WEAK TYPE A_p ESTIMATE FOR BILINEAR CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. In this paper, we investigate the boundedness of bilinear Calderón-Zygmund operators T from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^{p,\infty}(v_{\vec{w}})$ with the stopping time method, where $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 < \infty$ and $\vec{w}$ is a multiple $A_{\vec{P}}$ weight. Specifically, we study the exponent α of $A_{\vec{P}}$ constant in formula

$$\|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leqslant C_{m,n,\vec{P},T}[\vec{w}]^{\alpha}_{A_{\vec{P}}} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

Surprisingly, we show that when $p \geqslant \frac{3+\sqrt{5}}{2}$ or $\min\{p_1, p_2\} > 4$, the exponent α in the above estimate can be less than 1, which is different from the linear scenario.

1. INTRODUCTION AND MAIN RESULTS

In recent years, the theory of Calderón-Zygmund operators has attracted widespread attention. There have been many advances in the optimal control of weighted operator norms with A_p weights.

In the linear case, in 2012, Hytönen proved the A_2 conjecture in [3] and obtained

$$\|T(f)\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\{1, \frac{p'}{p}\}} \|f\|_{L^p(w)}.$$

Just one year later, in [6], Lerner proved that Calderón-Zygmund operators can be controlled by sparse operators and provided an alternative proof of the A_2 theorem. We recommend interested readers to learn about the history of A_2 theorem in the above two papers and the references therein.

For the weak weighted operator norms of Calderón-Zygmund operators, in [4], Hytönen, Lacey, Martikainen, Orponen, Reguera, Sawyer, and Uriarte-Tuero obtained

$$\|T(f)\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p} \|f\|_{L^p(w)}.$$

In the multilinear case, in [10], Li, Moen, and Sun proved that when $1 < p, p_1, p_2 < \infty$,

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{\max\{1, \frac{p_1'}{p}, \frac{p_2'}{p}\}} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}, \tag{1.1}$$

and they provided a beautiful example to show that their result is optimal. When $p < 1$, the estimate (1.1) still holds since the Calderón-Zygmund operators can be controlled by sparse operators pointwise, as shown in [1, 7, 2]. For weak norms, it is generally believed that the optimal exponent in $A_{\vec{P}}$ estimate is 1, which is the same as the linear case. Li and Sun gave a mixed $A_p - A_\infty$ estimate in [11], just in terms of the $A_{\vec{P}}$ constant, with the exponent larger than 1. In Li's master's thesis [9], he established a Coifman-Fefferman

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type inequality to prove that the exponent of the $A_{\vec{P}}$ constant for the weak type estimate can be $1+1/p$.

In this paper, we show that the exponent in weak $A_{\vec{P}}$ estimate for bilinear Calderón-Zygmund operators can be less than 1 for certain $\vec{P}$. Specifically, we prove the following theorem.

Theorem 1.1. *Let T be a bilinear Calderón-Zygmund operator, $\vec{P}=(p_1,p_2)$ with $1/p_1+1/p_2=1/p$ and $1<p,p_1,p_2<\infty$. Suppose $\vec{w}=(w_1,w_2)\in A_{\vec{P}}$, then*

$$\|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})}\leqslant C_{m,n,\vec{P},T}[\vec{w}]^{\alpha}_{A_{\vec{P}}}\|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}, \tag{1.2}$$

where

$$\alpha=\min\{\beta,\gamma\},\beta=\frac{1}{p}+\max\big\{\min\{\frac{1}{p_1'},\frac{1}{p_1'}\frac{p_2'}{p}\},\min\{\frac{1}{p_2'},\frac{1}{p_2'}\frac{p_1'}{p}\}\big\},\gamma=\max\{1,\frac{p_1'}{p},\frac{p_2'}{p}\}.$$

It should be noted that the exponent γ in the theorem comes from the strong type estimate (1.1), so we only need to prove estimate (1.2) with exponent $\alpha=\beta$.

Remark 1.2. Note that $\beta\leqslant\max\big\{\frac{1}{p}+\frac{1}{p_1'},\frac{1}{p}+\frac{1}{p_2'}\big\}<1+\frac{1}{p}$, so our results improve the one from [9].

Remark 1.3. We can apply the extrapolation techniques demonstrated in [12, Theorem 4.1], to generalize Theorem 1.1 to the case of $p<1$. In particular, if we use $\vec{P}=(p_1,p_2)$ with $2<p_1=p_2<\sqrt{2}+1$ as the starting point, we can obtain better results than the strong type estimate (1.1), and further details are left for interested readers.

