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Low-regularity integrator for the Davey-Stewartson II system

Cui Ning · Xiaomin Kou · Yaohong Wang

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Abstract We consider the Davey-Stewartson system in the hyperbolic-elliptic case (DS-II) in two dimensional case. It is a mass-critical equation, and was proved recently by Nachman, Regev and Tataru [1] the global well-posedness and scattering in L^2. In this paper, we give the numerical study on this model and construct a first order low-regularity integrator for the DS-II in the periodic case. It only requires the boundedness of one additional derivative of the solution to get the first order convergence. The Fast Fourier Transform is exploited to speed up the numerical implementation. By rigorous error analysis, we prove that the numerical scheme provides first order convergence in $H^\gamma(\mathbb{T}^2)$ for rough initial data in $H^{\gamma+1}(\mathbb{T}^2)$ with $\gamma > 1$. The optimality of the convergence is conformed by numerical experience.

Keywords Hyperbolic-elliptic Davey-Stewartson system · low-regularity integrator · first order convergence

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C. Ning
School of Financial Mathematics and Statistics, Guangdong University of Finance, Guangzhou, Guangdong 510521, China
E-mail: cuiningmath@sina.com

X. Kou
Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China
E-mail: 15733169190@163.com

Y. Wang
Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China
E-mail: yaohong@tju.edu.cn

1 Introduction

The Davey-Stewartson(DS) systems originated from fluid mechanics was first proposed by Davey and Stewartson [2] in the context of water waves in 1974, which describe the dynamic evolution process of wave packet in finite depth water, and waves travel in the main direction under the forces of gravity and surface tension. The DS systems have extensive applications in fluid mechanics [3,4], nonlinear optics [5,6], ferromagnetism [7], plasma physics [8]. In dimensionless form, they are generally read as the following systems for the amplitude $u(t;x_1,x_2)$ and the mean velocity potential $v(t;x_1,x_2)$:

$$\begin{cases} iu_t + \lambda u_{x_1x_1} + \mu u_{x_2x_2} = \mu_1|u|^2u + \mu_2 u v_{x_1}, \\ \nu v_{x_1x_1} + v_{x_2x_2} = \mu_3 \partial_{x_1}\left(|u|^2\right), \end{cases} \tag{1}$$

where λ, μ, ν, μ_1, μ_2, μ_3 are real constants and $\lambda > 0$, $\mu_2\mu_3 > 0$. These systems can be classified as elliptic-elliptic, elliptic-hyperbolic (DS-I), hyperbolic-elliptic (DS-II), hyperbolic-hyperbolic according to the signs of μ, ν: $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$. The Cauchy problems for the DS systems are widely studied. By using the functional analytic method, Ghidaglia and Saut [9] proved the existence, uniqueness and continuous dependence with initial value in H^1 for the elliptic-elliptic, DS-I and DS-II cases.

The DS systems in elliptic-elliptic case is a special case of nonlinear Schrödinger equations:

$$iu_t + \Delta u = \pm R_1^2(|u|^2)u,$$

where $R_1 = \partial_{x_1}|\nabla|^{-1}$ is the Riesz transformation. There are a large amount of works investigated this model and we only mention a few of them. The Cauchy problem for DS system in this case was studied by Gan and Zhang [10] who gave a sharp threshold for global existence and blowup of the solution. Wang and Guo [11] considered the scattering for a generalized DS systems. Lu and Wu [12] used a variational approach to give a dichotomy of the blow-up and the scattering for the generalized DS system.

The DS systems in elliptic-hyperbolic case, usually named by DS-I, is also extensively studied. In particular, Linares and Ponce [13] established the local well-posedness of small initial value problems for the DS-I and the hyperbolic-hyperbolic cases. Tsutsumi [14] obtained the L^p-decay estimates of solutions of the DS-I case for $2 < p < \infty$. Then, Hayashi and Saut [15] used the dispersive method to establish the local well-posedness and the global existence of the DS-I and the hyperbolic-hyperbolic cases for general large initial values.

The analytical work on the DS systems in hyperbolic-elliptic case, which was named by DS-II, is much limited. Ozawa [16] obtained the exact blow-up solution of the DS-II system. An important progress was made recently by Nachman, Regev and Tataru [1], who proved the global well-posedness and scattering for the defocusing DS-II system in L^2 by applying a Plancherel theorem.

Extensive numerical methods have been proposed for nonlinear Schrödinger equation and Korteweg-de Vries equation, including finite difference methods

[17–19], finite element methods [20–22], splitting methods [23,24], spectral methods [25–27], discontinuous Galerkin methods [28,29], exponential integrators [30,31], low-regularity integrators [32–38], etc. For the DS systems, White and Weideman [39,40] presented the numerical study of dromions for the DS-I and the DS-II by the split-step Fourier method. Besse, Mauser and Stimming [41] showed numerical results for the various blow-up phenomena of E-E and for the exact soliton type solutions of DS-II by the time splitting spectral method. Klein and Stoilov [42] discussed spectral algorithms for the numerical scattering for the defocusing DS-II with initial data having the compact support on a disk. However, all of these methods require high smoothness of the initial data.

In practical applications, due to randomness, noise and measurement, the initial data usually can't meet the requirements of the high smoothness. The convergence of a certain order can be achieved by imposing sufficient additional regularity, when the solution of the equation is not smooth enough. More precisely, to get

$$\|u^n - u(t_n)\|_{H^\gamma} \leq C\tau^\alpha,$$

where u^n is the numerical solution and $u(t_n)$ is the exact solution, τ denotes the time step, γ describes roughness of the solution, and α denotes the order of convergence, the initial value $u_0 = u(0, \boldsymbol{x})$ should belong to some $H^{\gamma+s}$ space, where the index s denotes the order of spatial derivatives of the solution that has been lost in the numerical implementation. For the cubic nonlinear Schrödinger equation, several studies of numerical algorithms with rough initial data have been carried out. Lubich [43] showed first-order convergence in H^γ for initial value in $H^{\gamma+2}$ by Strange splitting method. Ostermann and Schratz [44] proved first-order convergence in H^γ for initial value in $H^{\gamma+1}$ with $\gamma > \frac{d}{2}$ by introducing the low regularity exponential-type integrators. Wu and Yao [45] presented a Fourier integrator, which shows the first-order convergence in H^γ for initial value in H^γ for $\gamma > \frac{3}{2}$ in one dimension without any derivative loss. Ostermann, Wu and Yao [46] proposed a new exponential-type integrator on the torus $\mathbb{T}^d$, which proves the second-order convergence in $H^\gamma(\mathbb{T}^d)$ for initial data in $H^{\gamma+2}(\mathbb{T}^d)$ for any $\gamma > \frac{d}{2}$.

