Feedback law to stabilize linear infinite-dimensional systems*

Yaxing Ma† Gengsheng Wang‡ Huaiqiang Yu§

Abstract

We design a new feedback law to stabilize the linear infinite-dimensional control system, where the state operator generates a C_0-group and the control operator is unbounded. Our feedback law is based on the integration of a mutated Gramian operator-valued function. In the structure of the aforementioned mutated Gramian operator, we utilize the weak observability inequality in [21, 14] and borrow some idea used to construct generalized Gramian operators in [11, 23, 24]. Unlike most related works where the exact controllability is required, we only assume the above-mentioned weak observability inequality which is equivalent to the stabilizability of the system.

Keywords. Stabilizability, feedback law, unbounded control operator, weak observability inequality

2010 AMS Subject Classifications. 93B52, 93D05, 93D15

1 Introduction

1.1 Notation

Let $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{N} := \{0, 1, \ldots\}$ and $\mathbb{N}^+ := \{1, 2, \ldots\}$. Given a Hilbert space X, we use X', $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_X$ to denote its dual space, norm and inner product respectively; write $\langle\cdot,\cdot\rangle_{X,X'}$ for the dual product between X and X'; identify X'' (the dual space of X') with X; denote by $C(\mathbb{R}^+; X)$ the space of all continuous functions from $\mathbb{R}^+$ to X; write I for the identity operator on X. When L is a densely defined and closed linear operator on a Hilbert space X, we let $D(L) := \{x \in X : Lx \in X\}$ and $\|x\|_{D(L)} := (\|x\|_X^2 + \|Lx\|_X^2)^{\frac{1}{2}}$ $(x \in D(L))$, which are the domain of L and the graph norm on $D(L)$ respectively; use L^* to denote its adjoint operator, i.e., $\langle Lx, y\rangle_{X,X'} = \langle x, L^*y\rangle_{X,X'}$ $(x \in D(L), y \in D(L^*))$ (see [15, Chapter 1, Section 1.10]); use $\rho(L)$ to denote its resolvent set. When X_1 and X_2 are two Hilbert spaces, we write $\mathcal{L}(X_1; X_2)$ for the space of all linear and bounded operators from X_1 to X_2, and further write $\mathcal{L}(X_1) := \mathcal{L}(X_1; X_1)$. Given $F \in \mathcal{L}(X_1; X_2)$, we use $\|F\|_{\mathcal{L}(X_1;X_2)}$ and $F^* \in \mathcal{L}(X_2'; X_1')$ to denote its operator norm and adjoint operator respectively. We use $C(\cdots)$ to denote a constant that depends on what is enclosed in the brackets.

1.2 Motivation

Stabilization is one of the most important objectives in control theory. There are two important subjects on stabilization for linear control systems: The first one is to find sufficient conditions/equivalent conditions on stabilizability, such as resolvent conditions (see, for instance, [7, 8, 13, 16]) and weak observability inequalities (see [14, 21, 26]). The second one is to design feedback laws (see, for instance, [6, 10, 11, 12, 18, 19, 23, 24, 27]). For the latter, we would like to mention two usual methods: using Riccati equations (see, for instance, [6, 12, 27]); using Gramian operators (see, for instance, [10, 11, 18, 19, 23, 24]).

*This work was partially supported by the National Natural Science Foundation of China under grant 11971022.
†School of Mathematics, Tianjin University, Tianjin 300354, China (yaxingma@yeah.net).
‡Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China (wanggs@yeah.net).
§School of Mathematics, Tianjin University, Tianjin 300354, China (huaiqiangyu@tju.edu.cn, huaiqiangyu@yeah.net).

They all have their own advantages and disadvantages. We aim to design feedback laws by the way using Gramian operators.

It is well known that when matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ $(n, m \in \mathbb{N}^+)$ satisfy the Kalman controllability condition, the matrix $K = -B^\top G_T^{-1}$ (where $G_T := \int_0^T e^{-At}BB^\top e^{-A^\top t}dt$, with $T > 0$, is called a Gramian operator) is a feedback law stabilizing the system: $y'(t) = Ay(t) + Bu(t)$, $t \geq 0$. (See, for instance, [20, Chapter 5, Section 5.7].) Such idea has been extended to the infinite-dimensional settings (see, for instance, [3, 4, 11, 13, 17, 19, 23, 24]). To our best knowledge, all existing papers, which use Gramian operators to design feedback laws, need the following hypothesis (which is mainly used to ensure the invertibility of Gramian operators):

$(\widehat{H})$ The system is exactly controllable at some time $T > 0$.

However, when researching how to design feedback laws, it is more natural to use stabilizability instead of controllability as an assumption. Indeed, there are many systems which are not controllable but are stabilizable, even completely stabilizable (see examples in [20, Chapters 5] for finite-dimensional settings and in [9, 14, 21] for infinite-dimensional settings). On the other hand, the stabilizability of a linear control system is equivalent to some weak observability inequality for its dual system (see [14, 21]). *These motivate us to design feedback laws via modified Gramian operators, under assumption that the aforementioned weak observability inequality holds.*

1.3 System, hypotheses and definitions

System and hypotheses. Let H and U be two Hilbert spaces. Consider the control system:

$$x'(t) = Ax(t) + Bu(t), \ \ t > 0, \tag{1.1}$$

where $u \in L^2(\mathbb{R}^+; U)$ and the pair (A, B) verifies the following hypotheses:

(H_1) The linear operator $A : D(A) \subset H \to H$ is the generator of a C_0-group $S(\cdot)$ on H;

(H_2) The operator B belongs to the space $\mathcal{L}(U; D(A^*)')$;

(H_3) For any $T > 0$, there is a constant $C(T) > 0$ such that

$$\int_0^T \|B^*S^*(t)\varphi\|^2_{U'}dt \leq C(T)\|\varphi\|^2_{H'} \ \text{ for any } \ \varphi \in D(A^*). \tag{1.2}$$

Remark 1.1. *Several notes on the above hypotheses and the system are given.*

(i) In general, the method by using Gramian operators to design feedback laws works only for the linear control systems where the state operators generate C_0-groups. That is why we make the assumption (H_1).

(ii) The assumption (H_2) has appeared in many literature, where some specific examples were given. (See, for instance, [11, 12, 22]). From this assumption, we have $B^* \in \mathcal{L}(D(A^*); U')$. (Here, we identify $D(A^*)''$ with $D(A^*)$.) The latter is equivalent to the existence of $\lambda \in \mathbb{C}$ and $E \in \mathcal{L}(U; H)$ such that $B^* = E^*(\lambda I + A)^*$ (see, for instance, [22, 24]). Indeed, we can choose $\lambda \in \rho(-A)$ and set $E := (B^*((\lambda I + A)^{-1})^*)^*$. From now on, we fix $(\lambda, E) \in \mathbb{C} \times \mathcal{L}(U; H)$ satisfying $B^* = E^*(\lambda I + A)^*$.

(iii) The assumption (H_3) is called a regularity property in [4, 12] or an admissibility condition in [22]. With respect to this assumption, we have the following facts:

(iii_1) The condition (H_3) is equivalent to the existence of $T > 0$ and $C(T) > 0$ so that (1.2) holds (see [4, Chapter 2, Section 2.3]);

(iii_2) *If* (H_1)*-*(H_3) *are true, then for any* $T > 0$*, there is* $C(T) > 0$ *so that*

$$\int_{-T}^{T} \|B^* S^*(t)\varphi\|_{U'}^2 dt \le C(T)\|\varphi\|_{H'}^2 \quad \text{for any } \varphi \in D(A^*). \tag{1.3}$$

Consequently, the mapping $(\varphi \in D(A^*)) \to (t \to B^* S^*(t)\varphi) \in L^2_{loc}(\mathbb{R}; U')$ *can be extended, in a unique way, to a continuous operator, denoted by* $\widetilde{B^* S^*}(\cdot)$*, from* H' *to* $L^2_{loc}(\mathbb{R}; U')$ *(see [22, Chapter 4, Section 4.3]). Here,* $L^2_{loc}(\mathbb{R}; U')$ *is regarded as a Fréchet space with the seminorms:* $\{\|\cdot\|_{L^2(-n,n;U')} : n \in \mathbb{N}^+\}$*.*

(iv) *The solutions of (1.1) will be defined in the sense of transposition: a function* $x(\cdot) \in C(\mathbb{R}^+; H)$ *is called a solution of (1.1), with* $u \in L^2(\mathbb{R}^+; U)$ *and* $x(0) = x_0 \in H$*, if*

$$\langle x(t), \varphi\rangle_{H,H'} = \langle x_0, S^*(t)\varphi\rangle_{H,H'} + \int_0^t \langle u(s), B^* S^*(t-s)\varphi\rangle_{U,U'} ds \quad \text{for any } \varphi \in D(A^*),\ t \in \mathbb{R}^+.$$

It deserves mentioning that under (H_1)*-*(H_3)*, for any* $x_0 \in H$ *and* $u \in L^2(\mathbb{R}^+; U)$*, the system (1.1), with initial condition* $x(0) = x_0$*, has a unique solution (see, for instance, [11, Lemma 2.1]). We denote it by* $x(\cdot; x_0, u)$*.*

Stabilizability and weak observability inequality. We first recall that the operator A has a unique extension $\widetilde{A} \in \mathcal{L}(H; D(A^*)')$ (see [22, Proposition 2.10.4] or [24, Lemma 3.4]), which is defined by

$$\langle \varphi, A^*\psi\rangle_{H,H'} = \langle \widetilde{A}\varphi, \psi\rangle_{D(A^*)',D(A^*)}, \quad \varphi \in H, \psi \in D(A^*). \tag{1.4}$$

We now give the definition on the stabilizability for the system (1.1) which is quoted from [14].

Definition 1.2. *The system (1.1) is said to be exponentially stabilizable (stabilizable, for simplicity), if there is a constant* $\omega > 0$*, a* C_0*-semigroup* $\mathcal{S}(\cdot)$ *on* H *(with the generator* $\mathcal{A} : D(\mathcal{A}) \subset H \to H$*) and an operator* $K \in \mathcal{L}(D(\mathcal{A}); U)$ *so that*

(i) *there is a constant* $C_1 \ge 1$ *such that* $\|\mathcal{S}(t)\|_{\mathcal{L}(H)} \le C_1 e^{-\omega t}$ *for all* $t \in \mathbb{R}^+$*;*

(ii) *for any* $x \in D(\mathcal{A})$*,* $\mathcal{A}x = (\widetilde{A} + BK)x$*, with* $\widetilde{A}$ *given by (1.4);*

(iii) *there is a constant* $C_2 \ge 0$ *so that* $\|K\mathcal{S}(\cdot)x\|_{L^2(\mathbb{R}^+;U)} \le C_2\|x\|_H$ *for any* $x \in D(\mathcal{A})$*.*

The above K *and* ω *are called respectively a feedback law and a stabilization decay rate (a decay rate, for simplicity). When the above* ω*,* $\mathcal{S}(\cdot)$ *and* K *exist, we also say that* K *is a feedback law stabilizing the system (1.1) with the decay rate* ω*.*

Remark 1.3. *Several notes on Definition 1.2 are as follows:*

(i) *The above definition comes originally from [6] which shows that the finite cost condition of the LQ problem:* $\inf_{u\in L^2(\mathbb{R}^+;U)} \int_0^\infty [\|x(t; x_0, u)\|_H^2 + \|u(t)\|_U^2]dt$ *implies the stabilizability in the sense of Definition 1.2. It deserves mentioning that the above stabilizability is equivalent to the above finite cost condition (see [14, Proposition 3.9]).*

(ii) *In general,* $\widetilde{A} + BK$ *is not the generator of the semigroup* $\mathcal{S}(\cdot)$*, except for the case* $B \in \mathcal{L}(U; H)$*. This operator is only a densely defined restriction of such a generator (see [5, 6, 11, 12, 25]). The detailed explanation is given in the proof of our main theorem.*

(iii) *It was proved in [21, Section 3.3] (see also [14, Theorem 3.4] for the complete stabilizability) that the stabilizability of the system (1.1) is equivalent to the following weak observability inequality for the dual system of (1.1):*

There exists $\delta \in (0,1)$, $T>0$ *and* $C(\delta,T)\geq 0$ *such that*

$$\|S^*(T)\varphi\|^2_{H'} \leq C(\delta,T)\int_0^T \|B^*S^*(s)\varphi\|^2_{U'}ds + \delta\|\varphi\|^2_{H'} \text{ for any } \varphi\in D(A^*). \tag{1.5}$$

On the other hand, (1.5) *is equivalent to what follows (see Proposition 5.1 in Section 5.1):*
There exists $\alpha>0$, $C_1(\alpha)\geq 0$ *and* $C_2(\alpha)\geq 1$ *such that*

$$\|S^*(t)\varphi\|^2_{H'} \leq C_1(\alpha)\int_0^t \|B^*S^*(s)\varphi\|^2_{U'}ds + C_2(\alpha)e^{-\alpha t}\|\varphi\|^2_{H'} \text{ for any } t>0,\ \varphi\in D(A^*), \tag{1.6}$$

which is also called a weak observability inequality.

