ISRAEL JOURNAL OF MATHEMATICS **TBD** (2024), 1–39
DOI: 10.1007/s11856-024-2708-8

AGE-STRUCTURED MODELS WITH NONLOCAL DIFFUSION OF DIRICHLET TYPE. II: GLOBAL DYNAMICS*

BY

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* Research was partially supported by National Natural Science Foundation of China (No. 12301259 and No. 12371169) and National Science Foundation (DMS-2052648 and DMS-2424605).
** H. Kang would like to acknowledge the region Normandie for financial support during his postdoctoral study at UNIHAVRE in 2000–2022.
Received June 27, 2022 and in revised form December 31, 2022

ABSTRACT

Age-structured models with nonlocal diffusion arise naturally in describing the population dynamics of biological species and the transmission dynamics of infectious diseases in which individuals disperse nonlocally and interact each other, and the age structure of individuals matters. In the second part of this series of papers, we study the effects of principal eigenvalues on the global dynamics of the equation with monotone nonlinearity on the birth rate. More precisely, we analyze the existence and uniqueness of a nontrivial equilibrium and its stability for the age-structured model with nonlocal diffusion under certain assumptions via the sign of spectral bound of a linearized operator. Moreover, we investigate the asymptotic properties of the nontrivial equilibrium with respect to the diffusion rate and diffusion range.

1. Introduction

In this paper, we continue to study the global dynamics of age-structured models with nonlocal diffusion of Dirichlet type and monotone nonlinearity on the birth rate. More precisely, we are interested in the following age-structured model with nonlocal diffusion of Dirichlet type:

$$(1.1)\begin{cases}(\partial_t+\partial_a)u(t,a,x)=D[\int_\Omega J(x-y)u(t,a,y)dy-u(t,a,x)]-\mu(a,x)u(t,a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,a,x)\in(0,\infty)\times(0,\hat{a})\times\overline{\Omega}, \\ u(t,0,x)=f(\int_0^{\hat{a}}\beta(a,x)u(t,a,x)da), \qquad (t,x)\in(0,\infty)\times\overline{\Omega}, \\ u(0,a,x)=u_0(a,x), \qquad (a,x)\in(0,\hat{a})\times\overline{\Omega},\end{cases}$$

where $u(t,a,x)$ denotes the density of a population at time t, age a and position x, J is the dispersal kernel and f is a monotone type nonlinearity describing the birth rate of the population. The background and motivation of studying the above model were given in our first paper, the interested readers can refer to [5]. We only mention that the nonlocal diffusion operator in (1.1) corresponds to zero Dirichlet boundary condition, which indicates that the region outside the habitat, $\mathbb{R}^N\setminus\overline{\Omega}$, is so hostile that the population cannot survive there, see Hutson et al. [6].

Here $\hat{a}\in(0,\infty]$ represents the maximum age and $\Omega\subset\mathbb{R}^N$ is a bounded domain. Moreover, $D>0$ is the diffusion rate and the diffusion kernel J satisfies the following assumption.

Assumption 1.1: The kernel $J \in C(\mathbb{R}^N)$ is nonnegative and supported in $B(0,r)$ for some $r > 0$, where $B(0,r) \subset \mathbb{R}^N$ is the open ball centered at 0 with radius r. In addition, J satisfies $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$.

Next we present assumptions on the birth rate $\beta = \beta(a,x)$ and the death rate $\mu = \mu(a,x)$. Define

$$\underline{\mu}(a) := \min_{x\in\overline{\Omega}} \mu(a,x), \quad \overline{\mu}(a) := \max_{x\in\overline{\Omega}} \mu(a,x),$$
$$\underline{\beta}(a) := \min_{x\in\overline{\Omega}} \beta(a,x), \quad \overline{\beta}(a) := \max_{x\in\overline{\Omega}} \beta(a,x).$$

Assumption 1.2: The birth rate $\beta(a,x)$ and the death rate $\mu(a,x)$ satisfy the following conditions:

(i) $\beta \in C(\mathbb{R}^N, L^\infty_+(0,\hat{a}))$;
(ii) $\mu \in C(\mathbb{R}^N, L^\infty_{\text{loc},+}[0,\hat{a}))$;
(iii) there exists a constant $\widetilde{\mu} > 0$ such that $\underline{\mu}(a) \geq \widetilde{\mu} > 0$ a.e. $a \in (0,\hat{a})$;
(iv) for any $x \in \mathbb{R}^N$ and almost every $a \in (0,\hat{a})$,

$$\underline{\beta}(a) \leq \beta(a,x) \quad \text{and} \quad \mu(a,x) \leq \overline{\mu}(a).$$

The interested readers can refer to [5] for more comments on the above assumptions about β and μ. In addition, we make some assumptions on the nonlinear function $f = f(u)$ as follows.

Assumption 1.3: The function f satisfies the following conditions:

(i) $f \in C^1([0,\infty))$;
(ii) $f'(u) > 0$ for all $u \in [0,\infty)$;
(iii) $f(0) \equiv 0$ and $\frac{f(u)}{u}$ is decreasing with respect to u;
(iv) there exists a constant $L > 0$ such that $f(u) \leq L$ for all $u \in [0,\infty)$.

A typical example of such nonlinearity is

$$f(u) = \frac{u}{1+Au}, \quad u \geq 0,$$

with $A > 0$ being a constant or $f(u) = 1 - e^{-u}, u \geq 0$. We would like to mention that Assumption 1.3(ii) is used to avoid the existence of periodic solutions, which is common in age-structured models; see Magal and Ruan [8, 9], Liu et al. [7] and the references cited there. Moreover, it seems necessary in our problem, in particular in the monotone iterative scheme (using comparison principles), to obtain the nontrivial positive equilibrium of (1.1) (see Theorem 4.4).

Note that Assumptions 1.1–1.3 will be required throughout the paper, thus in the following we will assume that these assumptions hold everywhere without repeating them. However, we will mention any additional assumptions where they are needed.

As mentioned in our first paper [5], the important tool in studying the global dynamics of (1.1) is to investigate the spectrum set of the linearized operator of (1.1) at some equilibrium, and then use the information of the spectrum set (for example, the sign of the spectral bound or principal eigenvalue, if it exists) to study the long time behavior of (1.1). In [5], we have studied the existence of principal eigenvalues of the linearized operator associated to (1.1) at some equilibrium and their limiting properties with respect to the diffusion rate and diffusion range. In this paper, we are interested in the global dynamics of (1.1). More precisely, we analyze the existence and uniqueness of a nontrivial equilibrium and its stability for the age-structured model with nonlocal diffusion (1.1) under Assumption 1.3 via the sign of the spectral bound of a linearized operator. Moreover, we investigate the asymptotic properties of the nontrivial equilibrium with respect to the diffusion rate and diffusion range.

The paper is organized as follows. In Section 2, we recall the notations introduced in [5], which are also used in this paper. In Section 3, we first recall some established results in [5], in particular on the existence of principal eigenvalues and their asymptotic behavior with respect to the diffusion rate and diffusion range. In Section 4, we investigate the existence, uniqueness, regularity and stability of a nontrivial equilibrium of (1.1) via the sign of the spectral bound. In Section 5, we continue to study the global dynamics of (1.1) in terms of the diffusion rate and diffusion range.

2. Notations

In this section, we recall our notations employed in [5], which will be also used in this paper. We denote by X and X_+ respectively the Banach space $X = C(\overline{\Omega})$ and its positive cone or the Banach space $X = L^1(\Omega)$ and its positive cone. Here recall that $\Omega \subset \mathbb{R}^N$ is a given bounded domain. Recall that for both cases X_+ is a normal and generating cone. In addition, we denote by I the identity operator.

Then we define the following function spaces:

$$\mathcal{X} = X \times L^1((0,\hat{a}), X), \quad \mathcal{X}_0 = \{0_X\} \times L^1((0,\hat{a}), X),$$

endowed with the product norms and the positive cones:

$$\mathcal{X}^+ = X_+ \times L^1_+((0,\hat{a}), X) = X_+ \times \{u \in L^1((0,\hat{a}), X) : u(a,\cdot) \in X_+, \text{ a.e. in } (0,\hat{a})\},$$
$$\mathcal{X}_0^+ = \mathcal{X}^+ \cap \mathcal{X}_0.$$

We also define the linear positive and bounded operator $K \in \mathcal{L}(X)$ by

$$[K\varphi](\cdot) = \int_\Omega J(\cdot - y)\varphi(y)dy, \quad \forall \varphi \in X. \tag{2.1}$$

Note that due to Assumption 1.1 one has

$$\|K\|_{\mathcal{L}(X)} \leq \begin{cases} \sup_{y\in\Omega} \int_\Omega J(x-y)dx & \text{if } X = L^1(\Omega) \\ \sup_{x\in\overline{\Omega}} \int_\Omega J(x-y)dy & \text{if } X = C(\overline{\Omega}) \end{cases} \leq \int_{\mathbb{R}^N} J(z)dz = 1. \tag{2.2}$$

2.1. Evolution family without diffusion. We consider the following problem posed in X for $0 \leq \tau \leq a < \hat{a}$:

$$\begin{cases} \partial_a v(a) = -\mu(a,\cdot)v(a), & \tau < a < \hat{a}, \\ v(\tau) = \eta \in X. \end{cases} \tag{2.3}$$

This problem generates an evolution family on X, denoted by Π, that is explicitly given for $0 \leq \tau \leq a < \hat{a}$ and $\eta \in X$ by

$$\begin{aligned} &\Pi(\tau,a)\eta = \pi(\tau,a,\cdot)\eta \\ &\text{with } \pi(\tau,a,x) := \exp\left(-\int_\tau^a \mu(s,x)ds\right) \text{ for } 0 \leq \tau \leq a < \hat{a} \text{ and } x \in \overline{\Omega}. \end{aligned} \tag{2.4}$$

Observe that one has

$$\|\Pi(\tau,a)\|_{\mathcal{L}(X)} \leq \exp\left(-\int_\tau^a \underline{\mu}(s)ds\right) \leq e^{-\widetilde{\mu}(a-\tau)} \leq 1, \quad \forall\, 0 \leq \tau \leq a < \hat{a}. \tag{2.5}$$

We also define the following family of bounded linear operators

$$\{W_\lambda\}_{\lambda > -\widetilde{\mu}} \subset \mathcal{L}(\mathcal{X}, \mathcal{X}_0)$$

for $(\eta, g) \in \mathcal{X}$ by

$$\begin{aligned} &W_\lambda(\eta,g) = (0,h) \\ &\text{with } h(a) = e^{-\lambda a}\Pi(0,a)\eta + \int_0^a e^{-\lambda(a-s)}\Pi(s,a)g(s)ds. \end{aligned} \tag{2.6}$$

We will show that this provides a family of positive pseudoresolvents. With this aim, one can make some computations to obtain

$$\begin{aligned}W_\nu W_\lambda(\eta,g) = &\int_0^a e^{-\nu(a-s)}\Pi(s,a)e^{-\lambda s}\Pi(0,s)\eta ds\\ &+\int_0^a e^{-\nu(a-s)}\Pi(s,a)\int_0^s e^{-\lambda(s-\tau)}\Pi(\tau,s)g(\tau)d\tau ds\\ =&\int_0^a e^{-\nu a}e^{-(\lambda-\nu)s}ds\Pi(0,a)\eta\\ &+\int_0^a\int_0^s e^{\lambda\tau-\nu a}e^{-(\lambda-\nu)s}\Pi(\tau,a)g(\tau)d\tau ds.\end{aligned}$$

Hence for $\nu\neq\lambda$, we have

$$\begin{aligned}W_\nu W_\lambda(\eta,g) =&\frac{1}{\nu-\lambda}(e^{-\lambda a}-e^{-\nu a})\Pi(0,a)\eta\\ &+\frac{1}{\nu-\lambda}(e^{-(\lambda-\nu)a}-e^{-(\lambda-\nu)\tau})\int_0^a e^{\lambda\tau-\nu a}\Pi(\tau,a)g(\tau)d\tau\\ =&\frac{1}{\nu-\lambda}(W_\lambda-W_\nu)(\eta,g).\end{aligned}$$

Moreover, one sees (for example, Magal and Ruan [9, Lemma 3.8.3]) that for all $\lambda>-\widetilde{\mu}$,

$$W_\lambda(\eta,g)=0 \text{ only occurs if } \eta=0, g=0$$

and

$$\lim_{\lambda\to\infty}\lambda W_\lambda(0,g)=(0,g),\quad \forall(0,g)\in\mathcal{X}_0.$$

Furthermore, one has

$$\|W_\lambda\|_{\mathcal{L}(\mathcal{X},\mathcal{X}_0)}\le\frac{1}{\lambda+\widetilde{\mu}}.$$

Thus, by Pazy [11, Section 1.9] there exists a unique closed Hille–Yosida operator $\widetilde{B}_1:\mathrm{dom}(\widetilde{B}_1)\subset\mathcal{X}\to\mathcal{X}$ with $\overline{\mathrm{dom}(\widetilde{B}_1)}=\mathcal{X}_0$ such that

$$(\lambda I-\widetilde{B}_1)^{-1}=W_\lambda\quad\text{for all }\lambda>-\widetilde{\mu}. \tag{2.7}$$

