with $\alpha \in (0,1)$, of the total birth rate, which is also assumed to be L-periodic in x. The main feature of problem (1.1) is the combination of nonlinearity and nonlocality.

Structured models bridge the gap between the individual level and the population level, allowing us to study the population dynamics by examining the characteristics of individuals [41, 46, 52]. These models typically involve parameterizing the state of individuals based on their physiological or physical conditions, among which the age is an important characteristic. The age-structured model was proposed in the pioneering work of McKendrick and Lotka during 1920–1940 [38, 45] and has attracted intensive attentions in both theoretical and empirical investigations [1, 29, 30, 61].

In biological modeling, the spatial dispersal of individuals plays a crucial role. Since individuals need to be mature enough chronologically to disperse, age-structured population models incorporating diffusion arise naturally in biological investigations. The diffusive problem (1.1) was first proposed by Gurtin [26] and has been extensively studied in the literature [10, 25, 35, 39, 58, 60], with a focus on the dynamics of problem (1.1) in spatially bounded domains. Until 2007, the dynamics in unbounded domains was investigated via analyzing the existence of traveling wave solutions, see e.g. [13, 16–18]. However, the global dynamics and spatial spreading properties of problem (1.1) are left open.

The focus of this paper is two-fold. The first aim is to provide a complete characterization of the global dynamics for problem (1.1), as motivated by biological questions regarding the persistence of age-structured populations in spatially periodic environments. Specifically, we obtain the existence, uniqueness, and stability of stationary solutions for problem (1.1). This leads to a necessary and sufficient condition for the population to survive. The second aim is to establish the spatial propagation dynamics of problem (1.1) and derive the formula for the asymptotic spreading speed, as motivated by the invasion of age-structured populations. It turns out that the propagation fronts of populations is asymptotically unified for all age groups, and the age structure only affects the common spreading speed. Our approach focuses on the development of the homogenization theory for problem (1.1) in order to overcome new challenges arising from the nonlocal age component.

1.1 Global dynamics of Cauchy problem (1.1)

The stationary equation of problem (1.1) can be written as

$$\begin{cases} \partial_a u(a,x) = D\partial_{xx} u(a,x) - \mu(a,x)u(a,x), & (a,x) \in (0,a_m) \times \mathbb{R}, \\ u(0,x) = f\left(x, \int_0^{a_m} \beta(a,x)u(a,x)da\right), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Under some assumptions to be specified later, the positive solutions of (1.2) turn out to be periodic in x, while the periodicity assumption is not imposed a priori.

The dynamics of problems (1.1) and (1.2) are related to the following spatially periodic eigenvalue problem

$$\begin{cases} \partial_a \varphi = D\partial_{xx}\varphi - \mu(a,x)\varphi - \lambda\varphi, & (a,x) \in (0,a_m) \times \mathbb{R}, \\ \varphi(0,x) = f_u(x,0)\int_0^{a_m} \beta(a,x)\varphi(a,x)da, & x \in \mathbb{R}, \\ \varphi(a,x) = \varphi(a,x+L), & (a,x) \in (0,a_m) \times \mathbb{R}, \end{cases} \tag{1.3}$$

which can be regarded as the linearization of problem (1.2) at the trivial equilibrium $u = 0$. We first impose the following assumptions for the age-specific death rate μ and birth rate β.

Assumption 1.1 (i) There exists $\mu_{\inf} > 0$ such that $\mu(a,x) \geq \mu_{\inf}$ a.e. in $(0,a_m)\times [0,L]$, and $\int_0^{a_m} \mu(a,x)da = +\infty$ for all $x \in [0,L]$.
(ii) There exists $a_c \in (0,a_m)$ such that $\beta \equiv 0$ on $[a_c,a_m) \times [0,L]$ and $\int_a^{a_c} \min_{x\in[0,L]} \beta(s,x)ds > 0$ for any $a \in [0,a_c)$.

Remark 1.1 Assumption 1.1 is usually imposed in the age-structured models [29, 30, 41, 61]. The assumption $\int_0^{a_m} \mu(a,x)da = +\infty$ in part (i) is often employed to guarantee that the population density reaches zero at the maximal age. Part (ii) serves to obtain the simplicity of the principal eigenvalue and strict positivity of the principal eigenfunction for problem (1.3). We refer to Engel and Nagel [19, Theorem 4.4] or Ducrot et al. [14] for more details. The assumption $\beta \equiv 0$ nearby the maximal age means that the birth rate becomes zero when the age of the individuals approaches the maximal age, which is biologically reasonable. This allows us to consider problem (1.1) on a restricted interval $[0,a_+]$ for any $a_+ \in [a_c,a_m)$; see Remark 2.1 for further details. Such cutoff of age interval guarantees the principal eigenfunction to be positive everywhere in $[0,a_+] \times \mathbb{R}$ and avoids the singularity of the logarithm of the principal eigenfunction evaluated at the maximal age.

Next we make some assumptions on the nonlinear function $f = f(x,u)$ as follows:

Assumption 1.2 (i) $f_u(x,u) > 0$ for all $u \in [0,\infty)$ and $x \in \mathbb{R}$.
(ii) $f(x,0) \equiv 0$ and $\frac{f(x,u)}{u}$ is decreasing with respect to u for all $x \in \mathbb{R}$.

(iii) There exists $M > 0$ such that $f(x,u) \leq M$ for all $u \in [0,\infty)$ and $x \in \mathbb{R}$.

Remark 1.2 Two typical examples satisfying Assumption 1.2 are $f(x,u) = \frac{u}{1+Au}$ with some constant $A > 0$, commonly known as Holling's type II response [27, 28], and $f(x,u) = 1 - e^{-u}$, referred to as the logistic function and used to model the population growth [43]. Assumption 1.2-(i) is imposed to eliminate the existence of oscillatory solutions, which is commonly observed in age-structured models as in [40, 41] and the references therein. In combination with Assumption 1.2-(ii), it is necessary for our problem, in particular in the monotone iterative scheme (using comparison principles) to obtain the nontrivial positive equilibrium of problem (1.1) (see Proposition 3.5). Assumption 1.2-(iii) means that the birth rate of the population is bounded, which guarantees that the solution of (1.1) is uniformly bounded.[1]

Under Assumptions 1.1 and 1.2, the existence of the principal eigenvalue for problem (1.3) is established in Sect. 2.1. Note that Assumptions 1.1 and 1.2 will be required throughout the paper. Henceforth, we will assume their validity without repeating them, while we will indicate the additional assumptions where necessary.

Our first main result is given as follows.

Theorem 1.1 *Let $u(t,a,x)$ be the solution of (1.1) with any initial value $u_0 \geq\not\equiv 0$. Denote by H(0)*[2] *the principal eigenvalue of (1.3).*

(i) *If $H(0) > 0$, then there exists a unique positive solution to (1.2), denoted by $u^*(a,x)$, which is L-periodic in x and is globally asymptotically stable in the sense that $u(t,a,x) \to u^*(a,x)$ in $C_{\text{loc}}([0,a_m) \times \mathbb{R})$ as $t \to +\infty$. Furthermore, for any $a_+ \in [a_c, a_m)$, if $\inf_{(a,x)\in[0,a_+)\times\mathbb{R}} u_0(a,x) > 0$, then $u(t,a,x) \to u^*(a,x)$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$.*
(ii) *If $H(0) \leq 0$, then any nonnegative solution of (1.2) is identically zero and $u(t,a,x) \to 0$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$ for any $a_+ \in [a_c, a_m)$.*

Theorem 1.1 provides a complete description for the global dynamics of problem (1.1), which indicates the occurrence of the hair-trigger effect when $H(0) > 0$. Biologically, Theorem 1.1 formulates a necessary and sufficient condition for the persistence of an invading species in terms of the principal eigenvalue of (1.3), which reflects many crucial information regarding the interaction between age structure, species movement, and environmental heterogeneity. Similar results for age-structured models with diffusion in bounded domains have been found in [35, 39, 58]. However, establishing such global dynamics in unbounded domains is more intricate compared to bounded domains, even in the absence of age structure, see e.g. [31, 49].

[1] Indeed, if *f* is unbounded, the solution of problem (1.1) may blow up at infinity. For instance, assume that $f(x,u) = u(1+e^{-u})$, which satisfies parts (i) and (ii) in Assumption 1.2 but is not bounded. If $u_0(a,x)$ has a positive lower bound, then up to multiplication by a positive constant, it can be verified that $\underline{u}(t,a,x) = e^{H(0)t}\varphi(a,x)$ serves as a sub-solution of (1.1), where $(H(0),\varphi)$ is the principal eigenpair of problem (1.3) with $f(x,u) = u$. Hence, the solution of (1.1) is not uniformly bounded whenever $H(0) > 0$.

[2] The notation "$H(0)$" is adopted to keep consistency with the principal eigenvalue $H(\lambda)$ of problem (1.5) below.

Specifically, it is essential to prove a strong persistence result such that the solutions of problem (1.1) have a uniformly positive lower bound, which is of importance in analyzing the global dynamics by constructing suitable sub-solutions. Moreover, due to the presence of the operator ∂_a, problem (1.1) resembles a hybrid PDE, exhibiting parabolic and hyperbolic properties simultaneously. The classical parabolic estimates for (1.1) have not been established. This and the nonlocal term in age boundary condition lead to some difficulties in obtaining Theorem 1.1.

Remark 1.3 To prove Theorem 1.1, it is necessary to establish the relation between the principal eigenvalue of (1.3) and that of the following eigenvalue problem:

$$\begin{cases} \partial_a u(a,x) = D\partial_{xx}u - \mu(a,x)u(a,x) - \lambda u(a,x), & (a,x) \in (0,a_m)\times\mathbb{R}, \\ u(0,x) = f_u(x,0)\int_0^{a_m}\beta(a,x)u(a,x)da, & x\in\mathbb{R}, \end{cases} \tag{1.4}$$

for which the eigenfunction is not necessarily periodic in x. Such a problem itself is of independent interest. Note that the principal eigenvalue of (1.4) may not exist. This motivates us to define its generalized principal eigenvalue, which can be obtained through a limiting procedure of principal eigenvalues associated with problem (1.4) on bounded domains with Dirichlet boundary conditions (Lemma 2.5). Due to the symmetry property of our operator with respect to spatial diffusion (without advection term), one can establish the relation between such generalized principal eigenvalues and the principal eigenvalue of (1.3) (Proposition 2.6). In the aforementioned argument, the presence of the age-structure term distinguishes our eigenvalue problem from the time-periodic one as studied in [47–49]. In particular, our analysis requires special consideration on the nonlocal boundary condition at $a = 0$.

1.2 Spreading properties of Cauchy problem (1.1)

Next we consider the spreading properties of problem (1.1). For any $\lambda \in \mathbb{R}$, consider the following weighted eigenvalue problem:

$$\begin{cases} \partial_a\varphi = D\partial_{xx}\varphi - 2D\lambda\partial_x\varphi + D\lambda^2\varphi - \mu(a,x)\varphi - H(\lambda)\varphi, & (a,x)\in(0,a_m)\times\mathbb{R}, \\ \varphi(0,x) = f_u(x,0)\int_0^{a_m}\beta(a,x)\varphi(a,x)da, & x\in\mathbb{R}, \\ \varphi(a,x) = \varphi(a,x+L), & (a,x)\in(0,a_m)\times\mathbb{R}. \end{cases} \tag{1.5}$$

Let $H(\lambda)$ denote the principal eigenvalue of (1.5), for which the existence is established in Sect. 2.1. Our second main result is stated as follows.

Theorem 1.2 *Let $u(t,a,x)$ be the solution of (1.1) with any compactly supported initial value $u_0 \geq\not\equiv 0$. Assume $H(0) > 0$ and denote by $u^*(a,x)$ the unique positive solution to (1.2) as given by Theorem 1.1, then there exists $c^* > 0$ such that*

$$\begin{cases} \lim\limits_{t\to+\infty}\sup\limits_{|x|\geq ct}\sup\limits_{a\in[0,a_+]} u(t,a,x) = 0, & \text{for all } c > c^* \text{ and } a_+ \in [a_c,a_m), \\ \lim\limits_{t\to\infty}\sup\limits_{|x|\leq ct}\sup\limits_{a\in[0,a_+]} |u(t,a,x) - u^*(a,x)| = 0, & \text{for all } 0\leq c < c^* \text{ and } a_+\in[a_c,a_m). \end{cases}$$

More precisely, the spreading speed c^* can be determined by

$$c^* = \min_{\lambda>0} \frac{H(\lambda)}{\lambda}, \tag{1.6}$$

where the minimum can be attained by a unique positive λ.

The spreading properties shown in Theorem 1.2 are uniform in $a \in [0, a_m)$. One possible biological interpretation is that, since the maximal age of the population is finite, for all individuals born at the propagation front at time $t > 0$, the spatial distance diffused by the population throughout their lifespan remains finite. Hence, the positions of all age groups can be expressed as $c^*t + O(1)$, i.e. all individuals with different ages will spread at the same speed up to $O(1)$ terms. To our best knowledge, this phenomenon appears to be first proved in reaction-diffusion models with additional structures.

Theorem 1.2 is established via the homogenization method inspired by the works of Berestycki and Nadin [5, 6]. Such an approach was originally introduced by Freidlin [22] utilizing probabilistic arguments, and was generalized by Evans and Souganidis [20] by means of PDE arguments. The ideas for our problem are outlined as follows:

1. Consider the transformations:

$$u_\epsilon(t, a, x) := u\left(\frac{t}{\epsilon}, a, \frac{x}{\epsilon}\right) \quad \text{and} \quad z_\epsilon := \epsilon \ln\left(\frac{u_\epsilon(t, a, x)}{M}\right),$$

   where $M > 0$ is the uniform upper bound of the solution u to problem (1.1). Note that the variable a has not be rescaled. Then u_ϵ and z_ϵ satisfy the following equation:

$$\begin{cases} \partial_t z_\epsilon + \frac{1}{\epsilon}\partial_a z_\epsilon = D\epsilon\partial_{xx} z_\epsilon + D|\partial_x z_\epsilon|^2 - \mu\left(a, \frac{x}{\epsilon}\right), & (a, x) \in (0, a_m) \times \mathbb{R},\ t > 0, \\ \epsilon\partial_t u_\epsilon + \partial_a u_\epsilon = \epsilon^2 D\partial_{xx} u_\epsilon - \mu\left(a, \frac{x}{\epsilon}\right) u_\epsilon, & (a, x) \in (0, a_m) \times \mathbb{R},\ t > 0, \\ u_\epsilon(t, 0, x) = f\left(\frac{x}{\epsilon}, \int_0^{a_c} \beta\left(a, \frac{x}{\epsilon}\right) u_\epsilon(t, a, x) da\right), & x \in \mathbb{R},\ t > 0. \end{cases} \tag{1.7}$$

   The presence of the singular term "$\frac{1}{\epsilon}\partial_a z_\epsilon$" and the nonlocal boundary condition in (1.7) leads to some challenges in analyzing the limits of u_ϵ and z_ϵ.

2. Define the half-relaxed limit:

$$u_*(t, a, x) := \liminf_{\substack{\epsilon\to 0 \\ (t',a',x')\to(t,a,x)}} u_\epsilon(t', a', x').$$

   We establish that $\{u_* = 0\} = \{(t, x) : x \geq \bar{c}t, t > 0\} \times [0, a_m)$ for some $\bar{c} > 0$, which implies that the support of the limiting function u_* is age-independent (Proposition 4.1). This is a key ingredient of our homogenization method in this paper.
3. For any $a_+ \in [a_c, a_m)$, define the half-relaxed limits:

$$Z_* := \liminf_{\epsilon\to 0} \inf_{a\in[0,a_+]} z_\epsilon(t,a,x) \quad \text{and} \quad Z^* := \limsup_{\epsilon\to 0} \sup_{a\in[0,a_+]} z_\epsilon(t,a,x).$$

Let the critical value critical value $\bar{c} > 0$ be defined in Step 2. We prove that Z_* and Z^* constitute, respectively, a lower semi-continuous viscosity super-solution and an upper semi-continuous viscosity sub-solution of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t Z - H(\partial_x Z) = 0, & x > \bar{c}t,\ t > 0, \\ Z(t,\bar{c}t) = 0, & t > 0, \end{cases} \tag{1.8}$$

where the Hamiltonian $H(\partial_x Z)$ is the principal eigenvalue of problem (1.5) with $\lambda = \partial_x Z$ (Lemmas 4.2 and 4.3).

4. Define the half-relaxed limit:

$$z_*(t,a,x) := \liminf_{\substack{\epsilon\to 0\\ (t',a',x')\to(t,a,x)}} z_\epsilon(t',a',x').$$

By applying Step 3 and the comparison principle for Hamilton-Jacobi equations, we establish that $z_* = z_*(t,x)$ is independent of $a \in [0,a_m)$ and constitutes a viscosity solution of the Hamilton-Jacobi equation (1.8). Based on this, we prove that $u_* > 0$ in $\mathrm{Int}\{z_* = 0\} \times [0,a_+]$ for any $a_+ \in [a_c,a_m)$ (Lemma 4.5).

5. We complete our analysis by showing that the boundary point $\bar{c}$ serves as the spreading speed of (1.1) and coincides with c^* as defined by (1.6).

Mathematically, there are two main difficulties in proving Theorem 1.2. First, the presence of age structure in problem (1.1) brings the nonlocal effect in determining limits of u_ϵ and z_ϵ as $\epsilon \to 0$. This forces us to establish the uniform estimates with respect to the age variable a. Second, the nonlinear effects appear in the boundary condition at $a = 0$, which are different from the usual positions in the classical reaction-diffusion equations. This leads to some difficulties in applications of the theory of the viscosity solutions for Hamilton-Jacobi equations. Finally, to our best knowledge, while the homogenization method has been successfully employed in the studies of spreading properties for reaction-diffusion equations with random dispersal [5, 6, 37], nonlocal dispersal [8, 36] as well as time delays [34], its extension to problems with additional structures remains unexplored. We believe that our analytical framework developed in this paper may be applicable to other structured equations.

To conclude this section, we mention that our approach can be applied, with minor modifications, to the spreading problem of age-structured species in higher dimensions. Here, we focus on the one-dimensional case for the sake of clarity and simplicity in our presentation. For related works on the asymptotic spreading of a single population without age structure in heterogeneous environments, we refer to [3, 11, 21, 33, 48, 63] for the one-dimensional case, and refer to [4, 6, 50, 54, 62] for higher-dimensional case. In addition, our analysis specifically addresses the problem (1.1) with multiple age groups, which distinguishes from previous studies. In those works, the authors integrate the population from maturation age to maximal age, and trans-form (1.1) into a time-delayed reaction-diffusion equation under some specific death and birth rate functions. We refer to [12, 23, 31, 55, 57] for the related results.

The paper is organized as follows. In Sect. 2, we provide the existence of principal eigenvalues of (1.4) and (1.5) along with their properties, which are used to define the spreading speed and study the global dynamics of (1.1). In Sect. 3, we obtain the global dynamics of (1.1) including the existence, uniqueness, and global stability of positive equilibrium, which proves Theorem 1.1. In Sect. 4, we study the spreading properties of (1.1) and establish Theorem 1.2.

2 Theory of the principal eigenvalue

2.1 The existence and qualitative properties

In this subsection, we shall investigate the existence and some qualitative properties of the principal eigenvalue for problem (1.5). We first introduce the following notations:

$$\begin{aligned} \underline{\mu}(a) &:= \min_{x\in[0,L]} \mu(a,x), & \overline{\mu}(a) &:= \max_{x\in[0,L]} \mu(a,x), \\ \underline{\beta}(a) &:= \min_{x\in[0,L]} \beta(a,x), & \overline{\beta}(a) &:= \max_{x\in[0,L]} \beta(a,x), \\ \underline{f}(0) &:= \min_{x\in[0,L]} f_u(x,0), & \overline{f}(0) &:= \max_{x\in[0,L]} f_u(x,0). \end{aligned} \tag{2.1}$$

Lemma 2.1 *Let Assumption 1.1 hold. Then there exists a unique principal eigenvalue of (1.5), which is algebraically simple, and the corresponding eigenfunction can be positive.*

Proof The proof can follow by [25, Theorem 3] and [59, Lemma 2.6]. Here we only provide a sketch proof for completeness. Denote by X the Banach space

$$X = C_{\text{per}}(\mathbb{R}) := \{\phi \in C(\mathbb{R}) : \phi(x+L) = \phi(x)\}, \tag{2.2}$$

and denote its positive cone by X_+. Observe that X_+ is a normal and generating cone. Define the following function spaces:

$$\mathcal{X} = X \times L^1((0,a_m),X), \qquad \mathcal{X}_0 = \{0_X\} \times L^1((0,a_m),X),$$

endowed with the product norms and the positive cones:

$$\mathcal{X}^+ = X_+ \times \{u \in L^1((0,a_m),X) : u(a,\cdot) \in X_+ \text{ a.e. in } (0,a_m)\}, \quad \mathcal{X}_0^+ = \mathcal{X}^+ \cap \mathcal{X}_0.$$

We consider the following problem posed in X for $0 \le \tau \le a < a_m$:

$$\begin{cases} \partial_a v(a) = Dv_{xx}(a) - 2D\lambda\partial_x v(a) + D\lambda^2 v(a) - \mu(a,\cdot)v(a), & \tau < a < a_m, \\ v(\tau) = \eta \in X. \end{cases} \tag{2.3}$$

To avoid introducing more notations involving λ, we omit λ in the following notations. It follows that problem (2.3) generates an evolution family on X, denoted by $\{\mathcal{U}(a,\tau)\}_{0\le\tau\le a<a_m}$. In fact, such $\mathcal{U}$ can be given by a Green's function G:

$$(\mathcal{U}(a,\tau)\eta)(x) = \int_{\mathbb{R}} G(a,\tau;x-y)\eta(y)dy, \quad \forall\, 0\le\tau\le a<a_m. \tag{2.4}$$

Moreover, there exist $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|\mathcal{U}(a,\tau)\|_{\mathcal{L}(X)} \le Me^{\omega(a-\tau)}, \quad \forall\, 0\le\tau\le a<a_m. \tag{2.5}$$

In addition, we also define the following family of bounded linear operators $\{W_\lambda\}_{\lambda>\omega} \subset \mathcal{L}(\mathcal{X},\mathcal{X}_0)$ for $(\eta,g)\in\mathcal{X}$ by

$$W_\lambda(\eta,g) = \left(0,\ e^{-\lambda a}\mathcal{U}(a,0)\eta + \int_0^a e^{-\lambda(a-\tau)}\mathcal{U}(a,\tau)g(\tau)d\tau\right). \tag{2.6}$$

Following the argument in Thieme [56, Section 6], we can prove that this provides a family of positive pseudoresolvents. Hence, by Pazy [51, Section 1.9], there exists a unique closed Hille-Yosida operator $B : dom(B) \subset \mathcal{X} \mapsto \mathcal{X}$ with $\overline{dom(B)} = \mathcal{X}_0$ such that

$$(\lambda I - B)^{-1} = W_\lambda \quad \text{for all } \lambda > \omega, \tag{2.7}$$

where $I : \mathcal{X} \mapsto \mathcal{X}$ denotes the identity operator.