Inspired by the note (iii) in Remark 1.3, we further make the following hypothesis:

(H_4) There exists $\alpha>0$, $C_1(\alpha)\geq 0$ and $C_2(\alpha)\geq 1$ such that (1.6) holds.

1.4 Main result

To state our main result, we need to introduce some operators. First of all, we let $J_1: U'\to U$ and $J_2: H'\to H$ be the canonical isomorphisms given by Riesz-Fréchet representation theorem (see, for instance, [2, Chapter 5, Section 5.2]). It should be noticed that J_1 and J_2 are conjugate-linear operators. Next, we let $\alpha>0$, $C_1(\alpha)\geq 0$ and $C_2(\alpha)\geq 1$ be given in (H_4). Now, for any $\varepsilon\in[0,\alpha)$, $T>0$ and $t\in[0,T]$, we define an operator $\Lambda_{\alpha,\varepsilon,T}(t): H'\to H$ by

$$\begin{aligned}\langle \Lambda_{\alpha,\varepsilon,T}(t)\varphi,\psi\rangle_{H,H'} &= C_1(\alpha)e^{\alpha T}\int_0^t e^{-(\alpha-\varepsilon)s}\langle J_1\widetilde{B^*S^*}(-s)\varphi,\widetilde{B^*S^*}(-s)\psi\rangle_{U,U'}ds\\ &\quad + C_2(\alpha)e^{-(\alpha-\varepsilon)t}\langle J_2S^*(-t)\varphi,S^*(-t)\psi\rangle_{H,H'} \text{ for any } \varphi,\psi\in H',\end{aligned} \tag{1.7}$$

where $\widetilde{B^*S^*}(\cdot)$ is given in (iii_2) of Remark 1.1, and then define another operator $\Pi_{\alpha,\varepsilon,T}: H'\to H$ via

$$\langle \Pi_{\alpha,\varepsilon,T}\varphi,\psi\rangle_{H,H'} := \int_0^T \langle\Lambda_{\alpha,\varepsilon,T}(t)\varphi,\psi\rangle_{H,H'}dt \text{ for any } \varphi,\psi\in H'. \tag{1.8}$$

It is clear that both $\Lambda_{\alpha,\varepsilon,T}(t)$ and $\Pi_{\alpha,\varepsilon,T}$ are conjugate-linear. Moreover, we can show that $\Lambda_{\alpha,\varepsilon,T}(t)$ and $\Pi_{\alpha,\varepsilon,T}$ are bounded, and $\Pi^{-1}_{\alpha,\varepsilon,T}$ exists (see Lemma 2.1). Finally, for each T satisfying

$$T\in\mathcal{I}_\alpha := (\alpha^{-1}\ln[C_2(\alpha)],+\infty), \tag{1.9}$$

we write $\widehat{\varepsilon} := T^{-1}\ln[C_2(\alpha)]$, and then define an operator $K_T: \Pi_{\alpha,\widehat{\varepsilon},T}[D(A^*)]\to U$ via

$$K_T := -TC_1(\alpha)e^{\alpha T}J_1B^*\Pi^{-1}_{\alpha,\widehat{\varepsilon},T}. \tag{1.10}$$

The main result of this paper is as follows:

Theorem 1.4. *Assume that* (H_1)*-*(H_4) *are true. Then for each* T *satisfying (1.9), the operator* K_T*, defined by* (1.10)*, is a feedback law stabilizing the system* (1.1) *with the decay rate* $\frac{1}{2}\left(\alpha - T^{-1}\ln[C_2(\alpha)]\right)$.

Remark 1.5. *Some notes on Theorem 1.4 are given.*

- (i) *Theorem 1.4 gives a family of feedback laws* $\{K_T\}_{T\in\mathcal{I}_\alpha}$ *stabilizing the system* (1.1)*, and the decay rate corresponding to each* K_T *has an explicit expression. All coefficients in the weak observability inequality* (1.6) *appear in the expression of* K_T.
- (ii) *In* (1.10)*, we only need* $\Pi^{-1}_{\alpha,\varepsilon,T}$ *with* $\varepsilon := T^{-1}\ln[C_2(\alpha)]$*, but in the proof of the main theorem, we will use the family* $\{\Pi^{-1}_{\alpha,\varepsilon,T}\}_{\varepsilon\in[0,\alpha)}$.

(*iii*) *We now explain our design of the feedback law K_T as follows: First, based on the weak observability inequality in (H_4), we define an operator $\Lambda_{\alpha,\varepsilon,T}(t)$ (given by (1.7)), which can be treated as a kind of mutated Gramian operator. Thus, when ε is fixed, $t \to \Lambda_{\alpha,\varepsilon,T}(t)$, $t \in [0,T]$, is a mutated Gramian operator-valued function. Second, the operator $\Pi_{\alpha,\varepsilon,T}$ (given by (1.8)) can be viewed as the integration of the aforementioned function. Thus, each $\Lambda_{\alpha,\varepsilon,T}(t)$ can be treated as a slice of $\Pi_{\alpha,\varepsilon,T}$. Third, the feedback K_T (given by (1.10)) is built up with the aid of $\Pi_{\alpha,\varepsilon,T}$.*

It deserves mentioning the following two points: First, our structure is based on the assumption (H_4) quantitatively. However, the feedback laws given in [11, 18, 19, 23, 24] depend on the assumption of the exact controllability of (1.1) qualitatively. (The latter will be explained in more detail in the next subsection.) Second, unlike works [11, 18, 19, 23, 24], we are not able to design a feedback law by only one slice $\Lambda_{\alpha,\varepsilon,T}(t)$. The main reason is that our assumption (H_4) is weaker than the assumption of the observability used in [11, 18, 19, 23, 24].

(*iv*) *By the proof of Theorem 1.4, we would obtain a more general result (see Theorem 3.1).*

(*v*) *The family $\{K_T\}_{T\in\mathcal{I}_\alpha}$ gives an approximate decay rate $\alpha/2$, where α has been fixed in (H_4). Thus, it seems that our way to design feedback law can only give a fixed decay rate. Fortunately, this is not true. Indeed, we will show, in Section 4, what follows: For each $\mu \in (0,\omega^*)$, where*

$$\omega^* := \sup\{\omega \in \mathbb{R}^+ : \textit{the system (1.1) is stabilizable with decay rate } \omega\}, \tag{1.11}$$

we can use our way to design a feedback law stabilizing the system (1.1) with the decay rate μ (see Theorem 4.2).

1.5 Novelty and comparison with related works

For the studies relevant to our current work, we recall the main results in [11, 23, 24]. The papers [11, 24] build up, for each $\omega > 0$, a generalized Gramian operator $G_{T,\omega} : H \to H'$ (with $T > 0$) via

$$\langle G_{T,\omega}\varphi,\psi\rangle_{H,H'} := \int_0^{T_\omega} e_\omega(s)\langle J_1\widetilde{B^*S^*}(-s)\varphi, \widetilde{B^*S^*}(-s)\psi\rangle_{U,U'}ds \ \text{ for any } \ \varphi,\psi \in H', \tag{1.12}$$

where $T_\omega := T + (2\omega)^{-1}$ and

$$e_\omega(s) := \begin{cases} e^{-2\omega s}, & \text{if } \ s \in [0,T], \\ 2\omega e^{-2\omega T}(T_\omega - s), & \text{if } \ s \in [T,T_\omega], \end{cases}$$

and prove that $K := -J_1B^*G_{T,\omega}^{-1}$ is a feedback law stabilizing the system (1.1) with the decay rate ω, where $G_{T,\omega}$ satisfies a Riccati equation. The paper [23] designs, for each $\omega > 0$ large enough, a generalized Gramian operator (which is originally from [18] for some finite-dimensional systems):

$$\langle G_\omega\varphi,\psi\rangle_{H,H'} := \int_0^\infty e^{-2\omega s}\langle J_1\widetilde{B^*S^*}(-s)\varphi, \widetilde{B^*S^*}(-s)\psi\rangle_{U,U'}ds \ \text{ for any } \ \varphi,\psi \in H', \tag{1.13}$$

and shows that $K := -J_1B^*G_\omega^{-1}$ is a feedback law stabilizing the system (1.1) with the decay rate $(2\omega - g(-A))$ (where $g(-A) := \inf_{t>0}\frac{1}{t}\ln\|S(-t)\|_{\mathcal{L}(H)}$), where G_ω satisfies a Lyapunov equation.

In the above-mentioned papers [11, 23, 24], the assumption $\widehat{(H)}$ (i.e., the system (1.1) is exactly controllable at some time $T > 0$) is necessary to ensure the invertibility of the above generalized Gramian operators, while the corresponding observability inequality is not fully utilized, more precisely, the coefficients in the observability inequality does not appear in the design of the feedback laws. Besides, either $G_{T,\omega}$ or G_ω corresponds to a slice $\Lambda_{\alpha,\varepsilon,T}(t)$.

The novelties of this paper are as follows:

- Our assumption (H_4) is more natural and weaker than the above-mentioned $\widehat{(H)}$. Since (H_4) cannot ensure the invertibility of the Gramian operators $G_{T,\omega}$ and G_ω (given by (1.12) and (1.13) respectively), the method to design feedback laws in [11, 23, 24] does not work for our case.
- Our method to design feedback laws seems to be new from two perspective as follows: First, we replace the generalized Gramian operator (in [11, 23, 24]) with the integration of a mutated Gramian operator-valued function. It deserves mentioning that though each slice $\Lambda_{\alpha,\varepsilon,T}(t)$ is invertible (see Lemma 2.1), it does not work to replace $\Pi_{\alpha,\varepsilon,T}$ by one slice $\Lambda_{\alpha,\varepsilon,T}(t)$ in (1.10). Second, we use all information of the weak observability inequality.
- From perspective of stability, our design for feedback laws is reasonable in the sense: When $S(\cdot)$ is stable, i.e., for some $\omega > 0$ and $\widehat{C}(\omega) > 0$, $\|S(t)\|_{\mathcal{L}(H)} \le \widehat{C}(\omega)e^{-\omega t}$ for all $t \in \mathbb{R}^+$, the feedback law should be 0. This is consistent with our design. Indeed, in this case, we have (H_4) with $\alpha = 2\omega$, $C_1(\alpha) = 0$ and $C_2(\alpha) = (\widehat{C}(\omega))^2$), which, along with (1.10), gives $K_T = 0$.

1.6 Plan of this paper

The rest of the paper is organized as follows: Section 2 shows some preliminaries; Section 3 proves the main result; Section 4 presents further studies; Section 5 is appendix.

2 Preliminaries

In this section, we suppose that (H_1)-(H_4) hold and let $\alpha > 0$, $C_1(\alpha) \ge 0$ and $C_2(\alpha) \ge 1$ be given in (H_4).

Lemma 2.1. *Given $T > 0$ and $\varepsilon \in [0, \alpha)$. Let the operators $\Lambda_{\alpha,\varepsilon,T}(t)$ (with $t \in [0, T]$) and $\Pi_{\alpha,\varepsilon,T}$, be defined by (1.7) and (1.8) respectively. Then, the following statements hold:*

(i) There is $C_0(T) > 0$ so that $\|\Lambda_{\alpha,\varepsilon,T}(t)\|_{\mathcal{L}(H';H)} \le C_0(T)$ for all $t \in [0, T]$;

(ii) The operator $\Pi_{\alpha,\varepsilon,T}$ is bounded;

(iii) Both $\Lambda_{\alpha,\varepsilon,T}(t)$ and $\Pi_{\alpha,\varepsilon,T}$ are invertible. Moreover,

$$\langle \Lambda_{\alpha,\varepsilon,T}(t)\varphi, \varphi\rangle_{H,H'} \ge e^{\varepsilon t}\|\varphi\|^2_{H'} \text{ and } \langle \Pi_{\alpha,\varepsilon,T}\varphi, \varphi\rangle_{H,H'} \ge T\|\varphi\|^2_{H'} \text{ for all } \varphi \in H'. \tag{2.1}$$

Proof. Arbitrarily fix $T > 0$ and $\varepsilon \in [0, \alpha)$. We begin with proving (i). Arbitrarily fix $\varphi, \psi \in D(A^*)$ and $t \in [0, T]$. Since $\widetilde{B^*S^*}(\cdot) = B^*S^*(\cdot)$ on $D(A^*)$ (see the note (iii_2) in Remark 1.1), it follows from (1.7) and (1.3) that

$$\begin{aligned} & |\langle \Lambda_{\alpha,\varepsilon,T}(t)\varphi, \psi\rangle_{H,H'}| \\ \le\ & C_1(\alpha)e^{\alpha T}\Big(\int_0^T \|B^*S^*(-s)\varphi\|^2_{U'}ds\Big)^{\frac{1}{2}}\Big(\int_0^T \|B^*S^*(-s)\psi\|^2_{U'}ds\Big)^{\frac{1}{2}} + C_2(\alpha)\|S^*(-t)\varphi\|_{H'}\|S^*(-t)\psi\|_{H'} \\ \le\ & \Big(C_1(\alpha)e^{\alpha T}C(T) + C_2(\alpha)\Big(\sup_{s\in[0,T]} \|S^*(-s)\|_{\mathcal{L}(H';H')}\Big)^2\Big)\|\varphi\|_{H'}\|\psi\|_{H'}. \end{aligned}$$

This, along with the density of $D(A^*)$ in H', leads to (i) with

$$C_0(T) := C_1(\alpha)e^{\alpha T}C(T) + C_2(\alpha)\big(\sup_{s\in[0,T]} \|S^*(-s)\|_{\mathcal{L}(H';H')}\big)^2.$$

To show (ii), we arbitrarily fix $\varphi, \psi \in H'$. It follows from (1.7) that the function $t \to \langle \Lambda_{\alpha,\varepsilon,T}(t)\varphi, \psi\rangle_{H,H'}$, $t \in [0, T]$, is continuous, so is integrable. This, along with (1.8) and the property (i) in this lemma, yields

$$|\langle \Pi_{\alpha,\varepsilon,T}\varphi, \psi\rangle_{H,H'}| \le TC_0(T)\|\varphi\|_{H'}\|\psi\|_{H'},$$

which leads to (ii).

