RIGIDITY OF THE DELAUNAY TRIANGULATIONS OF THE PLANE

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Abstract. We proved a rigidity result for Delaunay triangulations of the plane under Luo's discrete conformal change, extending previous results on hexagonal triangulations. Our result is a discrete analogue of the conformal rigidity of the plane. We followed Zhengxu He's analytical approach in his work on the rigidity of disk patterns, and developed a discrete Schwarz lemma and a discrete Liouville theorem. The main tools include conformal modulus, discrete extremal length, and maximum principles in discrete conformal geometry.

Contents

1. Introduction

A fundamental property in conformal geometry is that a conformal embedding of the plane $\mathbb{R}^2$ to itself must be a similar transformation. In this paper we discretize the plane by geodesic triangulations and prove a similar rigidity result under the notion of discrete conformal change introduced by Luo [Luo04].

Let $T = (V, E, F)$ be a topological triangulation of a surface with or without boundary, where V is the set of vertices, E is the set of edges and F is the set of faces. Denote $|T|$ as the underlying space of the complex T. A *PL (piecewise linear) metric* on T is a function, $l : E \to \mathbb{R}_+$ such that every triangle $\triangle ijk \in F$

could form a Euclidean triangle under the length l. Luo [Luo04] introduced the following notion of *discrete conformality.*

Definition 1.1 ([Luo04])**.** *Two PL metrics l, l' on $T=(V,E,F)$ are discretely conformal if there exists a function $u: V \to \mathbb{R}$ such that for any edge $ij \in E$,*

$$l'_{ij} = e^{\frac{1}{2}(u_i+u_j)} l_{ij}.$$

In this case, u is called a discrete conformal factor, and we denote $l' = u * l$.

Given a PL metric l on T, let θ^i_{jk} denote the inner angle at the vertex i in the Euclidean triangle $\triangle ijk$ under the metric l. The PL metric l is called

(a) *uniformly nondegenerate* if there exists a constant $\epsilon > 0$ such that $\theta^i_{jk} \geq \epsilon$ for all $\triangle ijk$ in T, and
(b) *Delaunay* if $\theta^{k_1}_{ij} + \theta^{k_2}_{ij} \leq \pi$ for any pair of adjacent triangles $\triangle ijk_1$ and $\triangle ijk_2$ in T, and
(c) *uniformly Delaunay* if there exists a constant $\epsilon > 0$ such that $\theta^{k_1}_{ij} + \theta^{k_2}_{ij} \leq \pi - \epsilon$ for any pair of adjacent triangles $\triangle ijk_1$ and $\triangle ijk_2$ in T.

Remark 1.2. *The Delaunay condition is equivalent to that for every pair of adjacent triangles $\triangle ijk_1, \triangle ijk_2 \in F$, if the Euclidean quadrilateral (ik_1jk_2) is isometrically embedded in $\mathbb{C}$, then $k_2 \notin \mathrm{int}(D_{ijk_1})$, where $\mathrm{int}(D_{ijk_1})$ is the interior of the circumscribed disk of $\triangle ijk_1$.*

A map $\phi : |T| \to \mathbb{C}$ is called a *geodesic embedding* if for every $ij \in E$, ϕ maps ij to a segment connecting $\phi(i)$ and $\phi(j)$, and ϕ maps $|T|$ homeomorphically to its image. If further ϕ is surjective, we call ϕ is a *geodesic homeomorphism* or a *geodesic triangulation.* It is clear that a geodesic embedding ϕ gives a PL metric $l(\phi)$, or l for short, by using the Euclidean distance. A geodesic embedding ϕ is called *(uniformly) Delaunay* if $l(\phi)$ is (uniformly) Delaunay. The main result of the paper is the following.

Theorem 1.3. *Suppose $\phi : |T| \to \mathbb{C}$ is a geodesic homeomorphism and $\phi' : |T| \to \mathbb{C}$ is a geodesic embedding with the induced PL metric l, l' respectively, such that*
(a) l, l' are both uniformly nondegenerate,
(b) l is uniformly Delaunay and l' is Delaunay,
(c) l is discretely conformal to l', i.e., $l' = u * l$ for some $u \in \mathbb{R}^V$.
Then l and l' differ by a constant scaling, i.e., u is constant on V.

Wu-Gu-Sun [WGS15] first proved Theorem 1.3 for the special case where $\phi(T)$ is a regular hexagonal triangulation and $\phi'(T)$ satisfies the uniformly acute condition, i.e. all the inner angles are no more than $\frac{\pi}{2} - \epsilon$ for some constant $\epsilon > 0$. Luo-Sun-Wu [LSW22] and Dai-Ge-Ma [DGM22] generalized Wu-Gu-Sun's result by allowing l' to be only Delaunay rather than uniformly acute. All these works essentially rely on the lattice structure of the regular hexagonal triangulation, and apparently cannot be generalized to triangulations without translational invariance.

Remark 1.4. *In* [BPS15]*, Bobenko-Pinkall-Springborn observed the relation between Luo's discrete conformality and the hyperbolic polyhedra in $\mathbb{H}^3$. In fact the rigidity problem in this paper corresponds to the Cauchy rigidity of certain hyperbolic polyhedra and the Delaunay condition corresponds to the convexity of the hyperbolic polyhedra.*

To prove Theorem 1.3, we follow the approach developed by Zhengxu He in his state-of-the-art work on the rigidity of disk patterns [He99]. Theorem 1.3 immediately follows from the following two propositions.

Proposition 1.5. *Under the conditions of Theorem 1.3, the discrete conformal factor u is bounded on V. Furthermore, the condition (b) could be relaxed to that both l, l' are just Delaunay.*

Proposition 1.6. *Under the conditions of Theorem 1.3, if the discrete conformal factor u is bounded on V, then it is constant on V.*

The proof of Proposition 1.5 relies on a discrete Schwarz lemma, and estimating conformal moduli for annuli. The proof of Proposition 1.6 is by constructing a discretely conformal geometric flow from l toward l', keeping the surface flat. The derivative of the conformal factor in this flow is known to be discrete harmonic, and thus constant by a discrete Liouville theorem.

1.1. **Notations and conventions.** Given $0 < r < r'$, denote $D_r = \{z \in \mathbb{C} : |z| < r\}$ and $A_{r,r'} = \{z \in \mathbb{C} : r < |z| < r'\}$. We also denote $D = D_1$ as the unit open disk. Given a subset X of $\mathbb{C}$, X^c denotes the complement $\mathbb{C}\backslash X$ and ∂X denotes the boundary of X in $\mathbb{C}$ and $\text{int}(X)$ denotes the interior of X in $\mathbb{C}$ and

$$\text{diam}(X) = \sup\{|z - z'| : z, z' \in X\},$$

denotes the diameter of X. Given two subsets X, Y of $\mathbb{C}$, the distance between X, Y is denoted by

$$d(X, Y) = \inf\{|z - z'| : z \in X, z' \in Y\}.$$

Given $i \in V$, denote $\deg(i)$ as the number of neighbors of i and N_i as set of neighbors of i, i.e., $N_i = \{j \in V : ij \in E\}$. Furthermore, we denote R_i as the union of the triangles in T containing i. R_i is always viewed as the underlying space of the subcomplex generated by the triangles containing i. Such R_i is called a *1-ring neighborhood* of i if R_i is homeomorphic to a closed disk with vertex i mapped to the center of the disk. Given a subset V_0 of V, a vertex $i \in V_0 \subseteq V$ is called an *inner point* of V_0, if $N_i \subseteq V_0$ and R_i is a 1-ring neighborhood of i. We denote $\text{int}(V_0)$ as the set of inner points of V_0, and $\partial V_0 = V_0 - \text{int}(V_0)$. In particular, ∂V is the set of vertices of T that are on the boundary of the surface $|T|$.