Furthermore, we define $C \in \mathcal{L}(\mathcal{X}_0,\mathcal{X})$ by

$$C(0,h) = \left(f_u(\cdot,0)\int_0^{a_m} \beta(a,\cdot)h(a)da,\ 0\right), \quad (0,h)\in\mathcal{X}_0, \tag{2.8}$$

and $A : dom(A) \subset \mathcal{X} \mapsto \mathcal{X}$ by

$$A = B + C \quad \text{with} \quad dom(A) = dom(B) \subset \mathcal{X}_0. \tag{2.9}$$

Then it suffices to prove the existence of the principal eigenvalue for operator A.

To this end, for each $\Lambda \in \mathbb{R}$, we define a linear operator $\mathcal{M}_\Lambda : X \mapsto X$ by

$$\mathcal{M}_\Lambda\phi = f_u(\cdot,0)\int_0^{a_m} \beta(a,\cdot)e^{-\Lambda a}\mathcal{U}(a,0)\phi\, da, \quad \forall\phi\in X. \tag{2.10}$$

In fact, $\mathcal{M}_\Lambda$ is obtained by plugging the resolvent of B into the integral initial condition (1.5), and we refer to [25, 59] for the derivation. By Assumption 1.1-(ii), it follows from [25] that $\mathcal{M}_\Lambda$ is a compact and nonsupporting operator in X, where *nonsupporting* is a generalization of strong positivity when working on a Banach space with a positive cone which has empty interior; see [44] or [53] for the complete definition. Thus by the Krein-Rutman theorem, for each $\Lambda \in \mathbb{R}$, the spectral radius $r(\mathcal{M}_\Lambda)$ of operator $\mathcal{M}_\Lambda$ is the principal eigenvalue, which is algebraically simple, and the corresponding eigenfunction can be positive.

Note from (2.10) that $\Lambda \mapsto r(\mathcal{M}_\Lambda)$ is continuous and strictly decreasing. It follows that such Λ satisfying $r(\mathcal{M}_\Lambda) = 1$ indeed exists and is unique, denoted by $H(\lambda)$. By definition, such $H(\lambda)$ is an eigenvalue of operator A. Moreover, if $\Lambda' > H(\lambda)$, then $r(\mathcal{M}_{\Lambda'}) < r(\mathcal{M}_{H(\lambda)}) = 1$, which implies that $(I - \mathcal{M}_{\Lambda'})^{-1}$ exists, and so does $(\Lambda' I - A)^{-1}$. This prevents Λ' to be an eigenvalue of A and therefore $H(\lambda)$ is the principal eigenvalue of A. Furthermore, the algebraic simplicity follows from that of $r(\mathcal{M}_{H(\lambda)})$. By the classical parabolic estimates, the principal eigenfunction of A associated with $H(\lambda)$ is belonging to $W^{1,1}((0, a_m), C^2_{\text{per}}(\mathbb{R}))$. The proof is now complete. □

Remark 2.1 Due to $\beta \equiv 0$ on $[a_c, a_m) \times \mathbb{R}$ as in Assumption 1.1-(ii), the characteristic equation (2.10) can be rewritten as follows:

$$\mathcal{M}_\Lambda \phi = f_u(\cdot, 0) \int_0^{a_c} \beta(a, \cdot) e^{-\Lambda a} \mathcal{U}(a, 0) \phi \, da, \quad \forall \phi \in X.$$

We observe from the proof of Lemma 2.1 that the principal eigenvalue of (1.5) is the unique value such that $r(\mathcal{M}_{H(\lambda)}) = 1$. Hence, for any $a_+ \in [a_c, a_m)$, $H(\lambda)$ is also the principal eigenvalue of (1.5) with $[0, a_m)$ replaced by $[0, a_+]$. In the followings, we are focused on the eigenvalue problem (1.5) posed on $[0, a_+] \times \mathbb{R}$ instead of $[0, a_m) \times \mathbb{R}$ for any $a_+ \in [a_c, a_m)$.

Next we collect some useful properties of the principal eigenvalues for problem (1.5).

Proposition 2.2 *Let Assumption 1.1 hold. Denote by $H(\lambda)$ the principal eigenvalue of problem (1.5) as given in Lemma 2.1. Then the following assertions hold.*

(i) *The map $\lambda \mapsto H(\lambda)$ is analytic, convex, and even in $\mathbb{R}$.*
(ii) *The infimum of $\frac{H(\lambda)}{\lambda}$ can be attained at some finite value in $(0, +\infty)$ provided that $H(0) > 0$.*

Proof The adjoint problem of (1.5) can be written as

$$\begin{cases} -\partial_a \psi(a,x) = D\partial_{xx}\psi(a,x) + 2D\lambda\partial_x\psi(a,x) + D\lambda^2\psi(a,x) - \mu(a,x)\psi(a,x) \\ -H(\lambda)\psi(a,x) + f_u(x,0)\beta(a,x)\psi(0,x), \qquad (a,x) \in (0, a_m) \times \mathbb{R}, \\ \psi(a,x) = \psi(a, x+L), \qquad (a,x) \in (0, a_m) \times \mathbb{R}. \end{cases} \tag{2.11}$$

Then $H(\lambda)$ is the principal eigenvalue of (2.11). Due to $\beta \equiv 0$ in $[a_c, a_m) \times \mathbb{R}$ as imposed in Assumption 1.1-(ii), problem (2.11) can be reduced to a parabolic equation on $[a_c, a_m) \times \mathbb{R}$, which admits a unique solution for the given initial value at $a = a_c$. Note from Lemma 2.1 that the principal eigenfunction of (2.11) is unique up to some multiplier. We can restrict problem (2.11) on $[0, a_c] \times \mathbb{R}$, for which the principal eigenvalue is also $H(\lambda)$ with the same eigenfunction on $[0, a_c] \times \mathbb{R}$ (Indeed, the principal eigenfunction of problem (2.11) restricted on $[0, a_c] \times \mathbb{R}$ can be uniquely extended to be the principal eigenfunction of problem (2.11)). Set

$$C^2_{\text{per}}(\mathbb{R}) := \{\phi \in C^2(\mathbb{R}) : \phi(x + L) = \phi(x)\}. \tag{2.12}$$

We define the adjoint operator

$$\mathcal{A}_\lambda : W^{1,\infty}((0, a_c), C^2_{\text{per}}(\mathbb{R})) \mapsto L^\infty((0, a_c), C_{\text{per}}(\mathbb{R})),$$

which is restricted on $[0, a_c] \times \mathbb{R}$ such that

$$\begin{aligned}[\mathcal{A}_\lambda \psi](a, x) :=& \partial_a \psi(a, x) + D\partial_{xx}\psi(a, x) + 2D\lambda\partial_x\psi(a, x) + D\lambda^2\psi(a, x)\\ &- \mu(a, x)\psi(a, x) + f_u(x, 0)\beta(a, x)\psi(0, x).\end{aligned} \tag{2.13}$$

We first show that

$$H(\lambda) = \sup_{\substack{\phi \in W^{1,\infty}((0,a_c),C^2_{\text{per}}(\mathbb{R}))\\ \phi>0}} \inf_{(0,a_c)\times\mathbb{R}} \frac{\mathcal{A}_\lambda\phi}{\phi}, \tag{2.14}$$

for which the proof can follow by Griette and Matano [24, Proposition 2.2-(ii)]. Indeed, since $\mathcal{A}_\lambda$ admits an eigenfunction ϕ_λ in $W^{1,\infty}((0, a_c), C^2_{\text{per}}(\mathbb{R}))$, it holds

$$H(\lambda) \le H^*(\lambda) := \sup_{\substack{\phi \in W^{1,\infty}((0,a_c),C^2_{\text{per}}(\mathbb{R}))\\ \phi>0}} \inf_{(0,a_c)\times\mathbb{R}} \frac{\mathcal{A}_\lambda\phi}{\phi}. \tag{2.15}$$

Let us show the converse inequality. Given any $\epsilon > 0$, by definition of $H^*(\lambda)$ there exists a positive function $\phi \in W^{1,\infty}((0, a_c), C^2_{\text{per}}(\mathbb{R}))$ such that

$$-\mathcal{A}_\lambda\phi(a, x) + (H^*(\lambda) - \epsilon)\phi(a, x) \le 0, \quad \forall (a, x) \in (0, a_c] \times \mathbb{R}. \tag{2.16}$$

Following Berestycki et al. [7], we define the generalized principal eigenvalue as follows:

$$\begin{aligned}\underline{H}(\mathcal{A}_\lambda) := \sup\big\{\Lambda \in \mathbb{R} : \exists\, \phi \in W^{1,\infty}((0, a_c), C^2_{\text{per}}(\mathbb{R}))\ \text{s.t.}\ \phi > 0\\ \text{and}\ (-\mathcal{A}_\lambda + \Lambda)\phi \le 0\ \text{in}\ [0, a_c] \times \mathbb{R}\big\}.\end{aligned}$$

As stated in Sect. 2.1, $H(\lambda)$ is the eigenvalue of $\mathcal{A}_\lambda$ associated with a positive eigenfunction in $W^{1,\infty}((0, a_c), C^2_{\text{per}}(\mathbb{R}))$. We can apply the same argument as in Ducrot et al. [14, Proposition 5.2] to deduce that $\underline{H}(\mathcal{A}_\lambda) = H(\lambda)$. Hence, it follows from

(2.16) that $H^*(\lambda) - \epsilon \le \underline{H}(\mathcal{A}_\lambda) = H(\lambda)$, so that $H^*(\lambda) \le H(\lambda)$ due to the arbitrariness of $\epsilon > 0$. This together with (2.15) gives $H^*(\lambda) = H(\lambda)$, so that the supremum is attained at the principal eigenfunction. Hence, (2.14) is proved.

For part (i), we use Kato's perturbation theory to prove the analyticity. Note that the family of operators $\mathcal{A}_\lambda$ depends analytically on λ in the sense of Kato, which is called *holomorphic of type (A)*; see [32, Paragraph 2.1 on page 375] for details. Since the principal eigenvalue is isolated in the spectrum by the Krein-Rutman theorem, the principal eigenvalue $H(\lambda)$ is analytic with respect to λ; see [32, Remark 2.9 on page 379].

Next we follow the proof of [24, Proposition 2.2] (or Nadin [47, Proposition 2.10]) to prove the convexity. We first remark that (2.14) can be written as follows:

$$H(\lambda) = \sup_{\substack{e^{-\lambda x}\psi \in W^{1,\infty}((0,a_c), C^2_{\mathrm{per}}(\mathbb{R})) \\ \psi > 0}} \inf_{(0,a_c)\times\mathbb{R}} \frac{\mathcal{A}_0 \psi}{\psi}. \tag{2.17}$$

Fix any $\lambda_2 > \lambda_1$ and $\alpha \in (0,1)$. Choose ψ_1 and ψ_2 such that $e^{\lambda_1 x}\psi_1(a,x)$ and $e^{\lambda_2 x}\psi_2(a,x)$ are L-periodic in x. Define $z_i = \ln \psi_i, i = 1, 2,$ and $z = \alpha z_1 + (1-\alpha) z_2$, and finally $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2$. Elementary computations then show that $\psi(a,x) := e^{z(a,x)}$ satisfies that

$$\frac{\partial_a \psi}{\psi} = \alpha \frac{\partial_a \psi_1}{\psi_1} + (1-\alpha)\frac{\partial_a \psi_2}{\psi_2}.$$

By the Hölder's inequality, we have

$$\begin{aligned}
\frac{\partial_{xx}\psi}{\psi} &= \partial_{xx} z + |\partial_x z|^2 \\
&= \alpha \frac{\partial_{xx}\psi_1}{\psi_1} - \alpha \frac{|\partial_x \psi_1|^2}{\psi_1^2} + (1-\alpha)\frac{\partial_{xx}\psi_1}{\psi_1} - (1-\alpha)\frac{|\partial_x \psi_2|^2}{\psi_2^2} + \left(\alpha \frac{\partial_x \psi_1}{\psi_1} + (1-\alpha)\frac{\partial_x \psi_2}{\psi_2}\right)^2 \\
&\le \alpha \frac{\partial_{xx}\psi_1}{\psi_1} + (1-\alpha)\frac{\partial_{xx}\psi_2}{\psi_2}.
\end{aligned}$$

It follows from the Young's inequality that

$$\begin{aligned}
f_u(x,0)\beta(a,x)\frac{\psi(0,x)}{\psi(a,x)} &= \left(f_u(x,0)\beta(a,x)\frac{\psi_1(0,x)}{\psi_1(a,x)}\right)^{\alpha}\left(f_u(x,0)\beta(a,x)\frac{\psi_2(0,x)}{\psi_2(a,x)}\right)^{(1-\alpha)} \\
&\le \alpha f_u(x,0)\beta(a,x)\frac{\psi_1(0,x)}{\psi_1(a,x)} + (1-\alpha) f_u(x,0)\beta(a,x)\frac{\psi_2(0,x)}{\psi_2(a,x)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{\mathcal{A}_0\psi}{\psi} &= \frac{\partial_a\psi + D\partial_{xx}\psi}{\psi} - \mu(a,x) + f_u(x,0)\beta(a,x)\frac{\psi(0,x)}{\psi(a,x)} \\ &\leq \alpha\left(\frac{\partial_a\psi_1 + D\partial_{xx}\psi_1}{\psi_1} - \mu(a,x) + f_u(x,0)\beta(a,x)\frac{\psi_1(0,x)}{\psi_1(a,x)}\right) \\ &\quad + (1-\alpha)\left(\frac{\partial_a\psi_2 + D\partial_{xx}\psi_2}{\psi_2} - \mu(a,x) + f_u(x,0)\beta(a,x)\frac{\psi_2(0,x)}{\psi_2(a,x)}\right).\end{aligned}$$

By (2.17), this implies that $H(\alpha\lambda_1 + (1-\alpha)\lambda_2) = H(\lambda) \leq \alpha H(\lambda_1) + H(1-\alpha)H(\lambda_2)$, which proves that $\lambda \mapsto H(\lambda)$ is convex.

Next set $\widehat{\phi}(a,x) := \phi(a,-x)$ for any $\phi \in W^{1,\infty}((0,a_c), C^2_{\text{per}}(\mathbb{R}))$, then

$$\partial_{xx}\phi(a,-x) + 2\lambda\partial_x\phi(a,-x) = \partial_{xx}\widehat{\phi}(a,x) - 2\lambda\partial_x\widehat{\phi}(a,x), \quad \forall (a,x) \in (0,a_c] \times \mathbb{R}.$$

It follows from the max-min characterization in (2.14) that

$$\begin{aligned} H(\lambda) &= \sup_{\substack{\phi \in W^{1,\infty}((0,a_c), C^2_{\text{per}}(\mathbb{R})) \\ \phi(a,-x)>0}} \inf_{(0,a_c)\times\mathbb{R}} \frac{\mathcal{A}_\lambda \phi(a,-x)}{\phi(a,-x)} \\ &= \sup_{\substack{\phi \in W^{1,\infty}((0,a_c), C^2_{\text{per}}(\mathbb{R})) \\ \widehat{\phi}>0}} \inf_{(0,a_c)\times\mathbb{R}} \frac{\mathcal{A}_{-\lambda}\widehat{\phi}}{\widehat{\phi}} = H(-\lambda). \end{aligned}$$

Hence, $\lambda \mapsto H(\lambda)$ is even. The proof of part (i) is complete.

For part (ii), due to $H(0) > 0$, we first observe that $H(\lambda)/\lambda \to +\infty$ as $\lambda \to 0^+$. Let us prove that $H(\lambda)/\lambda \to +\infty$ as $\lambda \to +\infty$. To this end, we consider the following eigenvalue problem

$$\begin{cases} \partial_a v(a) = D\lambda^2 v(a) - \overline{\mu}(a)v(a) - \underline{H}(\lambda)v(a), & a \in (0,a_c), \\ v(0) = \underline{f}(0)\int_0^{a_c} \underline{\beta}(a)v(a)da, \end{cases} \tag{2.18}$$

By the classical theory of age-structured operators, there exists a unique principal eigenvalue $\underline{H}(\lambda)$ of (2.18), which satisfies the following characteristic equation

$$\underline{f}(0)\int_0^{a_c} \underline{\beta}(a)e^{-\underline{H}(\lambda)a}e^{-\int_0^a \overline{\mu}(s)ds}e^{D\lambda^2 a}da = 1,$$

and the corresponding adjoint eigenfunction is denoted by $\tilde{v}(a) > 0$. Observe that $\underline{H}(\lambda)$ is increasing with respect to λ and moreover, $\underline{H}(\lambda) \approx \mathcal{O}(\lambda^2) \to +\infty$ as $\lambda \to +\infty$. Note that $(\underline{H}(\lambda), \tilde{v})$ is a test pair of the operator $\mathcal{A}_\lambda$ satisfying

$$(-\mathcal{A}_\lambda + \underline{H}(\lambda))\tilde{v} \leq 0 \quad \text{in } [0,a_c] \times \mathbb{R}.$$

It follows from (2.17) that $H(\lambda) \geq \underline{H}(\lambda)$, so that $H(\lambda)/\lambda \to +\infty$ as $\lambda \to +\infty$. Hence, $\frac{H(\lambda)}{\lambda}$ attains its minimum at some finite value. The proof is now complete. □

2.2 Connection to the generalized principal eigenvalue

In this subsection, we study the relation between the periodic principal eigenvalue of (1.3) and some generalized principal eigenvalue. Under Assumption 1.1-(ii), for any $a_+ \in [a_c, a_m)$, we rewrite the eigenvalue problem (1.3) as follows:

$$\begin{cases} \partial_a \varphi = D\partial_{xx}\varphi - \mu(a,x)\varphi - \lambda\varphi, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \varphi(0,x) = f_u(x,0)\int_0^{a_c} \beta(a,x)\varphi(a,x)da, & x \in \mathbb{R}, \\ \varphi(a,x) = \varphi(a,x+L), & (a,x) \in (0,a_+] \times \mathbb{R}, \end{cases} \tag{2.19}$$

for which the principal eigenvalue is also $H(0)$ as stated in Remark 2.1. Next consider the eigenvalue problem

$$\begin{cases} \partial_a \phi = D\partial_{xx}\phi - \mu(a,x)\phi - \lambda\phi & \text{in } (0,a_+] \times \mathbb{R} \\ \phi(0,x) = f_u(x,0)\int_0^{a_c} \beta(a,x)\phi(a,x)\,da & \text{in } \mathbb{R}. \end{cases} \tag{2.20}$$

The generalized principal eigenvalue associated with (2.20) is defined as follows:

$$\lambda_1 := \inf \left\{ \begin{array}{c} \lambda : \exists\, \phi \in W^{1,1}((0,a_+),C^2(\mathbb{R})) \text{ s.t. } \phi > 0 \text{ in } [0,a_+] \times \mathbb{R}, \\ \partial_a\phi - D\partial_{xx}\phi + \mu\phi \geq -\lambda\phi \text{ in } [0,a_+] \times \mathbb{R}, \\ \text{and } \phi(0,x) \geq f_u(x,0)\int_0^{a_c} \beta(a,x)\phi(a,x)\,da \end{array} \right\}. \tag{2.21}$$

Lemma 2.3 *The generalized principal eigenvalue λ_1 in (2.21) is well-defined and $\lambda_1 < +\infty$.*

Proof Let $\underline{\lambda}$ be the principal eigenvalue of (2.20) on any fixed bounded interval with Dirichlet boundary condition. By the maximum principle, it can be verified that $\lambda_1 \geq \underline{\lambda} > -\infty$, so that λ_1 is well-defined. We next show that $\lambda_1 < +\infty$. Consider the homogeneous eigenvalue problem

$$\begin{cases} \partial_a \phi(a) = -\underline{\mu}(a)\phi(a) - \lambda\phi(a), & a \in (0,a_+], \\ \phi(0) = \overline{f}(0)\int_0^{a_c} \overline{\beta}(a)\phi(a)da, \end{cases} \tag{2.22}$$

where we used the notations in (2.1). By the classical theory of age-structured models, there exists a unique principal eigenvalue $\overline{\lambda} \in \mathbb{R}$ of (2.22). We denote by $\overline{\phi} = \overline{\phi}(a) > 0$ the associated principal eigenfunction. Then it follows from (2.22) that

$$\begin{cases} \partial_a \overline{\phi} \geq D\partial_{xx}\overline{\phi} - \mu(a,x)\overline{\phi} - \overline{\lambda\phi} & \text{in } (0,a_+] \times \mathbb{R} \\ \overline{\phi}(0) \geq f_u(x,0) \int_0^{a_c} \beta(a,x)\overline{\phi}(a)\, da & \text{in } \mathbb{R}. \end{cases}$$

Choosing $\overline{\phi}$ as a test function in (2.21), we find $\lambda_1 \leq \overline{\lambda} < +\infty$, which concludes the proof. □

We next prove a type of Harnack inequality for problem (2.20).