Recently, Ning and Wang [47] proposed a numerical integrator for elliptic-elliptic DS system and proved that the algorithm can achieve first-order convergence in $H^\gamma(\mathbb{T}^d)$ for rough initial data in $H^{\gamma+1}(\mathbb{T}^d)$ with $r > \frac{d}{2}$.

In this paper, we focus on the DS-II system with the rough initial data on a torus:

$$\begin{cases} iu_t + u_{x_1x_1} - u_{x_2x_2} = \mu_1|u|^2u + \mu_2 uv_{x_1}, \quad t > 0, \boldsymbol{x} = (x_1, x_2) \in \mathbb{T}^2, \\ v_{x_1x_1} + v_{x_2x_2} = \partial_{x_1}\left(|u|^2\right), \end{cases} \tag{2}$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\mathbb{T} = (0, 2\pi)$. Here $u = u(t, \boldsymbol{x}) : \mathbb{R}^+ \times \mathbb{T}^2 \longrightarrow \mathbb{C}$ is the (complex) amplitude of the wave and v is the (real) velocity potential of the wave movement, and $u_0(\boldsymbol{x}) = u(0, \boldsymbol{x}) \in H^\gamma(\mathbb{T}^2)$ with some $0 \leq \gamma < \infty$ is a given initial data.

Compared to the elliptic-elliptic DS system, the structure of the DS-II system is more complicated. The phase functions of the two systems are completely different, since the principal operator of (2) is not elliptic which makes the resonant set much larger.

The corresponding phase of the nonlinearity term in equation(2) is

$$\xi^2 - \eta^2 + (\xi_1^2 - \eta_1^2) - (\xi_2^2 - \eta_2^2) - (\xi_3^2 - \eta_3^2),$$

where $\xi = \xi_1 + \xi_2 + \xi_3, \eta = \eta_1 + \eta_2 + \eta_3$. This phase function cannot be dealt with in the same way as the elliptic-elliptic case. In previous work[47], it was treated in a certain manner that resulted in the principal part being reduced to $\xi_1^2 - \eta_1^2$. However, this approach does not lead to the development of an efficient algorithm. Hence, the method proposed for the elliptic-elliptic case [47] is not directly applicable to the DS-II system. In order to address this issue, we utilize the characteristic transformation to simplified the operator $\partial_{x_1x_1} - \partial_{x_2x_2}$ to $\partial_{x_1x_2}$. Then the phase function reduces to

$$\xi\eta + \xi_1\eta_1 - \xi_2\eta_2 - \xi_3\eta_3.$$

We can further simplify the above function by the following approximation

$$\xi\eta + \xi_1\eta_1 - \xi_2\eta_2 - \xi_3\eta_3 = 2\xi_1\eta_1 + O(\sum_{j\neq k} \xi_j\eta_k).$$

See Section 2.2 below. This allows us to construct a class of low-regularity algorithm below which only requires the boundedness of one additional derivative of the solution with rough initial data to get the first-order convergence.

Theorem 1 *Let u^n and v^n be the numerical solution of the DS system (2) obtained from the schemes (28) and (29) up to some fixed time $T > 0$. Under the assume $u_0(\boldsymbol{x}) \in H^{\gamma+1}(\mathbb{T}^2)$ for some $\gamma > 1$, there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \leq \tau_0$, we have*

$$\|u(t_n) - u^n\|_{H^\gamma} \leq C\tau, \quad \|v(t_n) - v^n\|_{H^{\gamma+1}} \leq C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}, \qquad (3)$$

where the constants τ_0 and C depend only on T and $\sup_{0\leq t\leq T} \|u(t)\|_{H^{\gamma+1}}$.

This paper is organized as follows. The detailed numerical integrator is introduced in section 2. The first-order convergence analysis are given in section 3. The numerical experiments are presented to validate the numerical scheme in section 4, and concluding remarks are made in section 5.

2 Construction of the scheme

In this section, we shall firstly present some useful definitions and properties, and then derive our numerical scheme.

2.1 Definitions and Properties.

We use $A \lesssim B$ or $B \gtrsim A$ to denote $A \leq CB$ for some large absolute constant $C > 0$, and we denote $A \sim B$ for $A \lesssim B \lesssim A$.

For $\boldsymbol{k} := (k_1, k_2) \in \mathbb{Z}^2, \boldsymbol{x} := (x_1, x_2) \in \mathbb{T}^2$, we denote

$$\boldsymbol{k} \cdot \boldsymbol{x} = k_1 x_1 + k_2 x_2, \qquad |\boldsymbol{k}|^2 = |k_1|^2 + |k_2|^2.$$

The Fourier transform of a function f on $\mathbb{T}^2$ is defined by

$$\hat{f}_{\boldsymbol{k}} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

For $f \in L^2(\mathbb{T}^2)$, we denote its Fourier expansion by

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \hat{f}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{4}$$

Let $H^s(\mathbb{T}^2)$ be the classic Sobolev space of H^s functions defined on the 2-dimensional torus $\mathbb{T}^2$. Its norm $\|\cdot\|_{H^s}$ is defined by

$$\|f\|^2_{H^s(\mathbb{T}^2)} = \|J^s f\|^2_{L^2(\mathbb{T}^2)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} (1+|\boldsymbol{k}|)^{2s} \left|\hat{f}_{\boldsymbol{k}}\right|^2, \tag{5}$$

where the operator J^s is defined by

$$J^s = (1-\Delta)^{\frac{s}{2}}, \quad s \in \mathbb{R}.$$

Furthermore, we define the operator $(-\Delta)^{-1}$ for some function $f(\boldsymbol{x})$ as

$$\widehat{(-\Delta)^{-1} f} = \begin{cases} |\boldsymbol{k}|^{-2} \hat{f}_{\boldsymbol{k}}, & \text{if } \ \boldsymbol{k} \neq 0, \\ 0, & \text{if } \ \boldsymbol{k} = 0, \end{cases} \tag{6}$$

and define the operator $\partial_{x_1 x_2}$ as

$$\widehat{\partial_{x_1 x_2} f} = -k_1 k_2 \hat{f}_{\boldsymbol{k}}, \qquad \boldsymbol{k} = (k_1, k_2). \tag{7}$$

Throughout the paper, we will exploit the well known bilinear estimate

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s} \|g\|_{H^s},$$

which hold for $\gamma > \frac{d}{2}$. Moreover, we will make frequent use of the isometric property of the free Schrödinger group $e^{i\partial_{xy} t}$

$$\left\|e^{i\partial_{xy} t} f\right\|_{H^s} = \|f\|_{H^s},$$

for all $f \in H^\gamma$ and $t \in \mathbb{R}$.

