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On zero-sum subsequences of cross number 1

by

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Abstract. For an additive finite abelian group G, let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G and $\mathsf{k}(S) = \mathrm{ord}(g_1)^{-1} + \cdots + \mathrm{ord}(g_l)^{-1}$ be its cross number. Let $\mathsf{T}(G)$ be the smallest integer t such that every sequence of t elements (repetition allowed) from G has a zero-sum subsequence T with $\mathsf{k}(T) = 1$. We study the relation of the new invariant $\mathsf{T}(G)$ to several classical invariants such as the Erdős–Ginzburg–Ziv constant $\mathsf{s}(G)$, $\eta(G)$, and $\mathsf{t}(G)$, a recently defined invariant (see the introduction). We also determine $\mathsf{T}(G)$ for some special abelian groups, including some cyclic groups.

1. Introduction. Let G be a finite abelian group, written additively. We denote by C_n a cyclic group of order n. For a general finite abelian group G, we can decompose G as a direct sum of cyclic groups $C_{n_1} \oplus \cdots \oplus C_{n_r}$ such that $1 < n_1 \mid \cdots \mid n_r \in \mathbb{N}$ (if $n_1 = \cdots = n_r = n$, it will be abbreviated as C_n^r), where r and n_r are respectively called the *rank* and *exponent* of G. Usually, the exponent of G is simply denoted by $\exp(G)$. The order of an element g of G will be written as $\mathrm{ord}(g)$.

Given a sequence $S = g_1 \cdot \ldots \cdot g_l$ over G, we denote by $S_{(d)}$ the subsequence of S consisting of all terms of S of order d, and S_H the subsequence of S consisting of all terms of S belonging to a subgroup H of G. We denote by $\mathsf{k}(S)$ the *cross number* of S, defined as follows:

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)}.$$

The cross number is an important concept in factorization theory; for more information we refer to [4, 8, 9, 11, 12, 13, 14, 15]. Recent progress on cross numbers and the EGZ-constant was achieved in [1, 16].

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In the development of zero-sum theory, many invariants have been defined in terms of the existence of zero-sum subsequences of restricted lengths. For example, let $\mathsf{s}(G)$ (resp. $\eta(G)$) be the smallest integer t such that every sequence of t elements (repetition allowed) from G has a zero-sum subsequence W of length $|W| = \exp(G)$ (resp. $1 \leq |W| \leq \exp(G)$). The invariant $\mathsf{s}(G)$ is called the *Erdős–Ginzburg–Ziv constant* and $\eta(G)$ is also a classical invariant in zero-sum theory.

Let $\mathsf{T}(G)$ (resp. $\mathsf{t}(G)$) be the smallest integer t such that every sequence of t elements (repetition allowed) from G has a zero-sum subsequence W with $\mathsf{k}(W) = 1$ (resp. $\mathsf{k}(W) \leq 1$). The invariant $\mathsf{t}(G)$ was defined in 1989 [14]. The inequality $\mathsf{t}(G) \leq |G|$ was first proved in [8] and new proofs were given in [2, 4]. It has been proved that $\mathsf{t}(G) = \eta(G)$ when G is cyclic or a certain special abelian group of rank 2 (see [7, 14, 18]).

As a zero-free sequence over a finite abelian group cannot be arbitrarily long, its cross number cannot be arbitrarily large. A related invariant, $\kappa(G)$, was introduced in [10, Definition 5.7.12] and is the smallest positive integer l such that every sequence $S \in \mathcal{F}(G)$ satisfying $\exp(G)\mathsf{k}(S) \geq l$ has a non-empty zero-sum subsequence T with $\mathsf{k}(T) \leq 1$. Then $\kappa(G)/\exp(G)$ (resp. $\mathsf{t}(G)$) is the smallest cross number (resp. length) of a sequence $S \in \mathcal{F}(G)$ which must have a non-empty zero-sum subsequence T with $\mathsf{k}(T) \leq 1$. Further, $\mathsf{T}(G)$ is the smallest length of a sequence $S \in \mathcal{F}(G)$ which must have a zero-sum subsequence T with $\mathsf{k}(T) = 1$. However, we do not discuss $\mathsf{t}(G)$ or $\kappa(G)$ in this paper.

The motivation for introducing $\mathsf{T}(G)$ comes from the study of $\mathsf{s}(G)$ and $\mathsf{t}(G)$. Just like $\mathsf{t}(G) = \eta(G)$ holds for all finite elementary abelian p-groups, from the definitions of $\mathsf{T}(G)$ and $\mathsf{s}(G)$ we can prove that $\mathsf{T}(G) = \mathsf{s}(G)$ holds for all finite elementary abelian p-groups, where p is an odd prime. We state it as follows.

THEOREM 1.1. *For a prime p and an integer $r \geq 1$, we have*

$$\mathsf{T}(C_p^r) = \begin{cases} 2^r & \text{if } p = 2, \\ \mathsf{s}(C_p^r) & \text{if } p \geq 3. \end{cases}$$

We can also prove $\mathsf{T}(G) = \mathsf{s}(G)$ for the following groups G, for which $\mathsf{s}(G)$ has been determined.

THEOREM 1.2. *Let p, q be odd primes (not necessarily distinct) and r be a positive integer. Then*

$$\mathsf{T}(C_{pq}^r) = \mathsf{s}(C_{pq}^r)$$

in each of the following cases:

(1) $r = 1$.
(2) $r = 2$.
(3) $r = 3$ *and* $p, q \in \{3, 5\}$.

For a cyclic p-groups G, we completely determine $\mathsf{T}(G)$.

THEOREM 1.3. *For a prime p and an integer $a \geq 1$, we have*

$$\mathsf{T}(C_{p^a}) = \begin{cases} 2^a + 2^{a-1} - 1 < \mathsf{s}(C_{2^a}) & \text{if } p = 2, \\ 2p^a - 1 = \mathsf{s}(C_{p^a}) & \text{if } p \geq 3. \end{cases}$$

We can also determine $\mathsf{T}(G)$ for some other cyclic groups G of odd order, which all satisfy $\mathsf{T}(G) = \mathsf{s}(G)$.