Proposition 2.4 *Assume that (λ, ϕ) is the principal eigenpair of (2.20) with $\phi > 0$. Then for any $R > 0$ and $a_+ \in [a_c, a_m)$, there exists some $C_{R,a_+} > 0$ depending only on R and a_+ such that*

$$\sup_{(a,x)\in[0,a_+]\times[-R,R]} \phi(a,x) \leq C_{R,a_+} \inf_{(a,x)\in[0,a_+]\times[-R,R]} \phi(a,x).$$

Proof For each R, a_+ and $\eta > 0$, applying the classical Harnack inequality for parabolic equations, there exists $C_{R,a_+,\eta} > 0$ such that

$$\sup_{(a,x)\in[\eta,a_+]\times[-R,R]} \phi(a,x) \leq C_{R,a_+,\eta} \inf_{(a,x)\in[\eta,a_+]\times[-R,R]} \phi(a,x).$$

By considering a super-solution of (2.20) defined by $\|\phi(0,\cdot)\|_{L^\infty(-R,R)}\, e^{-(\mu_{\inf}+\lambda)a}$, we have

$$\|\phi(a,\cdot)\|_{L^\infty(-R,R)} \leq \|\phi(0,\cdot)\|_{L^\infty(-R,R)}\, e^{-(\mu_{\inf}+\lambda)a}, \quad \forall a \in [0,a_+].$$

Hence, there exists some constant $C_{R,a_+} > 0$, depending only on μ, β, λ and f_u (which may change from line to line but is always independent of $(a,x) \in [0,a_+] \times [-R,R]$), such that

$$\begin{aligned} \sup_{(a,x)\in[0,a_+]\times[-R,R]} \phi(a,x) \leq\ & C_{R,a_+} \sup_{x\in[-R,R]} \phi(0,x) \\ \leq\ & C_{R,a_+} \int_\eta^{a_c} \overline{\beta}(a) \sup_{x\in[-R,R]} \phi(a,x) da \\ & + C_{R,a_+} \sup_{(a,x)\in[0,a_c]\times[-R,R]} \phi(a,x) \int_0^\eta \overline{\beta}(a) da. \end{aligned}$$

We choose $\eta > 0$ sufficiently small such that

$$\begin{aligned} \sup_{(a,x)\in[0,a_+]\times[-R,R]} \phi(a,x) &\leq C_{R,a_+} \int_\eta^{a_c} \overline{\beta}(a) \sup_{x\in[-R,R]} \phi(a,x) da \\ &\leq C_{R,a_+} \inf_{(a,x)\in[\eta,a_+]\times[-R,R]} \phi(a,x). \end{aligned} \tag{2.23}$$

This implies that

$$\begin{aligned}\inf_{x\in[-R,R]}\phi(0,x) &\geq \underline{f}(0)\int_0^{a_c}\underline{\beta}(a)\inf_{x\in[-R,R]}\phi(a,x)da\\ &\geq \underline{f}(0)\int_\eta^{a_c}\underline{\beta}(a)\inf_{(a,x)\in[\eta,a_+]\times[-R,R]}\phi(a,x)da\\ &\geq C_{R,a_+}\sup_{(a,x)\in[0,a_+]\times[-R,R]}\phi(a,x).\end{aligned}\tag{2.24}$$

Similarly, by choosing $e^{-(\mu_{\sup}+\lambda)a}\inf_{x\in[-R,R]}\phi(0,x)$, with $\mu_{\sup}=\|\overline{\mu}\|_{L^\infty(0,a_+)}$, as a sub-solution of (2.20), we can derive from (2.24) that

$$\begin{aligned}\inf_{(a,x)\in[0,\eta]\times[-R,R]}\phi(a,x) &\geq e^{-(\mu_{\sup}+\lambda)\eta}\inf_{x\in[-R,R]}\phi(0,x)\\ &\geq C_{R,a_+}\sup_{(a,x)\in[0,a_+]\times[-R,R]}\phi(a,x).\end{aligned}\tag{2.25}$$

Finally, combining (2.23) and (2.25), we obtain that

$$\sup_{(a,x)\in[0,a_+]\times[-R,R]}\phi(a,x)\leq C_{R,a_+}\inf_{(a,x)\in[0,a_+]\times[-R,R]}\phi(a,x).$$

This completes the proof. □

Next we show that the generalized principal eigenvalue defined by (2.21) is indeed the principal eigenvalue of (2.20) corresponding to a positive eigenfunction.

Lemma 2.5 *Let λ_1 be the generalized principal eigenvalue defined in (2.21). Then for any $a_+\in[a_c,a_m)$, there exists a positive eigenfunction in $W^{1,1}((0,a_+),C^2(\mathbb{R}))$ of problem (2.20) associated with λ_1 and*

$$\lambda_1=\lim_{R\to+\infty}\lambda_R,$$

where for each $R>0$, λ_R denotes the principal eigenvalue of the problem

$$\begin{cases}\partial_a\phi=D\partial_{xx}\phi-\mu(a,x)\phi-\lambda\phi & in\ \ (0,a_+]\times(-R,R),\\ \phi(0,x)=f_u(x,0)\int_0^{a_c}\beta(a,x)\phi(a,x)\,da & on\ \ (-R,R),\\ \phi(a,-R)=\phi(a,R)=0 & in\ \ (0,a_+].\end{cases}\tag{2.26}$$

Proof By the same argument as in Lemma 2.1 and the subsequent Remark 2.1, one can see that λ_R exists and is independent of the choice of $a_+\in[a_c,a_m)$. Recalling the definition of λ_1 in (2.21), by the maximum principle it is easily seen that $\lambda_R\leq\lambda_1$ for all $R>0$. Since the principal eigenvalue λ_R of (2.26) is nondecreasing in R, λ_R converges as $R\to+\infty$ and

$$\lambda_\infty:=\lim_{R\to+\infty}\lambda_R\leq\lambda_1.\tag{2.27}$$

Fix any $(a_0, x_0) \in (0, a_+) \times (-R, R)$. Let $\phi_R > 0$ be the principal eigenfunction of (2.26) associated with λ_R, which is normalized by $\phi_R(a_0, x_0) = 1$. Note that λ_R is uniformly bounded in $R > 0$ by (2.27) and Lemma 2.3. Applying the Harnack inequality in Proposition 2.4, we derive that for any $R_0 > 0$, there exists some constant $C_{R_0,a_+} > 0$ such that

$$\begin{aligned} \sup_{(a,x)\in[0,a_+]\times[-R_0,R_0]} \phi_R(a,x) &\leq C_{R_0,a_+} \inf_{(a,x)\in[0,a_+]\times[-R_0,R_0]} \phi_R(a,x) \\ &\leq C_{R_0,a_+}, \quad \forall R > R_0. \end{aligned} \tag{2.28}$$

By the standard Schauder estimates for parabolic equations, we deduce that for all R_0, by passing a subsequence if necessary, ϕ_R converges in $C^{1,2}([0, a_+] \times [-R_0, R_0])$ to some function $\phi_\infty \in C^{1,2}([0, a_+] \times \mathbb{R})$, which satisfies

$$\begin{cases} \partial_a \phi_\infty = D\partial_{xx}\phi_\infty - \mu(a,x)\phi_\infty - \lambda_\infty \phi_\infty & \text{in} \quad (0, a_+] \times (-R_0, R_0), \\ \phi_\infty(0,x) = f_u(x,0) \int_0^{a_c} \beta(a,x)\phi_\infty(a,x)\, da & \text{on} \quad (-R_0, R_0). \end{cases} \tag{2.29}$$

Using a diagonal extraction method, we can find a particular subsequence of $\{\phi_R\}_{R>R_0}$ converging to ϕ_∞ in $C^{1,2}_{\text{loc}}([0, a_+] \times \mathbb{R})$. Furthermore, $\phi_\infty(a_0, x_0) = 1$, $\phi_\infty \geq 0$, and ϕ_∞ satisfies (2.20) with $\lambda = \lambda_\infty$. Then the strong maximum principle yields that $\phi_\infty > 0$ in $[0, a_+] \times \mathbb{R}$. Choosing ϕ_∞ as a test function in (2.21), we find $\lambda_1 \leq \lambda_\infty$, which together with (2.27) implies $\lambda_1 = \lambda_\infty$, namely $\lambda_R \to \lambda_1$ as $R \to +\infty$. Hence, ϕ_∞ serves as a principal eigenfunction of (2.20) associated with λ_1. The proof is now complete. □

We conclude this section by showing that the principal eigenvalue with periodic boundary condition is the limit of the principal eigenvalue with Dirichlet boundary condition.

Proposition 2.6 *Let $H(0)$ be the principal eigenvalue of (2.19). Then there holds*

$$H(0) = \lambda_1 = \lim_{R\to+\infty} \lambda_R,$$

where λ_1 is defined by (2.21) and λ_R is the principal eigenvalue of problem (2.26).

Proof It is proved by Lemma 2.5 that $\lambda_R \to \lambda_1$ as $R \to +\infty$. Comparing problems (2.26) with (2.19), it can be verified by the maximum principle that $H(0) \geq \lambda_R$ for all $R > 0$. Hence, it remains to show $\lambda_1 \geq H(0)$.

To this end, for any $\alpha \in (0, a_c)$, consider the following perturbed eigenvalue problem

$$\begin{cases} \partial_a \varphi = D\partial_{xx}\varphi - \mu(a,x)\varphi - \lambda\varphi & \text{in } (0, a_+] \times \mathbb{R}, \\ \varphi(0,x) = f_u(x,0) \int_\alpha^{a_c} \beta(a,x)\varphi(a,x)\, da & \text{in } \mathbb{R}. \end{cases} \tag{2.30}$$

Similar to the definition of λ_1 in (2.21), we denote by $\lambda_{1\alpha}$ the generalized principal eigenvalue of (2.30), which is given by

$$\lambda_{1\alpha} := \inf \left\{ \begin{array}{c} \lambda : \exists\, \phi \in W^{1,1}((0,a_+), C^2(\mathbb{R})) \text{ s.t. } \phi > 0, \text{ in } [0,a_+] \times \mathbb{R}, \\ \partial_a \phi - D\partial_{xx}\phi + \mu\phi \geq -\lambda\phi \text{ in } [0,a_+] \times \mathbb{R}, \\ \text{and } \phi(0,x) \geq f_u(x,0) \int_\alpha^{a_c} \beta(a,x)\phi(a,x)\, da \end{array} \right\}. \quad (2.31)$$

By Lemma 2.5, there exists a principal eigenfunction $\varphi > 0$ associated with $\lambda_{1,\alpha}$, that is φ solves (2.30) with $\lambda = \lambda_{1,\alpha}$. We first claim that $\alpha \mapsto \lambda_{1\alpha}$ is nonincreasing. Indeed, choose any $0 \leq \alpha_1 \leq \alpha_2$ and $\lambda > \lambda_{1\alpha_1}$. By the above definition, there exists $0 < \phi \in W^{1,1}((0,a_+), C^2(\mathbb{R}))$ such that $\partial_a \phi - D\partial_{xx}\phi + \mu\phi > -\lambda\phi$ in $[0,a_+] \times \mathbb{R}$ and

$$\phi(0,x) \geq f_u(x,0) \int_{\alpha_1}^{a_c} \beta(a,x)\phi(a,x)da \geq f_u(x,0) \int_{\alpha_2}^{a_c} \beta(a,x)\phi(a,x)da,$$

which implies that (λ, ϕ) is a test eigenpair in (2.31) with $\alpha = \alpha_2$. Hence, $\lambda \geq \lambda_{1\alpha_2}$ follows by definition. Since this holds for any $\lambda > \lambda_{1\alpha_1}$, we arrive at $\lambda_{1\alpha_1} \geq \lambda_{1\alpha_2}$. This proves the monotonicity of $\alpha \mapsto \lambda_{1\alpha}$.

Next set $\psi(a,x) := \frac{\varphi(a,x+L)}{\varphi(a,x)}$. Applying the classical Harnack inequality to (2.30) yields that ψ is globally bounded. Define $m := \sup_{[0,a_+]\times\mathbb{R}} \psi > 0$. We choose the sequence $(a_n, x_n) \in [0,a_+] \times \mathbb{R}$ such that $\psi(a_n,x_n) \to m$ as $n \to \infty$ and there exists $y_n \in [0,L]$ such that $x_n - y_n \in L\mathbb{Z}$ for all n. One may assume that $y_n \to y_\infty \in [0,L]$ and $a_n \to a_\infty \in [0,a_+]$.

Set $\psi_n(a,x) := \psi(a,x+x_n)$ and $\phi_n(a,x) := \frac{\varphi(a,x+x_n)}{\varphi(0,x_n)}$. By (2.30) we have

$$\begin{cases} \partial_a \phi_n = D\partial_{xx}\phi_n - \mu(a,x+y_n)\phi_n - \lambda_{1\alpha}\phi_n, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \phi_n(0,x) = f_u(x+y_n,0) \int_\alpha^{a_c} \beta(a,x+y_n)\phi_n(a,x)da, & x \in \mathbb{R}. \end{cases}$$

Using the classical parabolic estimates, we may suppose, up to extraction, that $\phi_n \to \phi_\infty$ in $C^{1,2}_{\text{loc}}([0,a_+] \times \mathbb{R})$ and function ϕ_∞ satisfies

$$\begin{cases} \partial_a \phi_\infty = D\partial_{xx}\phi_\infty - \mu(a,x+y_\infty)\phi_\infty - \lambda_{1\alpha}\phi_\infty, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \phi_\infty(0,x) = f_u(x+y_\infty,0) \int_\alpha^{a_c} \beta(a,x+y_\infty)\phi_\infty(a,x)da, & x \in \mathbb{R}, \\ \phi_\infty > 0, \quad \phi_\infty(0,0) = 1. \end{cases} \quad (2.32)$$

On the other hand, by the definition of ψ_n, direct calculations from (2.30) yield

$$\partial_a \psi_n = D\partial_{xx}\psi_n - 2\frac{\partial_x \phi_n(a,x)}{\phi_n(a,x)}\partial_x \psi_n, \quad (a,x) \in (0,a_+] \times \mathbb{R},$$

and for all $x \in \mathbb{R}$,

$$\begin{aligned}\psi_n(0,x) &= \frac{\varphi(0,x+x_n+L)}{\varphi(0,x+x_n)}\\ &= \frac{\int_\alpha^{a_c}\beta(a,x+y_n)\varphi\left(a,x+x_n+L\right)da}{\int_\alpha^{a_c}\beta(a,x+y_n)\varphi\left(a,x+x_n\right)da}\\ &= \frac{\int_\alpha^{a_c}\beta(a,x+y_n)\varphi\left(a,x+x_n\right)\psi_n(a,x)da}{\int_\alpha^{a_c}\beta(a,x+y_n)\varphi\left(a,x+x_n\right)da}.\end{aligned} \tag{2.33}$$

Applying the classical estimates for parabolic equations once again yields that up to extraction, $\psi_n \to \psi_\infty$ in $C^{1,2}_{\text{loc}}([0,a_+]\times\mathbb{R})$, where ψ_∞ satisfies

$$\partial_a\psi_\infty = D\partial_{xx}\psi_\infty - 2\frac{\partial_x\phi_\infty}{\phi_\infty}\partial_x\psi_\infty, \quad (a,x)\in(0,a_+]\times\mathbb{R}. \tag{2.34}$$

Furthermore, $\psi_\infty \le m$ and $\psi_n(a_n,0)=\psi(a_n,x_n)\to m$ as $n\to\infty$, which implies $\psi_\infty(a_\infty,0)=m$.

We claim that the sequence $(a_n,x_n)\in[0,a_+]\times\mathbb{R}$ given above can be chosen such that $a_\infty>0$. If not, then $\psi_\infty(0,0)=m$ and $\psi_\infty(a,0)<m$ for all $a\in(0,a_+]$. Thus, by (2.33) one has

$$\begin{aligned} m &= \psi_\infty(0,0)\\ &= \lim_{n\to+\infty}\frac{\int_\alpha^{a_c}\beta(a,y_n)\varphi\left(a,x_n\right)\psi_n(a,0)da}{\int_\alpha^{a_c}\beta(a,y_n)\varphi\left(a,x_n\right)da}\\ &\le \lim_{n\to+\infty}\left[\frac{\int_\alpha^{a_c}\beta(a,y_n)\varphi\left(a,x_n\right)\psi_\infty(a,0)da}{\int_\alpha^{a_c}\beta(a,y_n)\varphi\left(a,x_n\right)da} + \|\psi_n(a,0)-\psi_\infty(a,0)\|_{L^\infty([\alpha,a_c])}\right]\\ &= \lim_{n\to+\infty}\frac{\int_\alpha^{a_c}\beta(a,y_n)\varphi\left(a,x_n\right)\psi_\infty(a,0)da}{\int_\alpha^{a_c}\beta(a,y_n)\varphi\left(a,x_n\right)da}\\ &\le \max_{a\in[\alpha,a_c]}\psi_\infty(a,0) < m,\end{aligned}$$

which is a contradiction. Hence, we may assume $a_\infty>0$.

Therefore, we apply the strong maximum principle to (2.34) and obtain $\psi_\infty(a,x)\equiv m$ for all $(a,x)\in(0,a_+]\times\mathbb{R}$. By definitions, we note that $\phi_n(a,x+L)/\phi_n(a,x)=\psi_n(a,x)$, and thus $\phi_\infty(a,x+L)/\phi_\infty(a,x)\equiv m$ for all $(a,x)\in(0,a_+]\times\mathbb{R}$. As $m>0$, we can define $\gamma := \frac{1}{L}\ln m$. Then the function $\tilde\phi_\infty := \phi_\infty\exp(-\gamma x)$ is L-periodic in x. By (2.32) we calculate that $\tilde\phi_\infty$ solves

$$\begin{cases} \partial_a\tilde\phi_\infty = D\partial_{xx}\tilde\phi_\infty - 2D\gamma\partial_x\tilde\phi_\infty + D\gamma^2\tilde\phi_\infty - \mu(a,x+y_\infty)\tilde\phi_\infty - \lambda_{1\alpha}\tilde\phi_\infty, & (a,x)\in(0,a_+]\times\mathbb{R},\\ \tilde\phi_\infty(0,x) = f_u(x+y_\infty,0)\int_\alpha^{a_c}\beta(a,x+y_\infty)\tilde\phi_\infty(a,x)da, & x\in\mathbb{R},\\ \tilde\phi_\infty(a,x) = \tilde\phi_\infty(a,x+L), & (a,x)\in(0,a_+]\times\mathbb{R}.\end{cases}$$

Since the periodic principal eigenvalue is invariant under a translation in x of the coefficients, it holds that $\lambda_{1\alpha} = H_\alpha(\gamma)$, where for any $\lambda \in \mathbb{R}$, $H_\alpha(\lambda)$ denotes the principal eigenvalue of

$$\begin{cases} \partial_a\varphi = D\partial_{xx}\varphi - 2D\lambda\partial_x\varphi + D\lambda^2\varphi - \mu(a,x)\varphi - H_\alpha(\lambda)\varphi, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \varphi(0,x) = f_u(x,0)\int_\alpha^{a_c} \beta(a,x)\varphi(a,x)da, & x \in \mathbb{R}, \\ \varphi(a,x) = \varphi(a,x+L), & (a,x) \in (0,a_+] \times \mathbb{R}. \end{cases} \tag{2.35}$$

For all $\lambda \in \mathbb{R}$, let $\varphi_\lambda > 0$ be the principal eigenfunction of (2.35) associated with $H_\alpha(\lambda)$. Then the function $v_\lambda := \varphi_\lambda e^{-\lambda x}$ satisfies the problem (2.30) with the eigenvalue $H_\alpha(\lambda)$. Taking v_λ as a test function in (2.31), one finds $\lambda_{1\alpha} \le H_\alpha(\lambda)$ for all $\lambda \in \mathbb{R}$. This together with $\lambda_{1\alpha} = H_\alpha(\gamma)$ implies that $\lambda_{1\alpha} = \min_{\lambda\in\mathbb{R}} H_\alpha(\lambda)$. Note that $H_\alpha(\lambda) \to H(\lambda)$ as $\alpha \to 0^+$, where $H(\lambda)$ denotes the principal eigenvalue of (1.5). Due to $\lambda_1 \ge \lambda_{1\alpha}$, letting $\alpha \to 0^+$ yields $\lambda_1 \ge \min_{\lambda\in\mathbb{R}} H(\lambda)$. Since $\lambda \mapsto H(\lambda)$ is convex and even by Proposition 1.6-(ii), one has $H(0) = \min_{\lambda\in\mathbb{R}} H(\lambda)$, and thus $\lambda_1 \ge H(0)$. The proof is now complete. □

3 Global dynamics

In this section, we are concerned with the global dynamics of problem (1.1) and prove Theorem 1.1. First we show that the solution of (1.1) exists globally in time and provide the weak comparison principle.

