Classification of solutions of the 2D steady Navier-Stokes equations with separated variables in cone-like domains

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Abstract

We investigate the problem of classification of solutions for the steady Navier-Stokes equations in any cone-like domain. In the form of separated variables,

$$u(x,y) = \begin{pmatrix} \varphi_1(r)v_1(\theta) \\ \varphi_2(r)v_2(\theta) \end{pmatrix},$$

where $x = r\cos\theta$ and $y = r\sin\theta$ in the polar coordinates, we obtain the expressions of all smooth solutions with C^0 Dirichlet boundary condition. In particular, we find some solutions which are Hölder continuous on the boundary but their gradients blow up at the corner, show that all solutions in the entire plane $\mathbb{R}^2$ must be polynomials, and prove a sharp Liouville type theorem.

Keywords: classification of solutions; Liouville type theorem; steady Navier-Stokes equations; separation of variables.

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1 Introduction

Consider the Navier-Stokes equations

$$\begin{cases} \partial_t u - \triangle u + u\cdot\nabla u + \nabla p = 0, \\ \text{div } u = 0, \end{cases} \tag{1}$$

for $x \in \mathbb{R}^n$ and $t \geq 0$. Here $u(x,t) = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big)$ denotes the unknown velocity of the fluid, and the scalar function $p(x,t)$ denotes the unknown pressure.

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It is well-known that one of the seven most important unsolved problems of the Clay Mathematics Institute is whether the existence and smoothness of solutions hold for 3D Navier-Stokes equations (1) with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3,$$

where $u_0(x)$ is a smooth, divergence-free vector field decaying sufficiently fast as $x \to \infty$; see [8].

In a seminal paper [17], Leray proved the global existence of weak (or generalized) solutions in a suitable function space. In 3D, the problems of uniqueness and regularity of weak solutions are of great significance in mathematical fluid mechanics and still open. Here the regularity problem usually refers to the smoothness problem of solutions. In the context of Navier-Stokes equations, sometimes local boundedness is sufficient for smoothness; see for example [15, Proposition 15.1, p. 147].

Recently, for the axi-symmetric Navier-Stokes equations, important progress has been made by Chen-Strain-Yau-Tsai [6, 7] and Koch-Nadirashvili-Seregin-Šverák [13], respectively. They showed that solutions do not develop type I singularity. We call a singularity of type I for a Navier-Stokes solution u at time T if

$$\sup_x |u(x,t)| \le \frac{C}{\sqrt{T-t}}.$$

The innovative idea in [13] is that they transform the regularity problem into Liouville type theorems by the classic rescaling and blow-up procedure. If a Liouville type theorem for some kind of solutions is available (they proved that this is true for the axi-symmetric case), then finite-time singularities of type I can be ruled out.

Here, a Liouville type theorem means a theorem asserting that equations only have constant or trivial solutions, as the classical Liouville theorem asserts that any bounded entire analytic function must be constant.

Regarding Liouville theorems for Navier-Stokes equations, they said ([13], p. 84): *"The case of general 3-dimensional fields is, as far as we know, completely open. In fact, it is open even in the steady-state case (u independent of t)."*

Motivated by their question in [13], we consider here the case of the incompressible steady Navier-Stokes equations on the whole space $\mathbb{R}^n$:

$$\begin{cases} -\triangle u + u \cdot \nabla u + \nabla p = 0, \\ \text{div } u = 0, \end{cases} \tag{2}$$

and investigate the classification of solutions of (2). First, let us review some major developments on this topic. In the 3D steady-state case, Galdi [10] proved the Liouville theorem that u must be constant under the assumptions that $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ and the Dirichlet integral is finite, namely

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx < \infty. \tag{3}$$

A very challenging open problem is whether there exists a nontrivial solution under the assumption of finite Dirichlet integral (3) without the condition $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$. This uniqueness problem (or equivalently the Liouville type problem), can date back to Leray's celebrated paper [16], and is explicitly written in Galdi's book [10, Remark X. 9.4, p. 729];

see also Tsai's book [21, p. 23]. Later, Chae-Wolf [4] gave an improvement for Galdi's result by replacing the condition $u \in L^{\frac{9}{2}}(\mathbb{R}^3)$ by $\int_{\mathbb{R}^3} |u|^{\frac{9}{2}}\{\ln(2+\frac{1}{|u|})\}^{-1}dx < \infty$. For more references, we refer to [20, 5, 3] and the references therein. Note that the above Dirichlet integral condition (3) implies that the solution is bounded. However, it is still unknown whether the Liouville theorem holds for bounded solutions.

In fact, even for the 2D case, the problem of classification of solutions for the steady Navier-Stokes equations is still not solved, while some Liouville type theorems were proved. We give some examples here. Gilbarg-Weinberger [11] proved the Liouville theorem by assuming the Dirichlet integral condition (3) alone. They used the fact that the vorticity function $w = \partial_y u_1 - \partial_x u_2$ satisfies the fine elliptic equation

$$\triangle w - u \cdot \nabla w = 0 \tag{4}$$

and applied a maximum principle to (4). If $\nabla u \in L^q(\mathbb{R}^2)$ with $1 < q < \infty$, the first author [22] proved that u is a constant vector by using the growth estimate of the functions whose gradients belong to the L^q space; see also [14] for another approach. When the solution u is bounded, a Liouville theorem was obtained by Koch-Nadirashvili-Seregin-Šverák in [13] as a byproduct of their work on the non-steady case. More generally, if $\limsup_{|x|\to\infty} |x|^{-\alpha}|u(x)| \le C$ with $\alpha \in [0, \frac{1}{7})$, Fuchs-Zhong [9] proved the constancy of u. Later, the exponent α is improved to $\alpha < \frac{1}{3}$ in [1] with the help of the vorticity equation (4). It seems that α could be improved to 1, as suggested by Fuchs-Zhong in [9]:

"Suppose that $\lim_{|x|\to\infty} |x|^{-1}|u(x)| = 0$. *Does the constancy of u follow?"*

This is true for harmonic functions. It follows from the well-known fact that a harmonic function on $\mathbb{R}^n$ having polynomial growth is necessarily a polynomial; see also Yau [23] and Li-Tam [18], where they considered the space of harmonic functions with linear growth on complete manifolds with nonnegative Ricci curvature.

In this paper, our purpose is to classify the solutions for the 2D steady Navier-Stokes equations by separating variables, and as a byproduct we prove a sharp Liouville theorem to answer the above question of Fuchs-Zhong in this setting.

Let Ω be the whole space $\mathbb{R}^2$, the half-space $\mathbb{R}^2_+$, or any cone domain of $\{(r,\theta); \alpha < \theta < \beta,\ 0 < r < \infty\}$ with $0 \le \alpha < \beta \le 2\pi$. Our first result is as follows.

