BOUNDED DIFFERENTIALS ON THE UNIT DISK AND THE ASSOCIATED GEOMETRY

SONG DAI[1] AND QIONGLING LI[2]

Abstract. For a harmonic diffeomorphism between the Poincaré disks, Wan in [Wan92] showed the equivalence between the boundedness of the Hopf differential and the quasi-conformality. In this paper, we will generalize this result from quadratic differentials to r-differentials. We study the relationship between bounded holomorphic r-differentials/$(r-1)$-differential and the induced curvature of the associated harmonic maps from the unit disk to the symmetric space $SL(r,\mathbb{R})/SO(r)$ arising from cyclic/subcyclic Higgs bundles. Also, we show the equivalence between the boundedness of holomorphic differentials and having a negative upper bound of the induced curvature on hyperbolic affine spheres in $\mathbb{R}^3$, maximal surfaces in $\mathbb{H}^{2,n}$ and J-holomorphic curves in $\mathbb{H}^{4,2}$. Benoist-Hulin and Labourie-Toulisse have previously obtained some of these equivalences using different methods.

Contents

1. Introduction

Consider a harmonic diffeomorphism between the unit disk $\mathbb{D}$ equipped with the Poincaré hyperbolic metric $g_{\mathbb{D}}$. By the work of Wan [Wan92], the following are equivalent: (i) the harmonic map is quasi-conformal; (ii) the energy density is bounded; (iii) its Hopf differential is bounded with respect to $g_{\mathbb{D}}$.

The results of Wan on holomorphic quadratic differentials have been generalized to holomorphic cubic differentials. Benoist and Hulin [BH14] established that the following conditions are equivalent for a hyperbolic affine sphere in $\mathbb{R}^3$, whose Blaschke metric has conformal type as $\mathbb{D}$: (i) its Blaschke metric has curvature bounded above by a negative constant; (ii) the Blaschke metric is conformally bounded with respect to $g_{\mathbb{D}}$; (iii) its Pick differential is bounded with respect to $g_{\mathbb{D}}$.

The associated equations in these two settings belong to a type of single vortex equation and can be dealt with simultaneously. The single vortex equation can be fit into a broader framework known as the nonabelian Hodge correspondence for Higgs bundles.

Recall that a Higgs bundle on a Riemann surface Σ consists of a pair: a holomorphic vector bundle E over Σ and a Higgs field, represented as an $\mathrm{End}(E)$-valued holomorphic 1-form. A Hermitian metric h on E is called harmonic if it satisfies the Hitchin equation. A harmonic metric h gives rise to an equivariant harmonic map from the universal cover of Σ to the symmetric space $GL(r,\mathbb{C})/U(r)$.

Similar to the case of a compact surface, one can define the Hitchin section consisting of Higgs bundles over $\mathbb{D}$, denoted as $s(q_2,\cdots,q_n)$, parametrized by a tuple of differentials $q_2,\cdots,q_r$. The image of $s(0,\cdots,0,q_r)/s(0,\cdots,0,q_{r-1},0)$ is referred to as cyclic/subcyclic Higgs bundles in the Hitchin section. In particular, the two cases discussed at the beginning belong to the category of cyclic Higgs bundles in the Hitchin section for $r=2,3$. For the (sub-)cyclic Higgs bundles, the Hitchin equation coincides with the (variant) Toda system.

The second author and Mochizuki in [LM20a] showed that for a Higgs bundle over $\mathbb{D}$, the spectral curve is bounded, i.e., $|\mathrm{tr}(\phi^i)|_{g_{\mathbb{D}}}(i=1,\cdots,r)$ are bounded, if and only if the norm of Higgs field $|\phi|_{h,g_{\mathbb{D}}}$ is bounded. This result can be viewed as a generalization of the equivalence between (ii) and (iii) for rank two and three cyclic Higgs bundles.

Similar to the case for quadratic differentials and cubic differentials, we aim to investigate whether the following holds: for a Higgs bundle in the Hitchin section, by choosing an appropriate harmonic metric, it has a bounded spectral curve if and only if the induced curvature of the harmonic map is bounded above by a negative constant. It is worth noting that this question restricts to the negative curvature conjecture for compact hyperbolic surface case in [DL19]. We will address this question for cyclic and subcyclic Higgs bundles in the Hitchin section.

For $r\geq 2$, given a holomorphic r-differential q on $\mathbb{D}$, it has been shown by the second author and Mochizuki [LM20a] that there exists a unique strongly complete solution (see Definition 3.7) to the Hitchin equation on the cyclic rank r Higgs bundle $s(0,\cdots,0,q)$. One may also define a similar notion of strongly complete solution to the Hitchin equation for subcyclic rank $(r-1)$ Higgs bundles. Consequently, one can associate a holomorphic r-differential or a holomorphic $(r-1)$-differential with a harmonic map f from $\mathbb{D}$ to the symmetric space $N:=SL(r,\mathbb{R})/SO(r)$ arising from the strongly complete solution.

We would like to investigate the relationship between bounded r-differentials or $(r-1)$-differentials with the geometry of harmonic maps. Our main result is the following theorem.

Theorem 1.1 (part of Theorem 4.8). *Suppose $f:\mathbb{D}\to\Sigma\subset N=SL(r,\mathbb{R})/SO(r)$ is a minimal immersion induced by a holomorphic r-differential q for $r\geq 3$ or a holomorphic $(r-1)$-differential q for $r\geq 4$ arising from the strongly complete solution. Then the following are equivalent:*

(1) *q is bounded with respect to $g_{\mathbb{D}}$.*
(2) *The induced metric of Σ is mutually bounded with $g_{\mathbb{D}}$.*
(3) *There exists a constant $\delta>0$ such that $K^N_\sigma\leq-\delta$ for every tangent plane σ of Σ.*
(4) *The induced curvature on Σ is bounded from above by a negative constant.*

Remark 1.2. (1) *The equivalence between (1) and (2) follows from the equivalence between the bounded spectral curve and the bounded norm of the Higgs field in Li-Mochizuki* [LM20a].
(2) *According to the definition of strongly completeness for both cases, the sectional curvature K^N_σ of the tangent plane σ of f satisfies $K^N_\sigma<0$ and the induced curvature is negative.*

We also have a version of Theorem 1.1 dealing with the cyclic case in the case of $r = 2, 3$, as stated in Proposition 4.2. For the case of $r = 2$, Proposition 4.2 corresponds to Wan's result on harmonic diffeomorphisms between $\mathbb{D}$. In fact, the main technique in our proof of Proposition 4.2 is adapted from Wan's proof. Wan's result has also been generalized to harmonic diffeomorphisms between pinched Hadamard spaces in [LTW95]. Building upon Wan's proof, we introduce the key Lemmas 2.2 and 2.5, which will be frequently used in this paper. As a direct corollary of Proposition 4.2 for the case $r = 3$, we provide an alternative proof of Benoist-Hulin's theorem [BH14] on hyperbolic affine spheres in $\mathbb{R}^3$. This theorem shows that the Blaschke metric has curvature bounded above by a negative constant if and only if its Pick differential is bounded with respect to the hyperbolic metric. Detailed discussions can be found in Section 5 and 6.

Holomorphic quartic differentials and sextic differentials naturally appear in the structure data of immersed surfaces in pseudo-hyperbolic spaces. We establish results analogous to those for quadratic and cubic differentials.

For a space-like maximal surface in $\mathbb{H}^{2,n}$, one can associate with a holomorphic quartic differential. According to [CTT19], such a maximal surface corresponds to a conformal $SO_0(2, n+1)$-Higgs bundle over a domain in $\mathbb{C}$ along with a real harmonic metric that is compatible with the group structure. In Section 7, we prove the following theorem, previously shown in Labourie-Toulisse [LT23], with the exception of part (i).

Theorem 1.3. *(Theorem 7.9) For a complete maximal surface X in $\mathbb{H}^{2,n}$, its induced curvature is either negative or constantly zero. In the latter case, X is conformal to the complex plane.*

Furthermore, assuming that X is conformal to $\mathbb{D}$, the following conditions are equivalent:

1. *The quartic differential is bounded with respect to the hyperbolic metric.*
2. *The induced metric is conformally bounded with respect to the hyperbolic metric.*
3. *The induced metric has curvature bounded above by a negative constant.*

Remark 1.4. *Our technique is different from the one in* [LT23]. *In fact, the method used in* [LT23] *shares a similar spirit with the one in* [BH14] *for affine spheres and cubic differentials.*

For space-like J-holomorphic curves in $\mathbb{H}^{4,2}$ with nonvanishing second fundamental form and a timelike osculation line, we can associate a holomorphic sextic differential q_6. According to [Nie22], such J-holomorphic curves correspond to subcyclic Higgs bundle with certain harmonic metric. In Section 8, we prove the following theorem.

Theorem 1.5. *(Theorem 8.4) Let X be a complete space-like J-holomorphic curve in $\mathbb{H}^{4,2}$ such that its second fundamental form never vanishes and has timelike osculation line. Then, its induced curvature is either negative or constantly zero. In the latter case, X is conformal to the complex plane.*

Furthermore, assume X is conformal to $\mathbb{D}$, the following are equivalent:

1. *The sextic differential is bounded with respect to the hyperbolic metric.*
2. *The induced metric is conformally bounded with respect to the hyperbolic metric.*
3. *The induced metric has curvature bounded above by a negative constant.*

Regarding Toda equations and related geometry on complex plane or more general Riemann surfaces, there is an extensive body of literature on solutions and corresponding geometry. Regarding harmonic maps between surfaces, see, for example, [SY78, Wol89, WA94, Han96, HTTW95, Gup21]. For hyperbolic affine spheres in $\mathbb{R}^3$, see, for example, [Cal72, Lof01, Lab07, BH13, BH14, DW15, Nie23]. For maximal surfaces in $\mathbb{H}^{2,n}$, see [CTT19, TW20, LTW20]. For J-holomorphic curves in $\mathbb{H}^{4,2}$ (or equivalently $S^{2,4}$), see [Bar10, Eva22]. For cyclic Higgs bundles of general rank, see, for example,[Bar15, GL14, GIL15, Moc, Moc14, DL19, DL20, LM20b].

Recall that the universal Teichmüller space $\mathcal{T}(\mathbb{D})$ is the space of quasisymmetric homeomorphisms of S^1 fixing three points. Combining Wan's result with the work of [LT23, Mar17], there is a bijection between the space of bounded quadratic differentials with the universal Teichmüller space. Recent work by Labourie and Toulisse [LT23] constructs an analogue of the universal Teichmüller space as a subspace of the space of maximal surfaces in $\mathbb{H}^{2,n}$ that relates to the space of bounded quartic differentials. Consider a 2-dimensional real vector space denoted as V. Denote by $\mathcal{QS}_n$ the space of quasisymmetric maps from $\mathbb{P}(V)$ to $\partial_\infty \mathbb{H}^{2,n}$ equipped with the C^0 topology, up to the action of $SO(2, n+1)$. The space $\mathcal{QS}_n$ can be viewed as a higher rank analogue of the universal Teichmüller space $\mathcal{T}(\mathbb{D})$. Denote by $H^0_b(\mathbb{D}, K^4)$ the space of bounded quartic differentials on $\mathbb{D}$. They construct a natural map $\mathcal{H}$ from $\mathcal{QS}_n$ to the product space $\mathcal{T}(\mathbb{D}) \times H^0_b(\mathbb{D}, K^4)$. At the end of Section 7, we provide a concise proof of the properness of this map $\mathcal{H}$, as posed by Labourie and Toulisse.

Organization of this paper. Section 2 gathers some useful tools, particularly Lemma 2.2, which is the key tool. We then present preliminaries on Higgs bundles and the Toda equation in Section 3. In Section 4, we prove the main Theorem 4.8. In Section 5, we study the application of Proposition 4.2 to harmonic maps between surfaces and bounded quadratic differentials. Section 6 explores the application of Proposition 4.2 to hyperbolic affine spheres and bounded cubic differentials. Section 7 is dedicated to the study of maximal surfaces in $\mathbb{H}^{2,n}$ and bounded quartic differentials. We also discuss an analogous universal Teichmüller space in Section 7. In Section 8, we explore J-holomorphic curves in $\mathbb{H}^{4,2}$ and bounded sextic differentials.

Acknowledgements. We want to thank Francois Labourie and Jeremy Toulisse for sending us their preprint and helpful discussion. The second author thanks Xin Nie for helpful explanations on the associated G_2-geometry. Li is supported by the National Key R&D Program of China No. 2022YFA1006600, the Fundamental Research Funds for the Central Universities and the Nankai Zhide foundation. Dai is supported by NSF of China (No.11871283, No.11971244 and No.12071338).

2. Main tools

In this section, we introduce several useful lemmas. We begin by recalling the following mean-value inequality, which is used to establish the key Lemma 2.2.

Proposition 2.1. *(Mean-value inequality,* [CT90, Lemma 2.5]*) Let (M, g) be a complete Riemannian manifold of dimension n. Consider constants $c \geq 0$, $R_0 > 0$, and $x_0 \in M$. Assume that (i) the Poincaré and Sobolev inequalities hold with constant c_p and c_s for functions supported in $B_g(x_0, R_0)$; and (ii) $Vol(B_g(x_0, r)) \leq c_2 r^n$, for all $r \leq R_0$, where c_2 is a positive constant. Then there exist constants $p_0 = p_0(n, c, R_0, c_p, c_s, c_2) > 0$ and $C = C(n, c, R_0, c_p, c_s, c_2) > 0$ such that, for any nonnegative $W^{1,2}$ supersolution u satisfying $\triangle_g u \leq cu$ in $B_g(x_0, R_0)$ and $p \in (0, p_0)$, the following inequality holds:*

$$\inf_{x \in B_g(x_0, R_0/4)} u(x) \geq C\Big(\int_{B_g(x_0, R_0/2)} u^p dV_g\Big)^{\frac{1}{p}}.$$

In particular, there exist constants $C > 0$ and $0 < p < 1$ such that

$$u(x_0) \geq C\Big(\int_{B_g(x_0, R_0/2)} u^p dV_g\Big)^{\frac{1}{p}}.$$

The following is the key lemma we will use in this paper. It follows from the proof in Wan [Wan92, Theorem 13], where he showed that bounded quadratic differentials correspond to quasi-conformal maps between the unit disk.

Lemma 2.2. *Let g be a Riemannian metric on $\mathbb{D}$ satisfying (i) g is equivalent to the hyperbolic metric $g_{\mathbb{D}}$, i.e. $C_1^{-1}g \le g_{\mathbb{D}} \le C_1 g$ for some constant $C_1 \ge 1$; (ii) the Gaussian curvature K_g of g satisfies $-H \le K_g \le 0$ for some positive constant H. Let u be a smooth function on $\mathbb{D}$ satisfying $-C_2^{-1}K_g \le u \le -C_2 K_g$ for some constant $C_2 \ge 1$.*

Assume that u satisfies

$$\triangle_g u \le cu$$

for some positive constant c, then there exist a positive constant δ, depending on C_1, H, C_2, and c, such that $u \ge \delta$ and $K_g \le -C_2^{-1}\delta$.

Proof. Since g is equivalent to $g_{\mathbb{D}}$, noting that $g_{\mathbb{D}}$ has exponential volume growth and is homogeneous, there exist constants $\epsilon, \eta, M > 0$, depending only on C_1, such that $\mathrm{Vol}(B_g(x,r)) \ge \epsilon e^{\eta r} - M$ for every ball $B_g(x,r)$ with respect to the metric g. Therefore, there exists a positive constant R, depending on C_1, such that for every $x \in \mathbb{D}$, there exists $r \in (0,R)$ satisfying $\frac{d^2}{dr^2}\mathrm{Vol}(B_g(x,r)) \ge 2\pi + 1$. Since $K_g \le 0$, and $\mathbb{D}$ is simply connected, the exponential map is a diffeomorphism. Then,

$$\frac{d^2}{dr^2}\mathrm{Vol}(B_g(x,r)) = \frac{d}{dr}\mathrm{L}(\partial B_g(x,r)) = \int_{\partial B_g(x,r)} k_g ds_g,$$

where L is the length functional and k_g is the geodesic curvature of $\partial B_g(x,r)$ with respect to the metric g. By the Gauss-Bonnet-Chern formula, we have

$$\frac{d^2}{dr^2}\mathrm{Vol}(B_g(x,r)) = 2\pi - \int_{B_g(x,r)} K_g dV_g.$$

Thus, for every $x \in \mathbb{D}$, there is a constant $r \in (0,R)$ such that $\int_{B_g(x,r)} -K_g dV_g \ge 1$. Since $u \ge -C_2^{-1}K_g$, we have

$$\int_{B_g(x,r)} u dV_g \ge C_2^{-1}.$$

Now we apply Proposition 2.1 to obtain the L^p estimate of u. In fact, let $R_0 = 2R$, where R is defined above depending on C_1. Since g is equivalent to $g_{\mathbb{D}}$, there is a positive constant c_2, depending on C_1, such that $\mathrm{Vol}(B_g(x,r)) \le c_2 r^n$, $\forall r \le R_0, \forall x \in \mathbb{D}$. Since $K_g \ge -H$, from [Li12, Theorem 5.9], the Poincaré inequality holds for a constant c_p depending on H. To see the Sobolev constant, since it is equivalent to the isoperimetric constant and g is equivalent to $g_{\mathbb{D}}$, we see that the Sobolev inequality holds for a constant c_s depending on C_1. It follows from Proposition 2.1 that there exist constants $C > 0$, $0 < p < 1$ depending on C_1, H, C_2, c such that for every $x \in \mathbb{D}$,

$$u(x) \ge C\left(\int_{B_g(x,R)} u^p dV_g\right)^{\frac{1}{p}}.$$

From the assumption, $0 \le u \le C_2 H$ and thus $u^p \ge (C_2H)^{p-1}u$ for $0 < p < 1$. Together with the L^1 estimate we obtained above, there exist a positive constant δ, depending only on C_1, H, C_2, and c, such that

$$u(x) \ge C\left(\int_{B_g(x,R)} u^p dV_g\right)^{\frac{1}{p}} \ge C\cdot(C_2H)^{\frac{p-1}{p}}\left(\int_{B_g(x,R)} u dV_g\right)^{\frac{1}{p}} \ge C\cdot(C_2H)^{\frac{p-1}{p}}\cdot C_2^{-\frac{1}{p}} =: \delta.$$

Since $u \le -C_2 K_g$, then $K_g \le -C_2^{-1}\delta$. This concludes the proof. □

The next lemma is the Cheng-Yau maximum principle, which allows us to deal with complete Riemannian manifolds whose curvature is bounded from below.