Lemma 3.1 *Given any bounded and nonnegative initial value u_0, problem (1.1) admits a unique global solution $u(t,a,x)$, which is nonnegative and uniformly bounded in $[0,a_m) \times \mathbb{R}$ for all $t > 0$. Furthermore, let u and v be the solutions of problem (1.1) with initial data u_0 and v_0, respectively. If $u_0 \ge v_0$, then $u(t,a,x) \ge v(t,a,x)$ for all $t > 0$ and $(a,x) \in [0,a_m) \times \mathbb{R}$.*

Proof Let us first modify the definitions of $\mathcal{X}_0$ and $\mathcal{X}$ in the proof of Lemma 2.1 without introducing more notations. To consider the general initial data (which may not be periodic), we define $X = C_b(\mathbb{R})$ denoting the space of continuous and bounded functions and modify $\mathcal{X}$ and $\mathcal{X}_0$ correspondingly. Define the map $F : \mathcal{X}_0 \mapsto \mathcal{X}$ such that

$$F(0,\psi) = \left(f\left(\cdot, \int_0^{a_c} \beta(a,\cdot)\psi(a)da\right), 0\right), \quad \forall\,(0,\psi) \in \mathcal{X}_0.$$

Observe that F is well-defined due to Assumptions 1.1 and 1.2. Let $u(t) = u(t,a,x)$ be the solution of (1.1) with initial value u_0. By identifying $U(t) = (0,u(t))$, one can rewrite problem (1.1) as the following abstract problem:

$$\begin{cases} \frac{dU}{dt} = BU + F(U), \\ U(0) = U_0, \end{cases} \quad \text{with } U_0 = (0,u_0) \in \mathcal{X}_0, \tag{3.1}$$

where $B : dom(B) \subset \mathcal{X} \mapsto \mathcal{X}$ is the unique closed Hille-Yosida operator defined in the same manner as that in the proof of Lemma 2.1 (Note that the Hille-Yosida estimate (2.5) also holds for $C_b(\mathbb{R})$). It follows from Assumption 1.2 that f is Lipschitz continuous, so that the solution U of (3.1) exists globally by a standard semigroup method, see Thieme [56] or Magal and Ruan [41].

Next, thanks to the definition of B, there holds that B is resolvent positive. Moreover, F is monotone due to Assumption 1.2-(ii), i.e. $0 \le U \le V \Rightarrow 0 \le F(U) \le F(V)$. Thus, by Magal et al. [42, Theorem 4.5], we can conclude that the weak comparison principle holds for (3.1).

Finally, we show the uniform boundedness of the unique solution u of (1.1). Indeed, set $V(t,x) := \int_0^{a_m} u(t,a,x)da$. Due to $u(t,a_m,x)=0$, by (1.1) we calculate that

$$\partial_t V = D\partial_{xx}V - \int_0^{a_m} \mu(a,x)u(t,a,x)da + f\Big(x, \int_0^{a_c} \beta(a,x)u(t,a,x)da\Big), \quad \forall t>0,\ x\in\mathbb{R},$$

and $V(0,x) = \int_0^{a_m} u_0(a,x)da$. It follows by Assumption 1.1-(i) and Assumption 1.2-(iii) that

$$\partial_t V \le D\partial_{xx}V - \mu_{\inf} V + M, \qquad \forall t>0,\ x\in\mathbb{R}.$$

Since $V(0,x)$ is bounded, by the comparison principle it is easily seen that $V(t,x)$ is uniformly bounded for all $x \in \mathbb{R}$ and $t>0$, which in turn implies the uniform boundedness of u. The proof is complete. □

We also present the strong comparison principle for (1.1), for which the proof is omitted. The interested readers can refer to Ducrot et. al. [15] for more details.

Lemma 3.2 *(Strong Comparison Principle) Assume that $u_0 \in C([0,a_m)\times\mathbb{R})$ and $u_0(a,x) \ge 0$ but $u_0(a,x) \not\equiv 0$. Let $u(t,a,x)$ be the unique solution to (1.1) established in Lemma 3.1. Then $u(t,a,x) > 0$ for any $t>0$ and $(a,x)\in[0,a_m)\times\mathbb{R}$.*

Under Assumption 1.1-(ii), the stationary equation of (1.1) can be rewritten as

$$\begin{cases} \partial_a u(a,x) = D\partial_{xx}u(a,x) - \mu(a,x)u(a,x), & (a,x)\in(0,a_m)\times\mathbb{R}, \\ u(0,x) = f\left(x, \int_0^{a_c}\beta(a,x)u(a,x)da\right), & x\in\mathbb{R}. \end{cases} \tag{3.2}$$

To proceed further, we present the definitions of sub-solutions and super-solutions to the stationary equation (3.2), where the periodicity assumption is not imposed a priori.

Definition 3.3 Function $u \in W^{1,1}((0,a_m), C^2(\mathbb{R}))$ is called as a *sub-solution* to (3.2) if

$$\begin{cases} \partial_a u(a,x) \le D\partial_{xx}u(a,x) - \mu(a,x)u(a,x), & (a,x)\in(0,a_m)\times\mathbb{R}, \\ u(0,x) \le f\left(x, \int_0^{a_c}\beta(a,x)u(a,x)da\right), & x\in\mathbb{R}. \end{cases} \tag{3.3}$$

Similarly, $u \in W^{1,1}((0,a_m), C^2(\mathbb{R}))$ is called as a *super-solution* if "$\leq$" in (3.3) is replaced by "$\geq$".
We first establish a comparison principle for (3.2).

Lemma 3.4 *Let $u, v \in W^{1,1}((0,a_m), C^2(\mathbb{R}))$ be respectively a sub-solution and super-solution of (3.2) as defined in Definition 3.3. Assume that for any $a_+ \in [a_c, a_m)$,*

$$\inf_{(a,x)\in[0,a_+]\times\mathbb{R}} u(a,x) > 0 \quad \text{and} \quad \inf_{(a,x)\in[0,a_+]\times\mathbb{R}} v(a,x) > 0. \tag{3.4}$$

Then $u \leq v$ in $[0,a_m)\times\mathbb{R}$.

Proof Set $\alpha_* := \sup\{\alpha > 0 : \alpha u \leq v \text{ in } [0,a_+]\times\mathbb{R}\}$. By (3.4), the number α_* is well-defined and positive. If $\alpha_* \geq 1$, then Lemma 3.4 follows due to the arbitrariness of $a_+ \in [a_c, a_m)$. It remains to consider the case $\alpha_* < 1$. Set $w := v - \alpha_* u$, then $w \geq 0$ in $[0,a_+]\times\mathbb{R}$. Denote $a_0 := \min\{a \in [0,a_+] : \exists x \in \mathbb{R}, \text{ s.t. } w(a_0,x) = 0\}$. It follows from the definition of α_* that there exists $x_0 \in \mathbb{R}$ such that $w(a_0,x_0) = 0$.

We first consider the case $a_0 \in (0,a_+]$. Observe that *w* satisfies

$$\partial_a w(a,x) \geq D\partial_{xx} w(a,x) - \mu(a,x)w(a,x), \quad (a,x) \in (0,a_+]\times\mathbb{R}.$$

Considering the above inequality at (a_0,x_0), we can reach a contradiction by applying the strong maximum principle for parabolic equations.

We next consider the case $a_0 = 0$, namely $w(0,x_0) = 0$. By Assumption 1.1-(ii), we have

$$\int_0^{a_c} \beta(a,x_0)u(a,x_0)da > 0. \tag{3.5}$$

Then it follows by Assumption 1.2-(ii) that

$$\begin{aligned} w(0,x_0) &= v(0,x_0) - \alpha_* u(0,x_0) \\ &\geq f\left(x_0, \int_0^{a_c} \beta(a,x_0)v(a,x_0)da\right) - \alpha_* f\left(x_0, \int_0^{a_c} \beta(a,x_0)u(a,x_0)da\right) \\ &> f\left(x_0, \int_0^{a_c} \beta(a,x_0)v(a,x_0)da\right) - f\left(x_0, \alpha_* \int_0^{a_c} \beta(a,x_0)u(a,x_0)da\right) \\ &\geq 0, \end{aligned} \tag{3.6}$$

where we used (3.5) and $\alpha_* < 1$ for the strict inequality. It is a contradiction with the fact that $w(0,x_0) = 0$. Therefore, $\alpha_* \geq 1$ and the proof is complete. □

Next we give the existence and uniqueness of the positive equilibrium of (3.2).

Proposition 3.5 *(Existence and Uniqueness) Assume $H(0) > 0$, then there exists a unique positive solution $u^*(a,x)$ of (3.2) belonging to $W^{1,1}((0,a_m), C^2_{\text{per}}(\mathbb{R}))$.*

Proof It suffices to prove the existence of positive L-periodic solution of (3.2), since the uniqueness is a direct consequence of the comparison principle in Lemma 3.4. The existence can be proved by the following two steps.

Step 1. Construction of super/sub-solutions. Set $\overline{u} \equiv M$ with $M > 0$ being defined in Assumption 1.2-(iii). Note that

$$\partial_a \overline{u} - D\partial_{xx}\overline{u} + \mu(a,x)\overline{u} = M\mu(a,x) \geq 0, \quad \forall (a,x) \in (0,a_m) \times \mathbb{R},$$

as well as

$$\overline{u}(0,x) = M \geq f\left(x, \int_0^{a_c} \beta(a,x)\overline{u}(a,x)da\right).$$

Hence, $\overline{u} \equiv M$ is indeed a super-solution of (3.2).

Next, we construct a sub-solution of (3.2). For any $\delta > 0$ sufficiently small, by Assumption 1.2 we can find some constant $\epsilon = \epsilon(\delta) > 0$ such that

$$f(x,u) \geq (f_u(x,0) - \delta)u \quad \text{for all} \;\; 0 < u \leq \epsilon \text{ and } x \in \mathbb{R}. \tag{3.7}$$

For any $a_+ \in [a_c, a_m)$, let $H_\delta(0) \in \mathbb{R}$ be the principal eigenvalue of the problem

$$\begin{cases} \partial_a \phi = D\partial_{xx}\phi - \mu(a,x)\phi - H_\delta(0)\phi, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \phi(0,x) = (f_u(x,0) - \delta)\int_0^{a_c} \beta(a,x)\phi(a,x)da, & x \in \mathbb{R}, \\ \phi(a,x) = \phi(a,x+L), & (a,x) \in (0,a_+] \times \mathbb{R}, \end{cases} \tag{3.8}$$

and the corresponding eigenfunction is denoted by $\phi_\delta > 0$, which can be normalized such that $\|\phi_\delta\|_{L^\infty((0,a_+)\times(0,L))} = 1$. By the continuity of $H_\delta(0)$ in δ, we have $H_\delta(0) \to H(0)$ as $\delta \to 0$. Hence, due to $H(0) > 0$ one can choose δ further small if necessary such that $H_\delta(0) > 0$.

Set $\underline{u}(a,x) := \epsilon\phi_\delta(a,x)$, so that $\underline{u} \leq \epsilon$. Then by (3.7) and (3.8) it can be verified that

$$\partial_a \underline{u} - D\partial_{xx}\underline{u} + \mu(a,x)\underline{u} = -H_\delta(0)\underline{u} \leq 0 \quad \text{in} \;\; (0,a_+] \times \mathbb{R},$$

and for all $x \in \mathbb{R}$,

$$\underline{u}(0,x) = (f_u(x,0) - \delta)\int_0^{a_c} \beta(a,x)\underline{u}(a,x)da \leq f\left(x, \int_0^{a_c} \beta(a,x)\underline{u}(a,x)da\right).$$

This implies that the constructed $\underline{u}$ is a L-periodic sub-solution of (3.2).

Step 2. Existence via iterative scheme. We choose ϵ small such that $\underline{u} \leq \overline{u}$. By a basic iterative scheme, we can establish the existence of a positive nontrivial solution u of (3.2). For completeness, we provide the iterative scheme as follows.

Set $u_0 := \underline{u}$ and denote u_n for $n \geq 1$ by the solution of the linear problem

$$\begin{cases} \partial_a u_n(a,x) = D\partial_{xx} u_n(a,x) - \mu(a,x)u_n(a,x), & (a,x) \in (0,a_+] \times \mathbb{R}, \\ u_n(0,x) = f\left(x, \int_0^{a_c} \beta(a,x)u_{n-1}(a,x)da\right), & x \in \mathbb{R}. \end{cases} \tag{3.9}$$

Note that u_n is well-defined and is belonging to $W^{1,1}((0,a_+), C^2_{\text{per}}(\mathbb{R}))$ (where the periodicity follows by that of $\underline{u}$ and $\overline{u}$). We will show that

$$\underline{u} \le u_1 \le \cdots \le u_n \le \cdots \le \overline{u} \quad \text{in} \quad [0,a_+] \times \mathbb{R}. \tag{3.10}$$

Indeed, taking $w := u_1 - \underline{u}$, it follows from Assumption 1.2-(ii) that

$$\partial_a w \ge D\partial_{xx} w - \mu(a,x)w, \quad \forall (a,x) \in (0,a_+] \times \mathbb{R}, \qquad w(0,x) \ge 0, \quad \forall x \in \mathbb{R}.$$

Using the comparison principle for parabolic equations, we conclude that $w \ge 0$, that is $u_1 \ge \underline{u}$ in $[0,a_+] \times \mathbb{R}$. Similarly, we can derive $u_1 \le \overline{u}$ in $[0,a_+] \times \mathbb{R}$. By induction, we can obtain the desired result (3.10).

Hence, for each $(a,x) \in [0,a_+] \times \mathbb{R}$, $u_n(a,x)$ has a limit as $n \to \infty$, denoted by $u^*(a,x)$, namely $u_n(a,x) \to u^*(a,x)$ as $n \to \infty$. Due to $\mu \in C^{\frac{\alpha}{2},\alpha}([0,a_+] \times \mathbb{R})$, by the classical parabolic estimates, we derive that $u_n \to u^*$ in $C^{1,2}_{\text{loc}}([0,a_+] \times \mathbb{R})$ as $n \to \infty$ and $u^* \in W^{1,1}((0,a_+), C^2(\mathbb{R}))$. Moreover, the continuity of f yields that for any $x \in \mathbb{R}$,

$$f\left(x, \int_0^{a_c} \beta(a,x)u_n(a,x)da\right) \to \left(x, \int_0^{a_c} \beta(a,x)u^*(a,x)da\right) \quad \text{as} \quad n \to \infty.$$

Due to the arbitrariness of $a_+ \in [a_c, a_m)$, we see that $u^* \in W^{1,1}((0,a_m), C^2_{\text{per}}(\mathbb{R}))$ solves (3.2) and is L-periodic in x, which proves the existence. This completes the proof. □

Next, we study the global stability of $u^*(a,x)$ with initial data having a positive lower bound.

Proposition 3.6 *(Stability I) Let $u(t,a,x)$ be the unique solution of (1.1) with initial value u_0. Assume that $H(0) > 0$, then for each $a_+ \in [a_c, a_m)$, if $\inf_{(a,x)\in[0,a_+]\times\mathbb{R}} u_0(a,x) > 0$, then $u(t,a,x) \to u^*(a,x)$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$.*

Proof By assumption, there exists a positive constant δ such that $u_0(a,x) \ge \delta$ in $[0,a_+] \times \mathbb{R}$. Since $H(0) > 0$, the function $\epsilon\underline{u}$ defined in the proof of Proposition 3.5 is a sub-solution of (3.2) for small ϵ. Since $u_0 \ge \delta$ and $\underline{u}$ is bounded, we choose ϵ small if necessary such that $\epsilon\underline{u} \le u_0$. Denote by $\underline{U}(t,a,x)$ the solution of (1.1) with initial value $\epsilon\underline{u}$. By the comparison principle in Lemma 3.1, $\underline{U}(t,a,x) \ge \epsilon\underline{u}(a,x)$ for all $t \ge 0$. Given $s \ge 0$, let $z^s(t,a,x) := \underline{U}(t+s,a,x) - \underline{U}(t,a,x)$, which satisfies $z^s(0,a,x) \ge 0$ and

$$\begin{cases} \partial_t z^s + \partial_a z^s = D\partial_{xx} z^s - \mu(a,x) z^s, & (t,a,x) \in (0,\infty) \times (0,a_+] \times \mathbb{R}, \\ z^s(t,0,x) = f_u\left(x, \xi(t,s,x)\right) \int_0^{a_c} \beta(a,x) z^s(t,a,x) da, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ z^s(0,a,x) = \underline{U}(s,a,x) - \epsilon \underline{u} \geq 0, & (a,x) \in (0,a_+] \times \mathbb{R} \end{cases} \tag{3.11}$$

for some continuous function ξ. Applying the weak comparison principle (Lemma 3.1) to (3.11) yields that $z^s \geq 0$ for all $s \geq 0$, which implies that $\underline{U}(t,a,x)$ is a non-decreasing function of the time and $\underline{U}(t,a,x) \leq u(t,a,x)$ for all $t \geq 0$ and $(a,x) \in [0,a_+] \times \mathbb{R}$.

On the other hand, as in the proof of Proposition 3.5, we can choose constant $M > 0$ large to be a super-solution of (3.2). Let $\overline{U}(t,a,x)$ denote the solution of (1.1) with initial value $\overline{U}(0,a,x) = M$. Using the comparison principle, the similar arguments as above show that $\overline{U}$ is a non-increasing function of t. The comparison principle in Lemma 3.1 implies that $u(t,a,x) \leq \overline{U}(t,a,x)$ for all $t \geq 0$ and $(a,x) \in [0,a_+] \times \mathbb{R}$.

In summary, we have defined the monotonic sub-solution $\underline{U}$ and super-solution $\overline{U}$, which are periodic in x, such that for all $t \geq 0$ and $(a,x) \in [0,a_+] \times \mathbb{R}$,

$$\epsilon \underline{u} \leq \underline{U}(t,a,x) \leq u(t,a,x) \leq \overline{U}(t,a,x) \leq M. \tag{3.12}$$

The monotonicity of $\underline{U}$ and $\overline{U}$ implies $\underline{U}(t,a,x) \nearrow U_*(a,x)$ and $\overline{U}(t,a,x) \searrow U^*(a,x)$ pointwise as $t \to +\infty$ for some L-periodic functions $U_* \leq U^*$. In what follows, we shall show that $U_*, U^* \in W^{1,1}((0,a_+), C^2_{\text{per}}(\mathbb{R}))$ are the solutions to (3.2). For now, we acknowledge it to be true and postpone its proof behind. Hence, the uniqueness in Proposition 3.5 implies $U_* = U^* = u^*$. Further, due to the periodicity of $\underline{U}$ and $\overline{U}$ in x, we apply the Dini's theorem to derive that $\underline{U} \nearrow U_*$ and $\overline{U} \searrow U^*$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$, which together with (3.12) implies that $u(t,a,x) \to u^*$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$.

Now let us finish the proof of $U_*, U^* \in W^{1,1}((0,a_+), C^2_{\text{per}}(\mathbb{R}))$. Define

$$\underline{V}(t,a,x) := \int_0^a \underline{U}(t,s,x) ds \quad \text{for all } t > 0 \text{ and } (a,x) \in (0,a_+] \times \mathbb{R},$$

which is L-periodic in x. Since $\underline{U}$ solves (1.1), direct calculation yields

$$\begin{cases} \partial_t \underline{V} = D\partial_{xx} \underline{V} - \int_0^a \mu(s,x) \underline{U}(t,s,x) ds \\ \qquad -\underline{U}(t,a,x) + f\left(x, \int_0^{a_c} \beta(a,x) \underline{U}(t,a,x) da\right), & t > 0, \; x \in \mathbb{R}, \\ \underline{V}(t,a,x) = \underline{V}(t,a,x+L), & t > 0, \; x \in \mathbb{R}, \\ \underline{V}(0,a,x) = \int_0^a \underline{U}(0,s,x) ds, & x \in \mathbb{R}. \end{cases} \tag{3.13}$$

By the boundedness of $\underline{U}$ in (3.12), which is independent of $a \in [0,a_+]$ and $t > 0$, we apply the L^p estimates for parabolic equations to (3.13) and deduce that for any $p > 1$,

$$\sup_{a \in [0,a_+]} \sup_{t \geq 1} \|\underline{V}(\cdot,a,\cdot)\|_{W^{1,2}_p((t,t+1)\times(0,L))} < +\infty. \tag{3.14}$$

By the monotone convergence theorem, one has $\underline{V}(t,a,x) \nearrow \int_0^a U_*(s,x)ds$ pointwise as $t \to +\infty$, which together with (3.14) yields that for each $a \in [0, a_+]$, $\underline{V}(t,a,x) \nearrow \int_0^a U_*(s,x)ds$ as $t \to +\infty$ weakly in $W^{2,p}((0,L))$ and strongly in $C^1([0,L])$ by the Sobolev embedding. This implies that $\int_0^a U_*(s,x)ds \in C^1_{\text{per}}(\mathbb{R})$ for each $a \in [0,a_+]$ and is a strong solution to the problem

$$\begin{aligned} U_*(a,x) =& D\partial_{xx}\int_0^a U_*(s,x)ds - \int_0^a \mu(s,x)U_*(s,x)ds \\ &+ f\left(x, \int_0^{a_c} \beta(a,x)U_*(a,x)da\right), \qquad x \in \mathbb{R}, \end{aligned} \tag{3.15}$$

where we have used the monotone convergence theorem in the integral terms. Moreover, applying Schauder estimates of elliptic equations to (3.15) yields that $\int_0^a U_*(s,x)ds \in C^2_{\text{per}}(\mathbb{R})$ for any $a \in [0,a_+]$. Hence, we conclude that $U_*(a,x)$ is Lipschitz continuous in $a \in [0,a_+]$, so that $U_* \in W^{1,1}((0,a_+), C^2_{\text{per}}(\mathbb{R}))$ is a solution to (3.2). By the same arguments one can deduce that $U^* \in W^{1,1}((0,a_+), C^2_{\text{per}}(\mathbb{R}))$ is also a solution to (3.2), which completes the proof. □

Finally, we establish the global stability of $u(t,a,x)$ with general initial data.