In addition, we need the following operator ω:

$$\omega(z) = \begin{cases} \dfrac{e^z - 1}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{8}$$

2.2 Construction of the numerical integrator

For simplicity of notations, in the following we shall omit the spatial variable $\boldsymbol{x}$ of the involved time-space dependent functions, e.g. $u(t) = u(t, \boldsymbol{x})$. In addition, we denote $\tau > 0$ as the time step and $t_n = n\tau$ as the time grids.

For the DS-II system

$$\begin{cases} iu_t + u_{x_1x_1} - u_{x_2x_2} = \mu_1|u|^2u + \mu_2 uv_{x_1}, \\ v_{x_1x_1} + v_{x_2x_2} = \partial_{x_1}\left(|u|^2\right), \end{cases}$$

where μ_1, μ_2 are real constants, and $\mu_1 > 0$, $\mu_2 > 0$.
By introducing the following variable substitution

$$\begin{cases} \xi_1 = \dfrac{1}{2}(x_1 + x_2), \\ \xi_2 = \dfrac{1}{2}(x_1 - x_2), \end{cases}$$

that is

$$\begin{cases} u(x_1, x_2) = \phi(\xi_1, \xi_2), \\ v(x_1, x_2) = \psi(\xi_1, \xi_2), \end{cases} \tag{9}$$

we can obtain the equation of ϕ and ψ

$$\begin{cases} i\phi_t + \phi_{\xi_1\xi_2} = \mu_1|\phi|^2\phi + \dfrac{1}{2}\mu_2\phi\left(\psi_{\xi_1} + \psi_{\xi_2}\right), \\ \psi_{\xi_1\xi_1} + \psi_{\xi_2\xi_2} = \left(\partial_{\xi_1} + \partial_{\xi_2}\right)\left(|\phi|^2\right). \end{cases} \tag{10}$$

We can reduce the equation (10) as follows

$$\begin{cases} i\phi_t + \partial_{\xi_1\xi_2}\phi - \phi E(|\phi|^2) = 0, \\ \psi = -(-\Delta)^{-1}(\partial_{\xi_1} + \partial_{\xi_2})(|\phi|^2), \end{cases} \tag{11}$$

where the operator E is defined by

$$Ef = (\tilde{\mu}_1 + \mu_2\frac{\partial_{\xi_1\xi_2}}{\Delta})f, \qquad \tilde{\mu}_1 = \mu_1 + \frac{1}{2}\mu_2. \tag{12}$$

Using the Duhamel's formula, we have

$$\phi(t) = e^{i\partial_{\xi_1\xi_2}t}\phi_0(\mathbf{x}) - i\int_0^t e^{i\partial_{\xi_1\xi_2}(t-s)}[\phi(s)E(|\phi(s)|^2)]ds.$$

We introduce the twisted variable

$$\varphi(t) = e^{-i\partial_{\xi_1\xi_2}t}\phi(t). \tag{13}$$

Note that the twisted variable $\varphi(t)$ satisfies

$$\partial_t\varphi(t) = -ie^{-i\partial_{\xi_1\xi_2}t}[e^{i\partial_{\xi_1\xi_2}t}\varphi(t) \cdot E(\left|e^{i\partial_{\xi_1\xi_2}t}\varphi(t)\right|^2)], \tag{14}$$

with the mild solution given by

$$\varphi(t_{n+1}) = \varphi(t_n) - i\int_0^\tau e^{-i\partial_{\xi_1\xi_2}(t_n+s)}\left[e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n+s)\cdot E(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n+s)\right|^2)\right]ds. \tag{15}$$

Since we only need first order convergent scheme, we can simplify the above scheme using $\varphi(t_n+s)\approx\varphi(t_n)$, and get

$$\varphi(t_{n+1}) = \varphi(t_n) - \Phi^n(\varphi(t_n)) - \mathcal{R}_1^n, \tag{16}$$

where

$$\Phi^n(\varphi(t_n)) = i\int_0^\tau e^{-i\partial_{\xi_1\xi_2}(t_n+s)}\left[e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n)\cdot E(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n)\right|^2)\right]ds,$$

and

$$\begin{aligned}\mathcal{R}_1^n = i\int_0^\tau e^{-i\partial_{\xi_1\xi_2}(t_n+s)}&\Big[e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n+s)\cdot E(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n+s)\right|^2)\\ &-e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n)\cdot E(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n)\right|^2)\Big]ds.\end{aligned} \tag{17}$$

Here, $\mathcal{R}_1^n$ can be treated as a high-order term without additional regularity assumption. Next, using Fourier expansion, we get

$$\Phi^n(\varphi(t_n)) = i\sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}}\left[\tilde{\mu}_1+\mu_2\frac{(j_1+k_1)(j_2+k_2)}{|\boldsymbol{j}+\boldsymbol{k}|^2}\right]\hat{\bar{\varphi}}_{\boldsymbol{j}}\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}_{\boldsymbol{l}}e^{i\boldsymbol{\Omega}\cdot\boldsymbol{\xi}}\cdot\int_0^\tau e^{i(\Omega_1\Omega_2+j_1j_2-k_1k_2-l_1l_2)s}ds, \tag{18}$$

where we set $\hat{\bar{\varphi}}_{\boldsymbol{j}}=\hat{\bar{\varphi}}_{\boldsymbol{j}}(t_n)$, $\hat{\varphi}_{\boldsymbol{k}}=\hat{\varphi}_{\boldsymbol{k}}(t_n)$, $\hat{\varphi}_{\boldsymbol{l}}=\hat{\varphi}_{\boldsymbol{l}}(t_n)$ for short. And we denote $\alpha=\Omega_1\Omega_2+j_1j_2-k_1k_2-l_1l_2$ and $\beta=j_1(k_2+l_2)+k_1(j_2+l_2)+l_1(j_2+k_2)$. Then, we have

$$\alpha = 2j_1j_2+\beta.$$

For the integration in (18), it cannot be transformed into the physical space directly. Inspired by [44], we only choose the dominant quadratic term $2j_1j_2$, so that the integration can be carried out fully in Fourier space as

$$\int_0^\tau e^{2ij_1j_2s}ds = \tau\omega(2i\tau j_1j_2). \tag{19}$$

Hence, $\Phi^n(\varphi(t_n))$ can be written as

$$\begin{aligned}\Phi^n(\varphi(t_n)) &= i\sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}}e^{it_n\alpha}\left[\tilde{\mu}_1+\mu_2\frac{(j_1+k_1)(j_2+k_2)}{|\boldsymbol{j}+\boldsymbol{k}|^2}\right]\hat{\bar{\varphi}}_{\boldsymbol{j}}\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}_{\boldsymbol{l}}\cdot e^{i\boldsymbol{\Omega}\cdot\boldsymbol{\xi}}\tau\omega(2i\tau j_1j_2)+\mathcal{R}_2^n\\ &= i\tau e^{-i\partial_{\xi_1\xi_2}t_n}\left[e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)\cdot E\left(\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)\right)\right]+\mathcal{R}_2^n,\end{aligned}$$

where

$$\mathcal{R}_2^n = i \sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2} \sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} e^{it_n\alpha}\left[\tilde{\mu}_1 + \mu_2\frac{(j_1+k_1)(j_2+k_2)}{|\boldsymbol{j}+\boldsymbol{k}|^2}\right]\hat{\bar{\varphi}}_{\boldsymbol{j}}\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}_{\boldsymbol{l}}e^{i\boldsymbol{\Omega}\cdot\boldsymbol{\xi}}\cdot\int_0^\tau e^{2isj_1j_2}(e^{is\beta}-1)ds. \tag{20}$$

Here, $\mathcal{R}_2^n$ can also be treated as a high-order term, however, one loss of spatial derivative comes when we drop this term.