Lemma 2.3 (Cheng-Yau Maximum Principle [CY75])**.** *Suppose (M,g) is a complete manifold with Ricci curvature bounded from below. Let u be a C^2-function defined on M such that $\triangle_g u \ge f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ is a function. Suppose there is a continuous positive function $h(t) : [a,\infty) \to \mathbb{R}_+$ such that*

(i) h is non-decreasing;

(ii) $\liminf_{t\to+\infty} \frac{f(t)}{h(t)} > 0$;
(iii) $\int_b^\infty (\int_a^t h(\tau)d\tau)^{-\frac{1}{2}} dt < \infty$, *for some* $b \geq a$.
Then the function u *is bounded from above. Furthermore, if* f *is lower semi-continuous,* $f(\sup u) \leq 0$.

In particular, for $\alpha > 1$ and a positive constant c_0, one can check if $f(t) \geq c_0 t^\alpha$ for t large enough, $h(t) = t^\alpha$ satisfy the above three conditions (i)(ii)(iii).

Lemma 2.4. *Let* g_0 *and* g *be two conformal Riemannian metrics on a Riemann surface* Σ. *Assume that* g_0 *is complete and there exist two positive constants* a *and* b *such that* $K_g \leq -b$ *and* $K_{g_0} \geq -a$. *Then,* $g \leq \frac{a}{b} g_0$.

Proof. Regard g_0 as a Hermitian metric, locally, denote $g_0 = \tilde{g}_0 dz \otimes d\bar{z}$ and $g = g_0 \cdot e^u$,

$$-b \geq K_g = -\frac{2}{e^u \tilde{g}_0} \partial_z \partial_{\bar{z}} \log(e^u \tilde{g}_0).$$

Then,

$$-\frac{2}{e^u} \triangle_{g_0} u + \frac{1}{e^u} K_{g_0} \leq -b.$$

So

$$\triangle_{g_0} u \geq \frac{b}{2} e^u + \frac{K_{g_0}}{2} \geq \frac{b}{2} e^u - \frac{a}{2}.$$

Then, by the Cheng-Yau maximum principle, $e^u \leq \frac{a}{b}$. □

Combining Lemmas 2.2, 2.3, and 2.4, we show the following lemma, which will be used frequently throughout this paper.

Lemma 2.5. *Suppose* (Σ, g) *is a complete surface with curvature bounded from below. Suppose there exist positive constants* c *and* d *such that:*

$$\triangle_g K_g \geq c K_g (K_g + d). \tag{1}$$

Then $K_g < 0$ *or* $K_g \equiv 0$, *in which case* Σ *is parabolic.*

When Σ *is hyperbolic, let* g_{hyp} *be the unique complete conformal hyperbolic metric. If* $g \leq C g_{hyp}$ *for some positive constant* C, *then there exists a positive constant* δ *such that* $K_g \leq -\delta$.

Proof. Since g is complete and K_g has a lower bound, the background metric is enough to apply the Cheng-Yau maximum principle. Since the right hand side of the equation has quadratic growth, from the Cheng-Yau maximum principle (Lemma 2.3), K_g has an upper bound, and $c \sup K_g (\sup K_g + d) \leq 0$. Therefore, $\sup K_g \leq 0$. By the strong maximum principle, either $K_g < 0$ or $K_g \equiv 0$.

From Lemma 2.4, we have $g \geq C_1 g_{hyp}$ for some positive constant C_1. Together with the assumption $g \leq C g_{hyp}$, g is equivalent to g_{hyp}. Now, lift (Σ, g) to the cover $(\widetilde{\Sigma} \cong \mathbb{D}, \widetilde{g})$, $\widetilde{g}$ is equivalent to $g_{\mathbb{D}}$. we can apply Lemma 2.2 to

$$\triangle_{\widetilde{g}} (-K_{\widetilde{g}}) \leq -c K_{\widetilde{g}} (K_{\widetilde{g}} + d) \leq -cd K_{\widetilde{g}}.$$

As a result, we conclude that $K_{\widetilde{g}} \leq -\delta$ for some positive constant δ. The same conclusion holds for K_g. □

3. Preliminaries on Higgs bundles

In this section, we will review some facts on Higgs bundles used in this article. For more detailed information, readers may refer to [Li19].

3.1. **Higgs bundles over Riemann surfaces.** Let Σ be a Riemann surface and K denote the canonical line bundle of Σ.

Definition 3.1. *A Higgs bundle over a Riemann surface Σ is a pair (E, ϕ) where E is a holomorphic bundle over Σ of rank r, and $\phi : E \to E \otimes K$ is a holomorphic bundle map. Additionally, an $SL(r,\mathbb{C})$-Higgs is a Higgs bundle with $\det E = \mathcal{O}$ and $\operatorname{tr}\phi = 0$.*

Now, let h be a Hermitian metric on the bundle E. We will also use h to denote the induced Hermitian metric on $\operatorname{End}(E)$.

Definition 3.2. *A Hermitian metric h on a Higgs bundle is called harmonic if it satisfies the Hitchin equation*

$$F(h) + [\phi, \phi^{*_h}] = 0,$$

where $F(h)$ is the curvature of the Chern connection ∇ with respect to h, ϕ^{*_h} is the adjoint of ϕ with respect to the metric h, and the bracket $[,]$ is the Lie bracket on $\operatorname{End}(E)$-valued 1-forms.

Locally, $\phi = f dz$ for a local holomorphic section f of $\operatorname{End}(E)$. We have the following inequality as a slight modification of [Moc16].

Lemma 3.3. *For a holomorphic section s of $\operatorname{End}(E)$, locally, we have*

$$\partial_z\partial_{\bar z}\log|s|_h^2 \geq \frac{|[s, f^{*_h}]|_h^2 - |[s,f]|_h^2}{|s|_h^2}.$$

Proof. We start by denoting $\partial_{z,h}$ as the $(1,0)$ part of the Chern connection on $\operatorname{End}(E)$ with respect to h. Since s is holomorphic, we can write: $\partial_z\partial_{\bar z}|s|_h^2 = |\partial_{z,h}s|^2 + h(s, \partial_{\bar z}\partial_{z,h}s)$. This allows us to derive:

$$\partial_z\partial_{\bar z}\log|s|_h^2 = \frac{\partial_z\partial_{\bar z}|s|_h^2}{|s|_h^2} - \frac{\partial_z|s|_h^2}{|s|_h^2}\frac{\partial_{\bar z}|s|_h^2}{|s|_h^2} = \frac{h(s, \partial_{\bar z}\partial_{z,h}s)}{|s|_h^2} + \frac{|\partial_{z,h}s|^2}{|s|_h^2} - \frac{\partial_z|s|_h^2}{|s|_h^2}\frac{\partial_{\bar z}|s|_h^2}{|s|_h^2}.$$

We also have:

$$\partial_z|s|_h^2 \cdot \partial_{\bar z}|s|_h^2 = |\partial_{\bar z}|s|_h^2|^2 = |h(\partial_{z,h}s, s)|^2 \leq |\partial_{z,h}s|_h^2 \cdot |s|_h^2.$$

Therefore,

$$\partial_z\partial_{\bar z}\log|s|_h^2 \geq \frac{h(s, \partial_{\bar z}\partial_{z,h}s)}{|s|_h^2} = \frac{h(s, (\partial_{\bar z}\partial_{z,h} - \partial_{z,h}\partial_{\bar z})s)}{|s|_h^2}.$$

By the Hitchin equation, locally we have $F(h) + [\phi, \phi^{*_h}] = F(h) + [f, f^{*_h}]dz \wedge d\bar z = 0$. So, we can write

$$(\partial_{\bar z}\partial_{z,h} - \partial_{z,h}\partial_{\bar z})s = [[f, f^{*_h}], s].$$

This leads to

$$\partial_z\partial_{\bar z}\log|s|_h^2 \geq \frac{h(s, [[f, f^{*_h}], s])}{|s|_h^2} = \frac{h(s, [f, [f^{*_h}, s]])}{|s|_h^2} - \frac{h(s, [f^{*_h}, [f, s]])}{|s|_h^2}.$$

Using the formula $h(u, [v, w]) = h([v^{*_h}, u], w)$, we obtain $\partial_z\partial_{\bar z}\log|s|_h^2 \geq \frac{|[s,f^{*_h}]|_h^2 - |[s,f]|_h^2}{|s|_h^2}$. □

The Hitchin equation, together with the holomorphicity of ϕ, gives a ρ-equivariant harmonic map. This map, denoted as $f : \widetilde{\Sigma} \to N := SL(r,\mathbb{C})/SU(r)$, is defined on the universal cover of Σ, and ρ represents the holonomy representation of the flat connection $D = \nabla + \phi + \phi^{*_h}$.

We consider a Kähler metric $g = \tilde g dz \otimes d\bar z$ on Σ. The Kähler form ω is given by $\omega = \frac{\sqrt{-1}}{2}\tilde g dz \wedge d\bar z$ with respect to $\operatorname{Re}(g) = \frac{1}{2}(g + \bar g)$. (Notice that in some literature the Kähler form ω is to $g + \bar g$.) We also regard the Kähler metric g as a Hermitian metric on K^{-1}. Respectively, the Laplacian with respect to g, the Gaussian curvature of $\mathrm{Re}(g)$, and the square norm of q with respect to g are locally given by:

$$\triangle_g = \frac{1}{\tilde{g}}\partial_z\partial_{\bar{z}}, \quad K_g = -\frac{2}{\tilde{g}}\partial_z\partial_{\bar{z}}\log\tilde{g}.$$

Let q be a holomorphic k-differential q. Locally, it is of the form $q(z)dz^k$. Then q defines a singular flat metric, denoted as $|q|^{\frac{2}{k}}$, locally of the form $|q(z)|^{\frac{2}{k}}|dz|^2$. The square norm of q with respect to g is denoted by $|q|_g^2$, locally given by $\frac{q(z)\bar{q}(z)}{\tilde{g}^r}$.

The Hopf differential of f is $\mathrm{Hopf}(f) = 2r\mathrm{tr}(\phi^2)$ and the energy density of f is $e(f) = 2r\mathrm{tr}(\phi\phi^*)/g$.

3.2. **Toda system.** The cyclic Higgs bundles in the Hitchin component have the following form:

$$E = K^{\frac{r-1}{2}} \oplus K^{\frac{r-3}{2}} \oplus \cdots \oplus K^{\frac{3-r}{2}} \oplus K^{\frac{1-r}{2}}, \quad \phi = \begin{pmatrix} & & & q \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

where q is a holomorphic r-differential on Σ.

The Hermitian metric g induces the Hermitian metrics $(g^{-1})^{\otimes(r+1-2i)/2}$ on $K^{(r+1-2i)/2}$, and a diagonal Hermitian metric $\bigoplus_{i=1}^{r}(g^{-1})^{\otimes(r+1-2i)/2}$ on E. For any $\mathbb{R}$-valued function $\boldsymbol{w} = (w_1, \ldots, w_r)$, we define a Hermitian metric on E,

$$h(g, \boldsymbol{w}) := \bigoplus_{i=1}^{r} e^{w_i}(g^{-1})^{\otimes(r+1-2i)/2}.$$

Remark 3.4. *For a smooth function f on Σ, consider another Kähler metric $g' = e^f g$. Let $\boldsymbol{w}' = (w_1 + \frac{r-1}{2}f, \ldots, w_r + \frac{1-r}{2}f)$. Then $h(g, \boldsymbol{w}) = h(g', \boldsymbol{w}')$.*

Through direct calculation, it can be shown that the Hermitian metric $h(g, \boldsymbol{w})$ is harmonic if and only if it satisfies the Toda system. One may refer to [LM20a].

Proposition 3.5. *The real Hermitian metric $h(g, \boldsymbol{w})$ is harmonic if and only if $\boldsymbol{w}$ satisfies the following Toda system:*

$$\begin{aligned} \triangle_g w_1 &= |q|_g^2 e^{w_1 - w_r} - e^{w_2 - w_1} - \frac{r-1}{4}K_g, \\ \triangle_g w_i &= e^{w_i - w_{i-1}} - e^{w_{i+1} - w_i} - \frac{r+1-2i}{4}K_g, \quad i = 2, \ldots, r-1 \\ \triangle_g w_r &= e^{w_r - w_{r-1}} - |q|_g^2 e^{w_1 - w_r} - \frac{1-r}{4}K_g. \end{aligned} \tag{2}$$

and $\sum_{i=1}^{r} w_i = 0$. In particular, in the case $r = 2$, the Hermitian metric $h(g, \boldsymbol{w})$ is harmonic if and only if $\boldsymbol{w} = (w_1, -w_1)$ satisfies

$$\triangle_g w_1 = |q|_g^2 e^{2w_1} - e^{-2w_1} - \frac{1}{4}K_g. \tag{3}$$

Definition 3.6. *A solution $(w_1, \cdots, w_r)$ to the Toda system (2) is said to be real if $w_i + w_{r+1-i} = 0$ for $i = 1, \cdots, r$.*

Denote $n = [\frac{r}{2}]$. A real solution to the Toda system satisfies

$$
\begin{aligned}
\triangle_g w_1 &= e^{2w_1}|q|_g^2 - e^{-w_1+w_2} - \frac{r-1}{4}K_g,\\
\triangle_g w_i &= e^{-w_{i-1}+w_i} - e^{-w_i+w_{i+1}} - \frac{r+1-2i}{4}K_g, \quad 2 \le i \le n-1,\\
\triangle_g w_n &= e^{-w_{n-1}+w_n} - e^{-(2n+2-r)w_n} - \frac{r+1-2n}{4}K_g.
\end{aligned} \tag{4}
$$

We also consider the subcyclic Higgs bundles in the Hitchin component, which have the form:

$$
E = K^{\frac{r-1}{2}} \oplus K^{\frac{r-3}{2}} \oplus \cdots \oplus K^{\frac{3-r}{2}} \oplus K^{\frac{1-r}{2}}, \quad \phi = \begin{pmatrix} & & & q & 0\\ 1 & & & & q\\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix},
$$

where q is a holomorphic $(r-1)$-differential on Σ. Similar to the cyclic case, the diagonal harmonic metric $h(g,\boldsymbol{w})$ gives a variant Toda system as follows:

$$
\begin{aligned}
\triangle_g w_1 &= e^{w_1+w_2}|q|_g^2 - e^{-w_1+w_2} + \frac{r-1}{4}K_g,\\
\triangle_g w_2 &= e^{w_1+w_2}|q|_g^2 + e^{-w_1+w_2} - e^{-w_2+w_3} + \frac{r-3}{4}K_g,\\
\triangle_g w_i &= e^{-w_{i-1}+w_i} - e^{-w_i+w_{i+1}} + \frac{r+1-2i}{4}K_g, \quad 3 \le i \le r-2,\\
\triangle_g w_{r-1} &= e^{-w_{r-2}+w_{r-1}} - e^{w_1+w_2}|q|_g^2 - e^{-w_{r-1}+w_r} + \frac{3-r}{4}K_g,\\
\triangle_g w_r &= e^{-w_{r-1}+w_r} - e^{w_1+w_2}|q|_g^2 + \frac{1-r}{4}K_g.
\end{aligned} \tag{5}
$$

If the solution is real, i.e., $w_i + w_{r+1-i} = 0$ for $i = 1, \cdots, r$, then the variant Toda system becomes

$$
\begin{aligned}
\triangle_g w_1 &= e^{w_1+w_2}|q|_g^2 - e^{-w_1+w_2} - \frac{r-1}{4}K_g,\\
\triangle_g w_2 &= e^{w_1+w_2}|q|_g^2 + e^{-w_1+w_2} - e^{-w_2+w_3} - \frac{r-3}{4}K_g,\\
\triangle_g w_i &= e^{-w_{i-1}+w_i} - e^{-w_i+w_{i+1}} - \frac{r+1-2i}{4}K_g, \quad 3 \le i \le n-1,\\
\triangle_g w_n &= e^{-w_{n-1}+w_n} - e^{-(2n+2-r)w_n} - \frac{r+1-2n}{4}K_g.
\end{aligned} \tag{6}
$$

For both the systems (2) and (5), denote

$$
g(h)_i = e^{-w_i+w_{i+1}}g, \quad i = 1, \cdots, r-1.
$$

We consider the strongly complete solution to the systems (2) or (5) in the following sense.

Definition 3.7.