THEOREM 1.4. *Let $G = C_n$ be the cyclic group of order n. If $4\sum_{p|n}\frac{1}{p} \leq 1$, and $p(n) > 2\omega(n)$, where p runs over all distinct prime divisors of n, $p(n)$ denotes the smallest prime divisor of n and $\omega(n)$ denotes the number of distinct prime divisors of n, then*

$$\mathsf{T}(G) = 2n - 1 = \mathsf{s}(G).$$

A classical invariant $\mathsf{K}(G)$ relating to cross numbers was formulated by Krause [15] in 1984, defined to be the maximal cross number of minimal zero-sum sequences over G. We remark that problems involving cross numbers are usually not easy even for cyclic groups. For example, the determination of $\mathsf{K}(G)$ for finite cyclic groups is far from complete. The equality $\mathsf{t}(C_n) = n$ was conjectured by Erdős and confirmed by Kleitman and Lemke [14, 12]. Determining $\mathsf{T}(G)$ for all finite cyclic groups seems to be a challenging problem.

The paper is organized as follows. Section 2 provides some notation and concepts which will be used in what follows. In Section 3, we prove some preliminary lemmas which are needed in the proofs of Theorems 1.2–1.4. In Section 4, we provide the proofs of Theorems 1.1 and 1.2. And in Section 5, we prove Theorems 1.3 and 1.4. The final section is devoted to some concluding remarks.

2. Notation and preliminaries. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 \mid a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively.

Let G be an additive finite abelian group with rank r. An r-tuple $(e_1, \ldots, e_r)$ in $G \setminus \{0\}$ is called a *basis* of G if $G = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$. We denote by $\mathcal{F}(G)$ the free (abelian, multiplicative) monoid with basis G. The elements of $\mathcal{F}(G)$ are called *sequences* over G. We write a sequence $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)} \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, and we say that S *contains* g if $\mathsf{v}_g(S) > 0$. A sequence S' is called a *subsequence* of S, denoted by $S' \mid S$, if

$\mathsf{v}_g(S') \le \mathsf{v}_g(S)$ for all $g \in G$, and SS'^{-1} denotes the subsequence obtained from S by deleting S'. Two subsequences S_1 and S_2 of S are called *disjoint* if $S_1 \mid SS_2^{-1}$. The unit element $\emptyset \in \mathcal{F}(G)$ is called the *empty* sequence.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S,
- $\operatorname{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G$ the *support* of S,
- S a *zero-sum sequence* if $\sigma(S) = 0 \in G$,
- S a *zero-sum free sequence* if there is no non-empty zero-sum subsequence of S,
- S a *short zero-sum sequence* if S is zero-sum and $1 \le |S| \le \exp(G)$,
- S a *tiny zero-sum sequence* if S is a non-empty zero-sum sequence and $\mathsf{k}(S) \le 1$,
- $b + S = (b + g_1) \cdot \ldots \cdot (b + g_l)$, where $b \in G$.

Every map of abelian groups $\varphi : G \to H$ extends to a homomorphism $\varphi : \mathcal{F}(G) \to \mathcal{F}(H)$, where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. If φ is a homomorphism, then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

3. Auxiliary lemmas. In this section, we present several auxiliary lemmas that are needed in the proofs of Theorems 1.2–1.4.

Lemma 3.1.

(1) *Let n be an odd integer and g be a generator of C_n. Let*

$$S_1 = g^{n-1} \cdot (-g)^{n-1} \in \mathcal{F}(C_n).$$

Then S_1 has no zero-sum subsequence of length n. That is, S_1 has no non-empty subsequence T with $\sigma(T) = 0$ and $\mathsf{k}(T) \in \mathbb{Z}$.

(2) *Let n be an even integer and g be a generator of C_n. Let*

$$S_2 = g^{n-1} \cdot (-g)^{n/2-1} \in \mathcal{F}(C_n).$$

Then S_2 has no zero-sum subsequence of length n. That is, S_2 has no non-empty subsequence T with $\sigma(T) = 0$ and $\mathsf{k}(T) \in \mathbb{Z}$.

Proof. (1) Any subsequence $R \mid S_1 = g^{n-1} \cdot (-g)^{n-1}$ with $|R| = n$ has the form $R = g^{n-b} \cdot (-g)^b$ with $1 \le b \le n-1$. It follows that

$$\sigma(R) = (n-b) \cdot g + b \cdot (-g) = b \cdot (-2g) \ne 0$$

since $\operatorname{ord}(-2g) = n$ and $b \in [1, n-1]$.

That is, if $\emptyset \neq T \mid S_1$ and $\sigma(T) = 0$, then $|T| \in [1, 2n-2] \setminus \{n\}$. So $\mathsf{k}(T) = |T|/n \notin \mathbb{Z}$.

(2) If $n = 2$, the result is obvious. Therefore, below we assume $n \geq 4$.

Any subsequence $R \mid S_2 = g^{n-1} \cdot (-g)^{n/2-1}$ with $|R| = n$ has the form $R = g^{n-b} \cdot (-g)^b$ with $1 \leq b \leq n/2 - 1$. It follows that

$$\sigma(R) = (n-b) \cdot g + b \cdot (-g) = 2b \cdot (-g) \neq 0$$

since $\mathrm{ord}(-g) = n$ and $2b \in [2, n-2]$.

That is, if $\emptyset \neq T \mid S_2$ and $\sigma(T) = 0$, then $|T| \in [1, n + n/2 - 1] \setminus \{n\}$. So $\mathsf{k}(T) = |T|/n \notin \mathbb{Z}$. ■

The following problem appears in the study of $\mathsf{T}(G)$ by induction and is of independent interest.

CONJECTURE 3.2. Let p be a prime and $S = \prod_{i=1}^{2p+1} a_i \in \mathcal{F}(\mathbb{Z})$ with $p \nmid a_i$ for all $i \in [1, 2p+1]$. Then there is a subsequence T of S such that $|T| = p$, $p \mid \sigma(T)$ but $p^2 \nmid \sigma(T)$.

The following example shows that the bound $2p+1$ in the above conjecture could not be improved for an odd prime p. Let $p > 2$ be a prime and

$$W = 1^{p-1}(1-p)(-1)^{p-1}(-1+p).$$

Note that $W \pmod{p} = 1^p(-1)^p$ and it has exactly two subsequences 1^p and $(-1)^p$ with sum zero mod p and length p. Therefore, W has exactly two subsequences $1^{p-1}(1-p)$ and $(-1)^{p-1}(-1+p)$ with sum zero mod p and length p. But each of these two subsequences has sum zero mod p^2.

To make some progress on Conjecture 3.2 we need the following well known result.