Given $l \in \mathbb{R}^E$ and $u \in \mathbb{R}^V$, if $u * l$ is a PL metric then

(a) $\theta^i_{jk}(u) = \theta^i_{jk}(u, l)$ denotes the inner angle of $\triangle ijk$ at i under $u * l$, and
(b) $K_i(u) = K_i(u, l)$ denotes the discrete curvature at i for $i \in \text{int}(V)$

$$K_i(u) = 2\pi - \sum_{jk:\triangle ijk \in F} \theta^i_{jk}(u).$$

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2. Preparations for the Proof

2.1. **Hyperbolic discrete conformality.** The notion of the hyperbolic discrete conformality was first introduced by Bobenko-Pinkall-Springborn in [BPS15]. A *piecewise hyperbolic (PH) metric* on T is represented by a function $l^h : E \to \mathbb{R}_+$ such that every $\triangle ijk \in F$ could form a hyperbolic triangle under the length l^h.

Let $l^h, l^{h\prime}$ be two PH metrics on T. We say l^h is *hyperbolic discretely conformal* to $l^{h\prime}$ if there exists a function $u^h : V \to \mathbb{R}$ such that for any edge $ij \in E$

$$\sinh \frac{l^h_{ij}}{2} = e^{\frac{1}{2}(u^h_i + u^h_j)} \sinh \frac{l^{h\prime}_{ij}}{2}.$$

In this case, u is called a *hyperbolic discrete conformal factor* and we denote $l^{h\prime} = u^h *^h l^h$.

Denote $\mathbb{D}$ as the hyperbolic space, represented in the Poincaré disk model. A map $\phi^h : |T| \to \mathbb{D}$ is called a *hyperbolic geodesic embedding* if for every $ij \in E$, ϕ^h maps ij to a hyperbolic geodesic segment connecting $\phi^h(i)$ and $\phi^h(j)$, and ϕ^h maps $|T|$ homeomorphically to its image. It is clear that a hyperbolic geodesic embedding ϕ^h gives a PH metric $l^h(\phi^h)$, or l^h for short, by using the hyperbolic distance. A hyperbolic geodesic embedding ϕ^h is called *Delaunay* if for any pair of adjacent triangles $\triangle ijk_1, \triangle ijk_2$ in T, $\phi^h(k_2) \notin \mathrm{int}(D_{ijk_1})$ where D_{ijk_1} is the circumscribed disk of $\phi^h(\triangle ijk_1)$. Notice that in the Poincaré disk model, a hyperbolic disk in $\mathbb{D}$ is also a Euclidean disk in $\mathbb{C}$.

Given $i \in \mathrm{int}(V)$ and a Euclidean geodesic embedding $\phi : R_i \to D$, we say that ϕ *induces* the hyperbolic geodesic embedding $\phi^h : R_i \to \mathbb{D}$ if $\phi(j) = \phi^h(j)$ for all $j \in \{i\} \cup N_i$. Such ϕ^h exists if $l(\phi)$ is non-degenerate and small, and $\phi(R_i)$ is away from ∂D.

Lemma 2.1. *Let $\phi : R_i \to D$ be a (Euclidean) geodesic embedding where all the inner angles in are at least $\epsilon > 0$. Suppose*

$$l_{ij} \leq (1 - |\phi(i)|^2) \sin \epsilon, \qquad \textit{for every } j \in N_i. \tag{2.1}$$

Then there exists a hyperbolic geodesic embedding $\phi^h : R_i \to \mathbb{D}$ such that ϕ^h coincides with ϕ on the set of vertices, i.e., $\phi(j) = \phi^h(j)$ for any $j \in N_i \cup \{i\}$.

Proof. Denote $m = \deg(i)$ and $z_0 = \phi(i)$. Let $z_1, ..., z_m$ be the points in $\phi(N_i)$ such that $z_1 - z_0, ..., z_m - z_0$ are counterclockwise. Let $\exp_{z_0}$ be the exponential map with respect to the hyperbolic metric at z_0. Identifying $T_{z_0}\mathbb{D}$ with $\mathbb{C}$ by translating z_0 to the origin, denote $v(z) = \exp_{z_0}^{-1} z \in \mathbb{C}$ for any $z \in \mathbb{D}$. Then we only need to show the following claims.

$$\arg(\frac{v(z_{k+1})}{v(z_k)}) \in (0, \pi), \quad k = 1, \cdots, m, \tag{2.2}$$

and

$$\sum_{k=1}^{m} \arg(\frac{v(z_{k+1})}{v(z_k)}) = 2\pi, \tag{2.3}$$

where $z_{m+1} = z_1$.

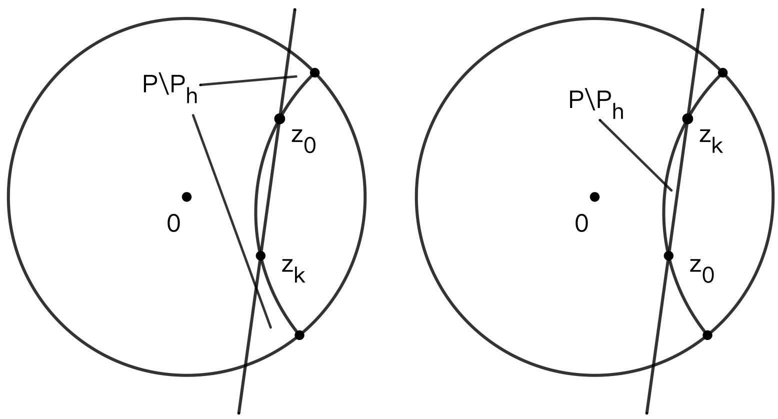
We first show the claim (2.2). Fix $k \in \{1, \cdots, m\}$, denote

$$P = \{z \in \mathbb{C} : \arg(\frac{z - z_0}{z_k - z_0}) \in (0, \pi)\},$$

and

$$P_h = \{z \in \mathbb{D} : \arg(\frac{v(z)}{v(z_k)}) \in (0, \pi)\}.$$

See Figure 1 for illustrations. Since ϕ is a geodesic embedding and $z_1 - z_0, ..., z_m - z_0$ are counterclockwise, we have $z_{k+1} \in P$. We need to show $z_{k+1} \in P_h$. Let γ_h be the entire geodesic connecting z_0 and z_k with respect to the hyperbolic metric. If γ_h is



(A) Case 1 (B) Case 2

FIGURE 1

a straight line, then $z_{k+1} \in P_h = P$ and we are done. Otherwise, γ_h is a Euclidean circular arc and orthogonal to the boundary of the unit disk D. We denote z_* and R as the Euclidean center and the radius, respectively, or this circular arc. Then

$$R^2 + 1 = |z_*|^2 \leq (|z_0| + R)^2.$$

So

$$1 - |z_0|^2 \leq 2R|z_0| < 2R.$$

Then by Equation (2.1)

$$\sin \frac{\angle z_0 z_* z_k}{2} = \frac{|z_k - z_0|}{2R} \leq \frac{(1 - |z_0|^2) \sin \epsilon}{2R} < \sin \epsilon.$$

So

$$\angle z_0 z_{k+1} z_k \geq \epsilon > \frac{1}{2} \angle z_0 z_* z_k,$$

and

$$\angle z_0 z_{k+1} z_k \leq \pi - \epsilon < \pi - \frac{1}{2} \angle z_0 z_* z_k.$$

Then from the knowledge of the plane geometry, we see $z_k \in P_h$. We finish the proof of claim (2.2).

