OPTIMAL LOWER BOUND FOR THE BLOW-UP RATE OF THE MAGNETIC ZAKHAROV SYSTEM WITHOUT THE SKIN EFFECT

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ABSTRACT. We focus on the Cauchy problem of the magnetic Zakharov system in two-dimensional space:

$$\begin{cases} iE_{1t} + \Delta E_1 - nE_1 + \eta E_2 \left(E_1\overline{E_2} - \overline{E_1}E_2\right) = 0, \\ iE_{2t} + \Delta E_2 - nE_2 + \eta E_1 \left(\overline{E_1}E_2 - E_1\overline{E_2}\right) = 0, \\ n_t + \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}_t + \nabla n + \nabla \left(|E_1|^2 + |E_2|^2\right) = 0, \end{cases} \tag{$G-Z$}$$

$$(E_1, E_2, n, \mathbf{v})(0, x) = (E_{10}, E_{20}, n_0, \mathbf{v}_0)(x). \tag{$G-Z-I$}$$

System (G-Z) describes the spontaneous generation of a magnetic field without the skin effect in a cold plasma, and $\eta > 0$ is the magnetic coefficient. The nonlinear cubic coupling terms $E_2\left(E_1\overline{E_2} - \overline{E_1}E_2\right)$ and $E_1\left(\overline{E_1}E_2 - E_1\overline{E_2}\right)$ generated by the cold magnetic field bring additional difficulties compared with the classical Zakharov system. If the initial mass meets a presettable condition

$$\frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1+\eta} < ||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} < \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{\eta},$$

where Q is the unique radially positive solution of the equation $-\Delta V + V = V^3$, we prove that there is a constant $c > 0$ depending only on the initial data such that for t near T (the blow-up time),

$$\|(E_1, E_2, n, \mathbf{v})\|_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \geqslant \frac{c}{T-t}.$$

As the magnetic coefficient η tends to 0, the blow-up rate recovers the result for the classical 2-D Zakharov system due to Merle [16]. On the other hand, for any positive η, the result of this paper reveals a rigorous justification that the optimal lower bound of the blow-up rates is not affected by the presence of magnetic field without the skin effect in a cold plasma.

Keywords: Blow-up rate; Magnetic Zakharov system; Skin effect; Optimal lower bound

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1. Introduction and main results

The magnetic Zakharov system

$$\begin{cases} i\mathbf{E}_t + \nabla\nabla\cdot\mathbf{E} - n\mathbf{E} - \alpha\nabla\times(\nabla\times\mathbf{E}) + i(\mathbf{E}\wedge\mathbf{B}) = 0, \\ \partial_t n = -\nabla\cdot\mathbf{v}, \\ \partial_t\mathbf{v} = -\nabla n - \nabla|\mathbf{E}|^2, \\ \Delta\mathbf{B} - i\eta\nabla\times\nabla\times\left(\mathbf{E}\wedge\overline{\mathbf{E}}\right) + \mathbf{A} = 0, \end{cases} \tag{I}$$

is proposed to describe the spontaneous generation of a magnetic field in a cold plasma [12]. Here, $\mathbf{E} = (E_1, E_2, E_3) \in \mathbb{C}^3$ denotes the slowly varying complex amplitudes of the high-frequency electric field, n the fluctuation of the electron density from its equilibrium, $\mathbf{B}$ the self-generated magnetic field in a cold plasma, $\mathbf{A} = \delta\mathbf{B}$, $\delta \le 0$, $\eta > 0$ and $\alpha \ge 1$ are all physical constants. In particular, the Zakharov system (I) is a Schrödinger-wave coupled system with different scalings. It keeps two conservation laws including the *total mass*

$$\|\mathbf{E}\|^2_{L^2(\mathbb{R}^2)} = \|E_1\|^2_{L^2(\mathbb{R}^2)} + \|E_2\|^2_{L^2(\mathbb{R}^2)} + \|E_3\|^2_{L^2(\mathbb{R}^2)}, \tag{II}$$

as well as the *total energy*

$$\begin{aligned} \mathcal{H} := \ & ||\nabla\times\mathbf{E}||^2_{L^2(\mathbb{R}^2)} + ||\nabla\cdot\mathbf{E}||^2_{L^2(\mathbb{R}^2)} \\ & + \frac{1}{2}||n||^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}||\mathbf{v}||^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n|\mathbf{E}|^2 dx \\ & + \frac{\eta}{2}\int_{\mathbb{R}^2} \frac{1}{|\xi|^2 - \delta}\left(\left|\xi\cdot\mathcal{F}(\mathbf{E}\times\overline{\mathbf{E}})\right|^2 - |\xi|^2\left|\mathcal{F}(\mathbf{E}\times\overline{\mathbf{E}})\right|^2\right) d\xi. \end{aligned} \tag{III}$$

In the cold plasma, the term $\delta\mathbf{B}$ describes the classical (collisionless) skin effect, α is related to the velocity of electrons and the plasma frequency [12]. In the present paper, we shall mainly focus on the case of $\delta = 0$ and $\alpha = 1$, that is, the skin effect would not be involved and the velocity of electrons increases synchronously with the frequency of plasma.

From a physical point of view, the two-dimensional case for $\mathbf{E}$ is essential since the electric fields usually traverse straightly with the reference planes. Let

$$\mathbf{E}(t,x) = (E_1(t,x), E_2(t,x), 0), \quad x \in \mathbb{R}^2.$$

Through standard computations, one obtains

$$\mathbf{B}(t,x) = (0, 0, B_3(t,x)) = \left(0, 0, -i\eta\left(E_1\overline{E_2} - \overline{E_1}E_2\right)\right)$$

by the fact that $\nabla\cdot\left(\mathbf{E}\wedge\overline{\mathbf{E}}\right) = 0$ and the vectorial identity $\Delta\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla\times\nabla\times\mathbf{E}$. On the other hand, the interaction term involving the electronic and magnetic fields enjoys the following expression:

$$i\mathbf{E}\wedge\mathbf{B} = \eta\mathbf{E}\wedge\mathbf{E}\wedge\overline{\mathbf{E}} = \eta\left(E_2\left(E_1\overline{E_2} - \overline{E_1}E_2\right), E_1\left(\overline{E_1}E_2 - E_1\overline{E_2}\right), 0\right).$$

Then (I) becomes a system

$$\begin{cases} i\partial_t E_1 + \Delta E_1 - nE_1 + \eta E_2 \left(E_1\overline{E}_2 - \overline{E}_1 E_2\right) = 0, & (1.1-1) \\ i\partial_t E_2 + \Delta E_2 - nE_2 + \eta E_1 \left(\overline{E}_1 E_2 - E_1\overline{E}_2\right) = 0, & (1.1-2) \\ \partial_t n + \nabla\cdot\mathbf{v} = 0, & (1.1-3) \\ \partial_t \mathbf{v} + \nabla n + \nabla\left(|E_1|^2 + |E_2|^2\right) = 0, & (1.1-4) \end{cases} \tag{1.1}$$

where $E_1(t,x), E_2(t,x) : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{C},\ n(t,x) : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{R},\ \mathbf{v}(t,x) : \mathbb{R}^+\times\mathbb{R}^2 \to \mathbb{R}^2$ are physical quantities mentioned earlier.

We supplement (1.1) with the initial condition:

$$\begin{cases} E_1(0,x) = E_{10}(x), \quad E_2(0,x) = E_{20}(x), \\ n(0,x) = n_0(x), \qquad \mathbf{v}(0,x) = \mathbf{v}_0(x). \end{cases} \tag{1.2}$$

Due to the identity (III), the Hamiltonian for (1.1) can be schematically written in the form

$$\begin{aligned} &\mathcal{H}(E_1, E_2, n, \mathbf{v}) \\ &= \|\nabla E_1\|_{L^2}^2 + \|\nabla E_2\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{1}{2}\|\mathbf{v}\|_{L^2}^2 \\ &\quad + \int_{\mathbb{R}^2} n\left(|E_1|^2 + |E_2|^2\right)dx - \frac{\eta}{2}\int_{\mathbb{R}^2}\left|E_1\overline{E}_2 - E_2\overline{E}_1\right|^2 dx. \end{aligned} \tag{1.3}$$

Clearly it is a well-defined functional on the energy space

$$\mathbb{H}_1 := H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2). \tag{1.3$*$}$$

The blow-up dynamics of the two dimensional classical Zakharov system have been studied in detail by several authors [10, 11, 15, 16]. For the Zakharov system with magnetic field effect (I), Laurey in [13] proved the global existence of weak solutions for small initial data as well as the local existence and uniqueness of smooth solutions in both two-dimensional and three-dimensional spaces. Based on Laurey's work [13], over the last decade, finite time blow-up dynamics for (I) were considered. Gan, Guo and Huang in [7] constructed a family of blow-up solutions in two-dimensional space, proved the existence of self-similar blow-up solutions and established the instability and the concentration property of a class of periodic solutions. In [6], the authors studied the Virial type blow-up solutions of the Cauchy problem for (I). Later, the authors in [8] established the space-time integral estimate of the blow-up rate for the finite time blow-up solutions to (I) in three dimensional space. These arguments naturally carry on system (1.1).

To our best knowledge, very few results on the lower bound of the blow-up rates are known for Zakharov system, especially for (1.1). Our aim here is to establish the (essentially optimal) lower bound of the blow-up rates for the finite time blow-up solutions to system (1.1).

Let us state a few preliminary results.

Firstly, with the methods used in [1, 2, 3, 9], the local well-posedness of mild solutions of (1.1) in the energy space can be established:

Proposition 1.1. The two dimensional magnetic Zakharov system (1.1) is locally well-posed in the energy space $\mathbb{H}_1$, which is defined by (1.3*). That is, there exists a unique solution $(E_1, E_2, n, \mathbf{v})$ satisfying

$$\|(E_1, E_2, n, \mathbf{v})(t)\|_{\mathbb{H}_1} \leq C, \ \ \forall t \in [0, T),$$

where T is the maximal existence time of the solution and C is a constant depending only on initial data. □

Next, following Gan, Guo, Han and Zhang [6], we claim the virial-type blow-up result for the Cauchy problem (1.1)-(1.2).

Proposition 1.2. Let $\eta > 0$ and let the solutions $(E_1, E_2, n, \mathbf{v})(t)$ to the Cauchy problem (1.1)-(1.2) be radially symmetric functions on $\mathbb{R}^2$ for all time. If $\mathcal{H}(E_{10}, E_{20}, n_0, \mathbf{v}_0) < 0$, then the following alternative holds:

(i) $(E_1, E_2, n, \mathbf{v})(t)$ blows up in finite time,

(ii) $(E_1, E_2, n, \mathbf{v})(t)$ blows up at infinity. That is, $(E_1, E_2, n, \mathbf{v})(t)$ is defined for all t and

$$\lim_{t \to +\infty} \|(E_1, E_2, n, \mathbf{v})\|_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} = +\infty. \qquad \square$$

With these results, it is natural to expect more quantitative descriptions on the behaviour of solutions as t near T, where $T < \infty$ is the blow-up time. Compared with the classical Zakharov system, the two extra nonlinear cubic coupling terms $E_2\left(E_1\overline{E_2} - \overline{E_1}E_2\right)$ and $E_1\left(\overline{E_1}E_2 - E_1\overline{E_2}\right)$ in (1.1) generated by the cold magnetic field without the skin effect do bring a further challenge. If the initial mass meets the presettable condition:

$$\frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1+\eta} < ||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} < \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{\eta}, \tag{1.4}$$

where Q is the unique radially positive solution of the equation

$$-\Delta V + V = V^3, \tag{1.5}$$

we can prove that there is a constant $c > 0$ depending only on the initial data such that for t near T (the blow-up time),

$$\|(E_1, E_2, n, \mathbf{v})\|_{\mathbb{H}_1} \geqslant \frac{c}{T-t}.$$

The main result of this paper is as follows:

Theorem 1.3. *Let $(E_1, E_2, n, \mathbf{v})(t)$ be the finite time blow-up solution of the Cauchy problem (1.1)-(1.2), and $T < \infty$ be the blow-up time. Suppose that the initial data (E_{10}, E_{20}) satisfies the condition (1.4), where Q is the unique radial positive solution of equation (1.5), then there exist constants $c > 0, \tilde{c} > 0$ depending only on initial data, such that as t near T,*

(1)

$$\|(E_1, E_2, n, \boldsymbol{v})(t)\|_{\mathbb{H}_1} \geqslant \frac{c}{T-t}, \tag{1.6}$$

$$\left(\|\nabla E_1(t)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2(t)\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \geqslant \frac{\tilde{c}}{T-t}, \tag{1.7}$$

$$||n(t)||_{L^2(\mathbb{R}^2)} \geqslant \frac{\tilde{c}}{T-t}. \tag{1.8}$$

More precisely, we have

(2)

$$\left(||\nabla E_1(t)||^2_{L^2(\mathbb{R}^2)} + ||\nabla E_2(t)||^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}$$
$$\geqslant \frac{c}{\left(||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} - \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1+\eta}\right)^{\frac{1}{2}}} \frac{1}{T-t}, \tag{1.9}$$

$$\|n(t)\|_{L^2(\mathbb{R}^2)} \geqslant \frac{c}{\left(||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} - \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1+\eta}\right)^{\frac{1}{2}}} \frac{1}{T-t}. \tag{1.10}$$

The assumption on the lower bound for the initial mass is natural, especially for small $\eta > 0$, as it is basically the minimal mass needed for blow-up to occur. Indeed, as the magnetic coefficient η tends to 0, the blow-up rate recovers the result of the classical 2-D Zakharov system due to Merle [16]. Whether or not the upper bound given in (1.4) is optimal remains open, but the assumption satisfies automatically for prescribed finite mass if η is sufficiently small. For large η, on the other hand, our result is in some sense more intriguing. It provides a mass band in which one can get more precise information for blow-ups. Indeed, under the presettable condition (1.4), Theorem 1.3 provides a rigorous justification that the presence of magnetic effects without the skin effect in the cold plasma does not change the optimal lower bound for the blow-up rates.

Remark 1.4. In [7], Gan, Guo and Huang constructed a family of blow-up solutions to the Cauchy problem (1.1)-(1.2):

$$\begin{cases} E_1(t,x) = \frac{\omega}{T-t} e^{i\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{\omega^2}{-T+t}\right)} \frac{\tilde{P}(\frac{x\omega}{T-t})}{\sqrt{2}}, \\ E_2(t,x) = -iE_1(t,x), \\ n(t,x) = \frac{\omega^2}{(T-t)^2} \tilde{N}\left(\frac{x\omega}{T-t}\right), \end{cases} \tag{1.11}$$

where $\tilde{P}(x) = \tilde{P}(|x|), \tilde{N}(x) = \tilde{N}(|x|)$ are radial functions on $\mathbb{R}^2$, $\theta \in \mathbb{R}$ and $\omega > 0$. Let $\tilde{P} = \frac{P}{(1+\eta)^{\frac{1}{2}}}$, $\tilde{N} = \frac{N}{1+\eta}$, then (P, N) satisfies

$$\begin{cases} \Delta P - P + \frac{\eta}{\eta+1} P^3 = \frac{1}{\eta+1} NP, \\ \frac{1}{\omega^2}(r^2 N_{rr} + 6rN_r + 6N) - \Delta N = \Delta |P|^2. \end{cases} \tag{1.12}$$

A straightforward calculation yields

$$\begin{cases} ||\nabla E_1(t)||_{L^2(\mathbb{R}^2)} = \frac{\omega}{T-t} ||\nabla \tilde{P}||_{L^2(\mathbb{R}^2)}, \\ ||\nabla E_2(t)||_{L^2(\mathbb{R}^2)} = \frac{\omega}{T-t} ||\nabla \tilde{P}||_{L^2(\mathbb{R}^2)}, \\ ||n(t)||_{L^2(\mathbb{R}^2)} = \frac{\omega}{T-t} ||N||_{L^2(\mathbb{R}^2)}, \\ ||v(t)||_{L^2(\mathbb{R}^2)} = \frac{\omega c(P,N)}{T-t}. \end{cases} \tag{1.13}$$

These estimates imply that the lower bound estimates of blow-up rate (1.6), (1.7) and (1.8) in Theorem 1.3 are indeed optimal. On the other hand, letting $\omega \to +\infty$

and $(P,N) \to (Q,-Q^2)$, it yields that (1.9) and (1.10) in Theorem 1.3 are also optimal. $\square$

Let us emphasize that the Zakharov system (1.1) is a Hamiltonian system and admits conservations of the total mass and total energy:

$$\|E_1\|^2_{L^2(\mathbb{R}^2)} + \|E_2\|^2_{L^2(\mathbb{R}^2)} = \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)}, \tag{1.14}$$

$$\mathcal{H}(E_1, E_2, n, \mathbf{v}) = \mathcal{H}(E_{10}, E_{20}, n_0, \mathbf{v}_0) = \mathcal{H}_0, \tag{1.15}$$

where $\mathcal{H}(E_1, E_2, n, \mathbf{v})$ is defined in (1.3).

In stark contrast to the Zakharov system without the magnetic field effect, the presence of extra nonlinear terms $E_2\left(E_1\overline{E_2} - \overline{E_1}E_2\right)$ and $E_1\left(\overline{E_1}E_2 - E_1\overline{E_2}\right)$ (generated by the cold magnetic field) in (1.1) brings new difficulties in establishing the lower bound of the blow-up rates. To this end, motivated by Merle's heuristic arguments covering the geometrical estimate, the non-vanishing estimate and the compactness argument, involved in [16], we need to establish additional a priori estimates for the extra nonlinear terms. On the other hand, we also need additional techniques from those adopted in [16]. In particular, to obtain the optimal lower bound of blow-up rate, the initial mass needs to satisfy (1.4) so that we can establish indispensable a priori estimates involving higher order nonlinear terms. As mentioned earlier, the mass condition (1.4) is natural from both physical and mathematical points of view. Indeed, by Lemma 2.2 in Section 2, it is standard to conclude that the mild (no blow up) solution of (1.1) is globally well-defined if the initial data (E_{10}, E_{20}) satisfies

$$||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} < \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1+\eta}.$$

In [7] the authors pointed out that there is no mass-concentration at a finite time provided

$$||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} = \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1+\eta}.$$

Remark 1.5. The condition (1.4) is consistent with the blow-up dynamics of the classical Zakharov system [11] when $\eta \to 0$. The upper bound

$$||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)} < \frac{||Q||^2_{L^2(\mathbb{R}^2)}}{\eta}.$$

means that multiple bubbles blow-ups may occur, even the blow-up rates are of different orders. It is a very interesting issue for further study. $\square$

The outline of the paper is as follows. The Section 2 (Preliminary) is mainly devoted to some preparatory materials and technical results. Section 3 concerns the analysis of the rescaled Zakharov system. In section 4 we prove the optimal lower-bound of finite time blow-up rate (Theorem 1.3).

2. Preliminaries

In this section, we give some notations and basic estimates. Firstly we recall several lemmas in [4, 5, 17].

Lemma 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geqslant 2$ and $\frac{1}{p}+\frac{1}{q}=1$. If $f_k \rightharpoonup f$ in $L^p(\Omega)$, and $g_k \to g$ in $L^q(\Omega)$ as $k \to +\infty$, then*

$$\int_\Omega f_k g_k dx \to \int_\Omega fg dx \text{ as } k \to +\infty. \qquad \square$$

Lemma 2.2. *(Weinstein [17]) For all $u \in H^1(\mathbb{R}^2)$,*

$$\frac{1}{2}\|u\|^4_{L^4(\mathbb{R}^2)} \leqslant \left(\frac{\|u\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\right)\|\nabla u\|^2_{L^2(\mathbb{R}^2)},$$

where Q is the unique positive solution of (1.5). $\square$

Since (1.1-1)-(1.1-2) and (1.1-3)-(1.1-4) have the same scaling on the spatial structure but different space-time structures, it will conserve (1.1-1)-(1.1-2) and (1.1-3)-(1.1-4), respectively. Taking a suitable space-time scaling to the Zakharov system (1.1) can yield a rescaled system.