LEMMA 3.3 (Cauchy–Davenport). *Let p be a prime, and let $A_1, \ldots, A_k$ be non-empty subsets of C_p. Then*

$$|A_1 + \cdots + A_k| \geq \min\{p, |A_1| + \cdots + |A_k| - k + 1\}.$$

LEMMA 3.4. *Let p be a prime and $S = \prod_{i=1}^{2p+1} a_i \in \mathcal{F}(\mathbb{Z})$ with $p \nmid a_i$ for all $i \in [1, 2p+1]$. If there are two terms a_i, a_j of S such that $a_i \equiv a_j \pmod{p}$ and $a_i \not\equiv a_j \pmod{p^2}$, then there is a subsequence T of S such that $|T| = p$, $p \mid \sigma(T)$ but $p^2 \nmid \sigma(T)$.*

Proof. By renumbering, let

$$a_{2p} \equiv a_{2p+1} \pmod{p} \quad \text{and} \quad a_{2p} \not\equiv a_{2p+1} \pmod{p^2}.$$

If there is one $m \in [1, p-1]$ such that $a_i \equiv m \pmod{p}$ for at least $p+1$ numbers $i \in [1, 2p-1]$, then we may assume, without loss of generality, that

$$a_1 \equiv \cdots \equiv a_{p+1} \equiv m \pmod{p}.$$

Assume to the contrary that the conclusion of this lemma is false for S. Then every subsequence T of $\prod_{i=1}^{p+1} a_i$ with length $|T| = p$ has sum $\sigma(T) \equiv 0 \pmod{p^2}$. This implies that

$$\sum_{i=1}^{p+1} a_i \equiv a_1 \equiv \cdots \equiv a_{p+1} \pmod{p^2}.$$

Hence, $a_1 + \cdots + a_p \equiv pa_1 \equiv pm \not\equiv 0 \pmod{p^2}$, a contradiction.

Next, we suppose that, for every $m \in [1, p-1]$, there are at most p numbers $i \in [1, 2p-1]$ such that $a_i \equiv m \pmod{p}$, and furthermore there are at most $p-1$ numbers $i \in [1, 2p-2]$ such that $a_i \equiv m \pmod{p}$. By renumbering, we may assume that $a_i \not\equiv a_{i+p-1} \pmod{p}$ for each $i \in [1, p-1]$. For any integer n, let $\overline{n} \in \mathbb{Z}/p\mathbb{Z} = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$ be the residue class of n modulo p and $A_i = \{\overline{a_i}, \overline{a_{i+p-1}}\}$ for each $i \in [1, p-1]$.

Since $\mathbb{Z}/p\mathbb{Z} \cong C_p$, by Lemma 3.3 we obtain

$$|A_1 + \cdots + A_{p-1}| \geq \min\{p, 2(p-1) - (p-1) + 1\} = p.$$

This forces $A_1 + \cdots A_{p-1} = \mathbb{Z}/p\mathbb{Z}$. In particular, $\overline{-a_{2p}} \in A_1 + \cdots + A_{p-1}$. Therefore, there is a subsequence W of $\prod_{i=1}^{2p-2} a_i$ with $|W| = p-1$ such that

$$a_{2p} + \sigma(W) \equiv 0 \pmod{p}.$$

Since $a_{2p} \equiv a_{2p+1} \pmod{p}$, we also have

$$a_{2p+1} + \sigma(W) \equiv 0 \pmod{p}.$$

Noting that $a_{2p} \not\equiv a_{2p+1} \pmod{p^2}$, we find that either $a_{2p} + \sigma(W) \not\equiv 0 \pmod{p^2}$ or $a_{2p+1} + \sigma(W) \not\equiv 0 \pmod{p^2}$, completing the proof. ∎

Lemma 3.5. *Conjecture* 3.2 *is true for* $p = 2$ *and for* $p = 3$.

Proof. Let $S = \prod_{i=1}^{2p+1} a_i \in \mathcal{F}(\mathbb{Z})$ with $p \nmid a_i$ for all $i \in [1, 2p+1]$. We need to prove that there is a subsequence T of S such that $|T| = p$, $p \mid \sigma(T)$ but $p^2 \nmid \sigma(T)$.

By Lemma 3.4, we may assume that if $a_i \equiv a_j \pmod{p}$ for some $i, j \in [1, 2p+1]$, then $a_i \equiv a_j \pmod{p^2}$. Let $A_m = \{i : a_i \equiv m \pmod{p}\}$ for every $m \in [1, p-1]$. Noting that $p = 2$ or 3, we find that

$$\max_{1 \leq m \leq p-1} |A_m| \geq \frac{2p+1}{p-1} > p.$$

So there is some $m_0 \in [1, p-1]$ such that $|A_{m_0}| \geq p+1$. Let T be a subsequence of S with $|T| = p$ and with every term of T equal to $m_0 \pmod{p}$. Then, by assumption,

$$\sigma(T) \equiv pm_0 \not\equiv 0 \pmod{p^2},$$

completing the proof. ∎

REMARK 3.6. (1) Conjecture 3.2 has been confirmed quite recently for $p = 5$ and for $p = 7$ by Yiming Wang, one of the postgraduate students of the first author, via a computer program.

(2) Let p be a prime, $a \geq 2$ be an integer and $S = \prod_{i=1}^{2p+1} a_i \in \mathcal{F}(C_{p^a})$ with $a_i \in C_{p^a} \setminus pC_{p^a}$ for every $i \in [1, 2p+1]$. If Conjecture 3.2 is true for p, then there is a subsequence T of S such that $|T| = p$, $\sigma(T) \in pC_{p^a} \setminus p^2C_{p^a}$. That is,

$$\mathsf{k}(T) = \frac{|T|}{p^a} = \frac{1}{p^{a-1}} = \frac{1}{\operatorname{ord}(\sigma(T))} = \mathsf{k}(\sigma(T)).$$

(3) For $p = 2$, the bound $2p + 1 = 5$ in the conjecture can actually be improved to 3. The proof is very easy and we omit it.

LEMMA 3.7 ([6, Lemma 2.2]). *Let n, k, t be three positive integers with $2 \leq t < n/2+1$, and let S be a sequence over C_n of length $|S| = (k+1)n-t$. Suppose that S contains no zero-sum subsequence of length kn. Then there exist two distinct elements $x, y \in C_n$ such that*

$$\mathsf{v}_x(S) + \mathsf{v}_y(S) \geq (k+1)n - 2t + 2.$$

4. On the groups C_p^r and C_{pq}^r. In this section, we give the proofs of Theorems 1.1 and 1.2.

We begin with the elementary abelian p-groups C_p^r.