Next we show the claim (2.3). We have

$$\arg\big(\frac{v(z_{k+1})}{v(z_k)}\big) + \arg\big(\frac{v(z_k)}{z_k - z_0}\big) = \arg\big(\frac{z_{k+1} - z_0}{z_k - z_0}\big) + \arg\big(\frac{v(z_{k+1})}{z_{k+1} - z_0}\big) + 2n\pi, \tag{2.4}$$

for some integer n. From the knowledge of plane geometry, we have that

$$\arg\big(\frac{v(z_k)}{z_k - z_0}\big) \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Then together with the claim (2.2), both the right hand side and the left hand side of Equation (2.4) lies in $(-\frac{\pi}{2}, \frac{3\pi}{2})$, so $n = 0$. Then we obtain

$$\sum_{k=1}^{m} \arg\big(\frac{v(z_{k+1})}{v(z_k)}\big) = \sum_{k=1}^{m} \arg\big(\frac{z_{k+1} - z_0}{z_k - z_0}\big) = 2\pi.$$

We finish the proof. $\square$

Remark 2.2. *One can show that such hyperbolic geodesic embedding ϕ^h exists, if the Euclidean geodesic embedding ϕ maps each triangle in R_i to an acute triangle.*

The discrete conformality and the hyperbolic discrete conformality are related as follows.

Lemma 2.3. *Let $\phi, \phi' : |T| \to D$ be two geodesic embeddings with the induced PL metrics l, l' respectively. Suppose both ϕ, ϕ' induce hyperbolic geodesic embeddings $\phi^h, \phi^{h\prime} : |T| \to \mathbb{D}$ with the induced hyperbolic PH metrics $l^h, l^{h\prime}$ respectively. Then $l' = u * l$ if and only if $l^{h\prime} = u^h *^h l^h$ where u and u^h are related by*

$$u_i^h = u_i + \ln \frac{1 - |z_i|^2}{1 - |z_i'|^2},$$

with $z_i = \phi(i)$, $z_i' = \phi'(i)$.

Proof. Denote d_h as the hyperbolic distance function on $\mathbb{D}$. Then the lemma follows from the formula

$$\sinh \frac{d_h(z_1, z_2)}{2} = \frac{|z_1 - z_2|}{\sqrt{(1 - |z_1|^2)(1 - |z_2|^2)}},$$

where $z_1, z_2 \in \mathbb{D}$. See [And05]. □

Remark 2.4. *In the smooth setting, Lemma 2.3 is interpreted as follows. Let Ω be a domain in D and f be a smooth map from Ω to D, $w = f(z)$. Denote $g_0 = |dz|^2$ as the Euclidean metric on D and $g_{-1} = \frac{4}{(1-|z|^2)^2}|dz|^2$ as the hyperbolic metric on D. Suppose f is conformal with respect to g_0, i.e. $f^* g_0 = e^{2u} g_0$ for some smooth function $u = u(z)$. Then f is also conformal with respect to g_{-1}. In fact*

$$f^* g_{-1} = f^*\left(\frac{4}{(1 - |w|^2)^2}|dw|^2\right) = \frac{4}{(1 - |f(z)|^2)^2} e^{2u} |dz|^2$$
$$= \left(\frac{1 - |z|^2}{1 - |f(z)|^2}\right)^2 e^{2u} g_{-1} = e^{2(u + \ln \frac{1-|z|^2}{1-|f(z)|^2})} g_{-1}.$$

2.2. **Discrete Laplacian on graphs.** Let $G = (V, E)$ be a connected simple graph, and $\mu : E \to [0, +\infty)$ be a function on the set of edges. We call $G_\mu = (V, E, \mu)$ a *weighted graph*, or an *electrical network*. The *discrete Laplacian operator* $\Delta_\mu : \mathbb{R}^V \to \mathbb{R}^V$ is defined as

$$(\Delta_\mu u)_i = \sum_{j: ij \in E} \mu_{ij}(u_j - u_i).$$

We say that u is *harmonic* at $i \in V$, if $(\Delta_\mu u)_i = 0$. In this case we have the average property

$$u_i = \sum_{j: ij \in E} \frac{\mu_{ij}}{\sum\limits_{k: ik \in E} \mu_{ik}} u_j$$

if $\mu_{jk} > 0$ for all $jk \in E$. So we have the maximum principle.

Lemma 2.5 (Maximum principle for discrete harmonic functions)**.** *If $\mu_{ij} > 0$ for all $ij \in E$ and V_0 is a finite proper subset of V and u is harmonic at every point in V_0, then u achieves its maximum and minimum on $V - V_0$.*

We also have the following well-posedness result of the discrete Laplace equation with Dirichlet boundary condition.

Lemma 2.6. *Suppose $\mu_{ij} > 0$ for all $ij \in E$ and V_0 is a finite proper subset of V and f is a given funciton on $V - V_0$. Then the following equation of $u \in \mathbb{R}^V$*

$$\Delta_\mu u = 0 \text{ in } V_0, \qquad u = f \text{ on } (V - V_0)$$

has a unique solution. Furthermore, the map $(\mu, f) \mapsto u$ is smooth.

Lemma 2.6 is well-known. Solving $u \in \mathbb{R}^V$ here is indeed solving a diagonal dominant linear system.

Let $T = (V, E, F)$ be a triangulation and l be a PL metric. Recall that $K_i(u) = 2\pi - \sum_{jk:\triangle ijk \in F} \theta^i_{jk}$ is the curvature at $i \in V$ under the metric $u * l$. The operator K, $u \mapsto K(u)$, is smooth with respect to u. From the direct calculation or [Luo04], one have

$$dK_i = -\sum_{j:ij\in E} \mu_{ij}(du_j - du_i), \tag{2.5}$$

where

$$\mu_{ij} = \mu_{ij}(u) = \frac{1}{2}(\cot \theta^{k_1}_{ij}(u) + \cot \theta^{k_2}_{ij}(u)) \tag{2.6}$$

for adjacent triangles $\triangle ijk_1, \triangle ijk_2 \in F$. It is not difficult to check that if $u * l$ is uniformly Delaunay then $\mu_{ij} \geq \epsilon$ for some constant $\epsilon = \epsilon(T, l, u) > 0$. From Equation (2.5), we see that the linearization of $-K$ is in fact a discrete Laplacian operator with respect to $G_\mu = (V, E, \mu)$.

2.3. **Maximum principles for discrete curvature.** Maximum principle plays a very important role in partial differential equations and geometric analysis. For the discrete conformal geometry, the curvature K, which is clearly nonlinear as an operator on the set of conformal factors, also satisfies the (strong) maximum principle.

The following lemma is a corollary of Theorem 3.1 in [LSW22]. In [DGM22] there is another proof of the maximum principle for a special case. Let $T = (V, E, F)$ be a triangulated surface.

Lemma 2.7. *Suppose $i \in V$ and R_i is a 1-ring neighborhood. Let $\phi, \phi' : R_i \to \mathbb{C}$ be two Delaunay geodesic embeddings with induced PL metric l, l' respectively. If $l' = u * l$, then*

$$u_i \leq \max_{j\in N_i} u_j, \qquad u_i \geq \min_{j \in N_i} u_j.$$

Furthermore, if $u_i = \max_{j\in N_i} u_j$ or $u_i = \min_{j\in N_i} u_j$, then u is a constant on $\{i\} \cup N_i$.

As a direct corollary, we have the following maximum principle.

Lemma 2.8. *Let $T = (V, E, F)$ be a triangulation of a compact surface with boundary. Let $\phi, \phi' : |T| \to \mathbb{C}$ be two Delaunay geodesic embeddings with induced PL metric l, l' respectively. Suppose ϕ and ϕ' are discretely conformal, $l' = u * l$. Then*

$$\max_{j\in V} u_j = \max_{j\in \partial V} u_j, \qquad \min_{j\in V} u_j = \min_{j \in \partial V} u_j.$$

Furthermore, if $\max_{j\in \mathrm{int}(V)} u_j = \max_{j\in V} u_j$ or $\min_{j\in \mathrm{int}(V)} u_j = \min_{j\in V} u_j$, then u is a constant on V.