Recalling (2.1) we also define a bounded linear operator $\mathcal{B}_2\in\mathcal{L}(\mathcal{X}_0)$ by

$$\mathcal{B}_2(0,g)=(0,DKg(\cdot)),\quad\forall(0,g)\in\mathcal{X}_0.$$

2.2. Evolution family with diffusion. Now consider the following evolution equation for $\eta \in X$ and $0 \leq \tau \leq a < \hat{a}$:

$$\begin{cases} \partial_a u(a) = D(K-I)u(a) - \mu(a,\cdot)u(a), \quad \tau < a < \hat{a}, \\ u(\tau) = \eta \in X. \end{cases} \tag{2.8}$$

Define the evolution family $\{\mathcal{U}(\tau,a)\}_{0\leq\tau\leq a<\hat{a}} \subset \mathcal{L}(X)$ associated with (2.8). Using the constant of variation formula, $\mathcal{U}$ becomes for all $0 \leq \tau \leq a < \hat{a}$ the solution of the equation

$$\begin{cases} \mathcal{U}(\tau,a) = e^{-D(a-\tau)}\Pi(\tau,a) + D\int_\tau^a e^{-D(a-l)}\Pi(l,a)K\,\mathcal{U}(\tau,l)dl, \\ \mathcal{U}(\tau,\tau) = I_X, \end{cases} \tag{2.9}$$

where I_X is the identity operator in X. Note that the right-hand side of (2.8) is linear and bounded with respect to u, thus the existence and uniqueness of $\{\mathcal{U}(\tau,a)\}_{0\leq\tau\leq a<\hat{a}}$ can be obtained from the general semigroup theory (see Pazy [11]). Next we prove that $\{\mathcal{U}(\tau,a)\}_{0\leq\tau\leq a<\hat{a}}$ is exponentially bounded.

With this aim fix $\eta \in X$, $\tau \in [0,\hat{a})$ and set $u(a) = \mathcal{U}(\tau,a)\eta$. Then one has

$$\|u(a)\|_X \leq e^{-(D+\widetilde{\mu})(a-\tau)}\|\eta\|_X + D\|K\|_{\mathcal{L}(X)}\int_\tau^a e^{-(D+\widetilde{\mu})(a-l)}\|u(l)\|_X dl.$$

Next, Gronwall's inequality yields

$$\|u(a)\|_X\, e^{(D+\widetilde{\mu})(a-\tau)} \leq \|\eta\|_X\, e^{D\|K\|_{\mathcal{L}(X)}(a-\tau)},$$

which implies, due to (2.2), that

$$\|\mathcal{U}(\tau,a)\|_{\mathcal{L}(X)} \leq e^{-\widetilde{\mu}(a-\tau)}.$$

As a consequence $\{\mathcal{U}(\tau,a)\}_{0\leq\tau\leq a<\hat{a}}$ is positive and exponentially bounded in X and satisfies

$$\|\mathcal{U}(a,a+t)\|_{\mathcal{L}(X)} \leq e^{-\widetilde{\mu}t}, \quad \forall t \geq 0, 0 \leq a < \hat{a} - t. \tag{2.10}$$

Now we define the family of bounded linear operators $\{R_\lambda\}_{\lambda>-\widetilde{\mu}} \subset \mathcal{L}(\mathcal{X},\mathcal{X}_0)$ as follows:

$$\begin{aligned} &R_\lambda(\eta,g) = (0,h) \\ &\text{with } h(a) = e^{-\lambda a}\mathcal{U}(0,a)\eta + \int_0^a e^{-\lambda(a-s)}\mathcal{U}(s,a)g(s)ds. \end{aligned} \tag{2.11}$$

Moreover, for any $\lambda > -\widetilde{\mu}$, one has

$$\|R_\lambda\|_{\mathcal{L}(\mathcal{X},\mathcal{X}_0)} \leq \frac{1}{\lambda+\widetilde{\mu}}.$$

Then by the same procedure as in the case without diffusion, we can prove that this provides a family of positive pseudoresolvents. Again by Pazy [11, Section 1.9] there exists a unique closed Hille–Yosida operator $\mathcal{B} : \mathrm{dom}(\mathcal{B}) \subset \mathcal{X} \to \mathcal{X}$ with $\overline{\mathrm{dom}(\mathcal{B})} = \mathcal{X}_0$ such that

$$(\lambda I - \mathcal{B})^{-1} = R_\lambda \quad \text{for all } \lambda > -\widetilde{\mu}.$$

Now we define the part of $\mathcal{B}$ in $\mathcal{X}_0$, denoted by $\mathcal{B}_0$, that is,

$$\mathcal{B}_0 x = \mathcal{B}x, \forall x \in \mathrm{dom}(\mathcal{B}_0), \quad \text{with } \mathrm{dom}(\mathcal{B}_0) := \{x \in \mathrm{dom}(\mathcal{B}) : \mathcal{B}x \in \mathcal{X}_0\}.$$

Note that $\mathcal{B}_0$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on $\mathcal{X}_0$, denoted by $\{T_{\mathcal{B}_0}(t)\}_{t\geq 0}$. Moreover, it satisfies the following estimate

$$\|T_{\mathcal{B}_0}(t)\|_{\mathcal{L}(\mathcal{X}_0)} \leq e^{-\widetilde{\mu}t}, \quad \forall t \geq 0.$$

Observe that we have $\widetilde{B}_1 + \mathcal{B}_2 - DI = \mathcal{B}$. From now on for the sake of convenience, we denote $\mathcal{B}_1 := \widetilde{B}_1 - DI$.

On the other hand, we define $\mathcal{C} \in \mathcal{L}(\mathcal{X}_0, \mathcal{X})$ by

$$\mathcal{C}(0,h) = \left(\int_0^{\hat{a}} \beta(a,\cdot)h(a)da,\ 0 \right), \quad (0,h) \in \mathcal{X}_0,$$

and $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ by

$$\begin{cases} \mathrm{dom}(\mathcal{A}) = \mathrm{dom}(\mathcal{B}) \subset \mathcal{X}_0, \\ \mathcal{A} = \mathcal{B} + \mathcal{C}. \end{cases} \tag{2.12}$$

This shows that $\mathcal{A}$ is not densely defined in $\mathcal{X}$.

Remark 2.1: In addition, for each fixed $x \in \overline{\Omega}$, following the above procedures one can obtain the age-structured operator, denoted by $\mathcal{B}_1^x + \mathcal{C}^x$, defined on $\mathbb{R} \times L^1(0,\hat{a})$.

Finally define a map $F : \mathcal{X}_0 \to \mathcal{X}$ by

$$F(0,\psi) = \left(f\left(\int_0^{\hat{a}} \beta(a,\cdot)\psi(a)da \right), 0 \right)$$

Then by identifying $U(t) = (0,u(t))$, one can rewrite problem (1.1) as the following abstract Cauchy problem:

$$\begin{cases} \frac{dU}{dt} = \mathcal{B}U + F(U), \\ U(0) = U_0, \end{cases} \quad \text{with } U_0 = (0,u_0) \in \mathcal{X}_0. \tag{2.13}$$

3. Preliminaries

3.1. Existence of principal eigenvalues. In this subsection we list some useful results which will be used later, and the interested readers can refer to [5] for more details. Note that all results in this section are under Assumptions 1.1 and 1.2. We first recall a sufficient condition to make the spectral bound

$$s(\mathcal{A}) := \sup\{\mathrm{Re}\lambda \in \mathbb{R}; \lambda \in \sigma(\mathcal{A})\}$$

become the principal eigenvalue. Here we say that $\lambda \in \sigma(T) \cap \mathbb{R}$ is the **principal eigenvalue** of a linear operator T if it is larger than the real parts of all other eigenvalues of T and associated with a positive eigenfunction.

Proposition 3.1 (Ducrot et al. [5, Propositions 3.3 and 3.5]): *The spectral bounds of $\mathcal{B}_1 + \mathcal{C}$ and $\mathcal{A}$ satisfy $s(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$ and $s(\mathcal{A}) = \lambda_0$ respectively, which are given as follows:*

$$\begin{aligned} r(\mathcal{G}_{\alpha^{**}}) &:= r\left(\int_0^{\hat{a}} \beta(a,\cdot) e^{-(\alpha^{**}+D)a} \Pi(0,a) da \right) \\ &= \max_{x\in\overline{\Omega}} \int_0^{\hat{a}} \beta(a,x) e^{-(\alpha^{**}+D)a} \pi(0,a,x) da = 1, \\ r(\mathcal{M}_{\lambda_0}) &:= r\left(\int_0^{\hat{a}} \beta(a,\cdot) e^{-\lambda_0 a} \mathcal{U}(0,a)\, da \right) = 1. \end{aligned} \tag{3.1}$$

Here $\mathcal{G}_\alpha$ is a linear and bounded operator from X to X, and $r(A)$ denotes the spectral radius of a linear bounded operator A, which is defined by

$$r(A) = \sup\{|\lambda| \in \mathbb{R}; \lambda \in \sigma(A)\}.$$

Theorem 3.2 (Ducrot et al. [5, Theorem 4.1]): *Assume that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. Then $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.*

Next we recall the result on the algebraic simplicity of the principal eigenvalue $s(\mathcal{A})$.

Assumption 3.3: There exists no a_0 such that $\underline{\beta}(a) = 0$ a.e. $[a_0, \hat{a})$. Moreover, it is equivalent to

$$\int_a^{\hat{a}} \underline{\beta}(l) dl > 0, \quad \forall a \in [0, \hat{a}).$$

Theorem 3.4 (Ducrot et al. [5, Theorem 4.6]): *Let Assumption 3.3 hold. Assume that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. Then the principal eigenvalue of $\mathcal{A}$, i.e., $s(\mathcal{A})$, is algebraically simple.*

Next we recall two established sufficient and relatively easily verifiable conditions ensuring that $s(\mathcal{A})$ is the the principal eigenvalue of $\mathcal{A}$.

Theorem 3.5 (Existence of principal eigenvalues. I, Ducrot et al. [5, Theorem 4.8]): *Assume that $\mu_{\max} := \sup_{a\in(0,\hat{a})} \overline{\mu}(a) < \infty$ and*

$$x \to \frac{1}{1 - G_{\alpha^{**}}(x)} \notin L^1_{\text{loc}}(\overline{\Omega}). \tag{3.2}$$

Then $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, where

$$\alpha^{**} = s(\mathcal{B}_1 + \mathcal{C})$$

and $G_\alpha : \overline{\Omega} \to \mathbb{R}$ is defined by

$$G_\alpha(x) = \int_0^{\hat{a}} \beta(a,x)e^{-(\alpha+D)a}\pi(0,a,x)da, \quad \forall x \in \overline{\Omega}. \tag{3.3}$$

Assumption 3.6: There exists $a_2 \in (0,\hat{a})$ such that $\beta \equiv 0$ in $[a_2,\hat{a}) \times \overline{\Omega}$.

Theorem 3.7 (Existence of principal eigenvalues. II, Ducrot et al. [5, Theorem 4.11]): *Let Assumption 3.6 hold. Assume that*

$$x \to \frac{1}{\alpha^{**} - \alpha(x)} \notin L^1_{\text{loc}}(\overline{\Omega}), \tag{3.4}$$

and that for each $x \in \overline{\Omega}$, the operator $\mathcal{B}_1^x + \mathcal{C}^x$ possesses a positive eigenvector $\phi \in W^{1,1}(0,a_2)$ corresponding to $\alpha(x)$. Then $s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Here $x \to \alpha(x) : \overline{\Omega} \to \mathbb{R}$ is continuous such that for any $x \in \overline{\Omega}$, the following equation

$$\begin{cases} \partial_a\phi(a,x) = -(D+\mu(a,x))\phi(a,x) - \alpha(x)\phi(a,x), & a \in (0,a_2), \\ \phi(0,x) = \int_0^{a_2} \beta(a,x)\phi(a,x)da \end{cases} \tag{3.5}$$

has a positive solution $a \to \phi(a,x) \in W^{1,1}(0,\hat{a})$ and $\mathcal{B}_1^x + \mathcal{C}^x$ is defined in Remark 2.1 in $(0,a_2)$.

3.2. Limiting properties. In this subsection we recall the asymptotic behavior of the principal eigenvalue with respect to diffusion rate and diffusion range. Throughout this subsection we let Assumption 3.6 hold. Before proceeding, let us first clarify the strict positivity in X.

$$f > 0 \text{ in } X = C(\overline{\Omega}) \quad \text{means that } f(x) > 0 \text{ for all } x \in \overline{\Omega},$$

$$f > 0 \text{ in } X = L^1(\Omega) \quad \text{means that } \int_\Omega f^*(x)f(x)dx > 0 \text{ for any } f^* \in L^\infty_+(\Omega)\setminus\{0\}.$$

Our tool in investigating the limiting properties is generalized principal eigenvalues, which are given as follows.