LEMMA 4.1. *Let G be a finite abelian group of odd order $|G| > 1$. Then there is always a sequence S of length $|S| = \mathsf{s}(G) - 1$ such that $0 \nmid S$ and S has no zero-sum subsequence of length $\exp(G)$.*

Proof. Let $m = \exp(G)$. Then $m > 1$ is odd. By the definition of $\mathsf{s}(G)$, let T be a sequence over G of length $|T| = \mathsf{s}(G) - 1$ and with no zero-sum subsequence of length m. If $0 \nmid T$, then we are done, so assume that $0 \mid T$. Let $g \in G$ with $\operatorname{ord}(g) = m$.

Since m is odd, it follows that $0 \cdot g \cdot (2g) \cdot \ldots \cdot (m-1)g$ is a zero-sum sequence of length m. Therefore, $ig \nmid T$ for some $i \in [1, m-1]$. Let $S = (-ig) + T$. Then $0 \nmid S$, $|S| = |T| = \mathsf{s}(G) - 1$ and S has no zero-sum subsequence of length m, as desired. ∎

Proof of Theorem 1.1. Notice that every element in $C_p^r \setminus \{0\}$ has the same order p. So for any sequence S over $C_p^r \setminus \{0\}$, $1 = \mathsf{k}(S) = |S|/p$ if and only if $|S| = p$.

CASE 1: $p = 2$. Let $S^* = \prod_{g \in C_2^r \setminus \{0\}} g$. Then $|S^*| = 2^r - 1$ and S^* has no zero-sum subsequence of length 2. This implies that $\mathsf{T}(C_2^r) \geq |S^*| + 1 = 2^r$. Next we need to prove that $\mathsf{T}(C_2^r) \leq 2^r$.

Let $S \in \mathcal{F}(C_2^r)$ be a sequence of length $|S| = 2^r$. If $0 \mid S$, then S has a subsequence $T = 0$ such that $\sigma(T) = 0$ and $\mathsf{k}(T) = \frac{1}{\operatorname{ord}(0)} = 1$ as desired. If

$0 \nmid S$, then S is over $C_2^r \setminus \{0\}$. Since $|S| > 2^r - 1$, there exists an element $x \mid S$ such that $\mathsf{v}_x(S) \geq 2$. Hence $T = x^2$ is a zero-sum subsequence of S with $\mathsf{k}(T) = 1$. Thus $\mathsf{T}(C_2^r) \leq 2^r$.

CASE 2: $p \geq 3$. By Lemma 4.1, let S^{**} be a sequence over $C_p^r \setminus \{0\}$ of length $|S^{**}| = \mathsf{s}(C_p^r) - 1$ and with no zero-sum subsequence of length p. This implies that $\mathsf{T}(C_p^r) \geq |S^{**}| + 1 = \mathsf{s}(C_p^r)$. Next we need to prove that $\mathsf{T}(C_p^r) \leq \mathsf{s}(C_p^r)$.

Let $S \in \mathcal{F}(C_p^r)$ be a sequence of length $|S| = \mathsf{s}(C_p^r)$. If $0 \mid S$, then S has a subsequence $T = 0$ satisfying $\sigma(T) = 0$ and $\mathsf{k}(T) = \frac{1}{\mathrm{ord}(0)} = 1$ as desired. If $0 \nmid S$, then S is over $C_p^r \setminus \{0\}$. By the definition of $\mathsf{s}(C_p^r)$, S has a zero-sum subsequence T of length $|T| = p$. So T is a zero-sum subsequence of S with $\mathsf{k}(T) = 1$. Thus $\mathsf{T}(C_p^r) \leq \mathsf{s}(C_p^r)$.

This completes the proof of Theorem 1.1. ∎

Next, we turn to the groups C_{pq}^r and we first prove three lemmas.

LEMMA 4.2. *Let p, q be primes (not necessarily distinct) and r be a positive integer. If there exists an integer constant $c = c(r, p, q)$ such that $s(C_n^r) = c(n-1) + 1$ for each $n \in \{p, q, pq\}$, then*

$$\mathsf{T}(C_{pq}^r) \leq \mathsf{s}(C_{pq}^r).$$

Proof. Here we just prove the case $p \neq q$. The much easier case $p = q$ can be proved in the same way.

Let $G = C_{pq}^r \cong C_p^r \oplus C_q^r$ and $H_p \cong C_p^r$, $H_q \cong C_q^r$ be the subgroups of G. For any $S \in \mathcal{F}(G)$ with $|S| = \mathsf{s}(C_{pq}^r) = c(pq-1) + 1$ and $0 \nmid S$, we divide S into three disjoint subsequences,

$$S = S_{(p)} S_{(q)} S_{(pq)},$$

where $\mathrm{ord}(g) = p$ for $g \mid S_{(p)}$, $\mathrm{ord}(g) = q$ for $g \mid S_{(q)}$ and $\mathrm{ord}(g) = pq$ for $g \mid S_{(pq)}$. Let

$$m_p = |S_{(p)}|, \quad m_q = |S_{(q)}|, \quad m = m_p + m_q.$$

Notice that $\mathrm{ord}(g) = p$ if and only if $g \in H_p$, and $\mathrm{ord}(g) = q$ if and only if $g \in H_q$. By Theorem 1.1, if $m_p \geq \mathsf{T}(C_p^r) = \mathsf{s}(C_p^r) = c(p-1) + 1$ or $m_q \geq \mathsf{T}(C_q^r) = \mathsf{s}(C_q^r) = c(q-1) + 1$, then we are done. Next, we assume that $m_p \leq c(p-1)$ and $m_q \leq c(q-1)$. Thus

$$|S_{(pq)}| = |S| - m_p - m_q \geq \max\{\mathsf{s}(C_p^r), \mathsf{s}(C_q^r)\}.$$

Let $\varphi_p : G \to G/H_p$ be the canonical homomorphism. Then $\varphi_p(S_{(pq)}) \in \mathcal{F}(G/H_p) \cong \mathcal{F}(C_q^r)$. Therefore, we can choose $T_1, \ldots, T_{w_p} \mid S_{(pq)}$ such that

$$\sigma(\varphi_p(T_i)) = 0 + H_p \quad \text{and} \quad |\varphi_p(T_i)| = \exp(G/H_p) = q \quad \text{for } i = 1, \ldots, w_p,$$

and

$$\begin{aligned} w_p &\geq 1 + \left\lfloor \frac{|\varphi(S_{(pq)})| - \mathsf{s}(G/H_p)}{\exp(G/H_p)} \right\rfloor \\ &= 1 + \left\lfloor \frac{[c(pq-1)+1-m] - [c(q-1)+1]}{q} \right\rfloor \\ &= 1 + c(p-1) + \lfloor -m/q \rfloor. \end{aligned}$$