For convenience, let's denote

$$\Psi^n(f) = i\tau e^{-i\partial_{\xi_1\xi_2}t_n}\left[e^{i\partial_{\xi_1\xi_2}t_n}f\cdot E\left(\omega\left(-2i\partial_{\xi_1\xi_2}\tau\right)\overline{e^{i\partial_{\xi_1\xi_2}t_n}f}\cdot e^{i\partial_{\xi_1\xi_2}t_n}f\right)\right]. \tag{21}$$

Then, we have

$$\Phi^n(\varphi(t_n)) = \Psi^n\left(\varphi\left(t_n\right)\right) + \mathcal{R}_2^n. \tag{22}$$

So far, we can obtain the representation of the exact solution $\varphi\left(t_{n+1}\right)$

$$\varphi\left(t_{n+1}\right) = \varphi\left(t_n\right) - \Psi^n\left(\varphi\left(t_n\right)\right) - \mathcal{R}_1^n - \mathcal{R}_2^n. \tag{23}$$

By dropping high order terms, we would get

$$\varphi\left(t_{n+1}\right) \approx \varphi\left(t_n\right) - \Psi^n\left(\varphi\left(t_n\right)\right). \tag{24}$$

Hence we finish the construction of the first order numerical integrator of φ

$$\varphi^{n+1} = \varphi^n - i\tau e^{-i\partial_{\xi_1\xi_2}t_n}\left[e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n\cdot E\left(\omega\left(-2i\partial_{\xi_1\xi_2}\tau\right)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n\right)\right], n\geq 0, \tag{25}$$

where $\varphi^0 = \varphi(0,\xi_1,\xi_2) = \phi(0,\xi_1,\xi_2) = u(0,x_1,x_2)$. By reversing the twisted variable (13) in (25), we deduce the scheme of the first order low-regularity integrator for solving the DS system (11): denote $\phi^n = \phi^n(\boldsymbol{\xi})$ as the numerical solution, for $n = 1,2,3,...$¡¡

$$\phi^n = e^{i\partial_{\xi_1\xi_2}\tau}\phi^{n-1} - i\tau e^{i\partial_{\xi_1\xi_2}\tau}\left[\phi^{n-1}\cdot E\left(\omega\left(-2i\partial_{\xi_1\xi_2}\tau\right)\overline{\phi^{n-1}}\cdot\phi^{n-1}\right)\right], \quad n = 1,2,3,..., \tag{26}$$

with ω in (8) and E in (12). Based on the DS system (11), we can write the numerical solution of ψ: denote $\psi^n = \psi^n(\boldsymbol{\xi})$ as the numerical solution, for $n = 1,2,3,...,$

$$\psi^n = -(-\Delta)^{-1}(\partial_{\xi_1}+\partial_{\xi_2})(|\phi^n|^2), \quad n = 1,2,3,\ldots. \tag{27}$$

In order to obtain an approximation to the original solution $u(t_n)$ of the DS-II equation (2), we then substitute the variable back again by setting $u^n(\boldsymbol{x}) = \phi^n(\boldsymbol{\xi})$. This yields the following integrator for the DS-II equation (2):

$$u^n = e^{i(\partial_{x_1}^2-\partial_{x_2}^2)\tau}u^{n-1} - i\tau e^{i(\partial_{x_1}^2-\partial_{x_2}^2)\tau}\left[u^{n-1}\cdot\hat{E}\left(\omega\left(-2i(\partial_{x_1}^2-\partial_{x_2}^2)\tau\right)\overline{u^{n-1}}\cdot u^{n-1}\right)\right], \tag{28}$$

where $n = \{1,2,3,...\}$ and $\hat{E}f = (\mu_1 + \frac{1}{2}\mu_2 + \mu_2\frac{\partial_{x_1}^2-\partial_{x_2}^2}{\Delta})f$.

Based on the DS system (2), we can write the numerical solution of v:

$$v^n = -(-\Delta)^{-1}\partial_{x_1}(|u^n|^2), \quad n = 1,2,3,\ldots. \tag{29}$$

3 The first-order convergence analysis

In this section, we will provide the rigorous proof of the convergence result. Because of the variable substitution (9), the conclusions about the equation (2) and (10) are the same, so we are going to prove ϕ. Since the twisted variable in (13) is isometric in the Sobolev space H^γ, that is

$$\|\phi(t_n)-\phi^n\|_{H^\gamma}=\|e^{i\partial_{\xi_1\xi_2}t}\varphi(t_n)-e^{i\partial_{\xi_1\xi_2}t}\varphi^n\|_{H^\gamma}=\|\varphi(t_n)-\varphi^n\|_{H^\gamma},$$

we will prove the first-order convergence theorem for the mild solution φ^n and $\varphi(t_n)$.

Recall the exact solution in Section 2

$$\varphi(t_{n+1})=\varphi(t_n)-\Psi^n(\varphi(t_n))-\mathcal{R}_1^n-\mathcal{R}_2^n,$$

and numerial solution

$$\varphi^{n+1}=\varphi^n-\Psi^n(\varphi^n).$$

The error of the two solutions is

$$\varphi(t_{n+1})-\varphi^{n+1}=\mathcal{L}^n+\mathcal{S}^n, \tag{30}$$

where

$$\mathcal{L}^n=-(\mathcal{R}_1^n+\mathcal{R}_2^n),$$

and

$$\mathcal{S}^n=\varphi(t_n)-\Psi^n(\varphi(t_n))-\varphi^n+\Psi^n(\varphi^n).$$

Then, we will analyze the local error and the stability of the numerical propagator in the following.

3.1 Local error estimation

Firstly, we give the following residual estimate, and then present local error estimate.