Theorem 1.1. *Suppose that* $(u,p) \in C^3(\Omega) \times C^1(\Omega)$ *is a solution of (2),* $u \in C^0(\bar{\Omega})$ *and* u *has the form*

$$u(x,y) = \varphi(r)\begin{pmatrix} v_1(\theta) \\ v_2(\theta) \end{pmatrix}.$$

Then, (u,p) *can only be expressed in one of the following types:*

(i)

$$u = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad p = C_3.$$

(ii)

$$u = \begin{pmatrix} C_1 x + C_2 y \\ C_3 x - C_1 y \end{pmatrix}, \quad p = -\frac{1}{2}(C_1^2 + C_2 C_3)(x^2+y^2) + C_4. \tag{5}$$

(iii)

$$u = \begin{pmatrix} (C_2+C_3)x^2 + (3C_2 - C_3)y^2 + 2(C_1+C_4)xy \\ (C_4 - 3C_1)x^2 - (C_1+C_4)y^2 - 2(C_2+C_3)xy \end{pmatrix},$$
$$p = \frac{1}{2}(C_1^2 + C_2^2 - C_3^2 - C_4^2)(x^2+y^2)^2 + 8C_2 x - 8C_1 y + C_5,$$

with C_1, C_2, C_3, C_4 satisfying

$$\begin{cases} C_1C_3 + C_2C_4 - 2C_1C_2 = 0, \\ C_1C_4 - C_2C_3 + C_1^2 - C_2^2 = 0. \end{cases} \tag{6}$$

(iv)

$$u = r^\lambda \begin{pmatrix} C_1\cos(\lambda\theta) + C_2\sin(\lambda\theta) \\ C_2\cos(\lambda\theta) - C_1\sin(\lambda\theta) \end{pmatrix}, \quad p = -\frac{1}{2}(C_1^2 + C_2^2)r^{2\lambda} + C_3.$$

Here, if $\Omega = \mathbb{R}^2$, $\lambda \geq 3$ and $\lambda \in \mathbb{N}$, otherwise $\lambda \in (0,1) \cup (1,2) \cup (2,\infty)$.

(v) If $\Omega \neq \mathbb{R}^2$,

$$u = (C_1 + C_2 \ln r)\begin{pmatrix} -y \\ x \end{pmatrix},$$

$$p = \frac{1}{4}r^2\left[2C_2^2\ln^2 r + (4C_1C_2 - 2C_2^2)\ln r + 2C_1^2 - 2C_1C_2 + C_2^2\right] + 2C_2\theta + C_3.$$

Remark 1.2.

1. *In type (iii), the coefficients of u_1 are proportional to those of u_2, since equations (6) are equivalent to*

$$\frac{C_2 + C_3}{C_4 - 3C_1} = \frac{3C_2 - C_3}{-(C_1 + C_4)} = \frac{(C_1 + C_4)}{-(C_2 + C_3)}.$$

2. *(The boundary blow-up phenomenon) The solutions in type (v) show that*

$$\nabla u(x,y) = \begin{pmatrix} -C_2\sin\theta\cos\theta & -(C_1 + C_2\ln r + C_2\sin^2\theta) \\ C_1 + C_2\ln r + C_2\cos^2\theta & C_2\sin\theta\cos\theta \end{pmatrix},$$

which blow up at the corner of $r = 0$. However, $u \in C^\gamma(\bar{\Omega})$ locally for any $0 < \gamma < 1$. This is different from the case in [12], where the authors consider a class of Hölder continuous boundary data on the time and prove that there exist unbounded gradients at the boundary. These solutions also show that the singularity of the solutions does not depend on the regularity of the boundary (for example, the case of $\mathbb{R}^2_+$ when $\alpha = 0, \beta = \pi$).

3. *The examples in type (iv) show that: for any $0 < \gamma < 1$ some solutions of steady Navier-Stokes equations have the boundary C^γ regularity similar to their nontrivial boundary data, while one can prove a uniform boundary C^{γ_0} with $\gamma_0 > 0$ regularity result for 6D steady Navier-Stokes equations with zero-Dirichlet boundary data (for example, see [19]). Therefore the boundary data plays an important role in the boundary regularity theory for steady Navier-Stokes equations.*

More generally, let u be the form of

$$u(x,y) = \begin{pmatrix} \varphi_1(r)v_1(\theta) \\ \varphi_2(r)v_2(\theta) \end{pmatrix}, \tag{7}$$

and we have the following conclusions.

Theorem 1.3. *Suppose that $(u,p) \in C^3(\Omega) \times C^1(\Omega)$ is a solution of (2) with the form of (7) and $u \in C^0(\bar{\Omega})$. Then, (u,p) can only be expressed as one of the forms of (i), (ii), (iii), (iv) and (v) in Theorem 1.1, or one of the following two types:*

$$u = \begin{pmatrix} C_1 \\ C_2 x \end{pmatrix}, \quad p = -C_1 C_2 y + C_3; \tag{8}$$

and

$$u = \begin{pmatrix} C_1 y \\ C_2 \end{pmatrix}, \quad p = -C_1 C_2 x + C_3. \tag{9}$$

This theorem immediately leads to the following conclusion.

Corollary 1.4. *Suppose that (u,p) satisfies the assumptions of Theorem 1.3 with $\Omega = \mathbb{R}^2$, then u and p must be polynomials, which is similar to harmonic functions on the whole space.*

As another application of Theorem 1.3, we obtain a sharp Liouville theorem for (2) in any cone domain when u has the form (7), which answers the question in [9] in this setting.

Corollary 1.5. *Suppose that (u,p) satisfies the assumptions of Theorem 1.3 with $u \in C^1(\bar{\Omega})$ and*

$$\lim_{|x|\to\infty} |x|^{-1}|u(x)| = 0, \tag{10}$$

then (u,p) must be constant.

Remark 1.6. *Condition (10) says that the growth of u is less than $|x|$. This condition is sharp for the constancy of u. If (10) does not hold, there exist nontrivial solutions, such as (5), (8) and (9).*

2 Preliminaries

Before proving the main theorems, we state some preliminary lemmas, which play important roles in our arguments.

In this part, we let I and J be intervals in $\mathbb{R}$.

Lemma 2.1. *Suppose that*

$$A(\theta)f(r) = B(\theta)g(r), \qquad \theta \in I, \quad r \in J. \tag{11}$$

If $g(r) \not\equiv 0$, then either

$$A(\theta) = B(\theta) \equiv 0,$$

or, there exists a constant λ such that

$$B(\theta) = \lambda A(\theta), \qquad f(r) = \lambda g(r).$$

Proof. Assume that $g(r_0) \neq 0$ without loss of generality.

Case 1: $A(\theta) \equiv 0$. Then $B(\theta)g(r) \equiv 0$ for $\theta \in I$ and $r \in J$, which implies $B(\theta) \equiv 0$ for $\theta \in I$ due to $g(r_0) \neq 0$.

Case 2: $A(\theta) \not\equiv 0$. We assume that $A(\theta_0) \neq 0$ for some $\theta_0 \in I$, then

$$f(r) = \frac{B(\theta_0)}{A(\theta_0)} g(r) =: \lambda g(r), \qquad r \in J, \tag{12}$$

where $\lambda = \frac{B(\theta_0)}{A(\theta_0)}$. Substituting equation (12) into (11), there holds

$$\lambda A(\theta) g(r) = B(\theta) g(r), \qquad \theta \in I, \quad r \in J,$$

which yields $B(\theta) = \lambda A(\theta)$ for $\theta \in I$ by taking $r = r_0$. The proof is complete. □

Lemma 2.2. *Let $v = v(\theta),\ \theta \in I$.*
(1) If $v \in C^1(I)$ and satisfies $\sin\theta v + \cos\theta v' = 0$, then $v = C\cos\theta$.
(2) If $v \in C^2(I)$ and satisfies $2\sin\theta v + \cos\theta v' = 0$, then $v = C\cos^2\theta$.