Proposition 3.7 *(Stability II) Let $u^*(a,x)$ be the unique solution of (1.1) with nonnegative initial value $u_0 \not\equiv 0$. Assume $H(0) > 0$, then $u(t,a,x) \to u^*(a,x)$ in $C_{\text{loc}}([0,a_m) \times \mathbb{R})$ as $t \to +\infty$.*

Proof Note that $u(1,a,x) > 0$ by the strong comparison principle in Lemma 3.2. For any $a_+ \in [a_c, a_m)$, let us consider the following auxiliary problem:

$$\begin{cases} \partial_t \underline{u}_n + \partial_a \underline{u}_n = D\partial_{xx}\underline{u}_n - \mu(a,x)\underline{u}_n, & (t,a,x) \in (0,\infty) \times (0,a_+] \times (-n,n), \\ \underline{u}_n(t,0,x) = f\left(x, \int_0^{a_c} \beta(a,x)\underline{u}_n(t,a,x)da\right), & (t,x) \in (0,\infty) \times (-n,n), \\ \underline{u}_n(t,a,-n) = \underline{u}_n(t,a,n) = 0, & (t,a) \in (0,\infty) \times (0,a_+], \\ \underline{u}_n(0,a,x) = \underline{u}_n^0(a,x), & (a,x) \in [0,a_+] \times (-n,n), \end{cases}$$

where the initial data $\{\underline{u}_n^0(a,x)\}_{n\geq 1}$ satisfy

(1) $\text{supp}(\underline{u}_n^0) \subset [0,a_+] \times (-n,n)$ for all $n \geq 1$;
(2) $\underline{u}_1^0(a,x) \leq \cdots \leq \underline{u}_n^0(a,x) \leq \cdots \leq u(1,a,x)$ for all $(a,x) \in [0,a_+] \times \mathbb{R}$.

Then by the comparison principle, one has

$$\underline{u}_1(t,a,x) \leq \cdots \leq \underline{u}_n(t,a,x) \leq \cdots \leq u(1,a,x) \quad \text{ for all } t > 0 \text{ and } (a,x) \in [0,a_+] \times \mathbb{R}.$$

Let λ_n be the principal eigenvalue of (2.26) with $R = n$. Due to $H(0) > 0$, applying Proposition 2.6, we can choose some n_* large such that $\lambda_n > 0$ for all $n \geq n_*$. Thus by the similar arguments as in the proof of Proposition 3.5, for $n \geq n_*$, there is a unique positive solution $\underline{u}_n^*(a,x)$ satisfying

$$\begin{cases} \partial_a \underline{u}_n^* = D\partial_{xx}\underline{u}_n^* - \mu(a,x)\underline{u}_n^*, & (a,x) \in (0,a_+] \times (-n,n), \\ \underline{u}_n^*(0,x) = f\left(x, \int_0^{a_c} \beta(a,x)\underline{u}_n^*(a,x)da\right), & x \in (-n,n), \\ \underline{u}_n^*(a,-n) = \underline{u}_n^*(a,n) = 0, & a \in (0,a_+]. \end{cases}$$

Similar to Proposition 3.6, one can apply the theory of monotone dynamical systems to deduce

$$\underline{u}_n(t,a,x) \to \underline{u}_n^*(a,x) \quad \text{in } C([0,a_+] \times [-n,n]) \quad \text{as } t \to +\infty; \tag{3.16}$$

see also Ducrot et al. [15, Theorem 4.11] for the case of nonlocal dispersal.

This implies that

$$\underline{u}_{n_*}^*(a,x) \le \cdots \le \underline{u}_n^*(a,x) \le \cdots \le \liminf_{t\to\infty} u(t,a,x) \quad \text{for all } (a,x) \in [0,a_+] \times \mathbb{R}.$$

By the monotone convergence theorem and parabolic estimates, there exists $\underline{u}^* \in C^{1,2}([0,a_+] \times \mathbb{R})$ satisfying (3.2) such that $\underline{u}_n^* \to \underline{u}^*$ in $C^{1,2}_{\text{loc}}([0,a_+] \times \mathbb{R})$ as $n \to +\infty$. It follows that

$$\underline{u}^*(a,x) \le \liminf_{t\to\infty} u(t,a,x) \quad \text{for all } (a,x) \in [0,a_+] \times \mathbb{R}. \tag{3.17}$$

Next, we shall prove $\underline{u}^* \equiv u^*$ in $[0,a_+] \times \mathbb{R}$ with $u^*(a,x)$ being the unique solution of (3.2). To this end, for any $\delta > 0$, let $H_\delta(0)$ denote the principal eigenvalue of the perturbed problem

$$\begin{cases} \partial_a \phi = D\partial_{xx}\phi - \mu(a,x)\phi - H_\delta(0)\phi, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \phi(0,x) = (f_u(x,0) - \delta) \int_0^{a_c} \beta(a,x)\phi(a,x)da, & x \in \mathbb{R}, \\ \phi(a,x) = \phi(a,x+L), & (a,x) \in (0,a_+] \times \mathbb{R}. \end{cases}$$

Due to $H(0) > 0$, we can choose $\delta > 0$ small such that $H_\delta(0) > 0$. Motivated by Nadin [49], for any $y \in \mathbb{R}$, we consider the following problem

$$\begin{cases} \partial_a \phi = D\partial_{xx}\phi - \mu(a,x+y)\phi - \lambda\phi, & (a,x) \in (0,a_+] \times (-n,n), \\ \phi(0,x) = (f_u(x+y,0) - \delta) \int_0^{a_c} \beta(a,x+y)\phi(a,x)da, & x \in (-n,n), \\ \phi(a,-n) = \phi(a,n) = 0, & a \in (0,a_+]. \end{cases}$$

Let (λ_n^y, ϕ_n^y) denote the corresponding principal eigenpair. The periodicity of μ, β and f implies that $y \mapsto \lambda_n^y$ is periodic and continuous. By Proposition 2.6 and the Dini's lemma, λ_n^y converges to $H_\delta(0)$ uniformly in any compact subset of $\mathbb{R}$ as $n \to \infty$, so that we can choose $n \ge 1$ large such that $\lambda_n^y > 0$ for all $y \in \mathbb{R}$ due to $H_\delta(0) > 0$.

Hence, for such $n \ge 1$, it follows that $\kappa\phi_n^y$ with sufficiently small $\kappa > 0$ is a subsolution of

$$\begin{cases} \partial_a u = D\partial_{xx}u - \mu(a, x+y)u, & (a,x) \in (0,a_+] \times (-n,n), \\ u(0,x) = f\left(x+y, \int_0^{a_c} \beta(a,x+y)u(a,x)da\right), & x \in (-n,n), \\ u(a,-n) = u(a,n) = 0, & a \in (0,a_+]. \end{cases} \tag{3.18}$$

Note that $\underline{u}^{*,y}(a,x) := \underline{u}^*(a,x+y)$ satisfies

$$\begin{cases} \partial_a u = D\partial_{xx}u - \mu(a, x+y)u, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ u(0,x) = f(x+y, \int_0^{a_c} \beta(a,x+y)u(a,x)da), & x \in \mathbb{R}, \end{cases}$$

which is a super-solution of (3.18). Thus the comparison principle applied in $[0,a_+] \times [-n,n]$ yields that $u^{*,y}(a,x) \geq \kappa\phi_n^y(a,x)$ for all $(a,x) \in [0,a_+] \times [-n,n]$, and in particular we deduce that for all $y \in \mathbb{R}$,

$$\underline{u}^*(a,y) = \underline{u}^{*,y}(a,0) \geq \kappa\phi_n^y(a,0) > 0 \quad \text{uniformly in } a \in [0,a_+].$$

Since $y \mapsto \phi_n^y(a,0)$ is periodic, we obtain

$$\underline{u}^*(a,x) \geq \kappa \inf_{(a,y)\in[0,a_+]\times\mathbb{R}} \phi_n^y(a,0) > 0, \quad \forall (a,x) \in [0,a_+] \times \mathbb{R}. \tag{3.19}$$

Hence one can apply Lemma 3.4 to both $\underline{u}^*$ and u^*, and then obtain $\underline{u}^* \equiv u^*$ in $[0,a_+] \times \mathbb{R}$.

Finally, by Proposition 3.6, it follows from (3.16) and (3.17) that $u(t,a,x)$ converges to $u^*(a,x)$ in $C_{\text{loc}}([0,a_+] \times \mathbb{R})$ as $t \to +\infty$. The proof is complete. □

To prove Theorem 1.1, we next consider the case $H(0) \leq 0$.

Lemma 3.8 *Assume $H(0) \leq 0$, then any nonnegative solution of (3.2) is identically zero.*

Proof For any $a_+ \in [a_c, a_m)$, assume by contradiction that there exists a nonnegative continuous solution u to (3.2) which is positive somewhere in $[0,a_+] \times \mathbb{R}$.

We first claim $u > 0$ in $[0,a_+] \times \mathbb{R}$. Assume by contradiction that $u(a_*,x_*) = 0$ for some $(a_*,x_*) \in [0,a_+] \times \mathbb{R}$. If $a_* \in (0,a_+]$, then we apply the strong maximum principle for parabolic equations to (3.2) and derive $u \equiv 0$ in $[0,a_+] \times \mathbb{R}$, which contradicts the fact that u is positive somewhere in $[0,a_+] \times \mathbb{R}$. Hence, $a_* = 0$, that is

$$u(0,x_*) = f\left(x_*, \int_0^{a_c} \beta(a,x_*)u(a,x_*)da\right) = 0,$$

which together with Assumption 1.2 implies that $\int_0^{a_c} \beta(a,x_*)u(a,x_*)da = 0$. This implies that $u(a^*,x_*) = 0$ at least for some $a^* \in (0,a_+]$, so that we have the same contradiction as above. Therefore, $u > 0$ in $[0,a_+] \times \mathbb{R}$.

Then it follows from Assumption 1.1-(ii) that there exists a positive constant c_0 such that $\int_0^{a_c} \beta(a,x)u(a,x)da \geq c_0$, $\forall x \in \mathbb{R}$. By Assumption 1.2, we have

$$\frac{f\left(x, \int_0^{a_c} \beta(a,x)u(a,x)da\right)}{\int_0^{a_c} \beta(a,x)u(a,x)da} \leq \frac{f(x, c_0)}{c_0} < f_u(x,0), \quad \forall x \in \mathbb{R}, \tag{3.20}$$

which implies that

$$\frac{f(x, c_0)}{c_0} \int_0^{a_c} \beta(a,x)u(a,x)da \geq f\left(x, \int_0^{a_c} \beta(a,x)u(a,x)da\right) = u(0,x), \quad \forall x \in \mathbb{R}. \tag{3.21}$$

Consider the following eigenvalue problem

$$\begin{cases} \partial_a \phi(a,x) = D\partial_{xx}\phi - \mu(a,x)\phi - \lambda\phi, & (a,x) \in (0,a_+] \times \mathbb{R}, \\ \phi(0,x) = \frac{f(x,c_0)}{c_0} \int_0^{a_c} \beta(a,x)\phi(a,x)da, & x \in \mathbb{R}, \\ \phi(a,x) = \phi(a,x+L), & (a,x) \in (0,a_+] \times \mathbb{R}. \end{cases} \tag{3.22}$$

Let $\lambda_{c_0} \in \mathbb{R}$ be the principal eigenvalue of (3.22) and the corresponding eigenfunction is denoted by $\phi_{c_0} > 0$. Due to (3.20), by the proof of Lemma 2.1, we can deduce $\lambda_{c_0} < H(0) \leq 0$.

Define $\alpha^* := \inf\{\alpha > 0 : u \leq \alpha\phi_{c_0}\}$. We conclude the proof by proving that $\alpha^* = 0$. Assume that $\alpha^* > 0$. Set $w := u - \alpha^*\phi_{c_0} \leq 0$ and $a_0 := \min\{a \in [0,a_+] : \exists x \in \mathbb{R}, \text{ s.t. } w(a,x) = 0\}$. The existence of such a_0 is due to the definition of α^*. Hence, $w \leq 0$ and there exists $x_0 \in \mathbb{R}$ such that $w(a_0,x_0) = 0$.

Next we claim that $w \equiv 0$ in $[0,a_+] \times \mathbb{R}$. Indeed, observe that *w* satisfies

$$\partial_a w \leq D\partial_{xx}w - \mu(a,x)w, \quad (a,x) \in (0,a_+] \times \mathbb{R}. \tag{3.23}$$

If $a_0 \in (0,a_+]$, the fact that $w \equiv 0$ in $[0,a_+] \times \mathbb{R}$ is a direct consequence of the strong maximum principle for (3.23); Otherwise, if $a_0 = 0$, then by (3.21) we have

$$\begin{aligned} 0 = w(0,x_0) &= u(0,x_0) - \alpha^*\phi_{c_0}(0,x_0) \\ &\leq \frac{f(x_0,c_0)}{c_0} \int_0^{a_c} \beta(a,x_0)u(a,x_0)da - \alpha^* \frac{f(x_0,c_0)}{c_0} \int_0^{a_c} \beta(a,x_0)\phi_{c_0}(a,x_0)da \\ &= \frac{f(x_0,c_0)}{c_0} \int_0^{a_c} \beta(a,x_0)w(a,x_0)da \leq 0. \end{aligned}$$

Hence, it follows by Assumption 1.1-(ii) that $w(a^*,x_0) = 0$ at least for some $a^* \in (0,a_+]$. Then applying the strong maximal principle again yields $w \equiv 0$ in $[0,a_+] \times \mathbb{R}$. By definition we have $u = \alpha^*\phi_{c_0}$ in $[0,a_+] \times \mathbb{R}$, which is a contradiction as follows,

$$0 = \partial_a u - D\partial_{xx}u + \mu u = \alpha^*\partial_a\phi_{c_0} - D\alpha^*\partial_{xx}\phi_{c_0} + \mu\alpha^*\phi_{c_0} \geq -\alpha^*\lambda_{c_0}\phi_{c_0} > 0.$$

Therefore, $\alpha^* = 0$ and the proof is complete. □

Finally, we present the proof of Theorem 1.1.

Proof (Proof of Theorem 1.1)

Theorem 1.1-(i) is a direct consequence of Propositions 3.5-3.7. To prove Theorem 1.1-(ii), by Lemma 3.8 it remains to show the global stability of the trivial solution 0 when $H(0) \leq 0$. Indeed, as in the proof of Proposition 3.6, we can choose some large M satisfying $u_0 \leq M$ to be a super-solution of (1.1). Let $\overline{U}$ still denote the solution of (1.1) with initial value M, which is L-periodic in x. Then there holds

$$0 \leq u(t,a,x) \leq \overline{U}(t,a,x) \leq M \quad \text{for all } (a,x) \in [0,a_+] \times \mathbb{R},\ t > 0. \tag{3.24}$$

Note that $\overline{U}$ is non-increasing in t for each $(a,x) \in [0,a_+] \times \mathbb{R}$. It holds that $\overline{U}$ converges in $C_{\text{loc}}([0,a_+] \times \mathbb{R})$ to some nonnegative solution of stationary equation (3.2) as $t \to +\infty$, which together with Lemma 3.8 yields $\overline{U} \to 0$ in $C_{\text{loc}}([0,a_+] \times \mathbb{R})$ as $t \to +\infty$. Since $\overline{U}$ is periodic in x, the convergence is uniform in x. Hence, by (3.24) we conclude that $u \to 0$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$. The proof of Theorem 1.1 is now complete. □

4 Spreading properties

In this section, we are concerned with the spreading properties of problem (1.1). To prove Theorem 1.2, we shall investigate the asymptotic spreading of the solution restricted on $[0,a_+] \times \mathbb{R}$ for any given $a_+ \in [a_c, a_m)$. By Assumption 1.1-(ii), the restricted solution is exactly that of the following equation:

$$\begin{cases} \partial_t u + \partial_a u = D\partial_{xx} u - \mu(a,x)u, & (a,x) \in (0,a_+] \times \mathbb{R},\ t > 0, \\ u(t,0,x) = f\left(x, \int_0^{a_c} \beta(a,x)u(t,a,x)da\right), & x \in \mathbb{R},\ t > 0, \\ u(0,a,x) = u_0(a,x), & (a,x) \in [0,a_+] \times \mathbb{R}, \end{cases} \tag{4.1}$$

where u_0 is any compactly supported nonnegative initial value.

4.1 Outer spreading

We first prove Theorem 1.2-(i), which follows by constructing appropriate super-solutions.

Proof (Proof of Theorem 1.2-(i)) Fix any $c > c^*$ with c^* being defined by (1.6). Choose $\lambda > 0$ such that $c\lambda \geq H(\lambda)$. Then we shall construct a super-solution to (4.1) in the form of $v(t,a,x) = v_0 e^{-\lambda(x-ct)}\phi_\lambda(a,x)$ for some positive constant v_0 to be chosen later, where $H(\lambda)$ is the principal eigenvalue of problem (1.5) and $\phi_\lambda > 0$ is the corresponding eigenfunction.

Due to $c\lambda \geq H(\lambda)$, direct calculation gives

$$\partial_t v + \partial_a v - D\partial_{xx} v + \mu(a,x)v = c\lambda v - H(\lambda)v \geq 0 \quad \text{for all } a \in (0, a_+] \text{ and } t > 0.$$

By the boundary condition of ϕ_λ in (1.5), one has

$$\begin{aligned} v(t,0,x) =& v_0 e^{-\lambda(x-ct)}\phi_\lambda(0,x) \\ =& v_0 e^{-\lambda(x-ct)} f_u(x,0) \int_0^{a_c} \beta(a,x)\phi_\lambda(a,x)da \\ =& f_u(x,0) \int_0^{a_c} \beta(a,x)v(t,a,x)da. \end{aligned}$$

Since u_0 is compactly supported, we next choose v_0 large enough such that

$$v(0,a,x) = v_0 e^{-\lambda x}\phi_\lambda(a,x) \geq u_0(a,x), \ \ \forall x \in \mathbb{R}, \ \ \text{uniformly in } [0, a_+].$$

Note from (4.1) and Assumption (1.2)-(ii) that

$$u(t,0,x) \leq f_u(x,0) \int_0^{a_c} \beta(a,x)u(t,a,x)da \quad \text{for all } x \in \mathbb{R} \text{ and } t > 0.$$

Set $w := v - u$. Then by the above discussion we derive that

$$\begin{cases} \partial_t w + \partial_a w \geq D\partial_{xx} w - \mu w, & (a,x) \in (0,a_+] \times \mathbb{R}, \ t > 0, \\ w(t,0,x) \geq f_u(x,0) \int_0^{a_c} \beta(a,x)w(t,a,x)da, & x \in \mathbb{R}, \ t > 0, \\ w(0,a,x) \geq 0, & (a,x) \in [0,a_+] \times \mathbb{R}. \end{cases}$$

It follows by the comparison principle in Lemma 3.1 that $w \geq 0$, which implies that

$$u(t,a,x) \leq v(t,a,x) = v_0 e^{-\lambda(x-ct)}\phi_\lambda(a,x) \quad \text{for all } (a,x) \in [0,a_+] \times \mathbb{R} \text{ and } t > 0.$$

Let c_1 be any real number such that $c_1 > c > c^*$. Then

$$\lim_{t\to+\infty} \sup_{|x|\geq c_1 t} \sup_{a\in[0,a_+]} u(t,a,x) \leq v_0 \lim_{t\to+\infty} \sup_{|x|\geq c_1 t} \sup_{a\in[0,a_+]} (e^{-\lambda(c_1-c)t}\phi_\lambda(a,x)) = 0.$$

By choosing $c > c^*$ to be arbitrarily close to c^*, we can prove Theorem 1.2-(i). □

4.2 The rescaled equation

To prove Theorem 1.2-(ii), we shall develop the homogenization method for problem (4.1) in this subsection, which is our key core and main contribution in analyzing spreading properties of problem (1.1). Motivated by [5, 6], for any $(a,x) \in [0,a_m) \times \mathbb{R}$ and $t > 0$, we define

$$u_\epsilon(t,a,x) := u\left(\frac{t}{\epsilon}, a, \frac{x}{\epsilon}\right) \quad \text{and} \quad z_\epsilon(t,a,x) := \epsilon \ln(u_\epsilon(t,a,x)/M), \tag{4.2}$$

where M is the uniform bound of the solution u as proved in Lemma 3.1, so that $z_\epsilon \le 0$ for all $\epsilon > 0$. By (4.1), for any $a_+ \in [a_c, a_m)$, direct calculations yield

$$\begin{cases} \partial_t z_\epsilon + \frac{1}{\epsilon}\partial_a z_\epsilon = \epsilon D \partial_{xx} z_\epsilon + D|\partial_x z_\epsilon|^2 - \mu\left(a, \frac{x}{\epsilon}\right), & (a,x) \in (0,a_+] \times \mathbb{R},\ t > 0, \\ \epsilon \partial_t u_\epsilon + \partial_a u_\epsilon = \epsilon^2 D \partial_{xx} u_\epsilon - \mu\left(a, \frac{x}{\epsilon}\right) u_\epsilon, & (a,x) \in (0,a_+] \times \mathbb{R},\ t > 0, \\ u_\epsilon(t,0,x) = f\left(\frac{x}{\epsilon}, \int_0^{a_c} \beta\left(a, \frac{x}{\epsilon}\right) u_\epsilon(t,a,x) da\right), & x \in \mathbb{R},\ t > 0, \\ u_\epsilon(0,a,x) = u_0\left(a, \frac{x}{\epsilon}\right), & (a,x) \in [0,a_+] \times \mathbb{R}. \end{cases} \tag{4.3}$$

Without loss of generality, we can assume that the compactly supported initial value u_0 satisfies $u_0(a,0) > 0$ for all $a \in [0,a_+]$. Indeed, since $u(1,a,x) > 0$ in $[0,a_+] \times \mathbb{R}$ by Lemma 3.2, we can choose the compactly supported function $v_0(a,x)$ such that $v_0(a,0) > 0$ in $[0,a_+]$ and $v_0 \le u(1,a,x)$. Then the solution v of (4.1) with initial value v_0 satisfies $v(t,a,x) \le u(t+1,a,x)$ for all $t > 0$. Thus we can establish Theorem 1.2-(ii) for v, which in turn implies the same result for u.