For the hyperbolic setting, we have the following modified version of the maximum principle.

Lemma 2.9. *Suppose $i \in V$ and R_i is a 1-ring neighborhood. Let $\phi^h, \phi^{h\prime} : R_i \to \mathbb{D}$ be two Delaunay hyperbolic geodesic embeddings with induced PH metric $l^h, l^{h\prime}$ respectively. Suppose ϕ^h and $\phi^{h\prime}$ are hyperbolic discretely conformal, $l^{h\prime} = u^h *^h l^h$. Then $u_i^h < 0$ implies that*

$$u_i^h > \min_{j \in N_i} u_j^h.$$

Proof. Since the hyperbolic discrete conformality and conformal factor are invariant under the hyperbolic isometric group, we may assume $\phi^h(i) = 0$, $\phi^{h\prime}(i) = 0$. Then $\phi^h, \phi^{h\prime}$ are induced by Euclidean geodesic embeddings ϕ, ϕ' with PL metric l, l' respectively. Notice that the hyperbolic isometric group preserves circles, so l, l' are also Delaunay. From Lemma 2.3, since $l^{h\prime} = u^h *^h l^h$, we have l and l' are also discretely conformal. Specifically $l' = u * l$ where

$$u_j = u_j^h - \ln \frac{1 - |z_j|^2}{1 - |z_j'|^2}.$$

In particular $u_i = u_i^h$. From the maximum principle Lemma 2.7, there exists $j_0 \in N_i$ such that $u_{j_0} \le u_i$. Suppose $u_i^h < 0$. Then

$$|z_{j_0}'| = l_{ij_0}' = e^{\frac{1}{2}(u_i + u_{j_0})} l_{ij_0} < l_{ij_0} = |z_{j_0}|.$$

Therefore

$$u_{j_0}^h = u_{j_0} + \ln \frac{1 - |z_{j_0}|^2}{1 - |z_{j_0}'|^2} \le u_i + \ln \frac{1 - |z_{j_0}|^2}{1 - |z_{j_0}'|^2} < u_i = u_i^h.$$

So

$$u_i^h > \min_{j \in N_i} u_j^h.$$

□

As a direct corollary, we have

Lemma 2.10. *Let $T = (V, E, F)$ be a triangulation of a compact surface with boundary. Let $\phi^h, \phi^{h\prime} : |T| \to \mathbb{D}$ be two Delaunay hyperbolic geodesic embeddings with induced PH metric $l^h, l^{h\prime}$ respectively. Suppose ϕ^h and $\phi^{h\prime}$ are hyperbolic discretely conformal, $l^{h\prime} = u^h *^h l^h$. Then $u_i^h < 0$ implies that*

$$u_i^h > \min_{j \in \partial V} u_j^h.$$

Remark 2.11. *In the smooth setting, the maximum principles above are interpreted as follows. Let Ω be a domain in $\mathbb{C}$. Let f be a conformal map from Ω to $\mathbb{C}$ with respect to the Euclidean metric $g_0 = |dz|^2$. Suppose $f^* g_0 = e^{2u} g_0$. From the curvature formula, for $g = g(z)|dz|^2$,*

$$K_g = -\frac{2}{g(z)} \partial_z \partial_{\bar{z}} \ln g(z),$$

we have $\triangle u = 0$. So u satisfies the maximum principle as in Lemma 2.8.

On the other hand, let Ω be a domain in D. Let f be a conformal map from Ω to D with respect to the hyperbolic metric $g_{-1} = \frac{4}{(1-|z|^2)^2}|dz|^2$. Suppose $f^* g_{-1} = e^{2u^h} g_{-1}$. From the curvature formula, we have

$$\triangle_{g_{-1}} u^h = e^{2u^h} - 1,$$

where $\triangle_{g_{-1}} = (1 - |z|^2)^2 \partial_z \partial_{\bar{z}}$. So u^h satisfies the maximum principle as in Lemma 2.10.

2.4. **Modulus of annuli.** We briefly review the notion of conformal modulus. The definitions and properties discussed here are mostly well-known. One may refer [Ahl10] and [LV73] for more comprehensive introductions.

A *closed annulus* is a subset of $\mathbb{C}$ that is homeomorphic to $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. An *(open) annulus* is the interior of a closed annulus. Given an annulus A, denote $\Gamma = \Gamma(A)$ as the set of smooth simple closed curves in A separating the two boundary components of A. A real-valued Borel measurable function f on A is called *admissible* if $\int_\gamma f ds \geq 1$ for all $\gamma \in \Gamma$. Here ds denotes the element of arc length. The *(conformal) modulus* of A is defined as

$$\text{Mod}(A) = \inf\{\|f\|_2^2 : f \text{ is admissible}\},$$

where $\|f\|_2^2$ denotes the integral of $(f(z))^2$ against the 2-dim Lebesgue measure on A. From the definition it is straightforward to verify that $\text{Mod}(A)$ is conformally invariant. Furthermore, if $f : A \to A'$ is a K-quasiconformal homeomorphism between two annuli, then

$$\frac{1}{K} \cdot \text{Mod}(A) \leq \text{Mod}(A') \leq K \cdot \text{Mod}(A).$$

Given $0 < r < r'$, denote $A_{r,r'}$ as the annulus $\{z \in \mathbb{C} : r < |z| < r'\}$. It is well-known that

$$\text{Mod}(A_{r,r'}) = \frac{1}{2\pi} \ln \frac{r'}{r}.$$

Intuitively, the modulus measures the relative thickness of an annulus. If an annulus A in $\mathbb{C}\backslash\{0\}$ contains $A_{r,r'}$, then it is "thicker" than $A_{r,r'}$ and we have the monotonicity

$$\text{Mod}(A) \geq \text{Mod}(A_{r,r'}) = \frac{1}{2\pi} \ln \frac{r'}{r}.$$

On the other hand, we have that

Lemma 2.12. *Suppose $A \subseteq \mathbb{C}\backslash\{0\}$ is an annulus separating 0 from the infinity. If Mod$(A) \geq 100$, then $A \supseteq A_{r,2r}$ for some $r > 0$.*

Proof. Deonte B as the bounded component of $\mathbb{C} - A$, and $r = \max\{|z| : z \in B\}$ and $r' = \min\{|z| : z \in (B \cup A)^c\}$. If $r' \geq 2r$ we are done. So we may assume $r' < 2r$.

Then $D_{2r} \cap \gamma \neq \emptyset$ for all $\gamma \in \Gamma(A)$. Let f be the function on A such that $f(z) = \frac{1}{r}$ on $A \cap D_{3r}$ and $f(z) = 0$ on $A \backslash D_{3r}$. If $\gamma \in \Gamma$ and $\gamma \subseteq D_{3r}$,

$$\int_\gamma f ds = s(\gamma) \cdot \frac{1}{r} \geq 2 \cdot \text{diam}(B) \cdot \frac{1}{r} \geq 2r \cdot \frac{1}{r} > 1.$$

If $\gamma \in \Gamma$ and $\gamma \nsubseteq D_{3r}$, then γ is a connected curve connecting D_{2r} and D_{3r}^c and

$$\int_\gamma f ds \geq d(D_{2r}, D_{3r}^c) \cdot \frac{1}{r} = r \cdot \frac{1}{r} = 1.$$

So f is admissible and

$$\text{Mod}(A) \leq \int_A f^2 = \frac{1}{r^2} \cdot \text{Area}(A \cap D_{3r}) \leq \frac{\pi(3r)^2}{r^2} = 9\pi < 100.$$

This contradicts with our assumption. □

Remark 2.13. *To some extend, Lemma 2.12 is a consequence of Teichmüller's result on extremal annuli (see Theorem 4-7 in* [Ahl10]*). The constant* 100 *is chosen for convenience and should not be optimal.*

2.5. **Recurrence of electrical networks.** Discrete harmonic functions are closely related to the theory of electrical networks. Let $G_\mu = (V, E, \mu)$ be a weighted graph, i.e. a graph $G = (V, E)$ with a function $\mu : E \to [0, \infty)$. A weighted graph G_μ could be viewed as an electrical network, where μ_{ij} denotes the conductance of the edge ij. Consider a function $u : V \to \mathbb{R}$, which denotes the electric potentials at the vertices. Then u is harmonic at i if and only if the outward electric flux at i is 0. The theory of electrical networks is closely related to discrete extremal length, originally introduced by Duffin [Duf62]. Here we briefly review the theory of discrete extremal length, adapted to our setting. All the definitions and properties here are well-known and one may read [He99] for references.