Let us consider the new sequence

$$W = \sigma(T_1) \cdot \ldots \cdot \sigma(T_{w_p}) \cdot S_{(p)}.$$

For $i = 1, \ldots, w_p$, $\sigma(T_i) \in H_p$ since $\sigma(\varphi_p(T_i)) = 0 + H_p$. So $W \in \mathcal{F}(H_p)$. If $|W| = w_p + m_p \geq \mathsf{s}(H_p)$, then by the definition of $\mathsf{s}(H_p)$, W has a subsequence

$$X = \sigma(T_{u_1}) \cdot \ldots \cdot \sigma(T_{u_v}) \cdot R,$$

where $\{u_1, \ldots, u_v\} \subseteq [1, w]$ and $R \mid S_{(p)}$, with the property that

$$\sigma(X) = 0 \text{ in } H_p \quad \text{and} \quad |X| = \exp(H_p) = p.$$

Finally, let

$$T = T_{u_1} \cdot \ldots \cdot T_{u_v} \cdot R.$$

Then $T \mid S$, $\sigma(T) = \sigma(X) = 0$ in G and

$$\mathsf{k}(T) = \sum_{i=1}^{v} \sum_{g \mid T_{u_i}} \frac{1}{\operatorname{ord}(g)} + \mathsf{k}(R) = v \cdot q \cdot \frac{1}{pq} + |R| \cdot \frac{1}{p} = |X| \cdot \frac{1}{p} = 1$$

as desired.

Similarly, replacing p by q above, we get w_q. If $w_q + m_q \geq \mathsf{s}(H_q)$, then we are done too.

We claim that $w_p + m_p \geq \mathsf{s}(H_p)$ or $w_q + m_q \geq \mathsf{s}(H_q)$. Otherwise, we have

$$\begin{aligned} &m = m_p + m_q, \\ &w_p \geq 1 + c(p-1) + \lfloor -m/q \rfloor, \\ &w_p + m_p \leq \mathsf{s}(H_p) - 1 = c(p-1), \\ &w_q \geq 1 + c(q-1) + \lfloor -m/p \rfloor, \\ &w_q + m_q \leq \mathsf{s}(H_q) - 1 = c(q-1). \end{aligned}$$

Then we would get

$$0 \geq 2 + m + \lfloor -m/q \rfloor + \lfloor -m/p \rfloor > m - m/q - m/p,$$

a contradiction with the fact that $m \geq 0$ and p, q are primes.

So Lemma 4.2 is proved. ■

LEMMA 4.3. *Let $K = C_{n_1} \oplus C_{n_2}$, where n_2 is odd and $1 < n_1 \mid n_2$, and (e_1, e_2) be a basis of K with* $\operatorname{ord}(e_1) = n_1$ *and* $\operatorname{ord}(e_2) = n_2$.

(1) (see [10]) *The sequence*

$$S = e_1^{n_1-1}(e_1+e_2)^{n_1-1}0^{n_2-1}e_2^{n_2-1}$$

does not contain any zero-sum subsequence of length n_2.

(2) (see [10]) $\mathsf{s}(C_{n_1} \oplus C_{n_2}) = 2n_1 + 2n_2 - 3$.

(3) $\mathsf{T}(C_{n_1} \oplus C_{n_2}) \geq 2n_1 + 2n_2 - 3$ *for odd* $n_1 n_2$.

Proof. (3) It is easy to check that the sequence

$$e_2 + S = (e_1+e_2)^{n_1-1}(e_1+2e_2)^{n_1-1}e_2^{n_2-1}(2e_2)^{n_2-1}$$

does not contain any zero-sum subsequence of length n_2 and every element in it has order n_2. Therefore, e_2+S does not contain any zero-sum subsequence of cross number 1 and so

$$\mathsf{T}(C_{n_1} \oplus C_{n_2}) \geq |S| + 1 = 2n_1 + 2n_2 - 3. \ ■$$

Lemma 4.4. *Let* $K = C_n^3$ *with odd* $n \geq 3$ *and* (e_1, e_2, e_3) *be a basis of* K. *For convenience, we write* (m_1, m_2, m_3) *for the element* $m_1e_1+m_2e_2+m_3e_3$.

(1) (*See* [3]) *Let*

$$T = (0,0,0)(0,0,1)(0,1,0)(0,1,1)(1,0,0)(1,0,1)(1,0,2)(1,1,2)(2,1,2)$$

and $S = T^{n-1}$. *Then* S *does not contain any zero-sum subsequence of length* n.

(2) (*See* [5]) $\mathsf{s}(C_n^3) = 9n - 8$, *where every prime divisor of* n *belongs to* $\{3, 5\}$.

(3) $\mathsf{T}(C_n^3) \geq 9n - 8$ *for odd* n.

Proof. (3) It is easy to check that the sequence $(1,0,0) + S$ does not contain any zero-sum subsequence of length n and every element in it has order n. Therefore, $(1,0,0) + S$ does not contain any zero-sum subsequence of cross number 1 and thus

$$\mathsf{T}(C_n^3) \geq |S| + 1 = 9n - 8. \ ■$$

We are now ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemmas 4.3(2) and 4.4(2), we know that, in each case of Theorem 1.2, there exists an integer constant $c = c(r,p,q)$ such that $s(C_n^r) = c(n-1)+1$ for each $n \in \{p, q, pq\}$. Thus by Lemma 4.2, we have $\mathsf{T}(C_{pq}^r) \leq \mathsf{s}(C_{pq}^r)$.

Therefore, we only need to prove $\mathsf{T}(C_{pq}^r) \geq \mathsf{s}(C_{pq}^r)$.

(1) The result follows immediately from Lemmas 3.1(1) and 4.3(2).

(2) The result follows from Lemma 4.3(2)(3).

(3) The result follows from Lemma 4.4(2)(3). ■

5. On the cyclic groups. In this section, we give the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3. We prove the theorem by induction on a. If $a = 1$, by Theorem 1.1 we have

$$\mathsf{T}(C_p) = \begin{cases} 2 & \text{if } p = 2, \\ 2p - 1 & \text{if } p \geq 3, \end{cases}$$

and the result holds.