Proof. Assume that $I = (0, 2\pi)$ for simplicity and denote

$$I_1 = \Big(0, \frac{\pi}{2}\Big), \quad I_2 = \Big(\frac{\pi}{2}, \frac{3\pi}{2}\Big), \quad I_3 = \Big(\frac{3\pi}{2}, 2\pi\Big),$$

then

$$\cos\theta \neq 0, \qquad \theta \in I_i, \quad i = 1, 2, 3.$$

(1) In $I_i,\ i = 1, 2, 3$,

$$\big(\cos^{-1}\theta v\big)' = \cos^{-2}\theta(\sin\theta v + \cos\theta v') = 0,$$

then

$$\cos^{-1}\theta v = C_i, \quad v = C_i\cos\theta, \qquad \theta \in I_i,$$

and

$$v' = -C_i\sin\theta, \qquad \theta \in I_i.$$

Since v' is continuous at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, then $C_1 = C_2 = C_3 =: C$, and thus $v = C\cos\theta$.

(2) The argument is simialr, and we omitted it.

The proof is complete. □

Lemma 2.3. *Suppose that $\varphi_1,\ \varphi_2 \in C^1\big((0, +\infty)\big)$ and satisfy*

$$\begin{cases} r\varphi_1'(r) = a\varphi_1 + b\varphi_2, \\ r\varphi_2'(r) = c\varphi_1 + d\varphi_2. \end{cases} \tag{13}$$

Let $\delta := (a-d)^2 + 4bc$. Then
(1) If $b = 0,\ d = a$, then

$$\begin{cases} \varphi_1 = C_1 r^a, \\ \varphi_2 = (cC_1 \ln r + C_2) r^a. \end{cases}$$

(2) If $b = 0,\ d \neq a$, then

$$\begin{cases} \varphi_1 = C_1 r^a, \\ \varphi_2 = \frac{c}{a-d} C_1 r^a + C_2 r^d. \end{cases}$$

(3) If $b \neq 0,\ \delta > 0$, then

$$\begin{cases} \varphi_1 = C_1 r^m + C_2 r^n, & (m > n) \\ \varphi_2 = \frac{m-a}{b} C_1 r^m + \frac{n-a}{b} C_2 r^n, \end{cases}$$

where m, n are two different real roots of the equation

$$\rho^2 - (a+d)\rho + ad - bc = 0. \tag{14}$$

(4) If $b \neq 0,\ \delta = 0$, then

$$\begin{cases} \varphi_1 = (C_1 \ln r + C_2) r^l, \\ \varphi_2 = \big[\frac{l-a}{b} C_1 \ln r + \frac{C_1 + (l-a)C_2}{b}\big] r^l, \end{cases}$$

where l is the unique real root of (14).
(5) If $b \neq 0,\ \delta < 0$, then

$$\begin{cases} \varphi_1 = \big[C_1 \cos(\mu \ln r) + C_2 \sin(\mu \ln r)\big] r^\lambda, \\ \varphi_2 = \big[\frac{(\lambda-a)C_1 + \mu C_2}{b} \cos(\mu \ln r) + \frac{(\lambda-a)C_2 - \mu C_1}{b} \sin(\mu \ln r)\big] r^\lambda, \end{cases}$$

where $\lambda \pm \mu i$ are the complex roots of (14).

Proof. Let $r = e^t$ and $D = \frac{d}{dt}$, then the equations (13) become

$$\begin{cases} D\varphi_1 = a\varphi_1 + b\varphi_2, \\ D\varphi_2 = c\varphi_1 + d\varphi_2. \end{cases} \tag{15}$$

Case 1: $b = 0$. The first equation of (15) becomes $D\varphi_1 = a\varphi_1$, then

$$\varphi_1 = C_1 e^{at} = C_1 r^a. \tag{16}$$

Substituting (16) into the second equation of (15), we get

$$D\varphi_2 - d\varphi_2 = cC_1 e^{at}.$$

Then

$$D\big(e^{-dt}\varphi_2\big) = e^{-dt}(D\varphi_2 - d\varphi_2) = cC_1 e^{(a-d)t}, \tag{17}$$

which can be divided into the following two situations.

Case 1.1: If $d = a$, there holds $e^{-at}\varphi_2 = cC_1 t + C_2$, and

$$\varphi_2 = (cC_1 t + C_2)e^{at} = (cC_1 \ln r + C_2) r^a.$$

Case 1.2: If $d \neq a$, it follows from (17) that $e^{-dt}\varphi_2 = \frac{c}{a-d} C_1 e^{(a-d)t} + C_2$, and

$$\varphi_2 = \frac{c}{a-d} C_1 e^{at} + C_2 e^{dt} = \frac{c}{a-d} C_1 r^a + C_2 r^d.$$

Case 2: $b \neq 0$. The first equation of (15) implies that

$$\varphi_2 = \frac{1}{b}(D\varphi_1 - a\varphi_1). \tag{18}$$

Substituting (18) into the second euqation of (15), we get

$$D^2\varphi_1 - (a+d)D\varphi_1 + (ad-bc)\varphi_1 = 0, \tag{19}$$

which has the characteristic equation (14).

Case 2.1: If $\delta > 0$, equation (14) has two different real roots m, n $(m > n)$ and the general solution of (19) is expressed as follows:

$$\varphi_1 = C_1e^{mt} + C_2e^{nt} = C_1r^m + C_2r^n.$$

Substituting this into (18), we get

$$\varphi_2 = \frac{m-a}{b}C_1e^{mt} + \frac{n-a}{b}C_2e^{nt} = \frac{m-a}{b}C_1r^m + \frac{n-a}{b}C_2r^n.$$

Case 2.2: If $\delta = 0$, equation (14) has a unique real root l and the general solution of (19) is

$$\varphi_1 = (C_1t + C_2)e^{lt} = (C_1\ln r + C_2)r^l.$$

Substituting this into (18), we get

$$\varphi_2 = \Big[\frac{l-a}{b}C_1\ln r + \frac{C_1 + (l-a)C_2}{b}\Big]r^l.$$

Case 2.3: If $\delta < 0$, equation (14) has complex roots $\lambda \pm \mu i$ $(\mu \neq 0)$ and the general solution of (19) is

$$\varphi_1 = \big[C_1\cos(\mu t) + C_2\sin(\mu t)\big]e^{\lambda t} = \big[C_1\cos(\mu\ln r) + C_2\sin(\mu\ln r)\big]r^\lambda.$$

Substituting this into (18), we get

$$\varphi_2 = \Big[\frac{(\lambda-a)C_1 + \mu C_2}{b}\cos(\mu\ln r) + \frac{(\lambda-a)C_2 - \mu C_1}{b}\sin(\mu\ln r)\Big]r^\lambda.$$

Thus the proof is complete. □

Lemma 2.4. *Suppose that $v_1,\ v_2 \in C^1(I)$ satisfying*

$$\begin{cases} a\cos\theta v_1 + c\sin\theta v_2 - \sin\theta v_1' = 0, \\ b\cos\theta v_1 + d\sin\theta v_2 + \cos\theta v_2' = 0, \end{cases} \tag{20}$$

and $v_1v_2 \equiv 0$. Then we have (i) $v_1 \equiv 0$ if $b \neq 0$; (ii) $v_2 \equiv 0$ if $c \neq 0$.