Let V_1, V_2 be two nonempty disjoint subsets of V. Denote $\Gamma(V_1, V_2)$ as the set of the paths joining V_1 and V_2. An *edge metric* is a function $m : V \to [0, \infty)$. An edge metric m is called $\Gamma(V_1, V_2)$-admissible if $\sum_{e\in\gamma} m(e) \geq 1$ for every $\gamma \in \Gamma(V_1, V_2)$. The *conductance* of $\Gamma(V_1, V_2)$ is defined as

$$\mathrm{Cond}(V_1, V_2) = \inf\{\sum_{e\in E} \mu(e)m^2(e) : \ m \text{ is } \Gamma(V_1, V_2)\text{-admissible.}\}.$$

The *resistance* of $\Gamma(V_1, V_2)$ is defined as

$$\mathrm{Res}(V_1, V_2) = \frac{1}{\mathrm{Cond}(V_1, V_2)}.$$

An electrical network G_μ is called *connected* if $G'_\mu = (V, E \setminus \{e \in E : \mu(e) = 0\})$ is connected. Let V_0 be a nonempty finite subset of vertices in G_μ. Then a connected electric network G_μ is called *recurrent* if $\mathrm{Res}(V_0, \infty) = \infty$, and *transient* otherwise. The recurrency of G_μ is independent of the choice of V_0. The following lemma is well-known and shows that the recurrency implies a discrete Liouville property. See Lemma 5.5 in [He99] for a proof.

Lemma 2.14. *Suppose G_μ is recurrent and u is a bounded function on V, if u is harmonic on every point of V, then u is constant.*

To check that an electrical network G_μ is recurrent, one may study the vertex extremal length of the graph G as follows.

Let $G = (V, E)$ be a graph. A *vertex metric* is a function $\eta : V \to [0, \infty)$. A vertex metric η is called $\Gamma(V_1, V_2)$-admissible if $\sum_{v\in\gamma} \eta(v) \geq 1$ for every $\gamma \in \Gamma(V_1, V_2)$. The *vertex modulus* of $\Gamma(V_1, V_2)$ is defined as

$$\mathrm{Mod}(V_1, V_2) = \inf\{\sum_{v\in V} \eta^2(v) : \ \eta \text{ is } \Gamma(V_1, V_2)\text{-admissible.}\}.$$

The *vertex extremal length* of $\Gamma(V_1, V_2)$ is defined as

$$\mathrm{VEL}(V_1, V_2) = \frac{1}{\mathrm{Mod}(V_1, V_2)}.$$

Let V_0 be a nonempty finite subset of vertices in a connected graph G. Then G is called *VEL-parabolic* if $\mathrm{VEL}(V_0, \infty) = \infty$, and *VEL-hyperbolic* otherwise. The definition is independent of the choice of V_0.

The relation between the VEL-parabolicity and the recurrence is as follows.

Lemma 2.15 (Lemma 5.4 in [He99])**.** *Let $C > 0$ be a constant. Suppose that for each vertex v, we have $\sum_{v \in e} \mu(e) \leq C$. Then for any mutually disjoint, nonempty subsets V_1 and V_2 of V, we have*

$$VEL(V_1, V_2) \leq 2C \cdot Res(V_1, V_2).$$

In particular, if G is VEL-parabolic and G_μ is connected, then G_μ is recurrent.

3. Proof of the Main Theorem

We will prove our main Theorem 1.3 by proving Propositions 1.5 and 1.6. We will first derive needed estimates on uniformly nondegenerate triangulations in Section 3.1. A discrete Schwarz lemma is developed in Section 3.2, for the proof of Proposition 1.5 in Section 3.3. A discrete Liouville theorem is developed in Section 3.4, for the proof of Proposition 1.6 in Section 3.5.

3.1. **Estimates on uniformly nondegenerate triangulations.**

Lemma 3.1. *Suppose $\phi : |T| \to \mathbb{C}$ is a geodesic embedding and all the inner angles in $l = l(\phi)$ are at least ϵ. Then we have the following.*

(a) For all $i \in V$, $\deg(i) \leq \frac{2\pi}{\epsilon}$.

(b) For all $\triangle ijk \in F$,

$$\sin\epsilon \leq \frac{l_{ij}}{l_{ik}} \leq \frac{1}{\sin\epsilon}.$$

(c) For all $\triangle ijk$,

$$\frac{\sin^2\epsilon}{2} l_{ij}^2 \leq \text{Area}(\phi(\triangle ijk)) \leq \frac{1}{2\sin\epsilon} l_{ij}^2.$$

(d) There exists a constant $\delta = \delta(\epsilon) > 0$ such that for all $\triangle ijk \in F$ with $i, j, k \in \text{int}(V)$,

$$d(U_{ijk}^c, \phi(\triangle ijk)) \geq \delta \cdot \text{diam}(\phi(\triangle ijk)),$$

where

$$U_{ijk} = \text{int}(\phi(R_i)) \cup \text{int}(\phi(R_j)) \cup \text{int}(\phi(R_k)).$$

(e) Suppose $a \in V$ and $\phi(a) = 0$. Assume $r > 0$ is such that

$$\phi(R_a) \subseteq D_r \subseteq \phi(|T|).$$

Denote $V_1 = \{i \in V : \phi(i) \in D_r\}$ and T_1 as the subcomplex generated by V_1. Then there exists a constant $C = C(\epsilon) > 0$ such that $D_{r/C} \subseteq \phi(|T_1|)$.

Proof. (a) It is from

$$2\pi \geq \sum_{jk:\triangle ijk \in F} \theta_{jk}^i \geq \sum_{jk:\triangle ijk \in F} \epsilon = \deg(i) \cdot \epsilon.$$

(b) This is by the sine law.

(c) This estimate is straightforward from the area formula

$$\text{Area}(\phi(\triangle ijk)) = \frac{1}{2} l_{ij} l_{ik} \sin\theta_{jk}^i = \frac{1}{2} l_{ij}^2 \frac{\sin\theta_{ik}^j}{\sin\theta_{ij}^k} \sin\theta_{jk}^i.$$

(d) We may normalize $\text{diam}(\phi(\triangle ijk)) = 1$ and $\phi(i) = 0$. By part (a), there are finitely many possible combinatorial structures of the natural triangulation of $R_i \cup R_j \cup R_k$. Fixing a combinatorial structure, $d(U_{ijk}^c, \phi(\triangle ijk))$ is positive and continuously determined by $\phi(a)$'s for $a \in N_i \cup N_j \cup N_k$. By the compactness $d(U_{ijk}^c, \phi(\triangle ijk))$ has a positive lower bound $\delta = \delta(\epsilon)$.

