Received: 23 May 2023 | Revised: 19 July 2023 | Accepted: 3 August 2023

DOI: 10.1112/blms.12919

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

On Boltyanski and Gohberg's partition conjecture

Jun Wang[1] | **Fei Xue**[2] | **Chuanming Zong**[3,4]

[1]School of Mathematical Sciences, Tianjin Normal University, Tianjin, China

[2]School of Mathematical Sciences, Nanjing Normal University, Nanjing, China

[3]Center for Applied Mathematics, Tianjin University, Tianjin, China

[4]Beijing International Center for Mathematical Research, Peking University, Beijing, China

Correspondence
Chuanming Zong, Center for Applied Mathematics, Tianjin University, Tianjin 300072, China.
Email: cmzong@tju.edu.cn

Funding information
National Natural Science Foundation of China, Grant/Award Numbers: NSFC12226006, NSFC11921001, NSFC12201307; Natural Key Research and Development Program of China, Grant/Award Number: 2018YFA0704701; Natural Science Foundation of Jiangsu Province, Grant/Award Number: BK20210555

Abstract

In 1965, Boltyanski and Gohberg made the following conjecture: Every bounded set in an n-dimensional normed space can be divided into 2^n subsets of smaller diameters. In this paper, we prove the following result: Every bounded set in an n-dimensional normed space can be divided into $2^{(1+o(1))n}$ subsets of smaller diameters.

MSC 2020
52C17, 46B20, 51K05 (primary)

1 | INTRODUCTION

Let $\mathbb{E}^n$ be the n-dimensional Euclidean space. For any bounded set $S \subseteq \mathbb{E}^n$, we call

$$d(S) = \sup\{\|\mathbf{x}, \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in S\}$$

the *diameter* of S, where $\|\cdot\|$ is the *Euclidean norm*. Let $b(S)$ be the smallest number such that S can be divided into $b(S)$ subsets with diameters strictly smaller than $d(S)$. In 1933, Borsuk [9] proved that an n-dimensional ball B^n in $\mathbb{E}^n$ cannot be partitioned into n parts of smaller diameters, which was announced in his ICM–Zurich talk (see [8]). Meanwhile, he proved that every bounded set in $\mathbb{E}^2$ can be divided into three subsets of smaller diameters. Based on this fact, he proposed the following problem. (Usually, the positive assertion of the problem is known as *Borsuk's partition conjecture.*)

Borsuk's problem. *Is it true that*

$$b(S) \leqslant n+1$$

holds for every bounded set S in $\mathbb{E}^n$?

In 1934, Bonnesen and Fenchel [7] proved that every bounded set S of $\mathbb{E}^2$ can be divided into three subsets S_1, S_2 and S_3 satisfying

$$d(S_i) \leqslant \frac{\sqrt{3}}{2} \cdot d(S), \quad i = 1, 2, 3.$$

In fact, the upper bound $\sqrt{3}/2$ can be attained at circular domains.

In 1947, Perkal [30] sketched a proof for a positive answer to the three-dimensional partition problem. Afterward, different proofs for this case were discovered by Eggleston [12], Grünbaum [14], Heppes [17], and others (see [15, 35]). In particular, Grünbaum proved that every bounded set S in $\mathbb{E}^3$ can be divided into four parts S_1, S_2, S_3, and S_4 satisfying

$$d(S_i) \leqslant 0.9887 \cdot d(S), \quad i = 1, 2, 3, 4.$$

In 1993, Kahn and Kalai [22] surprised the mathematical community by discovering counterexamples to Borsuk's conjecture in high dimensions. They proved that there exist sets S in $\mathbb{E}^n$ satisfying

$$b(S) \geqslant 1.07^{\sqrt{n}}.$$

Therefore, the first counterexample to Borsuk's conjecture occurs in $\mathbb{E}^{21801}$. Afterward, Kahn and Kalai's breakthrough was simplified by Alon [1] and improved by many authors, in particular by Hinrichs and Richter [18] to $n \geqslant 298$ in 2003. In 2014, Bondarenko [6] presented a 65-dimensional counterexample to Borsuk's conjecture. Soon after, Jenrich and Brouwer [21] discovered a 64-dimensional one. Up to now, Borsuk's problem is still open for $4 \leqslant n \leqslant 63$. Recently, Zong [38] proposed a computer proof program to this problem.