- *A solution $(w_1, \cdots, w_r)$ to the system (2) is said to be strongly complete if the following holds:*
  - *The metrics $g(h)_i$, $i = 1, \cdots, r-1$, are complete and mutually bounded.*
  - *It is real.*
  - *Set $w_0 = -w_1 - \log|q|_g^2$, $w_{n+1} = -(2n+1-r)w_n$. Then*

$$
\frac{e^{-w_{i-1}+w_i}}{e^{-w_i+w_{i+1}}} \le 1, \ \ i = 1, \cdots, n.
$$

- *A solution $(w_1, \cdots, w_r)$ to the system (5) is said to be strongly complete if the following holds:*

– *The metrics* $g(h)_i$*,* $i=1,\cdots,r-1$*, are complete and mutually bounded.*
– *It is real.*
– *Set* $w_0=-w_2-\log|q|_g^2$*,* $w_{n+1}=-(2n+1-r)w_n$*. Then*

$$\frac{e^{-w_{i-1}+w_i}}{e^{-w_i+w_{i+1}}}\le 1,\ i=1,\cdots,n,\ \text{and } e^{-w_0+w_1}+e^{-w_1+w_2}\le e^{-w_2+w_3}.$$

Remark 3.8. *Let* Σ *be a Riemann surface with a Kähler metric* g*.*

(1) *By* [LM20a]*, for any holomorphic* r*-differential* q *on* Σ*, there exists a unique strongly complete solution to the system (2). In fact, the uniqueness holds for a weaker condition.*
(2) *Before the first draft of this paper, Nathaniel Sagman emailed to the second author a note showing that for any holomorphic* $(r-1)$*-differential* q *on* Σ*, there uniquely exists a strongly complete solution to the system (5). The note uses a similar technique as in* [LM20a]*.*

4. Bounded differentials and harmonic maps

4.1. **Single equation.** We will first consider the case of a single equation. For the geometric applications, we will consider a slightly broader class of equations than (3). Let Σ be a Riemann surface with a Kähler metric g. Let $r\ge 2$ be a positive integer. We investigate the following equation for a smooth function w on Σ:

$$\triangle_g w=\kappa(e^w-e^{-(r-1)w}|q|_g^2)+\frac{1}{2}K_g. \tag{7}$$

Here, κ is a positive function on Σ satisfying $C_1\le\kappa\le C_2$ for two positive constants $C_1<C_2$.

Lemma 4.1. *Let* w *be a solution to Equation (7). Then* $K_{e^w g}$ *has a lower bound.*

Assume $e^w g$ *is complete, then either* $|q|_g^2e^{-rw}<1, K_{e^wg}<0$ *or* $|q|_g^2e^{-rw}\equiv 1, K_{e^wg}\equiv 0$*. In the latter case,* Σ *is parabolic.*

Furthermore, if $K_g\le -C_3$ *for some positive constant* C_3*, then* w *is bounded from below.*

Proof. Denote $g=\tilde g(z)dz\otimes d\bar z$. The curvature of the metric $e^w g$ is

$$\begin{aligned}K_{e^wg}&=\frac{-2}{e^w\tilde g}\partial_z\partial_{\bar z}\log(e^w\tilde g)=\frac{-2}{e^w}(\triangle_g w-\frac{K_g}{2})\\&=\frac{-2}{e^w}\kappa(e^w-e^{-(r-1)w}|q|_g^2)=-2\kappa(1-|q|_g^2e^{-rw})\\&\ge -2\kappa\ge -2C_2.\end{aligned}$$

Now, we assume that $e^w g$ is complete. Notice that for a Riemannian metric g and a smooth function f,

$$\triangle_g e^f=e^f(\triangle_g f+|\nabla_g f|_g^2)\ge e^f\,\triangle_g f. \tag{8}$$

Then, outside the zeros of q, we have

$$\begin{aligned}\triangle_g(|q|_g^2e^f)=\triangle_g e^{f+\log|q|_g^2}&\ge |q|_g^2e^f(\triangle_g f+\triangle_g\log|q|_g^2)\\&\ge |q|_g^2e^f(\triangle_g f+\frac{r}{2}K_g)\end{aligned}$$

outside the zeros of q. Since both sides of the above equation are smooth functions, by the continuity, the equation holds everywhere. So for $u = |q|_g^2 e^{-rw} - 1$,

$$\begin{aligned}\triangle_g u &= \triangle_g(|q|_g^2 e^{-rw}) \\ &\geq |q|_g^2 e^{-rw}(-r \triangle_g w + \frac{r}{2}K_g) \\ &= |q|_g^2 e^{-rw}(-r(\kappa(e^w - |q|_g^2 e^{-(r-1)w}) + \frac{1}{2}K_g) + \frac{r}{2}K_g) \\ &= r\kappa |q|_g^2 e^{-rw}(|q|_g^2 e^{-rw} - 1)e^w \\ &= r\kappa e^w u(u+1)\end{aligned}$$

implying that

$$\triangle_{e^w g} u \geq rC_1 u(u+1). \tag{9}$$

Since $e^w g$ is complete and has curvature bounded from below, the Cheng-Yau maximum principle (Lemma 2.3) implies that $u \leq 0$. By the strong maximum principle, we have two possibilities: either $u < 0$ or $u \equiv 0$. The curvature of $e^w g$ is given as follows:

$$K_{e^w g} = -2\kappa(1 - |q|_g^2 e^{-rw}) = 2\kappa u. \tag{10}$$

If $u \equiv 0$, then $e^w g$ is a complete flat metric, implying that Σ is parabolic. If $u < 0$, then $K_{e^w g} = 2\kappa u < 0$.

To show that w has a lower bound, consider the equation:

$$\triangle_{e^w g}(-w) = -\frac{1}{e^w}\triangle_g w = -\kappa(1 - |q|_g^2 e^{-rw}) - \frac{1}{2}K_g e^{-w} \geq \frac{C_3}{2}e^{-w} - C_2.$$

Using the Cheng-Yau maximum principle (Lemma 2.3), we conclude that $-w$ has an upper bound. □

Wan [Wan92] showed that the function u is bounded above by a negative constant for the equation (3) in the case where $r = 2$ and $\kappa = 1$. For our geometric applications, we are concerned with Equation (7). The proof is similar. For the convenience of our readers, we include the proof here. We will discuss the geometric applications in Section 5 and Section 6.

Proposition 4.2. *Let (Σ, g) be a complete hyperbolic surface and w be a solution to Equation (7). Assuming that $e^w g$ is complete, then the following statements are equivalent:*
(1) $|q|_g$ is bounded.
(2) $|w|$ is bounded.
(3) There exists a positive constant C such that $|q|_g^2 e^{-(r-1)w} + e^w \leq C$.
(4) There exists a positive constant δ such that the curvature of the metric $e^w g$ satisfies $K_{e^w g} \leq -\delta$.
(5) There exists a positive constant δ such that $|q|_g^2 e^{-rw} \leq 1 - \delta$.

Proof. (1)⇒(4): Let $u = |q|_g^2 e^{-rw} - 1$, and from (9), we have:

$$\triangle_{e^w g} u \geq rC_1 u(u+1).$$

From Lemma 2.5, we obtain $K_{e^w g} \leq -\delta$ for some $\delta > 0$.

(4)⇒(5): This is evident from formula (10).

(5)⇒(2): We only need to show w has an upper bound. In fact,

$$\triangle_g w = \kappa(e^w - |q|_g^2 e^{-(r-1)w}) - \frac{1}{2} \geq C_1 \delta e^w - \frac{1}{2}.$$

Then, from the Cheng-Yau maximum principle, w has an upper bound.

(2)⇒(3): It follows from Lemma 4.1 that $|q|_g^2 e^{-rw} \leq 1$. Then $|q|_g^2 e^{-(r-1)w} + e^w \leq 2e^w \leq C$.

(3)⇒(1): From the assumption, we have $|q|_g^2 e^{-(r-1)w} \leq C$, $e^w \leq C$. So $|q|_g^2$ is bounded. □

4.2. **Equation system.** We now consider the Toda system (4) and the variant Toda system (6). Since for lower values of r, the formulae cannot be written in a general form, we calculate the formulae case by case.

For the Toda system (4), set

$$f_0 = f_r = e^{2w_1}|q|_g^2, \quad f_i = e^{-w_i+w_{i+1}}(i = 1, \cdots, r-1).$$

Note that $\triangle_g \log |q|_g^2 = \frac{r}{2} K_g$.
For $r = 2, 3$, the Toda system (4) implies:

$$\triangle_g \log f_0 = 2f_0 - 2f_1 + \frac{1}{2}K_g, \text{ outside the zeros of } q$$
$$\triangle_g \log f_1 = a(f_1 - f_0) + \frac{1}{2}K_g, \quad a = \begin{cases} 2, & \text{for } r = 2 \\ 1, & \text{for } r = 3 \end{cases}$$

For $r \geq 4$, the Toda system (4) implies:

$$\triangle_g \log f_0 = 2f_0 - 2f_1 + \frac{1}{2}K_g, \text{ outside the zeros of } q,$$
$$\triangle_g \log f_i = 2f_i - f_{i-1} - f_{i+1} + \frac{1}{2}K_g, \quad i = 1, \cdots, r-1.$$

For the variant Toda system (6), set

$$f_0 = f_r = e^{w_1+w_2}|q|_g^2, \quad f_i = e^{-w_i+w_{i+1}}(i = 1, \cdots, r-1).$$

For $r = 3$, the variant Toda system (6) implies:

$$\triangle_g \log f_0 = f_0 - f_1 + \frac{1}{2}K_g, \text{ outside the zeros of } q$$
$$\triangle_g \log f_1 = f_1 - f_0 + \frac{1}{2}K_g.$$

For $r = 4, 5$, the variant Toda system (6) implies:

$$\triangle_g \log f_0 = 2f_0 - f_2 + \frac{1}{2}K_g, \text{ outside the zeros of } q$$
$$\triangle_g \log f_1 = 2f_1 - f_2 + \frac{1}{2}K_g,$$
$$\triangle_g \log f_2 = a(f_2 - f_0 - f_1) + \frac{1}{2}K_g, \quad a = \begin{cases} 2, & \text{for } r = 4 \\ 1, & \text{for } r = 5 \end{cases}$$

For $r \geq 6$, the variant Toda system (6) implies:

$$\triangle_g \log f_0 = 2f_0 - f_2 + \frac{1}{2}K_g, \text{ outside the zeros of } q$$
$$\triangle_g \log f_1 = 2f_1 - f_2 + \frac{1}{2}K_g,$$
$$\triangle_g \log f_2 = 2f_2 - f_0 - f_1 - f_3 + \frac{1}{2}K_g,$$
$$\triangle_g \log f_i = 2f_i - f_{i-1} - f_{i+1} + \frac{1}{2}K_g, \quad i = 3, \cdots, r-1.$$

Notice that for both (4) and (6), $f_i = f_{r-i}$, for $0 \leq i \leq r$.

Lemma 4.3. *For the Toda system (4) and the variant Toda system (6), let $(w_1, \cdots, w_r)$ be a real solution. Denote $g(h)_i = e^{-w_i + w_{i+1}} g$, $i = 1, \cdots, r-1$. Then the curvature $K_{g(h)_i}, i = 1, \cdots, r-1$, are bounded from below.*

Moreover, there exist positive constants c_1 and c_2 (which may be different in each case) such that:

$$\triangle_{g(h)_n} K_{g(h)_n} \geq c_1 K_{g(h)_n}(K_{g(h)_n} + c_2) \quad \text{for } r = 2,3 \text{ in (4)}, \ r = 3,4,5 \text{ in (6)}.$$
$$\triangle_{g(h)_{n-1}} K_{g(h)_{n-1}} \geq c_1 K_{g(h)_{n-1}}(K_{g(h)_{n-1}} + c_2) \quad \text{for } r = 4 \text{ in (4)}, \ r = 7 \text{ in (6)}.$$

Assume in addition $(w_1, \cdots, w_r)$ is strongly complete. Then:

$$\triangle_{g(h)_n} K_{g(h)_n} \geq c_1 K_{g(h)_n}(K_{g(h)_n} + c_2) \quad \text{for } r \geq 2 \text{ in (4)}, \ r \geq 3 \text{ in (6)}.$$
$$\triangle_{g(h)_{n-1}} K_{g(h)_{n-1}} \geq c_1 K_{g(h)_{n-1}}(K_{g(h)_{n-1}} + c_2) \quad \text{for } r \geq 4, \ r \neq 5 \text{ in (4)}, \ r \geq 6 \text{ in (6)}.$$

Proof. Locally, denote $g = \tilde{g}(z) dz \otimes d\bar{z}$. We first consider the Toda system (4).

For $r = 2, 3$,

$$\begin{aligned}
K_{g(h)_1} &= -\frac{2}{f_1 \tilde{g}} \partial_z \partial_{\bar{z}} \log(f_1 \tilde{g}) = 2a(\frac{f_0}{f_1} - 1). \\
\triangle_{g(h)_1} K_{g(h)_1} &= 2a \triangle_{g(h)_1} \frac{f_0}{f_1} \\
&\geq 2a \frac{f_0}{f_1} \triangle_{g(h)_1} \log \frac{f_0}{f_1} \quad \text{outside the zeros of } q \\
&= 2a(a+2) \frac{f_0}{f_1^2}(f_0 - f_1) = \frac{a+2}{2a} K_{g(h)_1}(K_{g(h)_1} + 2a).
\end{aligned}$$

Since q is holomorphic, its zeros are discrete. Then from the continuity of functions on both sides of the above equation, the inequality holds everywhere.

For $r = 4$, notice that $f_3 = f_1$,

$$\begin{aligned}
K_{g(h)_1} &= -\frac{2}{f_1 \tilde{g}} \partial_z \partial_{\bar{z}} \log(f_1 \tilde{g}) = 2(\frac{f_0 + f_2}{f_1} - 2). \\
\triangle_{g(h)_1} K_{g(h)_1} &= 2 \triangle_{g(h)_1} (\frac{f_0}{f_1} + \frac{f_2}{f_1}) \\
&\geq 2(\frac{f_0}{f_1} \triangle_{g(h)_1} \log \frac{f_0}{f_1} + \frac{f_2}{f_1} \triangle_{g(h)_1} \log \frac{f_2}{f_1}) \\
&= 2(\frac{f_0}{f_1^2}(3f_0 - 4f_1 + f_2) + \frac{f_2}{f_1^2}(f_0 - 3f_1 + 3f_2 - f_1)) \\
&= 2(3f_0^2 - 4f_0 f_1 + f_0 f_2 - 4f_1 f_2 + 3f_2^2 + f_0 f_2)/f_1^2 \\
&\geq 2(2(f_0 + f_2)^2 - 4(f_0 + f_2) f_1)/f_1^2 \\
&= K_{g(h)_1}(K_{g(h)_1} + 4).
\end{aligned}$$

Now, we consider the variant Toda system (6) similarly.

For $r = 3$,

$$\begin{aligned}
K_{g(h)_1} &= -\frac{2}{f_1 \tilde{g}} \partial_z \partial_{\bar{z}} \log(f_1 \tilde{g}) = 2(\frac{f_0}{f_1} - 1). \\
\triangle_{g(h)_1} K_{g(h)_1} &= 2 \triangle_{g(h)_1} \frac{f_0}{f_1} \geq 2 \frac{f_0}{f_1} \triangle_{g(h)_1} \log \frac{f_0}{f_1} = 4 \frac{f_0}{f_1^2}(f_0 - f_1) = K_{g(h)_1}(K_{g(h)_1} + 2).
\end{aligned}$$

For $r = 4,5$,

$$K_{g(h)_2} = -\frac{2}{f_2\tilde{g}}\partial_z\partial_{\bar{z}}\log(f_2\tilde{g}) = 2a\Big(\frac{f_0+f_1}{f_2}-1\Big).$$

$$\begin{aligned}
\triangle_{g(h)_2}K_{g(h)_2} &= 2a \triangle_{g(h)_2} \Big(\frac{f_0}{f_2}+\frac{f_1}{f_2}\Big)\\
&\geq 2a\Big(\frac{f_0}{f_2}\triangle_{g(h)_2}\log\frac{f_0}{f_2}+\frac{f_1}{f_2}\triangle_{g(h)_2}\log\frac{f_1}{f_2}\Big)\\
&= 2a\Big(\frac{f_0}{f_2^2}((2+a)f_0+af_1-(a+1)f_2)+\frac{f_1}{f_2^2}((2+a)f_1-(a+1)f_2+af_0)\Big)\\
&= 2a((2+a)f_0^2+2af_0f_1-(a+1)f_0f_2+(2+a)f_1^2-(a+1)f_1f_2)/f_2^2\\
&\geq 2a((a+1)(f_0+f_1)^2-(a+1)(f_0+f_1)f_2)/f_2^2\\
&= \frac{a+1}{2a}K_{g(h)_2}(K_{g(h)_2}+2a).
\end{aligned}$$

$$K_{g(h)_1} = -\frac{2}{f_1\tilde{g}}\partial_z\partial_{\bar{z}}\log(f_1\tilde{g}) = 2\Big(\frac{f_2}{f_1}-2\Big).$$

$$\triangle_{g(h)_1}K_{g(h)_1} = 2\triangle_{g(h)_1}\frac{f_2}{f_1} \geq 2\frac{f_2}{f_1}\triangle_{g(h)_1}\log\frac{f_2}{f_1} = 2\frac{f_2}{f_1^2}(-af_0-(2+a)f_1+(a+1)f_2).$$

Suppose the solution is strongly complete, then $f_2 \geq f_1 + f_0$. Therefore,

$$\triangle_{g(h)_1}K_{g(h)_1} \geq 2\frac{f_2}{f_1^2}(-2f_1+f_2) = \frac{1}{2}K_{g(h)_1}(K_{g(h)_1}+4).$$

For $r = 6,7$,

$$K_{g(h)_2} = -\frac{2}{f_2\tilde{g}}\partial_z\partial_{\bar{z}}\log(f_2\tilde{g}) = 2\Big(\frac{f_0+f_1+f_3}{f_2}-2\Big).$$

$$\begin{aligned}
\triangle_{g(h)_2}K_{g(h)_2} &= 2\triangle_{g(h)_2}\Big(\frac{f_0}{f_2}+\frac{f_1}{f_2}+\frac{f_3}{f_2}\Big)\\
&\geq 2\Big(\frac{f_0}{f_2}\triangle_{g(h)_2}\log\frac{f_0}{f_2}+\frac{f_1}{f_2}\triangle_{g(h)_2}\log\frac{f_1}{f_2}+\frac{f_3}{f_2}\triangle_{g(h)_2}\log\frac{f_3}{f_2}\Big)\\
&= 2\Big(\frac{f_0}{f_2^2}(3f_0+f_1+f_3-3f_2)+\frac{f_1}{f_2^2}(3f_1-3f_2+f_0+f_3)+\frac{f_3}{f_2^2}(f_0+f_1-3f_2+3f_3-f_4)\Big)\\
&= 2(3f_0^2+3f_1^2+2f_0f_1+2f_0f_3+2f_1f_3-3f_0f_2-3f_1f_2+3f_3^2-3f_2f_3-f_3f_4)/f_2^2\\
&= 2(2f_0^2+2f_1^2+(f_0+f_1)^2+2(f_0+f_1)f_3-3(f_0+f_1+f_3)f_2+3f_3^2-f_3f_4)/f_2^2\\
&= 2(2f_0^2+2f_1^2+2f_3^2+(f_0+f_1+f_3)^2-3(f_0+f_1+f_3)f_2-f_3f_4)/f_2^2\\
&\geq 2\Big(\frac{3}{2}(f_0+f_1+f_3)^2-3(f_0+f_1+f_3)f_2+f_3^2-f_3f_4\Big)/f_2^2\\
&= \frac{3}{4}K_{g(h)_2}(K_{g(h)_2}+4)+2\frac{f_3^2-f_3f_4}{f_2^2}
\end{aligned}$$