Definition 3.8: Define the **generalized principal eigenvalues** by

$$\begin{cases} \lambda_p(\mathcal{A}) := \sup\{\lambda\in\mathbb{R} : \exists\, \phi \in W^{1,1}((0,a_2),X) \\ \qquad\qquad \text{s.t. } \phi>0 \text{ and } (-\mathcal{A}+\lambda)(0,\phi)\le(0,0) \text{ in } [0,a_2]\}, \\ \lambda_p'(\mathcal{A}) := \inf\{\lambda\in\mathbb{R} : \exists\, \phi \in W^{1,1}((0,a_2),X) \\ \qquad\qquad \text{s.t. } \phi>0 \text{ and } (-\mathcal{A}+\lambda)(0,\phi)\ge(0,0) \text{ in } [0,a_2]\}. \end{cases} \tag{3.6}$$

Proposition 3.9 (Ducrot et al. [5, Proposition 5.2]): *Let Assumption 3.6 hold and, in addition, assume that $\lambda_1(\mathcal{A})$ is the eigenvalue of $\mathcal{A}$ associated with $(0,\phi_1) \in \mathrm{dom}(\mathcal{A})$ with $\phi_1 > 0$ in $[0,a_2]$. Then one has $\lambda_1(\mathcal{A}) = \lambda_p(\mathcal{A}) = \lambda_p'(\mathcal{A})$.*

3.2.1. *Without kernel scaling.* We first recall the theorem about the effects of the diffusion rate on $s(\mathcal{A})$. In the next result, we write $s^D(\mathcal{A})$ for $s(\mathcal{A})$ to highlight the dependence on D.

Theorem 3.10 (Ducrot et al. [5, Theorem 5.3]): *Let Assumption 3.6 hold and, in addition, assume that $s^D(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Then the function $D \to s^D(\mathcal{A})$ is continuous on $(0,\infty)$ and satisfies*

$$s^D(\mathcal{A}) \to \begin{cases} s(B_1+\mathcal{C}) & \text{as } D\to 0^+, \\ -\infty & \text{as } D \to \infty, \end{cases} \tag{3.7}$$

where B_1 is defined by

$$B_1(0,f) = (-f(0,\cdot), -\partial_a f - \mu f), \quad f \in W^{1,1}((0,a_2),X). \tag{3.8}$$

3.2.2. *With kernel scaling.* In this subsection we recall the effects of the diffusion rate and diffusion range on the principal eigenvalue. Define $K_{\gamma,\Omega}$ as follows:

$$[K_{\gamma,\Omega}f](\cdot) = \int_\Omega J_\gamma(\cdot - y)f(y)dy, \quad f \in X. \tag{3.9}$$

Here the kernel J_γ satisfies the scaling

$$J_\gamma(x) = \frac{1}{\gamma^N} J\Big(\frac{x}{\gamma}\Big) \quad \text{for } x \in \mathbb{R}^N,$$

where $\gamma > 0$ represents the diffusion range. Then the nonlocal diffusion operator is given as $\frac{D}{\gamma^m}[K_{\gamma,\Omega} - I]$, where $m \in [0, 2)$ denotes the cost parameter.

Write $\mathcal{A}_{\gamma,m,\Omega} = \mathcal{B}_{\gamma,m,\Omega} + \mathcal{C}$ for $\mathcal{A} = \mathcal{B} + \mathcal{C}$ to highlight the dependence on γ, m and Ω and further denote $\mathcal{B}^{\mu}_{\gamma,m,\Omega}$, $\mathcal{C}^{\beta}$ for $\mathcal{B}$, $\mathcal{C}$ to represent the dependence on μ and β respectively.

Theorem 3.11 (Ducrot et al. [5, Theorem 5.7]): *Let Assumption 3.6 hold. Assume that $s(\mathcal{A}_{\gamma,m,\Omega})$ is the principal eigenvalue of $\mathcal{A}_{\gamma,m,\Omega}$. Then:*

(i) *As $\gamma \to \infty$, there holds*

$$s(\mathcal{A}_{\gamma,m,\Omega}) \to \begin{cases} s(B_1 + \mathcal{C}) - D, & m = 0, \\ s(B_1 + \mathcal{C}), & m > 0, \end{cases} \tag{3.10}$$

where B_1 is defined in (3.8).

(ii) *Assume, in addition, that J is symmetric, i.e., $J(x) = J(-x)$, and $\mu, \beta \in C^2(\mathbb{R}^N, L^\infty_+(0, a_2))$. As $\gamma \to 0^+$, there holds*

$$s(\mathcal{A}_{\gamma,m,\Omega}) \to s(B_1 + \mathcal{C}), \quad \forall m \in [0, 2).$$

(iii) *In the case when $m = 0$, if Ω contains the origin and $\mu(a, x)$ is radially symmetric and radially non-decreasing with respect to x, namely,*

$$\mu(a, x) = \mu(a, y) \quad \text{if } |x| = |y|$$

and

$$\mu(a, x) \geq \mu(a, y) \quad \text{if } |x| \geq |y|$$

for all $a \in [0, \hat{a})$, then $\gamma \to s(\mathcal{A}_{\gamma,0,\Omega})$ is non-increasing.

4. Global dynamics in terms of the spectral bound

In this section, we extend the techniques and results in Coville [3] to investigate the existence, uniqueness and stability of a nontrivial equilibrium of our model (1.1). Throughout this section, let Assumptions 1.1, 1.2 and 1.3 hold. In addition, we make the following assumption.

Assumption 4.1: There exists $a_2 \in (0, \hat{a})$ such that $\beta \equiv 0$ on $[a_2, \hat{a}) \times \overline{\Omega}$ and

$$\int_a^{a_2} \underline{\beta}(l)dl > 0 \quad \text{for any } a \in [0, a_2).$$

For the sake of simplicity, we will not repeat Assumptions 1.1, 1.2, 1.3 and 4.1 in this section. However, we will indicate the additional assumptions where we need.

Before proceeding, we would like to mention that even for age-structured models with Laplacian diffusion, it is not easy to study the existence of a nontrivial equilibrium; see Walker [17–24] where the focuses were mainly on the existence of a nontrivial equilibrium of age-structured models with nonlinear Laplacian diffusion and with/without nonlinear birth and death rates. Walker assumed the maximum regularity of the nonlinear diffusion to obtain compactness of the semigroup and then used bifurcation method (Crandall and Rabinowitz [4], Rabinowitz [12]) to study them. However, the semiflow generated by nonlocal diffusion is not compact with respect to the compact open topology and solutions of nonlocal diffusion problems usually lose the spatial regularity, which means that we cannot apply Walker's methods directly to our case. Here we use the sub- and super-solution method to provide a criterion for the existence of a nontrivial equilibrium via the sign of the spectral bound of linearized equations motivated by Coville [3].

Let us first write down the equation that the equilibrium satisfies:

$$\begin{cases} \partial_a u(a,x) = D[\int_\Omega J(x-y)u(a,y)dy - u(a,x)] - \mu(a,x)u(a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a,x) \in (0,a_2) \times \overline{\Omega}, \\ u(0,x) = f(\int_0^{a_2} \beta(a,x)u(a,x)da), \qquad\qquad x \in \overline{\Omega}. \end{cases} \tag{4.1}$$

Definition 4.2: $u \in W^{1,1}((0,a_2), C(\overline{\Omega}))$ is called a **super-solution** (resp. **sub-solution**) of (4.1) if $=$ is replaced by $\geq$ (resp. $\leq$) in the two equations of (4.1).

4.1. Comparison principle. Now let us prove the comparison principle for (4.1).

Lemma 4.3: *Let* $0 < u \in W^{1,1}((0,a_2), C(\overline{\Omega}))$ *be a sub-solution of* (4.1) *and* $0 < v \in W^{1,1}((0,a_2), C(\overline{\Omega}))$ *be a super-solution of* (4.1). *Then* $u \leq v$ *in* $[0,a_2] \times \overline{\Omega}$.

Proof. Let $\alpha_* := \sup\{\alpha > 0 : \alpha u \leq v \,\text{in}\, [0, a_2] \times \overline{\Omega}\}$. By the assumption on u and v, the number α_* is well defined and positive. If $\alpha_* \geq 1$, then we are done. So we assume that $\alpha_* < 1$.

Set $w := v - \alpha_* u$, then $w \geq 0$. Moreover, set

$$a_0 := \min\{a \in [0, a_2] : \exists x \in \overline{\Omega}, \text{ s.t. } w(a_0, x) = 0\}.$$

Such a_0 exists due to the definition of α_*. It follows that there exists $x_0 \in \overline{\Omega}$ such that $w(a_0, x_0) = 0$.

If $a_0 \in (0, a_2]$, observe that w satisfies the following equation:

$$\partial_a w(a,x) \geq D\left[\int_\Omega J(x-y)w(a,y)dy - w(a,x)\right] - \mu(a,x)w(a,x), \qquad (a,x) \in (0, a_2] \times \overline{\Omega}.$$

Recalling the constant of variation formula (2.9), one has

$$w(a,x) \geq e^{-Da}\pi(0,a,x)w(0,x) + D\int_0^a e^{-D(a-l)}\pi(l,a,x)[Kw](l,x)dl. \tag{4.2}$$

Considering the above inequality at (a_0, x_0), we have a contradiction, since by the definition of a_0, $w(a,x) > 0$ for all $(a,x) \in [0, a_0) \times \overline{\Omega}$ implies that the right-hand side of (4.2) is positive.

If $a_0 = 0$, that is, $w(0, x_0) = 0$, thanks to Assumption 4.1 on β one has

$$\int_0^{a_2} \beta(a, x_0)\alpha_* u(a, x_0)da > 0.$$

On the other hand, by Assumption 1.3(iii) on f, one has that $w(0, x_0)$ satisfies

$$\begin{aligned} w(0,x_0) &= v(0,x_0) - \alpha_* u(0,x_0) \\ &\geq f\left(\int_0^{a_2} \beta(a,x_0)v(a,x_0)da\right) - \alpha_* f\left(\int_0^{a_2} \beta(a,x_0)u(a,x_0)da\right) \\ &> f\left(\int_0^{a_2} \beta(a,x_0)v(a,x_0)da\right) - f\left(\int_0^{a_2} \beta(a,x_0)\alpha_* u(a,x_0)da\right) \\ &\geq 0, \end{aligned}$$

where we used Assumption 1.3(iii) and $\alpha_* < 1$. It is a contradiction with $w(0, x_0) = 0$. Thus $\alpha_* \geq 1$ and the proof is complete. ∎

4.2. Existence and uniqueness of a positive equilibrium. Next let us define the linearized operator $\mathcal{A}^l$ which is obtained by linearizing (4.1) at $u=0$:

$$\mathcal{A}^l(0,\phi):=\left(-\phi(0,\cdot)+f'(0)\int_0^{a_2}\beta(a,\cdot)\phi(a,\cdot)da, -\partial_a\phi+D(K-I)\phi-\mu\phi\right), \quad (0,\phi)\in \mathrm{dom}(\mathcal{A}^l), \tag{4.3}$$

where $\mathrm{dom}(\mathcal{A}^l)=\{0\}\times W^{1,1}((0,a_2),C(\overline{\Omega}))$ and denote the spectral bound of $\mathcal{A}^l$ by λ_1^l. Recall from Proposition 3.1 that λ_1^l satisfies

$$r\left(f'(0)\int_0^{a_2}\beta(a,\cdot)e^{-\lambda_1^l a}\mathcal{U}(0,a)da\right)=1. \tag{4.4}$$

Theorem 4.4: *Assume that $\lambda_1^l>0$ is the principal eigenvalue of $\mathcal{A}^l$. Then there exists at least one positive solution $u^*(a,x)$ of (4.1) belonging to $W^{1,1}((0,a_2),L^1(\Omega))$.*

Proof. (a) Construction of super-/sub-solutions. Set $\overline{u}\equiv L$, where L is defined in Assumption 1.3(iv). Let us verify that $\overline{u}(a,x)$ is indeed a super-solution of (4.1):

$$\begin{aligned}\partial_a\overline{u}(a,x)-D\left[\int_\Omega J(x-y)\overline{u}(a,y)dy-\overline{u}(a,x)\right]+\mu(a,x)\overline{u}(a,x)&\\ =DL\left[1-\int_\Omega J(x-y)dy\right]+\mu(a,x)L\geq 0.&\end{aligned}$$

Moreover,

$$\overline{u}(0,x)=L\geq f\left(\int_0^{a_2}\beta(a,x)\overline{u}(a,x)da\right)$$

holds, since by Assumption 1.3(iv) one has $f(u)\leq L$ for all $u\geq 0$.