Proof. Multiplying $(20)_1$ by v_2, due to $v_1v_2 \equiv 0$ we get

$$cv_2^2 - v_1'v_2 = 0. \tag{21}$$

Similarly, multiplying $(20)_2$ by v_1, it follows that

$$bv_1^2 + v_1v_2' = 0. \tag{22}$$

(22) minus (21) tells us that $bv_1^2 - cv_2^2 + (v_1v_2)' = 0$. Then $bv_1^2 = cv_2^2$, since $v_1v_2 \equiv 0$. Consequently,

$$bv_1^3 = cv_2(v_1v_2) \equiv 0, \quad cv_2^3 = bv_1(v_1v_2) \equiv 0.$$

If $b \neq 0$, then $v_1 \equiv 0$. If $c \neq 0$, then $v_2 \equiv 0$. The proof is complete. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Let $w := \partial_2 u_1 - \partial_1 u_2$ be the vorticity of u, then w satisfies the equation

$$\triangle w - u\cdot\nabla w = 0. \tag{23}$$

Throughout this section, we write $v_i(\theta)$, $v_i'(\theta)$, $\varphi(r)$, $\varphi'(r)$ as v_i, v_i', φ, φ', $i = 1, 2$.

Direct computations show that

$$\begin{aligned}\text{div } u =& \big(\cos\theta v_1 + \sin\theta v_2\big)\varphi' - \big(\sin\theta v_1' - \cos\theta v_2'\big)\frac{\varphi}{r}\\ =: & A(\theta)\varphi' - B(\theta)\frac{\varphi}{r};\end{aligned} \tag{24}$$

$$w = \big(\sin\theta v_1 - \cos\theta v_2\big)\varphi' + \big(\cos\theta v_1' + \sin\theta v_2'\big)\frac{\varphi}{r}. \tag{25}$$

The equation div $u = 0$ and (24) yield that $A(\theta)\varphi' = B(\theta)\frac{\varphi}{r}$. By Lemma 2.1, we have either

$$A(\theta) = B(\theta) \equiv 0,$$

or

$$B(\theta) = \lambda A(\theta), \qquad \varphi' = \lambda\frac{\varphi}{r}.$$

Next, we discuss the two cases respectively.

Case 1: $A(\theta) = B(\theta) \equiv 0$. $A(\theta) \equiv 0$ implies that

$$v_1 = -\tan\theta v_2, \tag{26}$$

and $B(\theta) \equiv 0$ tells us that

$$-\tan\theta v_1' + v_2' = 0. \tag{27}$$

Substituting (26) into (27), we deduce that $\sin\theta v_2 + \cos\theta v_2' = 0$. Due to $v_2 \in C^1$, applying Lemma 2.2 we obtain that

$$v_2 = C\cos\theta, \tag{28}$$

and thus

$$v_1 = -C\sin\theta. \tag{29}$$

Without loss of generality, we assume that $C \neq 0$. Substituting (28) and (29) into (25), we have

$$w = -C\big(\varphi' + \frac{\varphi}{r}\big),$$

and then

$$\begin{aligned}\Delta w =& \big(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\big)w = -C\big(\partial_r^2 + \frac{1}{r}\partial_r\big)\big(\varphi' + \frac{\varphi}{r}\big)\\ =& -C\big(\varphi''' + \frac{2\varphi''}{r} - \frac{\varphi'}{r^2} + \frac{\varphi}{r^3}\big);\\ \partial_1 w =& -C\big(\varphi' + \frac{\varphi}{r}\big)'\cos\theta = -C\cos\theta\big(\varphi'' + \frac{\varphi'}{r} - \frac{\varphi}{r^2}\big);\\ \partial_2 w =& -C\sin\theta\big(\varphi'' + \frac{\varphi'}{r} - \frac{\varphi}{r^2}\big);\\ u\cdot\nabla w =& \varphi(v_1\partial_1 w + v_2\partial_2 w) = 0.\end{aligned} \tag{30}$$

Combining (23) and (30), we obtain

$$r^3\varphi''' + 2r^2\varphi'' - r\varphi' + \varphi = 0. \tag{31}$$

Let $r = e^t$ and denote $D = \frac{d}{dt}$, then the equation (31) becomes

$$(D+1)(D-1)^2\varphi = 0,$$

which has the general solution as

$$\varphi = C_1e^t + C_2te^t + C_3e^{-t} = C_1r + C_2r\ln r + C_3r^{-1}.$$

Recall that $u \in C^3(\Omega) \cap C^0(\bar{\Omega})$, then

$$\varphi = \begin{cases} C_1r, & \text{if } \Omega = \mathbb{R}^2; \\ C_1r + C_2r\ln r, & \text{if } \Omega \neq \mathbb{R}^2. \end{cases} \tag{32}$$

By (28), (29) and (32),

$$u = \begin{cases} C_1'\begin{pmatrix} -y \\ x \end{pmatrix}, & \text{if } \Omega = \mathbb{R}^2; \\ (C_1' + C_2'\ln r)\begin{pmatrix} -y \\ x \end{pmatrix}, & \text{if } \Omega \neq \mathbb{R}^2. \end{cases} \tag{33}$$

Case 2: $B(\theta) = \lambda A(\theta)$, $\varphi' = \lambda\frac{\varphi}{r}$. At this time, we have

$$\varphi = Cr^\lambda. \tag{34}$$

Without loss of generality, we assume that $C \neq 0$. Recall that $u \in C^0(\bar{\Omega})$, then $\lambda \geq 0$. Denote

$$L(\theta) = \sin\theta v_1 - \cos\theta v_2,$$

then it is easy to verify that

$$L' = A + B = (\lambda+1)A, \quad \cos\theta v_1' + \sin\theta v_2' = A' + L. \tag{35}$$

By (25), (34) and (35) we have

$$w = C\lambda r^{\lambda-1}L + Cr^{\lambda-1}(A' + L) = Cr^{\lambda-1}\big[A' + (\lambda+1)L\big] =: Cr^{\lambda-1}H, \tag{36}$$

where

$$H = A' + (\lambda+1)L.$$

Then we have

$$\begin{aligned} \Delta w =& \big(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\big)w \\ =& CH\big(\partial_r^2 + \frac{1}{r}\partial_r\big)r^{\lambda-1} + Cr^{\lambda-3}H'' \\ =& Cr^{\lambda-3}\big[H'' + (\lambda-1)^2H\big]; \\ u\cdot\nabla w =& \varphi(v_1\partial_1 w + v_2\partial_2 w) \\ =& \varphi\Big[v_1\Big(\partial_r w\cos\theta - \partial_\theta w\frac{\sin\theta}{r}\Big) + v_2\Big(\partial_r w\sin\theta + \partial_\theta w\frac{\cos\theta}{r}\Big)\Big] \\ =& \varphi\big(\partial_r wA - \frac{\partial_\theta w}{r}L\big) \\ =& C^2r^{2\lambda-2}\big[(\lambda-1)HA - H'L\big]. \end{aligned} \tag{37}$$

$\triangle w - u\cdot\nabla w = 0$ gives that

$$Cr^{\lambda+1}\big[(\lambda-1)HA - H'L\big] = H'' + (\lambda-1)^2 H.$$

Since $\lambda \geq 0$, the above equation is equivalent to

$$\begin{cases} (\lambda-1)HA - H'L = 0, & (38)\\ H'' + (\lambda-1)^2 H = 0. & (39)\end{cases}$$

We keep in mind that

$$L' = (\lambda+1)A, \tag{40}$$

$$H = A' + (\lambda+1)L. \tag{41}$$

The above two equations yield that

$$A'' + (\lambda+1)^2 A = H'. \tag{42}$$

First, we can solve H according to equation (39). If $\lambda = 1$, it is easy. If $\lambda \neq 1$, (39) has general solution

$$H = A' + (\lambda+1)L = C_1\cos\big((\lambda-1)\theta\big) + C_2\sin\big((\lambda-1)\theta\big), \qquad \lambda\neq 1. \tag{43}$$