As Borsuk's conjecture is not true in high dimensions, obtaining upper bounds for the partition number is an interesting problem. In 1955, Lenz [25] proved that

$$b(S) \leqslant (\sqrt{n}+1)^n$$

holds for every bounded set S in $\mathbb{E}^n$. This bound was successively improved by Danzer [11], Lassak [24], and Schramm [32]. The best-known upper bound is

$$b(S) \leqslant 5n^{\frac{3}{2}}(4+\log n)\left(\frac{3}{2}\right)^{\frac{n}{2}},$$

which was discovered by Schramm in 1988.

Let $(\mathbb{R}^n, \|\cdot\|^*)$ be an n-dimensional *normed space*, that is, an n-dimensional real linear space $\mathbb{R}^n$ with norm $\|\cdot\|^*$. It is well-known that $B^* = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^* \leqslant 1\}$ is a centrally symmetric convex body centered at the origin. Usually, a convex body means a convex compact set with nonempty interior and B^* is called the unit ball of $(\mathbb{R}^n, \|\cdot\|^*)$. On the other hand, if C is a centrally symmetric convex body centered at the origin and $\mathbf{v}$ is a vector in $\mathbb{R}^n$, then

$$\|\mathbf{v}\|_C = \min\{r : r \geqslant 0, \mathbf{v} \in rC\}$$

defines a norm on $\mathbb{R}^n$ and produces an n-dimensional normed space. Therefore, there is a one-to-one correspondence between n-dimensional normed spaces and n-dimensional centrally symmetric convex bodies centered at the origin. So, for convenience, we use $M_C = \{\mathbb{R}^n, \|\cdot\|_C\}$ to denote an n-dimensional normed space that takes C as the unit ball. In M_C, the distance between two points $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\|\mathbf{x}, \mathbf{y}\|_C = \|\mathbf{x} - \mathbf{y}\|_C.$$

For basic concepts and results in normed spaces, we refer to [28, 29].

Let S be a bounded set in M_C. We define

$$d_C(S) = \sup\{\|\mathbf{x}, \mathbf{y}\|_C : \mathbf{x}, \mathbf{y} \in S\}$$

to be the diameter of S and define $b_C(S)$ to be the smallest number such that S can be divided into $b_C(S)$ subsets, all of them having diameters strictly smaller than $d_C(S)$.

It is natural to study the analogies of Borsuk's problem in normed spaces. For convenience, let S be a bounded set in a normed space M_C and let $\overline{S}$ denote the closure of the *convex hull* of S. In 1957, Grünbaum [13] studied Borsuk's partition problem in normed planes. He showed that, for every bounded set S in a normed plane M_C,

$$b_C(S) \leqslant 4,$$

where the equality holds if and only if $\overline{S}$ and C are homothetic parallelograms. In 1965, Boltyanski and Gohberg [3, pp. 75, 92] made the following conjecture:

Boltyanski and Gohberg's conjecture. *For every bounded set S in an n-dimensional normed space M_C, we have*

$$b_C(S) \leqslant 2^n,$$

where the equality holds if and only if $\overline{S}$ and C are homothetic parallelotopes.

Remark 1.1. When C is an n-dimensional cube, it is well-known and easy to see that

$$b_C(C) = 2^n.$$

Let K denote a *convex body*, a compact and convex set with nonempty interior int(K), in $\mathbb{E}^n$. Let $h(K)$ denote the smallest number of translates of λK $(0 < \lambda < 1)$ (or int(K)) such that their union contains K. In 1957, Hadwiger [16] proposed the following conjecture, which has a close relation with the Boltyanski–Gohberg conjecture.