Next, we construct a sub-solution of (4.1) motivated by Coville [3, Theorem 1.6]. For any $\delta>0$ sufficiently small, we can find a small constant $\epsilon_1=\epsilon_1(\delta)>0$ such that $f(u)\geq(f'(0)-\delta)u$ for $0<u\leq\epsilon_1$. Such ϵ_1 can be achieved due to Assumption 1.3 on f. Moreover, due to $\lambda_1^l>0$ one can further reduce δ such that $\lambda_{1\delta}^l>0$ with $\lambda_{1\delta}^l$ satisfying

$$r\left((f'(0)-\delta)\int_0^{a_2}\beta(a,\cdot)e^{-\lambda_{1\delta}^l a}\mathcal{U}(0,a)da\right)=1.$$

Then we consider the following linear equation:

$$\begin{cases}\partial_a\phi(a,x)=-(D+\mu(a,x))\phi(a,x)-\alpha\phi(a,x), & a\in(0,a_2),\\ \phi(0,x)=(f'(0)-\delta)\int_0^{a_2}\beta(a,x)\phi(a,x)da.\end{cases} \tag{4.5}$$

By Ducrot et al. [5, Proposition 3.11], there exists a continuous function $x \to \alpha(x) : \mathbb{R}^n \to \mathbb{R}$ such that for any $x \in \mathbb{R}^n$, equation (4.5) with $\alpha = \alpha(x)$ has a positive solution $a \to \phi(a,x) \in W^{1,1}(0,a_2)$. Denote $\alpha^{**} = \max_{x\in\overline{\Omega}} \alpha(x)$. From the definition of α^{**} there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \overline{\Omega}$ and $|\alpha^{**} - \alpha(x_n)| \le \frac{1}{n}$. Thus, by the continuity of $\alpha(x)$, for each n there exists $\eta_n > 0$ such that for all $x \in B_{\eta_n}(x_n)$ we have $|\alpha^{**} - \alpha(x)| \le \frac{2}{n}$.

Now we consider a sequence of real numbers $\{\epsilon_n\}_{n\in\mathbb{N}}$ which converges to zero such that $\epsilon_n \le \frac{\eta_n}{2}$. Next let $\{\chi_n\}_{n\in\mathbb{N}}$ be the following sequence of cut-off functions:

$$\chi_n(x) := \chi\Big(\frac{|x-x_n|}{\epsilon_n}\Big)$$

where χ is a smooth function such that $0 \le \chi \le 1, \chi(x) = 0$ for $|x| \ge 2$ and $\chi(x) = 1$ for $|x| \le 1$.

Finally, let us consider the following sequence of continuous functions $\{\alpha_n\}_{n\in\mathbb{N}}$ defined by $\alpha_n(x) := \sup\{\alpha(x), \alpha^{**}\chi_n(x)\}$. Observe that by construction the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is such that $\|\alpha - \alpha_n\|_{C(\overline{\Omega})} \to 0$.

By construction, for each n, the function α_n satisfies $\max_{x\in\overline{\Omega}} \alpha_n = \alpha^{**}$ and $\alpha_n \equiv \alpha^{**}$ in $B_{\frac{\epsilon_n}{2}}(x_n)$. Therefore, the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies

$$\frac{1}{\alpha^{**} - \alpha_n} \notin L^1_{\text{loc}}(\Omega).$$

Next set

$$\mu_n(a,x) = \mu(a,x) - \alpha_n(x) + \alpha(x)$$

and consider equation (4.5) with μ replaced by μ_n. Then it can be checked that

$$(f'(0) - \delta)\int_0^{a_2} \beta(a,x)e^{-(D+\alpha_n(x))a}e^{-\int_0^a \mu_n(s,x)ds}da = 1.$$

It follows that α_n is a continuous function such that for any $x \in \mathbb{R}^n$, equation (4.5) with μ replaced by μ_n and with $\alpha = \alpha_n(x)$, has a positive solution $a \to \phi_n(a,x) \in W^{1,1}(0,a_2)$. Hence by Theorem 3.7, there exists a principal eigenpair (λ_1^n, ϕ_n) of the eigenvalue problem:

$$\begin{cases} \partial_a\phi(a,x) = D[\int_\Omega J(x-y)\phi(a,y)dy - \phi(a,x)] - \mu_n(a,x)\phi(a,x) - \lambda\phi(a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a,x) \in (0,a_2)\times\overline{\Omega}, \\ \phi(0,x) = (f'(0)-\delta)\int_0^{a_2}\beta(a,x)\phi(a,x)da, \qquad x \in \overline{\Omega} \end{cases}$$

such that $0 < \phi_n \in W^{1,1}((0,a_2), C(\overline{\Omega}))$.

Using the fact that $\|\mu - \mu_n\|_{C(\overline{\Omega}, L^\infty(0,a_2))} \to 0$ as $n \to \infty$, from Ducrot et al. [5, Proposition 5.6] it follows that for n big enough, say $n \geq n_0$, we have

$$\lambda_1^n > \frac{\lambda_{1\delta}^l}{2} > 0.$$

Moreover, by choosing n_0 bigger if necessary, we achieve for $n \geq n_0$ that

$$\lambda_1^n - \|\mu - \mu_n\|_{C(\overline{\Omega}, L^\infty(0,a_2))} \geq \frac{\lambda_{1\delta}^l}{4} > 0.$$

Now for $n \geq n_0$ fixed and $\psi = \epsilon\phi_n$ with $\epsilon > 0$ small enough such that

$$\int_0^{a_2} \beta(a,x)\psi(a,x)da \leq \epsilon_1,$$

we have

$$\begin{cases} \partial_a \psi(a,x) - D[\int_\Omega J(x-y)\psi(a,y)dy - \psi(a,x)] + \mu(a,x)\psi(a,x) \\ \qquad\qquad\qquad = -(\mu_n(a,x) - \mu(a,x) + \lambda_1^n)\psi \leq 0, \\ \psi(0,x) = (f'(0) - \delta)\int_0^{a_2} \beta(a,x)\psi(a,x)da \leq f(\int_0^{a_2} \beta(a,x)\psi(a,x)da), \end{cases}$$

where we used the fact that $f(u) \geq (f'(0) - \delta)u$ for $0 < u \leq \epsilon_1$. It implies that for $\epsilon > 0$ sufficiently small and n large enough, $\epsilon\phi_n$ is a sub-solution of (4.1). From now on, we fix an n large enough and denote $\underline{u} = \epsilon\phi_n$.

(b) Existence via an iterative scheme. Now it is clear that we can choose ϵ small enough such that $\underline{u} \leq \overline{u}$. Then by a basic iterative scheme we obtain the existence of a positive nontrivial solution u of (4.1). For completeness, we provide the iterative scheme in the following.

Let u_n for $n \geq 1$ be the solution of the following linear problem:

$$(4.6)\quad \begin{cases} \partial_a u_n(a,x) = D[\int_\Omega J(x-y)u_n(a,y)dy - u_n(a,x)] - \mu(a,x)u_n(a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad (a,x) \in (0,a_2) \times \overline{\Omega}, \\ u_n(0,x) = f(\int_0^{a_2} \beta(a,x)u_{n-1}(a,x)da), \qquad x \in \overline{\Omega}, \end{cases}$$

where $u_0 = \underline{u}$. First note that u_n is well defined and in $W^{1,1}((0,a_2), L^1(\Omega))$. Then we will show that u_n is increasing and that

$$(4.7)\qquad \underline{u} \leq u_1 \leq u_2 \leq \cdots \leq \overline{u}.$$

Indeed, taking $w := u_1 - \underline{u}$ and $v := \overline{u} - u_1$, by Assumption 1.3(ii) of f, they satisfy respectively

$$\begin{cases} \partial_a w(a,x) \geq D[\int_\Omega J(x-y)w(a,y)dy - w(a,x)] - \mu(a,x)w(a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a,x) \in (0,a_2)\times\overline{\Omega}, \\ w(0,x) \geq 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad x \in \overline{\Omega}, \end{cases}$$

and

$$\begin{cases} \partial_a v(a,x) \geq D[\int_\Omega J(x-y)v(a,y)dy - v(a,x)] - \mu(a,x)v(a,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (a,x) \in (0,a_2)\times\overline{\Omega}, \\ v(0,x) \geq 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad x \in \overline{\Omega}. \end{cases}$$

Using the comparison principle of nonlocal diffusion equations, we conclude that $w \geq 0$ and $v \geq 0$, that is $\underline{u} \leq u_1 \leq \overline{u}$. Now by induction, we can obtain the desired result (4.7).

Next, for $(a,x) \in (0,a_2)\times\Omega$ a.e., $u_n(a,x)$ has a limit as $n\to\infty$, denoted by $u^*(a,x)$, that is $u_n(a,x) \to u^*(a,x)$ in $(0,a_2)\times\Omega$ a.e. Thus, by the continuity of f we have that for a.e. $x\in\Omega$,

$$f\Big(\int_0^{a_2}\beta(a,x)u_n(a,x)da\Big) \xrightarrow{n\to\infty} f\Big(\int_0^{a_2}\beta(a,x)u^*(a,x)da\Big). \tag{4.8}$$

In addition, one has

$$\begin{aligned}\varphi_n(a,x) &:= D\Big[\int_\Omega J(x-y)u_n(a,y)dy - u_n(a,x)\Big] - \mu(a,x)u_n(a,x) \\ &\xrightarrow{n\to\infty} D\Big[\int_\Omega J(x-y)u^*(a,y)dy - u^*(a,x)\Big] - \mu(a,x)u^*(a,x) := \varphi(a,x),\end{aligned}$$

a.e. in $(0,a_2)\times\Omega$ and in $L^1((0,a_2)\times\Omega)$. Hence, for a.e. $x\in\Omega$ and $(\eta,\xi)\subset(0,a_2)$, one has

$$u_n(\xi,x) - u_n(\eta,x) = \int_\eta^\xi \varphi_n(a,x)da,$$

which implies that

$$u^*(\xi,x) - u^*(\eta,x) = \int_\eta^\xi \varphi(a,x)da.$$

It follows that $u^* \in W^{1,1}((0,a_2),L^1(\Omega))$ satisfies the first equation of (4.1) with $\partial_a u^* = \varphi$ a.e. in $(0,a_2)\times\Omega$. Further, $u^*(\cdot,x)$ is continuous in $[0,a_2]$ for a.e. $x \in \Omega$. Thus, one has $u_n(0,x) \to u^*(0,x)$ as $n \to \infty$ in Ω, which by (4.8) implies that

$$u^*(0,x) = f\bigg(\int_0^{a_2} \beta(a,x)u^*(a,x)da\bigg).$$

Hence, the proof is complete. ∎

Next we investigate the uniqueness of u^*. Before proceeding, we first study the regularity of u^* with respect to x. We make the following additional assumption.

Assumption 4.5: Assume that $F(x,u) := u - G_0(x)f(u)$ is strictly monotone with respect to u for any $x \in \overline{\Omega}$, where $G_0(x)$ is defined in (3.3) with $\alpha = 0$ and $\hat{a}$ replaced by a_2.

Assumption 4.5 with $G_0(x) \equiv 1$ is widely used to obtain the regularity of solutions of nonlocal diffusion problems; see Bates et al. [1] and Berestycki and Rodríguez [2].

Now let us revisit problem (4.1). Solving the first equation of (4.1), one obtains

$$u(a,x) = e^{-Da}\pi(0,a,x)u(0,x) + D\int_0^a e^{-D(a-l)}\pi(l,a,x)[Ku](l,x)dl.$$

Then plugging the above equality into the boundary condition one has

$$\begin{aligned}
\widetilde{u}(x) :&= \int_0^{a_2} \beta(a,x)u(a,x)da \\
&= \int_0^{a_2} \beta(a,x)e^{-Da}\pi(0,a,x)u(0,x)da \\
&\quad + D\int_0^{a_2} \beta(a,x)\int_0^a e^{-D(a-l)}\pi(l,a,x)[Ku](l,x)dlda \\
&=: G_0(x)f(\widetilde{u}(x)) + H(x),
\end{aligned} \tag{4.9}$$

where

$$H(x) = D\int_0^{a_2} \beta(a,x)\int_0^a e^{-D(a-l)}\pi(l,a,x)[Ku](l,x)dlda$$

is continuous, due to $Ku \in W^{1,1}((0,a_2),C(\overline{\Omega}))$ for any $u \in W^{1,1}((0,a_2),L^1(\Omega))$, by Assumption 1.1 on J. Now under Assumption 4.5, for any $x \in \overline{\Omega}$, one has $\widetilde{u}(x) = F^{-1}(x,H(x))$, where F^{-1} denotes the inverse of F with respect to u for any fixed $x \in \overline{\Omega}$. Thus $\widetilde{u}$ is continuous. It follows that $u(0,\cdot)$ is continuous and so is $u(a,\cdot)$. Thus, we now have $u^* \in W^{1,1}((0,a_2), C(\overline{\Omega}))$ under Assumption 4.5.