Second, we solve A according to equation (42). Substituting (43) into (42), we have

$$A'' + (\lambda+1)^2 A = (\lambda-1)\Big[C_2\cos\big((\lambda-1)\theta\big) - C_1\sin\big((\lambda-1)\theta\big)\Big], \qquad \lambda\neq 1.$$

This equation has general solution

$$\begin{aligned} A =& C_3\cos\theta + C_4\sin\theta + \frac{\theta}{2}(C_1\cos\theta - C_2\sin\theta)\\ =& \cos\theta\Big(\frac{C_1\theta}{2} + C_3\Big) + \sin\theta\Big(-\frac{C_2\theta}{2} + C_4\Big), \qquad \lambda = 0;\\ A =& C_3\cos\big((\lambda+1)\theta\big) + C_4\sin\big((\lambda+1)\theta\big)\\ &+ \frac{\lambda-1}{4\lambda}\Big[C_2\cos\big((\lambda-1)\theta\big) - C_1\sin\big((\lambda-1)\theta\big)\Big], \qquad \lambda\neq 0,1. \end{aligned} \tag{44}$$

Substituting (43) into equation (38), we have

$$\begin{aligned} &\Big[C_1\cos\big((\lambda-1)\theta\big) + C_2\sin\big((\lambda-1)\theta\big)\Big]A\\ -&\Big[C_2\cos\big((\lambda-1)\theta\big) - C_1\sin\big((\lambda-1)\theta\big)\Big]L = 0, \qquad \lambda\neq 1. \end{aligned} \tag{45}$$

Next, the classification of solutions is discussed in the following cases.

Case 2.1: $\lambda = 1$. In this easy case, equations (38) and (39) become

$$\begin{cases} H'L = 0,\\ H'' = 0,\end{cases}$$

then $H = a\theta + b$ and $aL = 0$. If $a\neq 0$, then $L = 0$. By (40), $A = \frac{1}{2}L' = 0$, $H = A' + 2L = 0$, this contradicts that $a \neq 0$. Therefore $a = 0$ and $H = b$. Namely $A' + 2L = b$, then $A'' + 2L' = 0$. By (40), $L' = 2A$, then $A'' + 4A = 0$. This equation has general solution

$$A = C_1\cos 2\theta + C_2\sin 2\theta. \tag{46}$$

Then

$$L = \frac{1}{2}(H - A') = \frac{b}{2} + C_1 \sin 2\theta - C_2 \cos 2\theta. \tag{47}$$

Combining (46) and (47), we obtain

$$\begin{cases} v_1 = C_1 \cos\theta + \left(C_2 + \frac{b}{2}\right)\sin\theta, \\ v_2 = \left(C_2 - \frac{b}{2}\right)\cos\theta - C_1 \sin\theta. \end{cases}$$

Then

$$u = Cr \begin{pmatrix} C_1 \cos\theta + \left(C_2 + \frac{b}{2}\right)\sin\theta \\ \left(C_2 - \frac{b}{2}\right)\cos\theta - C_1 \sin\theta \end{pmatrix} = \begin{pmatrix} C_1' x + C_2' y \\ C_3' x - C_1' y \end{pmatrix}. \tag{48}$$

Case 2.2: $\lambda = 0$. As shown in (44), in this case

$$A = \cos\theta\Big(\frac{C_1\theta}{2} + C_3\Big) + \sin\theta\Big(-\frac{C_2\theta}{2} + C_4\Big). \tag{49}$$

By (40), (43) and (49), we have

$$L = H - A' = \cos\theta\Big(\frac{C_1 + C_2\theta}{2} - C_4\Big) + \sin\theta\Big(\frac{C_1\theta - C_2}{2} + C_3\Big). \tag{50}$$

We substitute (49) and (50) into (45) and obtain that

$$\begin{aligned} &\Big[\frac{(C_1^2 - C_2^2)\theta - C_1C_2}{2} + C_1C_3 + C_2C_4\Big]\cos 2\theta \\ -&\Big[C_1C_2\theta - C_1C_4 + C_2C_3 + \frac{C_1^2 - C_2^2}{4}\Big]\sin 2\theta = 0. \end{aligned}$$

Applying Lemma 2.1 again, we get

$$\frac{(C_1^2 - C_2^2)\theta - C_1C_2}{2} + C_1C_3 + C_2C_4 = C_1C_2\theta - C_1C_4 + C_2C_3 + \frac{C_1^2 - C_2^2}{4} \equiv 0,$$

and thus

$$C_1 = C_2 = 0. \tag{51}$$

By (49), (50) and (51), we have

$$\begin{cases} A = C_3 \cos\theta + C_4 \sin\theta \\ L = C_3 \sin\theta - C_4 \cos\theta. \end{cases}$$

Then

$$v_1 = C_3, \quad v_2 = C_4,$$

and

$$u = C \begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}. \tag{52}$$

Moreover, for $\lambda \neq 0, 1$, as shown in (44) there holds

$$A = C_3 \cos\big((\lambda+1)\theta\big) + C_4 \sin\big((\lambda+1)\theta\big) + \frac{\lambda - 1}{4\lambda}\Big[C_2 \cos\big((\lambda-1)\theta\big) - C_1 \sin\big((\lambda-1)\theta\big)\Big], \tag{53}$$

and due to (40), (43) and (53) we have

$$\begin{aligned} L =& \frac{1}{\lambda+1}(H - A') \\ =& C_3 \sin\big((\lambda+1)\theta\big) - C_4 \cos\big((\lambda+1)\theta\big) \\ &+ \frac{\lambda+1}{4\lambda}\Big[C_1 \cos\big((\lambda-1)\theta\big) + C_2 \sin\big((\lambda-1)\theta\big)\Big]. \end{aligned} \tag{54}$$

Furthermore, by substituting (53) and (54) into (45) we obtain

$$\begin{aligned} &(C_1C_3 + C_2C_4)\cos 2\theta + (C_1C_4 - C_2C_3)\sin 2\theta \\ -&\frac{1}{2\lambda}\Big[C_1C_2 \cos\big((2\lambda-2)\theta\big) + \frac{C_2^2 - C_1^2}{2}\sin\big((2\lambda-2)\theta\big)\Big] = 0. \end{aligned} \tag{55}$$

<u>Case 2.3: $\lambda = 2$.</u> Then (55) becomes

$$\Big(C_1C_3 + C_2C_4 - \frac{C_1C_2}{4}\Big)\cos 2\theta + \Big(C_1C_4 - C_2C_3 + \frac{C_1^2 - C_2^2}{8}\Big)\sin 2\theta = 0, \tag{56}$$

then

$$\begin{cases} C_1C_3 + C_2C_4 - \frac{C_1C_2}{4} = 0, \\ C_1C_4 - C_2C_3 + \frac{C_1^2 - C_2^2}{8} = 0. \end{cases} \tag{57}$$