We now prove (iii). Because $S(\cdot)$ is a group (see (H_1)), the inequality (1.6) (which is true by (H_4)) is equivalent to

$$\|\varphi\|_{H'}^2 \leq C_1(\alpha)\int_0^t \|B^*S^*(-s)\varphi\|_{U'}^2 ds + C_2(\alpha)e^{-\alpha t}\|S^*(-t)\varphi\|_{H'}^2 \text{ for any } t\in\mathbb{R}^+,\ \varphi\in D(A^*).$$

It follows that when $\varepsilon\in[0,\alpha)$ and $t\in[0,T]$,

$$e^{\varepsilon t}\|\varphi\|_{H'}^2 \leq C_1(\alpha)e^{\alpha T}\int_0^t e^{-(\alpha-\varepsilon)s}\|B^*S^*(-s)\varphi\|_{U'}^2 ds + C_2(\alpha)e^{-(\alpha-\varepsilon)t}\|S^*(-t)\varphi\|_{H'}^2 \text{ for any } \varphi\in D(A^*),$$

which, together with (1.7), yields the first estimate in (2.1) with $\varphi\in D(A^*)$. This, along with the density of $D(A^*)$ in H', shows the first estimate in (2.1) with $\varphi\in H'$. Next, the second estimate in (2.1) follows from the first one and (1.8). Finally, it follows from (2.1), the claims (i) and (ii), and the Lax-Milgram theorem that both $\Lambda_{\alpha,\varepsilon,T}(t)$ and $\Pi_{\alpha,\varepsilon,T}$ are invertible.

Hence, we complete the proof of Lemma 2.1. □

Proposition 2.2. *Let $T>0$ and $\varepsilon\in[0,\alpha)$. Then the following conclusions are true:*

(i) Let the operator $\Pi_{\alpha,\varepsilon,T}$ be given by (1.8). If let $\mathcal{X}:=\Pi_{\alpha,\varepsilon,T}$, then $\mathcal{X}$ is a solution of the following Lyapunov equation:

$$\begin{aligned}&\langle\mathcal{X}A^*\varphi,\psi\rangle_{H,H'}+\langle\mathcal{X}\varphi,A^*\psi\rangle_{H,H'}-TC_1(\alpha)e^{\alpha T}\langle J_1B^*\varphi,B^*\psi\rangle_{U,U'}\\ =\ &-(\alpha-\varepsilon)\langle\mathcal{X}\varphi,\psi\rangle_{H,H'}-\langle Q_{\alpha,\varepsilon,T}\varphi,\psi\rangle_{H,H'} \text{ for any } \varphi,\psi\in D(A^*),\end{aligned}\tag{2.2}$$

where the bounded operator $Q_{\alpha,\varepsilon,T}$ is defined by

$$Q_{\alpha,\varepsilon,T}:=\Lambda_{\alpha,\varepsilon,T}(T)-C_2(\alpha)J_2,\quad \text{with } \Lambda_{\alpha,\varepsilon,T}(T) \text{ given by (1.7)}.\tag{2.3}$$

(ii) When (ε,T) verifies

$$\begin{cases}\varepsilon\in(0,\alpha) \text{ and } T\geq\varepsilon^{-1}\ln[C_2(\alpha)], & \text{if } C_2(\alpha)>1,\\ \varepsilon\in[0,\alpha) \text{ and } T>0, & \text{if } C_2(\alpha)=1,\end{cases}\tag{2.4}$$

the operator $Q_{\alpha,\varepsilon,T}$, given by (2.3), is non-negative in the sense of $\langle Q_{\alpha,\varepsilon,T}\varphi,\varphi\rangle_{H,H'}\geq 0$ for any $\varphi\in H'$.

Proof. We begin with showing (i). Let

$$\widehat{A}_{\alpha,\varepsilon}:=A^*+\frac{1}{2}(\alpha-\varepsilon)I,\ \text{ with } D(\widehat{A}_{\alpha,\varepsilon})=D(A^*).\tag{2.5}$$

Write $\widehat{S}_{\alpha,\varepsilon}(\cdot)$ for the C_0-group generated by $\widehat{A}_{\alpha,\varepsilon}$. Two observations are given in order: First, since $\widetilde{B^*S^*}(\cdot)=B^*S^*(\cdot)$ on $D(A^*)$ (see the note (iii_2) in Remark 1.1), it follows by (1.7) that

$$\begin{aligned}\langle\Lambda_{\alpha,\varepsilon,T}(t)\varphi,\psi\rangle_{H,H'} &= C_1(\alpha)e^{\alpha T}\int_0^t\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{U,U'}\\ &\quad +C_2(\alpha)\langle J_2\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{H,H'},\ \varphi,\psi\in D(A^*).\end{aligned}\tag{2.6}$$

Second, by the note (ii) of Remark 1.1 and the first observation above, we see that when $\varphi,\psi\in D((A^*)^2)$ and $s\in[0,T]$,

$$\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{H,H'}=\langle J_1E^*\widehat{S}_{\alpha,\varepsilon}(-s)(\lambda I+A)^*\varphi,E^*\widehat{S}_{\alpha,\varepsilon}(-s)(\lambda I+A)^*\psi\rangle_{H,H'},$$

from which, it follows that for any $\varphi,\psi\in D((A^*)^2)$, the function $s\to\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{H,H'}$ is continuously differentiable over $[0,T]$.

We now arbitrarily fix $\varphi,\psi\in D((A^*)^2)$. On the one hand, by the second observation above, we find

$$\begin{aligned}
& C_1(\alpha)e^{\alpha T}\int_0^t\frac{d}{ds}\Big(\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{U,U'}\Big)ds\\
= \;& C_1(\alpha)e^{\alpha T}\Big(\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{U,U'}-\langle J_1B^*\varphi,B^*\psi\rangle_{U,U'}\Big),\ t\in[0,T].
\end{aligned}$$

On the other hand, it follows from (2.6) that for each $t\in[0,T]$,

$$\begin{aligned}
& C_1(\alpha)e^{\alpha T}\int_0^t\frac{d}{ds}\Big(\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{U,U'}\Big)ds\\
= \;& -C_1(\alpha)e^{\alpha T}\bigg(\int_0^t\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\widehat{A}_{\alpha,\varepsilon}\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{U,U'}ds\\
& +\int_0^t\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-s)\widehat{A}_{\alpha,\varepsilon}\psi\rangle_{U,U'}ds\bigg)\\
= \;& -\langle\Lambda_{\alpha,\varepsilon,T}(t)\widehat{A}_{\alpha,\varepsilon}\varphi,\psi\rangle_{H,H'}-\langle\Lambda_{\alpha,\varepsilon,T}(t)\varphi,\widehat{A}_{\alpha,\varepsilon}\psi\rangle_{H,H'}\\
& +C_2(\alpha)\Big(\langle J_2\widehat{A}_{\alpha,\varepsilon}\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{H,H'}+\langle J_2\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,\widehat{A}_{\alpha,\varepsilon}\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{H,H'}\Big)\\
= \;& -\langle\Lambda_{\alpha,\varepsilon,T}(t)\widehat{A}_{\alpha,\varepsilon}\varphi,\psi\rangle_{H,H'}-\langle\Lambda_{\alpha,\varepsilon,T}(t)\varphi,\widehat{A}_{\alpha,\varepsilon}\psi\rangle_{H,H'}\\
& -C_2(\alpha)\left[\frac{d}{ds}\Big(\langle J_2\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{H,H'}\Big)\right]_{s=t}.
\end{aligned}$$

The above two equalities imply that for each $t\in[0,T]$,

$$\begin{aligned}
& C_1(\alpha)e^{\alpha T}\Big(\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{U,U'}-\langle J_1B^*\varphi,B^*\psi\rangle_{U,U'}\Big)\\
= \;& -\langle\Lambda_{\alpha,\varepsilon,T}(t)\widehat{A}_{\alpha,\varepsilon}\varphi,\psi\rangle_{H,H'}-\langle\Lambda_{\alpha,\varepsilon,T}(t)\varphi,\widehat{A}_{\alpha,\varepsilon}\psi\rangle_{H,H'}\\
& -C_2(\alpha)\left[\frac{d}{ds}\Big(\langle J_2\widehat{S}_{\alpha,\varepsilon}(-s)\varphi,\widehat{S}_{\alpha,\varepsilon}(-s)\psi\rangle_{H,H'}\Big)\right]_{s=t}.
\end{aligned}$$

Integrating the above equality with respect to t over $[0,T]$ and using (1.8), we obtain

$$\begin{aligned}
& \langle\Pi_{\alpha,\varepsilon,T}\widehat{A}_{\alpha,\varepsilon}\varphi,\psi\rangle_{H,H'}+\langle\Pi_{\alpha,\varepsilon,T}\varphi,\widehat{A}_{\alpha,\varepsilon}\psi\rangle_{H,H'}-TC_1(\alpha)e^{\alpha T}\langle J_1B^*\varphi,B^*\psi\rangle_{U,U'}\\
= \;& -C_1(\alpha)e^{\alpha T}\int_0^T\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{U,U'}dt\\
& -C_2(\alpha)\int_0^T\frac{d}{dt}\Big(\langle J_2\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{H,H'}\Big)dt.
\end{aligned}\tag{2.7}$$

Meanwhile, it follows from (2.6) that

$$\begin{aligned}
& C_2(\alpha)\int_0^T\frac{d}{dt}\Big(\langle J_2\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{H,H'}\Big)dt\\
= \;& C_2(\alpha)\langle J_2\widehat{S}_{\alpha,\varepsilon}(-T)\varphi,\widehat{S}_{\alpha,\varepsilon}(-T)\psi\rangle_{H,H'}-C_2(\alpha)\langle J_2\varphi,\psi\rangle_{H,H'}\\
= \;& \langle\Lambda_{\alpha,\varepsilon,T}(T)\varphi,\psi\rangle_{H,H'}-C_1(\alpha)e^{\alpha T}\int_0^T\langle J_1B^*\widehat{S}_{\alpha,\varepsilon}(-t)\varphi,B^*\widehat{S}_{\alpha,\varepsilon}(-t)\psi\rangle_{U,U'}dt-C_2(\alpha)\langle J_2\varphi,\psi\rangle_{H,H'}.
\end{aligned}$$

Replacing the above equality to (2.7), we get

$$\langle\Pi_{\alpha,\varepsilon,T}\widehat{A}_{\alpha,\varepsilon}\varphi,\psi\rangle_{H,H'}+\langle\Pi_{\alpha,\varepsilon,T}\varphi,\widehat{A}_{\alpha,\varepsilon}\psi\rangle_{H,H'}-TC_1(\alpha)e^{\alpha T}\langle J_1B^*\varphi,B^*\psi\rangle_{U,U'}$$

$$= \quad C_2(\alpha)\langle J_2\varphi,\psi\rangle_{H,H'} - \langle \Lambda_{\alpha,\varepsilon,T}(T)\varphi,\psi\rangle_{H,H'},$$

which, together with (2.5) and (2.3), shows that $\Pi_{\alpha,\varepsilon,T}$ verifies the equation (2.2), with $\varphi,\psi\in D((A^*)^2)$. This, along with the density of $D((A^*)^2)$ in $D(A^*)$, shows that $\mathcal{X} := \Pi_{\alpha,\varepsilon,T}$ is a solution of the equation (2.2).

We next show (ii). Indeed, in the case that $C_2(\alpha) > 1$, we see from (2.1) that for any $\varepsilon\in(0,\alpha)$ and $T\geq \varepsilon^{-1}\ln[C_2(\alpha)]$,

$$\langle Q_{\alpha,\varepsilon,T}\xi,\xi\rangle_{H,H'} = \langle \Lambda_{\alpha,\varepsilon,T}(T)\xi,\xi\rangle_{H,H'} - C_2(\alpha)\|\xi\|^2_{H'} \geq (e^{\varepsilon T} - C_2(\alpha))\|\xi\|^2_{H'} \geq 0,\ \ \xi\in H',$$

while in the case that $C_2(\alpha)=1$, we use (2.1) to get that for any $\varepsilon\in[0,\alpha)$ and $T>0$,

$$\langle Q_{\alpha,\varepsilon,T}\xi,\xi\rangle_{H,H'} = \langle \Lambda_{\alpha,\varepsilon,T}(T)\xi,\xi\rangle_{H,H'} - \|\xi\|^2_{H'} \geq 0,\ \ \xi\in H'.$$

These imply that $Q_{\alpha,\varepsilon,T}$, with (ε,T) satisfying (2.4), is non-negative.