THEOREM 4.6: *Under Assumption 4.5, the positive equilibrium u^* is unique.*

Proof. We prove the uniqueness by using a sliding argument. Let u and v be two positive bounded solutions of (4.1). Since they are bounded and strictly positive, the following quantity is well defined:

$$\kappa^* := \inf\{\kappa > 0 : \kappa u \geq v \text{ in } [0,a_2] \times \overline{\Omega}\}.$$

We claim that $\kappa^* \leq 1$. Indeed, assume by contradiction that $\kappa^* > 1$. We consider the following nonlocal problem

$$\partial_a w = D\left[\int_\Omega J(x-y)w(a,y)dy - w(a,x)\right] - \mu(a,x)w(a,x), \quad (a,x) \in (0,a_2) \times \overline{\Omega}. \tag{4.10}$$

By Shen and Zhang [14, Proposition 2.2] or Rawal et al. [13, Proposition 2.2] and Assumption 1.1 on J, solutions of equation (4.10) have the strong monotone property; i.e., for $\phi, \psi \in C_+(\overline{\Omega})$ with $\phi \geq \psi, \phi \not\equiv \psi, w(a,x;\phi) > w(a,x;\psi)$ for all $a > 0, x \in \overline{\Omega}$ at which both $w(a,x;\phi)$ and $w(a,x;\psi)$ exist, where $w(a,x;\phi)$ and $w(a,x;\psi)$ are solutions of (4.10) with initial data $w(0,x;\phi) = \phi$ and $w(0,x;\psi) = \psi$ respectively.

On one hand, from the integral boundary condition with non-negativeness of β, we have due to $\kappa^* > 1$ and assumptions of f that

$$\begin{aligned}
\kappa^* u_0 := \kappa^* u(0,x;u_0) &= \kappa^* f\left(\int_0^{a_2} \beta(a,x)u(a,x)da\right) \\
&> f\left(\int_0^{a_2} \beta(a,x)\kappa^* u(a,x)da\right) \\
&\geq f\left(\int_0^{a_2} \beta(a,x)v(a,x)da\right) \\
&= v(0,x;v_0) =: v_0.
\end{aligned}$$

It follows from the strong monotone property that

$$w(a,x;\kappa^* u_0) > w(a,x;v_0), \quad \forall (a,x) \in [0,a_2] \times \overline{\Omega}. \tag{4.11}$$

On the other hand, let $\phi(a,x) = \kappa^* w(a,x;u_0)$. Then $\phi(0,x) = \kappa^* u_0$ and

$$\partial_a \phi = D\left[\int_\Omega J(x-y)\phi(a,y)dy - \phi(a,x)\right] - \mu(a,x)\phi(a,x), \quad (a,x) \in (0,a_2) \times \overline{\Omega}.$$

By the uniqueness of solutions for nonlocal diffusion equations, we have

$$\kappa^* w(a,x;u_0) = w(a,x;\kappa^* u_0). \tag{4.12}$$

Combining (4.11) and (4.12) we have

$$\kappa^* u(a,x) = \kappa^* w(a,x;u_0) > w(a,x;v_0) = v(a,x), \quad \forall (a,x) \in [0,a_2] \times \overline{\Omega},$$

which is a contradiction with the definition of κ^*. We conclude that $u \geq v$. Now switching u and v in the above argument, we also have $v \geq u$, which shows the uniqueness of the solution. ∎

At last, we give a result on the nonexistence of positive continuous equilibria.

Proposition 4.7: *Assume that $\lambda_1^l \leq 0$ is the principal eigenvalue of the operator $\mathcal{A}^l$ defined in* (4.3). *Then any nonnegative continuous solution of* (4.1) *is identically zero.*

Proof. Assume by contradiction that $\lambda_1^l \leq 0$ and there exists a nonnegative continuous solution u to equation (4.1) which is positive somewhere in $[0,a_2]\times\overline{\Omega}$. Next, we have the following claim

Claim 4.8: *u is positive everywhere in $[0,a_2] \times \overline{\Omega}$.*

We first assume that the above claim holds. It follows from Assumption 4.1 that there exists a positive constant c_0 such that

$$\int_0^{a_2} \beta(a,x)u(a,x)da \geq c_0, \quad \forall x \in \overline{\Omega}.$$

From Assumption 1.3, we have

$$\frac{f(\int_0^{a_2} \beta(a,\cdot)u(a,\cdot)da)}{\int_0^{a_2} \beta(a,\cdot)u(a,\cdot)da} \leq \frac{f(c_0)}{c_0} < f'(0), \quad \text{in } \overline{\Omega}. \tag{4.13}$$

Moreover, since u is a solution of (4.1), we have for all $x \in \overline{\Omega}$ that

$$\begin{aligned}\frac{f(c_0)}{c_0}\int_0^{a_2} \beta(a,x)u(a,x)da &\geq \frac{f(\int_0^{a_2} \beta(a,x)u(a,x)da)}{\int_0^{a_2} \beta(a,x)u(a,x)da}\int_0^{a_2} \beta(a,x)u(a,x)da \\ &= f\left(\int_0^{a_2} \beta(a,x)u(a,x)da\right) = u(0,x).\end{aligned}$$

Next let us define an operator $\mathcal{A}^l_{c_0}$ as follows:

$$\mathcal{A}^l_{c_0}(0,\phi):=\left(-\phi(0,\cdot)+\frac{f(c_0)}{c_0}\int_0^{a_2}\beta(a,\cdot)\phi(a,\cdot)da,-\partial_a\phi+D(K-I)\phi-\mu\phi\right),\quad (0,\phi)\in\mathrm{dom}(\mathcal{A}^l_{c_0}),\tag{4.14}$$

where $\mathrm{dom}(\mathcal{A}^l_{c_0})=\{0\}\times W^{1,1}((0,a_2),C(\overline{\Omega}))$. Furthermore, due to (4.13), one has (by Claim 3.7 in Ducrot et al. [5] which states that $\lambda\to r(\mathcal{M}_\lambda)$ is strictly decreasing)

$$\lambda_p'(\mathcal{A}^l_{c_0})<\lambda_p'(\mathcal{A}^l)=\lambda_1^l\le 0.$$

Thus by Definition 3.8, for all negative $\lambda>\lambda_p'(\mathcal{A}^l_{c_0})$, there exists a positive continuous function ϕ_λ such that

$$(-\mathcal{A}^l_{c_0}+\lambda)(0,\phi_\lambda)\ge(0,0).$$

Arguing as above, we can see that $\phi_\lambda\ge\delta$ for some positive δ. Let us define the following quantity:

$$\gamma^*:=\inf\{\gamma>0:u\le\gamma\phi_\lambda\}.$$

We end the proof of the theorem by proving that $\gamma^*=0$. Assume that $\gamma^*>0$. Set $w:=u-\gamma^*\phi_\lambda$. Further, set

$$a_0=\min\{a\in[0,a_2]:\exists x\in\overline{\Omega},\ \text{s.t. } w(a,x_0)=0\}.$$

Such a_0 exists due to the definition of γ^*. It follows that there exists $(a_0,x_0)\in[0,a_2]\times\overline{\Omega}$ such that $w(a_0,x_0)=0$.

If $a_0\in(0,a_2]$, observe that w satisfies the following equation:

$$\partial_a w(a,x)\le D\left[\int_\Omega J(x-y)w(a,y)dy-w(a,x)\right]-\mu(a,x)w(a,x),\quad (a,x)\in(0,a_2]\times\overline{\Omega}.$$

Recalling the constant of variation formula (2.9), one has

$$w(a,x)\le e^{-Da}\pi(0,a,x)w(0,x)+D\int_0^a e^{-D(a-l)}\pi(l,a,x)[Kw](l,x)dl.\tag{4.15}$$

Considering the above inequality at (a_0,x_0), we have a contradiction, since by the definition of a_0, $w(a,x)<0$ for all $(a,x)\in[0,a_0)\times\overline{\Omega}$ implies the right hand side of (4.15) is negative.

If $a_0 = 0$, that is, $w(0, x_0) = 0$, by (4.13), one has that $w(0, x_0)$ satisfies

$$\begin{aligned} w(0,x_0) &= u(0,x_0) - \gamma^* \phi_\lambda(0,x_0) \\ &\leq \frac{f(c_0)}{c_0} \int_0^{a_2} \beta(a,x_0) u(a,x_0) da - \gamma^* \frac{f(c_0)}{c_0} \int_0^{a_2} \beta(a,x_0) \phi_\lambda(a,x_0) da \\ &= \frac{f(c_0)}{c_0} \int_0^{a_2} \beta(a,x_0) w(a,x_0) da \leq 0. \end{aligned}$$

It follows by Assumption 4.1 that $w(a, x_0) = 0$ at least in $[a_2 - \epsilon, a_2]$ for some $\epsilon > 0$ small enough. Then by the proof of Claim 4.8 in the following, we have $w \equiv 0$ in $[0, a_2] \times \overline{\Omega}$. Thus $u = \gamma^* \phi_\lambda$ and we get the following contradiction

$$0 = \partial_a u - D[K - I]u + \mu u = \partial_a \gamma^* \phi_\lambda - D[K - I]\gamma^* \phi_\lambda + \mu \gamma^* \phi_\lambda \geq -\lambda \gamma^* \phi_\lambda > 0.$$

Thus, $\gamma^* = 0$ and the proof is complete. ∎

Now let us prove Claim 4.8.

Proof of Claim 4.8. Assume by contradiction that u is zero somewhere, without loss of generality, say $u(a_0, x_0) = 0$. If $a_0 \in (0, a_2]$, recalling again the constant of variation formula (2.9), one has

$$u(a,x) = e^{-Da} \pi(0,a,x) u(0,x) + D \int_0^a e^{-D(a-l)} \pi(l,a,x) [Ku](l,x) dl. \tag{4.16}$$

Considering the above inequality at (a_0, x_0), it follows that for any $l \in [0, a_0]$, one has $[Ku](l, x_0) = 0$ and thus $u(l, x_1) = 0$ for all $x_1 \in B(x_0, r)$. Next consider (4.16) at (l, x_1), one has $u(l, x_2) = 0$ for all $x_2 \in B(x_1, r)$. Then by continuing this process, we can get $u(l, \cdot) \equiv 0$ in $\overline{\Omega} \cap B(x_0, nr)$ with some $n \in \mathbb{N}$ large enough for all $l \in [0, a_0]$. On the other hand, by the nonlocal equation, the solution starting at $u(a_0, \cdot) \equiv 0$ will be zero, i.e., $u(l, \cdot) \equiv 0$ when $l > a_0$, which implies $u \equiv 0$. This contradicts the fact that u is positive somewhere in $[0, a_2] \times \overline{\Omega}$.

If $a_0 = 0$, we have

$$u(0, x_0) = f\left(\int_0^{a_2} \beta(a, x_0) u(a, x_0) da \right) = 0.$$

It follows by the assumption on f that

$$\int_0^{a_2} \beta(a, x_0) u(a, x_0) da = 0.$$

This implies that $u(\widetilde{a}, x_0) = 0$ at least for $\widetilde{a} \in (0, a_2]$ due to Assumption 4.1. Considering equation (4.16) at $(\widetilde{a}, x_0)$ with $\widetilde{a} \in (0, a_2]$, we have the same contradiction as above. Hence, $u > 0$ in $[0, a_2] \times \overline{\Omega}$, which concludes the desired result. ∎

4.3. Stability. In this subsection we will show the global stability of the positive equilibrium u^* obtained in Theorem 4.4. First the existence of a solution $u(t, a, x)$ for (1.1) defined for all time $t \geq 0$ follows from a standard semigroup method by writing equation (1.1) as an abstract Cauchy problem (2.13), which is shown in the following:

$$\begin{cases} \frac{dU}{dt} = \mathcal{B}U + F(U), \\ U(0) = U_0, \end{cases} \qquad \text{with } U_0 = (0, u_0) \tag{4.17}$$

and based on the Lipshcitz assumption on f, see Thieme [15, 16] or Magal and Ruan [9]. Next, thanks to the definition of $\mathcal{B}$, we have that $\mathcal{B}$ is resolvent positive. Moreover, F is monotone due to Assumption 1.3(ii) on f, i.e.,

$$0 \leq U \leq V \Rightarrow 0 \leq F(U) \leq F(V).$$

Thus, by Magal et al. [10, Theorem 4.5], we can conclude that the weak comparison principle holds for (4.17), which is written as follows.

Lemma 4.9 (Weak comparison principle): *Assume that $U_0 \in \mathcal{X}_0$ and $U_0 \geq 0_{\mathcal{X}_0}$. Then the mild solution to* (4.17) *is $U(t) \geq 0_{\mathcal{X}_0}$ for any $t \geq 0$.*

It follows that the weak comparison principle also holds for (1.1). Now we give the strong comparison principle for (1.1).