2. Preliminaries

2.1. **Bilinear Calderón-Zygmund operators.** We call T a *bilinear Calderón-Zygmund operator* if it is originally defined on the product of Schwartz spaces and takes values in tempered distributions, meanwhile, for some $1<q_1,q_2<\infty$, it can be extended to a bounded bilinear operator from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/q_1+1/q_2=1/q$, and if there exists a function $K(y_0,y_1,y_2)$, defined off the diagonal $y_0=y_1=y_2$ in $(\mathbb{R}^n)^3$ that satisfies

$$T(f_1,f_2)(y_0)=\int_{(\mathbb{R}^n)^2}K(y_0,y_1,y_2)f_1(y_1)f_2(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2,\quad \forall y_0\notin \operatorname{supp} f_1\cap\operatorname{supp} f_2;$$

$$|K(y_0,y_1,y_2)|\leqslant\frac{C}{(|y_0-y_1|+|y_0-y_2|)^{2n}};$$

and for some $A,\varepsilon>0$, whenever $|h|\leqslant\frac{1}{2}\max\{|y_0-y_1|,|y_0-y_2|\}$,

$$\begin{aligned}|K(y_0+h,y_1,y_2)-K(y_0,y_1,y_2)|&+|K(y_0,y_1+h,y_2)-K(y_0,y_1,y_2)|\\&+|K(y_0,y_1,y_2+h)-K(y_0,y_1,y_2)|\\&\leqslant\frac{1}{(|y_0-y_1|+|y_0-y_2|)^{2n}}\omega\left(\frac{|h|}{|y_0-y_1|+|y_0-y_2|}\right),\end{aligned}$$

where ω is the modulus of Dini-continuity, i.e., an increasing function satisfies $\omega(0)=0$, $\omega(t+s)\leqslant\omega(t)+\omega(s)$, and

$$\|\omega\|_{\text{Dini}}:=\int_0^1\omega(t)\frac{\mathrm{d}t}{t}<\infty.$$

In [2], Damián, Hormozi, and Li proved that bilinear Calderón-Zygmund operators can be pointwise controlled by sparse operators, which will be introduced in section 2.3.

2.2. **Multiple $A_{\vec{P}}$ weight.** Recall that in the linear case, a *weight* is a non-negative locally integrable function. When $1 < p < \infty$, The set A_p is composed of weights that satisfy

$$[w]_{A_p} := \sup_{Q:\,\text{cube in}\,\mathbb{R}^n} \langle w\rangle_Q \langle w^{1-p'}\rangle_Q^{p-1} < \infty,$$

where $\langle w\rangle_Q := w(Q)/|Q|$. When $p = \infty$, $A_\infty := \bigcup_{1<p<\infty} A_p$ and the A_∞ constant $[w]_{A_\infty}$ is defined by

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q),$$

where M denotes the Hardy-Littlewood maximal function. Meanwhile, for any $w \in A_p$, we have $[w]_{A_\infty} \leqslant [w]_{A_p}$.

As is well known, in [8], Lerner, Ombrosi, Pérez, Torres, and Trujillo-González extended the above definition to the multilinear case, and defined *multiple $A_{\vec{P}}$ weights* as follows. Let $\vec{P} = (p_1, \cdots, p_m)$ with $1 < p_1, \ldots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Given $\vec{w} = (w_1, \cdots, w_m)$, set

$$v_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}.$$

The $A_{\vec{P}}$ constant is defined by

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \langle v_{\vec{w}}\rangle_Q \prod_{i=1}^m \langle \sigma_i\rangle_Q^{p/p_i'},$$

where $\sigma_i = w_i^{1-p_i'}$. We say that $\vec{w}$ satisfies the multilinear $A_{\vec{P}}$ condition if $[\vec{w}]_{A_{\vec{P}}} < \infty$. Particularly, in Theorem 3.6 of the aforementioned paper, they proved that

$$[v_{\vec{w}}]_{A_{mp}} \leqslant [\vec{w}]_{A_{\vec{P}}}, [\sigma_i]_{A_{mp_i'}} \leqslant [\vec{w}]_{A_{\vec{P}}}^{p_i'/p}, \quad \forall \vec{w} \in A_{\vec{P}}. \tag{2.1}$$

2.3. **Dyadic cubes system, sparse operators and stopping time argument.** The *dyadic cubes system $\mathscr{D}$* is a family of cubes with the following properties:

(1) for any $Q \in \mathscr{D}$, its sides are parallel to the coordinate axes and its sidelength is of the form of 2^k;

(2) $Q \cap R \in \{Q, R, \emptyset\}$, for any $Q, R \in \mathscr{D}$;

(3) the cubes of fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

A collection $\mathcal{S} \subset \mathscr{D}$ is called *sparse* if for each $Q \in \mathcal{S}$, there exists a subset $E_Q \subset Q$ such that $|E_Q| \geqslant \frac{1}{2}|Q|$ and the sets $\{E_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint. For a sparse family $\mathcal{S}$, we can define the *sparse operator $A_{\mathscr{D},\mathcal{S}}$* as follows:

$$A_{\mathscr{D},\mathcal{S}}(\vec{f}) = \sum_{Q\in\mathcal{S}} \langle f_1\rangle_Q \langle f_2\rangle_Q \chi_Q,$$

where $\vec{f} = (f_1, f_2)$.