Thus we finish the proof of Proposition 2.2. □

Remark 2.3. *First, in the proof of Proposition 2.2, we used the weak observability inequality in (H_4). Second, in the proof of Theorem 1.4, Proposition 2.2 plays an important role. Third, in the proof of Theorem 1.4, we also borrowed another idea, which was widely used in the related works (see, for instance, [6, 5, 24]) and can be explained as follows: By Proposition 2.2, $\Pi_{\alpha,\varepsilon,T}$ satisfies the Laypunov equation (2.2), which can be written formally as*

$$\Pi^{-1}_{\alpha,\varepsilon,T}\left(A - TC_1(\alpha)e^{\alpha T}BJ_1B^*\Pi^{-1}_{\alpha,T}\right)\Pi_{\alpha,\varepsilon,T} = -A^* - P_{\alpha,\varepsilon,T},$$

where

$$P_{\alpha,\varepsilon,T} := (\alpha-\varepsilon)I + \Pi^{-1}_{\alpha,\varepsilon,T}Q_{\alpha,\varepsilon,T}. \tag{2.8}$$

(The existence of $\Pi^{-1}_{\alpha,\varepsilon,T}$ is ensured by (iii) of Lemma 2.1.) In this sense, the operators $-A^*-P_{\alpha,\varepsilon,T}$ and $A-TC_1(\alpha,T)BJ_1B^*\Pi^{-1}_{\alpha,T}$ are "conjugated" each other. Thus one can obtain a C_0-group on H generated by $A-TC_1(\alpha)e^{\alpha T}BJ_1B^*\Pi^{-1}_{\alpha,T}$ formally, through using the C_0-group on H' generated by $-A^*-P_{\alpha,\varepsilon,T}$.

Now back to our case. Write $\mathcal{V}_{\alpha,\varepsilon,T}(\cdot)$ for the C_0-group on H', generated by $\Delta_{\alpha,\varepsilon,T} := -A^*-P_{\alpha,\varepsilon,T}$, with its domain $D(\Delta_{\alpha,\varepsilon,T})$ which is the same as $D(A^*)$. Here, we notice that $P_{\alpha,\varepsilon,T}\in\mathcal{L}(H')$. Then by the constant variation formula, we have

$$\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi = S^*(-t)\varphi - \int_0^t S^*(s-t)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi ds \ \text{ for any } t\in\mathbb{R},\ \varphi\in H'. \tag{2.9}$$

The next two lemmas will be used in the proof Theorem 1.4. For the first one, we did not find any exact version in published papers, while for the second one, a similar result was given in [24, Lemma 3.3], however, in its proof, there are some places that we do not understand. So we give their proofs in Section 5.

Lemma 2.4. *Given $\gamma>0$, $M\in\mathcal{L}(H')$ and $\varphi\in D(A^*)$, let $w(t;\varphi) := \int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi ds$, $t\in[-\gamma,\gamma]$. Then the following conclusions are true:*

(i) For any $t\in[-\gamma,\gamma]$, $w(t;\varphi)\in D(A^*)$;

(ii) There is a constant $C(\gamma)>0$ (independent of M and φ but depending on γ) so that

$$\int_{-\gamma}^{\gamma}\|B^*w(t;\varphi)\|^2_{U'}dt \leq C(\gamma)\Big(\|M\varphi\|^2_{H'} + \int_{-\gamma}^{\gamma}\big(\|M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi\|^2_{H'} + \|M\mathcal{V}_{\alpha,\varepsilon,T}(t)\Delta_{\alpha,\varepsilon,T}\varphi\|^2_{H'}\big)dt\Big). \tag{2.10}$$

Lemma 2.5. *For any $\varphi,\psi\in D(A^*)$ and $t\in\mathbb{R}$,*

$$\begin{aligned}\langle \Pi_{\alpha,\varepsilon,T}\varphi,\psi\rangle_{H,H'} &= \langle \Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi, S^*(-t)\psi\rangle_{H,H'}\\ &\quad + TC_1(\alpha)e^{\alpha T}\int_0^t\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, B^*S^*(-s)\psi\rangle_{U,U'}ds.\end{aligned} \tag{2.11}$$

3 Proof of main theorem

This section is devoted to prove Theorem 1.4.

Proof of Theorem 1.4. Arbitrarily fix $(\varepsilon,T)\in[0,\alpha)\times(0,+\infty)$ satisfying (2.4). Let $\mathcal{V}_{\alpha,\varepsilon,T}(\cdot)$ be the C_0-group on H' generated by $-A^*-P_{\alpha,\varepsilon,T}$, where $P_{\alpha,\varepsilon,T}$ is given by (2.8). Define

$$\mathcal{S}_{\alpha,\varepsilon,T}(t):=\Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\Pi_{\alpha,\varepsilon,T}^{-1},\quad t\in\mathbb{R}. \tag{3.1}$$

(The invertibility of $\Pi_{\alpha,\varepsilon,T}$ is ensured by (iii) of Lemma 2.1.) The rest of the proof is organized in several steps.

Step 1. We have the following conclusions:

(a_1) *The family $\{\mathcal{S}_{\alpha,\varepsilon,T}(t)\}_{t\in\mathbb{R}}$, given by (3.1), is a C_0-group on H;*

(a_2) *The generator of $\mathcal{S}_{\alpha,\varepsilon,T}(\cdot)$ is as: $\mathcal{A}_{\alpha,\varepsilon,T}:=\Pi_{\alpha,\varepsilon,T}(-A^*-P_{\alpha,\varepsilon,T})\Pi_{\alpha,\varepsilon,T}^{-1}$, with its domain $D(\mathcal{A}_{\alpha,\varepsilon,T})=\Pi_{\alpha,\varepsilon,T}[D(A^*)]$;*

(a_3) *For any $t\in\mathbb{R}^+$, $x\in D(\mathcal{A}_{\alpha,\varepsilon,T})$ and $\varphi\in D(A^*)$,*

$$\langle\mathcal{S}_{\alpha,\varepsilon,T}(t)x,\varphi\rangle_{H,H'}=\langle x,S^*(t)\varphi\rangle_{H,H'}-TC_1(\alpha)e^{\alpha T}\int_0^t\langle J_1B^*\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(s)x,B^*S^*(t-s)\psi\rangle_{U,U'}ds.$$

These can be proved by very similar methods used in the proof of [24, Theorem 3.2]. We omit the proofs.

Step 2. We have

$$(\widetilde{A}-TC_1(\alpha)e^{\alpha T}BJ_1B^*\Pi_{\alpha,\varepsilon,T}^{-1})x=\mathcal{A}_{\alpha,\varepsilon,T}x\ \ \textit{for all}\ \ x\in D(\mathcal{A}_{\alpha,\varepsilon,T}),$$

where $\widetilde{A}\in\mathcal{L}(H;D(A^*)')$ is the unique extension of A, defined by (1.4).

The very similar result has been proved in [24, Theorem 3.3] by using the conclusions in Step 1. Thus, we omit its proof.

Step 3. We prove that for any $\varphi,\psi\in D(A^*)$ and $t\in\mathbb{R}$,

$$\begin{aligned}\langle\Pi_{\alpha,\varepsilon,T}\varphi,\psi\rangle_{H,H'} &= TC_1(\alpha)e^{\alpha T}\int_0^t\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi,B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'}ds\\ &\quad+\langle\Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi,\mathcal{V}_{\alpha,\varepsilon,T}(t)\psi\rangle_{H,H'}+\int_0^t\langle\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi,\widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{H',H}ds,\end{aligned} \tag{3.2}$$

where $\widehat{Q_{\alpha,\varepsilon,T}}$ is defined by

$$\widehat{Q_{\alpha,\varepsilon,T}}:=\Pi_{\alpha,\varepsilon,T}P_{\alpha,\varepsilon,T}. \tag{3.3}$$

First of all, it follows from (2.9), the assumption (H_3) and Lemma 2.4 that the first term on the right hand of (3.2) makes sense.

We now arbitrarily fix $\varphi,\psi\in D(A^*)$ and $t\in\mathbb{R}$. Then by Lemma 2.5 and (2.9), we have

$$\begin{aligned}\langle\Pi_{\alpha,\varepsilon,T}\varphi,\psi\rangle_{H,H'} &= \langle\Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi,\mathcal{V}_{\alpha,\varepsilon,T}(t)\psi\rangle_{H,H'}\\ &\quad+TC_1(\alpha)e^{\alpha T}\int_0^t\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi,B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'}ds+\mathcal{W}_1(t)+\mathcal{W}_2(t),\end{aligned} \tag{3.4}$$

where

$$\begin{cases}\mathcal{W}_1(t):=\left\langle\Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi,\int_0^tS^*(s-t)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds\right\rangle_{H,H'},\\ \mathcal{W}_2(t):=TC_1(\alpha)e^{\alpha T}\int_0^t\left\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi,B^*\int_0^sS^*(\sigma-s)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\psi d\sigma\right\rangle_{U,U'}ds.\end{cases}$$

(It follows by the assumption (H_3), Lemma 2.4 and (2.9) that the term $\mathcal{W}_2(t)$ makes sense.)

Next, we will show

$$\mathcal{W}_1(t) = \int_0^t \langle \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, \widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{H',H} ds - \mathcal{W}_2(t). \tag{3.5}$$

When this is done, (3.2) follows from (3.4) and (3.5) at once.

To show (3.5), we let $n^* \in \mathbb{N}^+$ so that when $n \geq n^*$, $nI - A^*$ is invertible. We define, for each $n \geq n^*$,

$$\mathcal{R}_n := n(nI - A^*)^{-1} \tag{3.6}$$

and

$$\mathcal{K}_n(t) := \Big\langle \Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi, \int_0^t S^*(s-t)\mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds\Big\rangle_{H,H'}. \tag{3.7}$$

By [15, Chapter 1, Theorem 6.3], we can find two positive numbers c_1 and c_2 such that

$$\|\mathcal{R}_n\|_{\mathcal{L}(H')} \leq \frac{nc_1}{n-c_2} \ \text{ for each } \ n \geq n^*, \tag{3.8}$$

while by [15, Chapter 1, Lemma 3.2], we see that when $x \in H'$,

$$\mathcal{R}_n x \in D(A^*) \ \text{ and } \ \mathcal{R}_n x \to x \ \text{ in } \ H' \ \text{ as } \ n \to +\infty. \tag{3.9}$$

Then by (3.8) and (3.9), we can apply the dominated convergence theorem in (3.7) to get

$$\lim_{n\to+\infty} \mathcal{K}_n(t) = \mathcal{W}_1(t). \tag{3.10}$$

(Here, we used the definition of $\mathcal{W}_1(t)$.) Meanwhile, by (2.11) in Lemma 2.5 (where we replace φ and ψ by $\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi$ and $\mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi$, respectively), we have

$$\begin{aligned} \mathcal{K}_n(t) &= \int_0^t \langle \Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t-s)\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(s-t)\mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{H,H'} ds \\ &= \mathcal{K}_{n,1}(t) + \mathcal{K}_{n,2}(t), \end{aligned} \tag{3.11}$$

where

$$\begin{cases} \mathcal{K}_{n,1}(t) := \int_0^t \langle \Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, \mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{H,H'} ds, \\ \mathcal{K}_{n,2}(t) := -TC_1(\alpha)e^{\alpha T}\int_0^t\int_0^{t-s} \langle J_1 B^*\mathcal{V}_{\alpha,\varepsilon,T}(\sigma+s)\varphi, B^*S^*(-\sigma)\mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'} d\sigma ds. \end{cases}$$

With respect to $\mathcal{K}_{n,1}(t)$, we obtain, from (3.8), (3.9), (3.3) and the dominated convergence theorem, that

$$\lim_{n\to+\infty} \mathcal{K}_{n,1}(t) = \int_0^t \langle \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, \widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{H',H} ds. \tag{3.12}$$

With respect to $\mathcal{K}_{n,2}(t)$, we will claim

$$\lim_{n\to+\infty} \mathcal{K}_{n,2}(t) = -\mathcal{W}_2(t). \tag{3.13}$$

To this end, it suffices to show

$$\begin{aligned} &\int_0^t\int_0^{t-s} \langle J_1 B^*\mathcal{V}_{\alpha,\varepsilon,T}(\sigma+s)\varphi, B^*S^*(-\sigma)\mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'} d\sigma ds \\ =\ &\int_0^t \langle J_1 B^*\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi, B^*\int_0^\gamma S^*(s-\gamma)\mathcal{R}_n P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds\rangle_{U,U'} d\gamma, \end{aligned} \tag{3.14}$$

and

$$\lim_{n\to+\infty}\int_{-|t|}^{|t|}\Big\|B^*\int_0^\gamma S^*(s-\gamma)(\mathcal{R}_n-I)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds\Big\|_{U'}^2 d\gamma=0. \tag{3.15}$$

When these have been done, (3.13) follows from (3.14), (3.15) and the definitions of $\mathcal{K}_{n,2}(t)$ and $\mathcal{W}_2(t)$ at once.