In this case, we have by (53) and (54) that

$$\begin{cases} A = C_3 \cos 3\theta + C_4 \sin 3\theta + \frac{1}{8}\big(C_2 \cos\theta - C_1 \sin\theta\big), \\ L = C_3 \sin 3\theta - C_4 \cos 3\theta + \frac{3}{8}\big(C_1 \cos\theta + C_2 \sin\theta\big), \end{cases}$$

then

$$\begin{cases} v_1 = \big(C_3 + \frac{C_2}{8}\big)\cos^2\theta + \big(\frac{3C_2}{8} - C_3\big)\sin^2\theta + \big(C_4 + \frac{C_1}{8}\big)\sin 2\theta, \\ v_2 = \big(C_4 - \frac{3C_1}{8}\big)\cos^2\theta - \big(C_4 + \frac{C_1}{8}\big)\sin^2\theta - \big(C_3 + \frac{C_2}{8}\big)\sin 2\theta, \end{cases} \tag{58}$$

and

$$u = Cr^2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = C\begin{pmatrix} \big(C_3 + \frac{C_2}{8}\big)x^2 + \big(\frac{3C_2}{8} - C_3\big)y^2 + \big(2C_4 + \frac{C_1}{4}\big)xy \\ \big(C_4 - \frac{3C_1}{8}\big)x^2 - \big(C_4 + \frac{C_1}{8}\big)y^2 - \big(2C_3 + \frac{C_2}{4}\big)xy \end{pmatrix},$$

with C_1, C_2, C_3, C_4 satisfying equation (57). If we replace $\frac{CC_1}{8}$ by C_1, $\frac{CC_2}{8}$ by C_2, CC_3 by C_3, CC_4 by C_4, then

$$u = \begin{pmatrix} (C_2 + C_3)x^2 + (3C_2 - C_3)y^2 + 2(C_1 + C_4)xy \\ (C_4 - 3C_1)x^2 - (C_1 + C_4)y^2 - 2(C_2 + C_3)xy \end{pmatrix}, \tag{59}$$

with C_1, C_2, C_3, C_4 satisfying

$$\begin{cases} C_1C_3 + C_2C_4 - 2C_1C_2 = 0, \\ C_1C_4 - C_2C_3 + C_1^2 - C_2^2 = 0. \end{cases} \tag{60}$$

<u>Case 2.4: $\lambda \neq 0, 1, 2$.</u> By (55) we have

$$\begin{cases} C_1C_3 + C_2C_4 = 0, \\ C_1C_4 - C_2C_3 = 0, \\ C_1C_2 = 0, \\ C_2^2 - C_1^2 = 0, \end{cases}$$

namely

$$C_1 = C_2 = 0. \tag{61}$$

By (53), (54) and (61), we have

$$\begin{cases} A = C_3 \cos\big((\lambda+1)\theta\big) + C_4 \sin\big((\lambda+1)\theta\big), \\ L = C_3 \sin\big((\lambda+1)\theta\big) - C_4 \cos\big((\lambda+1)\theta\big), \end{cases}$$

then

$$\begin{cases} v_1 = C_3 \cos(\lambda\theta) + C_4 \sin(\lambda\theta), \\ v_2 = C_4 \cos(\lambda\theta) - C_3 \sin(\lambda\theta), \end{cases}$$

and

$$u = Cr^\lambda \begin{pmatrix} C_3 \cos(\lambda\theta) + C_4 \sin(\lambda\theta) \\ C_4 \cos(\lambda\theta) - C_3 \sin(\lambda\theta) \end{pmatrix} = r^\lambda \begin{pmatrix} C_1' \cos(\lambda\theta) + C_2' \sin(\lambda\theta) \\ C_2' \cos(\lambda\theta) - C_1' \sin(\lambda\theta) \end{pmatrix}, \tag{62}$$

where $\lambda \neq 0, 1, 2$.

Moreover, if $\Omega = \mathbb{R}^2$, it is necessary that $v_i(0) = v_i(2\pi), i = 1, 2$, namely

$$\begin{cases} C_1'[1 - \cos(2\lambda\pi)] - C_2' \sin(2\lambda\pi) = 0, \\ C_2'[1 - \cos(2\lambda\pi)] + C_1' \sin(2\lambda\pi) = 0. \end{cases}$$

Notice that the above equations are linear equations with respect to $1 - \cos(2\lambda\pi)$ and $\sin(2\lambda\pi)$, and the determinant

$$\begin{vmatrix} C_1' & -C_2' \\ C_2' & C_1' \end{vmatrix} = C_1'^2 + C_2'^2 \neq 0,$$

otherwise $u = 0$, which is included in the former case (52). By Cramer's Rule,

$$1 - \cos(2\lambda\pi) = \sin(2\lambda\pi) = 0,$$

thus $\lambda \in \mathbb{N}$. Recall that $\lambda \neq 0, 1, 2$, then $\lambda \geq 3$ and $\lambda \in \mathbb{N}$.

Finally, gathering (33), (48), (52), (59) and (62), we obtained that the solution u has five types of forms as in Theorem 1.1.

The pressure expressions. Next we substitute solutions of these types into equations (2) respectively. All these solutions satisfy

$$\text{div } u = 0,$$

then it is left to find the suitable pressure.

In type (i), it is easy to derive the solution

$$u = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad p = C_3.$$

In type (ii), direct computations give that

$$u = \begin{pmatrix} C_1 x + C_2 y \\ C_3 x - C_1 y \end{pmatrix}, \quad p = -\frac{1}{2}(C_1^2 + C_2 C_3)(x^2 + y^2) + C_4$$

is the solution.

In type (iii), by direct computations, we have that

$$\begin{aligned}
&\partial_1 p - 8C_2 + 2(C_3^2 + C_4^2 - 2C_1C_4 + 2C_2C_3 + C_2^2 - 3C_1^2)x^3 \\
&\quad + 2(C_3^2 + C_4^2 + 2C_1C_4 - 2C_2C_3 + C_1^2 - 3C_2^2)xy^2 \\
&\quad + 8(C_1C_3 + C_2C_4 - 2C_1C_2)x^2y = 0, \\
&\partial_2 p + 8C_1 + 2(C_3^2 + C_4^2 + 2C_1C_4 - 2C_2C_3 + C_1^2 - 3C_2^2)y^3 \\
&\quad + 2(C_3^2 + C_4^2 - 2C_1C_4 + 2C_2C_3 + C_2^2 - 3C_1^2)x^2y \\
&\quad + 8(C_1C_3 + C_2C_4 - 2C_1C_2)xy^2 = 0.
\end{aligned}$$

We apply (60) to the above equations, then

$$\begin{aligned}
&\partial_1 p - 8C_2 + 2(C_3^2 + C_4^2 - C_1^2 - C_2^2)(x^3 + xy^2) = 0, \\
&\partial_2 p + 8C_1 + 2(C_3^2 + C_4^2 - C_1^2 - C_2^2)(y^3 + x^2y) = 0.
\end{aligned}$$

Therefore

$$p = \frac{1}{2}(C_1^2 + C_2^2 - C_3^2 - C_4^2)(x^2 + y^2)^2 + 8C_2x - 8C_1y + C_5,$$

with C_1, C_2, C_3, C_4 satisfying equation (60).

In type (iv), note that $C_1 = C_2 = 0$ in (61), which implies $H = 0$ and $w \equiv 0$ due to (43) and (36). Using

$$(u_2, -u_1)^T w = u \cdot \nabla u - \nabla\Big(\frac{|u|^2}{2}\Big),$$

the pressure can be expressed by $-\frac{1}{2}|u|^2 + C$. Then

$$p = -\frac{1}{2}(C_1^2 + C_2^2)r^{2\lambda} + C_3.$$

In type (v), $\Omega \neq \mathbb{R}^2$ and

$$u = (C_1 r + C_2 r \ln r) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Direct computations give that

$$w = -(2C_2 \ln r + C_2 + 2C_1),$$

and

$$u \cdot \nabla u = (C_1 + C_2 \ln r)\partial_\theta u.$$

Then

$$\begin{aligned}\nabla p &= \triangle u - u\cdot\nabla u = \nabla^\top w - u\cdot\nabla u\\ &= \frac{2C_2}{r}\begin{pmatrix} -\sin\theta\\ \cos\theta\end{pmatrix} + (C_1+C_2\ln r)^2 r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},\end{aligned}$$

which implies

$$p = \frac{1}{4}r^2\Big[2C_2^2\ln^2 r + (4C_1C_2 - 2C_2^2)\ln r + 2C_1^2 - 2C_1C_2 + C_2^2\Big] + 2C_2\theta + C_3$$

by integration by parts.