As in [2, Section 6], we define the following half-relaxed limit:

$$u_*(t,a,x) = \liminf_{\substack{\epsilon \to 0 \\ (t',a',x') \to (t,a,x)}} u_\epsilon(t',a',x'). \tag{4.4}$$

The following result implies that the zero set of the half-relaxed limit u_* is uniform for age $a \in [0,a_m)$, which is a key ingredient for our homogenization method.

Proposition 4.1 *Let u_* be defined by (4.4). Assume that $u_*(t^*,a^*,x^*) = 0$ for some $(a^*,x^*) \in [0,a_m) \times \mathbb{R}$ and $t^* > 0$, then $u_*(t^*,a,x^*) \equiv 0$ for all $a \in [0,a_m)$.*

Proof The proof is divided into the following two steps.

Step 1. We show $u_*(t^*,a,x^*) \equiv 0$ for all $a \in [a^*,a_m)$. Define

$$\overline{U}_\epsilon(t,x) := \int_{a^*}^{a_m} u_\epsilon(t,a,x) da, \quad \forall (t,x) \in (0,+\infty) \times \mathbb{R}.$$

It follows by Lemma 3.1 that there exists some $M > 0$ such that $\overline{U}_\epsilon(t,x) \le M$ for all $\epsilon > 0$ and $(t,x) \in (0,+\infty) \times \mathbb{R}$. Integrating the equation of u_ϵ in (4.3) from a^* to a_m, one obtains

$$\epsilon \partial_t \overline{U}_\epsilon - u_\epsilon(t,a^*,x) = D\epsilon^2 \partial_{xx} \overline{U}_\epsilon - \int_{a^*}^{a_m} \mu\left(a, \frac{x}{\epsilon}\right) u_\epsilon(t,a,x) da. \tag{4.5}$$

We next show that

$$\liminf_{\substack{\epsilon\to 0\\(t,x)\to(t^*,x^*)}} \overline{U}_\epsilon(t,x)=0. \tag{4.6}$$

Due to $u_*(t^*,a^*,x^*)=0$, the definition of u_* in (4.4) implies that there exists a sequence $(\tilde{t}^\epsilon,\tilde{a}^\epsilon,\tilde{x}^\epsilon)\in(0,+\infty)\times[0,a_m)\times\mathbb{R}$ such that

$$(\tilde{t}^\epsilon,\tilde{a}^\epsilon,\tilde{x}^\epsilon)\to(t^*,a^*,x^*) \text{ and } u_\epsilon(\tilde{t}^\epsilon,\tilde{a}^\epsilon,\tilde{x}^\epsilon)\to 0 \quad \text{as } \epsilon\to 0. \tag{4.7}$$

Given any $\vartheta>0$, we define the test function

$$\phi_\epsilon(t,x):=\frac{(t-\tilde{t}^\epsilon)^2+(x-\tilde{x}^\epsilon)^2}{\vartheta\epsilon^2}, \quad \forall(t,x)\in(0,+\infty)\times\mathbb{R}. \tag{4.8}$$

By the uniform boundedness of $\overline{U}_\epsilon$ in ϵ, we can define the sequence $(t^\epsilon,x^\epsilon)\in(0,+\infty)\times\mathbb{R}$ such that $\overline{U}_\epsilon-\phi_\epsilon$ attains its maximum at (t^ϵ,x^ϵ). In particular, we have

$$\overline{U}_\epsilon(t^\epsilon,x^\epsilon)-\phi_\epsilon(t^\epsilon,x^\epsilon)\geq\overline{U}_\epsilon(\tilde{t}^\epsilon,\tilde{x}^\epsilon)-\phi_\epsilon(\tilde{t}^\epsilon,\tilde{x}^\epsilon)=\overline{U}_\epsilon(\tilde{t}^\epsilon,\tilde{x}^\epsilon).$$

Since $\overline{U}_\epsilon(t,x)\leq M$ holds for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ again, this and (4.8) together imply that

$$|t^\epsilon-\tilde{t}^\epsilon|+|x^\epsilon-\tilde{x}^\epsilon|\leq 2\epsilon\sqrt{M\vartheta}. \tag{4.9}$$

By combining (4.7) with (4.9), we conclude that as $\epsilon\to 0$, there hold $(t^\epsilon,x^\epsilon)\to(t^*,x^*)$ and

$$u_\epsilon(t^\epsilon,a^*,x^\epsilon)=u\left(\frac{t^\epsilon}{\epsilon},a^*,\frac{x^\epsilon}{\epsilon}\right)=u\left(\frac{\tilde{t}^\epsilon}{\epsilon}+O(1),a^*,\frac{\tilde{x}^\epsilon}{\epsilon}+O(1)\right)\to 0. \tag{4.10}$$

Here (4.10) follows from the comparison arguments involving the construction of appropriate super-solutions, and we omit the details here for brevity.

Then by evaluating (4.5) at the point (t^ϵ,x^ϵ), we derive that

$$\begin{aligned}
&u_\epsilon(t^\epsilon,a^*,x^\epsilon)-\int_{a^*}^{a_m}\mu\left(a,\frac{x^\epsilon}{\epsilon}\right)u_\epsilon(t^\epsilon,a,x^\epsilon)da\\
&=\epsilon\partial_t\overline{U}_\epsilon(t^\epsilon,x^\epsilon)-D\epsilon^2\partial_{xx}\overline{U}_\epsilon(t^\epsilon,x^\epsilon)\\
&\geq\epsilon\partial_t\phi_\epsilon(t^\epsilon,x^\epsilon)-D\epsilon^2\partial_{xx}\phi_\epsilon(t^\epsilon,x^\epsilon)=\frac{2(t^\epsilon-\tilde{t}^\epsilon)}{\vartheta\epsilon}-\frac{2D}{\vartheta}.
\end{aligned}$$

By (4.9) and (4.10), letting $\epsilon\to 0$ in the above inequality yields

$$\mu_{\inf}\liminf_{\epsilon\to 0}\overline{U}_\epsilon(t^\epsilon,x^\epsilon)\leq\limsup_{\epsilon\to 0}\int_{a^*}^{a_m}\mu\left(a,\frac{x^\epsilon}{\epsilon}\right)u_\epsilon(t^\epsilon,a,x^\epsilon)da\leq 4\sqrt{\frac{M}{\vartheta}}+\frac{2D}{\vartheta}.$$

Due to the arbitrariness of ϑ, letting $\vartheta \to +\infty$ yields that

$$\liminf_{\epsilon \to 0} \overline{U}_\epsilon(t^\epsilon, x^\epsilon) = 0,$$

which implies (4.6) directly. Therefore, we derive that

$$u_*(t^*, a, x^*) = \liminf_{\substack{\epsilon \to 0 \\ (t,s,x) \to (t^*, a, x^*)}} u_\epsilon(t, s, x) = 0 \quad \text{for all } a \in [a^*, a_m).$$

This completes the proof of Step 1.

Step 2. We prove $u_*(t^*, a, x^*) \equiv 0$ for all $a \in [0, a_m)$. By Step 1, if $a^* = 0$, then the proof is complete. It remains to consider the case $a^* > 0$ and $u_*(t^*, 0, x^*) > 0$. Set

$$a_* := \sup\{a \geq 0 : u_*(t^*, s, x^*) > 0, \ \forall s \in [0, a)\}.$$

It follows from Step 1 that $0 \leq a_* \leq a^*$. It suffices to show $a_* = 0$.

Assume by contradiction that $a_* > 0$, then $u_*(t^*, a, x^*) > 0$ for all $a \in [0, a_*)$. For any $s \in (0, a_*)$, we define

$$\underline{U}_\epsilon^s(t, x) := \int_s^{a_*} u_\epsilon(t, a, x) da, \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}.$$

Integrating the equation of u_ϵ in (4.3) from s to a_*, one obtains

$$\epsilon \partial_t \underline{U}_\epsilon^s + u_\epsilon(t, a_*, x) - u_\epsilon(t, s, x) = D\epsilon^2 \partial_{xx} \underline{U}_\epsilon^s - \int_s^{a_*} \mu\left(a, \frac{x}{\epsilon}\right) u_\epsilon(t, a, x) da. \tag{4.11}$$

By the definition of a_*, there exists some sequence $(\tilde{t}_\epsilon, \tilde{a}_\epsilon, \tilde{x}_\epsilon) \in (0, +\infty) \times [0, a_m) \times \mathbb{R}$ such that

$$(\tilde{t}_\epsilon, \tilde{a}_\epsilon, \tilde{x}_\epsilon) \to (t^*, a_*, x^*) \quad \text{and} \quad u_\epsilon(\tilde{t}_\epsilon, \tilde{a}_\epsilon, \tilde{x}_\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{4.12}$$

Let the test function ϕ_ϵ be defined in (4.8) with $(\tilde{t}^\epsilon, \tilde{x}^\epsilon)$ replaced by $(\tilde{t}_\epsilon, \tilde{x}_\epsilon)$. By the same arguments as in Step 1, there exists some sequence (t_ϵ, x_ϵ) such that $\underline{U}_\epsilon^s + \phi_\epsilon$ attains its minimum at (t_ϵ, x_ϵ) and

$$|t_\epsilon - \tilde{t}_\epsilon| + |x_\epsilon - \tilde{x}_\epsilon| \leq 2\epsilon\sqrt{M\vartheta}. \tag{4.13}$$

Similar to (4.10), by (4.12) and (4.13) we deduce that

$$(t_\epsilon, x_\epsilon) \to (t^*, x^*) \quad \text{and} \quad u_\epsilon(t_\epsilon, a_*, x_\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{4.14}$$

By evaluating (4.11) at (t_ϵ, x_ϵ), we calculate that

$$
\begin{aligned}
&u_\epsilon(t_\epsilon, s, x_\epsilon) - u_\epsilon(t_\epsilon, a_*, x_\epsilon) - \int_s^{a_*} \mu\left(a, \frac{x_\epsilon}{\epsilon}\right) u_\epsilon(t_\epsilon, a, x_\epsilon) da \\
&= \epsilon \partial_t \underline{U}_\epsilon^s(t_\epsilon, x_\epsilon) - D\epsilon^2 \partial_{xx} \underline{U}_\epsilon^s(t_\epsilon, x_\epsilon) \\
&\leq -\epsilon \partial_t \phi_\epsilon(t_\epsilon, x_\epsilon) + D\epsilon^2 \partial_{xx} \phi_\epsilon(t_\epsilon, x_\epsilon) = -\frac{2(t_\epsilon - \tilde{t}_\epsilon)}{\vartheta \epsilon} + \frac{2D}{\vartheta}.
\end{aligned}
$$

By (4.13) and (4.14), letting $\epsilon \to 0$ in the above inequality gives

$$
\limsup_{\epsilon \to 0} \left[u_\epsilon(t_\epsilon, s, x_\epsilon) - \int_s^{a_*} \mu\left(a, \frac{x_\epsilon}{\epsilon}\right) u_\epsilon(t_\epsilon, a, x_\epsilon) da \right] \leq 4\sqrt{\frac{M}{\vartheta}} + \frac{2D}{\vartheta}, \quad \forall s \in (0, a_*).
$$

By letting $\vartheta \to +\infty$ again, we derive that

$$
\overline{u}_*(s) := \limsup_{\epsilon \to 0} u_\epsilon(t_\epsilon, s, x_\epsilon) \leq \liminf_{\epsilon \to 0} \int_s^{a_*} \mu\left(a, \frac{x_\epsilon}{\epsilon}\right) u_\epsilon(t_\epsilon, a, x_\epsilon) da, \quad \forall s \in (0, a_*).
$$

This implies that

$$
\overline{u}_*(s) \leq \mu_{\sup} \int_s^{a_*} \overline{u}_*(a) da, \quad \forall s \in (0, a_*), \tag{4.15}
$$

where $\mu_{\sup} = \|\overline{\mu}\|_{L^\infty(0,a_*)} < +\infty$ since $\overline{\mu}(a) = \max_{x \in [0,L]} \mu(a, x)$ is bounded in $[0, a_*]$. Denote

$$
w(s) := \int_s^{a_*} \overline{u}_*(a) da \quad \text{and} \quad W(a) := \int_a^{a_*} w(s) ds.
$$

Then W is differentiable in a and $W'(a) = -w(a)$. It follows from (4.15) that

$$
w(a) \leq \mu_{\sup} \int_a^{a_*} w(s) ds = \mu_{\sup} W(a).
$$

Hence, one has $(\log W(a))' \geq -\mu_{\sup}$. Integrating this inequality from s to a_* yields

$$
-\mu_{\sup}(a_* - s) \leq \log W(a_*) - \log W(s), \quad \forall s \in (0, a_*),
$$

which is a contradiction due to $W(a_*) = 0$. Therefore, $a_* = 0$, and by Step 1 we conclude that $u_*(t^*, a, x^*) = 0$ for all $a \in (0, a_m)$. Noting that u_* is lower semi-continuous, we have $u_*(t^*, 0, x^*) = 0$, so that $u_*(t^*, a, x^*) \equiv 0$ for any $a \in [0, a_m)$. The proof is now complete. □

Recalling the definition of u_* in (4.4), we find that $u_*(t, a, x) = u_*(\alpha t, a, \alpha x)$ for any $\alpha > 0$. This, together with Proposition 4.1 and Theorem 1.2-(i), implies that

$$
\{u_* = 0\} = \{(t, x) : x \geq \bar{c}t, t > 0\} \times [0, a_m) \quad \text{for some} \quad 0 < \bar{c} \leq c^*. \tag{4.16}
$$

Recalling the definition of z_ϵ in (4.2), we define

$$z_*(t,a,x) := \liminf_{\substack{\epsilon\to 0\\(t',a',x')\to(t,a,x)}} z_\epsilon(t',a',x') \quad \text{for any } (a,x)\in[0,a_m)\times\mathbb{R} \text{ and } t>0, \tag{4.17}$$

which is well-defined by constructing the similar sub-solutions as in the proof of [37, Lemma 3.2]. To estimate the limit z_*, for any $a_+ \in [a_c, a_m)$ we define

$$\begin{aligned} Z_*(t,x) &:= \liminf_{\substack{\epsilon\to 0\\(t',x')\to(t,x)}} \inf_{a\in[0,a_+]} z_\epsilon(t',a,x') \\ Z^*(t,x) &:= \limsup_{\substack{\epsilon\to 0\\(t',x')\to(t,x)}} \sup_{a\in[0,a_+]} z_\epsilon(t',a,x'). \end{aligned} \tag{4.18}$$

Obviously, $Z_*(t,x) \le z_*(t,a,x) \le Z^*(t,x)$ for all $(a,x)\in[0,a_+]\times\mathbb{R}$ and $t>0$.

Next we shall prove that Z_* and Z^* are viscosity super-solution and viscosity sub-solution of a Hamilton-Jacobi equation, respectively. We refer to [2, 9] for the definitions of viscosity super-solutions and sub-solutions.

Lemma 4.2 *Let Z_* be defined in (4.18). Then $Z_*(t,x)$ is a lower semi-continuous viscosity solution of*

$$\begin{cases} \partial_t Z_* - H(\partial_x Z_*) \ge 0, & x>\bar{c}t,\ t>0,\\ Z_*(t,\bar{c}t)=0, & t>0, \end{cases}$$

where $\bar{c}\in(0,c^*]$ is defined in (4.16).

Proof **Step 1**. We first show $\partial_t Z_* - H(\partial_x Z_*) \ge 0$ for $x>\bar{c}t$ and $t>0$ in the sense of viscosity solutions. By the definition of viscosity solutions, we fix any $\varphi\in C^\infty((0,+\infty)\times\mathbb{R})$ and assume that $Z_*-\varphi$ attains a strict local minimum point (t_*,x_*) satisfying $x_*>\bar{c}t_*$. We must prove

$$\partial_t\varphi(t_*,x_*) - H(\partial_x\varphi(t_*,x_*)) \ge 0. \tag{4.19}$$

Set $\gamma := \partial_x\varphi(t_*,x_*)$. Assume by contradiction that (4.19) fails, namely

$$\partial_t\varphi(t_*,x_*) - H(\gamma) < 0. \tag{4.20}$$

For any $\delta>0$, denote $H_\delta(\gamma)\in\mathbb{R}$ by the principal eigenvalue of the problem

$$\begin{cases} \partial_a\phi = D\partial_{xx}\phi + 2D\gamma\partial_x\phi + D\gamma^2\phi - \mu(a,x)\phi - H_\delta(\gamma)\phi, & (a,x)\in(0,a_+]\times\mathbb{R},\\ \phi(0,x) = (f_u(x,0)-2\delta)\int_0^{a_c}\beta(a,x)\phi(a,x)da, & x\in\mathbb{R},\\ \phi(a,x)=\phi(a,x+L), & (a,x)\in(0,a_+]\times\mathbb{R}. \end{cases} \tag{4.21}$$

Again, as stated in Remark 2.1, $H_\delta(\gamma)$ exists and is independent of the choice of $a_+\in[a_c,a_m)$. Let $H(\gamma)$ be the principal eigenvalue of problem (1.5) with $\lambda=\gamma$. Noting that $H(\gamma)=H(-\gamma)$ as proved by Proposition 1.6-(i), we find $H_\delta(\gamma)\to H(\gamma)$ as $\delta\to 0$. By (4.20), we can choose $\delta>0$ small such that $\partial_t\varphi(t_*,x_*) < H_\delta(\gamma)$.

For the chosen $\delta > 0$, we define $\phi_\delta > 0$ as the principal eigenfunction of (4.21) associated with $H_\delta(\gamma)$. Note that $\phi_\delta > 0$ in $[0, a_+] \times \mathbb{R}$. We define the perturbed test function

$$\varphi_\epsilon(t, a, x) := \varphi(t, x) + \epsilon \ln \phi_\delta(a, x/\epsilon).$$

By direct calculations, we have

$$\begin{aligned}
&\partial_t \varphi_\epsilon + \frac{1}{\epsilon}\partial_a \varphi_\epsilon - \epsilon D \partial_{xx} \varphi_\epsilon - D|\partial_x \varphi_\epsilon|^2 + \mu(a, x/\epsilon) \\
=&\partial_t \varphi + \frac{\partial_a \phi_\delta}{\phi_\delta} - \epsilon D \partial_{xx} \varphi - D\frac{\partial_{xx}\phi_\delta}{\phi_\delta} + D\frac{|\partial_x \phi_\delta|^2}{\phi_\delta^2} - D\left(\partial_x \varphi + \frac{\partial_x \phi_\delta}{\phi_\delta}\right)^2 + \mu(a, x/\epsilon) \\
=&\partial_t \varphi - \epsilon D \partial_{xx} \varphi - H_\delta(\gamma) + D\gamma^2 - D|\partial_x \varphi|^2 + 2D(\gamma - \partial_x \varphi)\partial_x \ln \phi_\delta.
\end{aligned} \tag{4.22}$$

Due to $x_* > \bar{c}t_*$ and $\partial_t \varphi(t_*, x_*) < H_\delta(\gamma)$, we can choose some $r > 0$ small such that

$$B_r(t_*, x_*) \times [0, a_+] \subset \{u_* = 0\},$$

and for all $(t, x) \in B_r(t_*, x_*)$, there holds

$$\partial_t \varphi(t, x) - H_\delta(\gamma) - \epsilon D \partial_{xx} \varphi(t, x) + D(\gamma^2 - |\partial_x \varphi(t, x)|^2) + 2D(\gamma - \partial_x \varphi(t, x))\partial_x \ln \phi_\delta < 0,$$

provided that $\epsilon > 0$ is chosen small. Thus by (4.22) we arrive at

$$\partial_t \varphi_\epsilon + \frac{1}{\epsilon}\partial_a \varphi_\epsilon - \epsilon D \partial_{xx} \varphi_\epsilon - D|\partial_x \varphi_\epsilon|^2 + \mu(a, x/\epsilon) < 0 \ \text{ in } \ B_r(t_*, x_*) \times [0, a_+]. \tag{4.23}$$

On the other hand, recalling (4.3) one can see

$$\partial_t z_\epsilon + \frac{1}{\epsilon}\partial_a z_\epsilon - \epsilon D \partial_{xx} z_\epsilon - D|\partial_x z_\epsilon|^2 + \mu(a, x/\epsilon) = 0 \ \text{ in } \ B_r(t_*, x_*) \times [0, a_+]. \tag{4.24}$$

We shall claim that

$$\inf_{(t,x) \in B_r(t_*, x_*)} \inf_{a \in [0, a_+]} (z_\epsilon - \varphi_\epsilon) \geq \inf_{(t,x) \in \partial B_r(t_*, x_*)} \inf_{a \in [0, a_+]} (z_\epsilon - \varphi_\epsilon). \tag{4.25}$$

We assume (4.25) holds at the moment. By the definition of Z_*, it is easily seen that

$$\inf_{(t,x) \in B_r(t_*, x_*)} (Z_* - \varphi) \geq \inf_{(t,x) \in \partial B_r(t_*, x_*)} (Z_* - \varphi),$$

which is a contradiction since (t_*, x_*) is a strict local minimum of $Z_* - \varphi$. Hence, (4.19) holds.

It remains to prove (4.25). Indeed, if $z_\epsilon - \varphi_\epsilon$ attains its local minimum over $\overline{B}_r(t_*, x_*) \times [0, a_+]$ at some interior point $(t_\epsilon, a_\epsilon, x_\epsilon)$ with $(t_\epsilon, x_\epsilon) \in B_r(t_*, x_*)$ and $a_\epsilon \geq 0$, then we separate two cases to prove the claim.