Lemma 4.10 (Strong comparison principle): *Assume that $u_0 \in C([0, a_2] \times \overline{\Omega})$ and $u_0(a, x) \geq 0$ but $u_0(a, x) \not\equiv 0$. Then the strong solution to* (1.1) *is*

$$u(t, a, x) > 0 \quad \text{for any } t > 0 \text{ and } (a, x) \in [0, a_2] \times \overline{\Omega}.$$

Proof. Solving the problem (1.1) along the characteristic line $a - t = c$, where $c \in \mathbb{R}$, we now derive the formula for a solution to (1.1). For fixed $c \in \mathbb{R}$, we set $w(t) = u(t, t + c)$ for $t \in [\max(-c, 0), \infty)$. With $a = t + c$ one obtains for $t \in [\max(-c, 0), \infty)$ the equation

$$\partial_t w(t) = D[K - I]w - \mu(t + c, \cdot)w. \tag{4.18}$$

We first study the case $c \geq 0$. Clearly, $w(0) = u(0, c) = u(0, a - t) = u_0(a - t)$. Considering the equation (4.18) with initial data $w(0) \geq 0$ and $w(0) \not\equiv 0$, we have $w(t) > 0$ for $t > 0$ by the strong comparison principle of the nonlocal diffusion problem, due to $J(0) > 0$ in Assumption 1.1. It follows that $u(t, a) > 0$ for $a \geq t$. On the other hand, integrating (4.18) from 0 to t, one obtains

$$w(t) = \mathcal{U}(c, t + c)w(0),$$

and thus

$$u(t, a) = \mathcal{U}(a - t, a)u_0(a - t).$$

Next we consider the case $c < 0$. Integrating (4.18) from $-c$ to t, one gets

$$w(t) = \mathcal{U}(0, t + c)w(-c),$$

and hence

$$u(t, a) = \mathcal{U}(0, a)u(t - a, 0).$$

Thus now the solution to (1.1) reads as follows:

$$u(t,a) = \begin{cases} \mathcal{U}(a - t, a)u_0(a - t), & a \geq t, \\ \mathcal{U}(0, a)u(t - a, 0), & a < t. \end{cases} \tag{4.19}$$

Next we plug the explicit formula (4.19) into $u(t, 0)$ to obtain

$$\begin{aligned} u(t,0) = f\bigg(&\int_0^t \chi(a)\beta(a,\cdot)\mathcal{U}(0,a)u(t-a,0)da \\ &+ \int_t^{a_2} \chi(a)\beta(a,\cdot)\mathcal{U}(a-t,a)u_0(a-t)da \bigg), \end{aligned} \tag{4.20}$$

where $\chi(a)$ is the cutoff function satisfying $\chi(a) = 1$ when $a \in (0, a_2)$, otherwise $\chi(a) = 0$. Now we consider two cases.

Case 1. If $t < a_2$, (4.20) is written as follows:

$$\begin{aligned} u(t,0) = f\bigg(&\int_0^t \beta(a,\cdot)\mathcal{U}(0,a)u(t-a,0)da \\ &+ \int_t^{a_2} \beta(a,\cdot)\mathcal{U}(a-t,a)u_0(a-t)da \bigg). \end{aligned} \tag{4.21}$$

Since $u(t, a) = \mathcal{U}(a-t, a)u_0(a-t) > 0$ for $a \geq t$ and $\int_a^{a_2} \beta(l, \cdot)dl \geq \int_a^{a_2} \underline{\beta}(l)dl > 0$ a.e. for any $a \in [0, a_2)$ by Assumption 4.1 on β, the second term on the right-hand of (4.21) must be positive. It follows by Assumption 1.3 on f that we have $u(t, 0) > 0$. Thus $u(t, a) > 0$ for $a < t$ via (4.19).

Case 2. If $t \geq a_2$, (4.20) is written as follows:

$$u(t,0) = f\Big(\int_0^{a_2} \beta(a,\cdot)\mathcal{U}(0,a)u(t-a,0)da\Big). \tag{4.22}$$

We claim that $u(t,0,x) := [u(t,0)](x) > 0$ in $[a_2,\infty) \times \overline{\Omega}$. By contradiction, suppose that there exists $(t_0,x_0) \in [a_2,\infty) \times \overline{\Omega}$ such that $u(t_0,0,x_0) = 0$. By Assumption 1.3 on f, one obtains

$$\begin{aligned} 0 &= \int_0^{a_2} \beta(a,x_0)\mathcal{U}(0,a)u(t_0-a,0,x_0)da \\ &\geq \int_0^{a_2} \beta(a,x_0)e^{-\int_0^a (D+\overline{\mu}(s))ds}e^{DKa}u(t_0-a,0,x_0)da, \end{aligned}$$

where we used the fact that $e^{-\int_0^a (D+\overline{\mu}(s))ds}$ and e^{DKa} are commuting. By Assumption 4.1 on β, one has $\int_a^{a_2} \beta(l,x_0)dl \geq \int_a^{a_2} \underline{\beta}(l)dl > 0$ a.e. for any $a \in [0,a_2)$, then we can find one point $b_0 \in [a_2-\epsilon,a_2]$ such that $e^{DKb_0}u(t_0-b_0,0,x_0) = 0$, where $\epsilon > 0$ is small enough. By definition, one has

$$e^{DKb_0}u(t_0-b_0,0,x_0) = \sum_{n=0}^{\infty} \frac{(Db_0)^n}{n!} K^{*n} * u(t_0-b_0,0,x_0),$$

where K^{*n} denotes the n-fold convolution of K; that is $K^{*n} = K * \cdots * K$, n times. It follows that for each $n \in \mathbb{N}$,

$$K^{*n} * u(t_0-b_0,0,x_0) = 0.$$

However, by Assumption 1.1 on J, one has $J > 0$ in $B(0,r)$, which implies that

$$u(t_0-b_0,0,x) = 0, \quad \text{for all } x \in B(x_0,nr) \cap \overline{\Omega}.$$

When n is large enough, $B(x_0,nr) \cap \overline{\Omega}$ covers $\overline{\Omega}$, and thus $u(t_0-b_0,0,\cdot) \equiv 0$ in $\overline{\Omega}$.

Next replace t_0 by t_0-b_0 in (4.20). If t_0-b_0 falls in $[0,a_2)$, by the argument as in Case 1, one has $u(t_0-b_0,0) > 0$, which is a contradiction. Hence, t_0-b_0 must fall in $[a_2,\infty)$. Then by the same argument as in Case 2, one can find $b_1 \in [a_2-\epsilon,a_2]$ such that $u(t_0-b_0-b_1,0) = 0$. Now repeating the above process by induction, one can find a sequence $\{b_i\}_{i\geq 0}$ such that $u(t_0-\sum_{i=0}^M b_i,0) = 0$ for any $M \geq 0$. But we know every b_i is in $[a_2-\epsilon,a_2]$, so there always exists a minimal $M_0 > 0$ such that $t_0-\sum_{i=0}^{M_0} b_i < a_2$. Then by Case 1, one has

$$u\Big(t_0-\sum_{i=0}^{M_0} b_i,0\Big) > 0.$$

Now consider (4.22) at $t = t_0 - \sum_{i=0}^{M_0-1} b_i$, which is larger than or equal to a_2. We get a contradiction, since now the left hand side of (4.22) equals zero, while the right-hand side of (4.22) is larger than zero.

In summary, we cannot have $(t,x) \in (0,\infty)\times\overline{\Omega}$ such that $u(t,0,x) = 0$, which implies that $u(t,0,x) > 0$ and thus $u(t,a) > 0$ by (4.19). Hence, the proof is complete. ∎

Finally we present the following global stability result.

Theorem 4.11 (Stability): *Let Assumption 4.5 hold. Assume that $\lambda_1^l > 0$. Then the nontrivial equilibrium u^* is stable in the sense that $u(t,a,x) \to u^*(a,x)$ pointwise as $t \to \infty$, where $u(t,a,x)$ is a solution of (1.1) with initial data $u_0(a,x) \geq 0$ but $u_0(a,x) \not\equiv 0$ in $[0,a_2] \times \overline{\Omega}$.*

Proof. If $u_0(a,x) \geq 0$ but $u_0(a,x) \not\equiv 0$ in $[0,a_2]\times\overline{\Omega}$, using the strong comparison principle (Lemma 4.10), there exists a positive constant δ such that $u(1,a,x) > \delta$ in $[0,a_2] \times \overline{\Omega}$. Since $\lambda_1^l > 0$, we can allow $\epsilon\underline{u}$ defined in Theorem 4.4 to be a sub-solution of (4.1) for ϵ small enough. Since $u(1,a,x) \geq \delta$ and $\underline{u}$ is bounded, by choosing ϵ smaller if necessary we also achieve that $\epsilon\underline{u} \leq u(1,a,x)$. Now let us denote $\underline{U}(t,a,x)$ the solution of (1.1) with initial data $\epsilon\underline{u}$. By the weak comparison principle (Lemma 4.9), $\underline{U}(t,a,x) \geq \epsilon\underline{u}(a,x)$ for all $t \geq 0$. Given $s \geq 0$, let

$$z^s(t,a,x) := \underline{U}(t+s,a,x) - \underline{U}(t,a,x),$$

which satisfies $z^s(0,a,x) \geq 0$ by the above argument and

$$\begin{cases} \frac{dU}{dt} = \mathcal{B}U + GU, \\ U(0) = U_0, \end{cases} \qquad \text{with } U = (0, z^s) \tag{4.23}$$

on $(0,\infty) \times [0,a_2] \times \overline{\Omega}$ for some function G on $(0,\infty) \times [0,a_2] \times \Omega$ with

$$\|G\|_{L^\infty} \leq \|F'\|_{L^\infty}.$$

The weak comparison principle (Lemma 4.9) then applies to (4.23) to yield that $z^s \geq 0$ for all $s \geq 0$, from which follows that $\underline{U}(t,a,x)$ is a non-decreasing function of the time and $\underline{U}(t,a,x) \leq u(t+1,a,x)$.

On the other hand, L which is defined in the proof in Theorem 4.4 is a super-solution of (4.1) and u_0 is bounded; we also have $u(t,a,x) \leq \overline{U}(t,a,x)$ if necessary choosing L large enough, where $\overline{U}(t,a,x)$ denotes the solution of (1.1) with initial data $\overline{U}(0,a,x) = L \geq u_0$. A similar argument as above using the comparison principle shows that $\overline{U}$ is a non-increasing function of t. Thus, we have for all time $t \geq 0$ that

$$\epsilon \underline{u} \leq \underline{U}(t,a,x) \leq u(t+1,a,x) \leq \overline{U}(t+1,a,x).$$

The monotonicity of $\underline{U}$ and $\overline{U}$ implies

$$\underline{U}(t,a,x) \nearrow U_*(a,x) \quad \text{and} \quad \overline{U}(t,a,x) \searrow U^*(a,x)$$

pointwise as $t \to +\infty$ for some functions $U_* \leq U^*$. In what follows, we show that $U_*, U^* \in W^{1,1}((0,a_2), C(\overline{\Omega}))$ are solutions to (4.1).

For now, we acknowledge this claim and will give a prove later. Hence, the uniqueness in Theorem 4.6 implies $U_* = U^* = u^*$. Moreover, due to Dini's theorem, we can derive that $\underline{U} \nearrow U_*$ and $\overline{U} \searrow U^*$ in $C([0,a_2] \times \overline{\Omega})$ as $t \to +\infty$, which together with the above inequality implies that $u(t,a,x) \to u^*$ uniformly in $C([0,a_2] \times \overline{\Omega})$ as $t \to +\infty$.

Now let us finish the proof of $U_*, U^* \in W^{1,1}((0,a_2), C(\overline{\Omega}))$. To this end, we define

$$\underline{V}(t,a,x) := \int_0^a \underline{U}(t,s,x)ds \quad \text{for all } t > 0 \text{ and } (a,x) \in [0,a_2] \times \overline{\Omega}.$$

Since $\underline{U}$ solves (1.1), direct calculation yields

$$\begin{cases} \partial_t \underline{V} = D\int_\Omega J(x-y)\underline{V}(t,a,y)dy - D\underline{V} - \int_0^a \mu(s,x)\underline{U}(t,s,x)ds \\ \qquad -\underline{U}(t,a,x) + f(\int_0^{a_2} \beta(a,x)\underline{U}(t,a,x)da), t > 0, & x \in \overline{\Omega}, \\ \underline{V}(0,a,x) = \int_0^a \underline{U}(0,s,x)ds, & x \in \overline{\Omega}. \end{cases}$$

By the monotone convergence theorem, one has $\underline{V}(t,a,x) \nearrow \int_0^a U_*(s,x)ds$ pointwise as $t \to +\infty$. Next integrating the first equality of the above system with respect to t in $[t-\tau, t]$ for some $\tau > 0$ and letting $t \to +\infty$, we have

$$\begin{aligned} U_*(a,x) =& D\int_0^a \int_\Omega J(x-y)U_*(s,y)dyds - D\int_0^a U_*(s,x)ds \\ & - \int_0^a \mu(s,x)U_*(s,x)ds + f\left(\int_0^{a_2} \beta(a,x)U_*(a,x)da\right), \quad x \in \overline{\Omega}, \end{aligned}$$

where we have used the monotone convergence theorem in the integral terms. Hence, we conclude that $U_*(a,x)$ is Lipschitz continuous in $a \in [0,a_2]$ and that U_* satisfies

$$\begin{cases} \partial_a U_* = D\int_\Omega J(x-y)U_*(a,y)dy - DU_* - \mu(a,x)U_*(a,x), & (a,x)\in(0,a_+)\times\overline{\Omega},\\ U_*(0,x) = f(\int_0^{a_2}\beta(a,x)U_*(a,x)da), & x\in\overline{\Omega}. \end{cases}$$

Therefore, by using the same argument after Assumption 4.5, one has that $U_* \in W^{1,1}((0,a_2),C(\overline{\Omega}))$ is a solution to (4.1). By the same argument one can deduce that $U^* \in W^{1,1}((0,a_2),C(\overline{\Omega}))$ is also a solution to (4.1), which completes the proof. ∎

5. Global dynamics in terms of diffusion rate and diffusion range

In this section we give a similar result on the global dynamics of (1.1) by using the values of the diffusion rate D and diffusion range γ without and with kernel scaling, respectively. For the kernel scaling, Assumption 4.5 will be modified via replacing D by $\frac{D}{\gamma^m}$ and K by $K_{\gamma,\Omega}$.