Below, we will introduce the main technique of this paper, stopping time argument, which was introduced by Li and Sun in [11], and further improved by Damián, Hormozi, and Li in [2].

Let w be a weight and $f \in L^p(w)$ for some $0 < p < \infty$. Suppose that the sparse family $\mathcal{S}$ has a collection of maximal cubes, in other words, there exists a collecion of disjoint cubes $\{Q_i\}_{i\in\Lambda} \subset \mathcal{S}$, such that for any cube $Q \in \mathcal{S}$, there exists $i \in \Lambda$ such that $Q \subset Q_i$. Now we construct the *stopping time family* $\mathscr{F}$ from the pair (f, w). Let $\mathscr{F}_0 := \{Q_i\}_{i\in\Lambda}$ and

$$\mathscr{F}_k := \bigcup_{F\in\mathscr{F}_{k-1}} \{F' \subset F : F' \text{ is the maximal cube in } \mathcal{S} \text{ that satisfies } \langle f\rangle^w_{F'} > 2\,\langle f\rangle^w_F\},$$

where $\langle f\rangle^w_Q := \int_Q fw\,dx/w(Q)$, then the stopping time family is $\mathscr{F} := \bigcup_{k=0}^{\infty} \mathscr{F}_k$. It is easy to deduce from the construction above that

$$\sum_{F\in\mathscr{F}} \left(\langle f\rangle^w_F\right)^p w(F) \lesssim \|\,\mathrm{M}^w_{\mathscr{D}}(f)\|^p_{L^p(w)} \lesssim \|f\|^p_{L^p(w)}\,, \tag{2.2}$$

where $\mathrm{M}^w_{\mathscr{D}}(f)(x) := \sup_{x\in Q,Q\in\mathscr{D}} \langle f\rangle^w_Q$. We use $\pi_{\mathscr{F}}(Q)$ to represent the *stopping parents* of Q, that is, the minimal cube containing Q in $\mathscr{F}$. According to the definition, we have $\langle f\rangle^w_Q \leqslant 2\,\langle f\rangle^w_{\pi_{\mathscr{F}}(Q)}$.

3. Proof of the main results

In order to prove the main theorem, we need the following lemmas.

Lemma 3.1. ([10, lemma 3.1]) *Let $\vec{P} = (p_1, p_2)$ with $1/p_1+1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Then $\vec{w}^1 := (v_{\vec{w}}^{1-p'}, w_2) \in A_{\vec{P}^1}$, with $\vec{P}^1 = (p', p_2)$ and*

$$[\vec{w}^1]_{A_{\vec{P}^1}} = [\vec{w}]^{p_1'/p}_{A_{\vec{P}}}.$$

Lemma 3.2. ([11, lemma 4.5]) *Let $\vec{P} = (p_1, p_2)$ with $1/p_1+1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Suppose that $\mathscr{D}$ is a dyadic cubes system and $\mathcal{S}$ is a sparse family in $\mathscr{D}$. Then the following assertions are equivalent.*

(1) $\|A_{\mathscr{D},\mathcal{S}}(|f_1|\sigma_1, |f_2|\sigma_2)\|_{L^{p,\infty}(v_{\vec{w}})} \leqslant C \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}$.

(2) $\int_Q A_{\mathscr{D},\mathcal{S}}(|f_1|\sigma_1\chi_Q, |f_2|\sigma_2\chi_Q)v_{\vec{w}}\,\mathrm{d}x \leqslant C \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}\, v_{\vec{w}}(Q)^{1/p'}$ *for all dyadic cubes $Q \in \mathcal{S}$ and all functions $f_i \in L^{p_i}(\sigma_i)$, $i = 1, 2$.*

A careful read of [11, lemma 4.5] reveals that the constants C appearing in Lemma 3.2 are comparable.

Lemma 3.3. *Let $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Suppose that $\widetilde{Q}$ is a dyadic cube and* $\operatorname{supp} f_2 \subset \widetilde{Q}$, *then*

$$\begin{aligned}\|\chi_{\widetilde{Q}} A_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}}, |f_2|\sigma_2)\|_{L^p(v_{\vec{w}})} \lesssim &\max\left\{\min\left\{[\sigma_1]_{A_\infty}, [\sigma_2]_{A_\infty}\right\}^{1/p}, \min\left\{[\sigma_1]_{A_\infty}, [v_{\vec{w}}]_{A_\infty}\right\}^{1/p_2'}\right\}\\ &\times [\vec{w}]^{1/p}_{A_{\vec{P}}}\, \|f_2\|_{L^{p_2}(\sigma_2)}\, \sigma_1(\widetilde{Q})^{1/p_1}.\end{aligned}$$