Case 1: $a_\epsilon > 0$. In this case, $\partial_a z_\epsilon(t_\epsilon, a_\epsilon, x_\epsilon) = \partial_a \varphi_\epsilon(t_\epsilon, a_\epsilon, x_\epsilon)$. Combining (4.23) and (4.24), we evaluate them at $(t_\epsilon, a_\epsilon, x_\epsilon)$ to obtain $\partial_{xx} z_\epsilon(t_\epsilon, a_\epsilon, x_\epsilon) < \partial_{xx}\varphi_\epsilon (t_\epsilon, a_\epsilon, x_\epsilon)$, which is a contradiction since $(t_\epsilon, a_\epsilon, x_\epsilon)$ is a minimal point of $z_\epsilon - \varphi_\epsilon$.

Case 2: $a_\epsilon = 0$. In this case, we have

$$z_\epsilon(t_\epsilon, 0, x_\epsilon) - \epsilon \ln \phi_\delta(0, x_\epsilon/\epsilon) \leq z_\epsilon(t_\epsilon, a, x_\epsilon) - \epsilon \ln \phi_\delta(a, x_\epsilon/\epsilon), \quad \forall a \in [0, a_+],$$

which can be written as

$$\frac{u_\epsilon(t_\epsilon, 0, x_\epsilon)}{\phi_\delta(0, x_\epsilon/\epsilon)} = \min_{a \in [0, a_+]} \frac{u_\epsilon(t_\epsilon, a, x_\epsilon)}{\phi_\delta(a, x_\epsilon/\epsilon)} =: c_{\min}.$$

Recalling the integral boundary condition at $a_\epsilon = 0$, it follows that

$$\begin{aligned} &f\Big(x_\epsilon/\epsilon, \int_0^{a_c} \beta(a, x_\epsilon/\epsilon) u_\epsilon(t_\epsilon, a, x_\epsilon) da\Big) \\ =&c_{\min}\left(f_u\left(x_\epsilon/\epsilon, 0\right) - 2\delta\right) \int_0^{a_c} \beta\left(a, x_\epsilon/\epsilon\right) \phi_\delta\left(a, x_\epsilon/\epsilon\right) da. \end{aligned} \tag{4.26}$$

Due to $(t_\epsilon, a, x_\epsilon) \in \{u_* = 0\}$ for all $a \in [0, a_+]$, by Assumption 1.2-(ii) we can choose $\epsilon > 0$ further small if necessary such that

$$f\Big(x_\epsilon/\epsilon, \int_0^{a_c} \beta(a, x_\epsilon/\epsilon) u_\epsilon(t_\epsilon, a, x_\epsilon) da\Big) \geq \left(f_u\left(x_\epsilon/\epsilon, 0\right) - \delta\right) \int_0^{a_c} \beta\left(a, x_\epsilon/\epsilon\right) u_\epsilon(t_\epsilon, a, x_\epsilon) da.$$

By (4.26), we derive that

$$\begin{aligned} &\left(f_u\left(x_\epsilon/\epsilon, 0\right) - 2\delta\right) \int_0^{a_c} \beta\left(a, x_\epsilon/\epsilon\right) \phi_\delta\left(a, x_\epsilon/\epsilon\right) \Big(\frac{u_\epsilon(t_\epsilon, a, x)}{\phi_\delta\left(a, x_\epsilon/\epsilon\right)} - c_{\min}\Big) da \\ &+\delta \int_0^{a_c} \beta\left(a, x_\epsilon/\epsilon\right) u_\epsilon(t_\epsilon, a, x_\epsilon) da \leq 0, \end{aligned}$$

which concludes that

$$\int_0^{a_c} \beta\left(a, x_\epsilon/\epsilon\right) u_\epsilon(t_\epsilon, a, x_\epsilon) da \leq 0.$$

This is a contradiction, since $u_\epsilon(t_\epsilon, a, x_\epsilon) > 0$ for all $a \in [0, a_+]$ due to the strong maximum principle in Lemma 3.2. Therefore, (4.25) is proved and Step 1 is complete.

Step 2. We next show $Z_*(t, \bar{c}t) = 0$. Suppose on the contrary that $Z_*(t_*, \bar{c}t_*) < 0$ for some $t_* > 0$. Since $\{(t, x) : x \geq \bar{c}t, t > 0\} \times [0, a_+] \subset \{u_* = 0\}$ and $\{(t, \bar{c}t) : t > 0\} \times [0, a_+] \subset \partial\{u_* = 0\}$, by definitions it must hold

$$\liminf_{\substack{(t',x')\to(t_*,\bar{c}t_*)\\ x'/t'<\bar{c}}} Z_*(t',x') = 0 > Z_*(t_*,\bar{c}t_*). \tag{4.27}$$

We choose constant $M > 0$ large enough such that

$$M\bar{c} - DM^2 + 2M\,\|\partial_x \ln\phi_\delta\|_{L^\infty((0,a_+)\times\mathbb{R})} - H_\delta(0) + 1 < 0, \tag{4.28}$$

where $(H_\delta(0), \phi_\delta)$ denotes the principal eigenpair of (4.21) with $\gamma = 0$. By (4.27), we can define the function $\rho_M \in C^\infty(\mathbb{R})$ such that $Z_*(t,x) - \rho_M(x - \bar{c}t)$ attains a strict minimal point at $(t_*, \bar{c}t_*)$ and $\rho_M'(0) \le -M$. Then it follows from [2, Lemma 6.1] that

$$w_\epsilon(t,a,x) := z_\epsilon(t,a,x) - \epsilon\ln\phi_\delta(a, x/\epsilon) - \rho_M(x - \bar{c}t)$$

attains its minimal at some $(t_\epsilon, a_\epsilon, x_\epsilon) \in \mathbb{R}_+ \times [0, a_+] \times \mathbb{R}$, which satisfies $(t_\epsilon, x_\epsilon) \to (t_*, \bar{c}t_*)$ as $\epsilon \to 0$. Using the same arguments as in Step 1, we can deduce that $a_\epsilon > 0$. Hence, by evaluating the equation of z_ϵ in (4.3) at the point $(t_\epsilon, a_\epsilon, x_\epsilon)$, direct calculations yield

$$\begin{aligned}
&D\epsilon\partial_{xx}w_\epsilon(t_\epsilon,a_\epsilon,x_\epsilon)\\
=&D\epsilon\left[\partial_{xx}z_\epsilon(t_\epsilon,a_\epsilon,x_\epsilon) - \epsilon\partial_{xx}\ln\phi_\delta(a_\epsilon,x_\epsilon/\epsilon) - \rho_M''(x_\epsilon - \bar{c}t_\epsilon)\right]\\
=&\partial_t z_\epsilon + \frac{1}{\epsilon}\partial_a z_\epsilon - D|\partial_x z_\epsilon|^2 + \mu(a_\epsilon, x_\epsilon/\epsilon) - D\frac{\partial_{xx}\phi_\delta(a_\epsilon,x_\epsilon/\epsilon)}{\phi_\delta(a_\epsilon,x_\epsilon/\epsilon)}\\
&+ D\left(\partial_x\ln\phi_\delta(a_\epsilon,x_\epsilon/\epsilon)\right)^2 - D\epsilon\rho_M''(x_\epsilon - \bar{c}t_\epsilon)\\
=&-\bar{c}\rho_M'(x_\epsilon - \bar{c}t_\epsilon) - D\left|\rho_M'(x_\epsilon - \bar{c}t_\epsilon)\right|^2 - H_\delta(0)\\
&- 2\rho_M'(x_\epsilon - \bar{c}t_\epsilon)\partial_x\ln\phi_\delta(a_\epsilon,x_\epsilon/\epsilon) - D\epsilon\rho_M''(x_\epsilon - \bar{c}t_\epsilon),
\end{aligned}$$

where we used the equation of ϕ_δ in (4.21) with $\gamma = 0$. Due to $\rho_M'(0) \le -M$, by the choice of M in (4.28), we can choose $\epsilon > 0$ small if necessary such that $\partial_{xx}w_\epsilon(t_\epsilon, a_\epsilon, x_\epsilon) < 0$, which is a contradiction as $(t_\epsilon, a_\epsilon, x_\epsilon)$ is a minimal point of w_ϵ. This concludes the proof. □

Similarly, we can show the following lemma.

Lemma 4.3 *Let Z^* be defined in (4.18). Then $Z^*(t,x)$ is a upper semi-continuous viscosity solution of*

$$\begin{cases} \partial_t Z^* - H(\partial_x Z^*) \le 0, & t > 0,\ x \in \mathbb{R},\\ Z^*(t,\bar{c}t) = 0, & t > 0. \end{cases}$$

Proof By the definition of Z^* in (4.18), $Z^*(t,\bar{c}t) = 0$ is a direct consequence of

$$\{(t,\bar{c}t) : t > 0\} \times [0, a_+] \subset \partial\{u^* = 0\}.$$

It remains to show $\partial_t Z^* - H(\partial_x Z^*) \leq 0$ in the viscosity sense. The proof can follow by the ideas presented in Step 1 of Lemma 4.2 and we give a sketch for completeness.

Indeed, by the definition of viscosity sub-solutions, for any test function $\varphi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$, if $(t^*, x^*) \in \mathbb{R}_+ \times \mathbb{R}$ is a strict local maximum point of $Z^* - \varphi$, then we need to verify that

$$\partial_t \varphi(t^*, x^*) - H(\partial_x \varphi(t^*, x^*)) \leq 0. \tag{4.29}$$

Set $\gamma := \partial_x \varphi(t^*, x^*)$. Assume that (4.29) fails, namely $\partial_t \varphi(t^*, x^*) - H(\gamma) > 0$.

Let $\psi > 0$ be the principal eigenfunction of problem (1.5) with $\lambda = -\gamma$, and the associated principal eigenvalue is $H(-\gamma)$. We define

$$\varphi_\epsilon(t, a, x) := \varphi(t, x) + \epsilon \ln \psi(a, x/\epsilon).$$

Due to $H(-\gamma) = H(\gamma)$, by direct calculations as in (4.22), we have

$$\begin{aligned} &\partial_t \varphi_\epsilon + \frac{1}{\epsilon}\partial_a \varphi_\epsilon - \epsilon D \partial_{xx} \varphi_\epsilon - D|\partial_x \varphi_\epsilon|^2 + \mu(a, x/\epsilon) \\ =&\partial_t \varphi_t - \epsilon D \partial_{xx} \varphi - H(\gamma) + D\gamma^2 - D|\partial_x \varphi|^2 + 2D(\gamma - \partial_x \varphi)\partial_x \ln \psi. \end{aligned}$$

By the assumption that $\partial_t \varphi(t^*, x^*) > H(\gamma)$, we can choose $r > 0$ small such that

$$\partial_t \varphi_\epsilon + \frac{1}{\epsilon}\partial_a \varphi_\epsilon - \epsilon D \partial_{xx} \varphi_\epsilon - D|\partial_x \varphi_\epsilon|^2 + \mu(a, x/\epsilon) > 0 \quad \text{in} \quad B_r(t_*, x_*) \times [0, a_+], \tag{4.30}$$

provided that $\epsilon > 0$ is chosen small.

As in the proof of Lemma 4.2, we claim that

$$\sup_{(t,x)\in B_r(t^*,x^*)} \sup_{a\in[0,a_+]} (z_\epsilon - \varphi_\epsilon) \leq \sup_{(t,x)\in \partial B_r(t^*,x^*)} \sup_{a\in[0,a_+]} (z_\epsilon - \varphi_\epsilon). \tag{4.31}$$

If (4.31) holds, then by the definition of Z^* we have

$$\sup_{(t,x)\in B_r(t^*,x^*)} (Z^* - \varphi) \leq \sup_{(t,x)\in \partial B_r(t^*,x^*)} (Z^* - \varphi),$$

which is a contradiction since (t^*, x^*) is a strict local maximum of $Z^* - \varphi$. Hence, (4.29) holds and Lemma 4.3 is proved.

To prove (4.31), we assume that $z_\epsilon - \varphi_\epsilon$ attains its maximum at some interior point $(t^\epsilon, a^\epsilon, x^\epsilon)$ over $\overline{B}_r(t^*, x^*) \times [0, a_+]$ with $a^\epsilon \geq 0$. If $a^\epsilon > 0$, then comparing (4.30) with the equation of z_ϵ in (4.3) yields $\partial_{xx} z_\epsilon(t^\epsilon, a^\epsilon, x^\epsilon) > \partial_{xx} \varphi_\epsilon(t^\epsilon, a^\epsilon, x^\epsilon)$, which is a contradiction since $(t^\epsilon, a^\epsilon, x^\epsilon)$ is a maximum point of $z_\epsilon - \varphi_\epsilon$. It remains to consider the case $a^\epsilon = 0$. In this case,

$$z_\epsilon(t^\epsilon, 0, x^\epsilon) - \epsilon \ln \psi(0, x/\epsilon) \geq z_\epsilon(t^\epsilon, a, x^\epsilon) - \epsilon \ln \psi(a, x^\epsilon/\epsilon), \quad \forall a \in [0, a_+].$$

By the definition of z_ϵ, it can be written as

$$\frac{u_\epsilon(t^\epsilon,0,x^\epsilon)}{\psi(0,x^\epsilon/\epsilon)}=\max_{a\in[0,a_+]}\frac{u_\epsilon(t^\epsilon,a,x^\epsilon)}{\psi(a,x^\epsilon/\epsilon)}=:c_{\max}.$$

By the boundary conditions of z_ϵ and ψ in (4.3) and (1.5), it follows that

$$\begin{aligned}&f\Big(x^\epsilon/\epsilon,\int_0^{a_c}\beta\left(a,x^\epsilon/\epsilon\right)u_\epsilon(t^\epsilon,a,x^\epsilon)da\Big)\\&=c_{\max}f_u\left(x^\epsilon/\epsilon,0\right)\int_0^{a_c}\beta\left(a,x^\epsilon/\epsilon\right)\psi\left(a,x^\epsilon/\epsilon\right)da.\end{aligned}\tag{4.32}$$

Next by Assumption 1.2 again, one obtains

$$f\Big(x^\epsilon/\epsilon,\int_0^{a_c}\beta\left(a,x^\epsilon/\epsilon\right)u_\epsilon(t^\epsilon,a,x^\epsilon)da\Big)\le f_u\left(x^\epsilon/\epsilon,0\right)\int_0^{a_c}\beta\left(a,x^\epsilon/\epsilon\right)u_\epsilon(t^\epsilon,a,x^\epsilon)da,$$

which together with (4.32) implies that

$$0\le f_u\left(x^\epsilon/\epsilon,0\right)\int_0^{a_c}\beta\left(a,x^\epsilon/\epsilon\right)\psi\left(a,x^\epsilon/\epsilon\right)\Big(c_{\max}-\frac{u_\epsilon(t^\epsilon,a,x^\epsilon)}{\psi\left(a,x^\epsilon/\epsilon\right)}\Big)da\le 0.$$

Hence, there exists some $\bar{a}^\epsilon>0$ such that $u_\epsilon(t^\epsilon,\bar{a}^\epsilon,x^\epsilon)/\psi\left(\bar{a}^\epsilon,x^\epsilon/\epsilon\right)=c_{\max}$. Recalling the definition of z_ϵ, this implies that $z_\epsilon(t^\epsilon,\cdot,x^\epsilon)-\epsilon\ln\psi(\cdot,x^\epsilon/\epsilon)$ attains its maximum at $\bar{a}^\epsilon>0$, and thus we can obtain a contradiction as in the case of $a^\epsilon>0$. The proof is complete. □

The following result states that the function z_* defined by (4.17) is independent of $a\in[0,a_+]$.

Lemma 4.4 *Let function z_* be defined by (4.17). Then $z_*=z_*(t,x)$ is independent of $a\in[0,a_+]$ and is a viscosity solution of the Hamilton-Jacobi equation*

$$\begin{cases}\partial_t Z-H(\partial_x Z)=0, & x>\bar{c}t,\ t>0,\\ Z(t,\bar{c}t)=0, & t>0,\\ Z(0,0)=0,\\ Z(t,x)\to-\infty\ \ as & t\to 0, x>0,\end{cases}\tag{4.33}$$

where $\bar{c}\in(0,c_*]$ is defined by (4.16).

Proof Let Z be a viscosity solution of (4.33), which can be given by certain action functional as in [20]. Recall the definitions of Z_* and Z^* in (4.18). Since $u_0(a,0)>0$ for all $a\in[0,a_+]$, it follows that $Z_*(0,0)=Z^*(0,0)=0$. Based on the Lemmas 4.2 and 4.3, we can apply the similar arguments in [37, Lemmas 3.6 and 3.7] (see also [20, Lemma 3.1]) to show that

$$Z^*(t,x) \le Z(t,x) \le Z_*(t,x) \quad \text{for all } x > \bar{c}t \text{ and } t > 0,$$

for which the details are omitted here. In view of $Z_* \le Z^*$, this implies that

$$Z_*(t,x) = Z^*(t,x) \quad \text{for all } x > \bar{c}t \text{ and } t > 0.$$

Noting that $Z_* \le z_* \le Z^*$ by definition, this implies that $z_* \equiv Z_* \equiv Z^*$ in $\{u_* = 0\}$ and satisfies the Hamilton-Jacobi equation (4.33), which is independent of $a \in [0, a_+]$. Together with $z_* = 0$ in $\{u_* > 0\}$, we conclude that $z_* = z_*(t,x)$ is independent of $a \in [0, a_+]$. The proof is complete. □

We conclude this subsection by establishing the connection between z_* and u_ϵ defined by (4.17) and (4.2), respectively.

Lemma 4.5 *Assume $H(0) > 0$. Then for any $(t^*, x^*) \in \mathrm{Int}\{z_* = 0\}$ and $a_+ \in [a_c, a_m)$, there holds*

$$\liminf_{\substack{\epsilon \to 0 \\ (t,a,x)\to(t^*,s,x^*)}} u_\epsilon(t,a,x) > 0, \quad \forall s \in [0, a_+].$$

Proof Suppose that Lemma 4.5 fails, then a direct application of Proposition 4.1 yields

$$\liminf_{\substack{\epsilon \to 0 \\ (t,a,x)\to(t^*,s,x^*)}} u_\epsilon(t,a,x) = 0, \quad \forall s \in [0, a_+]. \tag{4.34}$$

Due to $(t^*, x^*) \in \mathrm{Int}\{z_* = 0\}$, there exists some $\eta > 0$ small such that $z_* = 0$ in $B_\eta(t^*, x^*)$, so that by (4.17), z_ϵ converges to zero uniformly in $B_\eta(t^*, x^*) \times [0, a_+]$ as $\epsilon \to 0$. Define the test function $\phi(t,x) := -(t-t^*)^2 - (x-x^*)^2$. For each $\delta > 0$, let $H_\delta(0)$ be the principal eigenvalue of

$$\begin{cases} \partial_a \varphi(a,x) = D\partial_{xx}\varphi - \mu(a,x)\varphi - H_\delta(0)\varphi, & (a,x) \in (0, a_+] \times \mathbb{R}, \\ \varphi(0,x) = (f_u(x,0) - 2\delta)\int_0^{a_c} \beta(a,x)\varphi(a,x)da, & x \in \mathbb{R}, \\ \varphi(a,x) = \varphi(a, x+L), & (a,x) \in (0, a_+] \times \mathbb{R}. \end{cases} \tag{4.35}$$

Due to $H(0) > 0$, we choose $\delta > 0$ small such that $H_\delta(0) > 0$.

We first claim that there exists the sequence $(t_\epsilon, x_\epsilon, a_\epsilon) \in B_\eta(t^*, x^*) \times [0, a_+]$ such that

$$w_\epsilon(t,a,x) := z_\epsilon(t,a,x) - \epsilon \ln \varphi(a, x/\epsilon) - \phi(t,x)$$

attains its local minimum at point $(t_\epsilon, x_\epsilon, a_\epsilon)$, and

$$(t_\epsilon, x_\epsilon, a_\epsilon) \to (t^*, x^*, a^*) \quad \text{and} \quad z_\epsilon(t_\epsilon, x_\epsilon, a_\epsilon) \to z_*(t^*, x^*) \quad \text{as } \epsilon \to 0, \tag{4.36}$$

for some $a^* \in [0, a_+]$, where $\varphi > 0$ denotes the principal eigenfunction of problem (4.35) associated with $H_\delta(0)$. Indeed, note that $z_\epsilon - \epsilon \ln \varphi \to 0$ uniformly in $B_\eta(t^*, x^*) \times [0, a_+]$ as $\epsilon \to 0$. For any $\sigma \in (0, \eta^2/8)$, there exists $\epsilon_0 > 0$ such that

$$\|z_\epsilon - \epsilon \ln \varphi\|_{L^\infty(B_\eta(t^*,x^*)\times[0,a_+])} \le \sigma, \quad \forall \epsilon \in (0, \epsilon_0).$$

Hence, for all $\epsilon \in (0, \epsilon_0)$ and $(t, x, a) \in (B_\eta(t^*, x^*) \setminus B_{2\sqrt{\sigma}}(t^*, x^*)) \times [0, a_+]$,

$$\begin{aligned} w_\epsilon(t, a, x) &= z_\epsilon(t, a, x) - \epsilon \ln \varphi(a, x/\epsilon) - \phi(t, x) \ge 3\sigma \\ &> z_\epsilon(t^*, a^*, x^*) - \epsilon \ln \varphi(a^*, x^*/\epsilon) + \sigma = w_\epsilon(t^*, a^*, x^*) + \sigma. \end{aligned}$$

This implies that w_ϵ has a local minimum at some point $(t_\epsilon, x_\epsilon, a_\epsilon) \in B_{2\sqrt{\sigma}}(t^*, x^*) \times [0, a_+]$, that is $(t_\epsilon - t^*)^2 + (x_\epsilon - x^*)^2 \le 4\sigma$, and thus (4.36) holds.