Proposition 2.3. Let $(E_1, E_2, n, \mathbf{v})$ be the finite-time blow-up solutions to the Zakharov system (1.1) and T be its blow-up time. For any $t \in [0, T)$, let

$$\begin{aligned}
\tilde{E}_1(t,s,x) &= \tfrac{1}{\lambda(t)} E_1\left(t + \tfrac{s}{\lambda(t)}, \tfrac{x}{\lambda(t)}\right),\\
\tilde{E}_2(t,s,x) &= \tfrac{1}{\lambda(t)} E_2\left(t + \tfrac{s}{\lambda(t)}, \tfrac{x}{\lambda(t)}\right),\\
\tilde{n}(t,s,x) &= \tfrac{1}{\lambda^2(t)} n\left(t + \tfrac{s}{\lambda(t)}, \tfrac{x}{\lambda(t)}\right),\\
\tilde{\mathbf{v}}(t,s,x) &= \tfrac{1}{\lambda^2(t)} \mathbf{v}\left(t + \tfrac{s}{\lambda(t)}, \tfrac{x}{\lambda(t)}\right),
\end{aligned} \tag{2.1}$$

where $s \in [0, \lambda(t)(T-t))$,

$$\begin{aligned}
\lambda^2(t) &= \|(E_1, E_2, n, \mathbf{v})\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)}\\
&= \int_{\mathbb{R}^2} |\nabla E_1(t,x)|^2\,dx + \int_{\mathbb{R}^2} |\nabla E_2(t,x)|^2\,dx\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} |n(t,x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{v}(t,x)|^2\,dx.
\end{aligned} \tag{2.2}$$

Then $(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}})(s)$ satisfies the following rescaled Zakharov system:

$$\begin{cases}
\frac{1}{\lambda} i\tilde{E}_{1s} + \Delta\tilde{E}_1 - \tilde{n}\tilde{E}_1 + \eta\tilde{E}_2\left(\tilde{E}_1\overline{\tilde{E}_2} - \overline{\tilde{E}_1}\tilde{E}_2\right) = 0, & (2.3a)\\
\frac{1}{\lambda} i\tilde{E}_{2s} + \Delta\tilde{E}_2 - \tilde{n}\tilde{E}_2 + \eta\tilde{E}_1\left(\overline{\tilde{E}_1}\tilde{E}_2 - \tilde{E}_1\overline{\tilde{E}_2}\right) = 0, & (2.3b)\\
\tilde{n}_s + \nabla\cdot\tilde{\mathbf{v}} = 0, & (2.3c)\\
\tilde{\mathbf{v}}_s + \nabla\left(\tilde{n} + |\tilde{E}_1|^2 + |\tilde{E}_2|^2\right) = 0. & (2.3d)
\end{cases} \tag{2.3}$$

In addition, there hold
(1)

$$\begin{aligned}
&\lim_{s\to\lambda(t)(T-t)}\left\|\left(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}}\right)(s)\right\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)}\\
&\quad=\lim_{s\to\lambda(t)(T-t)}\left(\left\|\nabla\tilde{E}_1(s)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\nabla\tilde{E}_2(s)\right\|^2_{L^2(\mathbb{R}^2)}\right.\\
&\qquad\qquad\left.+\tfrac{1}{2}||\tilde{n}(s)||^2_{L^2(\mathbb{R}^2)}+\tfrac{1}{2}||\tilde{\mathbf{v}}(s)||^2_{L^2(\mathbb{R}^2)}\right)\\
&\quad=+\infty,
\end{aligned}\tag{2.4}$$

(2)

$$\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1(t,0,x)\right|^2+\left|\nabla\tilde{E}_2(t,0,x)\right|^2+\frac{1}{2}|\tilde{n}(t,0,x)|^2+\frac{1}{2}|\tilde{\mathbf{v}}(t,0,x)|^2\right)dx=1,\tag{2.5}$$

(3)

$$\begin{aligned}
&\left\|\tilde{E}_1(t,s,x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}_2(t,s,x)\right\|^2_{L^2(\mathbb{R}^2)}\\
&\quad=\left\|\tilde{E}_1(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}_2(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}\\
&\quad=\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)},
\end{aligned}\tag{2.6}$$

and
(4)

$$\begin{aligned}
&\mathcal{H}(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}})\\
&=\left\|\nabla\tilde{E}_1\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\nabla\tilde{E}_2\right\|^2_{L^2(\mathbb{R}^2)}+\tfrac{1}{2}||\tilde{n}||^2_{L^2(\mathbb{R}^2)}+\tfrac{1}{2}||\tilde{\mathbf{v}}||^2_{L^2(\mathbb{R}^2)}\\
&\quad+\int_{\mathbb{R}^2}\tilde{n}\left(|\tilde{E}_1|^2+|\tilde{E}_2|^2\right)dx-\eta\int_{\mathbb{R}^2}|\tilde{E}_1|^2|\tilde{E}_2|^2\,dx\\
&\quad+\frac{\eta}{2}\int_{\mathbb{R}^2}\left(\left(\tilde{E}_1\right)^2\left(\overline{\tilde{E}_2}\right)^2+\left(\overline{\tilde{E}_1}\right)^2\left(\tilde{E}_2\right)^2\right)dx\\
&=\frac{1}{\lambda^2(t)}\mathcal{H}(E_1,E_2,n,\mathbf{v})\\
&=\frac{1}{\lambda^2(t)}\mathcal{H}(E_{10},E_{20},n_0,\mathbf{v}_0).
\end{aligned}\tag{2.7}$$

Proof. According to (2.1), direct calculation provides

$$\begin{cases}
\tilde{E}_{1s}=\frac{1}{\lambda^2(t)}E_{1t},\quad \Delta\tilde{E}_1=\frac{1}{\lambda^3(t)}\Delta E_1,\quad \nabla\left|\tilde{E}_1\right|^2=\frac{1}{\lambda^3(t)}\nabla|E_1|^2,\\
\tilde{E}_{2s}=\frac{1}{\lambda^2(t)}E_{2t},\quad \Delta\tilde{E}_2=\frac{1}{\lambda^3(t)}\Delta E_2,\quad \nabla\left|\tilde{E}_2\right|^2=\frac{1}{\lambda^3(t)}\nabla|E_2|^2,\\
\tilde{n}_s=\frac{1}{\lambda^3(t)}n_t,\quad \nabla\tilde{n}=\frac{1}{\lambda^3(t)}\nabla n,\quad \nabla\cdot\tilde{\mathbf{v}}=\frac{1}{\lambda^3(t)}\nabla\cdot\mathbf{v},\quad \tilde{\mathbf{v}}_s=\frac{1}{\lambda^3(t)}\mathbf{v}_t.
\end{cases}\tag{2.8}$$

Taking (2.8) into (1.1) yields the rescaled Zakharov system (2.3). Similarly, we deduces that

$$\begin{aligned}
&\left\|\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(t,s,x)\right\|_{\mathbb{H}_1}^2 \\
&= \left\|\nabla \tilde{E}_1(t,s,x)\right\|_{L^2(\mathbb{R}^2)}^2 + \left\|\nabla \tilde{E}_2(t,s,x)\right\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + \frac{1}{2}\left\|\tilde{n}(t,s,x)\right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2}\left\|\tilde{\mathbf{v}}(t,s,x)\right\|_{L^2(\mathbb{R}^2)}^2 \\
&= \left\|\frac{1}{\lambda^2(t)}\nabla E_1\right\|_{L^2(\mathbb{R}^2)}^2 + \left\|\frac{1}{\lambda^2(t)}\nabla E_2\right\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + \frac{1}{2}\left\|\frac{1}{\lambda^2(t)} n\right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2}\left\|\frac{1}{\lambda^2(t)}\mathbf{v}\right\|_{L^2(\mathbb{R}^2)}^2 \\
&= \frac{1}{\lambda^2(t)}\int_{\mathbb{R}^2}\left(|\nabla E_1|^2 + |\nabla E_2|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|\mathbf{v}|^2\right) \\
&\qquad\qquad \left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right) d\left(\frac{x}{\lambda(t)}\right).
\end{aligned} \tag{2.9}$$

Note that $t + \frac{s}{\lambda(t)} \to T$ as $s \to \lambda(t)(T-t)$, one has

$$\begin{aligned}
\lim_{s\to\lambda(t)(T-t)} \int_{\mathbb{R}^2} &\left(|\nabla E_1|^2 + |\nabla E_2|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|\mathbf{v}|^2\right) \\
&\left(t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)}\right) d\left(\frac{x}{\lambda(t)}\right) = +\infty.
\end{aligned} \tag{2.10}$$

That is,

$$\lim_{s\to\lambda(t)(T-t)} \left\|\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(t,s,x)\right\|_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)}^2 = +\infty, \tag{2.11}$$

this is the estimate (2.4). Furthermore, taking the inner product of (2.3a) with $\overline{\tilde{E}_1}$ and of (2.3b) with $\overline{\tilde{E}_2}$, integrating with respect to the spatial variable x, and taking the imaginary part of the resulting equations yield

$$\begin{aligned}
Im &\int_{\mathbb{R}^2}\left[\frac{i}{\lambda}\tilde{E}_{1s}\cdot\overline{\tilde{E}_1} + \Delta\tilde{E}_1\cdot\overline{\tilde{E}_1} - \tilde{n}\tilde{E}_1\cdot\overline{\tilde{E}_1} + \eta\overline{\tilde{E}_1}\tilde{E}_2\left(\tilde{E}_1\overline{\tilde{E}_2} - \overline{\tilde{E}_1}\tilde{E}_2\right)\right]dx \\
&= Re\int_{\mathbb{R}^2}\frac{1}{\lambda}\tilde{E}_{1s}\cdot\overline{\tilde{E}_1}\,dx - Im\int_{\mathbb{R}^2}\eta\left(\overline{\tilde{E}_1}\right)^2\left(\tilde{E}_2\right)^2 dx \\
&= \frac{1}{2\lambda}\frac{d}{ds}\int_{\mathbb{R}^2}\left|\tilde{E}_1\right|^2 dx - Im\int_{\mathbb{R}^2}\eta\left(\overline{\tilde{E}_1}\right)^2\left(\tilde{E}_2\right)^2 dx \\
&= 0.
\end{aligned} \tag{2.12}$$

That is,

$$\frac{1}{2\lambda}\frac{d}{ds}\int_{\mathbb{R}^2}\left|\tilde{E}_1\right|^2 dx - Im\int_{\mathbb{R}^2}\eta\left(\overline{\tilde{E}_1}\right)^2\left(\tilde{E}_2\right)^2 dx = 0. \tag{2.13}$$

A similar argument gives

$$\frac{1}{2\lambda}\frac{d}{ds}\int_{\mathbb{R}^2}\left|\tilde{E}_2\right|^2 dx - Im\int_{\mathbb{R}^2}\eta\left(\tilde{E}_1\right)^2\left(\overline{\tilde{E}_2}\right)^2 dx = 0. \tag{2.14}$$

(2.13) and (2.14) yield

$$\frac{d}{ds}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)dx=0. \tag{2.15}$$

Here we use the identity: $Im\left(f^2\overline{g}^2+\overline{f}^2g^2\right)=0,\ \forall f,g\in\mathbb{C}$. Hence there holds

$$\begin{aligned}&\left\|\tilde{E}_1(t,s,x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}_2(t,s,x)\right\|^2_{L^2(\mathbb{R}^2)}\\&\quad=\left\|\tilde{E}_1(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}_2(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.16}$$

In addition, from the mass conservation (1.14) it follows that

$$\begin{aligned}&\left\|\tilde{E}_1(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}_2(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}\\&\quad=\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1(t,0,x)\right|^2+\left|\tilde{E}_2(t,0,x)\right|^2\right)dx\\&\quad=\int_{\mathbb{R}^2}\frac{1}{\lambda^2(t)}\left(|E_1|^2+|E_2|^2\right)\left(t,\frac{x}{\lambda(t)}\right)dx\\&\quad=\int_{\mathbb{R}^2}\left(|E_1|^2+|E_2|^2\right)\left(t,\frac{x}{\lambda(t)}\right)d\left(\frac{x}{\lambda(t)}\right)\\&\quad=\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)}.\end{aligned} \tag{2.17}$$

Note that

$$\lambda^2(t)=\int_{\mathbb{R}^2}|\nabla E_1|^2\,dx+\int_{\mathbb{R}^2}|\nabla E_2|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^2}|n|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^2}|\mathbf{v}|^2\,dx, \tag{2.18}$$

one gets

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(|\nabla\tilde{E}_1(t,0,x)|^2+|\nabla\tilde{E}_2(t,0,x)|^2+\frac{1}{2}|\tilde{n}(t,0,x)|^2+\frac{1}{2}|\tilde{\mathbf{v}}(t,0,x)|^2\right)dx\\&=\frac{1}{\lambda^2(t)}\int_{\mathbb{R}^2}\left(|\nabla E_1(t,x)|^2+|\nabla E_2(t,x)|^2+\frac{1}{2}|n(t,x)|^2+\frac{1}{2}|\mathbf{v}(t,x)|^2\right)dx\\&=1.\end{aligned} \tag{2.19}$$

The above arguments imply (2.5) and (2.6).

Finally, taking the inner product of (2.3a) with $\overline{\tilde{E}_{1s}}$ and of (2.3b) with $\overline{\tilde{E}_{2s}}$, integrating with respect to spatial variable x, then taking the real part of the result equations, we have

$$Re\int_{\mathbb{R}^2}\left[\frac{i}{\lambda}\tilde{E}_{1s}\cdot\overline{\tilde{E}_{1s}}+\Delta\tilde{E}_1\cdot\overline{\tilde{E}_{1s}}-\tilde{n}\tilde{E}_1\cdot\overline{\tilde{E}_{1s}}+\eta\overline{\tilde{E}_{1s}}\tilde{E}_2\left(\tilde{E}_1\overline{\tilde{E}_2}-\overline{\tilde{E}_1}\tilde{E}_2\right)\right]dx=0,$$

that is,

$$\begin{aligned}
&Re\int_{\mathbb{R}^2}\frac{i}{\lambda}\left|\tilde{E}_{1s}\right|^2dx-\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^2}\left|\nabla\tilde{E}_1\right|^2dx-\int_{\mathbb{R}^2}\tilde{n}\frac{1}{2}\frac{d}{ds}\left|\tilde{E}_1\right|^2dx\\
&\quad+\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\tilde{E}_2\right|^2\cdot\frac{d}{ds}\left|\tilde{E}_1\right|^2dx-\frac{\eta}{2}Re\int_{\mathbb{R}^2}\left(\tilde{E}_2\right)^2\cdot\frac{d}{ds}\left(\overline{\tilde{E}_1}\right)^2dx\\
&=-\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^2}\left|\nabla\tilde{E}_1\right|^2dx-\frac{1}{2}\int_{\mathbb{R}^2}\tilde{n}\cdot\frac{d}{ds}\left|\tilde{E}_1\right|^2dx\\
&\quad+\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\tilde{E}_2\right|^2\frac{d}{ds}\left|\tilde{E}_1\right|^2dx-\frac{\eta}{2}Re\int_{\mathbb{R}^2}\left(\tilde{E}_2\right)^2\frac{d}{ds}\left(\overline{\tilde{E}_1}\right)^2dx\\
&=0.
\end{aligned}\tag{2.20}$$

Using an argument similar to the one in the derivation of (2.20), one obtains

$$Re\int_{\mathbb{R}^2}\left[\frac{i}{\lambda}\tilde{E}_{2s}\cdot\overline{\tilde{E}_{2s}}+\Delta\tilde{E}_2\cdot\overline{\tilde{E}_{2s}}-\tilde{n}\tilde{E}_2\cdot\overline{\tilde{E}_{2s}}+\eta\overline{\tilde{E}_{2s}}\tilde{E}_1\left(\overline{\tilde{E}_1}\tilde{E}_2-\tilde{E}_1\overline{\tilde{E}_2}\right)\right]dx=0,$$

and hence

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^2}|\nabla\tilde{E}_2|^2dx-\frac{1}{2}\int_{\mathbb{R}^2}\tilde{n}\cdot\frac{d}{ds}|\tilde{E}_2|^2dx\\
&\quad+\frac{\eta}{2}\int_{\mathbb{R}^2}|\tilde{E}_1|^2\frac{d}{ds}|\tilde{E}_2|^2dx-\frac{\eta}{2}Re\int_{\mathbb{R}^2}(\tilde{E}_1)^2\frac{d}{ds}\left(\overline{\tilde{E}_2}\right)^2dx\\
&=0.
\end{aligned}\tag{2.21}$$

Next, taking the inner product of (2.3c) with $\tilde{n}$ yields

$$\int_{\mathbb{R}^2}\left(\tilde{n}_s\tilde{n}+(\nabla\cdot\tilde{\mathbf{v}})\,\tilde{n}\right)dx=\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^2}|\tilde{n}|^2dx-\int_{\mathbb{R}^2}\tilde{\mathbf{v}}\cdot\nabla\tilde{n}dx=0,\tag{2.22}$$

On the other hand, taking the inner product of (2.3d) with $\tilde{v}$ implies

$$\begin{aligned}
\int_{\mathbb{R}^2}\left[\tilde{v}_s\cdot\tilde{\mathbf{v}}+\nabla\left(\tilde{n}+|\tilde{E}_1|^2+|\tilde{E}_2|^2\right)\cdot\tilde{\mathbf{v}}\right]dx\quad&=\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}|^2dx-\int_{\mathbb{R}^2}\tilde{n}\nabla\cdot\tilde{\mathbf{v}}dx\\
&\quad+\int_{\mathbb{R}^2}\tilde{n}_s\left(|\tilde{E}_1|^2+|\tilde{E}_2|^2\right)dx=0.
\end{aligned}\tag{2.23}$$

Combining (2.21) with (2.22) and (2.23) gives

$$\begin{aligned}
&\frac{d}{ds}\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2+\frac{1}{2}\left|\tilde{n}\right|^2+\frac{1}{2}\left|\tilde{\mathbf{v}}\right|^2+\tilde{n}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)\right)dx\\
&\quad-\frac{d}{ds}\int_{\mathbb{R}^2}\left[\eta\left|\tilde{E}_1\right|^2\left|\tilde{E}_2\right|^2+\frac{\eta}{2}\left(\left(\tilde{E}_1\right)^2\left(\overline{\tilde{E}_2}\right)^2+\left(\overline{\tilde{E}_1}\right)^2\left(\tilde{E}_2\right)^2\right)\right]dx=0,
\end{aligned}$$

which implies

$$\mathcal{H}\left(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}}\right)(t,s,x)=\mathcal{H}\left(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}}\right)(t,0,x).\tag{2.24}$$

According to the conservation of Hamiltonian (1.15), we have

$$\begin{aligned}
&\mathcal{H}\left(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}}\right)(t,0,x)\\
&\quad=\left\|\nabla\tilde{E}_1(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\nabla\tilde{E}_2(t,0,x)\right\|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\|\tilde{n}(t,0,x)\|_{L^2(\mathbb{R}^2)}^2+\frac{1}{2}\|\tilde{\mathbf{v}}(t,0,x)\|_{L^2(\mathbb{R}^2)}^2\\
&+\int_{\mathbb{R}^2}\tilde{n}(t,0,x)\left(\left|\tilde{E}_1(t,0,x)\right|^2+\left|\tilde{E}_2(t,0,x)\right|^2\right)dx\\
&-\eta\int_{\mathbb{R}^2}\left|\tilde{E}_1(t,0,x)\right|^2\left|\tilde{E}_2(t,0,x)\right|^2dx\\
&+\frac{\eta}{2}\int_{\mathbb{R}^2}\left(\left(\tilde{E}_1(t,0,x)\right)^2\left(\overline{\tilde{E}_2}(t,0,x)\right)^2\right.\\
&\qquad\qquad\left.+\left(\overline{\tilde{E}_1}(t,0,x)\right)^2\left(\tilde{E}_2(t,0,x)\right)^2\right)dx\\
=&\left\|\frac{1}{\lambda^2(t)}\nabla E_1(t)\right\|_{L^2(\mathbb{R}^2)}^2+\left\|\frac{1}{\lambda^2(t)}\nabla E_2(t)\right\|_{L^2(\mathbb{R}^2)}^2\\
&+\frac{1}{2}\left\|\frac{1}{\lambda^2(t)}n(t)\right\|_{L^2(\mathbb{R}^2)}^2+\frac{1}{2}\left\|\frac{1}{\lambda^2(t)}\mathbf{v}(t)\right\|_{L^2(\mathbb{R}^2)}^2\\
&+\int_{\mathbb{R}^2}\frac{1}{\lambda^4(t)}n(t)\left(|E_1(t)|^2+|E_2(t)|^2\right)dx\\
&-\eta\int_{\mathbb{R}^2}\frac{1}{\lambda^4(t)}|E_1(t)|^2|E_2(t)|^2dx\\
&+\frac{\eta}{2}\int_{\mathbb{R}^2}\frac{1}{\lambda^4(t)}\left((E_1(t))^2\left(\overline{E_2}(t)\right)^2+\left(\overline{E_1}(t)\right)^2(E_2(t))^2\right)dx\\
=&\frac{1}{\lambda^2(t)}\mathcal{H}(E_1,E_2,n,\mathbf{v})(t)\\
=&\frac{1}{\lambda^2(t)}\mathcal{H}(E_{10},E_{20},n_0,\mathbf{v}_0),
\end{aligned}
\tag{2.25}
$$

which is just the estimate (2.7). This finishes the proof of Proposition 2.3. □

3. Estimates for the rescaled Zakharov system (2.3)

In this section, we firstly establish some a priori estimates for the rescaled Zakharov system (2.3) in order to gain the optimal lower bound for the blow-up rate of the finite time blow-up solution to the Zakharov system (1.1). For simplicity, let

$$
\left(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}}\right)(s)\triangleq\left(\tilde{E}_1,\tilde{E}_2,\tilde{n},\tilde{\mathbf{v}}\right)(t,s,x). \tag{3.1}
$$

We claim some a priori estimates for the solution $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(s)$ to the rescaled Zakharov system (2.3).

Theorem 3.1. *(A priori estimates on $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\boldsymbol{v}}\right)(s)$)*
Let $(E_1, E_2, n, \mathbf{v})(t)$ be a solution of the Zakharov system (1.1), $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\boldsymbol{v}}\right)(s)$ be a solution of the rescaled Zakharov system (2.3) and T be the blow-up time. Suppose that the initial mass (E_{10}, E_{20}) satisfies (1.4), where Q is the unique radial positive solution of the equation (1.5), then there are constants $\theta_0 > 0$, and $A > 0$ depending only on the initial data such that for t near T,

$$\forall s \in [0, \theta_0), \ \left\| \left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\boldsymbol{v}}\right)(s) \right\|_{\mathbb{H}_1} \leqslant A. \tag{3.2}$$

Furthermore, we can choose

$$\theta_0 = \tilde{c}\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta} \right)^{-\frac{1}{2}}. \tag{3.3}$$

Remark 3.2. Theorem 3.1 is crucial to show the main result (Theorem 1.3) where the mass condition (1.4) is an essential ingredient. □

Before proving Theorem 3.1, we first establish the geometrical estimates on the solution $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)$ to the rescaled Zakharov system (2.3). These estimates concern Sobolev type estimates for $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(0)$, nonvanishing properties of $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}\right)(0)$ and compactness properties of $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}\right)(0)$. We will consider them with four portions:

♢ 3.1 Sobolev Estimates on $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{\mathbf{v}}(0)\right)$ for t near T;
♢ 3.2 Non-vanishing properties of the solutions to the rescaled Zakharov system as t near T;
♢ 3.3 Compactness of the solution to the rescaled Zakharov system (2.3);
♢ 3.4 Proof of Theorem 3.1.