To show (3.14), we first notice that by the note (ii) in Remark 1.1,

$$B^*\mathcal{R}_n=E^*(A+\lambda I)^*(n(nI-A^*)^{-1})=nE^*+(n^2+n\bar{\lambda})E^*(nI-A^*)^{-1},\ \text{ when }\ n\geq n^*,$$

(Here $\bar{\lambda}$ is the conjugate of λ.) which leads to

$$B^*\mathcal{R}_n\in\mathcal{L}(H')\ \text{ for all }\ n\geq n^*. \tag{3.16}$$

Next, since $\mathcal{R}_n=n\int_0^{+\infty}e^{-nt}S^*(t)dt$ (see the proof Theorem 3.1 in [15, Chapter 1])), we have $\mathcal{R}_nS^*(\cdot)=S^*(\cdot)\mathcal{R}_n$. From this, (3.16), Lemma 2.4 and [1, Lemma 11.45], we find

$$\begin{aligned}
&\int_0^t\int_0^{t-s}\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(\sigma+s)\varphi,B^*S^*(-\sigma)\mathcal{R}_nP_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'}d\sigma ds\\
=\ &\int_0^t\int_s^t\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi,B^*S^*(s-\gamma)\mathcal{R}_nP_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'}d\gamma ds\\
=\ &\int_0^t\int_0^\gamma\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi,B^*\mathcal{R}_nS^*(s-\gamma)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\rangle_{U,U'}dsd\gamma\\
=\ &\int_0^t\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi,B^*\mathcal{R}_n\int_0^\gamma S^*(s-\gamma)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds\rangle_{U,U'}d\gamma\\
=\ &\int_0^t\langle J_1B^*\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi,B^*\int_0^\gamma S^*(s-\gamma)\mathcal{R}_nP_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds\rangle_{U,U'}d\gamma,
\end{aligned}$$

which leads to (3.14).

To show (3.15), we let $z_n(\gamma):=\int_0^\gamma S^*(s-\gamma)(\mathcal{R}_n-I)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi ds$, then, by Lemma 2.4, we can find $C(|t|)>0$ such that

$$\begin{aligned}
\int_{-|t|}^{|t|}\|B^*z_n(\gamma)\|_{U'}^2d\gamma\ \leq\ & C(|t|)\Big(\|(\mathcal{R}_n-I)P_{\alpha,\varepsilon,T}\psi\|_{H'}^2+\int_{-|t|}^{|t|}\|(\mathcal{R}_n-I)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\psi\|_{H'}^2ds\\
&+\int_{-|t|}^{|t|}\|(\mathcal{R}_n-I)P_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(s)\Delta_{\alpha,\varepsilon,T}\psi\|_{H'}^2\Big)ds.
\end{aligned}$$

This, together with (3.8), (3.9) and the dominated convergence theorem, leads to (3.15).

Finally, (3.5) follows from (3.13), (3.10), (3.11) and (3.12) at once. This ends the proof of Step 3.

Step 4. We show that when $x,y\in D(\mathcal{A}_{\alpha,\varepsilon,T})$ and $t\in\mathbb{R}$,

$$\begin{aligned}
&\langle x,\Pi_{\alpha,\varepsilon,T}^{-1}y\rangle_{H,H'}\\
=\ &\langle\mathcal{S}_{\alpha,\varepsilon,T}(t)x,\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(t)y\rangle_{H,H'}+\int_0^t\langle\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(s)x,\widehat{Q_{\alpha,\varepsilon,T}}\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(s)y\rangle_{H',H}ds\\
&+TC_1(\alpha)e^{\alpha T}\int_0^t\langle J_1B^*\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(s)x,B^*\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(s)y\rangle_{U,U'}ds.
\end{aligned} \tag{3.17}$$

First of all, the third term on the right hand of (3.17) makes sense. The reason is as: it follows by (a_2) in Step 1 that when $z\in D(\mathcal{A}_{\alpha,\varepsilon,T})$, we have $\Pi_{\alpha,\varepsilon,T}^{-1}z\in D(A^*)$. Thus, it follows from (3.1), (2.9) and Lemma 2.4 that $B^*\Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}_{\alpha,\varepsilon,T}(\cdot)z=B^*\mathcal{V}_{\alpha,\varepsilon,T}(\cdot)\Pi_{\alpha,\varepsilon,T}^{-1}z\in L_{loc}^2(\mathbb{R};U')$.

Next, we arbitrarily fix $x, y \in D(\mathcal{A}_{\alpha,\varepsilon,T})$ and $t \in \mathbb{R}$. Then by the conclusion (a_2) in Step 1, we find $\Pi^{-1}_{\alpha,\varepsilon,T}x, \Pi^{-1}_{\alpha,\varepsilon,T}y \in D(A^*)$. This, along with (3.2) (where φ, ψ are replaced by $\Pi^{-1}_{\alpha,\varepsilon,T}x, \Pi^{-1}_{\alpha,\varepsilon,T}y$, respectively) and (3.1), yields (3.17).

Step 5. Let

$$\mathcal{S}(\cdot) := \mathcal{S}_{\alpha,\varepsilon,T}(\cdot) \ \ \textit{and} \ \ K := -TC_1(\alpha)e^{\alpha T} J_1 B^* \Pi^{-1}_{\alpha,\varepsilon,T}. \tag{3.18}$$

We show that K is a feedback law stabilizing (1.1) with the decay rate $\frac{1}{2}(\alpha - \varepsilon)$.

First of all, by the conclusions (a_1) and (a_2) in Step 1, we see that $\mathcal{S}(\cdot)$ is a C_0-group with the generator:

$$\mathcal{A} := \mathcal{A}_{\alpha,\varepsilon,T} = \Pi_{\alpha,\varepsilon,T}(-A^* - P_{\alpha,\varepsilon,T})\Pi^{-1}_{\alpha,\varepsilon,T}, \ \text{ with its domain } D(\mathcal{A}) = \Pi_{\alpha,\varepsilon,T}[D(A^*)]. \tag{3.19}$$

It follows by (3.19), (ii) in Remark 1.1 and the conclusion (a_2) in Step 1 that for each $x \in D(\mathcal{A})$, $\Pi^{-1}_{\alpha,\varepsilon,T}x \in D(A^*)$ and that

$$\begin{aligned} \|B^*\Pi^{-1}_{\alpha,\varepsilon,T}x\|_{U'} &= \|E^*(A^* + P_{\alpha,\varepsilon,T})\Pi^{-1}_{\alpha,\varepsilon,T}x + E^*(\bar{\lambda}I - P_{\alpha,\varepsilon,T})\Pi^{-1}_{\alpha,\varepsilon,T}x\|_{U'} \\ &= \|E^*\Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{A}x + E^*(\bar{\lambda}I - P_{\alpha,\varepsilon,T})\Pi^{-1}_{\alpha,\varepsilon,T}x\|_{U'} \\ &\leq \left(\|E^*\Pi^{-1}_{\alpha,\varepsilon,T}\|_{\mathcal{L}(H;U')} + \|E^*(\bar{\lambda}I - P_{\alpha,\varepsilon,T})\Pi^{-1}_{\alpha,\varepsilon,T}\|_{\mathcal{L}(H;U')}\right)\|x\|_{D(\mathcal{A})}. \end{aligned}$$

These, along with (3.18) and the conjugate-linearity of J_1 and $\Pi_{\alpha,\varepsilon,T}$, yields $K \in \mathcal{L}(D(\mathcal{A}); U)$.

Next, we will show that $\mathcal{S}(\cdot)$ (as well as $\mathcal{A}$) and K verify the conditions (i), (ii) and (iii) in Definition 1.2 (with $\omega = \frac{1}{2}(\alpha - \varepsilon)$) one by one.

Sub-step 5.1. We prove (i) in Definition 1.2 with $\omega = \frac{1}{2}(\alpha - \varepsilon)$.

We first claim

$$\langle \mathcal{S}(t)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(t)x\rangle_{H,H'} \leq e^{-(\alpha-\varepsilon)t}\langle x, \Pi^{-1}_{\alpha,\varepsilon,T}x\rangle_{H,H'} \ \text{ for all } \ x \in D(\mathcal{A}), \ t \in \mathbb{R}^+. \tag{3.20}$$

To this end, we arbitrarily fix $x \in D(\mathcal{A})$, t and σ with $t \geq \sigma \geq 0$. Then by (3.17), we have

$$\begin{aligned} \langle \mathcal{S}(\sigma)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(\sigma)x\rangle_{H,H'} &= \langle \mathcal{S}(t)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(t)x\rangle_{H,H'} + \int_\sigma^t \langle \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x, \widehat{Q_{\alpha,\varepsilon,T}}\Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{H',H} ds \\ &\quad + TC_1(\alpha)e^{\alpha T}\int_\sigma^t \langle J_1B^*\Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x, B^*\Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{U,U'} ds. \end{aligned} \tag{3.21}$$

Meanwhile, by (2.8) and (3.3) (the definitions of $P_{\alpha,\varepsilon,T}$ and $\widehat{Q_{\alpha,\varepsilon,T}}$), we see

$$\widehat{Q_{\alpha,\varepsilon,T}}\Pi^{-1}_{\alpha,\varepsilon,T} = (\alpha - \varepsilon)I + Q_{\alpha,\varepsilon,T}\Pi^{-1}_{\alpha,\varepsilon,T}, \tag{3.22}$$

where $Q_{\alpha,\varepsilon,T}$ is given by (2.3) and is non-negative (which follows from (ii) in Proposition 2.2, since (ε, T) verifies (2.4)). Now, by (3.22) and the non-negativity of $Q_{\alpha,\varepsilon,T}$, we find

$$\begin{aligned} &\int_\sigma^t \langle \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x, \widehat{Q_{\alpha,\varepsilon,T}}\Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{H',H} ds \\ = \ &(\alpha - \varepsilon)\int_\sigma^t \langle \mathcal{S}(s)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{H,H'} ds + \int_\sigma^t \langle Q_{\alpha,\varepsilon,T}\Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{H,H'} ds \\ \geq \ &(\alpha - \varepsilon)\int_\sigma^t \langle \mathcal{S}(s)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{H,H'} ds. \end{aligned} \tag{3.23}$$

From (3.23) and (3.21), it follows that

$$\langle \mathcal{S}(\sigma)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(\sigma)x\rangle_{H,H'} \geq \langle \mathcal{S}(t)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(t)x\rangle_{H,H'} + (\alpha - \varepsilon)\int_\sigma^t \langle \mathcal{S}(s)x, \Pi^{-1}_{\alpha,\varepsilon,T}\mathcal{S}(s)x\rangle_{H,H'} ds.$$

Since $x \in D(\mathcal{A})$ and $t \geq \sigma \geq 0$ were arbitrarily taken, we can apply the Gronwall inequality (see [11, Lemma 3.2]) in the above inequality to get (3.20).

We next claim that there exists $C(\alpha, \varepsilon, T) > 0$ such that

$$T(C(\alpha,\varepsilon,T))^{-2}\|y\|_H^2 \leq \langle y, \Pi_{\alpha,\varepsilon,T}^{-1} y\rangle_{H,H'} \leq T^{-1}\|y\|_H^2 \ \text{ for any } \ y \in H. \tag{3.24}$$

Indeed, according to the second inequality in (2.1) and the boundedness of $\Pi_{\alpha,\varepsilon,T}$ (see Lemma 2.1), there exists $C(\alpha,\varepsilon,T) > 0$ such that for all $\varphi \in H'$,

$$T\|\varphi\|_{H'}^2 \leq \langle \Pi_{\alpha,\varepsilon,T}\varphi, \varphi\rangle_{H,H'} \ \text{ and } \ \|\Pi_{\alpha,\varepsilon,T}\varphi\|_H \leq C(\alpha,\varepsilon,T)\|\varphi\|_{H'}.$$

These lead to (3.24).

Now it follows from (3.24) and (3.20) that

$$\|\mathcal{S}(t)x\|_H \leq \left(\frac{C(\alpha,\varepsilon,T)}{T}\right)^2 e^{-(\alpha-\varepsilon)t}\|x\|_H^2 \ \text{ for any } \ t \in \mathbb{R}^+, \ x \in D(\mathcal{A}).$$

This, together with the density of $D(\mathcal{A})$ in H, shows

$$\|\mathcal{S}(t)\|_{\mathcal{L}(H)} \leq \frac{C(\alpha,\varepsilon,T)}{T} e^{-\frac{1}{2}(\alpha-\varepsilon)t} \ \text{ for any } \ t \in \mathbb{R}^+,$$

i.e., (i) in Definition 1.2 with $\omega = \frac{1}{2}(\alpha - \varepsilon)$ is true.

Sub-step 5.2. We prove (ii) in Definition 1.2 with $\omega = \frac{1}{2}(\alpha-\varepsilon)$.

This follows from (3.18) and the conclusion in Step 2 at once.