Hadwiger's covering conjecture. *For every n-dimensional convex body K, we have*

$$h(K) \leqslant 2^n,$$

where the equality holds if and only if K is a parallelotope.

This conjecture has been studied by many authors, including Bezdek, Boltyanski, Lassak, Levi, Martini, Rogers, Soltan, Wu, and Zong. The two-dimensional case was solved by Levi [26] in 1954. However, the conjecture is still open for all $n \geqslant 3$. As the target of this paper is the Boltyanski–Gohberg conjecture, we will not go to the details of Hadwiger's conjecture. We refer the interested readers to the references of [2, 4, 10, 19, 37]. Here we only list two results that will be useful in this paper.

Lemma 1.1 (Boltyanski and Gohberg [3]). *For every bounded set S in a normed space* $M_C = \{\mathbb{R}^n, \|\cdot\|_C\}$*, we have*

$$b_C(S) \leqslant h(\overline{S}).$$

This lemma is simple, one can take it as an exercise.

Lemma 1.2 (Rogers and Zong [31]). *For every n-dimensional K, we have*

$$h(K) \leqslant \binom{2n}{n} n(\log n + \log\log n + 5).$$

For every n-dimensional centrally symmetric convex body C, we have

$$h(C) \leqslant 2^n n(\log n + \log\log n + 5).$$

There are some partial results on the Boltyanski–Gohberg conjecture for particular sets and norms (see [5, 20, 23, 27, 33, 34, 36]). However, it is still open for all $n \geqslant 3$ in general. As a consequence of Lemma 1.1 and the first part of Lemma 1.2, we have that

$$b_C(S) \leqslant \binom{2n}{n} n(\log n + \log\log n + 5)$$

holds for every bounded set S in an n-dimensional normed space M_C. Recently, this upper bound was improved by [10, 19] to

$$b_C(S) \leqslant \exp\left(-\Omega\left(\frac{n}{(\log n)^8}\right)\right) \cdot 4^n,$$

also by studying Hadwiger's conjecture. In this paper, we will prove the following theorem:

Theorem 1.1. *For every bounded set S in an n-dimensional normed space* M_C*, we have*

$$b_C(S) \leqslant 2^n(n+1)(\log(n+1) + \log\log(n+1) + 5) = 2^{(1+o(1))n}.$$

Remark 1.2. As usual, we write $f(n) = o(g(n))$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

2 | PROOF OF THE THEOREM

First, let us recall a basic concept in convex geometry. For every n-dimensional convex body K, we define

$$D(K) = K - K = \{\mathbf{x} - \mathbf{y} : \ \mathbf{x}, \ \mathbf{y} \in K\}.$$

Usually, it is known as the *difference body* of K. Clearly, $D(K)$ is centrally symmetric and convex. In fact the norm $\| \cdot \|_{D(K)}$ defined by $D(K)$ plays the key role of our proof.

Furthermore, for convenience, let $\mathcal{K}^n$ denote the set of all n-dimensional convex bodies K and let C^n denote the set of all n-dimensional centrally symmetric convex bodies C. It is easy to see that

$$d_C(S) = d_C(\overline{S})$$

and

$$b_C(S) \leqslant b_C(\overline{S}) \tag{2.1}$$

holds for all bounded sets S in M_C. Therefore, to study the Boltyanski–Gohberg conjecture, it is sufficient to deal with the convex bodies in $\mathcal{K}^n$.

Lemma 2.1. *In every n-dimensional normed space M_C, we have*

$$\max_{K\in\mathcal{K}^n} b_C(K) \leqslant \max_{K\in\mathcal{K}^n} b_{D(K)}(K).$$

Proof. Without loss of generality, let K be an n-dimensional convex body in M_C with $d_C(K) = 1$. Assume that $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$ are two points in $D(K)$, where all $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are points in K. Then, one can deduce that

$$\|\mathbf{x}, \mathbf{y}\|_C \leqslant \|\mathbf{x}_1 - \mathbf{x}_2, \mathbf{o}\|_C + \|\mathbf{y}_1 - \mathbf{y}_2, \mathbf{o}\|_C = \|\mathbf{x}_1, \mathbf{x}_2\|_C + \|\mathbf{y}_1, \mathbf{y}_2\|_C \leqslant 2$$

and therefore

$$d_C(D(K)) \leqslant 2 \tag{2.2}$$

(In fact, one has $d_C(D(K)) = 2$). Then, we obtain

$$D(K) \subseteq C$$

and, therefore,

$$d_C(S) \leqslant d_{D(K)}(S) \tag{2.3}$$

for all sets S in M_C.