3.1. Sobolev Estimates on $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{\mathbf{v}}(0)\right)$ for t near T.

The Sobolev estimates on $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{\mathbf{v}}(0)\right)$ is given as follows.

Proposition 3.3. Let $(E_1(t), E_2(t), n(t), \mathbf{v}(t))$ be the finite time blow-up solution to the Cauchy problem (1.1)-(1.2) on $t \in [0, T)$, and T be the blow-up time. Suppose that the initial mass satisfies (1.4), then there are constants

$$\delta_1 > 0, \ c_1 > 0, \ c_2 > 0, \ c_3 > 0 \tag{3.4}$$

depending only on $(E_{10}, E_{20}, n_0, \mathbf{v}_0)$, such that for $t \in [T - \delta_1, T)$, the solution $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(s)$ to the rescaled Zakharov system (2.3) admits

$$0 < c_1 \leqslant \left(\left\| \nabla \tilde{E}_1(0) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \nabla \tilde{E}_2(0) \right\|^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \leqslant c_2, \tag{3.5}$$

$$0 < c_1 \leqslant \|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)} \leqslant c_3, \tag{3.6}$$

$$0 \leqslant \|\tilde{\mathbf{v}}(0)\|_{L^2(\mathbb{R}^2)} \leqslant c_3. \tag{3.7}$$

Proof. From (2.5) it follows that

$$\begin{cases} \left(\left\| \nabla \tilde{E}_1(0) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \nabla \tilde{E}_2(0) \right\|^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \leqslant 1, \\ \|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)} \leqslant \sqrt{2}, \ \|\tilde{\mathbf{v}}(0)\|_{L^2(\mathbb{R}^2)} \leqslant \sqrt{2}. \end{cases} \tag{3.8}$$

Note that $\lambda(t) \to +\infty$ as $t \to T$, by (2.7), there exists $\delta_1 > 0$ such that for any $t \in [T - \delta_1, T)$,

$$\left| \mathcal{H} \left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{\mathbf{v}}(0) \right) \right| = \left| \frac{\mathcal{H}_0}{\lambda^2(t)} \right| \leqslant \frac{1}{64}. \tag{3.9}$$

Hence (2.5) and (2.7) yield

$$\begin{aligned} 1 \ &= \int_{\mathbb{R}^2} |\nabla \tilde{E}_1(0)|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{E}_2(0)|^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{\mathbf{v}}(0)|^2 dx \\ &\leqslant \frac{1}{64} - \int_{\mathbb{R}^2} \tilde{n}(0) \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right) dx \\ &\quad + \frac{\eta}{2} \int_{\mathbb{R}^2} \left| \overline{\tilde{E}_1}(0) \tilde{E}_2(0) - \tilde{E}_1(0) \overline{\tilde{E}_2}(0) \right|^2 dx. \end{aligned} \tag{3.10}$$

Since $\overline{\tilde{E}_1} \tilde{E}_2$ and $\tilde{E}_1 \overline{\tilde{E}_2}$ are conjugate complex-valued functions, we have the following estimate for the quartic term $\int_{\mathbb{R}^2} \left| \overline{\tilde{E}_1}(0) \tilde{E}_2(0) - \tilde{E}_1(0) \overline{\tilde{E}_2}(0) \right|^2 dx$:

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^2} \left| \overline{\tilde{E}_1}(0) \tilde{E}_2(0) - \tilde{E}_1(0) \overline{\tilde{E}_2}(0) \right|^2 dx \\ &\qquad \leqslant 2 \int_{\mathbb{R}^2} |\tilde{E}_1(0)|^2 |\tilde{E}_2(0)|^2 dx \\ &\qquad \leqslant 2 \int_{\mathbb{R}^2} \left(\frac{|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2}{2} \right)^2 dx \\ &\qquad = \frac{1}{2} \int_{\mathbb{R}^2} (|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2)^2 dx. \end{aligned} \tag{3.11}$$

On the other hand, it follows from Hölder's inequality that

$$\begin{aligned} &\int_{\mathbb{R}^2} -\tilde{n}(0) \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right) dx \\ &\quad \leqslant \left(b^2 \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx \right)^{\frac{1}{2}} \left(\frac{1}{b^2} \int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx \right)^{\frac{1}{2}} \\ &\quad \leqslant \frac{b^2}{2} \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx + \frac{1}{2b^2} \int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx. \end{aligned} \tag{3.12}$$

Let $b^2 = \frac{1}{2}$. Combining (3.10) with (3.11) and (3.12) yields

$$\frac{3}{4} \leqslant \frac{1}{4} \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx + \left(\frac{\eta}{2} + 1 \right) \int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx. \tag{3.13}$$

Due to $\|\tilde{n}(0)\|^2_{L^2(\mathbb{R}^2)} \leqslant 2$, (3.13) implies

$$\int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2\right)^2 dx > \frac{1}{4+2\eta}. \tag{3.14}$$

Using Gagliardo-Nirenberg inequality (Lemma 2.2), one gets the estimate for $\int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2\right)^2 dx$:

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2\right)^2 dx \\
&= \int_{\mathbb{R}^2} \left(|\tilde{E}_1(0)|^4 + |\tilde{E}_2(0)|^4\right) dx + 2\int_{\mathbb{R}^2} |\tilde{E}_1(0)|^2 |\tilde{E}_2(0)|^2 dx \\
&\leqslant \frac{2\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} |\nabla \tilde{E}_1(0)|^2 dx + \frac{2\|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} |\nabla \tilde{E}_2(0)|^2 dx \\
&\quad +2\left(\int_{\mathbb{R}^2} |\tilde{E}_1(0)|^4 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\tilde{E}_2(0)|^4 dx\right)^{\frac{1}{2}} \\
&\leqslant \frac{2\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} |\nabla \tilde{E}_1(0)|^2 dx + \frac{2\|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} |\nabla \tilde{E}_2(0)|^2 dx \\
&\quad +4\frac{\|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)} \\
&\leqslant \frac{2}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \\
&\quad \cdot \Big(\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}\Big) \\
&= \frac{2}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \left(\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\qquad \cdot \left(\|\nabla \tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla \tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned} \tag{3.15}$$

This together with (3.14) yields

$$\begin{aligned}
\frac{1}{8+4\eta} \quad &\leqslant \frac{\left\|\tilde{E}_1(0)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{E}_2(0)\right\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \\
&\quad \cdot \left(\left\|\nabla \tilde{E}_1(0)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\nabla \tilde{E}_2(0)\right\|^2_{L^2(\mathbb{R}^2)}\right),
\end{aligned} \tag{3.16}$$

and the conclusion (3.5) follows from (2.5) and the mass identity (2.6).

In the following we prove the conclusions (3.6) and (3.7). In view of condition (1.4), we can assume that there exists a sufficiently small δ_0 with $0 < \delta_0 < \frac{1}{1+\eta}$ such that

$$\|E_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|E_2(0)\|^2_{L^2(\mathbb{R}^2)} \leq \frac{1-\delta_0}{\eta} \|Q\|^2_{L^2(\mathbb{R}^2)}. \tag{3.17}$$

On the other hand, by (3.8) we deduce

$$
\begin{aligned}
&\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{\tilde{E}_1}(0)\tilde{E}_2(0)-\tilde{E}_1(0)\overline{\tilde{E}_2}(0)\right|^2dx\\
&\quad\leqslant 2\eta\int_{\mathbb{R}^2}|\tilde{E}_1(0)|^2|\tilde{E}_2(0)|^2dx\\
&\quad\leqslant 2\eta\left(\int_{\mathbb{R}^2}|\tilde{E}_1(0)|^4dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\tilde{E}_2(0)|^4dx\right)^{\frac12}\\
&\quad\leqslant 4\eta\frac{\|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}\|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}\|\nabla\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}\|\nabla\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \qquad (3.18)\\
&\quad\leqslant \eta\Bigg(\|\nabla\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}\Bigg)\\
&\qquad\cdot\left(\frac{\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}+\|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\right)\\
&\quad\leqslant 1-\delta_0.
\end{aligned}
$$

Combining (3.18) with (2.2), (2.5) and (2.7) gives

$$
\begin{aligned}
\delta_0 &\leqslant -\int_{\mathbb{R}^2}\tilde{n}(0)\left(|\tilde{E}_1(0)|^2+|\tilde{E}_2(0)|^2\right)dx\\
&\leqslant \left(\int_{\mathbb{R}^2}\tilde{n}^2(0)dx\right)^{\frac12}\left(\int_{\mathbb{R}^2}\left(|\tilde{E}_1(0)|^2+|\tilde{E}_2(0)|^2\right)^2dx\right)^{\frac12}.
\end{aligned} \qquad (3.19)
$$

Recalling (3.15), (3.17) yields

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\left(|\tilde{E}_1(0)|^2+|\tilde{E}_2(0)|^2\right)^2dx\\
&\quad\leqslant 2\Bigg(\|\nabla\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}\Bigg)\\
&\qquad\cdot\left(\frac{\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}+\|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\right)\\
&\quad\leq \frac{2-2\delta_0}{\eta},
\end{aligned} \qquad (3.20)
$$

In view of (3.19), for any fixed small $\delta_0>0$, there exists a constant $c_1>0$ such that

$$
c_1\leqslant\frac{\eta}{2}\frac{\delta_0^2}{1-\delta_0}\leqslant\int_{\mathbb{R}^2}|\tilde{n}(0)|^2dx. \qquad (3.21)
$$

Note that for $0<\delta_0<\frac{1}{1+\eta}$ and $\eta>0$, $\frac{\eta}{2}\frac{\delta_0^2}{1-\delta_0}<2$, the upper bounds for $\|\nabla\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}$, $\|\tilde{n}(0)\|^2_{L^2(\mathbb{R}^2)}$ and $\|\tilde{\mathbf{v}}(0)\|^2_{L^2(\mathbb{R}^2)}$ follow from (2.5).

The proof of Proposition 3.3 is completed. □

Due to the condition (2.5) concerning the rescaled Zakharov system (2.3), using Proposition 3.3 and the following scaling properties:

$$
\begin{aligned}
&\left\|\nabla \tilde{E}_j(0)\right\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)} \left\|\nabla E_j(t)\right\|_{L^2(\mathbb{R}^2)}, \ j = 1, 2, \\
&\|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)}\|n(t)\|_{L^2(\mathbb{R}^2)}, \ \|\tilde{\mathbf{v}}(0)\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)}\|\mathbf{v}(t)\|_{L^2(\mathbb{R}^2)},
\end{aligned}
\tag{3.22}
$$

we claim the following Sobolev-type estimates for the solution $(E_1(t), E_2(t), n(t), \mathbf{v}(t))$ to the Zakharov system (1.1).

Corollary 3.4. *Under the assumptions in Proposition 3.3, there exists constants $\delta_1 > 0$, c_1^*, c_2^* depending only on initial data (1.2) such that for $t \in [T - \delta_1, T)$, there hold:*

$$
c_1^*\|n(t)\|_{L^2(\mathbb{R}^2)} \leqslant \left(\|\nabla E_1(t)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2(t)\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \leqslant \frac{1}{c_1^*}\|n(t)\|_{L^2(\mathbb{R}^2)}, \tag{3.23}
$$

$$
\|v(t)\|_{L^2(\mathbb{R}^2)} \leqslant \frac{1}{c_1^*}\|n(t)\|_{L^2(\mathbb{R}^2)}, \tag{3.24}
$$

$$
\begin{aligned}
c_2^*\|n(t)\|_{L^2(\mathbb{R}^2)} \quad &\leqslant \|(E_1, E_2, n, v)(t)\|_{\mathbb{H}_1} \\
&\leqslant \tfrac{1}{c_2^*}\|n(t)\|_{L^2(\mathbb{R}^2)}.
\end{aligned}
\tag{3.25}
$$

Proof. Let $c^* = \max\{c_2, c_3\}$. From Proposition 3.3 it follows that $c_1 < c^*$. Taking $c_1^* = \frac{c_1}{c^*}$ and $c_2^* = \frac{\sqrt{3}}{3}\frac{c_1}{c^*}$ yields the conclusion of Corollary 3.4 due to Proposition 3.3. $\square$

3.2. Non-vanishing Properties of the Solutions to the Re-scaled Zakharov System as t near T.

We now consider the non-vanishing properties of the solution $\left(\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s)\right)$ of the rescaled magnetic Zakharov system (2.3) for $s = 0$, i.e., the non-vanishing properties of $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0)\right)$.

Proposition 3.5. For any $t \in [0, T)$, suppose that $(E_1(t), E_2(t), n(t), \mathbf{v}(t))$ (or $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{\mathbf{v}}(0)\right)$) is the finite time blow-up solution to the Cauchy problem (1.1)-(1.2), the initial data $(E_{10}(x), E_{20}(x))$ satisfies condition (1.4) and T is the blow-up time. Then we claim:

(1) There exist constants $R_1 > 0$ and $\beta_1 > 0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$ and $\|E_{20}\|_{L^2(\mathbb{R}^2)}$ such that for a sequence $x(t) \in \mathbb{R}^2$ one has

$$
\liminf_{t \to T}\left(\left\|\tilde{E}_1(0, x)\right\|^2_{L^2(|x - x(t)| \leqslant R_1)} + \left\|\tilde{E}_2(0, x)\right\|^2_{L^2(|x - x(t)| \leqslant R_1)}\right)^{\frac{1}{2}} \geqslant \beta_1, \tag{3.26}
$$

$$
\liminf_{t \to T}\|\tilde{n}(0, x)\|_{L^2(|x - x(t)| \leqslant R_1)} \geqslant \beta_1. \tag{3.27}
$$

(2) Let $\left(\tilde{E}_{1n}, \tilde{E}_{2n}, \tilde{n}_n\right)(s)$ be a sequence satisfying the following estimates:

$$
\left\|\tilde{E}_{1n}(0)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\tilde{E}_{2n}(0)\right\|^2_{L^2(\mathbb{R}^2)} \leqslant \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)}, \tag{3.28}
$$

$$c_1 \leqslant \int_{\mathbb{R}^2} \left|\nabla \tilde{E}_{1n}(0)\right|^2 dx + \int_{\mathbb{R}^2} \left|\nabla \tilde{E}_{2n}(0)\right|^2 dx \leqslant c_2, \tag{3.29}$$

$$c_1 \leqslant \int_{\mathbb{R}^2} |\tilde{n}_n(0)|^2 dx \leqslant c_2, \tag{3.30}$$

$$\limsup_{t\to T} \mathcal{H}\left(\tilde{E}_{1n}(0), \tilde{E}_{2n}(0), \tilde{n}_n(0), 0\right) \leqslant 0. \tag{3.31}$$

Then there exist $\beta_1 > 0$ and $R_1 > 0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$, $\|E_{20}\|_{L^2(\mathbb{R}^2)}$, $c_1 > 0$ and $c_2 > 0$ such that for a sequence $\{x_n\} \in \mathbb{R}^2$,

$$\lim_{n\to+\infty} \left(\left\|\tilde{E}_{1n}\right\|^2_{L^2(|x-x_n|\leqslant R_1)} + \left\|\tilde{E}_{2n}\right\|^2_{L^2(|x-x_n|\leqslant R_1)}\right)^{\frac{1}{2}} \geqslant \beta_1 > 0, \tag{3.32}$$

$$\lim_{n\to+\infty} \|\tilde{n}_n\|_{L^2(|x-x_n|\leqslant R_1)} \geqslant \beta_1 > 0. \tag{3.33}$$

The proof of this proposition is similar to that of Proposition 3.6 in [16], but the proof details of the treatment of Proposition 3.5 and Proposition 3.6 in [16] are different. In fact, Propositon 3.5 is much more complicated since the higher-order nonlinear terms are essentially involved in (1.1). Proposition 3.5 will be proved step by step later.

We firstly claim:

Proposition 3.6. Assume there is $m_k = m_k\left(\|E_{10}\|_{L^2(\mathbb{R}^2)}, \|E_{20}\|_{L^2(\mathbb{R}^2)}\right) > 0$ such that the sequences $(E_{1k}, E_{2k}, n_k, \mathbf{v}_k) \in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ satisfy

$$\|E_{1k}\|^2_{L^2(\mathbb{R}^2)} + \|E_{2k}\|^2_{L^2(\mathbb{R}^2)} = \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} > 0, \tag{3.34}$$

and there exist constants $R_0 > 0$ and $\delta_0' > 0$ such that

$$\sup_{y\in\mathbb{R}^2} \int_{|x-y|<R_0} \left(|E_{1k}(x)|^2 + |E_{2k}(x)|^2\right) dx \leqslant \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta} - \delta_0', \tag{3.35}$$

or

$$\sup_{y\in\mathbb{R}^2} \int_{|x-y|<R_0} |n_k(x)| dx \leqslant m_k - \delta_0'. \tag{3.36}$$

Then there exist constants $C_1 > 0,\ C_2 > 0$ such that

$$\begin{aligned} -C_1 + C_2 \int_{\mathbb{R}^2} &\left(|\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 + \frac{1}{2}|n_k|^2 + \frac{1}{2}|\mathbf{v}_k|^2\right) dx \\ &\leqslant \mathcal{H}(E_{1k}, E_{2k}, n_k, \mathbf{v}_k), \end{aligned} \tag{3.37}$$

where $\mathcal{H}$ is defined by (1.3).

Proof of Proposition 3.6.

We first define two functionals as follows:

$$\begin{aligned} \mathcal{E}(E_1, E_2) \quad &\triangleq \|\nabla E_1\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{2}\int_{\mathbb{R}^2} \left(|E_1|^2 + |E_2|^2\right)^2 dx \\ &- \frac{\eta}{2}\int_{\mathbb{R}^2} \left|\overline{E_1}E_2 - E_1\overline{E_2}\right|^2 dx, \end{aligned} \tag{3.38}$$

$$\mathcal{H}_1(E_1, E_2, n) \triangleq \mathcal{E}(E_1, E_2) + \frac{1}{2}\int_{\mathbb{R}^2} \left(n + |E_1|^2 + |E_2|^2\right)^2 dx. \tag{3.39}$$

Let

$$\begin{cases} \tilde{E}_{1k}(x) = \dfrac{1}{\lambda_k} E_{1k}\left(\dfrac{x}{\lambda_k}\right), \\ \tilde{E}_{2k}(x) = \frac{1}{\lambda_k} E_{2k}\left(\frac{x}{\lambda_k}\right), \\ \tilde{n}_k(x) = \frac{1}{\lambda_k^2} n_k\left(\frac{x}{\lambda_k}\right), \end{cases} \tag{3.40}$$

where

$$\lambda_k^2 = \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2}\|n_k\|_{L^2(\mathbb{R}^2)}^2. \tag{3.41}$$

We continue the proof of Proposition 3.6 through four steps.

Step 1. A non-vanishing property of $\left(\tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k\right)$

Lemma 3.7. *For the sequences (E_{1k}, E_{2k}, n_k) introduced in Proposition 3.6, assume there is a sequence $(\tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that as $k \to +\infty$, the following estimates hold:*

- $$\mathcal{H}\left(\tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k, 0\right) \leqslant 0, \tag{3.42}$$
- $$\int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2\right) dx \to c_1 > 0, \tag{3.43}$$
- $$\int_{\mathbb{R}^2} \left(|\nabla\tilde{E}_{1k}|^2 + |\nabla\tilde{E}_{2k}|^2\right) dx + \frac{1}{2}\int_{\mathbb{R}^2} |\tilde{n}_k|^2 dx \to c_2 > 0, \tag{3.44}$$
- $$\begin{aligned} &\int_{\mathbb{R}^2} \tilde{n}_k \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2\right) dx - \eta \int_{\mathbb{R}^2} |\tilde{E}_{1k}|^2 |\tilde{E}_{2k}|^2 dx \\ &\quad + \frac{\eta}{2}\int_{\mathbb{R}^2} \left(\left(\tilde{E}_{1k}\right)^2 \left(\overline{\tilde{E}_{2k}}\right)^2 + \left(\overline{\tilde{E}_{1k}}\right)^2 \left(\tilde{E}_{2k}\right)^2\right) dx \to -c_3 < 0. \end{aligned} \tag{3.45}$$

Then there exist a constant $c_4 = c_4(c_1, c_2, c_3) > 0$ and a sequence $x_k \in \mathbb{R}^2$ such that

$$\int_{|x-x_k|<1} |\tilde{n}_k| dx > c_4. \tag{3.46}$$

Proof. By (3.40), we claim that there exists a sequence $x_k \in \mathbb{R}^2$ such that

$$\begin{aligned} &\int_{C_k} -\tilde{n}_k \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2\right) dx + \eta \int_{C_k} \left|\tilde{E}_{1k}\right|^2 \left|\tilde{E}_{2k}\right|^2 dx \\ &\quad - \frac{\eta}{2}\int_{C_k} \left(\left(\tilde{E}_{1k}\right)^2 \left(\overline{\tilde{E}_{2k}}\right)^2 + \left(\overline{\tilde{E}_{1k}}\right)^2 \left(\tilde{E}_{2k}\right)^2\right) dx \\ &\geqslant q \cdot \int_{C_k} \left[\left(|\nabla\tilde{E}_{1k}|^2 + |\nabla\tilde{E}_{12k}|^2\right) + \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2\right) + \frac{1}{2}\left|\tilde{n}_k\right|^2\right] dx, \end{aligned} \tag{3.47}$$

for k large enough, where C_k is the square of center x_k and $q = \dfrac{c_3}{c_0(c_1+c_2)}$ with $c_0 > 1$ is a fixed constant. Otherwise, one would obtain

$$\begin{aligned}
&\int_{C_k} -\tilde{n}_k\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)dx+\eta\int_{C_k}\left|\tilde{E}_{1k}\right|^2\left|\tilde{E}_{2k}\right|^2dx\\
&\quad-\frac{\eta}{2}\int_{\mathbb{R}^2}\left(\left(\tilde{E}_{1k}\right)^2\left(\overline{\tilde{E}_{2k}}\right)^2+\left(\overline{\tilde{E}_{1k}}\right)^2\left(\tilde{E}_{2k}\right)^2\right)dx \qquad (3.48)\\
&< q\cdot\int_{C_k}\left[\left(|\nabla\tilde{E}_{1k}|^2+|\nabla\tilde{E}_{2k}|^2\right)+\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)+\frac{1}{2}|\tilde{n}_k|^2\right]dx.
\end{aligned}$$

Let $k\to+\infty$, (3.48) yields $c_3 < q(c_1+c_2) = \dfrac{c_3}{c_0}$ $(c_0 > 1)$, which is a contradiction.