Sub-step 5.3. We prove (iii) in Definition 1.2 with $\omega = \frac{1}{2}(\alpha-\varepsilon)$.

In the case that $C_1(\alpha) = 0$, we see from (3.18) that $K = 0$, thus (iii) holds for this case.

In the case that $C_1(\alpha) \neq 0$, it follows by (3.21), (3.23) (with $\sigma = 0$) and (3.18) that when $x \in D(\mathcal{A})$,

$$\langle x, \Pi_{\alpha,\varepsilon,T}^{-1}x\rangle_{H,H'} \geq \langle \mathcal{S}(t)x, \Pi_{\alpha,\varepsilon,T}^{-1}\mathcal{S}(t)x\rangle_{H,H'} + (TC_1(\alpha)e^{\alpha T})^{-1}\int_0^t \|K\mathcal{S}(s)x\|_{U'}^2 ds \ \text{ for each } \ t \in \mathbb{R}^+.$$

Letting $t \to +\infty$ in the above, using (i) and (3.24), we see

$$\int_0^{+\infty} \|K\mathcal{S}(s)x\|_{U'}^2 ds \leq C_1(\alpha)e^{\alpha T}\|x\|_H^2 \ \text{ for any } \ x \in D(\mathcal{A}),$$

which leads to (iii) for this case.

In summary, $\mathcal{S}(\cdot)$ (as well as $\mathcal{A}$) and K verify the conditions (i), (ii) and (iii) in Definition 1.2 with $\omega = \frac{1}{2}(\alpha-\varepsilon)$.

Step 6. We finish the proof.

Arbitrarily fix T satisfying (1.9). Let $\widehat{\varepsilon} := T^{-1}\ln[C_2(\alpha)]$. Then one can easily check that $(\widehat{\varepsilon}, T)$ satisfies (2.4). Thus by the conclusions in Step 5, we complete the proof of Theorem 1.4. $\square$

Our proof of Theorem 1.4 shows, indeed, the following more general result:

Theorem 3.1. *Assume that (H_1)-(H_4) are true. Then for each pair $(\varepsilon, T) \in [0,\alpha) \times (0,+\infty)$ verifying (2.4), the following operator (from $\Pi_{\alpha,\varepsilon,T}[D(A^*)]$ to U) is a feedback law stabilizing the system (1.1) with the decay rate $\frac{1}{2}(\alpha-\varepsilon)$:*

$$K_{\varepsilon,T} := -TC_1(\alpha)e^{\alpha T}J_1 B^* \Pi_{\alpha,\varepsilon,T}^{-1},$$

where $\Pi_{\alpha,\varepsilon,T}$ is defined by (1.8).

Remark 3.2. *(i) Several facts are given. First, it follows from (3.17) that the operator $\Pi^{-1}_{\alpha,\varepsilon,T}$, with $(\varepsilon,T)\in[0,\alpha)\times(0,+\infty)$, satisfies the following Riccati equation:*

$$\begin{aligned}&\langle A^*\mathcal{P}x,y\rangle_{H',H}+\langle x,A^*\mathcal{P}y\rangle_{H,H'}-TC_1(\alpha)e^{\alpha T}\langle JB^*\mathcal{P}x,B^*\mathcal{P}y\rangle_{H,H'}\\ =\ &-\left\langle\left((\alpha-\varepsilon)\Pi^{-1}_{\alpha,\varepsilon,T}+\Pi^{-1}_{\alpha,\varepsilon,T}Q_{\alpha,\varepsilon,T}\Pi^{-1}_{\alpha,\varepsilon,T}\right)x,y\right\rangle_{H',H},\quad x,y\in\Pi_{\alpha,\varepsilon,T}[D(A^*)],\end{aligned}\tag{3.25}$$

where $Q_{\alpha,\varepsilon,T}:=\Lambda_{\alpha,\varepsilon,T}(T)-C_2(\alpha)J_2$ is a conjugate-linear and bounded operator from H' to H. Second, if $Q_{\alpha,\varepsilon,T}$ is non-negative, then the solvability of the equation (3.25) is equivalent to the finite cost condition of the infinite-horizon LQ problem corresponding to (3.25) (see [6, Theorem 2.2]). Third, the finite cost condition of the aforementioned LQ problem is equivalent to the stabilizability of the system (1.1) (see [14, Proposition 3.9]). Finally, we can only show that when (ε,T) satisfies (2.4), the above $Q_{\alpha,\varepsilon,T}$ is non-negative. These facts explain why the pair (ε,T) needs to satisfy (2.4) in Theorem 3.1.

(ii) From the discussions in the note (i), we see that our method is to construct directly an operator which satisfies a Riccati equation (related to an infinite-horizon LQ problem) instead of solving a Riccati equation (the latter needs to prove the existence of solutions for the corresponding Riccati equation).

4 Further studies

The quantity ω^* defined in (1.11) is called as the best stabilization decay rate for the system (1.1). It is the same as that defined by [21, (4)]. When the system (1.1) is stabilizable, we have $\omega^*\in(0,+\infty]$. In particular, when $\omega^*=+\infty$, the system (1.1) is completely stabilizable. Before stating the main result of this section, we give the next proposition.

Proposition 4.1. *Suppose that (H_1)-(H_3) hold and the system (1.1) is stabilizable. Let $\omega^*\in(0,+\infty]$ be given by (1.11). Then for each $\theta\in(0,\omega^*)$, there is $\overline{C}_1(\theta)\geq 0$ and $\overline{C}_2(\theta)\geq 1$ so that (1.6) holds for $\alpha=\theta$, $C_1(\alpha)=\overline{C}_1(\theta)$ and $C_2(\alpha)=\overline{C}_2(\theta)$, i.e.,*

$$\|S^*(t)\varphi\|^2_{H'}\leq\overline{C}_1(\theta)\int_0^t\|B^*S^*(s)\varphi\|^2_{U'}ds+\overline{C}_2(\theta)e^{-2\theta t}\|\varphi\|^2_{H'},\quad t>0,\ \varphi\in D(A^*).\tag{4.1}$$

Proof. First of all, it follows by (1.11) and Definition 1.2 that for any $\theta\in(0,\omega^*)$, there is a C_0-semigroup $\mathcal{S}_\theta(\cdot)$ on H (with the generator $\mathcal{A}_\theta:D(\mathcal{A}_\theta)\subset H\to H$) and an operator $K_\theta\in\mathcal{L}(D(\mathcal{A}_\theta);U)$ so that

- (b_1) there exists $\widehat{C}_1(\theta)\geq 1$ such that $\|\mathcal{S}_\theta(t)\|_{\mathcal{L}(H)}\leq\widehat{C}_1(\theta)e^{-\theta t}$ for any $t\in\mathbb{R}^+$;
- (b_2) for any $x\in D(\mathcal{A}_\theta)$, $\mathcal{A}_\theta x=(\widetilde{A}+BK_\theta)x$;
- (b_3) there exists $\widehat{C}_2(\theta)\geq 0$ such that $\|K_\theta\mathcal{S}_\theta(\cdot)x\|_{L^2(\mathbb{R}^+;U)}\leq\widehat{C}_2(\theta)\|x\|_H$ for any $x\in D(\mathcal{A}_\theta)$.

Arbitrarily fix $x\in D(\mathcal{A}_\theta)$, $\varphi\in D(A^*)$ and $t>0$. Then it follows by (b_2) and (1.4) that

$$\begin{aligned}&\frac{d}{ds}\langle\mathcal{S}_\theta(s)x,S^*(t-s)\varphi\rangle_{H,H'}\\ =\ &\langle\mathcal{A}_\theta\mathcal{S}_\theta(s)x,S^*(t-s)\varphi\rangle_{H,H'}-\langle\mathcal{S}_\theta(s)x,A^*S^*(t-s)\varphi\rangle_{H,H'}\\ =\ &\langle(\widetilde{A}+BK_\theta)\mathcal{S}_\theta(s)x,S^*(t-s)\varphi\rangle_{D(A^*)',D(A^*)}-\langle\mathcal{S}_\theta(s)x,A^*S^*(t-s)\varphi\rangle_{H,H'}\\ =\ &\langle BK_\theta\mathcal{S}_\theta(s)x,S^*(t-s)\varphi\rangle_{D(A^*)',D(A^*)}=\langle K_\theta\mathcal{S}_\theta(s)x,B^*S^*(t-s)\varphi\rangle_{U,U'},\quad s\in(0,t).\end{aligned}$$

By integrating the above equality with respect to s over $[0,t]$, we get

$$\langle\mathcal{S}_\theta(t)x,\varphi\rangle_{H,H'}-\langle x,S^*(t)\varphi\rangle_{H,H'}=\int_0^t\langle K_\theta\mathcal{S}_\theta(s)x,B^*S^*(t-s)\varphi\rangle_{U,U'}ds.$$

This, together with (b_1) and (b_3), yields

$$|\langle x, S^*(t)\varphi\rangle_{H,H'}| \leq \widehat{C}_2(\theta)\|x\|_H \left(\int_0^t \|B^*S^*(t-s)\varphi\|_{U'}^2 ds\right)^{\frac{1}{2}} + \widehat{C}_1(\theta)e^{-\theta t}\|x\|_H\|\varphi\|_{H'}.$$

Since $t > 0$, $\varphi \in D(A^*)$ and $x \in D(\mathcal{A}_\theta)$ were arbitrarily taken, the above, along with the density of $D(\mathcal{A}_\theta)$ in H, gives

$$\|S^*(t)\varphi\|_{H'}^2 \leq 2(\widehat{C}_2(\theta))^2 \int_0^t \|B^*S^*(s)\varphi\|_{U'}^2 ds + 2(\widehat{C}_1(\theta))^2 e^{-2\theta t}\|\varphi\|_{H'}^2, \ \ t > 0, \ \varphi \in D(A^*),$$

which leads to (4.1) with $\overline{C}_1(\theta) = 2(\widehat{C}_2(\theta))^2$ and $\overline{C}_2(\theta) = 2(\widehat{C}_1(\theta))^2$. Thus, we complete the proof of Proposition 4.1. □

Theorem 4.2. *Assume that (H_1)-(H_3) are true and the system (1.1) is stabilizable. Let $\omega^* \in (0, +\infty]$ be given by (1.11). Let $\overline{C}_1(\theta) \geq 0$ and $\overline{C}_2(\theta) \geq 1$, with $\theta \in (0, \omega^*)$, be given by Proposition 4.1. Then for each $\mu \in (0, \omega^*)$, the following conclusions are true:*

- (i) *If $\omega^* \in (0, +\infty)$, then for each T satisfying*

$$(\omega^* - \mu)^{-1} \ln\left[\overline{C}_2\left(\overline{\theta}\right)\right] < T < +\infty, \ \ \textit{with} \ \overline{\theta} := \frac{1}{2}(\omega^* + \mu), \tag{4.2}$$

the following operator (from $\Pi_{2\overline{\theta},\overline{\varepsilon},T}[D(A^*)]$ to U) is a feedback law stabilizing the system (1.1) with the decay rate μ:

$$K_{\mu,T} := -T\overline{C}_1\left(\overline{\theta}\right)e^{2\overline{\theta}T}J_1B^*\Pi^{-1}_{2\overline{\theta},\overline{\varepsilon},T}, \tag{4.3}$$

where $\overline{\theta}$ is given in (4.2), $\overline{\varepsilon} := T^{-1}\ln\left[\overline{C}_2\left(\overline{\theta}\right)\right]$ and $\Pi_{2\overline{\theta},\overline{\varepsilon},T}$ is defined by (1.8) with

$$\alpha = 2\overline{\theta}; \ \varepsilon = \overline{\varepsilon}; \ C_1(\alpha) = \overline{C}_1\left(\overline{\theta}\right); \ C_2(\alpha) = \overline{C}_2\left(\overline{\theta}\right).$$

- (ii) *If $\omega^* = +\infty$, then for each T satisfying*

$$\mu^{-1}\ln\left[\overline{C}_2\left(\theta^*\right)\right] < T < +\infty, \ \ \textit{with} \ \theta^* := \frac{3\mu}{2}, \tag{4.4}$$

the following operator (from $\Pi_{2\theta^*,\varepsilon^*,T}[D(A^*)]$ to U) is a feedback law stabilizing the system (1.1) with the decay rate μ:

$$K'_{\mu,T} := -T\overline{C}_1\left(\theta^*\right)e^{2\theta^*T}J_1B^*\Pi^{-1}_{2\theta^*,\varepsilon^*,T}, \tag{4.5}$$

where θ^* is given in (4.4), $\varepsilon^* := T^{-1}\ln\left[\overline{C}_2\left(\theta^*\right)\right]$ and $\Pi_{2\theta^*,\varepsilon^*,T}$ is defined by (1.8) with

$$\alpha = 2\theta^*; \ \varepsilon = \varepsilon^*; \ C_1(\alpha) = \overline{C}_1\left(\theta^*\right); C_2(\alpha) = \overline{C}_2\left(\theta^*\right).$$

Proof. Arbitrarily fix $\mu \in (0, \omega^*)$. To show the conclusion (i), we arbitrarily fix T satisfying (4.2) and write $\overline{\theta} := \frac{1}{2}(\omega^* + \mu)$. Two observations are given in order. First, it follows from Proposition 4.1 that (H_4) holds for $\alpha = 2\overline{\theta}$, $C_1(\alpha) = \overline{C}_1\left(\overline{\theta}\right)$ and $C_2(\alpha) = \overline{C}_2\left(\overline{\theta}\right)$. Second, by (4.2), one can easily check that

$$\frac{1}{2}\left(2\overline{\theta} - T^{-1}\ln\left[\overline{C}_2\left(\overline{\theta}\right)\right]\right) \geq \mu \ \text{ and } \ (2\overline{\theta})^{-1}\ln\left[\overline{C}_2\left(\overline{\theta}\right)\right] < T < +\infty.$$

From the above two observations and Theorem 1.4, we see that the operator $K_{\mu,T} : \Pi_{2\overline{\theta},\overline{\varepsilon},T}[D(A^*)] \to U$ defined by (4.3) is a feedback law stabilizing the system (1.1) with the decay rate μ. This completes the proof of (i).