Suppose Lemma 3.3 is proven, referring to the method in [11], we can directly prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Using Lemma 3.1 and Lemma 3.3, for each $Q \in \mathcal{S}$, we have

$$v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathscr{D},\mathcal{S}}(|f_1|\sigma_1\chi_Q, |f_2|\sigma_2\chi_Q)v_{\vec{w}}\,\mathrm{d}x$$

$$
\begin{aligned}
&= v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathscr{D},\mathcal{S}}(v_{\vec{w}}\chi_Q, |f_2|\sigma_2\chi_Q)|f_1|\sigma_1 \,\mathrm{d}x \\
&\leqslant v_{\vec{w}}(Q)^{-1/p'} \left(\int_Q \left(A_{\mathscr{D},\mathcal{S}}(v_{\vec{w}}\chi_Q, |f_2|\sigma_2\chi_Q)\right)^{p_1'} \sigma_1 \,\mathrm{d}x \right)^{1/p_1'} \left(\int_Q |f_1|^{p_1}\sigma_1 \,\mathrm{d}x \right)^{1/p_1} \\
&\lesssim \max\Big\{ \min\{[v_{\vec{w}}]_{A_\infty}, [\sigma_2]_{A_\infty}\}^{\frac{1}{p_1'}}, \min\{[v_{\vec{w}}]_{A_\infty}, [\sigma_1]_{A_\infty}\}^{\frac{1}{p_2'}} \Big\} [\vec{w}^1]_{A_{\vec{P}1}}^{\frac{1}{p_1'}} \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)} \\
&\overset{(2.1)}{\leqslant} [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}+\max\left\{ \min\left\{\frac{1}{p_1'}, \frac{1}{p_1'}\frac{p_2'}{p}\right\}, \min\left\{\frac{1}{p_2'}, \frac{1}{p_2'}\frac{p_1'}{p}\right\}\right\}} \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)}.
\end{aligned}
$$

Finally, according to Lemma 3.2, we get the desired result. This finishes the proof. □

To prove Lemma 3.3, we need the following lemma.

Lemma 3.4. ([2, lemma 4.15]) *Let $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Then for any sparse family $\mathcal{S}$, we have*

$$
\Big\| \sum_{Q\in\mathcal{S}} \langle\sigma_1\rangle_Q \langle\sigma_2\rangle_Q \chi_Q \Big\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \Big(\sum_{Q\in\mathcal{S}} \langle\sigma_1\rangle_Q^{p/p_1} \langle\sigma_2\rangle_Q^{p/p_2} |Q| \Big)^{1/p}, \tag{3.1}
$$

$$
\Big\| \sum_{Q\in\mathcal{S}} \langle\sigma_1\rangle_Q \langle v_{\vec{w}}\rangle_Q \chi_Q \Big\|_{L^{p_2'}(\sigma_2)} \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \Big(\sum_{Q\in\mathcal{S}} \langle\sigma_1\rangle_Q^{p_2'/p_1} \langle v_{\vec{w}}\rangle_Q^{p_2'/p'} |Q| \Big)^{1/p_2'}. \tag{3.2}
$$

Proof of Lemma 3.3. In the first half of the proof, we will use a method similar to that in [2] and [11]. Since $\operatorname{supp} f_2 \subset \widetilde{Q}$, we have

$$
\begin{aligned}
A_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}}, |f_2|\sigma_2) &= \sum_{\substack{Q\in\mathcal{S} \\ Q\cap\widetilde{Q}\neq\emptyset}} \langle\sigma_1\chi_{\widetilde{Q}}\rangle_Q \langle|f_2|\sigma_2\rangle_Q \chi_Q \\
&= \sum_{\substack{Q\in\mathcal{S} \\ \widetilde{Q}\subset Q}} \langle\sigma_1\chi_{\widetilde{Q}}\rangle_Q \langle|f_2|\sigma_2\rangle_Q \chi_Q + \sum_{\substack{Q\in\mathcal{S} \\ Q\subset\widetilde{Q}}} \langle\sigma_1\rangle_Q \langle|f_2|\sigma_2\rangle_Q \chi_Q \\
&:= A^1_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}}, |f_2|\sigma_2) + A^2_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}}, |f_2|\sigma_2).
\end{aligned}
$$