(e) Pick δ as in part (d) and we claim that $C = 1 + 2/\delta$ is a desired constant. Let us prove by contradiction and assume that there exists $z \in D_{r/C} \backslash \phi(|T_1|)$. Then there exists a triangle $\triangle ijk \in F$ such that $z \in \triangle ijk$. Then $\triangle ijk$ is not a triangle in T_1 and we may assume $i \notin V_1$, so $|\phi(i)| \geq r$. Since $\phi(R_a) \subseteq D_r$, we have $ai \notin E$. So $a \neq i, j, k$, then $0 = \phi(a) \notin U_{ijk}$. Then

$$\begin{aligned} r/C &\geq |0 - z| \geq d(U_{ijk}^c, \triangle ijk) \geq \delta \cdot \text{diam}(\triangle ijk) \\ &\geq \delta \cdot |\phi(i) - z| \geq \delta \cdot (r - r/C) = (r/C) \cdot \delta(C - 1) = 2r/C \end{aligned}$$

and we get a contradiction. □

3.2. **A discrete Schwarz lemma.** Recall that the Schwarz lemma says that any holomorphic map $f : D \to D$ satisfies that $|f'(0)| \leq 1$ if $f(0) = 0$. Here we prove a discrete weaker version of the Schwarz lemma. Let T be a triangulated surface.

Proposition 3.2. *Suppose ϕ, ϕ' are geodesic embeddings of $|T|$ into $\mathbb{C}$ with induced PL metrics l, l'. Assume both l, l' satisfy the uniformly nondegenerate condition with constant $\epsilon > 0$ and the Delaunay condition. If $r, r' > 0$ and T_0 is a finite subcomplex of T satisfing that*

$$\phi(|T_0|) \subseteq D_r, \quad D_{r'} \subseteq \phi'(|T_0|),$$

then there exists a constant $M = M(\epsilon) > 0$ such that for every $i \in V$ satisfying $\phi'(i) \in D_{r'/2}$, we have

$$u_i \geq \log \frac{r'}{r} - M.$$

Proof. Without loss of generality, we assume $\epsilon \leq \pi/6$. By scaling, we may assume $r = \frac{1}{4} \sin^3 \epsilon \leq \frac{1}{4}$ and $r' = 1$. Denote

$$V_1 = \{i \in V : \phi'(i) \in D = D_1\}$$

and $T_1 = T(V_1)$ as the subcomplex generated by V_1. Then ϕ, ϕ' map $|T_1|$ into D. Let $z_i = \phi(i)$, $z_i' = \phi'(i)$. Denote

$$u_i^h = u_i + \ln \frac{1 - |z_i|^2}{1 - |z_i'|^2}.$$

We claim $u_i^h \geq 0$ for every $i \in V_1$. Just let i attain the minimum of u^h in V_1. Here $\text{int}(V_1)$ and ∂V_1 are with respect to T, and defined in Section 1.1.

(1) If $i \in \text{int}(V_1)$ and $l_{ij}' < (1 - |z_i'|^2) \sin \epsilon$, for every $ij \in E$, then from Lemma 2.1, ϕ' induces a hyperbolic geodesic embedding $\phi^{h\prime}$ from the 1-ring neighborhood R_i of i_0 into $\mathbb{D}$. Since ϕ' is Delaunay, $\phi^{h\prime}$ is also Delaunay. Since $\phi(|V_1|) \subset D_{\sin^3 \epsilon/4}$, for the same reason ϕ also induces a Delaunay hyperbolic geodesic embedding ϕ^h from R_i into $\mathbb{D}$. Then the hyperbolic maximum principle Lemma 2.9 implies $u_i^h \geq 0$.

(2) If $i \in \text{int}(V_1)$ and there exists $j \in V_1$, $ij \in E$ such that $l_{ij}' \geq (1 - |z_i'|^2) \sin \epsilon$, then from Lemma 3.1 (b),

$$e^{(u_i - u_j)/2} = \frac{e^{(u_i + u_k)/2}}{e^{(u_j + u_k)/2}} = \frac{l_{ik}'}{l_{ik}} \frac{l_{jk}}{l_{jk}'} \geq \sin^2 \epsilon$$

where $\triangle ijk \in F$. So

$$e^{u_i^h} = e^{u_i} \cdot \frac{1-|z_i|^2}{1-|z_i'|^2} = \frac{l_{ij}'}{l_{ij}} \cdot e^{(u_i-u_j)/2} \cdot \frac{1-|z_i|^2}{1-|z_i'|^2}$$
$$\geq \frac{l_{ij}'}{l_{ij}} \cdot \sin^2\epsilon \cdot \frac{1-|z_i|^2}{1-|z_i'|^2} \geq \sin^3\epsilon \cdot \frac{1-|z_i|^2}{l_{ij}} \geq \sin^3\epsilon \cdot \frac{1/2}{2r} = 1.$$

(3) If $i \in \partial V_1$, then there exists $j \in V$ such that $\phi'(j) \notin D$. Then $l_{ij}' \geq 1-|z_i'|$. Since we assume $\epsilon \leq \frac{\pi}{6}$, then $l_{ij}' \geq (1-|z_i'|^2)\sin\epsilon$. As the estimates above, we also have $e^{u_i^h} \geq 1$.

So $u_i^h \geq 0$ for every $i \in V_1$. Then for $i \in V$ satisfying $\phi'(i) \in D_{1/2}$, set $M = -\ln\frac{\sin^3\epsilon}{8}$, and we have

$$u_i = u_i^h - \ln\frac{1-|z_i|^2}{1-|z_i'|^2} \geq -\ln\frac{1-|z_i|^2}{1-|z_i'|^2} \geq \ln(1-|z_i'|^2) \geq \ln\frac{1}{2} = \ln\frac{r'}{r} - M.$$

□

3.3. **Proof of the boundedness of the conformal factor.** In this section, we prove that the discrete conformal factor u is bounded, i.e. Proposition 1.5.

Proof of Proposition 1.5. Without loss of generality, we may assume that $\phi' \circ \phi^{-1}$ is linear on each triangle $\phi(\triangle ijk)$. Then $\phi' \circ \phi^{-1}$ is K-quasiconformal for some constant $K = K(\epsilon) > 0$. We will prove the boundedness of u by showing that for every $j_1, j_2 \in V$,

$$|u_{j_1} - u_{j_2}| \leq 2M + 2\ln C + \ln C' - \ln 2,$$

where $M = M(\epsilon)$ is the constant given in Proposition 3.2 and $C = C(\epsilon)$ is the constant given in Lemma 3.1 (e) and $C' = C'(\epsilon) = e^{200\pi K}$.

Assume $j_1, j_2 \in V$. For convenience, let us assume $\phi(j_1) = \phi'(j_1) = 0$ by translations. Pick $r > 0$ sufficiently large such that $|\phi(j_2)| < r/(2C)$ and $\phi(R_{j_1}) \subseteq D_r$. Let

$$V_1 = \{i \in V : \phi(i) \in D_r\} \text{ and } V_2 = \{i \in V : \phi(i) \in D_{CC'r}\}.$$

Denote T_1, T_2 as the subcomplexes generated by V_1, V_2 respectively. Then by Lemma 3.1 (e) we have

$$\{\phi(j_1), \phi(j_2)\} \subseteq D_{r/(2C)} \subseteq D_{r/C} \subseteq \phi(|T_1|) \subseteq D_r. \tag{3.1}$$

And

$$D_{C'r} \subseteq \phi(|T_2|) \subseteq D_{CC'r}. \tag{3.2}$$

Recall that $A_{r_1,r_2} = \{z \in \mathbb{C} : r_1 < |z| < z_2\}$. Then $A = A_{r,C'r}$ separates $\phi(|T_1|)$ and $\phi(|T_2|)^c$, and then $A' = \phi' \circ \phi^{-1}(A)$ separates $\phi'(T_1)$ and $\phi'(T_2)^c$. Furthermore

$$\mathrm{Mod}(A') \geq \frac{1}{K} \cdot \mathrm{Mod}(A) = \frac{1}{K} \cdot \frac{1}{2\pi} \ln\frac{C'r}{r} = 100.$$