To show (ii), we arbitrarily fix T satisfying (4.4) and write $\theta^* := \frac{3\mu}{2}$. Two facts are given in order: First, it follows from Proposition 4.1 that (H_4) holds for $\alpha = 2\theta^*$, $C_1(\alpha) = \overline{C}_1(\theta^*)$ and $C_2(\alpha) = \overline{C}_2(\theta^*)$. Second, by (4.4), one can directly verify that

$$\frac{1}{2}\left(2\theta^* - T^{-1}\ln\left[\overline{C}_2(\theta^*)\right]\right) \geq \mu \text{ and } (2\theta^*)^{-1}\ln\left[\overline{C}_2(\theta^*)\right] < T < +\infty.$$

From these two facts and Theorem 1.4, we see that the operator $K'_{\mu,T} : \Pi_{2\theta^*,\varepsilon^*,T}[D(A^*)] \to U$ defined by (4.5) is a feedback law stabilizing the system (1.1) with the decay rate μ, i.e., (ii) holds.

Hence, we complete the proof of Theorem 4.2. □

5 Appendices

5.1 Appendix A

In this subsection, we present a direct proof for the equivalence of the inequalities (1.5) and (1.6).

Proposition 5.1. *The inequalities (1.5) and (1.6) are equivalent.*

Proof. We divide the proof by two steps.

Step 1. We first prove (1.6) ⇒ (1.5).

Let $\alpha > 0$, $C_1(\alpha) \geq 0$ and $C_2(\alpha) \geq 1$ be given in (1.6). Then there is $\widehat{T} > 0$ such that $\widehat{\delta} := C_2(\alpha)e^{-\alpha\widehat{T}} < 1$. Thus, by taking $t = \widehat{T}$ in (1.6), we get (1.5) with $T = \widehat{T}$, $C(\alpha, T) = C_1(\alpha)$ and $\delta = \widehat{\delta}$.

Step 2. We prove (1.5) ⇒ (1.6).

Let $T > 0$, $\delta \in (0, 1)$ and $C(\delta, T) \geq 0$ be given in (1.5). We first claim that for any $n \in \mathbb{N}^+$,

$$\|S^*(nT)\varphi\|^2_{H'} \leq C(\delta, T)\sum_{j=0}^{n-1}\delta^j \int_0^{nT} \|B^*S^*(s)\varphi\|^2_{U'}ds + \delta^n\|\varphi\|^2_{H'}. \tag{5.1}$$

Indeed, (1.5) gives (5.1) with $n = 1$. Suppose that (5.1), with $n = k$, is true. Then, by (1.5) and the time-invariance of the system (1.1), we have

$$\begin{aligned}\|S^*((k+1)T)\varphi\|^2_{H'} &\leq C(\delta, T)\sum_{j=0}^{k-1}\delta^j \int_T^{(k+1)T} \|B^*S^*(s)\varphi\|^2_{U'}ds + \delta^k\|S^*(T)\varphi\|^2_{H'} \\ &\leq C(\delta, T)\sum_{j=0}^{k}\delta^j \int_0^{(k+1)T} \|B^*S^*(s)\varphi\|^2_{U'}ds + \delta^{k+1}\|\varphi\|^2_{H'},\end{aligned}$$

which leads to (5.1) with $n = k + 1$. So by the induction, (5.1) holds for all $n \in \mathbb{N}^+$.

Next, we let $\alpha = -T^{-1}\ln\delta$ (which implies $\alpha > 0$ and $\delta = e^{-\alpha T}$). Then by (5.1), we have that for any $n \in \mathbb{N}^+$,

$$\begin{aligned}\|S^*(nT)\varphi\|^2_{H'} &\leq (1-\delta)^{-1}C(\delta, T)\int_0^{nT} \|B^*S^*(s)\varphi\|^2_{U'}ds + \delta^n\|\varphi\|^2_{H'} \\ &= (1-e^{-\alpha T})^{-1}C(e^{-\alpha T}, T)\int_0^{nT} \|B^*S^*(s)\varphi\|^2_{U'}ds + e^{-n\alpha T}\|\varphi\|^2_{H'}.\end{aligned} \tag{5.2}$$

We now arbitrarily fix $t \in \mathbb{R}^+$. Then there is $m \in \mathbb{N}$ such that

$$mT \leq t < (m+1)T. \tag{5.3}$$

In the case that $m = 0$ (i.e., $t \in [0, T)$), we have

$$\|S^*(t)\varphi\|_{H'}^2 \le \widehat{C}_2(\alpha, T)e^{-\alpha t}\|\varphi\|_{H'}^2, \text{ with } \widehat{C}_2(\alpha, T) := \big(\sup_{\sigma\in[0,T]} \|S^*(\sigma)\|_{\mathcal{L}(H')}\big)^2 e^{\alpha T}. \tag{5.4}$$

In the case that $m \in \mathbb{N}^+$, it follows by (5.2) and (5.3) that

$$\begin{aligned}\|S^*(t)\varphi\|_{H'}^2 &= \|S^*(t-mT)S^*(mT)\varphi\|_{H'}^2 \le \big(\sup_{\sigma\in[0,T]} \|S^*(\sigma)\|_{\mathcal{L}(H')}\big)^2 \|S^*(mT)\varphi\|_{H'}^2 \\ &\le \widehat{C}_1(\alpha, T)\int_0^t \|B^*S^*(s)\varphi\|_{U'}^2 ds + \widehat{C}_2(\alpha, T)e^{-\alpha t}\|\varphi\|_{H'}^2, \end{aligned} \tag{5.5}$$

where $\widehat{C}_2(\alpha, T)$ is given in (5.4) and

$$\widehat{C}_1(\alpha, T) := \big(\sup_{\sigma\in[0,T]} \|S^*(\sigma)\|_{\mathcal{L}(H')}\big)^2 (1-e^{-\alpha T})^{-1} C_1(e^{-\alpha T}, T).$$

Finally, (5.4) and (5.5) leads to (1.6) with $C_1(\alpha) = \widehat{C}_1(\alpha, T)$ and $C_2(\alpha) = \widehat{C}_2(\alpha, T)$. Hence, we finish the proof of Proposition 5.1. □

5.2 Appendix B

The proof of Lemma 2.4. Arbitrarily fix $\gamma > 0$, $M \in \mathcal{L}(H')$ and $\varphi \in D(A^*)$. The proof is divided into two steps.

Step 1. We show that for each $t \in [-\gamma, \gamma]$, $w(t; \varphi) \in D(A^*)$ and

$$A^*w(t;\varphi) = M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi - S^*(-t)M\varphi - \int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\Delta_{\alpha,\varepsilon,T}\varphi ds. \tag{5.6}$$

(Recall that $\mathcal{V}_{\alpha,\varepsilon,T}(\cdot)$ and $\Delta_{\alpha,\varepsilon,T}$ are given in Remark 2.3.)

To this end, we arbitrarily fix $t \in [-\gamma, \gamma]$. By the definition of $w(\cdot;\varphi)$, we have that for each $h \in (0, h_0)$ ($h_0 > 0$ is fixed arbitrarily),

$$\begin{aligned}\frac{S(h) - I}{h} w(t;\varphi) &= \frac{1}{h}\left(\int_0^t S^*(s+h-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi ds - \int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi ds\right) \\ &= \mathcal{I}_1(h) + \mathcal{I}_2(h) + \mathcal{I}_3(h), \end{aligned} \tag{5.7}$$

where

$$\begin{cases}\mathcal{I}_1(h) := \frac{1}{h}\int_t^{t+h} S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s-h)\varphi ds, \\ \mathcal{I}_2(h) := -\frac{1}{h}\int_0^h S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s-h)\varphi ds, \\ \mathcal{I}_3(h) := \frac{1}{h}\int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)(\mathcal{V}_{\alpha,\varepsilon,T}(-h) - I)\varphi ds.\end{cases}$$

With respect to $\mathcal{I}_1(h)$, we claim

$$\lim_{h\to 0^+} \mathcal{I}_1(h) = M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi. \tag{5.8}$$

Indeed, we have the following facts: First, it is obvious that

$$\mathcal{I}_1(h) = \frac{1}{h}\int_t^{t+h} S^*(s-t)M(\mathcal{V}_{\alpha,\varepsilon,T}(s-h) - \mathcal{V}_{\alpha,\varepsilon,T}(t))\varphi ds + \frac{1}{h}\int_t^{t+h} S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi ds. \tag{5.9}$$

Second, it follows from the strong continuity of $S^*(\cdot)$ that

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h} S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi ds = M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi. \tag{5.10}$$

Third, direct computations show

$$\begin{aligned}
&\left\|\frac{1}{h}\int_t^{t+h} S^*(s-t)M(\mathcal{V}_{\alpha,\varepsilon,T}(s-h)-\mathcal{V}_{\alpha,\varepsilon,T}(t))\varphi ds\right\|_{H'}\\
=&\left\|\frac{1}{h}\int_t^{t+h} S^*(s-t)M\int_{s-h}^t \mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\Delta_{\alpha,\varepsilon,T}\varphi d\sigma ds\right\|_{H'}\\
\leq& h\sup_{\sigma\in[0,h_0]}\|S^*(\sigma)\|_{\mathcal{L}(H')}\|M\|_{\mathcal{L}(H')}\sup_{\sigma\in[t-h_0,t]}\|\mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\|_{\mathcal{L}(H')}\|\Delta_{\alpha,\varepsilon,T}\varphi\|_{H'}\to 0 \text{ as } h\to 0^+.
\end{aligned} \tag{5.11}$$

(Here, we used that $\sup_{s\in[t,t+h]}|t-s+h|=h$.) Now, (5.8) follows by (5.9), (5.10) and (5.11) at once.

With respect to $\mathcal{I}_2(h)$, we can use a very similar way to that used in the proof of (5.8) to find

$$\lim_{h\to 0^+}\mathcal{I}_2(h)=-S^*(-t)M\varphi. \tag{5.12}$$

With respect to $\mathcal{I}_3(h)$, we claim

$$\lim_{h\to 0^+}\mathcal{I}_3(h)=-\int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\Delta_{\alpha,\varepsilon,T}\varphi ds. \tag{5.13}$$

For this purpose, several facts are given in order: First, since $\Delta_{\alpha,\varepsilon,T}$ is the generator of the C_0-group $\mathcal{V}_{\alpha,\varepsilon,T}(\cdot)$ and $\varphi\in D(A^*)(=D(\Delta_{\alpha,\varepsilon,T}))$, we have

$$\mathcal{I}_3(h)=-\int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\left(\frac{1}{h}\int_{-h}^0 \mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\Delta_{\alpha,\varepsilon,T}\varphi d\sigma\right)ds. \tag{5.14}$$

Second, direct computations show that for each $s\in[-|t|,|t|]$,

$$\begin{aligned}
&\left\|S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\left(\frac{1}{h}\int_{-h}^0 \mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\Delta_{\alpha,\varepsilon,T}\varphi d\sigma\right)\right\|_{H'}\\
\leq&\sup_{\sigma\in[-2|t|,2|t|]}\|S^*(\sigma)\|_{\mathcal{L}(H')}\|M\|_{\mathcal{L}(H')}\sup_{\sigma\in[-2|t|-h_0,2|t|]}\|\mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\|_{\mathcal{L}(H')}\|\Delta_{\alpha,\varepsilon,T}\varphi\|_{H'}.
\end{aligned} \tag{5.15}$$

Third, the strong continuity of $\mathcal{V}_{\alpha,\varepsilon,T}(\cdot)$ leads to

$$\lim_{h\to 0^+}\frac{1}{h}\int_{-h}^0 \mathcal{V}_{\alpha,\varepsilon,T}(\sigma)\Delta_{\alpha,\varepsilon,T}\varphi d\sigma=\Delta_{\alpha,\varepsilon,T}\varphi. \tag{5.16}$$

Now, by (5.15) and (5.16), we can apply the dominated convergence theorem in (5.14) to get (5.13).

Finally, it follows from (5.7), (5.8), (5.12) and (5.13) that

$$\lim_{h\to 0^+}\frac{S^*(h)-I}{h}w(t;\varphi) \text{ exists, i.e., } \quad w(t;\varphi)\in D(A^*)$$

(see [15, Chapter 1, Section 1.1]) and that (5.6) holds.