For $r = 6$, we have $f_4 = f_2$. Assume $(w_1,\cdots,w_r)$ is strongly complete, $f_2 \leq f_3$. Therefore,

$$\triangle_{g(h)_2}K_{g(h)_2} \geq \frac{3}{4}K_{g(h)_2}(K_{g(h)_2}+4).$$

For $r = 7$, we have $f_4 = f_3$. So

$$\triangle_{g(h)_2} K_{g(h)_2} \geq \frac{3}{4} K_{g(h)_2}(K_{g(h)_2} + 4).$$

For $r = 6, 7, 8, 9$, assume the solution is strongly complete, then $f_0 + f_1 \leq f_2$ and $f_4 \geq f_5$. Therefore,

$$\begin{aligned}
K_{g(h)_3} &= -\frac{2}{f_3 \tilde{g}} \partial_z \partial_{\bar{z}} \log(f_3 \tilde{g}) = 2\Big(\frac{f_2 + f_4}{f_3} - 2\Big).\\
\triangle_{g(h)_3} K_{g(h)_3} &= 2 \triangle_{g(h)_3} \Big(\frac{f_2}{f_3} + \frac{f_4}{f_3}\Big)\\
&\geq 2\Big(\frac{f_2}{f_3} \triangle_{g(h)_3} \log \frac{f_2}{f_3} + \frac{f_4}{f_3} \triangle_{g(h)_3} \log \frac{f_4}{f_3}\Big)\\
&= 2\Big(\frac{f_2}{f_3^2}(-f_0 - f_1 + 3f_2 - 3f_3 + f_4) - \frac{f_4}{f_3^2}(-f_2 + 3f_3 - 3f_4 + f_5)\Big)\\
&\geq 2(2f_2^2 + 3f_4^2 + 2f_2 f_4 - 3f_2 f_3 - 3f_4 f_3 - f_4 f_5)/f_3^2\\
&= 2\Big(f_2 + f_4 - 3f_3 + \frac{f_2^2 + f_4^2}{f_2 + f_4} + \frac{f_4^2 - f_4 f_5}{f_2 + f_4}\Big) \cdot \frac{f_2 + f_4}{f_3^2}\\
&\geq 2\Big(\frac{3}{2}(f_2 + f_4) - 3f_3\Big) \cdot \frac{f_2 + f_4}{f_3^2} + \frac{f_4^2 - f_4 f_5}{f_3^2}\\
&\geq \frac{3}{4} K_{g(h)_3}(K_{g(h)_3} + 4).
\end{aligned}$$

Finally, we study the remaining cases where $r \geq 4$, $2 \leq i \leq r - 2$ for (4) and $r \geq 8$, $4 \leq i \leq r - 4$ for (6).

$$\begin{aligned}
K_{g(h)_i} &= -\frac{2}{f_i \tilde{g}} \partial_z \partial_{\bar{z}} \log(f_i \tilde{g}) = -\frac{2}{f_i} \triangle_g \log f_i + \frac{1}{f_i} K_g = 2\Big(\frac{f_{i-1} + f_{i+1}}{f_i} - 2\Big).\\
\triangle_{g(h)_i} K_{g(h)_i} &= 2 \triangle_{g(h)_i} \Big(\frac{f_{i-1}}{f_i} + \frac{f_{i+1}}{f_i}\Big)\\
&\quad \text{using inequality (8)}\\
&\geq 2\Big(\frac{f_{i-1}}{f_i} \triangle_{g(h)_i} \log \frac{f_{i-1}}{f_i} + \frac{f_{i+1}}{f_i} \triangle_{g(h)_i} \log \frac{f_{i+1}}{f_i}\Big)\\
&= 2\Big(\frac{f_{i-1}}{f_i^2}(-f_{i-2} + 3f_{i-1} - 3f_i + f_{i+1}) - \frac{f_{i+1}}{f_i^2}(-f_{i-1} + 3f_i - 3f_{i+1} + f_{i+2})\Big)\\
&= 2(3f_{i-1}^2 + 3f_{i+1}^2 + 2f_{i-1}f_{i+1} - 3f_{i-1}f_i - 3f_{i+1}f_i - f_{i-2}f_{i-1} - f_{i+1}f_{i+2})/f_i^2\\
&= 2\Big(f_{i-1} + f_{i+1} - 3f_i + \frac{f_{i-1}^2 + f_{i+1}^2}{f_{i-1} + f_{i+1}} + \frac{f_{i-1}^2 + f_{i+1}^2 - f_{i-2}f_{i-1} - f_{i+1}f_{i+2}}{f_{i-1} + f_{i+1}}\Big) \cdot \frac{f_{i-1} + f_{i+1}}{f_i^2}\\
&\geq 2\Big(\frac{3}{2}(f_{i-1} + f_{i+1}) - 3f_i + \frac{f_{i-1}^2 + f_{i+1}^2 - f_{i-2}f_{i-1} - f_{i+1}f_{i+2}}{f_{i-1} + f_{i+1}}\Big) \cdot \frac{f_{i-1} + f_{i+1}}{f_i^2}\\
&= \frac{3}{4} K_{g(h)_i}(K_{g(h)_i} + 4) + 2\frac{f_{i-1}^2 + f_{i+1}^2 - f_{i-2}f_{i-1} - f_{i+1}f_{i+2}}{f_i^2}
\end{aligned}$$

For $i = n$, assume that the solution is strongly complete, then $f_{n-1} \geq f_{n-2}$ and $f_{n+1} \geq f_{n+2}$. Therefore,

$$\begin{aligned}
\triangle_{g(h)_n} K_{g(h)_n} &\geq \frac{3}{4} K_{g(h)_n}(K_{g(h)_n} + 4) + 2\frac{f_{n-1}^2 + f_{n+1}^2 - f_{n-2}f_{n-1} - f_{n+1}f_{n+2}}{f_n^2}\\
&\geq \frac{3}{4} K_{g(h)_n}(K_{g(h)_n} + 4).
\end{aligned}$$

For $i = n-1$, $r \geq 6$ in (4), $r \geq 10$ in (6), assume the solution is strongly complete, then $f_{n-2} \geq f_{n-3}$ and $f_n \geq f_{n+1}$. Therefore,

$$\begin{aligned}\triangle_{g(h)_{n-1}} K_{g(h)_{n-1}} &\geq \frac{3}{4} K_{g(h)_{n-1}} (K_{g(h)_{n-1}} + 4) + 2\frac{f_{n-2}^2 + f_n^2 - f_{n-3}f_{n-2} - f_n f_{n+1}}{f_{n-1}^2} \\ &\geq \frac{3}{4} K_{g(h)_{n-1}} (K_{g(h)_{n-1}} + 4).\end{aligned}$$

Lastly, the lower bounds of $K_{g(h)_i}$ are obvious from their formulae. □

For the rest of this section, assume Σ is hyperbolic and denote by g the unique complete hyperbolic metric on Σ.

Recall from Section 3 that a solution to the Toda system (4) or the variant Toda system (6) gives an equivariant harmonic map $f : \mathbb{D} \to N = SL(r, \mathbb{C})/SU(r)$. (In fact, the image of f lies in $SL(r, \mathbb{R})/SO(r)$, which is a totally geodesic submanifold in $SL(r, \mathbb{C})/SU(r)$.) If we assume $r \geq 3$ in (4) and $r \geq 4$ in (6), then f is both immersed and conformal. We denote e_f as the energy density, g_f as the pullback metric, and K_σ^N as the sectional curvature of a tangent plane σ in TN, where σ is the image of the tangent map at a point on $\mathbb{D}$. By a direct calculation (or see [DL19]), we have the formulae:

Lemma 4.4.

$$e_f \cdot g = g_f^{1,1} = 2r \sum_{i=0}^{r-1} g(h)_i,$$

$$K_\sigma^N = \begin{cases} -\frac{1}{2r} \frac{\sum_{i=1}^{r} (f_{i-1} - f_i)^2}{(\sum_{i=1}^{r} f_i)^2} & \textit{for (4)}, \\ -\frac{1}{2r} \frac{2(f_0 - f_1)^2 + 2(f_0 + f_1 - f_2)^2 + \sum_{i=3}^{r-2} (f_{i-1} - f_i)^2}{(\sum_{i=1}^{r} f_{i-1})^2} & \textit{for (6)}. \end{cases}$$

In the case of $r \geq 3$ in (4) or $r \geq 4$ in (6), f is conformal and thus $g_f = e_f \cdot g$.

For the variant Toda system, we obtain an estimate of $\frac{f_0}{f_1}$ under a weaker completeness condition.

Lemma 4.5. *Let $r \geq 4$ in (6). If $g(h)_1$ is complete, then either $\frac{f_0}{f_1} < 1$ or $\frac{f_0}{f_1} \equiv 1$.*

Proof. Away from zeros of q, $\triangle_g \log(\frac{f_0}{f_1}) = 2(f_0 - f_1)$,

$$\triangle_g (\frac{f_0}{f_1}) \geq (\frac{f_0}{f_1}) \cdot \triangle_g \log(\frac{f_0}{f_1}) = 2(\frac{f_0}{f_1})(f_0 - f_1).$$

Since both sides are smooth, the equation extends to the whole surface. So we have

$$\triangle_{g(h)_1} (\frac{f_0}{f_1}) = \triangle_{g \cdot f_1} \log(\frac{f_0}{f_1}) \geq 2(\frac{f_0}{f_1})(\frac{f_0}{f_1} - 1),$$

Applying the Cheng-Yau maximum principle and the assumption that $g(h)_1$ is complete, we obtain $\frac{f_0}{f_1} \leq 1$. By the strong maximum principle, either $\frac{f_0}{f_1} < 1$ or $\frac{f_0}{f_1} \equiv 1$. □

Now we study the boundedness of the geometric objects about the harmonic map. In [LM20a], Li-Mochizuki showed that for any Higgs bundle with a harmonic metric, the boundedness of the spectrum of the Higgs field is equivalent to the upper boundedness of the energy density of the corresponding harmonic map. Applying this theorem to the equation system (4) and (6), we obtain

Theorem 4.6. *(*[LM20a, Proposition 3.12]*) Consider the Toda system (4) for $r \geq 2$ and the variant Toda system (6) for $r \geq 3$ on a complete hyperbolic surface (Σ, g). Then, q is bounded with respect to g if and only if the energy density $e_f = \sum_{i=0}^{r-1} g(h)_i/g = \sum_{i=0}^{r-1} f_i$ is bounded from above.*

For the lower bound of the energy density, we have the following estimate.

Lemma 4.7. *Consider the Toda system (4) for $r \geq 2$ and the variant Toda system (6) for $r \geq 3$ on a complete hyperbolic surface (Σ, g). Suppose $g(h)_{i_0}$ is complete for some $i_0 \in \{1, \cdots, n\}$. Then there exists a positive constant C such that $g(h)_{i_0} \geq Cg$ and thus $e_f \geq C$.*

Proof. From Lemma 4.3, the curvature $K_{g(h)_{i_0}}$ is bounded from below. By Lemma 2.4 and the assumption that $g(h)_{i_0}$ is complete, we obtain $g(h)_{i_0} \geq C \cdot g$. □

Now we proceed to prove our main theorem.

Theorem 4.8. *Consider the Toda system (4) for $r \geq 3$ and the variant Toda system (6) for $r \geq 4$ on a complete hyperbolic surface (Σ, g). Let $(w_1, \cdots, w_n)$ be a strongly complete solution. Then for each $i \in \{1, \cdots, n\}$, there exists a constant $C > 0$ such that $g(h)_i \geq Cg$. Moreover, the following are equivalent:*

(1) q is bounded with respect to g.

(2) $|w_i|$, $i = 1, \cdots, n$, are bounded.

(3) There exists a constant $C > 0$ such that $g(h)_i \leq Cg$ for every $i \in \{1, \cdots, n\}$.

(4) There exists a constant $C > 0$ such that $g_f \leq Cg$.

(5) The Gaussian curvature K_{g_f} of the pullback metric g_f is bounded above by a negative constant.

(6) The sectional curvature K_σ^N is bounded above by a negative constant.

(7) The curvature $K_{g(h)_n}$ is bounded above by a negative constant.

(8) There exists a positive constant δ such that $g(h)_{i_0-1} \leq (1-\delta)g(h)_{i_0}$ for some $i_0 \in \{1, \cdots, n\}$, in the variant Toda system (6) *$g(h)_1 \leq (1-\delta)g(h)_2$ is replaced by $g(h)_0 + g(h)_1 \leq (1-\delta)g(h)_2$.*

(9) There exists a positive constant δ such that $g(h)_{i_0-1} \leq (1-\delta)g(h)_{i_0}$ for every $i_0 \in \{1, \cdots, n\}$, in the variant Toda system (6) *$g(h)_1 \leq (1-\delta)g(h)_2$ is replaced by $g(h)_0 + g(h)_1 \leq (1-\delta)g(h)_2$.*

(10) For $r \geq 4$, $r \neq 5$ in (4), $r \geq 6$ in (6), the curvature $K_{g(h)_{n-1}}$ is bounded above by a negative constant.

Proof. Note that in our case f is conformal, so $g_f = e_f \cdot g$. From Lemma 4.7 and the assumption that $(w_1, \cdots, w_n)$ is strongly complete, we have $f_i = e^{-w_i + w_{i+1}}$, $i = 1, \cdots, n$, is bounded from below by a positive constant. Therefore, $g(h)_i$ and g_f are bounded below by $C \cdot g$ for some positive constant C. From the lower bound of $f_n = e^{-(2n+2-r)w_n}$, we have the upper bound of w_n. By induction, we obtain the upper bound of all w_i, $i = 1, \cdots, n$.

Step 1: We show (1)(2)(3)(4) are equivalent.

By the same argument as in the beginning, the upper boundedness of all f_i, $i = 1, \cdots, n$, implies the lower boundedness of all w_i, $i = 1, \cdots, n$. Then by Proposition 4.6 and the assumption of strongly completeness, (1)(2)(3)(4) are equivalent.

Step 2: We show (7)⇒(6)⇒(5)⇒(4), (8)⇒(6)⇒(5)⇒(4).

(7)⇒(6), (8)⇒(6): Note that

$$K_{g(h)_n} = 2\left(\frac{f_{n-1} + f_{n+1}}{f_n} - 2\right) = \begin{cases} 2\left(\frac{f_{n-1}}{f_n} - 1\right) & r \text{ is odd and } f_{n+1} = f_n \\ 4\left(\frac{f_{n-1}}{f_n} - 1\right) & r \text{ is even and } f_{n-1} = f_{n+1} \end{cases}$$

So (8) is a restatement of (7) for the case $i_0 = n$. Therefore, it is enough to show (7)⇒(6). Recall from the assumption of strongly completeness, $f_i (i = 1, \cdots, r-1)$ are mutually bounded and $f_0 \leq f_1$.

For (4),

$$K_\sigma^N = -\frac{1}{2r}\frac{\sum_{i=1}^r (f_{i-1}-f_i)^2}{(\sum_{i=1}^r f_i)^2} \le -\frac{1}{2rC}\frac{\sum_{i=1}^r (f_{i-1}-f_i)^2}{f_{i_0}^2} \le -\frac{1}{2rC}\frac{(f_{i_0-1}-f_{i_0})^2}{f_{i_0}^2} \le -\delta'.$$

For (6), the proof is similar.

(6)⇒(5): From the Gauss equation, $K_{g_f} = K_\sigma^N + \det(II)$, where II is the second fundamental form. Since f is harmonic and conformal, f is minimal. So $\det(II) \le 0$. So $K_\sigma^N \le -\delta$ implies $K_{g_f} \le -\delta$.

(5)⇒(4): Since f is conformal, the upper bound of g_f follows from Lemma 2.4. From the argument in the beginning of the proof, the lower bound of g_f already follows from the assumption of strongly completeness.

Step 3: We show (2)⇒(9)⇒(8) and (2)⇒(9)⇒(7).

Again since (7) is a restatement of (8) for the case $i_0 = n$ and (8) is obvious from (9), it is enough to show (2)⇒(9). For each $i \in \{1,\cdots,n\}$, define the metric $g_i = e^{-\frac{2}{r+1-2i}w_i}g$. Since $|w_i|$ is bounded, g_i is complete. We calculate the curvature of g_i, denote $g = \tilde{g}dz\otimes d\bar{z}$,

$$\begin{aligned}
K_{g_i} &= -\frac{2}{e^{-\frac{2}{r+1-2i}w_i}\tilde{g}}\partial_z\partial_{\bar{z}}\log(e^{-\frac{2}{r+1-2i}w_i}\tilde{g})\\
&= -2e^{\frac{2}{r+1-2i}w_i}(-\frac{2}{r+1-2i}\triangle_g w_i + \triangle_g \log\tilde{g})\\
&= -2e^{\frac{2}{r+1-2i}w_i}(-\frac{2}{r+1-2i}\triangle_g w_i + \frac{1}{2}).
\end{aligned}$$

Then for (4) and (6), except $i=2$ in (6), we have

$$K_{g_i} = \frac{4e^{\frac{2}{r+1-2i}w_i}}{r+1-2i}(e^{-w_{i-1}+w_i} - e^{-w_i+w_{i+1}}), \quad 1\le i\le n.$$

For $i=2$ in (6),

$$K_{g_2} = \frac{4e^{\frac{2}{r-3}w_2}}{r-3}(e^{-w_0+w_1} + e^{-w_1+w_2} - e^{-w_2+w_3}).$$

Since $(w_1,\cdots,w_n)$ is strongly complete, $K_{g_i}\le 0$. From the assumption of (2), g_i is equivalent to g. Now set

$$u_i = 1 - e^{-w_{i-1}+w_i}/e^{-w_i+w_{i+1}},\ 1\le i\le n \text{ except } u_2 = 1-(e^{-w_0+w_1}+e^{-w_1+w_2})/e^{-w_2+w_3} \text{ for (6)}.$$

Then $u_i \ge 0$. From the assumption (2), $|w_i|$ are bounded and thus u_i is mutually bounded by $-K_{g_i}$.