On the other hand, for any pair of points $\mathbf{x}, \mathbf{y} \in K$, we have

$$\|\mathbf{x}, \mathbf{y}\|_{D(K)} = \frac{1}{2}\|\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{x}\|_{D(K)}$$

and therefore

$$d_{D(K)}(K) = 1. \tag{2.4}$$

If K can be divided into $b_{D(K)}(K)$ subsets $X_1, X_2, ..., X_{b_{D(K)}(K)}$ satisfying

$$d_{D(K)}(X_i) < 1, \quad i = 1, 2, \dots, b_{D(K)}(K),$$

it follows by (2.3) that

$$d_C(X_i) < 1, \quad i = 1, 2, \dots, b_{D(K)}(K)$$

and therefore

$$b_C(K) \leqslant b_{D(K)}(K).$$

Consequently, we get

$$\max_{K \in \mathcal{K}^n} b_C(K) \leqslant \max_{K \in \mathcal{K}^n} b_{D(K)}(K).$$

Lemma 2.1 is proved. □

Assume that K is a convex body in $\mathbb{R}^n$. We embed it into $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and create a centrally symmetric convex body $K^\bullet$ in $\mathbb{R}^{n+1}$. Setting K in the n-dimensional hyperplane

$$H = \{(x_1, x_2, \dots, x_{n+1}) : \ x_{n+1} = 0\}$$

of $\mathbb{R}^{n+1}$ and writing $\mathbf{e} = (0, 0, \dots, 0, 1)$, then we define

$$K^\bullet = \overline{(K + \mathbf{e}) \cup (-K - \mathbf{e})}.$$

Clearly, $K^\bullet$ is a centrally symmetric convex body in $\mathbb{R}^{n+1}$ and, therefore, $D(K^\bullet) = 2K^\bullet$.

Lemma 2.2.

$$b_{D(K^\bullet)}(K^\bullet) = b_{D(K^\bullet)}(K + \mathbf{e}) + b_{D(K^\bullet)}(-K - \mathbf{e}) = 2b_{D(K^\bullet)}(K + \mathbf{e}).$$

Proof. First of all, we note that

$$d_{D(K^\bullet)}(K^\bullet) = 1$$

and

$$\|\mathbf{x}, \mathbf{y}\|_{D(K^\bullet)} = 1,$$

whenever $\mathbf{x} \in K + \mathbf{e}$ and $\mathbf{y} \in -K - \mathbf{e}$. Thus, if $X_1, X_2, \ldots, X_{b_{D(K^\bullet)}(K^\bullet)}$ is a partition of $K^\bullet$ such that

$$d_{D(K^\bullet)}(X_i) < 1, \quad i = 1, 2, \ldots, b_{D(K^\bullet)}(K^\bullet),$$

none of the parts can contain two points $\mathbf{x} \in K + \mathbf{e}$ and $\mathbf{y} \in -K - \mathbf{e}$ simultaneously. Therefore, we have

$$b_{D(K^\bullet)}(K^\bullet) \geqslant b_{D(K^\bullet)}(K + \mathbf{e}) + b_{D(K^\bullet)}(-K - \mathbf{e}). \tag{2.5}$$

It is easy to see that

$$D(K^\bullet) \cap H = \{\mathbf{x} - \mathbf{y} : \mathbf{x}, \mathbf{y} \in K\} = D(K).$$