Step 2. We prove (2.10).

By the note (ii) in Remark 1.1 and Step 1, we have

$$\begin{aligned}
&\int_{-\gamma}^{\gamma}\|B^*w(t;\varphi)\|_{U'}^2dt=\int_{-\gamma}^{\gamma}\|E^*(\bar{\lambda}I+A^*)w(t;\varphi)\|_{U'}^2dt\\
\leq& 16\|E^*\|_{\mathcal{L}(H';U')}^2\int_{-\gamma}^{\gamma}\Big(|\lambda|^2\|w(t;\varphi)\|_{H'}^2+\|M\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi\|_{H'}^2+\|S^*(-t)M\varphi\|_{H'}^2\\
&+\left\|\int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\Delta_{\alpha,\varepsilon,T}\varphi ds\right\|_{H'}^2\Big)dt.
\end{aligned} \tag{5.17}$$

(Here, $\bar{\lambda}$ is the conjugate of λ.) Since

$$\begin{cases} \sup_{t\in[-\gamma,\gamma]} \|w(t;\varphi)\|_{H'} \leq \sup_{\sigma\in[-\gamma,\gamma]} \|S^*(\sigma)\|_{\mathcal{L}(H')} \int_{-\gamma}^{\gamma} \|M\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi\|_{H'} ds, \\ \sup_{t\in[-\gamma,\gamma]} \|S^*(-t)M\varphi\|_{H'} \leq \sup_{t\in[-\gamma,\gamma]} \|S^*(t)\|_{\mathcal{L}(H')} \|M\varphi\|_{H'}, \\ \sup_{t\in[-\gamma,\gamma]} \left\|\int_0^t S^*(s-t)M\mathcal{V}_{\alpha,\varepsilon,T}(s)\Delta_{\alpha,\varepsilon,T}\varphi ds\right\|_{H'} \\ \qquad \leq \sup_{\sigma\in[-\gamma,\gamma]} \|S^*(\sigma)\|_{\mathcal{L}(H')} \int_{-\gamma}^{\gamma} \|M\mathcal{V}_{\alpha,\varepsilon,T}(s)\Delta_{\alpha,\varepsilon,T}\varphi\|_{H'} ds, \end{cases}$$

applying the Hölder inequality in (5.17) leads to (2.10).

By Steps 1 and Step 2, we finish the proof of Lemma 2.4. □

5.3 Appendix C

The proof of Lemma 2.5. Arbitrarily fix $t \in \mathbb{R}$ and $\varphi, \psi \in D(A^*)$. The proof is divided into two steps.

Step 1. We prove

$$\begin{aligned} & \langle \Pi_{\alpha,\varepsilon,T}\varphi, \psi\rangle_{H,H'} - \mathcal{E}(t) \\ = \; & \langle \Pi_{\alpha,\varepsilon,T}\mathcal{V}_{\alpha,\varepsilon,T}(t)\varphi, S^*(-t)\psi\rangle_{H,H'} + TC_1(\alpha)e^{\alpha T}\int_0^t \langle J_1 B^*\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, B^*S^*(-s)\psi\rangle_{U,U'} ds, \end{aligned} \tag{5.18}$$

where $\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t) + \mathcal{E}_4(t)$ *with*

$$\begin{cases} \mathcal{E}_1(t) := \int_0^t \langle \Pi_{\alpha,\varepsilon,T} S^*(s-t)\Pi_{\alpha,\varepsilon,T}^{-1}\widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(-t)\psi\rangle_{H,H'} ds, \\ \mathcal{E}_2(t) := TC_1(\alpha)e^{\alpha T}\int_0^t \left\langle J_1 B^*\left(\int_0^s S^*(\gamma-s)\Pi_{\alpha,\varepsilon,T}^{-1}\widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi d\gamma\right), B^*S^*(-s)\psi\right\rangle_{U,U'} ds, \\ \mathcal{E}_3(t) := -\int_0^t \langle \widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(-s)\psi\rangle_{H,H'} ds, \\ \mathcal{E}_4(t) := -\int_0^t \left\langle \widehat{Q_{\alpha,\varepsilon,T}}\left(\int_0^s S^*(\gamma-s)\Pi_{\alpha,\varepsilon,T}^{-1}\widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi d\gamma\right), S^*(-s)\psi\right\rangle_{H,H'} ds. \end{cases}$$

(Here $\widehat{Q_{\alpha,\varepsilon,T}}$ *is defined by (3.3).)*

First of all, it is obvious that $\mathcal{E}_1(t)$, $\mathcal{E}_3(t)$ and $\mathcal{E}_4(t)$ are well defined since $\Pi_{\alpha,\varepsilon,T}^{-1}\widehat{Q_{\alpha,\varepsilon,T}} \in \mathcal{L}(H')$ and $\varphi, \psi \in D(A^*)$. Second, by Lemma 2.4 and the assumption (H_3) that $\mathcal{E}_2(t)$ is well defined since $\varphi, \psi \in D(A^*)$.

Next, it follows from Proposition 2.2 that for any $s \in \mathbb{R}$,

$$\begin{aligned} & \langle \Pi_{\alpha,\varepsilon,T}A^*S^*(-s)\varphi, S^*(-s)\psi\rangle_{H,H'} + \langle \Pi_{\alpha,\varepsilon,T}S^*(-s)\varphi, A^*S^*(-s)\psi\rangle_{H,H'} \\ = \; & TC_1(\alpha)e^{\alpha T}\langle J_1 B^*S^*(-s)\varphi, B^*S^*(-s)\psi\rangle_{U,U'} - \langle \widehat{Q_{\alpha,\varepsilon,T}}S^*(-s)\varphi, S^*(-s)\psi\rangle_{H,H'}. \end{aligned} \tag{5.19}$$

Then, since

$$\begin{aligned} & \langle \Pi_{\alpha,\varepsilon,T}A^*S^*(-s)\varphi, S^*(-s)\psi\rangle_{H,H'} + \langle \Pi_{\alpha,\varepsilon,T}S^*(-s)\varphi, A^*S^*(-s)\psi\rangle_{H,H'} \\ & \qquad = -\left[\frac{d}{d\gamma}\langle \Pi_{\alpha,\varepsilon,T}S^*(-\gamma)\varphi, S^*(-\gamma)\psi\rangle_{H,H'}\right]_{\gamma=s}, \quad s \in \mathbb{R}, \end{aligned}$$

we get, by integrating (5.19) with respect to s over $(0, t)$, that

$$\begin{aligned} & \langle \Pi_{\alpha,\varepsilon,T}\varphi, \psi\rangle_{H,H'} \\ = \; & \langle \Pi_{\alpha,\varepsilon,T}S^*(-t)\varphi, S^*(-t)\psi\rangle_{H,H'} + TC_1(\alpha)e^{\alpha T}\int_0^t \langle J_1 B^*S^*(-s)\varphi, B^*S^*(-s)\psi\rangle_{U,U'} ds \\ & -\int_0^t \langle \widehat{Q_{\alpha,\varepsilon,T}}S^*(-s)\varphi, S^*(-s)\psi\rangle_{H,H'} ds. \end{aligned} \tag{5.20}$$

Finally, (5.18) follows from (5.20) and (2.9).

Step 2. We show

$$\mathcal{E}(t) = 0. \tag{5.21}$$

When this is done, (2.11) follows from (5.21) and (5.18) at once.

The remainder is to show (5.21). Let $n^* \in \mathbb{N}^+$ be such that $nI - A^*$ is invertible for all $n \geq n^*$. We define, for each $n \geq n^*$,

$$\mathcal{F}_n(t) := \int_0^t \langle \Pi_{\alpha,\varepsilon,T} S^*(s-t)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(s-t)S^*(-s)\psi\rangle_{H,H'} ds,$$

where $\{\mathcal{R}_n\}_{n\geq n^*}$ are given by (3.6). By (5.20), we find

$$\mathcal{F}_n(t) = \mathcal{F}_{n,1}(t) + \mathcal{F}_{n,2}(t) + \mathcal{F}_{n,3}(t),\ n \geq n^*, \tag{5.22}$$

where

$$\begin{cases} \mathcal{F}_{n,1}(t) := \int_0^t \langle \Pi_{\alpha,\varepsilon,T} \mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(-s)\psi\rangle_{H,H'} ds, \\ \mathcal{F}_{n,2}(t) := -TC_1(\alpha)e^{\alpha T} \int_0^t \int_0^{t-s} \langle J_1 B^* S^*(-\gamma)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, B^* S^*(-(\gamma+s))\psi\rangle_{U,U'} d\gamma ds, \\ \mathcal{F}_{n,3}(t) := \int_0^t \int_0^{t-s} \langle \widehat{Q_{\alpha,\varepsilon,T}} S^*(-\gamma)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(-(\gamma+s))\psi\rangle_{H,H'} d\gamma ds. \end{cases}$$

Several facts are given in order: First, since $\{\mathcal{R}_n\}_{n\geq n^*}$ is uniformly bounded (see (3.8)), we can use (3.9) and the dominated convergence theorem to find

$$\lim_{n\to+\infty} \mathcal{F}_n(t) = \mathcal{E}_1(t) \text{ and } \lim_{n\to+\infty} \mathcal{F}_{n,1}(t) = -\mathcal{E}_3(t). \tag{5.23}$$

Second, direct computations show that when $n \geq n^*$,

$$\begin{aligned} \mathcal{F}_{n,3}(t) &= \int_0^t \int_s^t \langle \widehat{Q_{\alpha,\varepsilon,T}} S^*(s-\sigma)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(-\sigma)\psi\rangle_{H,H'} d\sigma ds \\ &= \int_0^t \int_0^\sigma \langle \widehat{Q_{\alpha,\varepsilon,T}} S^*(s-\sigma)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(s)\varphi, S^*(-\sigma)\psi\rangle_{H,H'} ds d\sigma. \end{aligned}$$

This, along with (3.9) and the dominated convergence theorem, yields

$$\lim_{n\to+\infty} \mathcal{F}_{n,3}(t) = -\mathcal{E}_4(t). \tag{5.24}$$

We now claim

$$\lim_{n\to+\infty} \mathcal{F}_{n,2}(t) = -\mathcal{E}_2(t). \tag{5.25}$$

To this end, we define, for each $n \geq n^*$,

$$\mathcal{H}_n(t) := \int_0^t \left\langle J_1 B^* \left(\int_0^s S^*(\gamma - s)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi d\gamma \right), B^* S^*(-s)\psi \right\rangle_{U,U'} ds. \tag{5.26}$$

Then we have two observations: First, since $\mathcal{R}_n S^*(\cdot) = S^*(\cdot)\mathcal{R}_n$ and $B^*\mathcal{R}_n \in \mathcal{L}(H')$ (see (3.16)), we have

$$\mathcal{H}_n(t) = \int_0^t \int_0^s \langle J_1 B^* S^*(\gamma - s)\mathcal{R}_n \Pi_{\alpha,\varepsilon}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi, B^* S^*(-s)\psi\rangle_{U,U'} d\gamma ds,\ n \geq n^*.$$

(See [1, Lemma 11.45].) By using some simple integral transformations in the above, we find

$$\mathcal{H}_n(t) = \int_0^t \int_0^{t-\gamma} \langle J_1 B^* S^*(-\sigma)\mathcal{R}_n \Pi_{\alpha,\varepsilon,T}^{-1} \widehat{Q_{\alpha,\varepsilon,T}} \mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi, B^* S^*(-(\gamma+\sigma))\psi\rangle_{U,U'} d\sigma d\gamma,\ n \geq n^*,$$

which implies

$$\mathcal{F}_{n,2}(t) = -TC_1(\alpha)e^{\alpha T}\mathcal{H}_n(t) \ \text{ for each } \ n \geq n^*. \tag{5.27}$$

Second, if we let, for each $n \geq n^*$,

$$w_n(s) := \int_0^s S^*(\gamma - s)(\mathcal{R}_n - I)\Pi^{-1}_{\alpha,\varepsilon,T}\widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi d\gamma, \tag{5.28}$$

then, similar to the proof of (3.15), we can show

$$\lim_{n\to+\infty}\int_{-|t|}^{|t|}\|B^*w_n(s)\|^2_{U'}ds = 0. \tag{5.29}$$

Now, it follows by the assumption (H_3), (5.26), (5.28) and (5.29) that

$$\lim_{n\to+\infty}\mathcal{H}_n(t) = \int_0^t \left\langle J_1B^*\left(\int_0^s S^*(\gamma - s)\Pi^{-1}_{\alpha,\varepsilon,T}\widehat{Q_{\alpha,\varepsilon,T}}\mathcal{V}_{\alpha,\varepsilon,T}(\gamma)\varphi d\gamma\right), B^*S^*(-s)\psi\right\rangle_{U,U'} ds, \tag{5.30}$$

which, along with the definition of $\mathcal{E}_2(t)$, (5.27) and (5.30), yields (5.25). Finally, (5.21) follows by (5.22), (5.23), (5.24) and (5.25).

Thus, we finish the proof of Lemma 2.5. □

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