Lemma 1 *Let $\gamma>1$. Assume that $\phi_0\in H^{\gamma+1}$, then there exist constants $\tau_0>0$ and $C>0$, such that for any $0<\tau\le\tau_0$, the following estimate hold:*

$$\|\mathcal{R}_1^n\|_{H^\gamma}+\|\mathcal{R}_2^n\|_{H^\gamma}\le C\tau^2, \tag{31}$$

where τ_0 and C depend only on T and $\sup\limits_{0\le t\le T}\|\varphi(t)\|_{H^{\gamma+1}}$.

Proof Recall the definition of $\mathcal{R}_1^n$ in (17)

$$\begin{aligned}\mathcal{R}_1^n=i\int_0^\tau e^{-i\partial_{\xi_1\xi_2}(t_n+s)}\Big[&e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n+s)\cdot E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n+s\right)\right|^2\right)\\&-e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\cdot E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\right|^2\right)\Big]ds,\end{aligned}$$

we have

$$
\begin{aligned}
\|\mathcal{R}_1^n\|_{H^\gamma} \leq & \int_0^\tau \Big\| e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi(t_n+s)\cdot E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n+s\right)\right|^2\right) \\
& \quad - e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\cdot E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\right|^2\right)\Big\|_{H^\gamma} ds. \\
\leq & \int_0^\tau \left\| e^{i\partial_{\xi_1\xi_2}(t_n+s)}\left(\varphi(t_n+s)-\varphi(t_n)\right)\cdot E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n+s\right)\right|^2\right)\right\|_{H^\gamma} ds \\
& + \int_0^\tau \left\| e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\cdot \left[E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n+s\right)\right|^2\right) - E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\right|^2\right)\right]\right\|_{H^\gamma} ds.
\end{aligned}
$$

Since $\|Ef\|_{H^\gamma} \lesssim \|f\|_{H^\gamma}$, together with bilinear estimate and isometric property, we have

$$
\begin{aligned}
\|\mathcal{R}_1^n\|_{H^\gamma} \leq & \int_0^\tau \left\|\varphi\left(t_n+s\right)-\varphi\left(t_n\right)\right\|_{H^\gamma}\cdot\left\|\varphi\left(t_n+s\right)\right\|_{H^\gamma}^2 ds \\
& + \int_0^\tau \left\|\varphi\left(t_n\right)\right\|_{H^\gamma}\left\|\left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n+s\right)\right|^2 - \left|e^{i\partial_{\xi_1\xi_2}(t_n+s)}\varphi\left(t_n\right)\right|^2\right\|_{H^\gamma} ds \\
\leq & \int_0^\tau \left\|\varphi(t_n+s)-\varphi\left(t_n\right)\right\|_{H^\gamma}\cdot\left\|\varphi\left(t_n+s\right)\right\|_{H^\gamma}^2 ds \\
& + \int_0^\tau \left\|\varphi\left(t_n\right)\right\|_{H^\gamma}\cdot\left(\left\|\varphi\left(t_n+s\right)\right\|_{H^\gamma}+\left\|\varphi\left(t_n\right)\right\|_{H^\gamma}\right)\left\|\varphi\left(t_n+s\right)-\varphi\left(t_n\right)\right\|_{H^\gamma} ds.
\end{aligned}
\tag{32}
$$

By (14), we can get

$$
\begin{aligned}
\|\varphi(t_n+s)-\varphi\left(t_n\right)\|_{H^\gamma} \leq & \int_0^s \|\partial_t\varphi(t_n+t)\|_{H^\gamma} dt \\
\leq & \int_0^s \left\| e^{i\partial_{\xi_1\xi_2}(t_n+t)}\varphi\left(t_n+t\right)\cdot E\left(\left|e^{i\partial_{\xi_1\xi_2}(t_n+t)}\varphi\left(t_n+t\right)\right|^2\right)\right\|_{H^\gamma} dt \\
\leq & \int_0^s \left\|\varphi\left(t_n+t\right)\right\|_{H^\gamma}\left\|\left|e^{i\partial_{\xi_1\xi_2}(t_n+t)}\varphi\left(t_n+t\right)\right|^2\right\|_{H^\gamma} dt \\
\leq & Cs \sup_{0\leq t\leq s}\left\|\varphi\left(t_n+t\right)\right\|_{H^\gamma}^3 .
\end{aligned}
\tag{33}
$$

Inserting (33) into (32), it follows that

$$
\|\mathcal{R}_1^n\|_{H^\gamma} \leq C\tau^2 \sup_{0\leq t\leq T}\|\varphi(t)\|_{H^\gamma}^5. \tag{34}
$$

Next, we consider $\mathcal{R}_2^n$. Recall the definition of $\mathcal{R}_2^n$

$$
\mathcal{R}_2^n = i\sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} e^{it_n\alpha}\left[\tilde{\mu}_1+\mu_2\frac{(j_1+k_1)(j_2+k_2)}{|\boldsymbol{j}+\boldsymbol{k}|^2}\right]\hat{\bar{\varphi}}_{\boldsymbol{j}}\hat{\varphi}_{\boldsymbol{k}}\hat{\varphi}_{\boldsymbol{l}}e^{i\boldsymbol{\Omega}\cdot\boldsymbol{\xi}}\cdot\int_0^\tau e^{2isj_1j_2}(e^{is\beta}-1)ds.
$$

Hence, we have

$$\begin{aligned}\|\mathcal{R}_2^n\|_{H^\gamma}^2 &= \sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2} (1+|\boldsymbol{\Omega}|)^{2\gamma} |\widehat{\mathcal{R}_2^n}(\boldsymbol{\Omega})|^2 \\ &= \sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2} (1+|\boldsymbol{\Omega}|)^{2\gamma} \Bigg| \sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2 \\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} e^{i\alpha t_n} \left[\tilde{\mu}_1 + \mu_2 \frac{(j_1+k_1)(j_2+k_2)}{|\boldsymbol{j}+\boldsymbol{k}|^2}\right] \\ &\qquad \cdot \hat{\bar{\varphi}}(t_n,\boldsymbol{j})\,\hat{\varphi}(t_n,\boldsymbol{k})\,\hat{\varphi}(t_n,\boldsymbol{l}) \int_0^\tau e^{2ij_1j_2s}(e^{is\beta}-1)ds \Bigg|^2. \quad (35)\end{aligned}$$

Since $\tilde{\mu}_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, then we know that

$$\left|\tilde{\mu}_1 + \mu_2 \frac{(j_1+k_1)(j_2+k_2)}{|\boldsymbol{j}+\boldsymbol{k}|^2}\right| \le |\tilde{\mu}_1| + |\mu_2| \le C,$$

and

$$\left|e^{i\beta s} - 1\right| \lesssim |s\beta|\,.$$

Hence, we have

$$\|\mathcal{R}_2^n\|_{H^\gamma}^2 \le C \sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2} (1+|\boldsymbol{\Omega}|)^{2\gamma} \Bigg| \sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2 \\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} \hat{\bar{\varphi}}_{\boldsymbol{j}} \hat{\varphi}_{\boldsymbol{k}} \hat{\varphi}_{\boldsymbol{l}} \int_0^\tau s\beta ds \Bigg|^2.$$