We now claim the following conclusion.

Conclusion I: There exist constants

$$c_1^* = \frac{2\sqrt{2}q^2}{1+q\eta}\|Q\|_{L^2(\mathbb{R}^2)} > 0,\ c_2^* = \frac{4q^3}{1+q\eta}\|Q\|_{L^2(\mathbb{R}^2)} > 0,$$

$$c_3^* = \varepsilon c_1^* > 0 \text{ with } \varepsilon = \frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta+1}$$

such that

$$\left[\int_{C_k}\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)^2dx\right]^{\frac{1}{2}} \geqslant c_1^* > 0, \qquad (3.49)$$

$$\begin{aligned}
&\int_{C_k} -\tilde{n}_k\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)dx+\eta\int_{C_k}\left|\tilde{E}_{1k}\right|^2\left|\tilde{E}_{2k}\right|^2dx\\
&\quad-\frac{\eta}{2}\int_{C_k}\left(\left(\tilde{E}_{1k}\right)^2\left(\overline{\tilde{E}_{2k}}\right)^2+\left(\overline{\tilde{E}_{1k}}\right)^2\left(\tilde{E}_{2k}\right)^2\right)dx \geqslant c_2^* > 0,
\end{aligned} \qquad (3.50)$$

$$\int_{C_k} -\tilde{n}_k\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)dx \geqslant c_3^* > 0. \qquad (3.51)$$

Proof of Conclusion I.

Lemma 2.2, Cauchy-Schwartz inequality: $2ab \leqslant \dfrac{(a+b)^2}{2}$ $(a,b>0)$ and Young's inequality give

$$\begin{aligned}
&\int_{C_k}\left(\left|\nabla\tilde{E}_{1k}\right|^2+\left|\nabla\tilde{E}_{2k}\right|^2\right)dx+\int_{C_k}\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)dx\\
&\quad\geqslant\sqrt{2}\|Q\|_{L^2(\mathbb{R}^2)}\left[\int_{C_k}\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)^2dx\right]^{\frac{1}{2}}.
\end{aligned} \qquad (3.52)$$

$$\begin{aligned}
&q \cdot \sqrt{2}\|Q\|_{L^2(C_k)} \left[\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)^2 dx\right]^{\frac{1}{2}} + \frac{q}{2}\|\tilde{n}_k\|^2_{L^2(C_k)} \\
&\leqslant \int_{C_k} -\tilde{n}_k \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right) dx + \eta \int_{C_k} \left|\tilde{E}_{1k}\right|^2 \left|\tilde{E}_{2k}\right|^2 dx \\
&\quad -\frac{\eta}{2}\int_{C_k} \left(\left(\tilde{E}_{1k}\right)^2 \left(\overline{\tilde{E}_{2k}}\right)^2 + \left(\overline{\tilde{E}_{1k}}\right)^2 \left(\tilde{E}_{2k}\right)^2\right) dx \qquad (3.53)\\
&\leqslant \frac{q}{2}\int_{C_k} |\tilde{n}_k|^2 dx + \frac{1}{2q}\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)^2 dx \\
&\quad +\frac{\eta}{2}\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)^2 dx.
\end{aligned}$$

That is,

$$\left(\frac{1+\eta q}{2q}\right)\left[\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)^2 dx\right]^{\frac{1}{2}} \geqslant \sqrt{2}q\|Q\|_{L^2(C_k)}. \qquad (3.54)$$

This yields (3.49). Similarly, (3.53) and (3.54) imply

$$\begin{aligned}
&\int_{C_k} -\tilde{n}_k \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right) dx + \eta \int_{C_k} \left|\tilde{E}_{1k}\right|^2 \left|\tilde{E}_{2k}\right|^2 dx \\
&\quad -\frac{\eta}{2}\int_{C_k} \left(\left(\tilde{E}_{1k}\right)^2 \left(\overline{\tilde{E}_{2k}}\right)^2 + \left(\overline{\tilde{E}_{1k}}\right)^2 \left(\tilde{E}_{2k}\right)^2\right) dx \\
&\geqslant q\sqrt{2}\|Q\|_{L^2(C_k)} \left[\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)^2 dx\right]^{\frac{1}{2}} \qquad (3.55)\\
&\geqslant \frac{4q^3\|Q\|^2_{L^2(C_k)}}{1+q\eta} = c_2^* > 0.
\end{aligned}$$

So (3.50) is true.

In the following we prove (3.51). By (3.47), one obtains

$$\begin{aligned}
&q\int_{C_k} \left[\left(\left|\nabla\tilde{E}_{1k}\right|^2 + \left|\nabla\tilde{E}_{2k}\right|^2\right) + \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)\right] dx \\
&\qquad + \frac{q}{2}\int_{C_k} |\tilde{n}_k|^2 dx \qquad (3.56)\\
&\leqslant \frac{\eta}{2}\int_{C_k} \left|\overline{\tilde{E}_{1k}}\tilde{E}_{2k} - \tilde{E}_{1k}\overline{\tilde{E}_{2k}}\right|^2 dx + \int_{C_k} -\tilde{n}_k \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right) dx.
\end{aligned}$$

From Lemma 2.2, it follows

$$\int_{C_k} \left(\left|\nabla\tilde{E}_{1k}\right|^2 + \left|\nabla\tilde{E}_{2k}\right|^2\right) dx \geqslant \frac{\|Q\|^2_{L^2(C_k)} \int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right)^2 dx}{2\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2\right) dx}.$$

Note that

$$\begin{aligned}&\int_{C_k}\left[\left(\left|\nabla\tilde{E}_{1k}\right|^2+\left|\nabla\tilde{E}_{2k}\right|^2\right)+\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)\right]dx\\&\geqslant 2\left(\int_{C_k}\left(\left|\nabla\tilde{E}_{1k}\right|^2+\left|\nabla\tilde{E}_{2k}\right|^2\right)dx\right)^{\frac{1}{2}}\left(\int_{C_k}\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)dx\right)^{\frac{1}{2}},\end{aligned}$$

(3.53),(3.54),(3.55) and (3.56) give

$$\begin{aligned}&\frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{\frac{\eta}{2}+\frac{1}{2q}}\sqrt{2}q\|Q\|_{L^2(C_k)}+\frac{q}{2}\int_{C_k}|\tilde{n}_k|^2\,dx\\&-\frac{\eta}{2}\int_{C_k}\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)^2dx\\&\leqslant\int_{C_k}-\tilde{n}_k\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)dx.\end{aligned}\tag{3.57}$$

Assume that there exists an $\varepsilon>0$ such that

$$\frac{\sqrt{2}q}{\frac{\eta}{2}+\frac{1}{2q}}\|Q\|_{L^2(C_k)}\leqslant\left(\int_{C_k}\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)^2dx\right)^{\frac{1}{2}}\leqslant\frac{\sqrt{2}q\|Q\|_{L^2(C_k)}-\varepsilon}{\frac{\eta}{2}}.\tag{3.58}$$

This yields

$$\eta\leqslant\frac{2\sqrt{2}q\|Q\|_{L^2(C_k)}-2\varepsilon}{\left(\int_{C_k}\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)^2dx\right)^{\frac{1}{2}}}\leqslant\frac{2\sqrt{2}q\|Q\|_{L^2(C_k)}-2\varepsilon}{\sqrt{2}q\|Q\|_{L^2(C_k)}}\left(\frac{\eta}{2}+\frac{1}{2q}\right),\tag{3.59}$$

that is,

$$\begin{aligned}\varepsilon\ &\leqslant\sqrt{2}q\|Q\|_{L^2(C_k)}\left(1-\frac{\eta}{\eta+\frac{1}{q}}\right)\\&=\sqrt{2}q\|Q\|_{L^2(C_k)}\frac{1}{q\eta+1}=\frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta+1}.\end{aligned}\tag{3.60}$$

Taking $\varepsilon=\dfrac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta+1}$ in (3.58), one obtains

$$\begin{aligned}&\sqrt{2}q\|Q\|_{L^2(C_k)}-\frac{\eta}{2}\left(\int_{C_k}\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)^2dx\right)^{\frac{1}{2}}\\&\geqslant\sqrt{2}q\|Q\|_{L^2(C_k)}-\frac{\sqrt{2}q\|Q\|_{L^2(C_k)}-\frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta+1}}{\frac{\eta}{2}}\cdot\frac{\eta}{2}\\&=\frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta+1}.\end{aligned}\tag{3.61}$$

Note that $\eta>0$, then there exists $\varepsilon>0$ small enough such that the following estimate holds:

$$\sqrt{2}q\|Q\|_{L^2(C_k)}-\frac{\eta}{2}\left[\int_{C_k}(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2)^2dx\right]^{\frac{1}{2}}\geqslant\varepsilon>0.\tag{3.62}$$

Combining (3.47), (3.52), (3.53) and (3.62) together yields

$$\begin{aligned}
&\int_{C_k} -\tilde{n}_k \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2 \right) dx \\
&\quad \geqslant \sqrt{2} q \|Q\|_{L^2(C_k)} \left(\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2 \right)^2 dx \right)^{\frac{1}{2}} \\
&\qquad + \frac{q}{2} \|\tilde{n}_k\|^2_{L^2(C_k)} - \frac{\eta}{2} \left(\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2 \right)^2 dx \right) \\
&\quad = \left[\sqrt{2} q \|Q\|_{L^2(C_k)} - \frac{\eta}{2} \left(\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2 \right)^2 dx \right)^{\frac{1}{2}} \right] \\
&\qquad \times \left(\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2 \right)^2 dx \right)^{\frac{1}{2}} + \frac{q}{2} \|\tilde{n}_k\|^2_{L^2(C_k)} \\
&\quad \geqslant \varepsilon \left(\int_{C_k} \left(\left|\tilde{E}_{1k}\right|^2 + \left|\tilde{E}_{2k}\right|^2 \right)^2 dx \right)^{\frac{1}{2}} + \frac{q}{2} \|\tilde{n}_k\|^2_{L^2(C_k)} \\
&\quad \geqslant \varepsilon \frac{2\sqrt{2} q^2 \|Q\|_{L^2(C_k)}}{q\eta + 1} = \varepsilon c_1^* = c_3^* > 0.
\end{aligned} \tag{3.63}$$

This is just the estimate (3.51). Hence Conclusion I follows from (3.54),(3.55) and (3.63). □

We finish the proof of Lemma 3.7 according to Conclusion I by contradiction.

Assume by contradiction that there exists a subsequence (still denoted by $\tilde{n}_k$) such that as $k \to +\infty$,

$$\int_{C_k} |\tilde{n}_k| dx \to 0. \tag{3.64}$$

Let

$$\tilde{n}_k(x_k + \cdot) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2), \tag{3.65}$$

and

$$\left(\tilde{E}_{1k}(x_k + \cdot), \tilde{E}_{2k}(x_k + \cdot) \right) \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2). \tag{3.66}$$

By Sobolev-type estimates, we have

$$\left(\tilde{E}_{1k}(x_k + \cdot), \tilde{E}_{2k}(x_k + \cdot) \right) \to (E_1', E_2') \text{ in } L^4_{loc}(\mathbb{R}^2) \times L^4_{loc}(\mathbb{R}^2), \tag{3.67}$$

and

$$\left(|\tilde{E}_{1k}(x_k + \cdot)|^2, |\tilde{E}_{2k}(x_k + \cdot)|^2 \right) \to \left(|E_1'|^2, |E_2'|^2 \right) \text{ in } L^2_{loc}(\mathbb{R}^2) \times L^2_{loc}(\mathbb{R}^2). \tag{3.68}$$

Indeed, for a bounded open domain C_k in $\mathbb{R}^2$, $H^1(C_k) \subset\subset L^p(C_k),\ p \in [2, +\infty)$, there holds:

$$\begin{aligned}
&\left\| \left|\tilde{E}_{1k}\right|^2 - |E_1'|^2 \right\|_{L^2(C_k)} \\
&\leqslant c \left(\int_{\mathbb{R}^2} \left|\tilde{E}_{1k} - E_1'\right|^4 dx \right)^{\frac{1}{4}} \cdot \left(\int_{\mathbb{R}^2} \left(\left|\tilde{E}_{1k}\right|^2 + |E_1'|^2 \right)^2 dx \right)^{\frac{1}{4}} \\
&\leqslant 2c \left(\int_{\mathbb{R}^2} \left|\tilde{E}_{1k} - E_1'\right|^4 dx \right)^{\frac{1}{4}} \cdot \left(\int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}|^4 + |E_1'|^4 \right) dx \right)^{\frac{1}{4}}.
\end{aligned} \tag{3.69}$$

From (3.64) it follows that as $k \to \infty$,

$$\tilde{n}_k(x_k + \cdot) \rightharpoonup 0 \text{ in } L^2(C_k). \tag{3.70}$$

On the other hand, by Lemma 2.1 there holds as $k \to +\infty$,

$$\begin{aligned}
&\int_{C_k} \tilde{n}_k \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx \\
&= \int_{C_0} \tilde{n}_k \left(x_k + x\right) \left(|\tilde{E}_{1k}\left(x_k + x\right)|^2 + |\tilde{E}_{2k}\left(x_k + x\right)|^2 \right) dx \\
&\to 0,
\end{aligned} \tag{3.71}$$

which is contradictory to (3.63). This completes the proof of Lemma 3.7. □

Step 2: An alternative form for (3.37)

Implementing similar arguments to those in the previous section, we follow from (3.63) that there exists a constant $c_4^* > 0$ such that

$$\int_{C_0} \left(\left|\tilde{E}_{1k}(x_k + x)\right|^2 + \left|\tilde{E}_{2k}(x_k + x)\right|^2 \right) dx \geqslant c_4^* > 0. \tag{3.72}$$

According to the definitions of $\mathcal{H}_1(E_1, E_2, n)$ (see (3.39)) and $\mathcal{H}(E_1, E_2, n, \mathbf{v})$(see (1.3)), for the sake of proving (3.37), it is sufficient to show that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$-C_1 + C_2 \int_{\mathbb{R}^2} \left(|\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 + \frac{1}{2}|n_k|^2 \right) dx \leqslant \mathcal{H}_1\left(E_{1k}, E_{2k}, n_k\right). \tag{3.73}$$

We will verify (3.73) by contradiction.
Assume that (3.73) would not hold for a subsequence (E_{1k}, E_{2k}, n_k). That is, for any constants $C_1 > 0$, $C_2 > 0$, there exists a subsequence (E_{1k}, E_{2k}, n_k) such that

$$-C_1 + C_2 \int_{\mathbb{R}^2} \left(|\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 + \frac{1}{2}|n_k|^2 \right) dx > \mathcal{H}_1\left(E_{1k}, E_{2k}, n_k\right). \tag{3.74}$$

Then the following conclusions would be true provided $k \to +\infty$:

$$\lambda_k^2 := \int_{\mathbb{R}^2} \left(|\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}^2} |n_k|^2 dx \to +\infty, \tag{3.75}$$

$$\limsup_{k \to +\infty} \frac{\mathcal{H}_1(E_{1k}, E_{2k}, n_k)}{\lambda_k^2} \leqslant 0. \tag{3.76}$$

Otherwise,

(1) If $\lambda_k \leqslant C$, then

$$
\begin{aligned}
&|\mathcal{H}_1(E_{1k}, E_{2k}, n_k)| \\
&= \|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} |n_k| \left(|E_{1k}|^2 + |E_{2k}|^2\right) dx \\
&\quad + \frac{\eta}{2} \int_{\mathbb{R}^2} \left|\overline{E}_{1k} E_{2k} - E_{1k} \overline{E}_{2k}\right|^2 dx + \frac{1}{2} \|n_k\|^2_{L^2(\mathbb{R}^2)} \\
&\le \lambda_k^2 + \frac{1}{2}\|n_k\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2} \int_{\mathbb{R}^2} \left(|E_{1k}|^2 + |E_{2k}|^2\right)^2 dx \\
&\qquad + \eta \int_{\mathbb{R}^2} \left(|E_{1k}|^2 + |E_{2k}|^2\right)^2 dx \\
&\leqslant \lambda_k^2 + \frac{1}{2}\|n_k\|^2_{L^2(\mathbb{R}^2)} + (1+2\eta) \frac{\left(\|E_{1k}\|^2_{L^2(\mathbb{R}^2)} + \|E_{2k}\|^2_{L^2(\mathbb{R}^2)}\right)}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \\
&\qquad \cdot \left(\|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\leqslant \lambda_k^2 + \frac{1}{2}\|n_k\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + (1+2\eta)\tfrac{1}{\eta} \left(\|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\leqslant \left(3 + \frac{1}{\eta}\right) \lambda_k^2 \leqslant C.
\end{aligned}
\tag{3.76*}
$$

This implies (3.73), which is contradictory to the assumption (3.74). Thus (3.75) holds true.

(2) If $\lim_{k\to+\infty} \dfrac{\mathcal{H}_1(E_{1k}, E_{2k}, n_k)}{\lambda_k^2} = C > 0$, then for $k_0 > 0$ large enough, there holds

$$
\begin{aligned}
&\mathcal{H}_1(E_{1k}, E_{2k}, n_k) \\
&\quad \geqslant \frac{C}{2}\lambda_k^2 \\
&\quad = \frac{C}{2}\left(\|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\|\nabla n_k\|^2_{L^2(\mathbb{R}^2)}\right),
\end{aligned}
\tag{3.77}
$$

which is a contradiction since (3.74) will be satisfied with $C_1 = 0$ and $C_2 = \frac{C}{2}$.

Step 3: Scaling discussion

The proof continues as follow. Let

$$
\begin{cases}
\tilde{E}_{1k}(x) = \frac{1}{\lambda_k} E_{1k}\left(\frac{x}{\lambda_k}\right), \\
\tilde{E}_{2k}(x) = \frac{1}{\lambda_k} E_{2k}\left(\frac{x}{\lambda_k}\right), \\
\tilde{n}_k(x) = \frac{1}{\lambda_k^2} n_k\left(\frac{x}{\lambda_k}\right).
\end{cases}
\tag{3.78}
$$

A straightforward calculation gives

$$\int_{\mathbb{R}^2}\left(\left|\tilde{E}_{1k}(x)\right|^2+\left|\tilde{E}_{2k}(x)\right|^2\right)dx=\int_{\mathbb{R}^2}\left(|E_{10}(x)|^2+|E_{20}(x)|^2\right)dx, \tag{3.79}$$

$$\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_{1k}(x)\right|^2+\left|\nabla\tilde{E}_{2k}(x)\right|^2\right)dx+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{n}_k(x)|^2dx=1. \tag{3.80}$$

Because of

$$\begin{aligned}\limsup_{k\to\infty}\Bigg(1+\int_{\mathbb{R}^2}\tilde{n}_k\left(\left|\tilde{E}_{1k}\right|^2+\left|\tilde{E}_{2k}\right|^2\right)dx\\ -\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{\tilde{E}_{1k}}\tilde{E}_{2k}-\tilde{E}_{1k}\overline{\tilde{E}_{2k}}\right|^2dx\Bigg)\\ =\limsup_{k\to+\infty}\mathcal{H}_1\left(\tilde{E}_{1k},\tilde{E}_{2k},\tilde{n}_k\right)\leqslant 0,\end{aligned} \tag{3.81}$$

by Hölder's inequality one has

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\tilde{n}_k\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)dx-\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{\tilde{E}_{1k}}\tilde{E}_{2k}-\tilde{E}_{1k}\overline{\tilde{E}_{2k}}\right|^2dx\right|\\ &\leqslant\frac{1}{2}\|\tilde{n}_k\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{2}\int_{\mathbb{R}^2}\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)^2dx\\ &\qquad+\frac{\eta}{2}\int_{\mathbb{R}^2}\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)^2dx\\ &\leqslant\frac{1}{2}\|\tilde{n}_k\|^2_{L^2(\mathbb{R}^2)}+(1+\eta)\frac{\left(\|\tilde{E}_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|\tilde{E}_{2k}\|^2_{L^2(\mathbb{R}^2)}\right)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\\ &\qquad\cdot\left(\|\nabla\tilde{E}_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\tilde{E}_{2k}\|^2_{L^2(\mathbb{R}^2)}\right)\\ &\leqslant C.\end{aligned} \tag{3.82}$$

Hence we can assume by (3.81) and (3.82) that as $k\to+\infty$,

$$\begin{aligned}\int_{\mathbb{R}^2}\tilde{n}_k\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)dx-\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{\tilde{E}_{1k}}\tilde{E}_{2k}-\tilde{E}_{1k}\overline{\tilde{E}_{2k}}\right|^2dx\\ \to c\leqslant -1.\end{aligned} \tag{3.83}$$

On the other hand, recalling (3.35) and (3.75), we have $\forall R>0$,

$$\liminf_{k\to+\infty}\left(\sup_y\int_{|x-y|<R}\left(|\tilde{E}_{1k}(x)|^2+|\tilde{E}_{2k}(x)|^2\right)dx\right)\leqslant\frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}-\delta_0', \tag{3.84}$$

or as $R\to 0$,

$$\liminf_{k\to+\infty}\left(\sup_y\int_{|x-y|<R}|\tilde{n}_k|dx\right)\to 0. \tag{3.85}$$

Because of Lemma 3.7, the alternative (3.85) does not hold. Therefore we need to consider case (3.84) only.