Therefore, we have

$$d_{D(K^\bullet)}(K + \mathbf{e}) = d_{D(K^\bullet)}(-K - \mathbf{e}) = 1. \tag{2.6}$$

By symmetry, we have

$$b_{D(K^\bullet)}(K + \mathbf{e}) = b_{D(K^\bullet)}(-K - \mathbf{e}). \tag{2.7}$$

Assume that $b_{D(K^\bullet)}(K + \mathbf{e}) = m$ and $K_1, K_2, \ldots, K_m$ is a partition of $K + \mathbf{e}$ satisfying

$$d_{D(K^\bullet)}(K_i) < 1, \quad i = 1, 2, \ldots, m.$$

Clearly, $-K_1, -K_2, \ldots, -K_m$ is a partition of $-K - \mathbf{e}$ satisfying

$$d_{D(K^\bullet)}(-K_i) < 1, \quad i = 1, 2, \ldots, m.$$

We define

$$H^+ = \{(x_1, \ldots, x_{n+1}) : x_{n+1} \geqslant 0\},$$

$$H^- = \{(x_1, \ldots, x_{n+1}) : x_{n+1} \leqslant 0\},$$

$$T_i = \overline{(K_i \cup (-K - \mathbf{e}))} \cap H^+$$

for $i = 1, 2, \ldots, m$, and

$$T_{m+i} = \overline{((-K_i) \cup (K + \mathbf{e}))} \cap H^-$$

for $i = 1, 2, \ldots, m$. It is obvious that

$$K^\bullet = \bigcup_{i=1}^{2m} T_i. \tag{2.8}$$

Next, we proceed to verify that

$$d_{D(K^\bullet)}(T_i) < 1$$

holds for all $i = 1, 2, \ldots, m$. For every pair of points $\mathbf{x}, \mathbf{y} \in T_i$, there exist two numbers $\lambda,\ \mu \in [0, \frac{1}{2}]$ and four points $\mathbf{x}_1, \mathbf{y}_1 \in K_i, \mathbf{x}_2, \mathbf{y}_2 \in -K - \mathbf{e}$ such that

$$\mathbf{x} = (1-\lambda)\mathbf{x_1} + \lambda\mathbf{x}_2$$

and

$$\mathbf{y} = (1-\mu)\mathbf{y_1} + \mu\mathbf{y}_2.$$

Hence, we have

$$\|\mathbf{x}_1, \mathbf{y}_1\|_{D(K^\bullet)} < 1,$$

$$\|\mathbf{y}_1, \mathbf{x}_2\|_{D(K^\bullet)} = 1$$

and

$$\|\mathbf{x}_2, \mathbf{y}_2\|_{D(K^\bullet)} \leqslant 1.$$

Without loss of generality, we assume that $\lambda > \mu$. Then, we get

$$\begin{aligned}\|\mathbf{x}, \mathbf{y}\|_{D(K^\bullet)} &= \|\mathbf{x} - \mathbf{y}\|_{D(K^\bullet)} \\ &= \|(1-\lambda)(\mathbf{x}_1 - \mathbf{y}_1) + (\mu - \lambda)(\mathbf{y}_1 - \mathbf{x}_2) + \mu(\mathbf{x}_2 - \mathbf{y}_2)\|_{D(K^\bullet)} \\ &\leqslant (1-\lambda)\|\mathbf{x}_1, \mathbf{y}_1\|_{D(K^\bullet)} + (\lambda - \mu)\|\mathbf{y}_1, \mathbf{x}_2\|_{D(K^\bullet)} + \mu\|\mathbf{x}_2, \mathbf{y}_2\|_{D(K^\bullet)} \\ &< 1 - \lambda + \lambda - \mu + \mu = 1\end{aligned}$$

and therefore

$$d_{D(K^\bullet)}(T_i) < 1, \quad i = 1, 2, \ldots, m. \tag{2.9}$$

Similarly, one can deduce that

$$d_{D(K^\bullet)}(T_i) < 1, \quad i = m+1, m+2, \ldots, 2m. \tag{2.10}$$

As a conclusion of (2.7), (2.8), (2.9), and (2.10), we get

$$b_{D(K^\bullet)}(K^\bullet) \leqslant 2m = b_{D(K^\bullet)}(K + \mathbf{e}) + b_{D(K^\bullet)}(-K - \mathbf{e}). \tag{2.11}$$