For $A^1_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}}, |f_2|\sigma_2)$, the calculation is not difficult,

$$
\begin{aligned}
\Big\| \chi_{\widetilde{Q}} A^1_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}}, |f_2|\sigma_2) \Big\|_{L^p(v_{\vec{w}})} &= \Big\| \sum_{\widetilde{Q}\subset Q} \frac{\sigma_1(Q\cap\widetilde{Q}) \int_{\widetilde{Q}} f_2(y_2)\sigma_2 \,\mathrm{d}y_2}{|Q|^2} \chi_{\widetilde{Q}} \Big\|_{L^p(v_{\vec{w}})} \\
&\lesssim \Big\| \frac{\sigma_1(\widetilde{Q}) \int_{\widetilde{Q}} f_2(y_2)\sigma_2 \,\mathrm{d}y_2}{|\widetilde{Q}|^2} \chi_{\widetilde{Q}} \Big\|_{L^p(v_{\vec{w}})} \\
&\leqslant \frac{\sigma_1(\widetilde{Q}) \|f_2\|_{L^{p_2}(\sigma_2)} \sigma_2(\widetilde{Q})^{1/p_2'}}{|\widetilde{Q}|^2} v_{\vec{w}}(\widetilde{Q})^{1/p} \\
&\leqslant [\vec{w}]_{A_{\vec{P}}}^{1/p} \|f_2\|_{L^{p_2}(\sigma_2)} \sigma_1(\widetilde{Q})^{1/p_1}.
\end{aligned}
$$

It remains to estimate $A^2_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}},|f_2|\sigma_2)$. By duality, we have

$$\begin{aligned}\left\|A^2_{\mathscr{D},\mathcal{S}}(\sigma_1\chi_{\widetilde{Q}},|f_2|\sigma_2)\right\|_{L^p(v_{\vec{w}})} &= \left\|\sum_{Q\subset\widetilde{Q}}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q\chi_Q\right\|_{L^p(v_{\vec{w}})}\\ &= \sup_{\|h\|_{L^{p'}(v_{\vec{w}})}=1}\sum_{Q\subset\widetilde{Q}}\langle f_2\rangle_Q^{\sigma_2}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q\int_Q h\,\mathrm{d}v_{\vec{w}}\\ &= \sup_{\|h\|_{L^{p'}(v_{\vec{w}})}=1}\sum_{Q\subset\widetilde{Q}}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{v_{\vec{w}}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q v_{\vec{w}}(Q).\end{aligned}$$

Let $\mathcal{S}' = \mathcal{S}\cap\widetilde{Q}$, then $\widetilde{Q}$ is the maximal cube in the sparse family $\mathcal{S}'$ and we can use the stopping time argument mentioned above. Let $\mathscr{F}_2$ and $\mathscr{H}$ represent the stopping time family constructed by (f_2,σ_2) and $(h,v_{\vec{w}})$ respectively, and write $\pi_{\mathscr{F}_2}(Q)=F_2$, and $\pi_{\mathscr{H}}(Q)=H$ together as $\pi(Q)=(F_2,H)$. Then,

$$\begin{aligned}\sum_{Q\in\mathcal{S}'}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{v_{\vec{w}}}\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q v_{\vec{w}}(Q) &= \sum_{F_2\in\mathscr{F}_2}\sum_{\substack{H\in\mathscr{H}\\ H\subset F_2}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{v_{\vec{w}}}\lambda_Q\\ &\quad+\sum_{H\in\mathscr{H}}\sum_{\substack{F_2\in\mathscr{F}_2\\ F_2\subset H}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\langle f_2\rangle_Q^{\sigma_2}\langle h\rangle_Q^{v_{\vec{w}}}\lambda_Q\\ &:= I_1+I_2,\end{aligned}$$

where $\lambda_Q=\langle\sigma_1\rangle_Q\langle\sigma_2\rangle_Q v_{\vec{w}}(Q)$.