Step 4: Proof of Proposition 3.6

Recalling the definitions of $\mathcal{E}$ (see (3.38)) and $\mathcal{H}_1$ (see (3.39)), there hold

$$\mathcal{H}_1\left(\tilde{E}_{1k},\tilde{E}_{2k},\tilde{n}_k\right)=\mathcal{E}\left(\tilde{E}_{1k},\tilde{E}_{2k}\right)+\frac{1}{2}\int_{\mathbb{R}^2}\left(\tilde{n}_k+|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)^2dx, \tag{3.86}$$

and

$$\limsup_{k\to+\infty}\mathcal{E}\left(\tilde{E}_{1k},\tilde{E}_{2k}\right)\leqslant\limsup_{k\to+\infty}\mathcal{H}_1\left(\tilde{E}_{1k},\tilde{E}_{2k},\tilde{n}_k\right)\leqslant 0. \tag{3.87}$$

Then by (3.79), (3.80) and Sobolev type estimates, we conclude that there exist $c_1>0$ and $c_2>0$ such that

$$c_1\leqslant\int_{\mathbb{R}^2}\left(\left|\tilde{E}_{1k}(x)\right|^2+\left|\tilde{E}_{2k}(x)\right|^2\right)^2dx\leqslant c_2, \tag{3.88}$$

$$c_1\leqslant\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_{1k}(x)\right|^2+\left|\nabla\tilde{E}_{2k}(x)\right|^2+\left|\tilde{E}_{1k}(x)\right|^2+\left|\tilde{E}_{2k}(x)\right|^2\right)dx\leqslant c_2. \tag{3.89}$$

Hence there exist a constant $\delta_1>0$ and a sequence $x_k^1\in\mathbb{R}^2$ such that

$$\int_{|x-x_k^1|<1}\left(\left|\tilde{E}_{1k}(x)\right|^2+\left|\tilde{E}_{2k}(x)\right|^2\right)dx\geqslant\delta_1. \tag{3.90}$$

In view of Lemma 3.7 and its proof, we introduce the following dichotomy

$$\begin{cases}\tilde{E}_{1k}(x)=\tilde{E}_{1k}^1(x)+\tilde{E}_{1k}^{1,R}(x),\\ \tilde{E}_{2k}(x)=\tilde{E}_{2k}^1(x)+\tilde{E}_{2k}^{1,R}(x).\end{cases} \tag{3.91}$$

Hence, for a sequence x_k^1,

$$\left(\tilde{E}_{1k}^1(x+x_k^1),\tilde{E}_{2k}^1(x+x_k^1)\right)\rightharpoonup(\psi_1,\psi_2)\text{ in }H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2), \tag{3.92}$$

and

$$\left(\int_{|x-x_k^1|<1}\left(|\tilde{E}_{1k}^1(x+x_k^1)|^2+|\tilde{E}_{2k}^1(x+x_k^1)|^2\right)^2dx\right)^{\frac14}\geqslant c>0. \tag{3.93}$$

By Sobolev estimates, there exists a constant $\delta_1>0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$ and $\|E_{20}\|_{L^2(\mathbb{R}^2)}$ such that

$$\left\|\tilde{E}_{1k}^1(x_k^1+\cdot)\right\|_{L^2(|x-x_k^1|<1)}^2+\left\|\tilde{E}_{2k}^1(x_k^1+\cdot)\right\|_{L^2(|x-x_k^1|<1)}^2\geq\delta_1>0. \tag{3.94}$$

Recalling (3.84), we also obtain for any $R>0$,

$$\liminf_{k\to+\infty}\left(\|\tilde{E}_{1k}^1(x_k^1+\cdot)\|_{L^2(B_R)}^2+\|\tilde{E}_{2k}^1(x_k^1+\cdot)\|_{L^2(B_R)}^2\right)\leqslant\frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1+\eta}-\delta_0'. \tag{3.95}$$

Furthermore, using the concentration compactness method (Lions [14]), one gets for a suitable choice for $\left(\tilde{E}_{1k},\tilde{E}_{2k}\right)$,

$$\begin{aligned}\left\|\tilde{E}_{1k}^1\right\|_{L^2(\mathbb{R}^2)}^2+\left\|\tilde{E}_{1k}^{1,R}\right\|_{L^2(\mathbb{R}^2)}^2+\left\|\tilde{E}_{2k}^1\right\|_{L^2(\mathbb{R}^2)}^2+\left\|\tilde{E}_{2k}^{1,R}\right\|_{L^2(\mathbb{R}^2)}^2&\\ \to\|E_{10}\|_{L^2(\mathbb{R}^2)}^2+\|E_{20}\|_{L^2(\mathbb{R}^2)}^2,&\end{aligned} \tag{3.96}$$

$$\delta_1\leqslant\lim_{k\to+\infty}\left(\|\tilde{E}_{1k}^1(x)\|_{L^2(\mathbb{R}^2)}^2+\|\tilde{E}_{2k}^1(x)\|_{L^2(\mathbb{R}^2)}^2\right)\leqslant\frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1+\eta}-\delta_0', \tag{3.97}$$

and

$$\begin{aligned}\mathcal{E}(\psi_1,\psi_2) &\leqslant \limsup_{k\to+\infty}\mathcal{E}\left(\tilde{E}^1_{1k},\tilde{E}^1_{2k}\right)+\limsup_{k\to+\infty}\mathcal{E}\left(\tilde{E}^{1,R}_{1k},\tilde{E}^{1,R}_{2k}\right)\\ &\leqslant \limsup_{k\to+\infty}\mathcal{E}\left(\tilde{E}_{1k},\tilde{E}_{2k}\right)\leqslant 0.\end{aligned} \tag{3.98}$$

Since

$$\delta_1 \leqslant \|\psi_1\|^2_{L^2(\mathbb{R}^2)}+\|\psi_2\|^2_{L^2(\mathbb{R}^2)} \leqslant \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}-\delta_0', \tag{3.99}$$

we have

$$\limsup_{k\to+\infty}\mathcal{E}\left(\tilde{E}^{1,R}_{1k},\tilde{E}^{1,R}_{2k}\right)\leq -\mathcal{E}\left(\psi_1,\psi_2\right)<0. \tag{3.100}$$

We now can extract a subsequence which is still denoted by $\left(\tilde{E}^{1,R}_{1k},\tilde{E}^{1,R}_{2k}\right)$ such that

$$\left\|\tilde{E}^{1,R}_{1k}(x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}^{1,R}_{2k}(x)\right\|^2_{L^2(\mathbb{R}^2)}\to c_1<\frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}-\delta_1', \tag{3.101}$$

$$\limsup_{k\to+\infty}\mathcal{E}\left(\tilde{E}^{1,R}_{1k},\tilde{E}^{1,R}_{2k}\right)\leqslant -\mathcal{E}(\psi_1,\psi_2)<0. \tag{3.102}$$

Then there exists a constant $k_0>0$ such that $\forall k\geqslant k_0$,

$$\mathcal{E}\left(\tilde{E}^{1,R}_{1k},\tilde{E}^{1,R}_{2k}\right)\leqslant \frac{-\mathcal{E}(\psi_1,\psi_2)}{2}<0. \tag{3.103}$$

Note that

$$\left\|\tilde{E}^{1,R}_{1k}(x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}^{1,R}_{2k}(x)\right\|^2_{L^2(\mathbb{R}^2)}\leqslant \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}, \tag{3.104}$$

then

$$\begin{aligned}\mathcal{E}\left(\tilde{E}^{1,R}_{1k},\tilde{E}^{1,R}_{2k}\right) &= \int_{\mathbb{R}^2}\left(|\nabla\tilde{E}^{1,R}_{1k}|^2+|\nabla\tilde{E}^{1,R}_{2k}|^2\right)dx\\ &\quad -\frac{1}{2}\int_{\mathbb{R}^2}\left(|\tilde{E}^{1,R}_{1k}|^2+|\tilde{E}^{1,R}_{2k}|^2\right)^2dx\\ &\quad -\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{\tilde{E}^{1,R}_{1k}}\tilde{E}^{1,R}_{2k}-\tilde{E}^{1,R}_{1k}\overline{\tilde{E}^{1,R}_{2k}}\right|^2dx\\ &\geqslant \int_{\mathbb{R}^2}\left(|\nabla\tilde{E}^{1,R}_{1k}|^2+|\nabla\tilde{E}^{1,R}_{1k}|^2\right)dx\\ &\quad -\frac{(1+\eta)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\left(\|\tilde{E}^{1,R}_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|\tilde{E}^{1,R}_{2k}\|^2_{L^2(\mathbb{R}^2)}\right)\\ &\qquad\cdot\int_{\mathbb{R}^2}\left(|\nabla\tilde{E}^{1,R}_{1k}|^2+|\nabla\tilde{E}^{1,R}_{2k}|^2\right)dx\\ &\geqslant 0,\end{aligned} \tag{3.105}$$

which is contradictory to (3.103), hence (3.104) does not hold. Therefore, we claim:
(2)

$$\left\|\tilde{E}^{1,R}_{1k}(x)\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\tilde{E}^{1,R}_{2k}(x)\right\|^2_{L^2(\mathbb{R}^2)}>\frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}. \tag{3.106}$$

In view of (3.103), there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}^{1,R}|^2 + |\tilde{E}_{2k}^{1,R}|^2\right)^2 dx > c. \tag{3.107}$$

Indeed, by $\mathcal{E}\left(\tilde{E}_{1k}^{1,R}, \tilde{E}_{2k}^{1,R}\right) < 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}^{1,R}|^2 + |\tilde{E}_{2k}^{1,R}|^2\right)^2 dx \\
&\quad > 2\int_{\mathbb{R}^2} \left(|\nabla \tilde{E}_{1k}^{1,R}|^2 + |\nabla \tilde{E}_{2k}^{1,R}|^2\right) dx - \eta \int_{\mathbb{R}^2} \left|\overline{\tilde{E}_{1k}^{1,R}} \tilde{E}_{2k}^{1,R} - \tilde{E}_{1k}^{1,R} \overline{\tilde{E}_{2k}^{1,R}}\right|^2 dx \\
&\quad > 2\int_{\mathbb{R}^2} \left(|\nabla \tilde{E}_{1k}^{1,R}|^2 + |\nabla \tilde{E}_{2k}^{1,R}|^2\right) dx - \eta \int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}^{1,R}|^2 + |\tilde{E}_{2k}^{1,R}|^2\right)^2 dx.
\end{aligned}$$

This yields the estimate (3.107).
We then iterate the same procedure as above and define

$$\begin{cases} \tilde{E}_{1k}^{1,R} = \tilde{E}_{1k}^{2} + \tilde{E}_{1k}^{2,R}, \\ \tilde{E}_{2k}^{1,R} = \tilde{E}_{2k}^{2} + \tilde{E}_{2k}^{2,R}, \end{cases} \tag{3.108}$$

where $\tilde{E}_{1k}^2$ and $\tilde{E}_{2k}^2$ satisfy for a sequence x_k^2,

$$\left\|\tilde{E}_{1k}^2(x_k^2 + \cdot)\right\|_{L^2(|x - x_k^2| < 1)}^2 + \left\|\tilde{E}_{2k}^2(x_k^2 + \cdot)\right\|_{L^2(|x - x_k^2| < 1)}^2 \geq \delta_1. \tag{3.109}$$

Defining p such that

$$-p\delta_1 + \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 < \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1+\eta}, \tag{3.110}$$

applying the same procedure at most p times, we find for an $i \leq p$ and k large, there exists a function $\left(\tilde{E}_{1k}^{i,R}, \tilde{E}_{2k}^{i,R}\right)$ such that

$$\left\|\tilde{E}_{1k}^{i,R}\right\|_{L^2(\mathbb{R}^2)}^2 + \left\|\tilde{E}_{2k}^{i,R}\right\|_{L^2(\mathbb{R}^2)}^2 \leqslant \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1+\eta}, \tag{3.111}$$

and

$$\mathcal{E}\left(\tilde{E}_{1k}^{i,R}, \tilde{E}_{2k}^{i,R}\right) \leqslant \frac{-\mathcal{E}(\psi_1, \psi_2)}{2} < 0. \tag{3.112}$$

Then by Lemma 2.2, (3.111) and (3.112) are contradictory.

In addition, from (3.4) one gets

$$\begin{aligned}
&\left\|\nabla \tilde{E}_{1k}\right\|_{L^2(\mathbb{R}^2)}^2 + \left\|\nabla \tilde{E}_{2k}\right\|_{L^2(\mathbb{R}^2)}^2 \\
&\qquad \geqslant \frac{\int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2\right)^2 dx \cdot \|Q\|_{L^2(\mathbb{R}^2)}^2}{2\left(\|\tilde{E}_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{E}_{2k}\|_{L^2(\mathbb{R}^2)}^2\right)} \\
&\qquad > \frac{\eta}{2} \int_{\mathbb{R}^2} \left(|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2\right)^2 dx \\
&\qquad \geqslant \frac{\eta}{2} \int_{\mathbb{R}^2} \left|\overline{\tilde{E}_{1k}} \tilde{E}_{2k} - \tilde{E}_{1k} \overline{\tilde{E}_{2k}}\right|^2 dx.
\end{aligned} \tag{3.113}$$

Since

$$\limsup_{k\to+\infty}\mathcal{H}_1\left(\tilde{E}_{1k},\tilde{E}_{1k},\tilde{E}_{2k},\tilde{n}_k\right)=\limsup_{k\to+\infty}\frac{\mathcal{H}_1\left(\tilde{E}_{1k},\tilde{E}_{1k},\tilde{E}_{2k},\tilde{n}_k\right)}{\lambda_k^2}\leq 0, \tag{3.114}$$

then

$$\int_{\mathbb{R}^2}\tilde{n}_k\left(|\tilde{E}_{1k}|^2+|\tilde{E}_{2k}|^2\right)dx\to -C<0. \tag{3.115}$$

It follows from Lemma 3.7 that there exist a constant $C'>0$ and a sequence x_k such that

$$\int_{|x-x_k|<1}|\tilde{n}_k|dx>C'>0. \tag{3.116}$$

Therefore, recalling (3.36) and the definition of $\tilde{n}_k$, we have as $R\to 0$,

$$\liminf_{k\to+\infty}\left(\sup_y\int_{|x-y|<R}|\tilde{n}_k|dx\right)\to 0, \tag{3.117}$$

which is contradictory to (3.116), thus the proof of Proposition 3.6 is complete. □

We now claim the following conclusions to prove the non-vanishing properties of $\left(\tilde{E}_1(0),\tilde{E}_2(0),\tilde{n}(0)\right)$.

Proposition 3.8. Let $(E_1,E_2,n,\mathbf{v})$ be the finite time blow-up solution of the Zakharov system (1.1) and T be its blowup time. That is, as $t\to T$,

$$\|E_1\|_{H^1(\mathbb{R}^2)}+\|E_2\|_{H^1(\mathbb{R}^2)}+\|n\|_{L^2(\mathbb{R}^2)}+\|\mathbf{v}\|_{L^2(\mathbb{R}^2)}\to+\infty. \tag{3.118}$$

Assume that the initial data satisfy (3.4), then

(1) If E_1,E_2,n are radially symmetric functions, then there exists a constant $m>0$ such that for any $R>0$,

$$\liminf_{t\to T}\left(\|E_1(t,x)\|^2_{L^2(B(0,R))}+\|E_2(t,x)\|^2_{L^2(B(0,R))}\right)>\frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)}, \tag{3.119}$$

$$\liminf_{t\to T}\|n(t,x)\|_{L^1(B(0,R))}\geqslant m. \tag{3.120}$$

(2) If E_1,E_2,n are non-radially symmetric functions, then there exists a sequence $x(t)\in\mathbb{R}^2$ and a constant $m>0$ depending only on initial data such that for any $R>0$,

$$\begin{aligned}\liminf_{t\to T}&\left(\|E_1(t,x)\|^2_{L^2(B(x(t),R))}+\|E_2(t,x)\|^2_{L^2(B(x(t),R))}\right)\\&>\frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)},\end{aligned} \tag{3.121}$$

$$\liminf_{t\to T}\|n(t,x)\|_{L^1(B(x(t),R))}\geqslant m. \tag{3.122}$$

Proof. We first show the case (1): $(E_1,E_2,n)\in H^1_r(\mathbb{R}^2)\times H^1_r(\mathbb{R}^2)\times L^2_r(\mathbb{R}^2)$.

Define two spaces:

$$H^1_r(\mathbb{R}^2)=\left\{f\in H^1(\mathbb{R}^2),f(x)=f(|x|)\right\},$$
$$L^2_r(\mathbb{R}^2)=\left\{f\in L^2(\mathbb{R}^2),f(x)=f(|x|)\right\}.$$

It follows from (1.3),(3.38) and (3.39) that

$$\mathcal{E}(E_1,E_2)\leqslant\mathcal{H}_1(E_1,E_2,n,\mathbf{v})=\mathcal{H}(E_1,E_2,n,\mathbf{v})-\frac{1}{2}\|\mathbf{v}\|^2_{L^2(\mathbb{R}^2)}. \tag{3.123}$$

We proceed our proof by contradiction.
Assume that there exist constants $\delta_0 > 0$, $R_0 > 0$ and a sequence $t_k \to T$ $(k \to +\infty)$ such that

$$\int_{|x|<R_0} \left(|E_1(t_k,x)|^2 + |E_2(t_k,x)|^2\right) dx \leqslant \frac{1}{(1+\eta)} \|Q\|^2_{L^2(\mathbb{R}^2)} - \delta_0, \tag{3.124}$$

or

$$\liminf_{k\to+\infty} \int_{|x|<R_0} |n\,(t_k,x)\,|dx = 0. \tag{3.125}$$

We then complete the proof of case (1) by scaling and compactness.
Let

$$\begin{cases} E_{1k}(x) = \dfrac{1}{\lambda_k} E_1\left(t_k, \dfrac{x}{\lambda_k}\right), \quad E_{2k}(x) = \dfrac{1}{\lambda_k} E_2\left((t_k, \dfrac{x}{\lambda_k}\right), \\ n_k(x) = \dfrac{1}{\lambda_k^2} n\left(t_k, \dfrac{x}{\lambda_k}\right), \quad v_k(x) = \dfrac{1}{\lambda_k^2} \mathbf{v}\left(t_k, \dfrac{x}{\lambda_k}\right), \end{cases} \tag{3.126}$$

where

$$\lambda_k^2 = \|\nabla E_1(t_k,x)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2(t_k,x)\|^2_{L^2(\mathbb{R}^2)}.$$

Direct calculation gives

$$\begin{cases} \displaystyle\int_{\mathbb{R}^2} |\nabla E_{1k}|^2 dx + \int_{\mathbb{R}^2} |\nabla E_{2k}|^2 dx = 1, \\ \displaystyle\int_{\mathbb{R}^2} |E_{1k}|^2 dx + \int_{\mathbb{R}^2} |E_{2k}|^2 dx = \int_{\mathbb{R}^2} |E_{10}|^2 dx + \int_{\mathbb{R}^2} |E_{20}|^2 dx, \\ \mathcal{E}\left(E_{1k}, E_{2k}\right) = \dfrac{1}{\lambda_k^2} \mathcal{E}\left(E_1\left(t_k,x\right), E_2\left(t_k,x\right)\right), \\ \mathcal{H}_1\left(E_{1k}, E_{2k}, n_k\right) = \dfrac{1}{\lambda_k^2} \mathcal{H}_1\left(E_1\left(t_k,x\right), E_2\left(t_k,x\right), n\left(t_k,x\right)\right), \\ \mathcal{H}(t_k) = \mathcal{H}(0). \end{cases} \tag{3.127}$$

Note that

$$\begin{aligned} \mathcal{H}(t_k) = \quad & \mathcal{E}\left(E_1(t_k,x), E_2(t_k,x)\right) + \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{v}(t_k)|^2 dx \\ & + \frac{1}{2}\int_{\mathbb{R}^2} \left[n(t_k) + \left(|E_1(t_k)|^2 + |E_2(t_k)|^2\right)\right]^2 dx. \end{aligned} \tag{3.128}$$

We then conclude

$$\mathcal{E}\left(E_1(t_k), E_2(t_k)\right) \leqslant \mathcal{H}_1\left(E_1(t_k), E_2(t_k), n(t_k)\right) \leqslant \mathcal{H}(t_k) = \mathcal{H}(0), \tag{3.129}$$

$$\mathcal{E}\left(E_{1k}, E_{2k}\right) \leqslant \mathcal{H}_1\left(E_{1k}, E_{2k}, n_k, \mathbf{v}_k\right) \leqslant \frac{1}{\lambda_k^2}\mathcal{H}(0) \overset{k\to+\infty}{\longrightarrow} 0. \tag{3.130}$$

Especially,

$$\limsup_{k\to+\infty} \mathcal{E}\left(E_{1k}, E_{2k}\right) \leqslant 0, \ \limsup_{k\to+\infty} \mathcal{H}_1\left(E_{1k}, E_{2k}, n_k\right) \leqslant 0. \tag{3.131}$$