Recall that $\beta = j_1(k_2+l_2) + k_1(j_2+l_2) + l_1(j_2+k_2)$, then we have

$$\begin{aligned}|\beta| &\le |\boldsymbol{j}|(|\boldsymbol{k}|+|\boldsymbol{l}|) + |\boldsymbol{k}|(|\boldsymbol{j}|+|\boldsymbol{l}|) + |\boldsymbol{l}|(|\boldsymbol{j}|+|\boldsymbol{k}|) \\ &\le 2(|\boldsymbol{j}||\boldsymbol{k}| + |\boldsymbol{j}||\boldsymbol{l}| + |\boldsymbol{k}||\boldsymbol{l}|) \\ &\le C(1+|\boldsymbol{j}|)(1+|\boldsymbol{k}|)(1+|\boldsymbol{l}|).\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\|\mathcal{R}_2^n\|_{H^\gamma}^2 &\le C\tau^4 \sum_{\boldsymbol{\Omega}\in\mathbb{Z}^2} (1+|\boldsymbol{\Omega}|)^{2\gamma} \Bigg| \sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^2 \\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} (1+|\boldsymbol{j}|)(1+|\boldsymbol{k}|)(1+|\boldsymbol{l}|)|\hat{\bar{\varphi}}_{\boldsymbol{j}}||\hat{\varphi}_{\boldsymbol{k}}||\hat{\varphi}_{\boldsymbol{l}}| \Bigg|^2 \\ &\le C\tau^4 \underset{0\le t\le T}{sup} \|\varphi(t)\|_{H^{\gamma+1}}^6. \quad (36)\end{aligned}$$

By (34) and (36), we can get

$$\|\mathcal{R}_1^n\|_{H^\gamma} + \|\mathcal{R}_2^n\|_{H^\gamma} \le C\tau^2,$$

where C depends on $\sup\limits_{0\le t\le T} \|\varphi(t)\|_{H^{\gamma+1}}$. This finishes the proof of the lemma.

By Lemma 1, we will show the local error $\mathcal{L}^n$ and have following lemma.

Lemma 2 *(Local error)* *Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}$, then there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \leq \tau_0$, the following inequality holds*

$$\|\mathcal{L}^n\|_{H^\gamma} \leq C\tau^2, \tag{37}$$

where τ_0 and C depend only on T and $\sup\limits_{0\leq t\leq T} \|\varphi(t)\|_{H^{\gamma+1}}$.

Proof Based on the definition of $\mathcal{L}^n$, we have

$$\|\mathcal{L}^n\|_{H^\gamma} \leq \|\mathcal{R}_1^n\|_{H^\gamma} + \|\mathcal{R}_2^n\|_{H^\gamma}.$$

Using Lemma 1, we have

$$\|\mathcal{R}_1^n\|_{H^\gamma} \leq C\tau^2, \quad \|\mathcal{R}_2^n\|_{H^\gamma} \leq C\tau^2,$$

then we get

$$\|\mathcal{L}^n\|_{H^\gamma} \leq C\tau^2,$$

where C depends on $\sup\limits_{0\leq t\leq T} \|\varphi(t)\|_{H^{\gamma+1}}$. This finishes the proof of lemma.

3.2 Stability analysis

For the numerical propagator φ^n defined in (25), we have the following stability result.

Lemma 3 *(Stability)* *Let $\gamma > 1$. Assume that $\phi_0 \in H^{\gamma+1}$, then there exist constants $\tau_0 > 0$ and $C > 0$, such that for any $0 < \tau \leq \tau_0$, the following inequality holds*

$$\|\mathcal{S}^n\|_{H^\gamma} \leq (1 + C\tau) \|\varphi(t_n) - \varphi^n\|_{H^\gamma} + C\tau \|\varphi(t_n) - \varphi^n\|^3_{H^\gamma}, \tag{38}$$

where τ_0 and C depend only on T and $\sup\limits_{0\leq t\leq T} \|\varphi(t)\|_{H^\gamma}$.

Proof According to the definition of $\mathcal{S}^n$, we have

$$\begin{aligned}\|\mathcal{S}^n\|_{H^\gamma} &= \|\varphi(t_n) - \Psi^n(\varphi(t_n)) - \varphi^n + \Psi^n(\varphi^n)\|_{H^\gamma}\\ &\leq \|\varphi(t_n) - \varphi^n\|_{H^\gamma} + \|\Psi^n(\varphi(t_n)) - \Psi^n(\varphi^n)\|_{H^\gamma}.\end{aligned} \tag{39}$$

Recall that

$$\Psi^n(f) = i\tau e^{-i\partial_{\xi_1\xi_2}t_n}\left[e^{i\partial_{\xi_1\xi_2}t_n}f \cdot E\left(\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}f} \cdot e^{i\partial_{\xi_1\xi_2}t_n}f\right)\right].$$

Then, we have

$$
\begin{aligned}
\|\Psi^n(\varphi(t_n)) - \Psi^n(\varphi^n)\|_{H^\gamma} \leq & \tau\Big\| e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)\cdot E\left(\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)\right) \\
& - e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n\cdot E\left(\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n\right)\Big\|_{H^\gamma} \\
\leq & \tau\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\cdot \Big\|E\left(\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)\right)\Big\|_{H^\gamma} \\
& + \tau\|\varphi^n\|_{H^\gamma}\cdot\Big\|E\big[\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n) \\
& \qquad - \omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n\big]\Big\|_{H^\gamma}.
\end{aligned}
$$

Recall that ω, we have

$$
|\omega(z)| \leq 1,\ \forall z = ib,\ b\in\mathbb{R}.
$$

Therefore, we obtain

$$
\|\omega(-2i\partial_{\xi_1\xi_2}\tau)f\|_{H^\gamma} \leq C\|f\|_{H^\gamma}. \tag{40}
$$