Thus the proof is complete. □

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Our strategy is to reduce the problem here to the one stated in Theorem 1.1.

Proof of Theorem 1.3. In this part, u has the form

$$u = \begin{pmatrix}\varphi_1(r)v_1(\theta)\\ \varphi_2(r)v_2(\theta)\end{pmatrix}.$$

In the following, we will only consider $v_1\not\equiv 0$, $v_2\not\equiv 0$, $\varphi_1\not\equiv 0$ and $\varphi_2\not\equiv 0$, for the cases $v_1\equiv 0$, $v_2\equiv 0$, $\varphi_1\equiv 0$ or $\varphi_2\equiv 0$ can be reduced to the problem in Theorem 1.1.

Direct computations show that

$$\begin{aligned}\text{div } u =&\Big(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\Big)(v_1\varphi_1) + \Big(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\Big)(v_2\varphi_2)\\ =&\cos\theta v_1\varphi_1' + \sin\theta v_2\varphi_2' + \cos\theta v_2'\frac{\varphi_2}{r} - \sin\theta v_1'\frac{\varphi_1}{r},\end{aligned}$$

and

$$\begin{aligned} w &= \partial_2 u_1 - \partial_1 u_2\\ &= \Big(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\Big)(v_1\varphi_1) - \Big(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\Big)(v_2\varphi_2)\\ &= \sin\theta v_1\varphi_1' - \cos\theta v_2\varphi_2' + \cos\theta v_1'\frac{\varphi_1}{r} + \sin\theta v_2'\frac{\varphi_2}{r}. \end{aligned} \tag{63}$$

Moreover, div $u = 0$ implies that

$$\cos\theta v_1\varphi_1' + \sin\theta v_2\varphi_2' + \cos\theta v_2'\frac{\varphi_2}{r} - \sin\theta v_1'\frac{\varphi_1}{r} = 0. \tag{64}$$

Next we discuss the problem according to whether $\cos\theta v_1$ and $\sin\theta v_2$ are linearly dependent.

Case 1: $\cos\theta v_1$ and $\sin\theta v_2$ are linearly dependent. There exists $\lambda\neq 0$, such that

$$\cos\theta v_1 = \lambda\sin\theta v_2,$$

since $v_1 \not\equiv 0$ and $v_2 \not\equiv 0$. Then

$$v_1 = \lambda \tan\theta v_2. \tag{65}$$

Substituting (65) into (64), we obtain that

$$\sin\theta v_2(\lambda\varphi_1' + \varphi_2') + \cos\theta v_2' \frac{\varphi_2}{r} - \lambda \sin\theta(\sec^2\theta v_2 + \tan\theta v_2')\frac{\varphi_1}{r} = 0. \tag{66}$$

Since $v_2 \not\equiv 0$, there exist an interval K, such that $v_2, \sin\theta, \cos\theta \neq 0$ when $\theta \in K$. Then we deduce from (66) that

$$\lambda\varphi_1' + \varphi_2' + \frac{v_2'}{\tan\theta v_2}\frac{\varphi_2}{r} - \lambda\Big(\sec^2\theta + \frac{\tan\theta v_2'}{v_2}\Big)\frac{\varphi_1}{r} = 0, \qquad \theta \in K.$$

Let $M(\theta) = \frac{v_2'}{\tan\theta v_2}$, $N(\theta) = \sec^2\theta + M(\theta)\tan^2\theta$, then we can rewrite the above formula as

$$\lambda\varphi_1' + \varphi_2' + M(\theta)\frac{\varphi_2}{r} - \lambda N(\theta)\frac{\varphi_1}{r} = 0, \qquad \theta \in K. \tag{67}$$

Case 1.1: If $M(\theta)$ is not a constant in K, namely there exists $\theta_1, \theta_2 \in K$, such that

$$M(\theta_1) \neq M(\theta_2),$$

then

$$\lambda\varphi_1' + \varphi_2' + M(\theta_1)\frac{\varphi_2}{r} - \lambda N(\theta_1)\frac{\varphi_1}{r} = 0,$$
$$\lambda\varphi_1' + \varphi_2' + M(\theta_2)\frac{\varphi_2}{r} - \lambda N(\theta_2)\frac{\varphi_2}{r} = 0.$$

In the above two equations, the first one minus the second gives that

$$\varphi_2 = \lambda\frac{N(\theta_1) - N(\theta_2)}{M(\theta_1) - M(\theta_2)}\varphi_1 =: C\varphi_1,$$

then $u = \begin{pmatrix} v_1\varphi_1 \\ Cv_2\varphi_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ Cv_2 \end{pmatrix}\varphi_1$, reducing to the problem in Theorem 1.1.

Case 1.2: If $N(\theta)$ is not a constant in K, this case is similar to the above case of Case 1.1.

Case 1.3: If both $M(\theta)$ and $N(\theta)$ are constants in K, it is easy to see that $M(\theta) = -1, N(\theta) = 1$, and (67) becomes

$$(\lambda\varphi_1 + \varphi_2)' = \frac{\lambda\varphi_1 + \varphi_2}{r},$$

then $\lambda\varphi_1 + \varphi_2 = C_1 r$, and

$$\varphi_2 = C_1 r - \lambda\varphi_1. \tag{68}$$

If $C_1 = 0$, this case can be reduced to the problem in Theorem 1.1, so we can assume that $C_1 \neq 0$. We substitute (68) into (66) and obtain that

$$(\sin\theta v_2 + \cos\theta v_2')\Big(C_1 - \lambda\sec^2\theta\frac{\varphi_1}{r}\Big) = 0.$$

Since $\lambda \neq 0$ and $\varphi_1 \not\equiv 0$, then we have

$$\sin\theta v_2 + \cos\theta v_2' = 0.$$

Applying Lemma 2.2, there holds

$$v_2 = C_2 \cos\theta, \qquad C_2 \neq 0.$$

By (65), we get $v_1 = \lambda C_2 \sin\theta$. We substitute

$$\begin{cases} v_1 = \lambda C_2 \sin\theta \\ v_2 = C_2 \cos\theta \\ \varphi_2 = C_1 r - \lambda \varphi_1 \end{cases}$$

into (63) and obtain that

$$w = \lambda C_2 \big(\varphi_1' + \frac{\varphi_1}{r}\big) - C_1 C_2,$$

then

$$\begin{aligned} \Delta w =& \big(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\big) w = \lambda C_2 \big(\partial_r^2 + \frac{1}{r}\partial_r\big)\big(\varphi_1' + \frac{\varphi_1}{r}\big) \\ =& \lambda C_2 \big(\varphi_1''' + \frac{2\varphi_1''}{r} - \frac{\varphi_1'}{r^2} + \frac{\varphi_1}{r^3}\big); \\ \partial_1 w =& \lambda C_2 \big(\varphi_1' + \frac{\varphi_1}{r}\big)' \cos\theta = \lambda C_2 \cos\theta \big(\varphi_1'' + \frac{\varphi_1'}{r} - \frac{\varphi_1}{r^2}\big); \\ \partial_2 w =& \lambda C_2 \sin\theta \big(\varphi_1'' + \frac{\varphi_1'}{r} - \frac{\varphi_1}{r^2}\big); \\ u \cdot \nabla w =& v_1 \varphi_1 \partial_1 w + v_2 \varphi_2 \partial_2 w = \lambda C_1 C_2^2 \sin\theta \cos\theta r \big(\varphi_1'' + \frac{\varphi_1'}{r} - \frac{\varphi_1}{r^2}\big). \end{aligned} \tag{69}$$