Using Cauchy-Schwartz inequality: $ab \leq \dfrac{(a+b)^2}{4}$ $(a>0,\ b>0)$, we obtain

$$\begin{aligned}
&\mathcal{E}(E_1, E_2)\\
&\quad\geqslant \|\nabla E_1\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2\|^2_{L^2(\mathbb{R}^2)}\\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^2}\left(|E_1|^2+|E_2|^2\right)^2 dx - 2\eta\int_{\mathbb{R}^2}|E_1|^2|E_2|^2 dx\\
&\quad\geqslant \|\nabla E_1\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2\|^2_{L^2(\mathbb{R}^2)} - \frac{1+\eta}{2}\int_{\mathbb{R}^2}\left(|E_1|^2+|E_2|^2\right)^2 dx.
\end{aligned} \tag{3.132}$$

Together with (3.127) and (3.131), this yields

$$\begin{aligned}
&\liminf_{k\to+\infty}\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2 dx\\
&\quad\geqslant \frac{2}{1+\eta}\liminf_{k\to+\infty}\left(\int_{\mathbb{R}^2}\left(|\nabla E_{1k}|^2+|\nabla E_{2k}|^2\right)dx - \mathcal{E}\left(E_{1k},E_{2k}\right)\right)\\
&\quad\geqslant \frac{2}{1+\eta}.
\end{aligned} \tag{3.133}$$

On the other hand, since

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(n_k+\left(|E_{1k}|^2+|E_{2k}|^2\right)\right)^2 dx - (1+\eta)\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2 dx\\
&\quad\leqslant \int_{\mathbb{R}^2}\left(n_k+\left(|E_{1k}|^2+|E_{2k}|^2\right)\right)^2 dx - \int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2 dx\\
&\qquad -\eta\int_{\mathbb{R}^2}\left|\overline{E_{1k}}E_{2k}-E_{1k}\overline{E_{2k}}\right|^2 dx\\
&\quad = 2\left(\mathcal{H}_1\left(E_{1k},E_{2k},n_k\right) - \|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)} - \|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)}\right),
\end{aligned}$$

one has

$$\begin{aligned}
\limsup_{k\to+\infty}&\int_{\mathbb{R}^2}\left(n_k+\left(|E_{1k}|^2+|E_{2k}|^2\right)\right)^2 dx\\
&-(1+\eta)\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2 dx \leqslant -2.
\end{aligned} \tag{3.134}$$

By Lemma 2.2, (3.4), (3.127) and

$$\begin{aligned}
\eta\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2 dx \quad &\leqslant \frac{2\eta\left(\|E_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|E_{2k}\|^2_{L^2(\mathbb{R}^2)}\right)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\\
&\quad\cdot\left(\|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\leqslant 2,
\end{aligned}$$

we get

$$\limsup_{k\to+\infty}\int_{\mathbb{R}^2}\left(n_k+\left(|E_{1k}|^2+|E_{2k}|^2\right)\right)^2 dx - \int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2 dx \leqslant 0.$$

In view of (3.38), one obtains

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^2}\left[\left(|E_1|^2+|E_2|^2\right)^2+\frac{\eta}{2}\left|\overline{E_1}E_2-E_1\overline{E_2}\right|^2\right]dx\\&=\|\nabla E_1\|^2_{L^2(\mathbb{R}^2)}+\|\nabla E_2\|^2_{L^2(\mathbb{R}^2)}-\mathcal{E}(E_1,E_2).\end{aligned}$$

Hence there holds

$$\begin{aligned}&\liminf_{k\to+\infty}\left(\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2dx+\eta\int_{\mathbb{R}^2}\left|\overline{E_{1k}}E_{2k}-E_{1k}\overline{E_{2k}}\right|^2dx\right)\\&=2\liminf_{k\to+\infty}\left(\|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)}-\mathcal{E}\left(E_{1k},E_{2k}\right)\right)\\&\geqslant 2.\end{aligned}$$

In addition, (3.39) yields

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^2}n_k^2dx\\&=\mathcal{H}_1\left(E_{1k},E_{2k},n_k\right)-\|\nabla E_{1k}\|^2_{L^2(\mathbb{R}^2)}-\|\nabla E_{2k}\|^2_{L^2(\mathbb{R}^2)}\\&\quad-\int_{\mathbb{R}^2}n_k\left(|E_{1k}|^2+|E_{2k}|^2\right)dx+\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{E_{1k}}E_{2k}-E_{1k}\overline{E_{2k}}\right|^2dx.\end{aligned}$$

Now for any $\epsilon\in(0,1)$, by Young's inequality, Hölder's inequality and Cauchy -Schwartz inequality: $ab\leq\dfrac{(a+b)^2}{4}$, $\forall\ a,b>0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}n_k^2dx\quad&\leqslant 2\int_{\mathbb{R}^2}-n_k\left(|E_{1k}|^2+|E_{2k}|^2\right)dx\\&\quad+\eta\int_{\mathbb{R}^2}\left|\overline{E_{1k}}E_{2k}-E_{1k}\overline{E_{2k}}\right|^2dx\\&\leqslant\epsilon\|n_k\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{\epsilon}\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2dx\\&\quad+\eta\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2dx.\end{aligned}$$

This yields that

$$\begin{aligned}\limsup_{k\to+\infty}\int_{\mathbb{R}^2}n_k^2dx\quad&\leqslant\frac{1+\eta\epsilon}{(1-\epsilon)\epsilon}\int_{\mathbb{R}^2}\left(|E_{1k}|^2+|E_{2k}|^2\right)^2dx\\&\leqslant\frac{2+2\eta\epsilon}{(1-\epsilon)\epsilon}\cdot\frac{\|E_{1k}\|^2_{L^2(\mathbb{R}^2)}+\|E_{2k}\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}}.\end{aligned}\tag{3.135}$$

Due to (3.124), (3.125) and $\lambda_k\stackrel{k\to+\infty}{\longrightarrow}+\infty$, one obtains $\forall R>0$,

$$\limsup_{k\to+\infty}\int_{|x|<R}\left(|E_{1k}|^2+|E_{2k}|^2\right)dx\leqslant\frac{1}{1+\eta}\|Q\|^2_{L^2}-\delta_0,\tag{3.136}$$

or

$$\forall R>0,\ \limsup_{k\to+\infty}\int_{|x|<R}|n_k|dx=0.\tag{3.137}$$

Our discussion is continued by the compactness argument.
By (3.127) and (3.135), there exists $(E_1', E_2', N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that

$$(E_{1k}, E_{2k}) \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2), \tag{3.138}$$

$$n_k \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2). \tag{3.139}$$

Since the embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L_r^p(\mathbb{R}^2)$ $(2 < p < +\infty)$ is compact, we obtain

$$(E_{1k}, E_{2k}) \rightharpoonup (E_1', E_2') \text{ in } L^4(\mathbb{R}^2) \times L^4(\mathbb{R}^2). \tag{3.140}$$

On one hand,

$$\left(E_{1k}^2, E_{2k}^2\right) \rightharpoonup \left(E_1'^2, E_2'^2\right) \text{ in } L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2). \tag{3.141}$$

On the other hand,

$$\int_{\mathbb{R}^2} \left(|E_1'|^2 + |E_2'|^2\right)^2 dx \geqslant \frac{1}{1+\eta}, \quad (E_1', E_2') \not\equiv (0,0). \tag{3.142}$$

Let $R \to +\infty$, it follows from (3.136) that

$$\int_{\mathbb{R}^2} \left(|E_1'|^2 + |E_2'|^2\right) dx < \frac{1}{1+\eta} \|Q\|_{L^2(\mathbb{R}^2)}^2, \tag{3.143}$$

or

$$N' = 0. \tag{3.144}$$

The boundedness of weakly convergent sequence implies

$$\begin{aligned} \int_{|x|<R} \left(|E_1'|^2 + |E_2'|^2\right) dx &\leqslant \liminf_{k \to +\infty} \int_{|x|<R} \left(|E_{1k}|^2 + |E_{2k}|^2\right) dx \\ &\leqslant \frac{1}{1+\eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 - \delta_0, \end{aligned} \tag{3.145}$$

or

$$\int_{|x|<R} |N'| dx \leqslant \liminf_{k \to +\infty} \int_{|x|<R} |n_k| dx = 0. \tag{3.146}$$

In addition, there hold:

$$\lim_{k \to +\infty} \int_{\mathbb{R}^2} n_k \left(|E_k|^2 + |E_{2k}|^2\right) dx = \lim_{k \to +\infty} \int_{\mathbb{R}^2} N' \left(|E_1'|^2 + |E_2'|^2\right) dx, \tag{3.147}$$

$$\lim_{k \to +\infty} \int_{\mathbb{R}^2} |E_{1k}|^2 |E_{2k}|^2 dx = \lim_{k \to +\infty} \int_{\mathbb{R}^2} \left|E_1'|^2 |E_2'\right|^2 dx, \tag{3.148}$$

$$\lim_{k \to +\infty} Re \int_{\mathbb{R}^2} (E_{1k})^2 \left(\overline{E_{2k}}\right)^2 dx = \lim_{k \to +\infty} Re \int_{\mathbb{R}^2} (E_1')^2 \left(\overline{E_2'}\right)^2 dx, \tag{3.149}$$

$$\lim_{k \to +\infty} \int_{\mathbb{R}^2} \left|\overline{E_{1k}} E_{2k} - E_{1k} \overline{E_{2k}}\right|^2 dx = \lim_{k \to +\infty} \int_{\mathbb{R}^2} \left|\overline{E_1'} E_2' - E_1' \overline{E_2'}\right|^2 dx. \tag{3.150}$$

(3.147) follow from Lemma 2.1 and (3.141).

Secondly, a direct calculation yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} |E_{1k}|^2|E_{2k}|^2dx - \int_{\mathbb{R}^2} \left|E_1'\right|^2\left|E_2'\right|^2 dx\right| \\
&\quad= \int_{\mathbb{R}^2} \left(|E_{1k}|^2 - |E_1'|^2\right)|E_{2k}|^2dx + \int_{\mathbb{R}^2} |E_1'|^2\left(|E_{2k}|^2 - |E_2'|^2\right)dx \\
&\quad\leqslant \left\|\left(|E_{1k}|^2 - |E_1'|^2\right)\right\|_{L^2(\mathbb{R}^2)} \left\||E_{2k}|^2\right\|_{L^2(\mathbb{R}^2)} \\
&\qquad+ \left\||E_1'|^2\right\|_{L^2(\mathbb{R}^2)} \left\||E_{2k}|^2 - |E_2'|^2\right\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Let $k \to +\infty$, one gets (3.148).

We next note that

$$\begin{aligned}
&\left|Re\int_{\mathbb{R}^2} (E_{1k})^2\left(\overline{E_{2k}}\right)^2 dx - Re\int_{\mathbb{R}^2} (E_1')^2\left(\overline{E_2'}\right)^2 dx\right| \\
&\quad\leqslant \int_{\mathbb{R}^2} \left|(E_{1k})^2\left(\overline{E_{2k}}\right)^2 - (E_1')^2\left(\overline{E_2'}\right)^2\right| dx \\
&\quad= \int_{\mathbb{R}^2} \left|\left[(E_{1k})^2 - (E_1')^2\right]\left(\overline{E_{2k}}\right)^2 + (E_1')^2\left[\left(\overline{E_{2k}}\right)^2 - \left(\overline{E_2'}\right)^2\right]\right| dx \\
&\quad\leqslant \left\|(E_{1k})^2 - (E_1')^2\right\|_{L^2(\mathbb{R}^2)} \left\||\overline{E_{2k}}|^2\right\|_{L^2(\mathbb{R}^2)} \\
&\qquad+ \left\|\, |E_1'|^2\right\|_{L^2(\mathbb{R}^2)} \left\|\left(\overline{E_{2k}}\right)^2 - \left(\overline{E_2'}\right)^2\right\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

As $k \to +\infty$, we get (3.149). Moreover, according to (3.148) and (3.149), (3.150) holds. We now use estimats (3.130), (3.147) and (3.150) to get

$$\mathcal{H}_1\left(E_1', E_2', N'\right) \leqslant \liminf_{k\to+\infty} \mathcal{H}_1\left(E_{1k}, E_{2k}, n_k\right) \leqslant 0, \tag{3.151}$$

that is,

$$\mathcal{E}\left(E_1', E_2'\right) + \frac{1}{2}\int_{\mathbb{R}^2}\left[N' + \left(|E_1'|^2 + |E_2'|^2\right)\right]^2 dx \leqslant 0. \tag{3.152}$$

According to $\int_{\mathbb{R}^2}\left(|E_1'|^2 + |E_2'|^2\right)dx < \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}$, (3.152) then yields

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(|\nabla E_1'|^2 + |\nabla E_2'|^2\right)dx \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^2}\left(|E_1'|^2 + |E_2'|^2\right)^2 dx + \frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{E'}_1E_2' - E_1'\overline{E'}_2\right|^2 dx \\
&\leqslant \frac{\eta+1}{2}\int_{\mathbb{R}^2}\left(|E_1'|^2 + |E_2'|^2\right)^2 dx \\
&\leqslant (1+\eta)\frac{\|E_1'\|^2_{L^2(\mathbb{R}^2)} + \|E_2'\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\left(\|\nabla E_1'\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2'\|^2_{L^2(\mathbb{R}^2)}\right) \\
&< \|\nabla E_1'\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2'\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

This is a contradiction.

On the other hand, if $N'=0$, (3.4) yields

$$
\begin{aligned}
&\mathcal{H}_1\left(E_1', E_2', 0\right) \\
&\qquad \geqslant \int_{\mathbb{R}^2}\left(\left|\nabla E_1'\right|^2+\left|\nabla E_2'\right|^2\right) dx-2\eta \int_{\mathbb{R}^2}\left|E_1'\right|^2\left|E_2'\right|^2 dx \\
&\qquad \geqslant \int_{\mathbb{R}^2}\left(\left|\nabla E_1'\right|^2+\left|\nabla E_2'\right|^2\right) dx-\frac{\eta}{2} \int_{\mathbb{R}^2}\left(\left|E_1'\right|^2+\left|E_2'\right|^2\right)^2 dx \\
&\qquad \geqslant \int_{\mathbb{R}^2}\left(\left|\nabla E_1'\right|^2+\left|\nabla E_2'\right|^2\right) dx \\
&\qquad\quad -\eta \frac{\left(\|E_1'\|_{L^2(\mathbb{R}^2)}^2+\|E_2'\|_{L^2(\mathbb{R}^2)}^2\right)}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \cdot\left(\|\nabla E_1'\|_{L^2(\mathbb{R}^2)}^2+\|\nabla E_2'\|_{L^2(\mathbb{R}^2)}^2\right) \\
&\qquad \geqslant \int_{\mathbb{R}^2}\left(\left|\nabla E_1'\right|^2+\left|\nabla E_2'\right|^2\right) dx\left(1-\frac{\eta\left(\|E_{10}\|_{L^2(\mathbb{R}^2)}^2+\|E_{20}\|_{L^2(\mathbb{R}^2)}^2\right)}{\|Q\|_{L^2(\mathbb{R}^2)}^2}\right) \\
&\qquad >0,
\end{aligned}
$$

which is contradictory to (3.151). Hence there exists a constant $m>0$ depending only on initial data such that for any $R>0$, (3.119) and (3.120) hold.

Now we turn to consider the non-radial case (2).

Assume that there exist constants $R_0>0,\ \delta_0>0$ and a sequence t_k such that as $t_k \rightarrow T\ (k \rightarrow+\infty)$,

$$
\liminf_{k \rightarrow+\infty}\left(\sup_y \int_{|x-y|<R_0}\left(|E_1(t_k, x)|^2+|E_2(t_k, x)|^2\right) dx\right) \leqslant \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1+\eta}-\delta_0,
$$

or

$$
\liminf_{k \rightarrow+\infty}\left(\sup_y \int_{|x-y|<R_0}|n(t_k, x)| dx\right) \leqslant m_n-\delta_0 .
$$

Then it follows from Lemma 3.7 that as $t_k \rightarrow T$,

$$
\int_{\mathbb{R}^2}\left(\left|\nabla E_1(t_k)\right|^2+\left|\nabla E_2(t_k)\right|^2+\left|n(t_k)\right|^2+\left|\mathbf{v}(t_k)\right|^2\right) dx \leqslant C .
$$

This is contradictory to the assumption that $(E_1, E_2, n, \mathbf{v})$ blows up at a finite time T. So (3.121) and (3.122) hold.

This finishes the proof of Proposition 3.8. $\square$

We are now in the position to prove Proposition 3.5 by utilizing Proposition 3.6, Lemma 3.7 and Proposition 3.8.

Proof of Proposition 3.5.

Due to (2.1), Proposition 3.8 implies the conclusion (1) in Proposition 3.5.

In fact, let $R_1>0$ be a fixed constant, then

$$
\begin{aligned}
&\left\|\tilde{E}_1(0, x)\right\|_{L^2(|x-x(t)| \leqslant R_1)}^2+\left\|\tilde{E}_2(0, x)\right\|_{L^2(|x-x(t)| \leqslant R_1)}^2 \\
&\quad=\|E_1(t, x)\|_{L^2\left(|x-x(t)| \leqslant \frac{R_1}{\lambda(t)}\right)}^2+\|E_2(t, x)\|_{L^2\left(|x-x(t)| \leqslant \frac{R_1}{\lambda(t)}\right)}^2 .
\end{aligned}
$$

Noting that $\lambda(t) \to +\infty$ as $t \to T$ and (3.119), we have

$$\begin{aligned}
&\liminf_{t\to T} \left(\|\tilde{E}_1(0,x)\|^2_{L^2(|x-x(t)|\leqslant R_1)} + \|\tilde{E}_2(0,x)\|^2_{L^2(|x-x(t)|\leqslant R_1)} \right) \\
&\quad \geqslant \liminf_{t\to T} \left(\|E_1(t,x)\|^2_{L^2\left(|x-x(t)|\leqslant \frac{R_1}{\lambda(t)}\right)} + \|E_2(t,x)\|^2_{L^2\left(|x-x(t)|\leqslant \frac{R_1}{\lambda(t)}\right)} \right) \\
&\quad \geqslant \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1+\eta}.
\end{aligned}$$

On the other hand, in view of Proposition 3.8, for any fixed $R_1 > 0$, Hölder's inequality yields

$$\begin{aligned}
&R_1^{\frac{1}{2}} \liminf_{t\to T} \|\tilde{n}(0,x)\|_{L^2(|x-x(t)|\leqslant R_1)} \\
&\quad \geqslant \liminf_{t\to T} \|\tilde{n}(0,x)\|_{L^1(|x-x(t)|\leqslant R_1)} \\
&\quad \geqslant \liminf_{t\to T} \|n(t,x)\|_{L^1\left(|x-x(t)|\leqslant \frac{R_1}{\lambda(t)}\right)} \\
&\quad \geqslant m_n.
\end{aligned}$$

Hence (3.26) and (3.27) hold.

We now show conclusion (2) in Proposition 3.5 by using the same scaling argument as that adopted in Proposition 3.8.

For a sequence $t_k \to T$ $(k \to +\infty)$, let

$$\begin{cases}
\hat{E}_{1n} = \dfrac{1}{\tilde{\lambda}_n} \tilde{E}_1\left(\dfrac{x}{\tilde{\lambda}_n}\right), \ \hat{E}_{2n} = \dfrac{1}{\tilde{\lambda}_n} \tilde{E}_2\left(\dfrac{x}{\tilde{\lambda}_n}\right), \\[2ex]
\hat{n}_n = \dfrac{1}{\tilde{\lambda}_n^2} \tilde{n}\left(t_n, \dfrac{x}{\tilde{\lambda}_n}\right), \hat{\mathbf{v}}_n = \dfrac{1}{\tilde{\lambda}_n^2} \tilde{\mathbf{v}}\left(t_n, \dfrac{x}{\tilde{\lambda}_n}\right)
\end{cases}$$

satisfy

$$\int_{\mathbb{R}^2} \left(\left|\nabla \hat{E}_{1n}\right|^2 + \left|\nabla \hat{E}_{2n}\right|^2 + \frac{1}{2}|\hat{n}_n|^2 + \frac{1}{2}|\hat{\mathbf{v}}_n|^2 \right) dx = 1.$$

Here,

$$\begin{aligned}
\tilde{\lambda}_n^2(t) = \int_{\mathbb{R}^2} \Bigg(&\left|\nabla \tilde{E}_1(0,x)\right|^2 + \left|\nabla \tilde{E}_2(0,x)\right|^2 \\
&+ \frac{1}{2}|\tilde{n}(0,x)|^2 + \frac{1}{2}|\tilde{v}(0,x)|^2 \Bigg) dx,
\end{aligned}$$

and $\tilde{\lambda}_n \to 1$ as $n \to +\infty$. Direct calculation gives $\left(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n\right)$ satisfy (3.28)-(3.31). Hence for a sequence $x_n(t) \in \mathbb{R}^2$ and $x_n(t) \to x(t)$ as $n \to +\infty$, there holds

$$\begin{aligned}
&\|\hat{E}_{1n}\|^2_{L^2(|x-x_n(t)|\leqslant R_1)} + \|\hat{E}_{2n}\|^2_{L^2(|x-x_n(t)|\leqslant R_1)} \\
&\quad = \|\tilde{E}_1(0,x)\|^2_{L^2\left(|x-x_n(t)|\leqslant \frac{R_1}{\tilde{\lambda}_n}\right)} + \|\tilde{E}_2(0,x)\|^2_{L^2\left(|x-x_n(t)|\leqslant \frac{R_1}{\tilde{\lambda}_n}\right)},
\end{aligned}$$

and

$$\|\hat{n}_n\|_{L^2(|x-x_n(t)|\leqslant R_1)} = \|\tilde{n}(0,x)\|_{L^2\left(|x-x_n(t)|\leqslant \frac{R_1}{\tilde{\lambda}_n}\right)}.$$

Letting $n \to +\infty$ yields (3.32) and (3.33) due to (3.121) and (3.122), which ends the proof of Proposition 3.5. $\square$

3.3. **Compactness of the Solution to the Rescaled Zakharov System (2.3).** Here, we discuss the compactness of $\left(\tilde{E}_1(0,x), \tilde{E}_2(0,x), \tilde{n}(0,x)\right)$.