If $a \geq 2$, we assume that the result holds for $a - 1$, i.e.,

$$\mathsf{T}(C_{p^{a-1}}) = \begin{cases} 2^{a-1} + 2^{a-2} - 1 & \text{if } p = 2, \\ 2p^{a-1} - 1 & \text{if } p \geq 3. \end{cases}$$

By Lemma 3.1, we have

$$\mathsf{T}(C_{p^a}) \geq \begin{cases} 2^a + 2^{a-1} - 1 & \text{if } p = 2, \\ 2p^a - 1 & \text{if } p \geq 3. \end{cases}$$

Therefore, we just need to prove that, for any $S \in \mathcal{F}(C_{p^a})$ with

$$|S| = \begin{cases} 2^a + 2^{a-1} - 1 & \text{if } p = 2, \\ 2p^a - 1 & \text{if } p \geq 3, \end{cases}$$

there always exists a non-empty subsequence $T \mid S$ with $\sigma(T)=0$ and $\mathsf{k}(T)=1$.

First we divide S into two disjoint subsequences,

$$S = S_{(p^a)} S_{(<p^a)},$$

where $\operatorname{ord}(g) = p^a$ for $g \mid S_{(p^a)}$ and $\operatorname{ord}(g) < p^a$ for $g \mid S_{(<p^a)}$. Let

$$m = |S_{(<p^a)}|.$$

Notice that $\operatorname{ord}(g) < p^a$ if and only if $g \in pC_{p^a} \cong C_{p^{a-1}}$. If $m \geq \mathsf{T}(C_{p^{a-1}})$, then by the induction hypothesis $S_{(<p^a)}$ has a non-empty zero-sum subsequence T with $\mathsf{k}(T) = 1$. Thus so does S and we are done.

If $m < \mathsf{T}(C_{p^{a-1}})$, we claim that either we can choose $T_1, \ldots, T_w \mid S_{(p^a)}$ such that

$$(5.1) \quad w + m \geq \mathsf{T}(C_{p^{a-1}}),\ |T_i| = p \text{ and } \mathsf{k}(T_i) = \mathsf{k}(\sigma(T_i)) \text{ for } i = 1, \ldots, w,$$

or $S_{(p^a)}$ contains a zero-sum subsequence T of length $|T| = p^a$.

We divide the proof of the claim into the following three cases with different prime p.

Case 1: $p = 2$. Here $|S| = 2^a + 2^{a-1} - 1$ and $m \leq 2^{a-1} + 2^{a-2} - 2$.

Then $|S_{(2^a)}| = |S| - |S_{(<2^a)}| = 2^a + 2^{a-1} - 1 - m \geq 3 \times 2^{a-2} + 1 > 3$. By Remark 3.6(3), we can choose $T_1, \ldots, T_w \mid S_{(2^a)}$ with

$$w = 1 + \left\lfloor \frac{|S_{(2^a)}| - 3}{2} \right\rfloor = 2^{a-1} + 2^{a-2} - 1 + \lfloor -m/2 \rfloor \geq 1$$

such that $|T_i| = 2$ and $\mathsf{k}(T_i) = \mathsf{k}(\sigma(T_i))$ for every $i = 1, \ldots, w$. Moreover,

$$\begin{aligned} w + m &= 2^{a-1} + 2^{a-2} - 1 + \lfloor -m/2 \rfloor + m \\ &= 2^{a-1} + 2^{a-2} - 1 + \lfloor m/2 \rfloor \\ &\geq 2^{a-1} + 2^{a-2} - 1 = \mathsf{T}(C_{2^{a-1}}). \end{aligned}$$

So our claim is proved in this case.

Case 2: $p = 3$. Here $|S| = 2 \times 3^a - 1$ and $m \leq 2 \times 3^{a-1} - 2$.

If $m = 0$, then $|S_{(3^a)}| = |S| = \mathsf{s}(C_{3^a})$. Hence $S_{(3^a)}$ contains a zero-sum subsequence T of length 3^a by the definition of $\mathsf{s}(C_{3^a})$.

If $1 \leq m \leq 2 \times 3^{a-1} - 2$, then $|S_{(3^a)}| = |S| - |S_{(<3^a)}| = 2 \times 3^a - 1 - m \geq 4 \times 3^{a-1} + 1 > 7$. By Lemma 3.5, we can choose $T_1, \ldots, T_w \mid S_{(3^a)}$ with

$$w = 1 + \left\lfloor \frac{|S_{(3^a)}| - 7}{3} \right\rfloor = 2 \times 3^{a-1} - 1 + \left\lfloor \frac{-m-2}{3} \right\rfloor \geq 1$$

such that $|T_i| = 3$ and $\mathsf{k}(T_i) = \mathsf{k}(\sigma(T_i))$ for every $i = 1, \ldots, w$. Moreover,

$$\begin{aligned} w + m &= 2 \times 3^{a-1} - 1 + \left\lfloor \frac{-m-2}{3} \right\rfloor + m \\ &= 2 \times 3^{a-1} - 1 + \left\lfloor \frac{2(m-1)}{3} \right\rfloor \\ &\geq 2 \times 3^{a-1} - 1 = \mathsf{T}(C_{3^{a-1}}). \end{aligned}$$

Again, our claim is true in this case.

Case 3: $p \geq 5$. Here $|S| = 2p^a - 1$ and $m \leq 2p^{a-1} - 2$.

If $m = 0$, then $|S_{(p^a)}| = |S| = \mathsf{s}(C_{p^a})$. Hence $S_{(p^a)}$ contains a zero-sum subsequence T of length p^a by the definition of $\mathsf{s}(C_{p^a})$.