For I_1, we have

$$\begin{aligned}I_1 &\leqslant 4\sum_{F_2\in\mathscr{F}_2}\langle f_2\rangle_{F_2}^{\sigma_2}\sum_{\substack{H\in\mathscr{H}\\ H\subset F_2}}\langle h\rangle_H^{v_{\vec{w}}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\lambda_Q\\ &\lesssim\sum_{F_2\in\mathscr{F}_2}\langle f_2\rangle_{F_2}^{\sigma_2}\int_{F_2}\sum_{\substack{H\in\mathscr{H}\\ H\subset F_2}}\langle h\rangle_H^{v_{\vec{w}}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\frac{\lambda_Q\chi_Q}{v_{\vec{w}}(Q)}\,\mathrm{d}v_{\vec{w}}\\ &\lesssim\sum_{F_2\in\mathscr{F}_2}\langle f_2\rangle_{F_2}^{\sigma_2}\int_{F_2}\Big(\sup_{\substack{H'\in\mathscr{H}\\ \pi_{\mathscr{F}_2}(H')=F_2}}\langle h\rangle_{H'}^{v_{\vec{w}}}\chi_{H'}\Big)\sum_{\substack{H\in\mathscr{H}\\ H\subset F_2}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\frac{\lambda_Q\chi_Q}{v_{\vec{w}}(Q)}\,\mathrm{d}v_{\vec{w}}\\ &\lesssim\sum_{F_2\in\mathscr{F}_2}\langle f_2\rangle_{F_2}^{\sigma_2}\left\|\sum_{\substack{H\in\mathscr{H}\\ H\subset F_2}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\frac{\lambda_Q\chi_Q}{v_{\vec{w}}(Q)}\right\|_{L^p(v_{\vec{w}})}\left\|\sup_{\substack{H'\in\mathscr{H}\\ \pi_{\mathscr{F}_2}(H')=F_2}}\langle h\rangle_{H'}^{v_{\vec{w}}}\chi_{H'}\right\|_{L^{p'}(v_{\vec{w}})}\\ &\leqslant\Bigg(\sum_{F_2\in\mathscr{F}_2}\Big(\langle f_2\rangle_{F_2}^{\sigma_2}\Big)^p\left\|\sum_{\substack{H\in\mathscr{H}\\ H\subset F_2}}\sum_{\substack{Q\in\mathcal{S}'\\ \pi(Q)=(F_2,H)}}\frac{\lambda_Q\chi_Q}{v_{\vec{w}}(Q)}\right\|_{L^p(v_{\vec{w}})}^p\Bigg)^{\frac{1}{p}}\\ &\quad\times\Bigg(\sum_{F_2\in\mathscr{F}_2}\sum_{\substack{H'\in\mathscr{H}\\ \pi_{\mathscr{F}_2}(H')=F_2}}\Big(\langle h\rangle_{H'}^{v_{\vec{w}}}\Big)^{p'}v_{\vec{w}}\left(H'\right)\Bigg)^{\frac{1}{p'}}\end{aligned}$$

$$\lesssim \Big(\sum_{F_2 \in \mathscr{F}_2} \Big(\langle f_2 \rangle_{F_2}^{\sigma_2} \Big)^p \Big\| \sum_{\substack{H \in \mathscr{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q) = (F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\vec{w}}(Q)} \Big\|_{L^p(v_{\vec{w}})}^p \Big)^{\frac{1}{p}}.$$

The last inequality is due to (2.2). By (3.1), we have

$$\Big\| \sum_{\substack{H \in \mathscr{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q) = (F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\vec{w}}(Q)} \Big\|_{L^p(v_{\vec{w}})} = \Big\| \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \frac{\lambda_Q \chi_Q}{v_{\vec{w}}(Q)} \Big\|_{L^p(v_{\vec{w}})}$$

$$\lesssim [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \Big(\sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle \sigma_1 \rangle_Q^{\frac{p}{p_1}} \langle \sigma_2 \rangle_Q^{\frac{p}{p_2}} |Q| \Big)^{\frac{1}{p}}.$$

Let $\varepsilon = \frac{1}{2^{11+d}[\sigma_1]_{A_\infty}}$. Hytönen and Pérez [5] proved the reverse Hölder inequality

$$\langle \sigma_1^{1+\varepsilon} \rangle_Q \lesssim \langle \sigma_1 \rangle_Q^{1+\varepsilon}, \quad \forall Q \subset \mathbb{R}^n.$$

Let $\gamma := \frac{p}{p_1} \frac{1}{1+\varepsilon}$, $\eta := \frac{p}{p_2}$, $\frac{1}{r} := \gamma + \eta$, $\frac{1}{s} := \gamma + \frac{1}{2}(1 - \frac{1}{r})$, $\frac{1}{s'} := 1 - \frac{1}{s}$. We have

$$\begin{aligned}
I_1 &\lesssim [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \Big(\sum_{F_2 \in \mathscr{F}_2} \Big(\langle f_2 \rangle_{F_2}^{\sigma_2} \Big)^p \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle \sigma_1 \rangle_Q^{\frac{p}{p_1}} \langle \sigma_2 \rangle_Q^{\frac{p}{p_2}} |Q| \Big)^{\frac{1}{p}} \\
&\leqslant [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \Big(\sum_{F_2 \in \mathscr{F}_2} \Big(\langle f_2 \rangle_{F_2}^{\sigma_2} \Big)^p \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle {\sigma_1}^{1+\varepsilon} \rangle_Q^{\gamma} \langle \sigma_2 \rangle_Q^{\eta} |Q| \Big)^{\frac{1}{p}} \\
&\leqslant [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \Big(\sum_{F_2 \in \mathscr{F}_2} \Big(\langle f_2 \rangle_{F_2}^{\sigma_2} \Big)^p \Big(\sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle {\sigma_1}^{1+\varepsilon} \rangle_Q^{s\gamma} |Q| \Big)^{\frac{1}{s}} \Big(\sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle \sigma_2 \rangle_Q^{s'\eta} |Q| \Big)^{\frac{1}{s'}} \Big)^{\frac{1}{p}} \\
&\leqslant [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \Big(\sum_{F_2 \in \mathscr{F}_2} \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle {\sigma_1}^{1+\varepsilon} \rangle_Q^{s\gamma} |Q| \Big)^{\frac{1}{sp}} \times \Big(\sum_{F_2 \in \mathscr{F}_2} \Big(\langle f_2 \rangle_{F_2}^{\sigma_2} \Big)^{s'p} \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle \sigma_2 \rangle_Q^{s'\eta} |Q| \Big)^{\frac{1}{s'p}} \\
&:= [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} J_1 \times J_2.
\end{aligned}$$