To apply Lemma 2.2, we calculate the Bochner formula for u_i, $i=1,\cdots,n$.

For (4), from the formula (8) in Lemma 4.1, we obtain

$$\begin{aligned}
\triangle_g u_1 &\le -e^{3w_1-w_2}|q|_g^2\triangle_g(3w_1 - w_2 - \frac{r}{2}),\\
\triangle_g u_i &\le -e^{-w_{i-1}+2w_i-w_{i+1}}\triangle_g(-w_{i-1}+2w_i-w_{i+1}), \quad 2\le i\le n.
\end{aligned}$$

Then

$$\begin{aligned}
\triangle_g u_1 &\le e^{3w_1-w_2}|q|_g^2(3e^{-w_1+w_2}u_1 - e^{-w_2+w_3}u_2),\\
\triangle_g u_i &\le e^{-w_{i-1}+2w_i-w_{i+1}}(-e^{-w_{i-1}+w_i}u_{i-1} + 2e^{-w_i+w_{i+1}}u_i - e^{-w_{i+1}+w_{i+2}}u_{i+1}), \quad 2\le i\le n.
\end{aligned}$$

From the assumption, $|w_i|$'s and then $|q|_g^2$ are bounded. So we obtain that there is a constant $c>0$ depending on C such that

$$\triangle_{g_i}u_i = e^{\frac{2}{r+1-2i}w_i}\triangle_g u_i \le cu_i, \quad i=1,\cdots,n.$$

Then Lemma 2.2 implies $u_i \geq \delta$ for some constant $\delta > 0$ depending on C, which means $g(h)_{i-1} \leq (1-\delta)g(h)_i$.
For (6), similarly,

$$\begin{aligned}
\triangle_g u_1 &\leq -e^{2w_1}|q|_g^2 \triangle_g (2w_1 - \frac{r}{2}),\\
\triangle_g u_2 &\leq -e^{w_1+2w_2-w_3}|q|_g^2 \triangle_g (w_1 + 2w_2 - w_3 - \frac{r}{2})\\
&\quad - e^{-w_1+2w_2-w_3} \triangle_g (-w_1 + 2w_2 - w_3),\\
\triangle_g u_i &\leq -e^{-w_{i-1}+2w_i-w_{i+1}} \triangle_g (-w_{i-1} + 2w_i - w_{i+1}), \quad 2 \leq i \leq n.
\end{aligned}$$

Then

$$\begin{aligned}
\triangle_g u_1 &\leq e^{2w_1}|q|_{g_{\mathbb{D}}}^2 (2e^{-w_1+w_2}u_1 - \frac{r-1}{2}),\\
\triangle_g u_2 &\leq e^{w_1+2w_2-w_3}|q|_g^2 (e^{-w_1+w_2}u_1 - \frac{r-1}{4} + 2e^{-w_2+w_3}u_2 - 2e^{w_1+w_2}|q|_g^2 - e^{-w_3+w_4}u_3)\\
&\quad + e^{-w_1+2w_2-w_3}(-e^{-w_1+w_2}u_1 + 2e^{-w_2+w_3}u_2 - 2e^{w_1+w_2}|q|_g^2 - e^{-w_3+w_4}u_3)\\
&\leq (e^{w_1}|q|_g^2 - e^{-w_1})e^{2w_2-w_3}e^{-w_1+w_2}u_1 + cu_2,\\
\triangle_g u_3 &\leq e^{-w_2+2w_3-w_4}(e^{w_1+w_2}|q|_g^2 + e^{-w_1+w_2} - e^{-w_2+w_3} + 2e^{-w_3+w_4}u_3 - e^{-w_4+w_5}u_4),\\
\triangle_g u_i &\leq e^{-w_{i-1}+2w_i-w_{i+1}}(-e^{-w_{i-1}+w_i}u_{i-1} + 2e^{-w_i+w_{i+1}}u_i - e^{-w_{i+1}+w_{i+2}}u_{i+1}), \quad 4 \leq i \leq n.
\end{aligned}$$

Assume the solution is strongly complete, then

$$e^{w_1+w_2}|q|_g^2 \leq e^{-w_1+w_2}, \quad e^{w_1+w_2}|q|_g^2 + e^{-w_1+w_2} \leq e^{-w_2+w_3}.$$

Therefore, we obtain $\triangle_g u_i \leq cu_i$, $i = 1, \cdots, n$. Then Lemma 2.2 implies the desired results.

Step 4: We show (10)⇔(3) .
(3)⇒(10): From the conditions in (3), $g(h)_{n-1}$ is complete and upper bounded by $C \cdot g$. Applying the equation of $K_{g(h)_{n-1}}$ in Lemma 4.3 to Lemma 2.5, we obtain the estimate.
(10)⇒(3): By Lemma 2.4, $g(h)_{n-1}$ is upper bounded by $C \cdot g$ for some positive constant C. Since $g(h)_i$'s are mutually bounded, we obtain the statement in (3). □

Notice that in Lemma 4.3, in some lower rank cases, to obtain the Bochner formula for curvatures, we only need the completeness of $g(h)_n$ or $g(h)_{n-1}$, but not the strongly completeness of the whole solution.

Theorem 4.9. *For the Toda system (4) of rank r, let (i_0, r) be $(1, 4)$; for the variant Toda system (6) of rank r, let (i_0, r) be $(2, 4), (2, 5)$, or $(2, 7)$. Consider these two systems over a complete hyperbolic surface (Σ, g). Let $(w_1, \cdots, w_r)$ be a real solution with $g(h)_{i_0}$ is complete. Then there exists a constant $C > 1$ such that for every $i \in \{0, \cdots, [\frac{r}{2}]\}$,*

$$C^{-1}g(h)_i \leq g(h)_{i_0}, \quad g(h)_{i_0} \geq Cg.$$

Moreover, the following are equivalent:
(1) q is bounded with respect to g.
(2) There is a constant $C > 0$ such that $g_f \leq Cg$.
(3) The curvature K_{g_f} of the pullback metric g_f is bounded above by a negative constant.
(4) The curvature K_σ^N is bounded above by a negative constant.
(5) There is a constant $C > 0$ such that $g(h)_{i_0} \leq Cg$.
(6) The curvature $K_{g(h)_{i_0}}$ is bounded above by a negative constant.

Proof. Since $g(h)_{i_0}$ is complete and has curvature bounded from below, by Lemma 2.4, we obtain $g(h)_{i_0} \geq Cg$.

We then claim that for $i \in \{0, \cdots, [\frac{r}{2}]\}$, $g(h)_i \leq Cg(h)_{i_0}$ for some constant $C > 0$.

For the Toda system: $(i_0, r) = (1, 4)$. By the equation of $g(h)_1$ in Lemma 4.3 and use the assumption $g(h)_1$ is complete, we obtain that $K_{g(h)_1} < 0$. Therefore, $g(h)_0 + g(h)_2 \leq 2g(h)_1$.

For the variant Toda system:

Case 1: $(i_0, r) = (2, 4), (2, 5)$. Then we only have $g(h)_0, g(h)_1, g(h)_2$. By the equation of $g(h)_2$ in Lemma 4.3 and use the assumption $g(h)_2$ is complete, we obtain that $K_{g(h)_2} < 0$. Therefore, $g(h)_0 + g(h)_1 \leq g(h)_2$.

Case 2: $(i_0, r) = (2, 7)$. Then we only have $g(h)_0, g(h)_1, g(h)_2, g(h)_3$. By the equation of $g(h)_2$ in Lemma 4.3 and use the assumption $g(h)_2$ is complete, we obtain that $K_{g(h)_2} < 0$. Therefore, $g(h)_0 + g(h)_1 + g(h)_3 \leq 2g(h)_2$.

So we finish proving the claim.

(1)⇔(2) again follows from Theorem 4.6.

(4)⇒(3)⇒(2) is identical to the proof of (6)⇒(5)⇒(4) in Theorem 4.8.

(5)⇔(2) is obvious since g_f is a linear combination of $g(h)_0, g(h)_1, g(h)_2$.

(6)⇔(5): (6)⇒(5) follows from Lemma 2.4. (5)⇒(6) follows from applying Lemma 2.5 to the equation for $K_{g(h)_{i_0}}$ in Lemma 4.3.

(5)⇒(4): From the claim, f_j, $j = 0, 1, 2$, are bounded by Cf_{i_0} for some positive constant C. Using the formula of K_σ^N in Lemma 4.4, we have:

For the Toda system (4): $(i_0, r) = (1, 4)$. The condition in (5) implies $\frac{f_0}{f_1} - 1 \leq -\delta$.

$$K_\sigma^N = -\frac{1}{2r}\frac{2(f_0 - f_1)^2 + 2(f_1 - f_2)^2}{(f_0 + 2f_1 + f_2)^2} \leq -\frac{1}{2rC}\frac{(f_0 - f_1)^2}{f_1^2} \leq -\delta'.$$

For the variant Toda system (6):

Case 1: $(i_0, r) = (2, 4)$. The condition in (5) implies $\frac{f_0 + f_1}{f_2} - 1 \leq -\delta$.

$$K_\sigma^N = -\frac{1}{2r}\frac{2(f_0 - f_1)^2 + 2(f_0 + f_1 - f_2)^2}{(f_0 + 2f_1 + f_2)^2} \leq -\frac{1}{2rC}\frac{(f_0 + f_1 - f_2)^2}{f_2^2} \leq -\delta'.$$

Case 2: $(i_0, r) = (2, 5)$. The condition in (5) implies $\frac{f_0 + f_1}{f_2} - 1 \leq -\delta$.

$$K_\sigma^N = -\frac{1}{2r}\frac{2(f_0 - f_1)^2 + 2(f_0 + f_1 - f_2)^2}{(f_0 + 2f_1 + 2f_2)^2} \leq -\frac{1}{2rC}\frac{(f_0 + f_1 - f_2)^2}{f_2^2} \leq -\delta'.$$

Case 3: $(i_0, r) = (2, 7)$. The condition in (5) implies $\frac{f_0 + f_1 + f_3}{f_2} - 2 \leq -\delta$.

$$\begin{aligned} K_\sigma^N &= -\frac{1}{2r}\frac{2(f_0 - f_1)^2 + 2(f_0 + f_1 - f_2)^2 + 2(f_2 - f_3)^2}{(f_0 + 2f_1 + 2f_2)^2} \\ &\leq -\frac{1}{C'}\frac{(f_0 + f_1 - f_2)^2 + (f_2 - f_3)^2}{f_2^2} \\ &\leq -\frac{1}{C''}\frac{(f_0 + f_1 + f_3 - 2f_2)^2}{f_2^2} \leq -\delta'. \end{aligned}$$

□

5. Harmonic maps between surfaces and bounded quadratic differentials

In this section, we discuss the Toda system for $r = 2$, which corresponds to the harmonic map equation between surfaces. First, we recall some calculations in [SY78]. Let $(\Sigma, g = \sigma(z)|dz|^2)$ and $(M, h = \mu(u)|du|^2)$ be two Riemann surfaces with Kähler metrics. Let f be a harmonic map between Σ and M.

We define $\mathcal{H} = |\partial u|^2_{g,h} = |u_z|^2 \frac{\mu}{\sigma}$ and $\mathcal{L} = |\bar\partial u|^2_{g,h} = |u_{\bar z}|^2 \frac{\mu}{\sigma}$. The Hopf differential is the $(2,0)$-part of the pullback metric $\mathrm{Hopf}(f) = u_z \bar u_z \mu dz \otimes dz$, denoted as q. So $|q|^2_g = \mathcal{H}\mathcal{L}$. We denote K_g and K_h as the Gaussian curvature of Σ and M, respectively. Then at nonzero point of H, (our $\triangle_g$ differs from the notation in [SY78] by a factor of 4)

$$4 \triangle_g \log \mathcal{H} = -2K_h H + 2K_h \mathcal{L} + 2K_g.$$

Let $w = -\frac{1}{2}\log \mathcal{H}$, and the above equation becomes

$$\triangle_g w = -\frac{K_h}{4}(|q|^2_g e^{2w} - e^{-2w}) - \frac{K_g}{4}, \tag{11}$$

which coincides with Equation (3) for $\Sigma = M = \mathbb{D}$, $g = 4h = g_{\mathbb{D}}$.

The Jacobian $J(f) = \mathcal{H} - \mathcal{L} = e^{-2w} - |q|^2_g e^{2w}$. The map f is called orientation-preserving if the Jacobian $J(f) \geq 0$. The map f is called quasi-conformal if $\frac{\mathcal{L}}{\mathcal{H}} = |q|^2_g e^{4w} \leq k$ for some constant $k < 1$. The energy density $e(f) := \frac{1}{2}|df|^2_{g,h} = \mathcal{H} + \mathcal{L} = e^{-2w} + |q|^2_g e^{2w}$. Let $|\partial f|^2 = \mathcal{H} \cdot g$.

Wan [Wan92] showed that for a given holomorphic quadratic differential q, there exists a unique orientation-preserving harmonic map $f : \mathbb{D} \to \mathbb{D}$ such that $|\partial f|^2$ is complete. Moreover, he showed the following theorem.

Theorem 5.1. *(*[Wan92]*) Let $f : \mathbb{D} \to \mathbb{D}$ be an orientation-preserving harmonic map such that the metric $|\partial f|^2$ is complete. Then the following are equivalent:*
(a) the Hopf differential is bounded with respect to $g_{\mathbb{D}}$.
(b) f is quasi-conformal.
(c) the energy density of f is bounded.
In such case, f is a diffeomorphism.

The universal Teichmüller space is the space of quasi-symmetric homeomorphism equipped with C^0 topology between S^1 that fix three points. According to the work of [LT93] and [Mar17], equivalently, the universal Teichmüller space $\mathcal{T}(\mathbb{D})$ is in bijection with the space of harmonic quasi-conformal homeomorphisms between $\mathbb{D}$, up to a $PSL(2,\mathbb{R})$-action. Therefore, by Theorem 5.1, there exists a bijection between the space of bounded quadratic differentials with the universal Teichmüller space $\mathcal{T}(\mathbb{D})$.

Li, Tam and Wang in [LTW95] generalized Wan's result to hyperbolic Hadamard surfaces using a similar technique. The hyperbolic Hadamard surfaces are characterized as complete, simply connected Riemannian surfaces with Gaussian curvature K satisfying $-\kappa \leq K \leq 0$ for some constant $\kappa > 0$ and they have positive lower bounds for their spectra. It is shown in [LTW95] that the metric of a hyperbolic Hadamard surface is equivalent to the uniformization hyperbolic metric, i.e. the conformal factor is bounded away from 0 and ∞.

Theorem 5.2. *(*[LTW95]*) Let S_1 and S_2 be two hyperbolic Hadamard surfaces. Let $f : S_1 \to S_2$ be a harmonic diffeomorphism. Then the following are equivalent:*
(a) The Hopf differential is bounded with respect to the uniformization hyperbolic metric.
(b) f is quasi-conformal.
(c) The energy density of f is bounded.

In Theorem 5.2, the specific assumption of the domain is not essential since the harmonicity depends solely on the conformal structure. In fact, we only need to consider $\mathbb{D}$.

From [Sch93], f being a diffeomorphism implies the metric $|\partial f|^2$ being complete. In fact, The assumption regarding the diffeomorphism in Theorem 5.2 can be replaced by the completeness of the metric $|\partial f|^2$.

Theorem 5.3. *Consider two Riemannian surfaces (Σ, g) and (M, h). Suppose g is conformal and mutually bounded by a hyperbolic metric. Suppose the curvature of h satisfies $-C_1 \le K_h \le -C_2$ for some constants $C_1 \ge C_2 > 0$. Let f be a harmonic map from (Σ, g) to (M, h) such that $|\partial f|^2$ is complete. Then f is a harmonic immersion.*

Moreover, the following are equivalent:
(a) The Hopf differential is bounded with respect to the uniformization hyperbolic metric.
(b) f is quasi-conformal.
(c) The energy density of f is bounded.

Proof. The harmonic Equation (11), taking the form of Equation (7), fulfills the assumptions stated in Lemma 4.1, implying that $J(f) > 0$. Since g is conformal and mutually bounded by the hyperbolic metric, we only need to examine the boundedness with respect to the hyperbolic metric. Consequently, Theorem 5.3 can be deduced from Proposition 4.2. $\square$

6. Hyperbolic affine spheres and bounded cubic differentials

In this section, we discuss the relationship between bounded cubic differentials and hyperbolic affine spheres in $\mathbb{R}^3$. First, we provide some background on hyperbolic affine spheres. For more details, readers can refer to [Lof01, BH14, DW15].

Consider a non-compact, simply connected 2-manifold M. Let $f : M \to \mathbb{R}^3$ be a locally strictly convex immersed hypersurface. Affine differential geometry associates a transversal vector field ξ, known as the affine normal, to such a locally convex hypersurface. An affine spherical immersion is characterized by its affine normals intersecting at a single point, known as the center. By applying a translation, the center of the affine sphere can be relocated to the origin and express $\xi(p) = -Hp$ for all $p \in f(M) \subset \mathbb{R}^3$, where H is a constant representing the affine curvature. When H is negative, we refer to the affine spherical immersion as hyperbolic. After renormalization, we obtain a hyperbolic affine spherical immersion centered at 0 with affine curvature -1, which is called a normalized hyperbolic affine sphere.