Then we can get

$$
\begin{aligned}
& \|\Psi^n(\varphi(t_n)) - \Psi^n(\varphi^n)\|_{H^\gamma} \qquad (41)\\
& \leq C\tau\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\cdot\|\varphi(t_n)\|^2_{H^\gamma} \\
& \quad + C\tau\|\varphi^n\|_{H^\gamma}\cdot\Big\|\left(\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)} - \omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\right)\cdot e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n) \\
& \qquad + \omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\cdot\left(e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n) - e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n\right)\Big\|_{H^\gamma} \\
& \leq C\tau\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|^2_{H^\gamma} \\
& \quad + C\tau\|\varphi^n\|_{H^\gamma}\cdot\Big[\Big\|\omega(-2i\partial_{\xi_1\xi_2}\tau)\left(\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)} - \overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\right)\Big\|_{H^\gamma}\Big\|e^{i\partial_{\xi_1\xi_2}t_n}\varphi(t_n)\Big\|_{H^\gamma} \\
& \qquad + \Big\|\omega(-2i\partial_{\xi_1\xi_2}\tau)\overline{e^{i\partial_{\xi_1\xi_2}t_n}\varphi^n}\Big\|_{H^\gamma}\Big\|e^{i\partial_{\xi_1\xi_2}t_n}(\varphi(t_n)-\varphi^n)\Big\|_{H^\gamma}\Big] \\
& \leq C\tau\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|^2_{H^\gamma} \\
& \quad + C\tau\|\varphi^n\|_{H^\gamma}\Big[\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|_{H^\gamma} + \|\varphi^n\|_{H^\gamma}\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\Big]. \qquad (42)
\end{aligned}
$$

Using $\|\varphi^n\|_{H^\gamma} \leq \|\varphi(t_n)-\varphi^n\|_{H^\gamma} + \|\varphi(t_n)\|_{H^\gamma}$, we get

$$
\begin{aligned}
& \|\Psi^n(\varphi(t_n)) - \Psi^n(\varphi^n)\|_{H^\gamma} \\
& \leq C\tau\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|^2_{H^\gamma} + C\tau(\|\varphi(t_n)-\varphi^n\|_{H^\gamma} + \|\varphi(t_n)\|_{H^\gamma}) \\
& \quad \cdot\Big[\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|_{H^\gamma} + (\|\varphi(t_n)-\varphi^n\|_{H^\gamma} + \|\varphi(t_n)\|_{H^\gamma})\cdot\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\Big] \\
& \leq C\tau\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|^2_{H^\gamma} + C\tau\big(3\|\varphi(t_n)-\varphi^n\|^2_{H^\gamma}\|\varphi(t_n)\|_{H^\gamma} \\
& \quad + 2\|\varphi(t_n)-\varphi^n\|_{H^\gamma}\|\varphi(t_n)\|^2_{H^\gamma} + \|\varphi(t_n)-\varphi^n\|^3_{H^\gamma}\big).
\end{aligned}
$$

Since the quadratic term can be controlled by the first term and third term, we can obtain

$$\|\Psi^n(\varphi(t_n)) - \Psi^n(\varphi^n)\|_{H^\gamma} \le C\tau\|\varphi(t_n) - \varphi^n\|_{H^\gamma} + C\tau\|\varphi(t_n) - \varphi^n\|_{H^\gamma}^3, \tag{43}$$

where C depends on $\sup\limits_{0\le t\le T} \|\varphi(t)\|_{H^\gamma}$.

Using (39) and (43), we have

$$\|\mathcal{S}^n\|_{H^\gamma} \le (1 + C\tau)\|\varphi(t_n) - \varphi^n\|_{H^\gamma} + C\tau\|\varphi(t_n) - \varphi^n\|_{H^\gamma}^3,$$

this proves this lemma.

Finally, we give a proof of the first-order convergence theorem.

3.3 The proof of theorem 1

Together with the local error estimate and the stability result, we give the proof of Theorem 1. From Lemma 2 and 3, we have

$$\left\|\varphi(t_{n+1}) - \varphi^{n+1}\right\|_{H^\gamma} \le C\tau^2 + (1 + C\tau)\left\|\varphi(t_n) - \varphi^n\right\|_{H^\gamma} + C\tau\left\|\varphi(t_n) - \varphi^n\right\|_{H^\gamma}^3, \tag{44}$$

where $n = 0, 1, 2, \ldots, \frac{T}{\tau} - 1$, C depends only on T and $\sup\limits_{0\le t\le T} \|\varphi(t)\|_{H^{\gamma+1}}$.

Followed by the argument in [38], we can get

$$\|\varphi(t_n) - \varphi^n\|_{H^{\gamma+1}} \le C\tau, \tag{45}$$

where C depends on T and $\sup\limits_{0\le t\le T} \|\phi(t)\|_{H^{\gamma+1}}$.

To prove it, we claim that there exists some $\tau_0 > 0$ (to be determined) such that for any $\tau \in (0, \tau_0]$ and any $n = 0, 1, \ldots, \frac{T}{\tau}$,

$$\|\varphi(t_n) - \varphi^n\|_{H^\gamma} \le C\tau^2 \sum_{j=0}^{n} (1 + 2C\tau)^j. \tag{46}$$

It trivially holds for $n = 0$. Now we assume that

$$\|\varphi(t_n) - \varphi^n\|_{H^\gamma} \le C\tau^2 \sum_{j=0}^{n} (1 + 2C\tau)^j, \text{ for any } 0 \le n \le n_0. \tag{47}$$

From (47), we have

$$\|\varphi(t_n) - \varphi^n\|_{H^\gamma} \le C_1\tau, \tag{48}$$

where $C_1 = 2\mathrm{e}^{2CT}$, for any $0 \le n \le n_0$. Then by (44), we get

$$\left\|\varphi(t_{n_0+1}) - \varphi^{n_0+1}\right\|_{H^\gamma} \le C\tau^2 + \left(1 + C\tau + CC_1^2\tau^3\right) \cdot C\tau^2 \sum_{j=0}^{n_0} (1 + 2C\tau)^j.$$

Now we can choose $\tau_0 > 0$ such that

$$CC_1^2\tau_0^2 \leq C,$$

then for any $\tau \in (0, \tau_0]$, it follows

$$\begin{aligned}\left\|\varphi(t_{n_0+1}) - \varphi^{n_0+1}\right\|_{H^\gamma} \leq & C\tau^2 + (1+2C\tau)\cdot C\tau^2\sum_{j=0}^{n_0}(1+2C\tau)^j \\ = & C\tau^2 + C\tau^2\sum_{j=0}^{n_0}(1+2C\tau)^{j+1} < C\tau^2\sum_{j=0}^{n_0+1}(1+2C\tau)^j.\end{aligned}$$

This finishes the induction and proves (46), which gives (45).