Next, by (4.34) we apply the same arguments of Case 2 in Lemma 4.2 to show $a_\epsilon > 0$, so that

$$\partial_a z_\epsilon(t_\epsilon, a_\epsilon, x_\epsilon) = \epsilon \partial_a \ln \varphi\,(a_\epsilon, x_\epsilon/\epsilon)\,.$$

By evaluating the equation of z_ϵ in (4.3) at $(t_\epsilon, a_\epsilon, x_\epsilon)$, we have

$$\begin{aligned} &\partial_t \phi + \partial_a \ln \varphi(a_\epsilon, x_\epsilon/\epsilon) \\ &\ge D\epsilon\partial_{xx}\phi + D(\partial_{xx}\varphi/\varphi - |\partial_x \ln \varphi|^2)\,(a_\epsilon, x_\epsilon/\epsilon) - \mu\,(a_\epsilon, x_\epsilon/\epsilon) \\ &\quad + D\left(\partial_x\phi + \partial_x \ln \varphi\,(a_\epsilon, x_\epsilon/\epsilon)\right)^2 \\ &\ge D\epsilon\partial_{xx}\phi + D\,(\partial_{xx}\varphi/\varphi)\,(a_\epsilon, x_\epsilon/\epsilon) + D|\partial_x\phi|^2 - \mu\,(a_\epsilon, x_\epsilon/\epsilon) \\ &\quad + 2D\partial_x\phi\partial_x \ln \varphi\,(a_\epsilon, x_\epsilon/\epsilon)\,. \end{aligned}$$

By the definition of φ in (4.35), we calculate that

$$\partial_t\phi - D\epsilon\partial_{xx}\phi \ge 2D\partial_x\phi\partial_x \ln \varphi\,(a_\epsilon, x_\epsilon/\epsilon) + D|\partial_x\phi|^2 + H_\delta(0) \quad \text{at} \;\; (t_\epsilon, a_\epsilon, x_\epsilon),$$

from which letting $\epsilon \to 0$, by (4.36) and the definition of test function ϕ, we deduce $H_\delta(0) \le 0$, contradicting our assumption that $H_\delta(0) > 0$. This completes the proof. □

4.3 Inner spreading

In this subsection, we continue to complete the proof of Theorem 1.2-(ii). To this end, we first give a lower bound on z_* motivated by Berestycki and Nadin [5, Lemma 4.4].

Lemma 4.6 *Let z_* be defined by (4.17). Then $z_*(t, x) \ge \min\{-tH^*(-x/t), 0\}$ for all $t > 0$ and $x > 0$, where H^* is the convex conjugate of H, defined by $H^*(q) = \sup_{\lambda\in\mathbb{R}}(q\lambda - H(\lambda))$.*

Proof Define $U(t,x) := -t^{-1}z_*(t,-tx)$ for all $t,x>0$. By (4.33) in Lemma 4.4, we have

$$\partial_t U(t,x) = -\frac{1}{t}U(t,x) - \frac{1}{t}H(\partial_x z_*(t,-tx)) + \frac{x}{t}\partial_x z_*(t,-tx), \quad x>\bar{c}t,\ t>0,$$

in the sense of viscosity solutions. As $H(\lambda)+H^*(x)\geq \lambda x$ for all $\lambda, x\in\mathbb{R}$, it follows that

$$\partial_t U(t,x) \leq -\frac{1}{t}U(t,x) + \frac{1}{t}H^*(x), \quad x>\bar{c}t,\ t>0. \tag{4.37}$$

By the definition of z_* in (4.17), we have $z_*(\alpha t,\alpha x)=\alpha z_*(t,x)$ for all $\alpha>0$. Hence, $U(t,x)=-z_*(1,-x)$ and in particular, $\partial_t U(t,x)=0$ in the sense of viscosity solutions for all $x>\bar{c}t$ and $t>0$. It follows from (4.37) that $U(t,x)\leq H^*(x)$.

Then we deduce that

$$z_*(t,x) = -tU(t,-x/t) \geq -tH^*(-x/t) \geq \min\{-tH^*(-x/t),0\}, \quad x>\bar{c}t,\ t>0. \tag{4.38}$$

Note from the definition of $\bar{c}$ in (4.16) that $z_*(t,x)=0$ for all $x\leq\bar{c}t$, which together with (4.38) completes the proof. □

For any $c_1<c_2$ and $t>0$, we define the set

$$S_t(c_1,c_2) := \{x\in\mathbb{R} : c_1 t < x < c_2 t\}. \tag{4.39}$$

To prove Theorem 1.2-(ii), we prepare the following result.

Lemma 4.7 *Let $\tau>0$ and $u_0(a,x), \tilde{u}_0(a,x)\geq 0$ for all $(a,x)\in[0,a_m)\times\mathbb{R}$. Assume that u and $\tilde{u}$ are the solutions of (1.1) with initial data $u(0,a,x)=u_0(a,x)$ and $\tilde{u}(0,a,x)=\tilde{u}_0(a,x)$, respectively. If $u_0(a,x)=\tilde{u}_0(a,x)$ for $(a,x)\in[0,a_m)\times S_\tau(c_1,c_2)$, then for any $\theta\in(0,(c_2-c_1)/2)$, there holds*

$$|u(t,a,x)-\tilde{u}(t,a,x)| \leq Me^{(D-\mu_{\inf})t}e^{-\theta\tau}, \quad \forall t\geq 0,\ (a,x)\in[0,a_m)\times S_\tau(c_1+\theta,c_2-\theta),$$

where $M>0$ is some constant independent of t and τ. In particular, fixing $T_1>0$, for any $\sigma>0$, one can find T such that if $u_0(a,x)=\tilde{u}_0(a,x)$ for $(a,x)\in[0,a_m)\times S_\tau(c_1,c_2)$ for some $\tau>T$, then

$$|u(T_1,a,x)-\tilde{u}(T_1,a,x)| \leq \sigma, \quad (a,x)\in[0,a_m)\times S_\tau(c_1+\theta,c_2-\theta).$$

Proof Fix any $z\in\mathbb{R}$, consider the space

$$X_z = \{\phi\in C(\mathbb{R}) : \sup_{x\in\mathbb{R}} e^{-|x-z|}|\phi(x)| < +\infty\}$$

equipped with the norm $\|\phi\|_{X_z} = \sup_{x\in\mathbb{R}} e^{-|x-z|}|\phi(x)|$. Recall that $\{\mathcal{U}_0(a,s)\}_{0\le s\le a<a_m}$ is defined in (2.4) for the case of $\lambda = 0$. It is easily seen that it is positive and bounded in X_z satisfying

$$\|\mathcal{U}_0(a,s)\|_{\mathcal{L}(X_z)} \le e^{(D-\mu_{\inf})(a-s)} \quad \text{for all} \ \ 0 \le s \le a < a_m. \tag{4.40}$$

Indeed, due to the property of Green's function defined in (2.4), for any $\eta \in X_z$ we have

$$\begin{aligned}
\|\mathcal{U}_0(a,s)\eta\|_{X_z} &= \sup_{x\in\mathbb{R}} e^{-|x-z|} \left| \int_{\mathbb{R}} G_0(a,s;x-y)\eta(y)dy \right| \\
&\le \sup_{x\in\mathbb{R}} e^{-|x-z|} \int_{\mathbb{R}} G_0(a,s;x-y)|\eta(y)|dy \\
&= \sup_{x\in\mathbb{R}} e^{-|x-z|} \int_{\mathbb{R}} G_0(a,s;y)|\eta(x-y)|dy \\
&\le e^{-\mu_{\inf}(a-s)} \sup_{x\in\mathbb{R}} e^{-|x-z|} \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4D(a-s)}}}{2\sqrt{D\pi(a-s)}} |\eta(x-y)|dy \\
&\le e^{-\mu_{\inf}(a-s)} \|\eta\|_{X_z} \int_{\mathbb{R}} \frac{e^{-\frac{y^2}{4D(a-s)}+|y|}}{2\sqrt{D\pi(a-s)}} dy \\
&= e^{-\mu_{\inf}(a-s)} \|\eta\|_{X_z} \int_{\mathbb{R}} \frac{e^{-\frac{(|y|-2D(a-s))^2}{4D(a-s)}+D(a-s)}}{2\sqrt{D\pi(a-s)}} dy \\
&= e^{(D-\mu_{\inf})(a-s)} \|\eta\|_{X_z},
\end{aligned}$$

where, in the second inequality, we used the fact that $D\Delta u - \mu u \le D\Delta u - \mu_{\inf} u$ for nonnegative $u \in X_z$. Hence, (4.40) is proved.

Set $w := u - \tilde{u}$. Direct calculation gives

$$\begin{cases}
\partial_t w + \partial_a w = D\partial_{xx} w - \mu w, & (a,x) \in (0,a_m)\times\mathbb{R}, \ t>0, \\
w(t,0,x) = c(t,x)\int_0^{a_c} \beta(a,x)w(t,a,x)da, & (a,x) \in (0,a_m)\times\mathbb{R}, \ t>0, \\
w(0,a,x) = w_0(a,x), & (a,x) \in (0,a_m)\times\mathbb{R},
\end{cases} \tag{4.41}$$

where $w_0 = u_0 - \tilde{u}_0$ and

$$c(t,x) = \int_0^1 f_u \left(x, (1-s)\int_0^{a_c} \beta(a,x)\tilde{u}(t,a,x)da + s\int_0^{a_c} \beta(a,x)u(t,a,x)da \right) ds.$$

Recall that, via the method of characteristics [61], the solution of (4.41) can be written as

$$w(t,a)=\begin{cases}\mathcal{U}_0(a,a-t)w_0(a-t), & a\ge t,\\ \mathcal{U}_0(a,0)w(t-a,0), & a<t,\end{cases} \tag{4.42}$$

where we write $w(t,a)=w(t,a,x)$ for simplicity.

Plugging (4.42) into the equation of $w(t, 0)$ in (4.41) yields that

$$w(t,0)=c(t,\cdot)\left[\int_0^t \chi(a)\beta(a,\cdot)\mathcal{U}_0(a,0)w(t-a,0)da+\int_t^{a_c}\chi(a)\beta(a,\cdot)\mathcal{U}_0(a,a-t)w_0(a-t)da\right], \tag{4.43}$$

whereas $\chi(a)$ denotes the cutoff function satisfying $\chi(a)=1$ for $a\in(0,a_c)$ and $\chi(a)=0$ for $a\in[a_c,a_m)$. Now we consider the following two cases.

Case 1. If $t<a_c$, then (4.43) can be written as

$$w(t,0)=c(t,\cdot)\left[\int_0^t \beta(a,\cdot)\mathcal{U}_0(a,0)w(t-a,0)da+\int_t^{a_c}\beta(a,\cdot)\mathcal{U}_0(a,a-t)w_0(a-t)da\right].$$

By Assumption 1.2, using (4.40) we have

$$\begin{aligned}\|w(t,0)\|_{X_z}\le&\overline{f}(0)\left[\int_0^t \overline{\beta}(a)e^{(D-\mu_{\inf})a}\|w(t-a,0)\|_{X_z}da+\int_t^{a_c}\overline{\beta}(a)e^{(D-\mu_{\inf})t}\|w_0(a-t)\|_{X_z}da\right]\\ \le&\overline{f}(0)\left[\int_0^t \overline{\beta}(t-s)e^{(D-\mu_{\inf})(t-s)}\|w(s,0)\|_{X_z}ds+e^{(D-\mu_{\inf})t}\|\overline{\beta}\|_{L^\infty(0,a_c)}\int_0^{a_c}\|w_0(a)\|_{X_z}da\right],\end{aligned}$$

where $\overline{f}(0)$ and $\overline{\beta}(a)$ are defined by (2.1). The Gronwall's inequality implies that

$$\|w(t,0)\|_{X_z}\le\overline{f}(0)\|\overline{\beta}\|_{L^\infty(0,a_c)}e^{(D-\mu_{\inf})t}e^{\overline{f}(0)\|\overline{\beta}\|_{L^1(0,a_c)}}\int_0^{a_c}\|w_0(a)\|_{X_z}da\quad\text{for}\quad t<a_c. \tag{4.44}$$

Case 2. If $t\ge a_c$, then (4.43) can be written as

$$w(t,0)=c(t,\cdot)\int_0^{a_c}\beta(a,\cdot)\mathcal{U}_0(a,0)w(t-a,0)da.$$

By the similar argument as in (4.44), we can use (4.40) to derive

$$\|w(t,0)\|_{X_z}\le\overline{f}(0)\int_0^t\overline{\beta}(t-s)e^{(D-\mu_{\inf})(t-s)}\|w(s,0)\|_{X_z}ds.$$

The Gronwall's inequality concludes that

$$\|w(t,0)\|_{X_z}=0\quad\text{for}\quad t\ge a_c. \tag{4.45}$$

Thanks to the above two cases, we have completed the estimates of $\|w(t,0)\|_{X_z}$ for any $(t,a)\in(0,\infty)\times[0,a_c]$. Now let us finish the estimates of $\|w(t,a)\|_{X_z}$ for any $(t,a)\in(0,\infty)\times[0,a_c]$ via (4.42). By (4.42), (4.44) and (4.45), we observe that

$$\|w(t,a)\|_{X_z} \le e^{(D-\mu_{\inf})a}\|w(t-a,0)\|_{X_z} \le \overline{f}(0)\|\overline{\beta}\|_{L^\infty(0,a_c)}e^{(D-\mu_{\inf})t}e^{\overline{f}(0)\|\overline{\beta}\|_{L^1(0,a_c)}}\int_0^{a_c}\|w_0(a)\|_{X_z}\,da, \quad \forall t \ge a. \tag{4.46}$$

It follows from (4.40) and (4.42) that

$$\|w(t,a)\|_{X_z} \le e^{(D-\mu_{\inf})t}\|w_0(a-t)\|_{X_z}, \quad \forall t < a. \tag{4.47}$$

In summary, for all $t < a$ and $(a,z) \in [0,a_m) \times S_\tau(c_1+\theta, c_2-\theta)$, by (4.47) we have

$$\begin{aligned}
|u(t,a,z) - \tilde{u}(t,a,z)| &\le e^{(D-\mu_{\inf})t}\sup_{x\in\mathbb{R}}\left(e^{-|x-z|}|u(0,a-t,x) - \tilde{u}(0,a-t,x)|\right)\\
&= e^{(D-\mu_{\inf})t}\sup_{x\in\mathbb{R}\setminus S_\tau(c_1,c_2)}\left(e^{-|x-z|}|u(0,a-t,x) - \tilde{u}(0,a-t,x)|\right)\\
&\le Ce^{(D-\mu_{\inf})t}\sup_{x\in\mathbb{R}\setminus S_\tau(c_1,c_2)} e^{-|x-z|}\\
&\le Ce^{(D-\mu_{\inf})t}e^{-\theta\tau},
\end{aligned}$$

where $C > 0$ depends only on u_0 and $\tilde{u}_0$. On the other hand, (4.46) implies that

$$\|w(t,0)\|_{X_z} \le Me^{(D-\mu_{\inf})t}\sup_{a\in[0,a_c]}\|w_0(a)\|_{X_z}, \quad \forall t \ge a,$$

where $M = a_c\overline{f}(0)\|\overline{\beta}\|_{L^\infty(0,a_c)}e^{\overline{f}(0)\|\overline{\beta}\|_{L^1(0,a_c)}}$. Then for all $t \ge a$ and $(a,z) \in [0,a_m) \times S_\tau(c_1+\theta, c_2-\theta)$, we derive that

$$\begin{aligned}
|u(t,a,z) - \tilde{u}(t,a,z)| &\le Me^{(D-\mu_{\inf})t}\sup_{x\in\mathbb{R}}\left(e^{-|x-z|}|u(0,a,x) - \tilde{u}(0,a,x)|\right)\\
&= Me^{(D-\mu_{\inf})t}\sup_{x\in\mathbb{R}\setminus S_\tau(c_1,c_2)}\left(e^{-|x-z|}|u(0,a,x) - \tilde{u}(0,a,x)|\right)\\
&\le CMe^{(D-\mu_{\inf})t}\sup_{x\in\mathbb{R}\setminus S_\tau(c_1,c_2)} e^{-|x-z|}\\
&\le CMe^{(D-\mu_{\inf})t}e^{-\theta\tau}.
\end{aligned}$$

Thus the proof is complete. □

We are in a position to complete the proof of Theorem 1.2-(ii).

Proof (Proof of Theorem 1.2-(ii))

Let c^* be defined by (1.6). We first show $\bar{c} = c^*$, where $\bar{c}$ is defined by

$$\{u_* = 0\} = \{(t,x) : x \ge \bar{c}t,\ t \ge 0\} \times [0,a_m).$$

Since $\bar{c} \le c^*$ by Theorem 1.2-(i), we assume by contradiction that $\bar{c} < c^*$. By the symmetry of $H(\lambda)$ proved by Proposition 1.6-(i), it follows that

$$c^* = \min_{\lambda>0} \frac{H(\lambda)}{\lambda} = \min_{\lambda>0} \frac{H(-\lambda)}{\lambda}.$$

Then for any $c \in (\bar{c}, c^*)$, there exists some $\epsilon > 0$ such that $H(-\lambda) \ge (1+\epsilon)\lambda c$ for all $\lambda > 0$. As $H(0) > 0$ and $\lambda \mapsto H(\lambda)$ is continuous, there exists $\delta > 0$ such that $H(-\lambda) \ge \lambda c + \delta$ for all $\lambda > 0$, which implies from the definition of convex conjugate that $-H^*(-c) > 0$. Lemma 4.6 together with the continuity of H^* yields that for all (t, x) in a neighborhood of $(1, c)$,

$$z_*(t,x) \ge \min\{-tH^*(-x/t),\ 0\} = 0,$$

which particularly implies $(1,c) \in \mathrm{Int}\{z_* = 0\}$ for all $c \in (\bar{c}, c^*)$. For any $a_+ \in [a_c, a_m)$, an application of Lemma 4.5 yields that

$$\liminf_{\epsilon\to 0} u_\epsilon(1,a,c) = \liminf_{\epsilon\to 0} u\left(\frac{1}{\epsilon}, a, \frac{c}{\epsilon}\right) = \liminf_{t\to\infty} u(t,a,ct) > 0, \quad \forall a \in [0,a_+],$$

which is a contradiction since $\{(t,ct) : t > 0\} \times [0,a_+] \subset \{u_* = 0\}$. Therefore, $\bar{c} = c^*$, namely

$$\{u_* = 0\} = \{(t,x) : x \ge c^* t,\ t \ge 0\} \times [0,a_+]. \tag{4.48}$$

Next, let us prove

$$\liminf_{t\to\infty} \sup_{a\in[0,a_+],0<x<ct} |u(t,a,x) - u^*(a,x)| = 0, \quad \forall c \in (0,c^*).$$

By (4.48), one can fix any $c \in (0, c^*)$ and define

$$\alpha := \frac{1}{2} \liminf_{t\to\infty} \inf_{(a,x)\in[0,a_+]\times S_t(0,c)} u(t,a,x) > 0, \tag{4.49}$$

where the set $S_t(0,c)$ is defined by (4.39). Let v be the solution of (1.1) with initial value $u_0 \equiv \alpha$. Then by the uniqueness of the solution to problem (1.1), $v(t, a, x)$ is periodic in x for all $t > 0$ and $a \in [0,a_+]$. Hence, Theorem 1.1-(i) implies that $v(t,a,x) \to u^*(a,x)$ in $C([0,a_+] \times \mathbb{R})$ as $t \to +\infty$. Hence, for any $\sigma > 0$, there exists $T_1 > 0$ such that

$$v(t,a,x) \ge u^*(a,x) - \sigma \quad \text{for all } t \ge T_1 \text{ and } (a,x) \in [0,a_+] \times \mathbb{R}. \tag{4.50}$$

Motivated by the proof of [36, Theorem 5.1], let $u^\tau(t,a,x)$ be the solution of (1.1) with initial value $u^\tau(0,a,x) = \min_{x\in\mathbb{R}}\{\alpha, u(\tau,a,x)\}$. It follows from (4.49) that $u^\tau(0,a,x) = \alpha$ for $x \in S_\tau(0,c)$, whenever $\tau > 0$ is large, say $\tau > T_2$ for some

$T_2 > 0$. Applying Lemma 4.7, for any $0 < \theta < c/2$, one can find a constant $T_3 \geq T_2$ such that for all $\tau > T_3$,

$$|v(T_1, a, x) - u^\tau(T_1, a, x)| \leq \sigma, \quad \forall (a, x) \in [0, a_+] \times S_\tau(\theta, c - \theta).$$

This together with (4.50) gives

$$u^\tau(T_1, a, x) \geq u^*(a, x) - 2\sigma \quad \text{for all } \tau > T_3 \text{ and } (a, x) \in [0, a_+] \times S_\tau(\theta, c - \theta). \tag{4.51}$$

Note that there exists some $T_4 > 0$ such that

$$S_\tau(2\theta, c - 2\theta) \subset S_{\tau - T_1}(\theta, c - \theta), \quad \forall \tau \geq T_4.$$

Now taking $T_0 = T_1 + T_3 + T_4$, it follows from (4.51) that for all $t > T_0$,

$$u^{t-T_1}(T_1, a, x) \geq u^*(a, x) - 2\sigma, \quad \forall (a, x) \in [0, a_+] \times S_t(2\theta, c - 2\theta).$$

On the other hand, comparison principle yields that for all $t, s \geq 0$,

$$u(s + t, a, x) \geq u^t(s, a, x), \quad \forall (a, x) \in [0, a_+] \times \mathbb{R}.$$

Therefore, for $t \geq T_0$, we have

$$u(t, a, x) \geq u^{t-T_1}(T_1, a, x) \geq u^*(a, x) - 2\sigma, \quad \forall (a, x) \in [0, a_+] \times S_t(2\theta, c - 2\theta).$$

Due to the arbitrariness of θ, it follows that

$$\lim_{t \to \infty} \sup_{0 \leq x < c' t} \sup_{a \in [0, a_+]} |u(t, a, x) - u^*(a, x)| = 0, \quad \forall c' \in (0, c).$$

The proof is now complete. □

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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