If $1 \leq m \leq 2p^{a-1} - 2$ and $S_{(p^a)}$ contains no zero-sum subsequence T of length p^a, then $|S_{(p^a)}| = 2p^a - (m+1)$ with $2 \leq m+1 \leq 2p^{a-1} - 1 < p^a/2 + 1$ since $p \geq 5$. By Lemma 3.7 (taking $n = p^a$, $k = 1$ and $t = m+1$), there exist two distinct elements $x, y \in S_{(p^a)} \mid S$ such that

$$\mathsf{v}_x(S) + \mathsf{v}_y(S) \geq 2p^a - 2m.$$

Therefore, we can choose $T_1, \ldots, T_w \mid x^{\mathsf{v}_x(S)} \cdot y^{\mathsf{v}_y(S)}$ with

$$\begin{aligned} w &= \left\lfloor \frac{\mathsf{v}_x(S)}{p} \right\rfloor + \left\lfloor \frac{\mathsf{v}_y(S)}{p} \right\rfloor \\ &\geq \frac{\mathsf{v}_x(S) - (p-1)}{p} + \frac{\mathsf{v}_y(S) - (p-1)}{p} \\ &= \frac{\mathsf{v}_x(S) + \mathsf{v}_y(S) + 2}{p} - 2 \\ &\geq 2p^{a-1} - \frac{2(m-1)}{p} - 2 \end{aligned}$$

such that $T_i = x^p$ or y^p for every $i = 1, \ldots, w$. It follows that $|T_i| = p$ and

$$\mathsf{k}(T_i) = \frac{|T_i|}{p^a} = \frac{1}{p^{a-1}} = \frac{1}{\mathrm{ord}(\sigma(T_i))} = \mathsf{k}(\sigma(T_i)).$$

Moreover,

$$\begin{aligned} w + m &\geq 2p^{a-1} - \frac{2(m-1)}{p} - 2 + m \\ &= 2p^{a-1} - 1 + \frac{(p-2)(m-1)}{p} \\ &\geq 2p^{a-1} - 1 = \mathsf{T}(C_{p^{a-1}}). \end{aligned}$$

This concludes the proof of our claim.

If $S_{(p^a)}$ contains a zero-sum subsequence T of length p^a, then $T \mid S$, $\sigma(T) = 0$ and $\mathsf{k}(T) = |T|/p^a = 1$ as desired.

If $S_{(p^a)}$ does not contain any zero-sum subsequence T of length p^a, we consider the sequence

$$W = \sigma(T_1) \cdot \ldots \cdot \sigma(T_w) \cdot S_{(<p^a)}.$$

Since $|W| = w + m \geq \mathsf{T}(C_{p^{a-1}})$ by (5.1) and W is over $pC_{p^a} \simeq C_{p^{a-1}}$ by the induction assumption, W has a subsequence

$$U = \sigma(T_{u_1}) \cdot \ldots \cdot \sigma(T_{u_v}) \cdot R,$$

where $\{u_1, \ldots, u_v\} \subseteq [1, w]$ and $R \mid S_{(<p^a)}$, with the property that

$$\sigma(U) = 0 \quad \text{and} \quad \mathsf{k}(U) = 1.$$

Finally, let

$$V = T_{u_1} \cdot \ldots \cdot T_{u_v} \cdot R.$$

Then $V \mid S$, $\sigma(V) = \sigma(U) = 0$ and by (5.1),

$$\mathsf{k}(V) = \sum_{i=1}^{v} \mathsf{k}(T_{u_i}) + \mathsf{k}(R) = \sum_{i=1}^{v} \mathsf{k}(\sigma(T_{u_i})) + \mathsf{k}(R) = \mathsf{k}(U) = 1$$

as desired.

This completes the proof of Theorem 1.3. ■

Proof of Theorem 1.4. By the hypothesis of the theorem, n is odd. Let $p_1, \ldots, p_s$ be all the distinct prime divisors of n. Then

$$4 \sum_{i=1}^{s} \frac{1}{p_i} \leq 1.$$

Let $d(n)$ denote the number of positive divisors (> 1) of n. We proceed by induction on $d(n)$. If $d(n) = 1$, then n is a prime, so by Theorem 1.1 the conclusion of the theorem holds. Next we suppose the conclusion is true for $d(n) < k$ ($k \geq 2$), and we need to prove it for $d(n) = k$.

By Lemma 3.1 we have $\mathsf{T}(G) \geq 2n-1$, so it suffices to prove that $\mathsf{T}(G) \leq 2n-1$.

Let S be a sequence over G of length $|S| = 2n-1$. We need to show that S has a zero-sum subsequence T with $\mathsf{k}(T) = 1$. Let

$$H_i = p_i C_n \cong C_{n/p_i}$$

be a subgroup of G, where $i \in [1, s]$, and let

$$S = S_{(<n)} S_{(n)}.$$

If $|S_{H_i}| \geq 2n/p_i - 1 = \mathsf{T}(C_{n/p_i})$ for some $i \in [1, s]$, then S_{H_i} contains a non-empty zero-sum subsequence T with $\mathsf{k}(T) = 1$. So does S. If $S_{(n)}$ contains a zero-sum subsequence T of length $|T| = n$ with $\mathsf{k}(T) = |T|/n = 1$, then S contains a zero-sum subsequence T of length n with $\mathsf{k}(T) = 1$.

Next we assume that

$$|S_{(n)}| \leq \mathsf{s}(G) - 1 = 2n - 2 \quad \text{and} \quad |S_{H_i}| \leq 2\frac{n}{p_i} - 2,$$

for all $i \in [1, s]$. Then

$$1 \leq |S_{(<n)}| \leq \sum_{i=1}^{s} |S_{H_i}| \leq \sum_{i=1}^{s} 2\frac{n}{p_i} - 2s.$$

Therefore,

$$2 \leq 1 + |S_{(<n)}| \leq \sum_{i=1}^{s} 2\frac{n}{p_i} - 2s + 1 < \frac{n}{2} + 1$$

because $4\sum_{i=1}^{s} \frac{1}{p_i} \leq 1$. Since $|S_{(n)}| = |S| - |S_{(<n)}| = 2n - 1 - |S_{(<n)}|$, it follows from Lemma 3.7 that there exist two distinct elements x, y in $S_{(n)}$ such that

$$\mathsf{v}_x(S_{(n)}) + \mathsf{v}_y(S_{(n)}) \geq 2n - 2|S_{(<n)}|.$$

Therefore, we can choose $T_1 \cdot \ldots \cdot T_{w_i} \mid x^{\mathsf{v}_x(S_{(n)})} y^{\mathsf{v}_y(S_{(n)})}$ with

$$w_i \geq \left\lceil \frac{2n - 2|S_{(<n)}| - (2p_i - 2)}{p_i} \right\rceil,$$

satisfying $T_j = x^{p_i}$ or y^{p_i} for $j \in [1, w_i]$ and $i \in [1, s]$. It follows that $|T_j| = p_i$, $\sigma(T_j) \in H_i$ and

$$\mathsf{k}(T_j) = \mathsf{k}(\sigma(T_j)).$$

If there exists some $i \in [1, s]$ such that $|S_{H_i}| + w_i \geq 2n/p_i - 1 = \mathsf{T}(C_{n/p_i})$, then

$$S_{H_i} \sigma(T_1) \cdot \ldots \cdot \sigma(T_{w_i})$$

has a non-empty zero-sum subsequence $U = S'_{H_i} \prod_{k \in K} \sigma(T_k)$ with $\mathsf{k}(U) = 1$, where $S'_{H_i} \mid S_{H_i}$ and $K \subset [1, w_i]$. Then $T = S'_{H_i} \prod_{k \in K} T_k$ is a zero-sum subsequence of S with

$$\mathsf{k}(T)=\mathsf{k}(S'_{H_i})+\sum_{k\in K}\mathsf{k}(T_k)=\mathsf{k}(S'_{H_i})+\sum_{k\in K}\mathsf{k}(\sigma(T_k))=\mathsf{k}(U)=1,$$

and we are done.