From (13), together with isometric property, we get

$$\|\phi(t_n) - \phi^n\|_{H^\gamma} = \|\varphi(t_n) - \varphi^n\|_{H^\gamma} \leq C\tau. \tag{49}$$

From the DS system (11), we have $\psi = -(-\Delta)^{-1}(\partial_{\xi_1} + \partial_{\xi_2})\left(|\phi|^2\right)$, then we get

$$\begin{aligned}\|\psi(t_n) - \psi^n\|_{H^{\gamma+1}} = & \left\|-(-\Delta)^{-1}(\partial_{\xi_1} + \partial_{\xi_2})\left(|\phi(t_n)|^2 - |\phi^n|^2\right)\right\|_{H^{\gamma+1}} \\ \leq & C\left\||\phi(t_n)|^2 - |\phi^n|^2\right\|_{H^\gamma} \\ \leq & C\|\phi(t_n) - \phi^n\|_{H^\gamma}\|\phi(t_n) + \phi^n\|_{H^\gamma} \\ \leq & C\|\phi(t_n) - \phi^n\|_{H^\gamma}\left(\|\phi(t_n) - \phi^n\|_{H^\gamma} + 2\|\phi(t_n)\|_{H^\gamma}\right).\end{aligned}$$

This together with (49), we have that

$$\|\psi(t_n) - \psi^n\|_{H^{\gamma+1}} \leq C\tau.$$

This finishes the proof of the first-order convergence theorem.

4 Numerical Experiments

In this section, we present the numerical experiments of the scheme (26) to justify the convergence theorem, including experimental parameter settings and experimental results. We consider the 2-dimension case, i.e. $d = 2$. Since ψ^n is calculated via equation (27), which won't lose any regularity and kept first order convergence (see Theorem 1), we only need to test ϕ^n in this section.

To get an initial data with the desired regularity, we construct $\phi_0(\boldsymbol{x})$ by the following strategy [44]. Choose $N \geq 0$ as an even integer and discrete the spatial domain $\mathbb{T}^2$ with grid points $x_{j,k,l} = (\frac{2j\pi}{N}, \frac{2k\pi}{N})$ for $j,k = 0,\ldots,N-1$ and $l = 1,2$. Take a uniformly distributed random array $\mathrm{rand}(N,N) \in [0,1]^N$, and an $N \times N \times 2$ array $\mathcal{U}$ whose elements are defined as

$$\mathcal{U}_{j,k,l} = \mathrm{rand}(N,N) + i\,\mathrm{rand}(N,N), \quad (j,k = 0,\ldots,N-1,\ l = 1,2).$$

In our numerical experiments, we set

$$\phi_0(\boldsymbol{x}) := \frac{|\partial_{\boldsymbol{x},N}|^{-\gamma}\mathcal{U}}{\||\partial_{\boldsymbol{x},N}|^{-\gamma}\mathcal{U}\|_{L^\infty}}, \quad \boldsymbol{x} \in \mathbb{T}^2, \tag{50}$$

where the pseudo-differential operator $|\partial_{\boldsymbol{x},N}|^{-\gamma}$ for $\gamma \geq 0$ reads as follows:

$$(|\partial_{\boldsymbol{x},N}|^{-\gamma})_{\boldsymbol{k}} = \begin{cases} |\boldsymbol{k}|^{-\gamma}, & \boldsymbol{k} \neq 0, \\ 0, & \boldsymbol{k} = 0, \end{cases}$$

for Fourier modes $\boldsymbol{k} = (k_1, k_2)$, and $k_l = -N/2, \ldots, N/2-1$ for $l = 1, 2$. Thus, we get $\phi_0 \in H^\gamma(\mathbb{T}^2)$ for any $\gamma \geq 0$.

We implement the spatial discretizations of the numerical methods within discussions by the Fourier pseudo-spectral method with a large number of grid points $N = 2^6$, $N = 2^7$, $N = 2^8$ in the torus domain $\mathbb{T}^2$. Since the scheme (26) requires $\gamma > 1$ and a derivative be added to the initial value, we consider $\gamma = 2$ and $\gamma = 3$ in this experiment. For initial data in $H^{\gamma+1}$, we shall present the relative error $\phi^n - \phi_{ref}$ in the L^2 and H^γ norm at the final time $t_n = T = 2.0$, where the "exact" solution is obtained numerically by scheme (26) with $\tau = 10^{-4}$. The results are shown below.

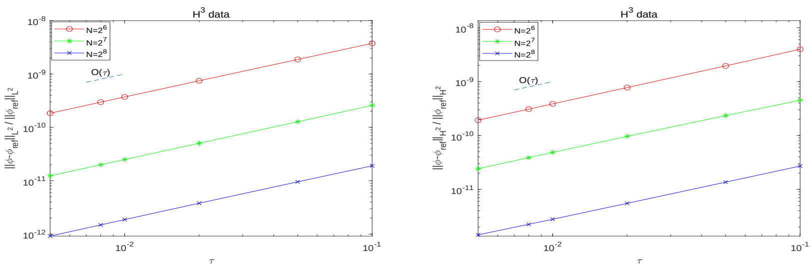

Fig. 1: Convergence of the scheme: relative error $\|\phi^n - \phi_{ref}\|_{L^2}/\|\phi_{ref}\|_{L^2}$ (left) and $\|\phi^n - \phi_{ref}\|_{H^2}/\|\phi_{ref}\|_{H^2}$ (right) at $t_n = T = 2.0$ with initial condition $\gamma = 2$.

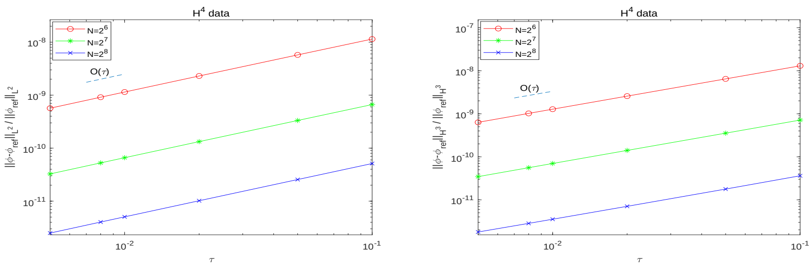

Fig. 2: Convergence of the scheme: relative error $\|\phi^n - \phi_{ref}\|_{L^2}/\|\phi_{ref}\|_{L^2}$ (left) and $\|\phi^n - \phi_{ref}\|_{H^3}/\|\phi_{ref}\|_{H^3}$ (right) at $t_n = T = 2.0$ with initial condition $\gamma = 3$.

The Figure 1 shows the first-order accuracy in L^2-norm and H^2-norm with data in H^3 space, and the Figure 2 shows the first-order accuracy in L^2-norm and H^3-norm with data in H^4 space. This verifies the conclusion of Theorem 1.

5 Conclusion

In this work, we have numerically studied the DS-II system on a torus under rough initial data. By some rigorous tools from harmonic analysis, we established the sharp convergence theorem of a low-regularity integrator. The theoretical result and experimental result show that the presented integrator can reach first-order accuracy in space H^γ with rough initial data from space $H^{\gamma+1}$ for any $\gamma > 1$.

Conflict of interest

The authors declare that they have no conflict of interest.

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