By (2.5) and (2.11), Lemma 2.2 is proved. □

Lemma 2.3.

$$b_{D(K^\bullet)}(K + \mathbf{e}) = b_{D(K)}(K).$$

Proof. Recalling the equation just below (2.5),

$$D(K^\bullet) \cap H = D(K),$$

when we measure the diameters of subsets of $K + \mathbf{e}$ by $\|\cdot\|_{D(K^{\bullet})}$, the real norm is $\|\cdot\|_{D(K)}$. Therefore, we get

$$b_{D(K^{\bullet})}(K + \mathbf{e}) = b_{D(K)}(K).$$

Lemma 2.3 is proved.

Proof of Theorem 1.1.. By Lemmas 2.2 and 2.3, we have

$$b_{D(K^{\bullet})}(K^{\bullet}) = 2b_{D(K^{\bullet})}(K + \mathbf{e}) = 2b_{D(K)}(K). \tag{2.12}$$

As $K^{\bullet} \in C^{n+1}$, by Lemma 1.1 and the second part of Lemma 1.2, we get

$$b_{D(K^{\bullet})}(K^{\bullet}) \leqslant h(K^{\bullet}) \leqslant 2^{n+1}(n+1)(\log(n+1) + \log\log(n+1) + 5).$$

Therefore, by (2.12) we have

$$\begin{aligned} b_{D(K)}(K) &= \frac{1}{2} b_{D(K^{\bullet})}(K^{\bullet}) \\ &\leqslant 2^{n}(n+1)(\log(n+1) + \log\log(n+1) + 5) \\ &= 2^{(1+o(1))n} \end{aligned}$$

for all $K \in \mathcal{K}^n$. Then, Theorem 1.1 follows from (2.1) and Lemma 2.1 □

ACKNOWLEDGMENTS

Wang and Zong are supported by the National Natural Science Foundation of China (NSFC12226006, NSFC11921001) and the Natural Key Research and Development Program of China (2018YFA0704701). Xue is supported by the National Natural Science Foundation of China (NSFC12201307) and Jiangsu Natural Science Foundation (BK20210555). The authors are grateful to the referee and Professor Senlin Wu for their helpful suggestions.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. N. Alon and A. Nilli, *On Borsuk's problem*, Contemp. Math. **178** (1994), 209–210.
2. K. Bezdek and M. A. Khan, *The geometry of homothetic covering and illumination*, Discrete geometry and symmetry, Springer Proc. Math. Stat., vol. 234, Springer, Cham, 2018, pp. 1–30.
3. V. G. Boltyanski and I. T. Gohberg, *Results and problems in combinatorial geometry*, Cambridge University Press, Cambridge, 1985; Nauka, Moscow, 1965.
4. V. G. Boltyanski, H. Martini, and V. P. Soltan, *Excursions into combinatorial geometry*, Springer, Berlin, 1997.
5. V. G. Boltyanski and V. P. Soltan, *Borsuk's problem* (in Russian), Mat. Zametki **22** (1977), 621–631.
6. A. Bondarenko, *On Borsuk's conjecture for two-distance sets*, Discrete Comput. Geom. **51** (2014), 509–515.
7. T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, Springer, Berlin, 1934.