Then by Lemma 2.12 there exists $r' > 0$ such that $A_{r',2r'} \subseteq A'$. So $A_{r',2r'}$ separates $\phi'(T_1)$ and $\phi'(T_2)^c$ and then

$$\phi'(|T_1|) \subseteq D_{r'} \tag{3.3}$$

and

$$\{\phi'(j_1), \phi'(j_2)\} \subseteq D_{r'} \subseteq D_{2r'} \subseteq \phi'(|T_2|). \tag{3.4}$$

By Proposition 3.2, Equation (3.2) and Equation (3.4), both u_{j_1}, u_{j_2} are at least

$$\ln \frac{2r'}{CC'r} - M = \ln \frac{r'}{r} + \ln \frac{2}{CC'} - M.$$

Again by Proposition 3.2, Equation (3.3) and Equation (3.1), both $-u_{j_1}$ and $-u_{j_2}$ are at least

$$\ln \frac{r/C}{r'} - M = \ln \frac{r}{r'} - \ln C - M.$$

So both u_{j_1} and u_{j_2} are in the interval

$$[\ln \frac{r'}{r} + \ln \frac{2}{CC'} - M, \ln \frac{r'}{r} + \ln C + M],$$

and $|u_{j_1} - u_{j_2}|$ is bounded by the length of this interval

$$2M + \ln C - \ln \frac{2}{CC'} = 2M + 2\ln C + \ln C' - \ln 2.$$

□

3.4. **A discrete Liouville theorem.** In this section, we prove the following discrete Liouville theorem.

Proposition 3.3. *Suppose $\phi : |T| \to \mathbb{C}$ is a geodesic homeomorphism, and l satisfies the uniformly nondegenerate condition and the Delaunay condition. Given the weight $\mu \in \mathbb{R}^E_{\geq 0}$ defined as in equation (2.6), then any bounded function u on V is constant if u is harmonic at every point of V.*

Proof. By Lemma 2.14, it suffices to show (V, E, μ) is recurrent. Let us assume l is uniformly nondegerate with constant $\epsilon > 0$. Then for all $i \in V$,

$$\sum_{j:ij\in E} \mu_{ij} \leq \deg(i) \cdot \cot \epsilon \leq \frac{2\pi \cot \epsilon}{\epsilon}.$$

Then by Lemma 2.15, it suffices to show (V, E) is VEL-parabolic and (V, E, μ) is connected.

First we show that if two subsets V_1, V_2 are separated by an annulus, then the vertex extremal length has a lower bound.

Lemma 3.4. *Let V_1, V_2 be two nonempty subsets of V such that $\phi(V_1) \subseteq D_{r_1}$, $\phi(V_2) \subseteq D^c_{r_2}$. Suppose for every $i \in V_1$, $\phi(|R_i|) \subseteq D_{r_2}$. Then there is a constant $C = C(\epsilon) > 0$ such that*

$$VEL(V_1, V_2) \geq C\big(1 - (\frac{r_1}{r_2})^2\big).$$

In particular, if $r_2 \geq 2r_1$, there is a constant $C = C(\epsilon) > 0$ such that $VEL(V_1, V_2) \geq C$.

Proof. Denote $d_M(i) = \max_{j:ji\in E} d(\phi(i), \phi(j))$. Consider the vertex metric η as follows. For $i \in V$, if $\phi(i) \in D^c_{r_2}$, then define $\eta(i) = 0$; if $\phi(i) \in D_{r_2}$ and there exists $j \in V$, $ji \in E$ such that $\phi(j) \in D_{r_2}$, then define $\eta(i) = \frac{d_M(i)}{r_2 - r_1}$; if $\phi(i) \in D_{r_2}$ and there is no $j \in V$, $ji \in E$ such that $\phi(j) \in D_{r_2}$, then define $\eta(v) = 0$.

First we check η is $\Gamma(V_1, V_2)$-admissible. Let $\gamma = \{i_0, ..., i_k\}$ be a path joining V_1 and V_2 such that $i_0, \cdots, i_{k-1} \in D_{r_2}$ and $i_k \in D^c_{r_2}$. Then from the assumptions

$$\sum_{s=0}^{k-1} \eta(i_s) = \frac{1}{r_2 - r_1} \sum_{s=0}^{k-1} d_M(i_s)$$
$$\geq \frac{1}{r_2 - r_1} \sum_{s=0}^{k-1} d(\phi(i_s), \phi(i_{s+1})) \geq \frac{1}{r_2 - r_1} d(\phi(i_0), \phi(i_{k+1})).$$

Since $\phi(i_0) \in V_1 \subseteq D_{r_1}$ and $\phi(i_{k+1}) \in D^c_{r_2}$, we obtain $\sum_{i\in\gamma} \eta(i) \geq \frac{1}{r_2-r_1}(r_2 - r_1) = 1$.

Next we estimate an upper bound of $\sum_{i\in V} \eta^2(i)$. We only need to consider the vertices where η are nonzero. For $i \in V$, $\eta(i) \neq 0$, since $\phi(i) \in D_{r_2}$ and there is $j \in V$ such that $\phi(j) \in D_{r_2}$ and $ji \in E$, we have $d(\phi(i), \phi(j)) \leq 2r_2$. Then from Lemma 3.1 (a) and (b), there is a constant $C_1 = C_1(\epsilon) > 0$ such that $d_M(i) \leq C_1 r_2$. So for every $i \in V$, $\eta(i) \neq 0$, we have $\phi(R_i) \subset D_{(1+C_1)r_2}$. Then from Lemma 3.1 (a) and (b) and (c), there is a constant $C_2 = C_2(\epsilon) > 0$ such that

$$\sum_{i\in V} \eta^2(i) = \sum_{\eta(i)\neq 0} \frac{d^2_M(i)}{(r_2 - r_1)^2} \leq \sum_{\eta(i)\neq 0} \frac{C_2 \text{Area}(\phi(R_i))}{(r_2 - r_1)^2}.$$

Since every triangle is calculated at most three times in $\phi(|R_i|)$, $i \in V$, we obtain

$$\sum_{i\in V} \eta^2(i) \leq \frac{3C_2 \text{Area}(D_{(1+C_1)r_2})}{(r_2 - r_1)^2} = \frac{1}{C} \frac{r_2^2}{(r_2 - r_1)^2}.$$

So $\text{Mod}(V_1, V_2) \leq \frac{1}{C} \frac{r_2^2}{(r_2-r_1)^2}$. We finish the proof. □

The vertex extremal length has a property of additivity, which is from Lemma 5.1 in [He99].

Lemma 3.5 (Lemma 5.1 in [He99]). *Let $V_1, V_2, \cdots, V_{2m}$ be mutually disjoint, non-void subsets of vertices such that for $i_1 < i_2 < i_3$, V_{i_2} separates V_{i_1} from V_{i_3}, i.e. every path joining V_{i_1} and V_{i_3} must pass through a vertex in V_{i_2}. We allow $V_{2m} = \infty$. Then we have*

$$VEL(V_1, V_{2m}) \geq \sum_{i=1}^{m} VEL(V_{2k-1}, V_{2k}).$$

Then combining Lemma 3.4 and Lemma 3.5, we can show the following lemma.