We can decompose the standard connection D of $\mathbb{R}^3$ into the tangent direction of $f(M)$ and the affine normal components:

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \quad \forall X, Y \in T_{f(p)} f(M).$$

The second fundamental form h of the image $f(M)$, relative to the affine normal ξ, defines a Riemannian metric h on M. This metric is known as the *Blaschke metric*. This metric induces a complex structure on M. Additionally, the decomposition defines an induced connection ∇ on TM. Let ∇^h be the Levi-Civita connection of the Blaschke metric h, and the Pick form $A(X, Y, Z) = h((\nabla - \nabla^h)_X Y, Z)$ is a 3-tensor that uniquely determines a cubic differential $q = q(z)dz^3$ such that $\operatorname{Re} q = A$, which is referred to as the *Pick differential*.

An affine sphere is said to be complete if its Blaschke metric is complete. The work of Cheng-Yau [CY77, CY86] and An-Min Li [Li92] establishes a correspondence between properly convex domains in $\mathbb{R}P^2$ and complete hyperbolic affine spheres. An open subset $\Omega \subset \mathbb{R}P^2$ is properly convex if, when restricted to an affine chart, it corresponds to a bounded convex domain in $\mathbb{R}^2$. For a properly convex domain $\Omega \subset \mathbb{R}P^2$, denote $C(\Omega)$ as one of the two open convex cones above Ω. Given a properly convex domain $\Omega \subset \mathbb{R}P^2$, there exists a unique normalized hyperbolic affine sphere asymptotic to the boundary of the cone $C(\Omega)$. Moreover, the Blaschke metric on the affine sphere is complete. Conversely, every normalized complete hyperbolic affine sphere is asymptotic to the boundary of a cone above a properly convex subset of $\mathbb{R}P^2$.

In this discussion, we focus on the affine spherical immersions where (M,h) is conformal to the hyperbolic disk $(\mathbb{D}, g_{\mathbb{D}} = \sigma(z)|dz|^2)$. In this case, the hyperbolic affine spherical immersion can be reparametrized as $f:\mathbb{D}\to\mathbb{R}^3$. Write the Blaschke metric $h = e^w\cdot g_{\mathbb{D}}$ and $q = q(z)dz^3$. According to Wang [Wan91] and Simon-Wang [SW93], the condition of f being an affine spherical immersion is equivalent to q being holomorphic and (w,q) satisfying the equation:

$$\triangle_{g_{\mathbb{D}}} w = 2e^w - 4|q|^2_{g_{\mathbb{D}}} e^{-2w} - 2. \tag{12}$$

Up to a change of constants, this equation coincides with the Toda equation in the case of $r=3$ for a real solution. The curvature of the Blaschke metric $h = e^w\cdot g_{\mathbb{D}}$ is given by

$$k_h = -\frac{1}{2}\triangle_h w = -1 + 2|q|^2_{g_{\mathbb{D}}} e^{-3w}.$$

Then Lemma 4.1 implies that the curvature of the Blaschke metric of a complete affine sphere is non-positive, a result originally proven by Calabi [Cal72].

As a direct corollary of Proposition 4.2 applied to $r=3$, we can recover the following theorem shown by Benoist and Hulin.

Theorem 6.1. *(Benoist-Hulin* [BH14]*) For a complete hyperbolic affine spherical immersion $f:\mathbb{D}\to\mathbb{R}^3$, the following are equivalent:*
(a) The Pick differential is bounded with respect to the hyperbolic metric.
(b) The Blaschke metric has curvature bounded above by a negative constant.
(c) The Blaschke metric is conformally bounded with respect to the hyperbolic metric.

Remark 6.2. *We briefly explain the original proof from (c) to (b) in Benoist-Hulin* [BH14]*. On a properly convex domain, one can also define the Hilbert metric. Their proof is by adding the following three equivalent conditions:*
(d) Ω equipped with the Blaschke metric is Gromov hyperbolic.
(e) Ω equipped with the Hilbert metric is Gromov hyperbolic.
(f) the closure of the orbit $\overline{SL(3,\mathbb{R})\cdot\Omega}$ does not contain the projective triangle.

Using these three equivalent conditions, the authors in [BH14] *show that for a properly convex domain Ω, the Pick differential is bounded with respect to the hyperbolic metric if and only if Ω is Gromov hyperbolic with respect to the Hilbert metric. One may view the moduli space of Gromov hyperbolic convex sets, in Hilbert metric or Blaschke metric, as a generalization of the universal Teichmüller space to rank* 3.

The proof of (c)⇒(b) in [BH14] *involves showing (c)⇒(d)⇒(e)⇒(f)⇒(b), which relies on the following two facts:*
(i) Benzécri's compactness, [Ben60]*: consider $\mathcal{E}$ the set of pairs (x,Ω) where $\Omega\subset\mathbb{R}P^2$ is a properly convex domain and x is a point in Ω. The natural action of $SL(3,\mathbb{R})$ on the space of $\mathcal{E}$, equipped with the Hausdorff topology is cocompact.*
(ii) Continuity dependence of the curvature function: the curvature of the Blaschke metric depends continuously on the pair $(x,\Omega)\in\mathcal{E}$, due to Benoist-Hulin [BH13].

(c) ⇒ (d) follows from the fact that quasi-isometry preserves Gromov hyperbolicity.

(d) ⇒ (e): It uses the fact that the densities $\mu_{Hilbert}$ and $\mu_{Blaschke}$ are uniformly bounded with respect to each other.

(e) ⇒ (f) follows from the fact that the limit of a sequence of Gromov δ-hyperbolic spaces is still Gromov δ-hyperbolic.

(f) ⇒ (b): It uses Benzécri's compactness and the continuity dependence of the curvature function.

Note that our proof here for Theorem 6.1 bypasses Benzécri's compactness and the continuity dependence of the curvature function, relying solely on Wang's equation.

7. Maximal surfaces in $\mathbb{H}^{2,n}$ and bounded quartic differentials

7.1. **Maximal surfaces in $\mathbb{H}^{2,n}$.** In this section, we investigate the relationship between bounded quartic differentials and complete maximal surfaces in $\mathbb{H}^{2,n}$. First, we provide some background on maximal surfaces in $\mathbb{H}^{2,n}$ and their relationships with $SO_0(2,n+1)$-Higgs bundles developed in [CTT19].

Let E be an $(n+3)$-dimensional real vector space equipped with a quadratic form of signature $(2,n+1)$, denoted by Q. The pseudo-hyperbolic space is defined as follows:

$$\mathbb{H}^{2,n} := \{x \in E, Q(x) = -1\}/\{\pm Id\}.$$

Here, the group $SO(2,n+1)$ acts isometrically on $\mathbb{H}^{2,n}$. A space-like surface in $\mathbb{H}^{2,n}$ is defined as an immersion of a connected 2-dimensional manifold into $\mathbb{H}^{2,n}$ whose induced metric is positive definite.

Let Σ be a Riemann surface with the fundamental group π_1, and let $\tilde{\Sigma}$ be its universal cover. Consider a representation $\rho : \pi_1 \to SO_0(2,n+1)$. Now, let $f : \tilde{\Sigma} \to \mathbb{H}^{2,n}$ be a space-like, conformal, and ρ-equivariant immersion. We also consider the trivial bundle $\tilde{\Sigma} \times \mathbb{R}^{2,n+1}$. At each point $p \in \tilde{\Sigma}$, the immersion f provides a decomposition into three components: $\mathbb{R}^{2,n+1} = \tilde{T}_{f(p)} \oplus \tilde{l}_{f(p)} \oplus \tilde{N}_{f(p)}$. Here $\tilde{T}_{f(p)}$ is the tangent space of $f(\tilde{\Sigma})$ at $f(p)$, $\tilde{l}_{f(p)}$ is the position line $\mathbb{R} \cdot f(p)$, and $\tilde{N}_{f(p)}$ is the normal space of $f(\tilde{\Sigma})$ in $T_{f(p)}\mathbb{H}^{2,n}$ at $f(p)$. Since f is ρ-equivariant, the fundamental group π_1 also acts on and preserves the decomposition $\tilde{\Sigma} \times \mathbb{R}^{2,n+1} = \tilde{T} \oplus \tilde{l} \oplus \tilde{N}$. Consequently, the decomposition of the vector bundle over $\tilde{\Sigma}$ descends to Σ and is denoted as $T \oplus l \oplus N$. Notice that $\tilde{T}$ is π_1-equivariantly isomorphic to $\tilde{l} \otimes T\tilde{\Sigma}$. Therefore, we construct a vector bundle over Σ, which is given by $T \oplus l \oplus N \cong (l \otimes T\Sigma) \oplus l \oplus N$. Denote by $\Pi \in \Omega^1(\Sigma, \mathrm{Hom}(T,N))$ the second fundamental form.

Let's introduce some definitions.

Definition 7.1. *(1) A space-like surface in $\mathbb{H}^{2,n}$ is called complete if its induced metric is complete.*
(2) A space-like surface in $\mathbb{H}^{2,n}$ is called maximal if $\mathrm{tr}_g \Pi = 0$ where g is the induced metric.

Remark 7.2. *As shown in* [LTW20, Proposition 3.10]*, a complete maximal space-like surface is an entire graph.*

Now, we relate maximal surfaces in $\mathbb{H}^{2,n}$ to $SO_0(2,n+1)$-Higgs bundles.

Definition 7.3. *An $SO_0(2,n+1)$-Higgs bundle over Σ is a tuple $(\mathcal{U}, q_{\mathcal{U}}, \mathcal{V}, q_{\mathcal{V}}, \eta)$ where*

- *$\mathcal{U}, \mathcal{V}$ are holomorphic vector bundles of rank 2 and $n+1$ over Σ, respectively, and they have trivial determinant line bundles, i.e., $\wedge^2\mathcal{U} \cong \mathcal{O}, \wedge^{n+1}\mathcal{V} \cong \mathcal{O}$.*
- *$q_{\mathcal{U}}, q_{\mathcal{V}}$ are non-degenerate holomorphic sections of $\mathrm{Sym}^2(\mathcal{U}^*)$ and $\mathrm{Sym}^2(\mathcal{V}^*)$.*
- *η is a holomorphic section of $\mathrm{Hom}(\mathcal{U},\mathcal{V}) \otimes K$.*

The associated $SL(n+3,\mathbb{C})$-Higgs bundle is denoted as $(\mathcal{E},\phi)$, with $\mathcal{E} = \mathcal{U} \oplus \mathcal{V}$ and $\phi = \begin{pmatrix} 0 & \eta^\dagger \\ \eta & 0 \end{pmatrix} :$ $\mathcal{U} \oplus \mathcal{V} \to (\mathcal{U} \oplus \mathcal{V}) \otimes K$, where $\eta^\dagger = q_{\mathcal{U}}^{-1} \circ \eta^* \circ q_{\mathcal{V}} \in H^0(\mathrm{Hom}(\mathcal{V},\mathcal{U}) \otimes K)$.

A Higgs bundle is said to be conformal if the corresponding harmonic map is conformal. According to the work in [CTT19], a maximal conformal $SO_0(2,n+1)$-Higgs bundle is determined by the tuple $(I, \mathcal{V}_0, q_{\mathcal{V}_0}, \beta)$, where

$$(\mathcal{U}, q_{\mathcal{U}}, \mathcal{V}, q_{\mathcal{V}}, \eta) = (IK \oplus IK^{-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I \oplus \mathcal{V}_0, \begin{pmatrix} 1 & 0 \\ 0 & q_{\mathcal{V}_0} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}),$$

with I as a holomorphic line bundle satisfying $I^2 = \mathcal{O}$, $\mathcal{V}_0$ as a holomorphic vector bundle of rank n satisfying $\wedge^n\mathcal{V}_0 = I$, $q_{\mathcal{V}_0}$ as a non-degenerate holomorphic section of $\mathrm{Sym}^2(\mathcal{V}_0^*)$, and $\beta \in$

$H^0(\Sigma, \mathrm{Hom}(IK^{-1}, \mathcal{V}_0) \otimes K)$. The original definition for "maximal Higgs bundles" was for compact Riemann surface of genus $g \geq 2$. Here, we adopt this definition for general (possibly non-compact) Riemann surfaces. When expressed as an $SL(n+3,\mathbb{C})$-Higgs bundle, the conformal maximal $SO(2,n+1)$-Higgs bundle takes the following form:

$$\mathcal{E} = IK \oplus IK^{-1} \oplus I \oplus \mathcal{V}_0, \quad \phi = \begin{pmatrix} 0 & 0 & 0 & \beta^\dagger \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}, \tag{13}$$

where $\beta^\dagger = \beta^* \circ q_{\mathcal{V}_0}$. We consider the Hermitian metric solving the Hitchin equation in the form:

$$h = \mathrm{diag}(h_{IK}, h_{IK}^{-1}, h_I, h_{\mathcal{V}_0}).$$

The Hitchin equation simplifies to:

$$\begin{aligned} F_{h_{IK}} + \beta^\dagger \wedge (\beta^\dagger)^{*_h} + 1^{*_h} \wedge 1 = 0 \\ F_{h_{\mathcal{V}_0}} + \beta \wedge (\beta)^{*_h} + (\beta^\dagger)^{*_h} \wedge \beta^\dagger = 0. \end{aligned}$$

Now, let $\tilde{\Sigma}$ be a maximal surface in $\mathbb{H}^{2,n}$. Based on the previous discussion, we construct a vector bundle $T \oplus l \oplus N$ over Σ, and we denote their induced metrics by g_T, g_l, and g_N. We also consider the complexification of the vector bundle, denoted as $T^{\mathbb{C}} \oplus l^{\mathbb{C}} \oplus N^{\mathbb{C}}$. Let $g_T^{\mathbb{C}}, g_l^{\mathbb{C}}, g_N^{\mathbb{C}}, h_T, h_l, h_N$ be the complex linear extensions and Hermitian extensions of g_T, g_l, g_N. The data of Higgs bundles are organized as follows: $T \cong l \otimes T\Sigma$, $T^{\mathbb{C}}\Sigma \cong \overline{K^{-1}} \oplus K^{-1}$, and $\overline{K^{-1}} \cong K$ by the Hermitian metric.

- $(l^{\mathbb{C}}, (\nabla_l^{\mathbb{C}})^{(0,1)}, h_l)$ gives $(I, \bar{\partial}_I, h_I)$.
- $(T^{\mathbb{C}}, (\nabla_T^{\mathbb{C}})^{(0,1)}, g_T^{\mathbb{C}}, h_T)$ gives the orthogonal Hermitian bundle
$$(IK \oplus IK^{-1}, (\bar{\partial}_I \otimes \bar{\partial}_K) \oplus (\bar{\partial}_I \otimes \bar{\partial}_{K^{-1}}), \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, h_{IK \oplus IK^{-1}}).$$
- $(N^{\mathbb{C}}, (\nabla_N^{\mathbb{C}})^{(0,1)}, g_N^{\mathbb{C}}, h_N)$ gives the orthogonal Hermitian bundle $(\mathcal{V}_0, \bar{\partial}_{\mathcal{V}_0}, q_{\mathcal{V}_0}, h_{\mathcal{V}_0})$.
- The $(1,0)$-part of the second fundamental form II gives β.

The maximality of surfaces implies the holomorphicity of the Higgs field ϕ. Furthermore, we have the standard connection $D^{\mathbb{C}} = \nabla_h + \phi + \phi^{*_h}$, where ∇_h is the Chern connection.

Denote by q_4 the holomorphic quartic differential $\beta^\dagger \beta = q_{\mathcal{V}_0}(\beta, \beta)$. Through direct calculation, we deduce that $\mathrm{tr}(\phi^j) = 0$ if $j \neq 0(\mathrm{mod} 4)$ and $\mathrm{tr}(\phi^{4j}) = 4q_4^j$. Hence, $q_4^j (j = 1, \cdots, [\frac{n+3}{4}])$ capture all spectral data of ϕ.

We now consider the induced metric and its curvature derived from the ρ-equivariant maximal surface $f : \tilde{\Sigma} \to \mathbb{H}^{2,n}$. Due to the ρ-equivariance, these quantities can be descended to Σ. We abuse the notation g and k for short when there is no ambiguity.

We first prove the following proposition ([LT23, Proposition 4.5 and Equation (20)]).

Proposition 7.4. *(*[LT23, Proposition 4.5 and Equation (20)]*) Let X be a maximal surface in $\mathbb{H}^{2,n}$. Let g and k be the induced metric and its curvature. Then,*

$$k = -1 + |\beta|^2_{h,g} = -1 + \frac{1}{2}||\mathrm{II}||^2 \geq -1. \tag{14}$$

$$\triangle_g k \geq k(k+1). \tag{15}$$

Here is a remark regarding the proof:

Remark 7.5. *The original proof in* [LT23] *uses the Gauss equation of a maximal surface and the Bochner formula. Here, we derive these two equations in terms of Higgs bundles for its own sake. Once we establishes the explicit correspondence between the standard connection in the Higgs bundle and the harmonic metric, these two methods are essentially equivalent.*

Proof. Suppose that, locally with respect to a holomorphic frame $e_1, e_2, \cdots, e_{n+3}$ of E, where e_1, e_2, e_3 are frames of IK, IK^{-1}, I respectively, and they are related by $e_1 \cdot \partial/\partial z = e_3 = e_2 dz$. The remaining frames, denoted as $\{e_4, \cdots, e_{n+3}\}$, form a frame of $\mathcal{V}_0$. Consequently, $\phi = f dz$ and $\beta = \gamma dz$, which implies:

$$f = \begin{pmatrix} 0 & 0 & 0 & \gamma^\dagger \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \end{pmatrix}, \quad f^{*_h} = \begin{pmatrix} 0 & 0 & 1^{*_h} & 0 \\ 0 & 0 & 0 & \gamma^{*_h} \\ 0 & 1^{*_h} & 0 & 0 \\ (\gamma^\dagger)^{*_h} & 0 & 0 & 0 \end{pmatrix}.$$

Here, $1 : IK \to I$ indicates the contraction with $\frac{\partial}{\partial z}$. Then, the induced metric $|1|_h^2 = h(\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$, and the induced Riemannian metric $g = h + \bar{h} = 2|1|_h^2 |dz|^2$.