8. K. Borsuk, *Über die Zerlegung einer n-dimensionalen Vollkugel in n Mengen*, Verh. Internat. Math.-Kongress Zürich, 1932, Bd. II, ed. W. Saxer, Orell Füssli, Zürich, 1932, p. 192.
9. K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933), 177–190.
10. M. Campos, P. van Hintum, R. Morris, and M. Tiba, *Towards Hadwiger's conjecture via Bourgain Slicing*, arXiv:2206.11227.
11. L. Danzer, *Über Durchschnittseigenschaften n-dimensionaler Kugelfamilien*, J. Reine Angew. Math. **208** (1961), 181–203.
12. H. G. Eggleston, *Covering a three-dimensional set with sets of smaller diameter*, J. Lond. Math. Soc. **30** (1955), 11–24.
13. B. Grünbaum, *Borsuk's partition conjecture in Minkowski planes*, Bull. Res. Council Israel **7** (1957), 25–30.
14. B. Grünbaum, *A simple proof of Borsuk's conjecture in three dimensions*, Proc. Cambridge Philos. Soc. **53** (1957), 776–778.
15. B. Grünbaum, *Borsuk's problem and related questions*, Proc. Symp. Pure Math. **7** (1963), 271–284.
16. H. Hadwiger, *Ungelöste Probleme Nr. 20*, Elem. Math. **12** (1957), 121.
17. A. Heppes, *On the partitioning of three-dimensional point-sets into sets of smaller diameter (Hungarian)*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **7** (1957), 413–416.
18. A. Hinrichs and C. Richter, *New sets with large Borsuk numbers*, Discrete Math. **270** (2003), 137–147.
19. H. Huang, B. A. Slomka, T. Tkocz, and B. Vritsiou, *Improved bounds for Hadwigers covering problem via thin-shell estimates*, J. Eur. Math. Soc. **24** (2021), 1431–1448.
20. M. Hujter and Z. Lángi, *On the multiple Borsuk numbers of sets*, Israel J. Math. **199** (2014), 219–239.
21. T. Jenrich and A. E. Brouwer, *A 64-dimensional counterexample to Borsuk's conjecture*, Electron. J. Combin. **21** (2014), 4.29.
22. J. Kahn and G. Kalai, *A counterexample to Borsuk's conjecture*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), 60–62.
23. Z. Lángi and M. Naszódi, *On multiple Borsuk numbers in normed spaces*, Studia Sci. Math. Hungar. **54** (2017), 13–26.
24. M. Lassak, *An estimate concerning Borsuk partition problem*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (1982), 449–451.
25. H. Lenz, *Zur Zerlegung von Punktmengen in solche kleineren Durchmessers*, Arch. Math. **6** (1955), 413–416.
26. F. W. Levi, *Ein geometrisches Überdeckungsproblem*, Arch. Math. **5** (1954), 476–478.
27. Y. Lian and S. Wu, *Partition bounded sets into sets having smaller diameters*, Results Math. **76** (2021), 116.
28. J. Lindenstrauss and V. D. Milman, *The local theory of normed spaces and its applications to convexity*, Handbook of convex geometry (P. M. Gruber and J. M. Wills, eds.), North-Holland, Amsterdam, 1993, pp. 1149–1220.
29. H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces: a survey. II*, Expo. Math. **22** (2004), 93–144.
30. J. Perkal, *Sur la subdivision des ensembles en parties de diamètre inférieur*, Colloq. Math. **1** (1947), 45.
31. C. A. Rogers and C. Zong, *Covering convex bodies by translates of convex bodies*, Mathematika **44** (1997), 215–218.
32. O. Schramm, *Illuminating sets of constant width*, Mathematika **35** (1988), 180–189.
33. J. Wang and F. Xue, *Borsuk's partition problem in four-dimensional ℓp space*, Math. Inequal. Appl. **25** (2023), 131–139.
34. L. Yu and C. Zong, *On the blocking number and the covering number of a convex body*, Adv. Geom. **9** (2009), 13–29.
35. C. Zong, *Strange phenomena in convex and discrete geometry*, Springer, New York, 1996.
36. C. Zong, *The kissing number, blocking number and covering number of a convex body*, Contemp. Math. **453** (2008), 529–548.
37. C. Zong, *A quantitative program for Hadwiger's covering conjecture*, Sci. China Math. **53** (2010), 2551–2560.
38. C. Zong, *Borsuk's partition conjecture*, Jpn. J. Math. **16** (2021), 185–201.