Next we suppose that $|S_{H_i}|+w_i\le 2n/p_i-2$ for all $i\in[1,s]$. It follows that

$$|S_{H_i}|+\left\lceil\frac{2n-2|S_{(<n)}|-(2p_i-2)}{p_i}\right\rceil\le 2\frac{n}{p_i}-2$$

for all $i\in[1,s]$. Thus,

$$p_i|S_{H_i}|-2|S_{(<n)}|+2\le 0.$$

Therefore,

$$\sum_{i=1}^{s}p_i|S_{H_i}|-2s|S_{(<n)}|+2s\le 0,$$

a contradiction with $p(n)>2s$.

This completes the proof of Theorem 1.4. ∎

6. Concluding remarks. It is interesting to study the relationship between $\mathsf{T}(G)$ and $\mathsf{s}(G)$ for other finite abelian groups G.

If there is a sequence S over G of length $|S|=\mathsf{s}(G)-1$ such that S has no zero-sum subsequence with length $\exp(G)$ and every term of S has order $\exp(G)$, then clearly S has no zero-sum subsequence with cross number 1. Therefore, $\mathsf{T}(G)\ge|S|+1=\mathsf{s}(G)$ for such groups G.

Let $\mathsf{D}(G)$ be the smallest integer t such that every sequence of t elements (repetition allowed) from G has a zero-sum subsequence. The invariant $\mathsf{D}(G)$ is called the *Davenport constant*. By the definition of $\mathsf{T}(G)$, the following result is obvious.

LEMMA 6.1. *Let H be a subgroup of G, and S a sequence over $G\setminus H$. If any subsequence R of S with $\sigma(R)\in H$ satisfies $\mathsf{k}(R)\notin\frac{1}{\exp(H)}\mathbb{Z}$, then*

$$\mathsf{T}(G)\ge\mathsf{T}(H)+|S|.$$

In particular, such an S always exists for $|S|=\mathsf{D}(G/H)-1$ and so

$$\mathsf{T}(G)\ge\mathsf{T}(H)+\mathsf{D}(G/H)-1\ge\mathsf{T}(H).$$

However, this relationship does not hold for $\mathsf{s}(G)$ and $\mathsf{s}(H)$. First, we present a 2010 result of Schmid and Zhuang.

THEOREM 6.2 ([17]). *Let p be an odd prime and G be a finite abelian p-group. If $\mathsf{D}(G)\le 2\exp(G)-1$, then*

$$\mathsf{s}(G)\le\mathsf{D}(G)+2\exp(G)-2.$$

Example 6.3. For an odd prime p and integers $r \geq 4$, $\alpha \geq 2$, let $G = C_p^{r-1} \oplus C_{p^\alpha}$ and $H = C_p^r$. Let $\beta = \frac{p^\alpha - 1}{p-1}$. Then

$$\mathsf{T}(G) > \mathsf{T}(H) = \mathsf{s}(H) > \mathsf{s}(G)$$

for $r \in \left[\frac{\ln \beta}{\ln 2} + 2, \beta + 1\right]$.

Since $r \leq \beta + 1$, we have

$$\mathsf{D}(G) = (r-1)(p-1) + (p^\alpha - 1) + 1 = p^\alpha + (r-1)(p-1) \leq p^\alpha + \beta(p-1) = 2p^\alpha - 1$$

and so

$$\mathsf{s}(G) \leq \mathsf{D}(G) + 2p^\alpha - 2 = 3p^\alpha + (r-1)(p-1) - 2$$

by Theorem 6.2. On the other hand, $\frac{\ln \beta}{\ln 2} + 2 \leq r \leq \beta + 1$ implies that $2^r \geq 4\beta \geq 3\beta + r - 1$. But these two inequalities cannot both hold since $\beta = \frac{p^\alpha - 1}{p-1} \geq 1 + p \geq 4$. Thus $2^r > 3\beta + r - 1$. By Theorem 1.1 and Lemma 6.1,

$$\mathsf{T}(G) > \mathsf{T}(H) = \mathsf{s}(H) \geq 2^r(p-1) + 1 > 3p^\alpha + (r-1)(p-1) - 2 \geq \mathsf{s}(G).$$

It is easy to check that there indeed exists an integer $r \in \left[\frac{\ln \beta}{\ln 2} + 2, \beta + 1\right]$ with $\beta = \frac{p^\alpha - 1}{p-1} \geq 4$ for any odd prime p and any integer $\alpha \geq 2$. Moreover, it is well known that there is always an odd prime $p \in (m/2, m)$ for any integer $m \geq 4$. Therefore, when $\alpha = 2$,

$$\bigcup_{p \in \mathbb{P} \setminus \{2\}} \left[\frac{\ln \beta}{\ln 2} + 2, \beta + 1\right] = \bigcup_{p \in \mathbb{P} \setminus \{2\}} \left[\frac{\ln(1+p)}{\ln 2} + 2, p + 2\right] = [4, \infty).$$

Corollary 6.4. *For any integer $r \geq 4$, there is always a finite abelian group G of rank r such that $\mathsf{T}(G) > \mathsf{s}(G)$.*

Conjecture 6.5. If n is an odd integer, then for any positive integer r,

$$\mathsf{T}(C_n^r) = \mathsf{s}(C_n^r).$$

Regarding Conjecture 3.2, we can consider the following general problem. Let p be a prime and α be a positive integer. Let $s(p, \alpha)$ be the smallest integer t such that every sequence over $\mathbb{Z} \setminus p\mathbb{Z}$ with length $|S| = t$ has a subsequence T with $|T| = p^\alpha$, and $\sigma(T) \equiv 0 \pmod{p^\alpha}$ but $\sigma(T) \not\equiv 0 \pmod{p^{\alpha+1}}$. Determine $s(p, \alpha)$.

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