Now, we apply Lemma 3.3 to a local holomorphic section $s_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Locally, we have:

$$[f^{*_h}, s_1] = \begin{pmatrix} 1^{*_h} \circ 1 & & & \\ & 0 & & \\ & & -1 \circ 1^{*_h} & \\ & & & 0 \end{pmatrix}, \quad [f, s_1] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 \circ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \circ \gamma^\dagger \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This leads us to the following equation:

$$\partial_z \partial_{\bar{z}} \log |1|_h^2 = \frac{|[s_1, f^{*_h}]|_h^2 - |[s_1, f]|_h^2}{|s_1|_h^2} = \frac{2|1|_h^4 - |1|_h^4 - |1|_h^2 \cdot |\gamma^\dagger|_h^2}{|1|_h^2} = |1|_h^2 - |\gamma|_h^2. \tag{16}$$

We should clarify why it is equality and not inequality. Notice that in Lemma 3.3, the inequality only occurs at the point where $|h(\partial_{z,h} s, s)|^2 \leq |\partial_{z,h} s|_h^2 |s|_h^2$. Therefore, we only need to demonstrate that $(\partial_{z,h})_{\frac{\partial}{\partial z}} s = \lambda s$ for some scalar function λ on X. Recall $\partial_{z,h}$ represents the $(1,0)$ part of the Chern connection ∇_h on $E = IK \oplus IK^{-1} \oplus I \oplus \mathcal{V}_0$. As both the complex structure and the Hermitian metric are diagonal, the Chern connection ∇_h, and consequently $\partial_{z,h}$ are also diagonal. Since $\mathrm{Hom}(IK, I)$ is 1-dimensional, it follows that $(\partial_{z,h})_{\frac{\partial}{\partial z}} s = \lambda s$ for a certain function λ on X.

Considering that $g = 2|1|_h^2 |dz|^2$ and $|dz|_g^2 = \frac{1}{|1|_h^2}$, we obtain

$$k = -\frac{2}{2|1|_h^2} \partial_z \partial_{\bar{z}} \log 2|1|_h^2 = -1 + \frac{|\gamma|_h^2}{|1|_h^2} = -1 + |\beta|_{h,g}^2.$$

Since β represents the $(1,0)$-part of the second fundamental form Π, we have $|\beta|_{h,g}^2 = \frac{1}{2}||\Pi||^2$. Thus, we have established the first equation.

Next, we apply Lemma 3.3 to a local holomorphic section $s_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \end{pmatrix}$. Locally, we have:

$$[f^{*_h}, s_2] = \begin{pmatrix} 0 & & & \\ & \gamma^{*_h} \gamma & & \\ & & 0 & \\ & & & -\gamma \gamma^{*_h} \end{pmatrix}, \quad [f, s_2] = \begin{pmatrix} 0 & \gamma^\dagger \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma \circ 1 & 0 \end{pmatrix}.$$

This leads to

$$\begin{aligned} \partial_z \partial_{\bar{z}} \log |\gamma|_h^2 &\geq \frac{|[s_2, f^{*_h}]|_h^2 - |[s_2, f]|_h^2}{|s_2|_h^2} = \frac{2\mathrm{tr}(\gamma \gamma^{*_h} \gamma \gamma^{*_h}) - |\gamma^\dagger \gamma|_h^2 - |\gamma \circ 1|_h^2}{|\gamma|_h^2} \\ &= \frac{2|\gamma|_h^4 - |\gamma^\dagger \gamma|_h^2 - |1|_h^2 \cdot |\gamma|_h^2}{|\gamma|_h^2} \geq |\gamma|_h^2 - |1|_h^2, \end{aligned} \tag{17}$$

where the last inequality follows from the Cauchy-Schwarz inequality $|\gamma^\dagger \gamma|_h \leq |\gamma|_h \cdot |\gamma^\dagger|_h = |\gamma|_h^2$.

By combining Equations (16) and (17), we conclude locally that:

$$\partial_z \partial_{\bar{z}} \log \frac{|\gamma|_h^2}{|1|_h^2} \geq 2(|\gamma|_h^2 - |1|_h^2).$$

Globally, we obtain:

$$\triangle_g \log \frac{|\gamma|_h^2}{|1|_h^2} = \frac{1}{2|1|_h^2} \partial_z \partial_{\bar{z}} \log \frac{|\gamma|_h^2}{|1|_h^2} \geq \frac{|\gamma|_h^2}{|1|_h^2} - 1 = k.$$

Then, away from zeros of γ, we have

$$\triangle_g k = \triangle_g \frac{|\gamma|_h^2}{|1|_h^2} \geq \frac{|\gamma|_h^2}{|1|_h^2} \cdot \triangle_g \log \frac{|\gamma|_h^2}{|1|_h^2} \geq k(k+1).$$

Since the equation is continuous on both sides, the equation holds over the entire surface. Thus, we have proven the second equation. □

Now, let us proceed to the following theorem, which is the curvature rigidity theorem in [LT23, Theorem A]. While the original proof in [LT23] relies on a compactness result, this proof will provide an alternative approach using the curvature formula mentioned above:

Proposition 7.6. *Let X be a complete maximal surface in $\mathbb{H}^{2,n}$. Then the intrinsic curvature k satisfies either $k < 0$ or identically zero ($k \equiv 0$). Equivalently, in terms of Higgs bundles, $|\beta|_{h,g}^2 < 1$ or $|\beta|_{h,g}^2 \equiv 1$.*

As a result, the induced metric g satisfies either $g > 2|q_4|^{\frac{1}{2}}$ or $g \equiv 2|q_4|^{\frac{1}{2}}$.

Proof. By applying Lemma 2.5 to Equation (15) in Proposition 7.4, we can obtain either $k < 0$ or $k \equiv 0$. Then, from Equation (14), we obtain that either $|\beta|_{h,g}^2 < 1$ or $|\beta|_{h,g}^2 \equiv 1$. This implies, as discussed in the proof of Proposition 7.4, that $|\gamma|_h \leq |1|_h$ and $q_4 = q(z)dz^4 = \beta^\dagger \beta = \gamma^\dagger \gamma dz^2$.

Consider $g_0 = |1|_h^2 dz \otimes d\bar{z}$. We have

$$|q_4|_{g_0}^2 = \frac{|q(z)|^2}{|1|_h^4} \leq \frac{|\gamma^\dagger \gamma|_h^2}{|1|_h^4} \leq \frac{|\gamma|_h^4}{|1|_h^4} \leq 1.$$

Then $g = 2|1|_h^2 |dz|^2 \geq 2|q_4|^{\frac{1}{2}}$. The rigidity follows from the one of k. □

We now present a different proof of the following theorem, originally shown by Labourie-Toulisse in [LT23].

Theorem 7.7. *(Labourie-Toulisse* [LT23]*) Consider a complete maximal surface X in $\mathbb{H}^{2,n}$ that is conformal to $\mathbb{D}$. The following conditions are equivalent:*
(a) The induced metric has curvature bounded above by a negative constant;
(b) The induced metric is conformally bounded with respect to the hyperbolic metric.

Proof. For the "only if" direction, we apply Lemma 2.5 to Equation (15) in Proposition 7.4, which yields $k \leq -\delta$ for a positive constant δ.

For the "if" direction, since the induced metric has curvature satisfying $-1 \leq k \leq -\delta$, according to Lemma 2.4, we have $g_{\mathbb{D}} \leq g \leq C g_{\mathbb{D}}$ for some constant $C = C(\delta) > 0$. □

Remark 7.8. *The proof from (b) to (a) is the same as in* [LT23]*, while the proof from (a) to (b) differs from the one in* [LT23]*. Let us briefly explain the original proof from (b) to (a) in Labourie-Toulisse* [LT23]*. Their proof involves adding the following two equivalent conditions, which were their main focus:*
(c) The induced metric is Gromov hyperbolic.
(d) Σ is quasiperiodic, meaning that the orbit closure $\overline{SO_0(2, n+1) \cdot X}$ in $\mathcal{M}(n)$ does not contain the Barbot surface, in which case the induced metric is flat.

The proof of (b)⇒ (a) proceeds by showing (b)⇒(c)⇒(d)⇒(a), relying on the following two facts:
(i) Compactness: Consider $\mathcal{M}(n)$ as the set of pairs (x, X), where X is a complete maximal surface in $\mathbb{H}^{2,n}$ and x is a point on X. The natural action of $SO_0(2, n)$ on the space of $\mathcal{M}(n)$, equipped

with the Hausdorff topology, is cocompact, as proved in [LTW20].
(ii) Continuity: The curvature $k_X(x)$ *depends continuously on the pair* $(x, X) \in \mathcal{M}(n)$.

(b) $\Rightarrow$ *(c) follows from the fact that quasi-isometry preserves Gromov hyperbolicity.*

(c) $\Rightarrow$ *(d) follows from the fact that the limit of a sequence of Gromov* δ*-hyperbolic spaces is still Gromov* δ*-hyperbolic.*

(d) $\Rightarrow$ *(a) uses the compactness as in (ii) above.*

Note that our proof here for Theorem 7.7 bypasses the need for cocompactness and instead derives directly from the equation itself.

For a maximal surface X in $\mathbb{H}^{2,n}$, we associate a holomorphic quartic differential $q_4 = g_N(\beta, \beta)$, or equivalently, we can define $q_4 = \frac{\mathrm{tr}(\phi^4)}{4}$ for the corresponding Higgs bundle (E, ϕ).

Theorem 7.9. *For a complete maximal surface* X *in* $\mathbb{H}^{2,n}$*, assuming that* X *is conformal to* $\mathbb{D}$*, the following conditions are equivalent:*

(1) *The quartic differential is bounded with respect to the hyperbolic metric.*
(2) *The induced metric has curvature bounded above by a negative constant.*
(3) *The induced metric is conformally bounded with respect to the hyperbolic metric.*

Proof. According to Theorem 7.7, conditions (2) and (3) are equivalent.

(3)$\Rightarrow$(1) follows from the fact that $g \geq 2|q_4|^{\frac{1}{2}}$ as stated in Proposition 7.6.

(1)$\Rightarrow$(3): If the quartic differential $q_4 = \frac{\mathrm{tr}(\phi^4)}{4}$ is bounded with respect to the hyperbolic metric $g_{\mathbb{D}}$, the spectra of the Higgs field ϕ are bounded with respect to $g_{\mathbb{D}}$. Proposition 3.12 in [LM20a] states that $|\phi|_{h,g_{\mathbb{D}}} \leq C$. Since the Hermitian metric h on E is diagonal (as shown in Equation (13)), we have $|\phi|^2_{h,g_{\mathbb{D}}} = 2|\beta|^2_{h,g_{\mathbb{D}}} + 2|1|^2_{h,g_{\mathbb{D}}}$. Therefore, $|1|^2_{h,g_{\mathbb{D}}} \leq C$. Consequently, we obtain $g \leq C g_{\mathbb{D}}$ for some constant C. □

7.2. **An analogue of the universal Teichmüller space.** In the work of F. Labourie and J. Toulisse, they introduce an analogue of the universal Teichmüller space $\mathcal{QS}_n$. They define a natural map from $\mathcal{QS}_n$ to the product of the universal Teichmüller space and the space of bounded quartic differentials. Let's briefly explain their construction in [LT23].

Consider a 2-dimensional real vector space denoted as V. For any quadruple of pairwise distinct points (x, y, z, w) in $\mathbb{P}(V)^4$, let's denote its cross ratio as $[x, y, z, w]$. Recall that a homeomorphism ϕ of $\mathbb{P}(V)$ is quasisymmetric if there exist constants A and B, both greater than 1, such that for any quadruple of pairwise distinct points in $\mathbb{P}(V)$, if $A^{-1} \leq |[x, y, z, t]| \leq A$, then

$$B^{-1} \leq |[\phi(x), \phi(y), \phi(z), \phi(t)]| \leq B.$$

Now, let QS_0 be the group of quasisymmetric homeomorphisms of $\mathbb{P}(V)$. The group $PSL(V)$ acts on QS_0 by post-composition, and the quotient is defined to be the universal Teichmüller space, denoted as $\mathcal{T}(\mathbb{H}^2)$.

The Einstein universe is the quadric associated to the quadratic form q:

$$\partial_\infty \mathbb{H}^{2,n} := \{x \in \mathbb{P}(E), q(x) = 0\}.$$

The group $SO_0(2, n+1)$ acts transitively on $\partial_\infty \mathbb{H}^{2,n}$. There is a natural generalization of the definition of cross ratio on $\partial_\infty \mathbb{H}^{2,n}$. Specifically, for a map ξ from $\mathbb{P}(V)$ to $\partial_\infty \mathbb{H}^{2,n}$ and a pair of constants A, B, both greater than 1, the map ξ is considered (A, B)-quasisymmetric if it is positive and for all quadruples $(x, y, z, t) \in \mathbb{P}(V)^4$, if $A^{-1} \leq |[x, y, z, t]| \leq A$, then

$$B^{-1} \leq |[\xi(x), \xi(y), \xi(z), \xi(w)]| \leq B.$$

Let QS_n be the space of quasisymmetric maps from $\mathbb{P}(V)$ to $\partial_\infty \mathbb{H}^{2,n}$, equipped with the C^0 topology. The quotient $\mathcal{QS}_n := QS_n / SO_0(2, n+1)$ forms a Hausdorff topological space. Note that

$\mathcal{QS}_0$ corresponds to $\mathcal{T}(\mathbb{H}^2)$. The space $\mathcal{QS}_n$ serves as an analogue of the universal Teichmüller space.

According to [LT23, Theorem B], a maximal surface Σ is quasiperiodic if and only if the boundary map is quasisymmetric. Additionally, [LT23, Theorem 7.1] establishes that for any element $\xi \in \mathcal{QS}_n$, there exists a unique reparametrization $\xi_{repar} \in \mathcal{QS}_n$ of the image of ξ. Moreover, if the image of ξ bounds a maximal surface with induced curvature bounded above by $-c$, then ξ_{repar} is (A, B)-quasisymmetric, where the constants (A, B) depend solely on c. The space $\mathcal{T}(\mathbb{H}^2)$ acts on $\mathcal{QS}_n$ by post-composition. They define a continuous map $\pi_{\mathbb{H}^2} : \mathcal{QS}_n \to \mathcal{T}(\mathbb{H}^2)$ by setting $\pi_{\mathbb{H}^2}(\xi) = \phi$, where ϕ is such that $\xi \circ \phi = \xi_{repar}$.

When Σ is quasiperiodic, as shown in Theorem 6.3 in [LT23], the uniformization gives a biLipschitz map between $\mathbb{H}^2$ and Σ, resulting in q_4 is bounded with respect to the hyperbolic metric. Denote by $H^0_b(\mathbb{H}^2, K^4)$ the vector space of holomorphic quartic differentials on $\mathbb{H}^2$ that are bounded with respect to the hyperbolic metric.

In summary, we obtain a map:

$$\begin{aligned} \mathcal{H}_{\mathbb{H}^2} : \mathcal{QS}_n &\to \mathcal{T}(\mathbb{H}^2) \times H^0_b(\mathbb{H}^2, K^4) \\ \xi &\mapsto (\pi_{\mathbb{H}^2}(\xi), q(\xi)). \end{aligned}$$

As posed by F. Labourie and J. Toulisse, we aim to prove that this map is proper.

Theorem 7.10. *The map $\mathcal{H}_{\mathbb{H}^2}$ is proper.*

Proof. Suppose $(\phi, q) \in \mathcal{T}(\mathbb{H}^2) \times H^0_b(\mathbb{H}^2, K^4)$ satisfies that ϕ is (A, B)-quasisymmetric and $|q|_{g_{\mathbb{D}}} \le M$. It is enough to show $\xi \in \mathcal{H}^{-1}_{\mathbb{H}^2}(\phi, q)$ is (A'', B'')-quasisymmetric, where (A'', B'') only depends on A, B, M. The properness follows from the cocompactness of the action of $PSL(V) \times SO_0(2, n+1)$ on the space of (A, B)-quasisymmetric maps, as shown in Theorem 3.12 in [LT23].

Given $|q|_{g_{\mathbb{D}}} \le M$, the induced curvature on Σ is bounded from above by a constant $-c$ for $c = c(M)$ as per Theorem 7.9. Therefore, Σ is quasiperiodic according to Theorem B in [LT23], which is discussed in Remark 7.8. By Theorem 7.1 in [LT23], since (x, Σ) is a quasiperiodic surface with curvature bounded above by $-c$, ξ_{repar} is (A', B')-quasisymmetric, where $(A', B') = (A', B')(c) > 1$.

Since ϕ is (A, B)-quasisymmetric, then ϕ^{-1} is also (A, B)-quasisymmetric. Therefore, $\xi = \xi_{repar} \circ \phi^{-1}$ is of (A'', B'')-quasisymmetric, where (A'', B'') only depends on A, B, M. This completes the poof of properness of $\mathcal{H}_{\mathbb{H}^2}$. □

8. J-holomorphic curve in $\mathbb{H}^{4,2}$ and bounded sextic differentials

In this section, we explore the relationship between bounded sextic differentials and J-holomorphic curves in $\mathbb{H}^{4,2}$.

We begin by considering the split octonion, represented by $\mathbb{O}'$. The automorphism group of $\mathbb{O}'$, denoted as $G'_2 = Aut(\mathbb{O}') \subset SO_0(3, 4)$, is a subgroup of $SO_0(3, 4)$ known as the split real G'_2. The imaginary split octonion $\Im\mathbb{O}'$, equipped with a natural quadratic form q, can be identified with $\mathbb{R}^{3,4}$. On $\mathbb{R}^{3,4}$, there exists a cross product defined as $x \times y := \Im(xy)$, which induces an almost complex structure J on the pseudosphere given by:

$$S^{2,4} = \{x \in \mathbb{R}^{3,4} | q(x, x) = 1\}$$

as follows:

$$J(X) := x \times X,$$

where $x \in S^{2,4}, X \in T_x S^{2,4} \cong x^\perp$.