Remark 3.9. From Proposition 3.5, it follows that $\left(\tilde{E}_1(0,x), \tilde{E}_2(0,x)\right)$ is bounded and weakly compact in $H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$. Then we can choose a sequence $t_n \to T$, and extract a subsequence (still denoted by t_n). Let

$$\left(\tilde{E}_1(0,x+x(t_n)), \tilde{E}_2(0,x+x(t_n))\right) \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2),$$

and

$$\tilde{n}(0,x+x(t_n)) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2).$$

Note that the embedding $H^1(\mathbb{R}^2)\hookrightarrow L^2_{loc}(\mathbb{R}^2)$ is compact, from Proposition 3.5, for a bounded domain $\Omega\subset\mathbb{R}^2$, there holds

$$\|E_1'\|^2_{L^2(\Omega)} + \|E_2'\|^2_{L^2(\Omega)} \geqslant c_1 > 0.$$

Under this case, one can not exclude the case of $N'\equiv 0$, that is, Proposition 3.5 can not guarantee the non-vanishing property of N'. □

To overcome the difficulty mentioned in Remark 3.9, we will investigate the relation of (E_1', E_2') and N' to obtain the compactness property of N' following the related information for (E_1', E_2').

We now claim:

Proposition 3.10. Let $t_n\to T$. There exists a subsequence (still denote t_n) such that for a sequence $x_n := x(t_n)\in\mathbb{R}^2$ and $(E_1', E_2', N')\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$, as $n\to+\infty$, the conclusions hold as below:

$$\left(\tilde{E}_1\left(0,x+x_n\right), \tilde{E}_2\left(0,x+x_n\right)\right) \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2), \tag{3.153}$$

and

$$\tilde{n}(0,x+x_n) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2). \tag{3.154}$$

Furthermore, there exist constants $\beta_1>0$ and $R_1>0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)},\ \|E_{20}\|_{L^2(\mathbb{R}^2)})$ such that

$$\left(\|E_1'\|^2_{L^2(|x|\leqslant R_1)} + \|E_2'\|^2_{L^2(|x|\leqslant R_1)}\right)^{\frac{1}{2}} \geqslant \beta_1, \tag{3.155}$$

$$\mathcal{H}\left(E_1', E_2', N', 0\right) \leqslant 0. \tag{3.156}$$

Proof. In view of Proposition 3.5, we will prove this proposition by implementing the classical iteration technique, Concentration-compactness principle and mathematical induction.

Let β_1 be defined as in (ii) of Proposition 3.5. Assume that $\left(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n, \hat{\mathbf{v}}_n\right)\in$

$H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ satisfies

$$\begin{cases} \hat{E}_{1n} = \dfrac{1}{\lambda_n(t_n)}\tilde{E}_1\left(t_n+\dfrac{s}{\lambda_n(t_n)},\dfrac{x}{\lambda_n(t_n)}\right), \\ \hat{E}_{2n} = \dfrac{1}{\lambda_n(t_n)}\tilde{E}_2\left(t_n+\dfrac{s}{\lambda_n(t_n)},\dfrac{x}{\lambda_n(t_n)}\right), \\ \hat{n}_n = \dfrac{1}{\lambda_n^2(t_n)}\tilde{n}\left(t_n+\dfrac{s}{\lambda_n(t_n)},\dfrac{x}{\lambda_n(t_n)}\right), \\ \hat{\mathbf{v}}_n = \dfrac{1}{\lambda_n^2(t_n)}\tilde{\mathbf{v}}\left(t_n+\dfrac{s}{\lambda_n(t_n)},\dfrac{x}{\lambda_n(t_n)}\right), \end{cases} \tag{3.157}$$

$$\lambda_n^2(t_n)=\int_{\mathbb{R}^2}\left|\nabla\tilde{E}_1\right|^2dx+\int_{\mathbb{R}^2}\left|\nabla\tilde{E}_2\right|^2dx+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{n}|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}|^2\,dx. \tag{3.158}$$

Direct calculation yields

$$\int_{\mathbb{R}^2}\left(\left|\hat{E}_{1n}\right|^2+\left|\hat{E}_{2n}\right|^2\right)dx\leqslant\int_{\mathbb{R}^2}\left(|E_{10}|^2+|E_{20}|^2\right)dx, \tag{3.159}$$

$$\lim_{n\to+\infty}\mathcal{H}\left(\hat{E}_{1n},\hat{E}_{2n},\hat{n}_n,\hat{\mathbf{v}}_n\right)=0, \tag{3.160}$$

$$\int_{\mathbb{R}^2}\left(\left|\nabla\hat{E}_{1n}\right|^2+\left|\nabla\hat{E}_{2n}\right|^2+\frac{|\hat{n}_n|^2}{2}+\frac{|\hat{\mathbf{v}}_n|^2}{2}\right)dx=1. \tag{3.161}$$

We will show that there exists $(E_1',E_2',N')\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ and a sequence $x_n\in\mathbb{R}^2$ such that as $n\to+\infty$,

$$\left(\hat{E}_{1n}(x_n+x),\hat{E}_{2n}(x_n+x)\right)\rightharpoonup(E_1'(x),E_2'(x))\ \text{ in } H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2), \tag{3.162}$$

$$\hat{n}_n(x_n+x)\rightharpoonup N'(x)\text{ in }L^2(\mathbb{R}^2), \tag{3.163}$$

and

$$\left(\int_{|x|\leqslant R_1}\left(|E_1'|^2+|E_2'|^2\right)dx\right)^{\frac{1}{2}}\geqslant\beta_1,\ \mathcal{H}\left(E_1',E_2',N'\right)\leqslant 0. \tag{3.164}$$

By Proposition 3.3 and Corollary 3.4, there exist constants $c_1>0,\ c_2>0$ such that

$$c_1\leqslant\int_{\mathbb{R}^2}\left(\left|\nabla\hat{E}_{1n}\right|^2+\left|\nabla\hat{E}_{2n}\right|^2\right)dx\leqslant c_2,\ c_1\leqslant\int_{\mathbb{R}^2}|\hat{n}_n|^2\,dx\leqslant c_2. \tag{3.165}$$

Let the integer k_0 be defined by

$$\left(\int_{\mathbb{R}^2}\left(\left|\hat{E}_{1n}\right|^2+\left|\hat{E}_{2n}\right|^2\right)dx\right)^{\frac{1}{2}}<(k_0+1)\beta_1, \tag{3.166}$$

where

$$k_0=1,2,...,E\left[\frac{\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)}}{\beta_1}-1\right]. \tag{3.167}$$

We then show Proposition 3.10 by induction on the integer k_0.

(1) For $k_0=1$, there holds $\dfrac{\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)}}{\beta_1}=2$, that is,

$$\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)}=2\beta_1. \tag{3.168}$$

From (3.159) it follows that

$$\left\|\hat{E}_{1n}\right\|_{L^2(\mathbb{R}^2)}^2+\left\|\hat{E}_{2n}\right\|_{L^2(\mathbb{R}^2)}^2 \leqslant \|E_{10}\|_{L^2(\mathbb{R}^2)}^2+\|E_{20}\|_{L^2(\mathbb{R}^2)}^2=2\beta_1. \tag{3.169}$$

Hence (3.166) holds for $k_0=1$. From compactness and the boundedness of weakly convergent sequence, a similar argument to the proof of Lemma 3.7 yields

$$\mathcal{H}\left(E_1', E_2', N'\right) \leqslant \lim_{n\to+\infty} \mathcal{H}\left(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n\right) \leqslant 0.$$

So (3.162)-(3.165) are true. Note that Proposition 3.5, Proposition 3.10 holds for $k_0=1$.

(2) Assume that (3.162)-(3.165) are true for $k_0>1$, we then show that they also hold for k_0+1.

Let $\left(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n\right)$ be the sequence satisfying (3.164), (3.165), (3.166) and

$$\lim_{n\to+\infty} \mathcal{H}\left(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n, 0\right) \leqslant 0.$$

In view of Proposition 3.5, we may assume that there exist a sequence $x_n\in\mathbb{R}^2$ and a constant $R=R(c_1,c_2)$ satisfying

$$\left(\int_{|x-x_n|\leqslant R}\left(\left|\hat{E}_{1n}\right|^2+\left|\hat{E}_{2n}\right|^2\right)dx\right)^{\frac{1}{2}} \geqslant \beta_1, \tag{3.170}$$

and $\left(\hat{E}_1, \hat{E}_2, \hat{N}\right)\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ such that

$$\left(\hat{E}_{1n}\left(x+x_n\right), \hat{E}_{2n}\left(x+x_n\right)\right) \rightharpoonup \left(\hat{E}_1, \hat{E}_2\right) \text{ in } H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2),$$

$$\hat{n}_n(x+x_n) \rightharpoonup \hat{N} \text{ in } L^2(\mathbb{R}^2).$$

We extract a subsequence still denoted by $\left(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n\right)$ for simplicity. We make the following decomposition for the extracted subsequence:

$$\begin{cases} \hat{E}_{1n}(x+x_n)=\hat{E}_{1n,1}(x+x_n)+\hat{E}_{1n,2}(x+x_n), \\ \hat{E}_{2n}(x+x_n)=\hat{E}_{2n,1}(x+x_n)+\hat{E}_{2n,2}(x+x_n), \\ \hat{n}_n(x+x_n)=\hat{n}_{n,1}(x+x_n)+\hat{n}_{n,2}(x+x_n). \end{cases}$$

Let $R_n\to+\infty$ as $n\to+\infty$. This decomposition admits the following properties:
(I)

$$\begin{aligned} &\hat{E}_{1n,1}(x)=\hat{E}_{2n,1}(x)=\hat{n}_{n,1}(x)=0, \quad |x|\leqslant \frac{R_n}{2}, \\ &\hat{E}_{1n,2}(x)=\hat{E}_{2n,2}(x)=\hat{n}_{n,2}(x)=0, \quad |x|\geqslant R_n. \end{aligned}$$

(II) As $n \to +\infty$,

$$\int_{\mathbb{R}^2} \left(\left|\hat{E}_{1n,1}\right|^2 + \left|\hat{E}_{1n,2}\right|^2 + \left|\hat{E}_{2n,1}\right|^2 + \left|\hat{E}_{2n,2}\right|^2 \right) dx - \int_{\mathbb{R}^2} \left(\left|\hat{E}_{1n}\right|^2 + \left|\hat{E}_{2n}\right|^2 \right) dx \to 0,$$

$$\int_{\mathbb{R}^2} \left(\left|\nabla\hat{E}_{1n,1}\right|^2 + \left|\nabla\hat{E}_{1n,2}\right|^2 + \left|\nabla\hat{E}_{2n,1}\right|^2 + \left|\nabla\hat{E}_{2n,2}\right|^2 \right) dx - \int_{\mathbb{R}^2} \left(\left|\nabla\hat{E}_{1n}\right|^2 + \left|\nabla\hat{E}_{2n}\right|^2 \right) dx \to 0,$$

$$\int_{\mathbb{R}^2} \left(|\hat{n}_{n,1}|^2 + |\hat{n}_{n,2}|^2 \right) dx - \int_{\mathbb{R}^2} |\hat{n}_n|^2 \, dx \to 0,$$

(III)

$$\lim_{n\to+\infty} \mathcal{H}\left(\hat{E}_{1n,1}, \hat{E}_{2n,1}, \hat{n}_{n,1}, 0\right) + \lim_{n\to+\infty} \mathcal{H}\left(\hat{E}_{1n,2}, \hat{E}_{2n,2}, \hat{n}_{n,2}, 0\right) \leqslant 0.$$

Note that as $n \to +\infty$,

$$\int_{\mathbb{R}^2} \left(\left|\hat{E}_{1n,1}\right|^2 + \left|\hat{E}_{2n,1}\right|^2 \right) dx \to \int_{\mathbb{R}^2} \left(\left|\hat{E}_1\right|^2 + \left|\hat{E}_2\right|^2 \right) dx.$$

Especially, $\int_{\mathbb{R}^2} \left(\left|\hat{E}_1\right|^2 + \left|\hat{E}_2\right|^2 \right) dx \geqslant \beta_1$. So when n large enough, there holds

$$\int_{\mathbb{R}^2} \left(\left|\hat{E}_{1n,2}\right|^2 + \left|\hat{E}_{2n,2}\right|^2 \right) dx < k_0\beta_1.$$

We proceed our argument by considering the following two cases:

Case 1:

$$\mathcal{H}\left(\hat{E}_1, \hat{E}_2, \hat{N}, 0\right) \leqslant \lim_{n\to+\infty} \mathcal{H}\left(\hat{E}_{1n,1}, \hat{E}_{2n,1}, \hat{n}_{n,1}, 0\right) \leqslant 0.$$

In this case, letting $E_1' = \hat{E}_1,\ E_2' = \hat{E}_2,\ N' = \hat{N}$, one can get the conclusion of Proposition 3.10.

Case 2:

$$\mathcal{H}\left(\hat{E}_1, \hat{E}_2, \hat{N}, 0\right) > 0.$$

In this case, let $P_1 = \lim_{n\to+\infty} \mathcal{H}\left(\hat{E}_{1n,1}, \hat{E}_{2n,1}, \hat{n}_{n,1}, 0\right) > 0$, and for n large enough,

$$\mathcal{H}\left(\hat{E}_{1n,2}, \hat{E}_{2n,2}, \hat{n}_{n,2}, 0\right) \leqslant -\frac{P_1}{2} < 0.$$

By inductive assumption, there exists a sequence $y_n \in \mathbb{R}^2$ such that

$$\left(\hat{E}_{1n,2}\left(+y_n\right), \hat{E}_{2n,2}\left(+y_n\right)\right) \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),$$

$$\hat{n}_{n,2}(+y_n) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2),$$

with

$$\left(\int_{|x|\leqslant R_1} \left(|E_1'|^2 + |E_2'|^2 \right) dx \right)^{\frac{1}{2}} \geqslant \beta_1, \quad \mathcal{H}\left(E_1', E_2', N', 0\right) \leqslant 0.$$

Using translation transformation finishes the proof of Proposition 3.10. $\square$

Corollary 3.11. *There exists a constant $c_* > 0$ such that*

$$c_* \leqslant \|N'\|_{L^2(\mathbb{R}^2)} \leqslant 1. \tag{3.171}$$

Proof. From Proposition 3.10 it follows that

$$\begin{aligned}
&\mathcal{H}\left(E_1', E_2', N'\right)\\
&= \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 + N' \left(\left|E_1'\right|^2 + \left|E_2'\right|^2 \right) + \frac{1}{2} N'^2 \right) dx\\
&\quad + \frac{\eta}{2} \int_{\mathbb{R}^2} \left[\left((E_1')^2 \left(\overline{E_2'}\right)^2 + (E_2')^2 \left(\overline{E_1'}\right)^2 \right) - 2\left|E_1'\right|^2 \left|E_2'\right|^2 \right] dx\\
&\leqslant 0,
\end{aligned} \tag{3.172}$$

and

$$\|E_1'\|^2_{L^2(|x|\leqslant R_1)} + \|E_2'\|^2_{L^2(|x|\leqslant R_1)} \geqslant \beta_1. \tag{3.173}$$

The Sobolev embedding theorem $\left(\forall q \in [n, +\infty),\ W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)\right)$ yields $H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$. In view of (3.161), there exists $c_1 > 0$ such that

$$0 < (c_1)^2 \leqslant \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 \right) dx \leqslant 1. \tag{3.174}$$

On one hand, by (3.172) one has

$$\begin{aligned}
&- \int_{\mathbb{R}^2} N' \left(\left|E_1'\right|^2 + \left|E_2'\right|^2 \right) dx\\
&\geqslant \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 \right) dx - \eta \int_{\mathbb{R}^2} \left|E_1'\right|^2 \left|E_2'\right|^2 dx\\
&\quad + \frac{\eta}{2} \int_{\mathbb{R}^2} \left((E_1')^2 \left(\overline{E_2'}\right)^2 + (E_2')^2 \left(\overline{E_1'}\right)^2 \right) dx\\
&\geqslant \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 \right) dx - \frac{\eta}{2} \int_{\mathbb{R}^2} \left(\left|E_1'\right|^2 + \left|E_2'\right|^2 \right)^2 dx\\
&\geqslant \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 \right) dx\\
&\quad - \frac{\eta}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \left(\left|E_1'\right|^2 + \left|E_2'\right|^2 \right) dx \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 \right) dx\\
&= \left[1 - \frac{\eta}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \left(\left|E_{10}\right|^2 + \left|E_{20}\right|^2 \right) dx \right]\\
&\qquad \cdot \int_{\mathbb{R}^2} \left(\left|\nabla E_1'\right|^2 + \left|\nabla E_2'\right|^2 \right) dx.
\end{aligned} \tag{3.175}$$

Let

$$c_0 = 1 - \frac{\eta}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \left(|E_{10}|^2 + |E_{20}|^2 \right) dx > 0. \tag{3.176}$$

According to (3.4), one has $0 < c_0 < \dfrac{1}{1+\eta}$. On the other hand,

$$\begin{aligned}
&-\int_{\mathbb{R}^2} N' \left(|E_1'|^2 + |E_2'|^2\right) dx \\
&\leqslant \left(\int_{\mathbb{R}^2} |N'|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \left(|E_1'|^2 + |E_2'|^2\right)^2 dx\right)^{\frac{1}{2}} \\
&\leqslant \sqrt{2}\, \|N'\|_{L^2(\mathbb{R}^2)} \frac{1}{\|Q\|_{L^2(\mathbb{R}^2)}} \left[\int_{\mathbb{R}^2} \left(|E_1'|^2 + |E_2'|^2\right) dx \right. \\
&\qquad \left. \cdot \int_{\mathbb{R}^2} \left(|\nabla E_1'|^2 + |\nabla E_2'|^2\right) dx\right]^{\frac{1}{2}} \\
&\leqslant \sqrt{2}\, \|N'\|_{L^2(\mathbb{R}^2)} \cdot \frac{1}{\sqrt{\eta}} \left(\int_{\mathbb{R}^2} \left(|\nabla E_1'|^2 + |\nabla E_2'|^2\right) dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining (3.174) with (3.175) and (3.176) yields

$$\|N'\|_{L^2(\mathbb{R}^2)} \geqslant \sqrt{\frac{\eta}{2}} c_0 c_1 = c^* > 0.$$

From (3.174) and (3.176) it follows that $0 < c_0 < \dfrac{1}{1+\eta}$, $0 < c_1 \leqslant 1$. We then conclude

$$c^* = \sqrt{\frac{\eta}{2}} c_0 c' = \sqrt{\frac{\eta}{2}} c_0 c_1 < \sqrt{\frac{\eta}{2}} \frac{1}{1+\eta} c_1 < 1.$$

So far, the proof of Corollary 3.11 is completed. □

3.4. **Proof of Theorem 3.1.** In this subsection, we shall establish some estimates for $\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(0)$ based on the estimates obtained in subsection 3.1, subsection 3.2 and subsection 3.3. Secondly, by considering the rescaled Zakharov system (2.3c)-(2.3d):

$$\tilde{n}_s = -\nabla \cdot \tilde{\mathbf{v}}, \tag{3.177}$$

$$\tilde{\mathbf{v}}_s = -\nabla \left(\tilde{n} + \left|\tilde{E}_1\right|^2 + \left|\tilde{E}_2\right|^2\right), \tag{3.178}$$

We then finish the proof of Theorem 3.1.

Proof of Theorem 3.1.

$\forall\, t > 0$, we consider $\left(\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s), \tilde{\mathbf{v}}(s)\right)$ for $[0, \lambda(t)(T-t))$. From (2.5) it follows that

$$\left\|\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{\mathbf{v}}(0)\right)\right\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} = 1,$$

$$\lim_{s\to\lambda(t)(T-t)} \left\|\left(\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s), \tilde{\mathbf{v}}(s)\right)\right\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} = +\infty.$$

Let $A > 1$ be a fixed constant. By the continuity of s, there exists $\theta(t) > 0$ such that $\forall s \in [0, \theta(t)]$,

$$\left\|\left(\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s), \tilde{\mathbf{v}}(s)\right)\right\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \leqslant A, \tag{3.179}$$

$$\left\|\left(\tilde{E}_1(\theta(t)), \tilde{E}_2(\theta(t)), \tilde{n}(\theta(t)), \tilde{\mathbf{v}}(\theta(t))\right)\right\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} = A. \tag{3.180}$$

We now claim that as $t \to T$, there exists a uniform lower bound $\theta_0 > 0$ for $\theta(t)$, that is,

$$\theta(t) \geqslant \theta_0. \tag{3.181}$$

We proceed our discussion through two steps. On one hand, some properties for $\left(\tilde{E}_1(\theta), \tilde{E}_2(\theta), \tilde{n}(\theta), \tilde{\mathbf{v}}(\theta)\right)$ will be established. On the other hand, in view of (3.177) and the compactness for $\tilde{n}$, we complete the proof of (3.181) by contradiction.