THEOREM 5.1: *Let Assumption 4.5 hold. Assume that $s(\mathcal{A}^l)$ coincides with the principal eigenvalue of $\mathcal{A}^l$ defined in* (4.3)*. Then equation* (1.1) *admits a unique positive equilibrium $u^* \in C([0,a_2]\times\overline{\Omega})$ that is stable for each $D>0$ sufficiently small if $s(B_1+\mathcal{C})>0$, where $s(B_1+\mathcal{C})=\alpha_2$ and α_2 satisfies*

$$\max_{x\in\overline{\Omega}} f'(0)\int_0^{a_2}\beta(a,x)e^{-\alpha_2 a}\pi(0,a,x)da = 1. \tag{5.1}$$

Proof. Note that $\mathcal{A}^l$ defined in (4.3) also satisfies all the properties of $\mathcal{A}$ discussed in Section 3. Then by Theorem 3.10, $s^D(\mathcal{A}^l)>0$ for all $D>0$ sufficiently small if $s(B_1+\mathcal{C})>0$. Thus, the result follows from Theorem 4.4, Theorem 4.6 and Theorem 4.11. ∎

THEOREM 5.2: *Let Assumption 4.5 hold. Assume that $s(\mathcal{A}^l)$ coincides with the principal eigenvalue of $\mathcal{A}^l$ defined in* (4.3)*, then we have the following results:*

1. *For each $m>0$, assume that $s(B_1+\mathcal{C})=\alpha_2>0$. Then there exists γ_1 sufficiently large such that for each $\gamma>\gamma_1$ equation* (1.1) *with kernel scaling defined in* (3.9) *admits a unique stable positive equilibrium $u^*\in C([0,a_2]\times\overline{\Omega})$.*
2. *Suppose that J is symmetric, i.e., $J(x)=J(-x)$, with*

$$\mu,\beta\in C^2(\mathbb{R}^N, L^\infty_+(0,a_2)).$$

For each $m \in [0,2)$, assume that $s(B_1 + \mathcal{C}) = \alpha_2 > 0$. Then there exists $\gamma_2 > 0$ sufficiently small such that for each $\gamma \in (0,\gamma_2)$ equation (1.1) *with kernel scaling defined in* (3.9) *admits a unique stable positive equilibrium $u^* \in C([0,a_2] \times \overline{\Omega})$.*

Proof. It follows from Theorems 3.11, 4.4, 4.6 and 4.11. ∎

At the end of this section, we investigate the asymptotic behavior of the equilibrium u^* in terms of D without kernel scaling and in terms of γ with kernel scaling respectively. In order to highlight the dependence of u^* on D or γ, we denote u^* by u^*_D or u^*_γ. Before proceeding, we first give a lemma about the solution of (4.1) without nonlocal diffusion, that is,

$$\begin{cases} \partial_a v(a,x) = -\mu(a,x)v(a,x), & (a,x) \in (0,a_2) \times \overline{\Omega}, \\ v(0,x) = f(\int_0^{a_2} \beta(a,x)v(a,x)da), & x \in \overline{\Omega}. \end{cases} \tag{5.2}$$

To proceed , we first define two sets as follows:

$$\begin{aligned} Q_> &:= \left\{x \in \overline{\Omega} : f'(0)\int_0^{a_2} \beta(a,x)\pi(0,a,x)da > 1\right\}, \\ Q_= &:= \left\{x \in \overline{\Omega} : f'(0)\int_0^{a_2} \beta(a,x)\pi(0,a,x)da = 1\right\}. \end{aligned} \tag{5.3}$$

Lemma 5.3: *Assume that $Q_>$ is nonempty. Then for any $x \in Q_>$, the equation*

$$\begin{cases} \partial_a v(a,x) = -\mu(a,x)v(a,x), & a \in (0,a_2) \\ v(0,x) = f(\int_0^{a_2} \beta(a,x)v(a,x)da) \end{cases} \tag{5.4}$$

has a unique positive solution, denoted by $v^*(a,x)$, which belongs to $W^{1,1}(0,a_2)$. Moreover, the function $x \to v^*(\cdot,x)$ is continuous from $Q_>$ to $W^{1,1}(0,a_2)$.

Proof. For any fixed $x \in Q_>$, the existence of solutions of (5.4) is transformed into finding solutions of the following equation:

$$v(0,x) = f\left(\int_0^{a_2} \beta(a,x)\pi(0,a,x)v(0,x)da\right).$$

It follows by $x \in Q_>$ that one has

$$\frac{f(v(0,x)\int_0^{a_2} \beta(a,x)\pi(0,a,x)da)}{v(0,x)\int_0^{a_2} \beta(a,x)\pi(0,a,x)da} = \frac{1}{\int_0^{a_2} \beta(a,x)\pi(0,a,x)da} < f'(0). \tag{5.5}$$

Due to Assumption 1.3(iii) on f, we can conclude that there exists a unique number $h(x) > 0$ such that $h(x) = v(0,x)\int_0^{a_2} \beta(a,x)\pi(0,a,x)da$.Thus the unique positive solution of (5.4) is given as follows

$$v^*(a,x) = \frac{h(x)\pi(0,a,x)}{\int_0^{a_2}\beta(a,x)\pi(0,a,x)da}.$$

It is obvious that $v^*(\cdot,x) \in W^{1,1}(0,a_2)$. Moreover, define $g(u) := \frac{f(u)}{u}$, then by Assumption 1.3(iii) again, the inverse of g exists and is continuous over $[0, f'(0)]$. It follows by (5.5) that

$$h(x) = g^{-1}\Big(\frac{1}{\int_0^{a_2}\beta(a,x)\pi(0,a,x)da}\Big)$$

is continuous with respect to $x \in \overline{\Omega}$ due to the assumptions on β and μ. Hence, we have that $x \to v^*(\cdot,x)$ is continuous from $Q_>$ to $W^{1,1}(0,a_2)$. Thus, our proof is complete. ∎

Before showing the asymptotic behavior of the positive equilibrium with respect to the diffusion rate and diffusion range, we show that the unique solution of (5.4) is zero when $x \notin Q_>$.

LEMMA 5.4: *Assume that $x \notin Q_>$. Then the unique solution of* (5.4) *is zero.*

Proof. CASE 1. $x \notin Q_> \cup Q_=$. In this case, we have

$$f'(0)\int_0^{a_2}\beta(a,x)\pi(0,a,x)da < 1.$$

Consider the linearized equation of (5.4) which is rewritten as follows:

$$\begin{cases} \partial_a w(a,x) = -\mu(a,x)w(a,x), & a \in (0,a_2), \\ w(0,x) = f'(0)\int_0^{a_2}\beta(a,x)w(a,x)da. \end{cases} \tag{5.6}$$

Recall by Assumption 1.3 that $f(u) \le f'(0)u$; we have $v^*(a,x) \le w(a,x)$ for all $a \in [0,a_2] \times \overline{\Omega}$. But by the second equation of (5.6), if $v^*(0,x) \neq 0$ for some $x \notin Q_> \cup Q_=$ is positive, one has

$$w(0,x) = f'(0)\int_0^{a_2}\beta(a,x)\pi(0,a,x)da\, w(0,x) < w(0,x),$$

which is a contradiction. Thus, $v^*(a,x) \equiv w(a,x) \equiv 0$ when $x \notin Q_> \cup Q_=$.

CASE 2. $x \in Q_=$. We have

$$v(0,x) = f\Big(\int_0^{a_2}\beta(a,x)v(a,x)da\Big) \le f'(0)\int_0^{a_2}\beta(a,x)v(a,x)da = v(0,x).$$

It implies that $v(a,x) \equiv 0$ by the assumptions on β and f. Hence, the proof is complete. ∎

Corollary 5.5: *The function v^* provided by Lemma 5.3 is continuous in $Q_> \cup Q_=$.*

Proof. By Lemma 5.3, we know that $x \to v^*(\cdot, x)$ is continuous in $Q_>$; it remains to show that $x \to v^*(\cdot, x)$ is continuous in $Q_=$. With this aim, choose $x_0 \in Q_=$ and a sequence $x_n \to x_0$ as $n \to \infty$, where $\{x_n\}_{n\geq 1} \subset Q_>$. Observing the first equation of (5.4), one has

$$\|\partial_a v^*(\cdot, x_n)\|_{L^1(0,a_2)} \leq C,$$

where $C > 0$ denotes some constant that may vary from line to line but is independent of $n \geq 0$. It follows that the sequence $\{v^*(\cdot, x_n)\}_{n\geq 0}$ is bounded in $W^{1,1}(0, a_2)$ which is continuously embedded into $L^\infty(0, a_2)$ so that

$$\|v^*(\cdot, x_n)\|_{L^\infty(0,a_2)} \leq C.$$

Again by the first equation of (5.4), one has

$$\|\partial_a v^*(\cdot, x_n)\|_{L^\infty(0,a_2)} \leq C.$$

Thus, we have $\|v^*(\cdot, x_n)\|_{W^{1,\infty}(0,a_2)} \leq C$. By the compact Sobolev embedding, we can find a limit, denoted by $\widehat{v}^*(\cdot) \in C([0, a_2])$, up to a subsequence such that

$$v^*(\cdot, x_n) \to \widehat{v}^*(\cdot) \quad \text{uniformly on } [0, a_2].$$

Since $x \to \mu(\cdot, x)$ is continuous from $\overline{\Omega}$ to $L^\infty(0, a_2)$, one has $\mu(\cdot, x_n) \to \mu(\cdot, x_0)$ in $L^\infty(0, a_2)$, and thus

$$\mu(\cdot, x_n)v^*(\cdot, x_n) \to \mu(\cdot, x_0)\widehat{v}^*(\cdot) \quad \text{in } L^\infty(0, a_2).$$

Applying the same argument to β and then passing to the limit on (5.4), one obtains

$$\begin{cases} \partial_a \widehat{v}^*(a) = -\mu(a, x_0)\widehat{v}^*(a), & a \in (0, a_2), \\ \widehat{v}^*(0) = f(\int_0^{a_2} \beta(a, x_0)\widehat{v}^*(a))da, \end{cases} \tag{5.7}$$

with $\widehat{v}^* \geq 0$. Next recalling Lemma 5.4, we have that when $x_0 \in Q_=$, the unique solution of (5.7) is zero. It follows that $\widehat{v}^*(\cdot) \equiv 0$. Thus, $v^*(\cdot, x_n) \to 0$ uniformly in $[0, a_2]$ as $n \to \infty$ and the function $x \to v^*(\cdot, x)$ is continuous in $Q_=$. Hence, the proof is complete. ∎

Theorem 5.6: *Let Assumption 4.5 hold. Assuming that $s(\mathcal{A}^l)$ coincides with the principal eigenvalue of $\mathcal{A}^l$ defined in* (4.3)*, and v^* is from Lemma 5.3, we have the following asymptotic results:*

(i) *Assume $u_D^*(a,x)$ is given by Theorem 5.1. Then*

$$\lim_{D\to 0^+} u_D^*(a,x) = \begin{cases} v^*(a,x) \text{ locally uniformly in } [0,a_2]\times Q_>, \\ 0 \text{ locally uniformly in } [0,a_2]\times \Omega\setminus Q_>. \end{cases} \tag{5.8}$$

(ii) *Assume $u_\gamma^*(a,x)$ is given by Theorem 5.2, $m\in[0,2)$ and J is symmetric, i.e., $J(x)=J(-x)$. Then*

$$\lim_{\gamma\to 0^+} u_\gamma^*(a,x) = \begin{cases} v^*(a,x) \text{ locally uniformly in } [0,a_2]\times Q_>, \\ 0 \text{ locally uniformly in } [0,a_2]\times \Omega\setminus Q_>. \end{cases} \tag{5.9}$$

(iii) *Assume $u_\gamma^*(a,x)$ is given by Theorem 5.2 and $m>0$. Then*

$$\lim_{\gamma\to \infty} u_\gamma^*(a,x) = \begin{cases} v^*(a,x) \text{ locally uniformly in } [0,a_2]\times Q_>, \\ 0 \text{ locally uniformly in } [0,a_2]\times \Omega\setminus Q_>. \end{cases} \tag{5.10}$$

Proof. We first prove (iii).