The pseudo-hyperbolic space $\mathbb{H}^{4,2}$ is the counter part of the pseudosphere $S^{2,4}$:

$$\mathbb{H}^{4,2} = \{x \in \mathbb{R}^{4,3} | q(x, x) = -1\}.$$

There is an obvious diffeomorphism $\mathbb{H}^{4,2} \cong S^{2,4}$ identifying the metric of the former with -1 times that of the latter.

In an almost complex manifold (M, J), a J-holomorphic curve is an immersed surface Σ whose tangent bundle $T\Sigma \subset TM$ is preserved by J.

Baraglia [Bar10] discovered that a subcyclic rank 7 Higgs bundle in the Hitchin section over a domain $\Omega \subset \mathbb{C}$, together with a harmonic metric $h = \mathrm{diag}(h_1, h_2, h_3, 1, h_3^{-1}, h_2^{-1}, h_1^{-1})$ satisfying $h_1 = 2h_2h_3$, gives rise to a J-holomorphic (also known as almost-complex) curve $\nu : \Omega \to \hat{S}^{2,4}$. The explicit relationship between the resulting surface and Higgs bundles has been further developed in Evans [Eva22] using harmonic sequences.

For a J-holomorphic curve, the osculation line is defined as the J-complex line in a normal space formed by the images of the second fundamental form. Nie [Nie22] showed that subcyclic Higgs bundles in the Hitchin section, together with a real harmonic metric $\mathrm{diag}(h_1, h_2, h_3, 1, h_3^{-1}, h_2^{-1}, h_3^{-1})$ satisfying $h_1 = 2h_2h_3$, are characterized as space-like J-holomorphic curves in $\mathbb{H}^{4,2}$ with nowhere vanishing second fundamental forms and timelike osculation lines. The holomorphic sextic differential q can be retrieved from the data of the structure equation of the immersion.

The induced Hermitian metric on the J-holomorphic curve is $h = |1|_h^2 dz \otimes d\bar{z} = h_3^{-1} dz \otimes d\bar{z}$, and the induced Riemannian metric is $g = h + \bar{h} = 2h_3^{-1}(dx^2 + dy^2)$(see [Eva22, Section 3.1]).

Definition 8.1. *We call a space-like J-holomorphic curve in $\mathbb{H}^{4,2}$ complete if its induced metric is complete.*

Remark 8.2. *(1) It is not clear if the completeness condition implies the surface is a proper embedding or entire.*
(2) One may construct plenty of examples of complete metrics which do not necessarily come from a strongly complete solution for the variant Toda system.

Lemma 8.3. *Let Σ be a space-like J-holomorphic curve in $\mathbb{H}^{4,2}$ with a nowhere vanishing second fundamental form and a timelike osculation line. Let g and k be the induced metric and its curvature, respectively, of Σ. Then $k \geq -1$. If Σ is complete, then*

$$\triangle_g k \geq 3k(k+1). \tag{18}$$

Proof. The surface data correspond to the Higgs bundle $s(0, \cdots, 0, q_6, 0)$ together with a diagonal harmonic metric

$$\mathrm{diag}(h_1, h_2, h_3, 1, h_3^{-1}, h_2^{-1}, h_1^{-1})$$

satisfying $h_1 = 2h_2h_3$.

Let $g_0 = \widetilde{g_0} dz \otimes d\bar{z}$ be a background Kähler metric on Σ. We employ the concepts introduced in 4.2. Let $h_i = e^{w_i} g_0^{i-4}$, where $i = 1, \cdots, 7$. Set

$$f_0 = e^{w_1+w_2}|q|_g^2, f_1 = e^{-w_1+w_2}, f_2 = e^{-w_2+w_3}, f_3 = e^{-w_3}.$$

From the relation $h_1 = 2h_2h_3$, we have $2f_1 = f_3$.

As in Section 4.2, from the Hitchin equation, we obtain the following system

$$\begin{aligned}
\triangle_{g_0} \log f_0 &= 2f_0 - f_2 + \frac{1}{2}K_{g_0} \quad \text{away from zero of } q \\
\triangle_{g_0} \log f_1 &= 2f_1 - f_2 + \frac{1}{2}K_{g_0} \\
\triangle_{g_0} \log f_2 &= 2f_2 - f_0 - f_1 - f_3 + \frac{1}{2}K_{g_0} \\
\triangle_{g_0} \log f_3 &= f_3 - f_2 + \frac{1}{2}K_{g_0}
\end{aligned}$$

Note that $h_3^{-1} = f_3 \cdot g_0$ and $g = f_3 \cdot 2\,\mathrm{Re}(g_0) = \mathrm{Re}(2f_3 \cdot \widetilde{g_0} dz \otimes f z\bar{z})$. The Gaussian curvature k of g is

$$\begin{aligned} k &= -\frac{2}{2f_3\widetilde{g_0}}\partial_z\partial_{\bar z}\log(2f_3\widetilde{g_0}) = -\frac{1}{f_3\widetilde{g_0}}\partial_z\partial_{\bar z}(\log f_3 + \log\widetilde{g_0}) \\ &= -\frac{1}{f_3}(\triangle_{g_0}\log f_3 + \triangle_{g_0}\log\widetilde{g_0}) = -\frac{1}{f_3}(f_3 - f_2) \\ &= \frac{f_2}{f_3} - 1 \geq -1. \end{aligned}$$

Now we assume g is complete, that is, $f_3 \cdot g_0$ is complete. Since $2f_1 = f_3$, we have $g(h)_1 = f_1 \cdot g_0$ is also complete. By Lemma 4.5, we obtain $f_0 \leq f_1$.

Therefore,

$$\begin{aligned} \triangle_g k &= \frac{1}{2f_3}\triangle_{g_0} k = \frac{1}{2f_3}\triangle_{g_0}\frac{f_2}{f_3} = \frac{f_2}{f_3^2}\triangle_{g_0}\log\frac{f_2}{f_3} \\ &= \frac{f_2}{f_3^2}(3f_2 - f_0 - f_1 - 2f_3) \\ &\text{using } 2f_1 = f_2 \text{ and } f_0 \leq f_1 \\ &\geq \frac{f_2}{f_3^2}(3f_2 - 3f_3) \\ &= 3k(k+1). \end{aligned}$$

□

Similar to the case of maximal surfaces in $\mathbb{H}^{2,4}$, we show the following result.

Theorem 8.4. *Let Σ be a complete space-like J-holomorphic curve in $\mathbb{H}^{4,2}$ with nowhere vanishing second fundamental form and a timelike osculation line. Let q be its associated holomorphic sextic differential. Then, its induced curvature is either strictly negative or constantly zero. Consequently, the induced metric satisfies either $g > 2\sqrt{2}|q|^{\frac{1}{3}}$ or $g \equiv 2\sqrt{2}|q|^{\frac{1}{3}}$.*

Next, assume (Σ, g) is conformal to $\mathbb{D}$. Then, $g \geq g_{\mathbb{D}}$. Moreover, the following statements are equivalent:

1. *q is bounded with respect to the hyperbolic metric.*
2. *The induced metric on Σ is conformally bounded with respect to the hyperbolic metric.*
3. *The induced curvature on Σ is bounded from above by a negative constant.*

Proof. Applying Lemma 2.5 to the equation of curvature k in Lemma 8.3 and using the fact $k \geq -1$, we obtain $k < 0$ or $k \equiv 0$.

Since $k \leq 0$, we obtain $f_2 \leq f_3$. Moreover, using the same notions in the proof of Lemma 8.3, we obtain $f_0 \leq f_1$. Also note that $2f_1 = f_3$. Thus we obtain

$$|q|^2_{g_0} = |q|^2_{g_0} e^{2w_1} \cdot e^{2(-w_1+w_2)} \cdot e^{2(-w_2+w_3)} \cdot e^{-2w_3} = f_0 f_1^2 f_2^2 f_3 \leq \frac{1}{8} f_3^6.$$

Therefore, $\sqrt{2}|q|^{\frac{1}{3}}_{g_0} \leq f_3$. Since locally $g = 2f_3\widetilde{g_0}|dz|^2$ and $|q|^2_{g_0} = \frac{|q(z)|^2}{\widetilde{g_0}^6}$, we obtain

$$|q|^{\frac{1}{3}} = |q(z)|^{\frac{1}{3}}|dz|^2 \leq \frac{1}{2\sqrt{2}} g.$$

The rigidity follows from the one of k.

Next, $g \geq g_{\mathbb{D}}$ follows from the fact $k \geq -1$, the assumption g is complete, and Lemma 2.4.

Finally, we show the equivalence between (1)(2)(3).

(1)⇒(2) follows from Theorem 4.6.

(2)⇒(1) follows from $\frac{1}{\sqrt{2}}|q|^{\frac{1}{3}} \le g$.

(2)⇒(3): Applying Lemma 2.5 to the equation of k in Lemma 8.3, we have $k \le -\delta$ for a positive constant δ.

(3)⇒(2): It follows from Lemma 2.4. □

References

[Bar10] David Baraglia, *G_2 Geometry and Integrable Systems*, arXiv:1002.1767 (2010).

[Bar15] ———, *Cyclic Higgs bundles and the affine Toda equations*, Geom. Dedicata **174** (2015), 25–42. MR 3303039

[Ben60] Jean-Paul Benzécri, *Sur les variétés localement affines et localement projectives*, Bull. Soc. Math. France **88** (1960), 229–332. MR 124005

[BH13] Yves Benoist and Dominique Hulin, *Cubic differentials and finite volume convex projective surfaces*, Geom. Topol. **17** (2013), no. 1, 595–620. MR 3039771

[BH14] ———, *Cubic differentials and hyperbolic convex sets*, J. Differential Geom. **98** (2014), no. 1, 1–19. MR 3238310

[Cal72] Eugenio Calabi, *Complete affine hyperspheres. I*, Symposia Mathematica, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971), Academic Press, London, 1972, pp. 19–38. MR 0365607 (51 #1859)

[CT90] Hyeong In Choi and Andrejs Treibergs, *Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space*, J. Differential Geom. **32** (1990), no. 3, 775–817. MR 1078162

[CTT19] Brian Collier, Nicolas Tholozan, and Jérémy Toulisse, *The geometry of maximal representations of surface groups into* $\mathrm{SO}_0(2,n)$, Duke Math. J. **168** (2019), no. 15, 2873–2949. MR 4017517

[CY75] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354. MR 0385749

[CY77] Shiu Yuen Cheng and Shing Tung Yau, *On the regularity of the Monge-Ampère equation* $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x,u)$, Comm. Pure Appl. Math. **30** (1977), no. 1, 41–68. MR 437805

[CY86] Shiu Yuen Cheng and Shing-Tung Yau, *Complete affine hypersurfaces. I. The completeness of affine metrics*, Comm. Pure Appl. Math. **39** (1986), no. 6, 839–866. MR 859275

[DL19] Song Dai and Qiongling Li, *Minimal surfaces for Hitchin representations*, J. Differential Geom. **112** (2019), no. 1, 47–77. MR 3948227

[DL20] ———, *On cyclic Higgs bundles*, Math. Ann. **376** (2020), no. 3-4, 1225–1260. MR 4081114

[DW15] David Dumas and Michael Wolf, *Polynomial cubic differentials and convex polygons in the projective plane*, Geom. Funct. Anal. **25** (2015), no. 6, 1734–1798. MR 3432157

[Eva22] Parker Evans, *Polynomial almost-complex curves in* $\mathbb{S}^{2,4}$, arXiv:2208.14409 (2022).

[GIL15] Martin A. Guest, Alexander R. Its, and Chang-Shou Lin, *Isomonodromy aspects of the* tt^* *equations of Cecotti and Vafa I. Stokes data*, Int. Math. Res. Not. IMRN (2015), no. 22, 11745–11784. MR 3456701

[GL14] Martin A. Guest and Chang-Shou Lin, *Nonlinear PDE aspects of the tt* equations of Cecotti and Vafa*, J. Reine Angew. Math. **689** (2014), 1–32. MR 3187927

[Gup21] Subhojoy Gupta, *Harmonic maps and wild Teichmüller spaces*, J. Topol. Anal. **13** (2021), no. 2, 349–393. MR 4284613

[Han96] Zheng-Chao Han, *Remarks on the geometric behavior of harmonic maps between surfaces*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, 1996, pp. 57–66. MR 1417948

[HTTW95] Zheng-Chao Han, Luen-Fai Tam, Andrejs Treibergs, and Tom Wan, *Harmonic maps from the complex plane into surfaces with nonpositive curvature*, Comm. Anal. Geom. **3** (1995), no. 1-2, 85–114. MR 1362649

[Lab07] François Labourie, *Flat projective structures on surfaces and cubic holomorphic differentials*, Pure Appl. Math. Q. **3** (2007), no. 4, Special Issue: In honor of Grigory Margulis. Part 1, 1057–1099. MR 2402597

[Li92] An Min Li, *Calabi conjecture on hyperbolic affine hyperspheres. II*, Math. Ann. **293** (1992), no. 3, 485–493. MR 1170522

[Li12] Peter Li, *Geometric analysis*, Cambridge Studies in Advanced Mathematics, vol. 134, Cambridge University Press, Cambridge, 2012. MR 2962229

[Li19] Qiongling Li, *An introduction to Higgs bundles via harmonic maps*, SIGMA Symmetry Integrability Geom. Methods Appl. **15** (2019), Paper No. 035, 30. MR 3947036

[LM20a] Qiongling Li and Takuro Mochizuki, *Complete solutions of Toda equations and cyclic higgs bundles over non-compact surfaces*, arXiv:2010.05401 (2020).

[LM20b] ———, *Isolated singularities of Toda equations and cyclic higgs bundles*, arXiv:2010.06129 (2020).

[Lof01] John C. Loftin, *Affine spheres and convex $\mathbb{RP}^n$-manifolds*, Amer. J. Math. **123** (2001), no. 2, 255–274. MR 1828223 (2002c:53018)

[LT93] Peter Li and Luen-Fai Tam, *Uniqueness and regularity of proper harmonic maps*, Ann. of Math. (2) **137** (1993), no. 1, 167–201. MR 1200080

[LT23] François Labourie and Jérémy Toulisse, *Quasicircles and quasiperiodic surfaces in pseudo-hyperbolic spaces*, Invent. Math. **233** (2023), no. 1, 81–168. MR 4601997

[LTW95] Peter Li, Luen-Fai Tam, and Jiaping Wang, *Harmonic diffeomorphisms between Hadamard manifolds*, Trans. Amer. Math. Soc. **347** (1995), no. 9, 3645–3658. MR 1308017

[LTW20] François Labourie, Jrmy Toulisse, and Michael Wolf, *Plateau problems for maximal surfaces in pseudo-hyperbolic spaces*, arXiv: 2006.12190 (2020).

[Mar17] Vladimir Markovic, *Harmonic maps and the Schoen conjecture*, J. Amer. Math. Soc. **30** (2017), no. 3, 799–817. MR 3630088

[Moc] Takuro Mochizuki, *Harmonic bundles and Toda lattices with opposite sign I*, to appear in RIMS Kôkyûroku Bessatsu (essentially the part I of arXiv:1301.1718).

[Moc14] ———, *Harmonic bundles and Toda lattices with opposite sign II*, Comm. Math. Phys. **328** (2014), no. 3, 1159–1198. MR 3201222

[Moc16] ———, *Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces*, J. Topol. **9** (2016), no. 4, 1021–1073. MR 3620459

[Nie22] Xin Nie, *Cyclic higgs bundles and minimal surfaces in pseudo-hyperbolic spaces*, arXiv:2206.13357 (2022).

[Nie23] ———, *Poles of cubic differentials and ends of convex $\mathbb{RP}^2$-surfaces*, J. Differential Geom. **123** (2023), no. 1, 67–140. MR 4564412

[Sch93] Richard M. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Complex geometry (Osaka, 1990), Lecture Notes in Pure and Appl. Math., vol. 143, Dekker, New York, 1993, pp. 179–200. MR 1201611

[SW93] U. Simon and C. P. Wang, *Local theory of affine 2-spheres*, Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 585–598. MR 1216648

[SY78] Richard Schoen and Shing Tung Yau, *On univalent harmonic maps between surfaces*, Invent. Math. **44** (1978), no. 3, 265–278. MR 0478219

[TW20] Andrea Tamburelli and Michael Wolf, *Planar minimal surfaces with polynomial growth in the* $\mathrm{Sp}(4,\mathbb{R})$*-symmetric space*, arXiv:2002.07295 (2020).

[WA94] Tom Yau-Heng Wan and Thomas Kwok-Keung Au, *Parabolic constant mean curvature spacelike surfaces*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 559–564. MR 1169052

[Wan91] Chang Ping Wang, *Some examples of complete hyperbolic affine 2-spheres in* $\mathbf{R}^3$, Global differential geometry and global analysis (Berlin, 1990), Lecture Notes in Math., vol. 1481, Springer, Berlin, 1991, pp. 271–280. MR 1178538

[Wan92] Tom Yau-Heng Wan, *Constant mean curvature surface, harmonic maps, and universal Teichmüller space*, J. Differential Geom. **35** (1992), no. 3, 643–657. MR 1163452

[Wol89] Michael Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **29** (1989), no. 2, 449–479. MR 982185 (90h:58023)

Song Dai, Center for Applied Mathematics, Tianjin University, No.92 Weijinlu Nankai District, Tianjin, P.R.China 300072

E-mail address: `song.dai@tju.edu.cn`

Qiongling Li, Chern Institute of Mathematics and LPMC, Nankai University, No. 94 Weijinlu Nankai District, Tianjin, P.R.China 300071

E-mail address: `qiongling.li@nankai.edu.cn`