Lemma 3.6. *(V, E) is VEL-parabolic.*

Proof. We construct infinitely many V_k by induction starting from a finite subset V_1 of V. Given a finite subset V_k of V, choose r large enough such that $\bigcup_{i\in V_k} \phi(R_i) \subseteq D_r$. Denote $\tilde{V} = \{i \in V : \phi(i) \in D_{2r}\}$. Consider $V' = \bigcup_{i\in\tilde{V}} R_i$, $V'' = \bigcup_{i\in V'} R_i$. Choose r' such that $\phi(V'') \subseteq D_{r'}$. Since the interior of $\phi(R_i)$, $i \in V$ cover the plane, we can choose a finite subset V'_{k+1} such that $\partial D_{r'} \subseteq \bigcup_{i\in V'_{k+1}} \text{int}(\phi(R_i))$, and we may assume $\partial D_{r'} \cap \text{int}(\phi(R_i)) \neq \emptyset$. Then set

$$V_{k+1} = \{i \in V : \text{ there exists } j \in V'_{k+1} \text{ such that } ji \in E\}.$$

We claim $\phi(V_{k+1}) \subseteq D_{2r}^c$. If not, then there exists $i \in V'_{k+1}$ such that $i \in V'$. Then $\phi(R_i) \subseteq D_{r'}$, which does not intersect with $\partial D_{r'}$. Contradiction.

As construction above, we obtain that for $i \in V_k$, $\phi(R_i) \subseteq D_r$ and $\phi(V_{k+1}) \subseteq D_{2r}^c$. Since $\partial D_{r'} \subseteq \bigcup_{i \in V'_{k+1}} \phi(R_i)$, it is clearly V_{k+1} separates V_k and V_{k+2}. So the conditions in Lemma 3.4 and Lemma 3.5 hold. □

It remains to show that $G_\mu = (V, E, \mu)$ is connected, i.e., G'_μ is connected. We notice that if $\mu_{ij} = 0$ then the points $\phi(i), \phi(k_1), \phi(j), \phi(k_2)$ are co-circle. So to obtain G'_μ, we just delete, for finitely many times, the interior edges of a polygon, whose vertices are co-circle. So G'_μ is connected. □

3.5. **Proof of that the conformal factor is constant.** In this section, we show that u is constant, i.e. Proposition 1.6, then finish the whole proof of main Theorem 1.3.

Proof of Proposition 1.6. In the proof, we will construct a deformation of discrete conformal factors $u(t)$. To avoid being confused with u and $u(t)$, we use the notation $\bar{u}$ instead of u in the statement of Proposition 1.6. In other words, we assume $l' = \bar{u} * l$ and want to prove that $\bar{u}$ is bounded.

Let us prove by contradiction and assume that $\bar{u}$ is not constant. Without loss of generality, we can do a scaling and assume

$$\inf_{i \in V} \bar{u}_i < 0 < \sup_{i \in V} \bar{u}_i$$

and

$$-\inf_{i \in V} \bar{u}_i = \sup_{i \in V} \bar{u}_i = |\bar{u}|_\infty.$$

By a standard compactness arguement, there exists a constant $\delta_0 = \delta_0(\epsilon)$, such that for every function $u : V \to \mathbb{R}$ satisfying $|u|_\infty < 2\delta_0$, $u * l$ satisfies the triangle inequalities and is uniformly nondegenerate and uniformly Delaunay.

Denote $\delta = \min\{\delta_0, |\bar{u}|_\infty\}$. Pick a sequence of increasing finite subsets V_n of V such that $\bigcup_{n=1}^{\infty} V_n = V$. We use the notation in Section 2.2. For each $n \in \mathbb{N}$, we will construct a smooth $\mathbb{R}^{V_n}$-valued function $u^{(n)}(t) = [u_i^{(n)}(t)]_{i \in V_n}$ on $(-2\delta, 2\delta)$ such that

(a) $u^{(n)}(0) = 0$, and
(b) $\dot{u}_i^{(n)}(t) = \bar{u}_i / |\bar{u}|_\infty$ if $i \in \partial V_n$, and
(c) if $i \in \text{int}(V_n)$ then

$$\sum_{j: ij \in E} \mu_{ij}(u^{(n)}(t))(\dot{u}_j^{(n)}(t) - \dot{u}_i^{(n)}(t)) = 0 \tag{3.5}$$

where $\mu_{ij}(u)$ is defined for all $ij \in E(V_n)$ as in Equation (2.6).

The conditions (b) and (c) give an autonomous ODE system on

$$\mathcal{U}_n = \{u \in \mathbb{R}^{V_n} : |u|_\infty < 2\delta\}.$$

Notice that $\mu_{ij}(u) > 0$ if $u \in \mathcal{U}_n$. Then by Lemma 2.6, $\dot{u}^{(n)}(t)$ is smoothly determined by $u^{(n)}(t)$ on $\mathcal{U}_n$. Given the initial condition $u^{(n)}(0) = 0$, assume the maximum existence interval for this ODE system on $\mathcal{U}_n$ is $(t_{\min}, t_{\max})$ where

$t_{\min} \in [-\infty, 0)$ and $t_{\max} \in (0, \infty]$. By the maximum principle Lemma 2.5, for all $i \in V_n$

$$|\dot{u}^{(n)}|_\infty \leq \max_{j \in \partial V_n} |\dot{u}_j^{(n)}| = \max_{j \in \partial V_n} |\bar{u}_j| / |\bar{u}|_\infty \leq 1.$$

So $|u^{(n)}(t)|_\infty \leq t \leq t_{\max}$ for all $t \in [0, t_{\max})$. By the maximality of $t_{\max}$, $t_{\max} \geq 2\delta$ and for a similar reason $t_{\min} \leq -2\delta$. So $u^{(n)}(t)$ is indeed well-defined on $(-2\delta, 2\delta)$. By Equation (2.5) and Equation (3.5), $K_i(u^{(n)}(t)) = 0$ for all $i \in \text{int}(V_n)$. Then by Lemma 2.5, for all $i \in V_n$

$$\begin{aligned} |\bar{u}_i - u_i^{(n)}(\delta)| &\leq \max_{j \in \partial V_n} |\bar{u}_j - u_j^{(n)}(\delta)| \\ &= \max_{j \in \partial V_n} \left(\bar{u}_j - \delta \cdot \frac{\bar{u}_j}{|\bar{u}|_\infty} \right) \leq (1 - \frac{\delta}{|\bar{u}|_\infty}) |\bar{u}|_\infty = |\bar{u}|_\infty - \delta. \end{aligned} \tag{3.6}$$

By picking a subsequence, we may assume that $u_i^{(n)}$ converge to u_i^* on $[0, \delta]$ uniformly for all $i \in V$. Then $u^* = [u_i^*]_{i \in V}$ satisfies the following properties.

(a) $u_i^*(t)$ is 1-Lipschitz for all $i \in V$. As a consequence, for all $i \in V$, $u_i^*(t)$ is differentiable at a.e. $t \in [0, \delta]$.

(b) $u^*(t) * l$ is uniformly nondegenerate and uniformly Delaunay for all $t \in [0, \delta]$.

(c) For all $i \in V$, $K_i(u^*(t)) = 0$. As a consequence for a.e. $t \in [0, \delta]$,

$$0 = \frac{d}{dt} K_i(u^*(t)) = \sum_{j: ij \in E} \mu_{ij}(u^*(t)) (\dot{u}_j^*(t) - \dot{u}_i^*(t)),$$

for all $i \in V$.

(d) By Proposition 3.3, $\dot{u}^*(t)$ is constant on V for a.e. $t \in [0, \delta]$. As a consequence $u_i^*(\delta)$ equals to a constant c independent on $i \in V$.

(f) By Equation (3.6),

$$|\bar{u}_i - c| = |\bar{u}_i - u_i^*(\delta)| \leq |\bar{u}|_\infty - \delta$$

for all $i \in V$. As a consequence we get the following contradiction

$$2|\bar{u}|_\infty = |\sup_{i \in V} \bar{u}_i - \inf_{i \in V} \bar{u}_i| \leq |\sup_{i \in V} \bar{u}_i - c| + |\inf_{i \in V} \bar{u}_i - c| \leq 2|\bar{u}|_\infty - 2\delta.$$

□

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