Firstly we claim that:

Proposition 3.12. There exist constants $c_1 > 0$ and $c_2 > 0$ independent of t and $A > 1$ such that
(1) $\forall s \in [0, \theta(t)]$,

$$\left(\left\|\nabla \tilde{E}_1(s)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\nabla \tilde{E}_2(s)\right\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \leqslant Ac_2, \tag{3.182}$$

$$\|\tilde{n}(s)\|_{L^2(\mathbb{R}^2)} \leqslant Ac_2, \ \|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)} \leqslant Ac_2. \tag{3.183}$$

(2) Let $t_n \to T$. Extracting a subsequence, still denoted by t_n, such that for sequences $x_n := x(t_n) \in \mathbb{R}^2$, $(E_1', E_2', N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, the following assertions hold

$$\begin{aligned}
&\left(\tilde{E}_1\left(t_n, \theta\left(t_n\right), x - x_n\right), \tilde{E}_2\left(t_n, \theta\left(t_n\right), x - x_n\right)\right) \\
&\quad = \left(\tilde{E}_1\left(\theta\left(t_n\right), x - x_n\right), \tilde{E}_2\left(\theta\left(t_n\right), x - x_n\right)\right) \\
&\quad \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\end{aligned} \tag{3.184}$$

$$\tilde{n}\left(t_n, \theta(t_n), x - x_n\right) = \tilde{n}\left(\theta(t_n), x - x_n\right) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2), \tag{3.185}$$

$$\left(\|\nabla E_1'\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2'\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \geqslant Ac_1, \quad \|N'\|_{L^2(\mathbb{R}^2)} \geqslant Ac_1. \tag{3.186}$$

Proof. It follows from (3.179) and (3.180) that

$$\int_{\mathbb{R}^2} \left(\left|\nabla \tilde{E}_1(s)\right|^2 + \left|\nabla \tilde{E}_2(s)\right|^2 + \frac{1}{2}\left|\tilde{n}(s)\right|^2 + \frac{1}{2}\left|\tilde{\mathbf{v}}(s)\right|^2\right) dx \leqslant A^2. \tag{3.187}$$

Note that (2.1), there holds

$$\begin{aligned}
&\left\|\left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(s)\right\|^2_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}^2} \left(\left|\nabla \tilde{E}_1(t,s)\right|^2 + \left|\nabla \tilde{E}_2(t,s)\right|^2 + \frac{1}{2}\left|\tilde{n}(t,s)\right|^2 + \frac{1}{2}\left|\tilde{\mathbf{v}}(t,s)\right|^2\right) dx \\
&= \frac{1}{\lambda^2(t)}\left[\int_{\mathbb{R}^2}\left(\left|\nabla E_1\left(t + \frac{s}{\lambda(t)}\right)\right|^2 + \left|\nabla E_2\left(t + \frac{s}{\lambda(t)}\right)\right|^2\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}\left|n\left(t + \frac{s}{\lambda(t)}\right)\right|^2 + \frac{1}{2}\left|\mathbf{v}\left(t + \frac{s}{\lambda(t)}\right)\right|^2\right) dx\right] \\
&= \left(\frac{\lambda\left(t + \frac{s}{\lambda(t)}\right)}{\lambda(t)}\right)^2.
\end{aligned} \tag{3.188}$$

So $\forall s \in [0, \theta(t)]$,

$$\frac{\lambda\left(t+\frac{s}{\lambda(t)}\right)}{\lambda(t)} \leqslant A, \quad \frac{\lambda\left(t+\frac{\theta(t)}{\lambda(t)}\right)}{\lambda(t)} = A. \tag{3.189}$$

This yields that

$$\begin{aligned}
\tilde{E}_1(t,\theta(t),x) \quad &= \frac{1}{\lambda(t)} E_1\left(t+\frac{\theta(t)}{\lambda(t)}, \frac{x}{\lambda(t)}\right) \\
&= \frac{A}{A\lambda(t)} E_1\left(t+\frac{\theta(t)}{\lambda(t)}, \frac{Ax}{A\lambda(t)}\right) \\
&= \frac{A}{\lambda\left(t+\frac{\theta(t)}{\lambda(t)}\right)} E_1\left(t+\frac{\theta(t)}{\lambda(t)}, \frac{Ax}{\lambda\left(t+\frac{\theta(t)}{\lambda(t)}\right)}\right) \\
&= A\tilde{E}_1\left(t+\frac{\theta(t)}{\lambda(t)}, 0, Ax\right).
\end{aligned} \tag{3.190}$$

A similar argument to (3.190) yields

$$\tilde{E}_2\left(t,\theta(t),x\right) = A\tilde{E}_2\left(t+\frac{\theta(t)}{\lambda(t)}, 0, Ax\right), \tag{3.191}$$

$$\tilde{n}\left(t,\theta(t),x\right) = A^2\tilde{n}\left(t+\frac{\theta(t)}{\lambda(t)}, 0, Ax\right). \tag{3.192}$$

As $t_n \to T$, there holds $t_n + \frac{\theta(t_n)}{\lambda(t_n)} \to T$. Hence from Proposition 3.10 and Corollary 3.11, it follows that there exists a subsequence (still denoted by t_n) such that for sequences $x_n \in \mathbb{R}^2$, $(E_1', E_2', N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, as $n \to +\infty$, the following conclusions hold:

$$\begin{aligned}
&\left(\tilde{E}_1\left(t_n+\frac{\theta(t_n)}{\lambda(t_n)}, 0, x+x_n\right), \tilde{E}_2\left(t_n+\frac{\theta(t_n)}{\lambda(t_n)}, 0, x+x_n\right)\right) \\
&\qquad \rightharpoonup (E_1', E_2') \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\end{aligned} \tag{3.193}$$

$$\tilde{n}\left(t_n+\frac{\theta(t_n)}{\lambda(t_n)}, 0, x+x_n\right) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2). \tag{3.194}$$

In addition, in view of Proposition 3.10 and Corollary 3.11, there exists a constant $c_1 > 0$ such that

$$\left(\|E_1'\|^2_{L^2(\mathbb{R}^2)} + \|E_2'\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \geqslant c_1, \quad \|N'\|_{L^2(\mathbb{R}^2)} \geqslant c_1. \tag{3.195}$$

Hence, combining (3.190)-(3.195) yields

$$\begin{aligned}
&\left(\tilde{E}_1\left(t_n, \theta\left(t_n\right), x+\frac{x_n}{A}\right), \quad \tilde{E}_2\left(t_n, \theta\left(t_n\right), x+\frac{x_n}{A}\right)\right) \\
&= \left(A\tilde{E}_1\left(t_n+\frac{\theta\left(t_n\right)}{\lambda\left(t_n\right)}, 0, Ax+x_n\right), A\tilde{E}_2\left(t_n+\frac{\theta\left(t_n\right)}{\lambda\left(t_n\right)}, 0, Ax+x_n\right)\right) \\
&\quad \rightharpoonup \left(AE_1'\left(Ax\right), AE_2'\left(Ax\right)\right) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\end{aligned} \tag{3.196}$$

$$\begin{aligned}
\tilde{n}\left(t_n, \theta(t_n), x+\tfrac{x_n}{A}\right) \quad &= A^2\tilde{n}\left(t_n+\frac{\theta(t_n)}{\lambda(t_n)}, 0, Ax+x_n\right) \\
&\rightharpoonup A^2 N'(Ax) \text{ in } L^2(\mathbb{R}^2),
\end{aligned} \tag{3.197}$$

and

$$\begin{aligned}
&\left(\|\nabla\left[AE_1'\left(Ax\right)\right]\|^2_{L^2(\mathbb{R}^2)}+\|\nabla\left[AE_2'\left(Ax\right)\right]\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}\\
&\quad=A\left(\|\nabla E_1'\|^2_{L^2(\mathbb{R}^2)}+\|\nabla E_2'\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}\\
&\quad\geqslant Ac_1,
\end{aligned}\tag{3.198}$$

$$\left\|A^2N'(Ax)\right\|_{L^2(\mathbb{R}^2)}=A\left\|N'\right\|_{L^2(\mathbb{R}^2)}\geqslant Ac_1.\tag{3.199}$$

This finishes the proof of Proposition 3.12. □

Remark 3.13. By the property of weak convergence, from (3.196)-(3.199) it follows that

$$\begin{aligned}
c_1\quad&\leqslant\left(\|\nabla E_1'\|^2_{L^2(\mathbb{R}^2)}+\|\nabla E_2'\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}\\
&\leqslant\liminf_{n\to+\infty}\left(\left\|\nabla\tilde{E}_1\right\|^2_{L^2(\mathbb{R}^2)}+\left\|\nabla\tilde{E}_2\right\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}},\\
c_1&\leqslant\|N'\|_{L^2(\mathbb{R}^2)}\leqslant\liminf_{n\to+\infty}\|\tilde{n}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

□

Next, we fixed A such that $Ac_1\geqslant 4$.

Remark 3.14. Note that $\|\tilde{n}(t,0)\|_{L^2(\mathbb{R}^2)}\leqslant\sqrt{2}$, we confine the value of A to distinguish $\|\tilde{n}(t,\theta(t))\|_{L^2(\mathbb{R}^2)}$ and $\|\tilde{n}(t,0)\|_{L^2(\mathbb{R}^2)}$. □

Remark 3.15. By Proposition 3.12, one can obtain more delicate estimates on $\tilde{\mathbf{v}}(s)$. Compared to the classical case, these estimates are uniform. □

Corollary 3.16. *For any $s\in[0,\theta(t)]$, there holds*

$$\|\tilde{\boldsymbol{v}}(s)\|_{L^2(\mathbb{R}^2)}\leqslant A\sqrt{\frac{2}{\eta}}.$$

Proof. From (2.7) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx\\
&\quad-\frac{\eta}{2}\int_{\mathbb{R}^2}\left|\overline{\tilde{E}_1}\tilde{E}_2-\tilde{E}_1\overline{\tilde{E}_2}\right|^2dx\\
&\geqslant\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1+\eta}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx\\
&\geqslant\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1+\eta}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\left(|E_{10}|^2+|E_{20}|^2\right)dx\\
&\qquad\cdot\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx\\
&=\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx\left(1-\frac{1+\eta}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\left(|E_{10}|^2+|E_{20}|^2\right)dx\right).
\end{aligned}$$

By (3.4) and (3.179), there holds

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left[\tilde{n}+\left(|\tilde{E}_1|^2+|\tilde{E}_2|^2\right)\right]^2dx+\|\tilde{\mathbf{v}}\|^2_{L^2(\mathbb{R}^2)}\\
&=\frac{2H_0}{\lambda^2}-2\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx+\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx\\
&\quad+\eta\int_{\mathbb{R}^2}\left|\overline{\tilde{E}_1}\tilde{E}_2-\tilde{E}_1\overline{\tilde{E}_2}\right|^2dx\\
&\leqslant\frac{2H_0}{\lambda^2}-\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx\\
&\qquad\cdot\left(2-\frac{2(1+\eta)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\left(|E_{10}|^2+|E_{20}|^2\right)dx\right)\\
&\leqslant\frac{2H_0}{\lambda^2}+\left(\frac{2(1+\eta)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\left(|E_{10}|^2+|E_{20}|^2\right)dx-2\right)A^2\\
&\leqslant\frac{2H_0}{\lambda^2}+\frac{2}{\eta}A^2.
\end{aligned}$$

Note that $\lambda\to+\infty$ as $t\to T$, one then obtains

$$\|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\leqslant\sqrt{\frac{2}{\eta}}A.$$

This finishes the proof of Corollary 3.16. □

Proposition 3.17. There exists a constant $c>0$ such that

$$\liminf_{t\to T}\int_0^{\theta(t)}\|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\,ds\geqslant c. \tag{3.200}$$

Proof. We argue it by contradiction. Assume that as $n\to+\infty$, there exists a sequence $t_n\to T$ such that

$$\int_0^{\theta(t_n)}\|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\,ds\to 0. \tag{3.201}$$

From (3.177) it follows that $\forall\psi(x)\in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\tilde{n}(t_n,\theta(t_n))\psi dx-\int_{\mathbb{R}^2}\tilde{n}(t_n,0)\psi dx\\
&=\int_0^{\theta(t_n)}\int_{\mathbb{R}^2}\left(-\nabla\cdot\tilde{\mathbf{v}}(s)\psi\right)\,dxds\\
&=\int_0^{\theta(t_n)}\int_{\mathbb{R}^2}\tilde{\mathbf{v}}(s)\cdot\nabla\psi\,dxds,
\end{aligned}\tag{3.202}$$

which yields

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\tilde{n}(t_n,\theta(t_n))\psi\,dx-\int_{\mathbb{R}^2}\tilde{n}(t_n,0)\psi\,dx\right|\\&\leqslant\left(\int_0^{\theta(t_n)}\|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\,ds\right)\|\nabla\psi\|_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{3.203}$$

By (3.185) and (3.186), taking a subsequence still denoted by t_n yields that there exist a sequence $x_n \in \mathbb{R}^2$, and $N' \in L^2(\mathbb{R}^2)$ such that

$$\tilde{n}\left(t_n,\theta(t_n),x-x_n\right)\rightharpoonup N' \text{ in } L^2(\mathbb{R}^2),$$

and

$$\|N'\|_{L^2(\mathbb{R}^2)}\geqslant Ac_1.$$

Let $\psi_0(x)\in C_0^\infty(\mathbb{R}^2)$ satisfy $\left(\int_{\mathbb{R}^2}\psi_0^2\,dx\right)^{\frac{1}{2}}=1$ and $\int_{\mathbb{R}^2}N'\psi_0\,dx\geqslant\frac{1}{2}\left(\int_{\mathbb{R}^2}N'^2\,dx\right)^{\frac{1}{2}}$.
In view of the assumptions (3.201) and (3.202), we have as $n\to+\infty$,

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\tilde{n}\left(t_n,\theta(t_n),x\right)\psi_0\left(x+x_n\right)dx-\int_{\mathbb{R}^2}\tilde{n}\left(t_n,0,x\right)\psi_0\left(x+x_n\right)dx\right|\\&\leqslant\left(\int_0^{\theta(t_n)}\|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\,ds\right)\|\nabla\psi_0\left(x+x_n\right)\|_{L^2(\mathbb{R}^2)}\to 0.\end{aligned}\tag{3.204}$$

On the other hand, one has

$$\begin{aligned}\int_{\mathbb{R}^2}\tilde{n}\left(t_n,\theta(t_n),x\right)\psi_0(x+x_n)\,dx=&\int_{\mathbb{R}^2}\tilde{n}\left(t_n,\theta(t_n),x-x_n\right)\psi_0(x)\,dx\\&\to\int_{\mathbb{R}^2}N'\psi_0dx\quad(n\to+\infty)\\&\geqslant\frac{1}{2}\left(\int_{\mathbb{R}^2}N'^2dx\right)^{\frac{1}{2}}\geqslant\frac{Ac_1}{2}\geqslant 2.\end{aligned}\tag{3.205}$$

However, by (2.5) one has

$$\left|\int_{\mathbb{R}^2}\tilde{n}(t_n,0)\psi_0\,dx\right|\leqslant\left(\int_{\mathbb{R}^2}|\tilde{n}(t_n,0)|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\psi_0^2\,dx\right)^{\frac{1}{2}}\leqslant\sqrt{2}.\tag{3.206}$$

It is obviously contradictory to (3.205).
This finishes the proof of Proposition 3.17. □

Remark 3.18. (3.200) gives the estimate for $\theta(t)$.
In fact, in view of (3.183), if

$$\int_0^{\theta(t)}\|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\,ds\leqslant\int_0^{\theta(t)}Ac_2\,ds=Ac_2\theta(t),\tag{3.207}$$

then (3.200) implies that there exists a constant $c>0$ such that

$$\liminf_{t\to T}Ac_2\theta(t)\geqslant c,$$

that is,

$$\liminf_{t\to T}\theta(t)\geq c.\tag{3.208}$$

On the other hand, by Corollary 3.16 one has

$$c \leqslant \int_0^{\theta(t)} \|\tilde{\mathbf{v}}\|_{L^2(\mathbb{R}^2)} ds \leqslant \theta(t)\sqrt{\frac{2}{\eta}}A, \tag{3.209}$$

namely, there exists $\tilde{c} = \dfrac{c}{\sqrt{2}A}$ such that

$$\theta(t) \geqslant \tilde{c}\sqrt{\eta}. \tag{3.210}$$

Therefore (3.208) and (3.210) imply that there exists a constant $\theta_0 > 0$ such that as $t \to T$, $\theta(t) \geq \theta_0$, and

$$\forall s \in [0, \theta), \quad \left\| \left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(s) \right\|_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \leqslant A. \tag{3.211}$$

This finishes the proof of Theorem 3.1. □

4. Proof of the main results(Theorem 1.3)

In this section, based on the estimates obtained in Section 2 and Section 3, we prove the main result (Theorem 1.3) of this paper.

We first show Conclusion (1) of Theorem 1.3.

By Theorem 3.1, as $t \to T$, there exist $\theta_0 = \theta_0\left(\|E_{10}\|_{L^2(\mathbb{R}^2)}, \|E_{20}\|_{L^2(\mathbb{R}^2)}\right)$ and $A > 0$ such that

$$\forall s \in [0, \theta_0), \quad \left\| \left(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\mathbf{v}}\right)(s) \right\|_{H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} \leqslant A. \tag{4.1}$$

In view of (2.2) and (2.4), one gets

$$\lambda(t)(T-t) \geq \theta_0. \tag{4.2}$$

This yields the estimate (1.6).

In addition, it follows from (2.1) that

$$\begin{aligned} &\left(\left\|\nabla\tilde{E}_1(0)\right\|^2_{L^2(\mathbb{R}^2)} + \left\|\nabla\tilde{E}_2(0)\right\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \\ &\quad = \frac{1}{\lambda(t)}\left(\|\nabla E_1(t)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2(t)\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

$$\|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)}\|n(t)\|_{L^2(\mathbb{R}^2)}. \tag{4.4}$$

Going back to Proposition 3.3, (4.3) and (4.4) yields

$$\left(\|\nabla E_1(t)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2(t)\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \geq c_1\lambda(t) \geq \frac{c_1\theta}{T-t} = \frac{\tilde{c}}{T-t}, \tag{4.5}$$

$$\|n(t)\|_{L^2(\mathbb{R}^2)} \geq c_1\lambda(t) \geq \frac{c_1\theta}{T-t} = \frac{\tilde{c}}{T-t}. \tag{4.6}$$

This completes the proof of (1) in Theorem 1.3.

Next we are going to prove the conclusion (2). Firstly we claim the following proposition:

Proposition 4.1. For θ_0 in Theorem 3.1, there exists a contant $c > 0$ such that

$$\theta_0 \geq \frac{c}{\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{\eta+1}\|Q\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}}}. \tag{4.7}$$

Proof. Due to the Hamiltonian given by (2.7), one gains

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^2}\left(n+\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}|^2\,dx \qquad (4.8)\\
&\quad\le\frac{\mathcal{H}_0}{\lambda^2(t)}+2\eta\int_{\mathbb{R}^2}\left|\tilde{E}_1\right|^2\left|\tilde{E}_2\right|^2dx,
\end{aligned}$$

which yields

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1+\eta}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^2}\left(n+\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}|^2\,dx \qquad (4.9)\\
&\quad\le\frac{\mathcal{H}_0}{\lambda^2(t)}.
\end{aligned}$$

A direct calculation then gives

$$\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1+\eta}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx\le\frac{\mathcal{H}_0}{\lambda^2(t)}, \qquad (4.10)$$

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}\left(n+\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}|^2\,dx\\
&\quad\le\frac{\mathcal{H}_0}{\lambda^2(t)}-\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx \qquad (4.11)\\
&\qquad+\frac{1+\eta}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(1-\frac{(\eta+1)\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)}\right)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\right)\\
&\quad\cdot\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx \qquad (4.12)\\
&\le\int_{\mathbb{R}^2}\left(\left|\nabla\tilde{E}_1\right|^2+\left|\nabla\tilde{E}_2\right|^2\right)dx-\frac{1+\eta}{2}\int_{\mathbb{R}^2}\left(\left|\tilde{E}_1\right|^2+\left|\tilde{E}_2\right|^2\right)^2dx,
\end{aligned}$$

then one obtains

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2}\left(n+|\tilde{E}_1|^2+|\tilde{E}_2|^2\right)^2+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{v}}|^2dx\\
&\quad\le\frac{\mathcal{H}_0}{\lambda^2(t)}+\left(\frac{(\eta+1)\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)}+\|E_{20}\|^2_{L^2(\mathbb{R}^2)}\right)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}-1\right) \qquad (4.13)\\
&\qquad\cdot\int_{\mathbb{R}^2}\left(|\nabla\tilde{E}_1|^2+|\nabla\tilde{E}_2|^2\right)dx.
\end{aligned}$$

Since $\lambda(t) \to \infty$ as $t \to T$, by taking $t \to T$ one obtains

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2} |\tilde{\mathbf{v}}|^2 dx \\
&\leq \frac{\eta+1}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\qquad \cdot \int_{\mathbb{R}^2}\left(|\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2\right) dx \\
&\leq \frac{A^2(\eta+1)}{\|Q\|^2_{L^2(\mathbb{R}^2)}}\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned} \tag{4.14}$$

Therefore, by Proposition 3.17, we obtain

$$\begin{aligned}
c &\leq \liminf_{t\to T}\int_0^{\theta(t)} \|\tilde{\mathbf{v}}(s)\|_{L^2(\mathbb{R}^2)}\, ds \\
&\leq \frac{A\sqrt{2(\eta+1)}}{\|Q\|_{L^2(\mathbb{R}^2)}}\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)}\right)^{\frac{1}{2}} \theta_0,
\end{aligned} \tag{4.15}$$

and

$$\theta_0 \geq c'\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)}\right)^{-\frac{1}{2}}. \tag{4.16}$$

This finishes the proof of Proposition 4.1. □

Using Proposition 4.1 and taking $\theta_0 = c'\left(\|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{1+\eta}\|Q\|^2_{L^2(\mathbb{R}^2)}\right)^{-\frac{1}{2}}$ in the proof of (1.7) and (1.8), we achieve (1.9) and (1.10).

This finishes the proof of Theorem 1.3. □

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