Since $\mathcal{S}'$ is sparse, for J_1, we have

$$\begin{aligned}
J_1 &\lesssim \Big(\sum_{F_2 \in \mathscr{F}_2} \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathscr{F}_2}(Q) = F_2}} \langle {\sigma_1}^{1+\varepsilon} \rangle_Q^{s\gamma} |E_Q| \Big)^{\frac{1}{sp}} \\
&\leqslant \Big(\int_{\widetilde{Q}} (M({\sigma_1}^{1+\varepsilon} \chi_{\widetilde{Q}}))^{s\gamma} \, \mathrm{d}x \Big)^{\frac{1}{sp}} \\
&= \| M({\sigma_1}^{1+\varepsilon} \chi_{\widetilde{Q}}) \|_{L^{s\gamma}\left(\frac{\mathrm{d}x}{|\widetilde{Q}|}\right)}^{\frac{\gamma}{p}} |\widetilde{Q}|^{\frac{1}{sp}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant [\sigma_1]_{A_\infty}^{\frac{1}{sp}} \left\| M(\sigma_1{}^{1+\varepsilon}\chi_{\widetilde{Q}}) \right\|_{L^{1,\infty}\left(\frac{dx}{|\widetilde{Q}|}\right)}^{\frac{\gamma}{p}} |\widetilde{Q}|^{\frac{1}{sp}} \\
&\lesssim [\sigma_1]_{A_\infty}^{\frac{1}{sp}} \langle \sigma_1{}^{1+\varepsilon}\rangle_{\widetilde{Q}}^{\frac{\gamma}{p}} |\widetilde{Q}|^{\frac{1}{sp}} \lesssim [\sigma_1]_{A_\infty}^{\frac{1}{sp}} \langle \sigma_1\rangle_{\widetilde{Q}}^{\frac{1}{p_1}} |\widetilde{Q}|^{\frac{1}{sp}}.
\end{aligned}
$$

The third inequality in the estimate above is due to Kolmogorov's inequality, that is, for any cube Q in $\mathbb{R}^n$ and $f \in L^{1,\infty}(Q)$,

$$
\|f\|_{L^p\left(\frac{dx}{|Q|}\right)} \leqslant \left(\frac{1}{p} + \frac{1}{1-p}\right)^{\frac{1}{p}} \|f\|_{L^{1,\infty}\left(\frac{dx}{|Q|}\right)}, \quad 0 < p < 1.
$$

Specifically,

$$
\left(\frac{1}{s\gamma} + \frac{1}{1-s\gamma}\right)^{\frac{1}{sp}} = \left(\frac{1}{1 - \frac{s}{2}\frac{\varepsilon}{1+\varepsilon}\frac{p}{p_1}} + \frac{2}{s}\frac{1+\varepsilon}{\varepsilon}\frac{p_1}{p}\right)^{\frac{1}{sp}} \lesssim [\sigma_1]_{A_\infty}^{\frac{1}{sp}}.
$$

For J_2, using the same method as for J_1 and (2.2), we obtain

$$
\begin{aligned}
J_2 &\lesssim \Big(\sum_{F_2 \in \mathscr{F}_2} \left(\langle f_2\rangle_{F_2}^{\sigma_2}\right)^{s'p} [\sigma_1]_{A_\infty} \langle \sigma_2\rangle_{F_2}^{s'\eta} |F_2| \Big)^{\frac{1}{s'p}} \\
&\leqslant [\sigma_1]_{A_\infty}^{\frac{1}{s'p}} \Big(\sum_{F_2 \in \mathscr{F}_2} \left(\langle f_2\rangle_{F_2}^{\sigma_2}\right)^{p_2} \langle \sigma_2\rangle_{F_2} |F_2| \Big)^{\frac{1}{p_2}} \Big(\sum_{F_2 \in \mathscr{F}_2} |F_2| \Big)^{\frac{1}{s'p} - \frac{1}{p_2}} \\
&\lesssim [\sigma_1]_{A_\infty}^{\frac{1}{s'p}} \|f_2\|_{L^{p_2}(\sigma_2)} |\widetilde{Q}|^{\frac{1}{s'p} - \frac{1}{p_2}}.
\end{aligned}
$$