Case 1: Converging to $v^*(a,x)$. We choose any nonempty set $W\subset Q_>$ such that $\overline{W}\subset Q_>$. Then it suffices to show that for each $0<\delta=\delta(W)\ll 1$, there exists $\gamma_\delta=\gamma_\delta(W)>0$ such that for each $\gamma\in(\gamma_\delta,\infty)$

$$(1-\delta)v^*(a,x)\le u_\gamma^*(a,x)\le (1+\delta)v^*(a,x),\quad (a,x)\in[0,a_2]\times\overline{W}.$$

We here outline the proof of the upper bound and the lower bound that follows from similar arguments. Denote

$$v := (1+\delta)v^*$$

and define $F_\gamma : C([0,a_2]\times\overline{W})\to C([0,a_2]\times\overline{W})$ as follows:

$$F_\gamma(\psi) := \frac{D}{\gamma^m}\Big[\int_W J_\gamma(x-y)\psi(a,y)dy - \psi(a,x)\Big].$$

By the same argument as in Theorem 3.11(i) (see Ducrot et al. [5] for more details), one can show that

$$\|F_\gamma\|_{\mathcal{L}(C([0,a_2]\times\overline{W}))}\to 0,\quad \text{as } \gamma\to\infty \text{ uniformly in } W\subset Q_>. \tag{5.11}$$

On the other hand, thanks to Assumption 4.1 on β, one has

$$\int_0^{a_2}\beta(a,x)v^*(a,x)da>0,\quad \forall x\in\overline{W}.$$

Since for each $(a,x) \in [0,a_2] \times \overline{W}$,

$$f\left(\int_0^{a_2} \beta(a,x)v(a,x)da\right) - (1+\delta)f\left(\int_0^{a_2} \beta(a,x)v^*(a,x)da\right)$$
$$=\int_0^{a_2}\beta(a,x)v(a,x)da\left[\frac{f(\int_0^{a_2}\beta(a,x)v(a,x)da)}{\int_0^{a_2}\beta(a,x)v(a,x)da}-\frac{f(\int_0^{a_2}\beta(a,x)v^*(a,x)da)}{\int_0^{a_2}\beta(a,x)v^*(a,x)da}\right]<0,$$

where we used Assumption 1.3(iii), there exists a sufficiently small positive constant $c = c(\delta, W)$, which satisfies $c(\delta, W) \to 0$ as $\delta \to 0$, such that

$$\sup_{[0,a_2]\times\overline{W}}\left[f\left(\int_0^{a_2}\beta(a,x)v(a,x)da\right)-(1+\delta)f\left(\int_0^{a_2}\beta(a,x)v^*(a,x)da\right)\right] \le -c < 0. \tag{5.12}$$

This implies that for any $\delta > 0$, we can find $\gamma = \gamma(\delta) > 0$ such that

$$|F_\gamma(v)| \le c(\delta, W)$$

for each $\gamma \in (\gamma(\delta), \infty)$.

Now fix this $\gamma(\delta)$, and let us show that for each $\gamma \in (\gamma(\delta), \infty)$ we have

$$u_\gamma^*(a,x) \le v(a,x) \quad \text{for all } (a,x) \in [0,a_2] \times \overline{W}.$$

To do that, fix any $\gamma \in (\gamma(\delta), \infty)$ and define

$$\alpha^* := \sup\{\alpha > 0 : \alpha u_\gamma^*(a,x) \le v(a,x) \,\text{in}\, [0,a_2] \times \overline{W}\}.$$

Since $\min_{[0,a_2]\times\overline{W}} u_\gamma^* > 0$ and $v(a,x)$ is bounded, the number α^* is well defined and positive. Due to the continuity of $v(a,x)$ and $u_\gamma^*(a,x)$, $v(a,x) \ge \alpha^* u_\gamma^*(a,x)$ for all $(a,x) \in [0,a_2] \times \overline{W}$.

Clearly, if $\alpha^* \ge 1$, then we are done. So we assume that $\alpha^* < 1$. Set $w := v - \alpha^* u_\gamma^*$, then $w \ge 0$. Further, set

$$a_0 := \min\{a \in [0,a_2] : \exists x \in \overline{W}, \text{ s.t. } w(a_0,x) = 0\}.$$

Such a_0 exists due to the definition of α^*. It follows that there exists $x_0 \in \overline{W}$ such that $w(a_0,x_0) = 0$.

SUB-CASE 1. If $a_0 = 0$, that is, $w(0, x_0) = 0$, one has by (5.12) that

$$\begin{aligned} v(0,x) =&(1+\delta)f\left(\int_0^{a_2}\beta(a,x)v^*(a,x)da\right) \\ >&(1+\delta)f\left(\int_0^{a_2}\beta(a,x)v^*(a,x)da\right)+f\left(\int_0^{a_2}\beta(a,x)v(a,x)da\right) \\ &-(1+\delta)f\left(\int_0^{a_2}\beta(a,x)v^*(a,x)da\right)+\frac{c}{2} \\ =&f\left(\int_0^{a_2}\beta(a,x)v(a,x)da\right)+\frac{c}{2}. \end{aligned}$$

Thus, $w(0, x_0)$ satisfies

$$\begin{aligned} w(0,x_0) &= v(0,x_0)-\alpha^* u_\gamma^*(0,x_0) \\ &> f\left(\int_0^{a_2}\beta(a,x_0)v(a,x_0)da\right)+\frac{c}{2}-\alpha^* f\left(\int_0^{a_2}\beta(a,x_0)u_\gamma^*(a,x_0)da\right) \\ &> f\left(\int_0^{a_2}\beta(a,x_0)v(a,x_0)da\right)+\frac{c}{2}-f\left(\int_0^{a_2}\beta(a,x_0)\alpha^* u_\gamma^*(a,x_0)da\right) \\ &\geq \frac{c}{2}, \end{aligned}$$

where we used Assumption 1.3(iii) and $\alpha^* < 1$. It is a contradiction with $w(0, x_0) = 0$.

SUB-CASE 2. If $a_0 \in (0, a_2]$, observe that w satisfies

$$\begin{aligned} \partial_a w(a,x) =&\frac{D}{\gamma^m}\left[\int_W J_\gamma(x-y)w(a,y)dy-w(a,x)\right]-\mu(a,x)w(a,x) \\ &-\frac{D}{\gamma^m}\left[\int_W J_\gamma(x-y)v(a,y)dy-v(a,x)\right] \\ =&\frac{D}{\gamma^m}\left[\int_W J_\gamma(x-y)w(a,y)dy-w(a,x)\right]-\mu(a,x)w(a,x)-F_\gamma(v). \end{aligned}$$

Again by the constant of variation formula (2.9) and recalling (3.9), one has

$$\begin{aligned} w(a,x) =&e^{-\frac{D}{\gamma^m}a}\pi(0,a,x)w(0,x) \\ &+\frac{D}{\gamma^m}\int_0^a e^{-\frac{D}{\gamma^m}(a-l)}\pi(l,a,x)[K_{\gamma,W}w](l,x)dl \\ &-\int_0^a e^{-\frac{D}{\gamma^m}(a-l)}\pi(l,a,x)[F_\gamma(v)](l,x)dl. \end{aligned} \tag{5.13}$$

Recall from Sub-case 1 that $w(0,x) > \frac{c(\delta,W)}{2}$. Now considering the above inequality (5.13) at (a_0,x_0), $w(a,x)>0$ for all $(a,x)\in[0,a_0)\times\overline{W}$ implies that

$$e^{-\frac{D}{\gamma^m}a}\pi(0,a,x)w(0,x) \geq \frac{c(\delta,W)}{4}e^{-\int_0^{a_2}\overline{\mu}(s)ds},$$
$$\frac{D}{\gamma^m}\int_0^a e^{-\frac{D}{\gamma^m}(a-l)}\pi(l,a,x)[K_{\gamma,W}w](l,x)dl \to 0$$
$$\text{as } \gamma\to\infty, \text{ uniformly in } (a,x)\in[0,a_2]\times\overline{W}.$$

These inequalities combining with (5.11) (up to increasing γ if necessary) imply that the right hand side of (5.13) is positive. But the left hand side $w(a_0,x_0)=0$ induces a contradiction. Thus, $\alpha^*\geq 1$. Since $W\subset Q_>$ is arbitrary, the proof of case $x\in Q_>$ in (iii) is complete.

Case 2: Converging to 0. We use a similar argument as in Proposition 4.7. Choose any $W\subset\Omega\setminus Q_>$ satisfying $\overline{W}\subset\Omega\setminus Q_>$. Remember that the same argument in Theorem 3.11(i) gives us

$$\|F_\gamma\|_{\mathcal{L}(C([0,a_2]\times\overline{W}))}\to 0 \quad \text{as } \gamma\to\infty \text{ uniformly in } W\subset\Omega\setminus Q_>. \tag{5.14}$$

Now let us recall that for any $0\leq\tau\leq a\leq a_2$, one has

$$\mathcal{U}(\tau,a)=\Pi(\tau,a)+\int_\tau^a \Pi(l,a)[F_\gamma(\mathcal{U})](\tau,l)dl,$$

which implies that

$$\|\mathcal{U}(0,\cdot)-\Pi(0,\cdot)\|_{\mathcal{L}(C(\overline{W}))}\to 0 \quad \text{as } \gamma\to\infty, \text{ uniformly in } a\in[0,a_2]. \tag{5.15}$$

On the other hand, recalling $x\in\overline{W}\subset\Omega\setminus Q_>$ implies that

$$r\left(f'(0)\int_0^{a_2}\beta(a,\cdot)\Pi(0,a)da\right)=\max_{x\in\overline{W}}f'(0)\int_0^{a_2}\beta(a,x)\pi(0,a,x)da\leq 1.$$

It follows from (5.15) that for any sufficiently large γ, one has

$$r\left(f'(0)\int_0^{a_2}\beta(a,\cdot)\mathcal{U}(0,a)da\right)\leq 1. \tag{5.16}$$

Then the principal eigenvalue of $\mathcal{A}^l$, defined in (4.4), satisfies $\lambda_1(\mathcal{A}^l)\leq 0$. Now assume by contradiction that $u_\infty^* := \lim_{\gamma\to\infty}u_\gamma^*(a,\cdot)$ is positive somewhere in W. Then the remaining proof is the same with Proposition 4.7, so we omit it. Hence, we have finished the proof of (iii).

For (i) we follow the lines as in the proof of (iii) except that we need to set $\gamma = 1$ and replace the limit

$$F_1(v) = \frac{D}{\gamma^m}\left[\int_W J_\gamma(x-y)v(a,y)dy - v(a,x)\right] \to 0$$
$$\text{as } \gamma \to \infty, \text{ for any } v \in C([0,a_2]\times\overline{W})$$

uniformly in $(a,x) \in [0,a_2]\times\overline{W}$ with both $\overline{W} \subset Q_>$ and $\overline{W} \subset \Omega \setminus Q_>$ by the following limit:

$$F_1(v) = D\left[\int_W J(x-y)v(a,y)dy - v(a,x)\right] \to 0 \quad \text{as } D \to 0^+,$$
$$\text{for any } v \in C([0,a_2]\times\overline{W})$$

uniformly in $(a,x) \in [0,a_2]\times\overline{W}$ with both $\overline{W} \subset Q_>$ and $\overline{W} \subset \Omega\setminus Q_>$, and (5.13) is replaced by the following equality:

$$\begin{aligned} w(a,x) =& e^{-Da}\pi(0,a,x)w(0,x) + D\int_0^a e^{-D(a-l)}\pi(l,a,x)[K_{1,W}w](l,x)dl \\ &- \int_0^a e^{-D(a-l)}\pi(l,a,x)[F_1(v)](l,x)dl. \end{aligned}$$

Then the remaining proof is the same as for (iii).

For (ii), note by the argument in Theorem 3.11(ii) (see Ducrot et al. [5] for more details) that

$$F_\gamma(v) = \frac{D}{\gamma^m}\left[\int_W J_\gamma(x-y)v(a,y)dy - v(a,x)\right] \le C\gamma^{2-m}\|v\|_{C^2(\overline{W},C([0,a_2]))}$$

uniformly in $(a,x) \in [0,a_2]\times\overline{W}$ with $\overline{W} \subset Q_>$ and $\overline{W} \subset \Omega \setminus Q_>$, where C is a positive constant independent of γ. Next we revisit (5.13). Observe that

$$\frac{D}{\gamma^m}K_{\gamma,W}w \le C\gamma^{-m}\|w\|_{C([0,a_2]\times\overline{W})}$$

uniformly in $(a,x) \in [0,a_2]\times\overline{W}$ with $\overline{W} \subset Q_>$ and $\overline{W} \subset \Omega \setminus Q_>$. Then the remaining proof is the same as for (iii), so we are finished. ∎

Acknowledgements. We thank the anonymous reviewer for their helpful comments and suggestions.

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