If we apply the reverse Hölder inequality for σ_2, we can obtain another bound similarly. Therefore, we get

$$
I_1 \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \min\{[\sigma_1]_{A_\infty}, [\sigma_2]_{A_\infty}\}^{1/p} \sigma_1(\widetilde{Q})^{1/p_1} \|f_2\|_{L^{p_2}(\sigma_2)}.
$$

The estimation of I_2 is similar to I_1, only by replacing formula (3.1) with (3.2). By combining the above estimates of I_1 and I_2, we conclude the proof of the theorem. □

At the end of this section, we use Python to draw a graph to compare the weak type estimate we obtained with the sharp strong type estimate (1.1). In particular, we show that when $p \geqslant \frac{3+\sqrt{5}}{2}$ or $\min\{p_1, p_2\} > 4$, the exponent we obtained is smaller than 1.

Without loss of generality, we assume $p_1 \leqslant p_2$ in the following caculations.

- When $p \geqslant \frac{3+\sqrt{5}}{2}$, it is obvious that $p_1' \leqslant p$. In this case, the exponent in Theorem 1.1 is $\frac{1}{p} + \frac{1}{p_2'}\frac{p_1'}{p}$. If it is greater than or equal to 1, we obtain

$$
\frac{1}{p} + \frac{1}{p_2'}\frac{p_1'}{p} \geqslant 1 \Rightarrow \frac{p_1'}{p_2'} \geqslant p - 1 \Rightarrow p_1' \geqslant p - 1 \Rightarrow \frac{1}{p_1} \geqslant \frac{p-2}{p-1}.
$$

  Since $p \geqslant \frac{3+\sqrt{5}}{2}$, we have $\frac{p-2}{p-1} \geqslant \frac{1}{p}$, which leads to a contradiction.
- When $\min\{p_1, p_2\} > 4$, we can also obtain $p_1' \leqslant p$, thus,

$$
\frac{1}{p} + \frac{1}{p_2'}\frac{p_1'}{p} = \frac{p_1'}{p}\Big(2 - \frac{1}{p}\Big) < 1 \Leftrightarrow \frac{2}{p} - \frac{1}{p^2} < \frac{1}{p_1'},
$$

  and this holds automatically since the left-hand side is always less than $\frac{3}{4}$, while the right-hand side is greater than it.

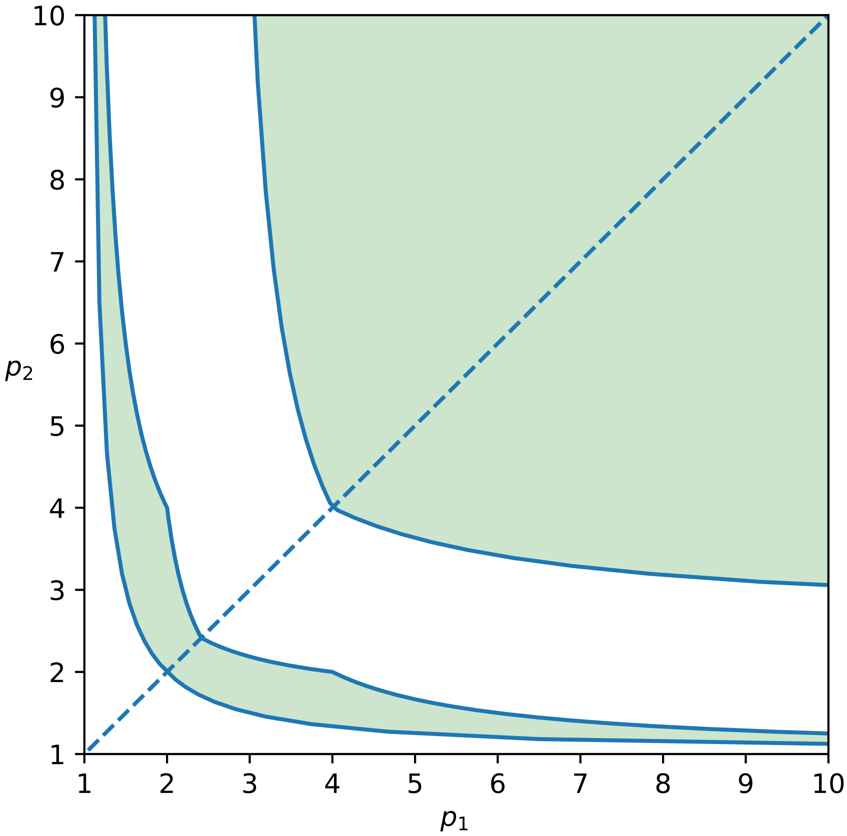

FIGURE 1. Compared to the sharp strong type estimate, our results are better in shaded areas.

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