Combine (23) and (69), and notice that $C_1 \neq 0$, $C_2 \neq 0$, $\lambda \neq 0$, then we obtain

$$\varphi_1''' + \frac{2\varphi_1''}{r} - \frac{\varphi_1'}{r^2} + \frac{\varphi_1}{r^3} = C_1 C_2 \sin\theta \cos\theta r \big(\varphi_1'' + \frac{\varphi_1'}{r} - \frac{\varphi_1}{r^2}\big),$$

which implies that

$$\begin{cases} \varphi_1'' + \frac{\varphi_1'}{r} - \frac{\varphi_1}{r^2} = 0, \\ \varphi_1''' + \frac{2\varphi_1''}{r} - \frac{\varphi_1'}{r^2} + \frac{\varphi_1}{r^3} = 0. \end{cases} \tag{70}$$

Notice that the second equation of (70) is namely (31), whose solutions are $\varphi_1 = C_3 r + C_4 r \ln r$. These solutions verify the first equation of (70) if and only if $C_4 = 0$, then $\varphi_1 = C_3 r$ are the solutions of (70). Therefore,

$$\begin{cases} \varphi_1 = C_3 r, \\ \varphi_2 = (C_1 - \lambda C_3) r. \end{cases}$$

Finally we have

$$u = C_2 \begin{pmatrix} \lambda C_3 r \sin\theta \\ (C_1 - \lambda C_3) r \cos\theta \end{pmatrix} = \begin{pmatrix} C_4 y \\ C_5 x \end{pmatrix}, \quad p = -\frac{1}{2} C_4 C_5 (x^2 + y^2) + C_6,$$

included in the type (ii) as shown in Theorem 1.1.

Case 2: $\cos\theta v_1$ and $\sin\theta v_2$ are linearly independent. Then there exist $\theta_1 \neq \theta_2$ such that the determinant

$$\mathbf{D}_1 := \begin{vmatrix} \cos\theta_1 v_1(\theta_1) & \sin\theta_1 v_2(\theta_1) \\ \cos\theta_2 v_1(\theta_2) & \sin\theta_2 v_2(\theta_2) \end{vmatrix} \neq 0.$$

We take $\theta = \theta_1$ and $\theta = \theta_2$ respectively in (64), then we obtain equations

$$\begin{cases} \cos\theta_1 v_1(\theta_1)\varphi_1' + \sin\theta_1 v_2(\theta_1)\varphi_2' = \sin\theta_1 v_1'(\theta_1)\frac{\varphi_1}{r} - \cos\theta_1 v_2'(\theta_1)\frac{\varphi_2}{r}, \\ \cos\theta_2 v_1(\theta_2)\varphi_1' + \sin\theta_2 v_2(\theta_2)\varphi_2' = \sin\theta_2 v_1'(\theta_2)\frac{\varphi_1}{r} - \cos\theta_2 v_2'(\theta_2)\frac{\varphi_2}{r}. \end{cases} \tag{71}$$

Notice that the determinant of the coefficients of (71) is exactly $\mathbf{D}_1$. Since $\mathbf{D}_1 \neq 0$, by Cramer's Rule we must have

$$\begin{cases} \varphi_1' = a\frac{\varphi_1}{r} + b\frac{\varphi_2}{r}, \\ \varphi_2' = c\frac{\varphi_1}{r} + d\frac{\varphi_2}{r}. \end{cases} \tag{72}$$

We substitute (72) into (64) and obtain that

$$(a\cos\theta v_1 + c\sin\theta v_2 - \sin\theta v_1')\varphi_1 + (b\cos\theta v_1 + d\sin\theta v_2 + \cos\theta v_2')\varphi_2 = 0.$$

Since $\varphi_2 \not\equiv 0$, by Lemma 2.1 we get either

$$\begin{cases} \varphi_1 = \lambda\varphi_2 \\ b\cos\theta v_1 + d\sin\theta v_2 + \cos\theta v_2' = -\lambda(a\cos\theta v_1 + c\sin\theta v_2 - \sin\theta v_1') \end{cases} \tag{73}$$

or

$$\begin{cases} a\cos\theta v_1 + c\sin\theta v_2 - \sin\theta v_1' = 0, \\ b\cos\theta v_1 + d\sin\theta v_2 + \cos\theta v_2' = 0. \end{cases} \tag{74}$$

In the first case (73), since $\varphi_1 = \lambda\varphi_2$, $u = \begin{pmatrix} \lambda v_1\varphi_2 \\ v_2\varphi_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ v_2 \end{pmatrix}\varphi_2$. This case can be reduced to the problem in Theorem 1.1.

Next we focus on the second case (74), which implies that

$$\begin{cases} v_1' = a\cot\theta v_1 + cv_2, \\ v_2' = -(bv_1 + d\tan\theta v_2). \end{cases} \tag{75}$$

Substituting (72) and (75) into (63), we obtain that

$$w = \frac{av_1}{\sin\theta}\frac{\varphi_1}{r} - \frac{dv_2}{\cos\theta}\frac{\varphi_2}{r}. \tag{76}$$

w satisfy the equation $\triangle w = u \cdot \nabla w$. First, we compute $u \cdot \nabla w$.

$$\begin{aligned} u \cdot \nabla w &= v_1\varphi_1\partial_1 w + v_2\varphi_2\partial_2 w \\ &= v_1\varphi_1\big(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\big)w + v_2\varphi_2\big(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\big)w \\ &= (\cos\theta v_1\varphi_1 + \sin\theta v_2\varphi_2)\partial_r w + \frac{1}{r}(-\sin\theta v_1\varphi_1 + \cos\theta v_2\varphi_2)\partial_\theta w. \end{aligned} \tag{77}$$

According to (76) and (72),

$$\begin{aligned} \partial_r w =& \frac{av_1}{\sin\theta}\Big(\frac{\varphi_1}{r}\Big)' - \frac{dv_2}{\cos\theta}\Big(\frac{\varphi_2}{r}\Big)' \\ =& \frac{av_1}{\sin\theta}r^{-2}\big[(a-1)\varphi_1 + b\varphi_2\big] - \frac{dv_2}{\cos\theta}r^{-2}\big[c\varphi_1 + (d-1)\varphi_2\big] \\ =& \Big[a(a-1)\frac{v_1}{\sin\theta} - cd\frac{v_2}{\cos\theta}\Big]r^{-2}\varphi_1 + \Big[ab\frac{v_1}{\sin\theta} - d(d-1)\frac{v_2}{\cos\theta}\Big]r^{-2}\varphi_2. \end{aligned} \tag{78}$$

According to (76) and (75),

$$
\begin{aligned}
\partial_\theta w =&\frac{a\varphi_1}{r}\Big(\frac{v_1}{\sin\theta}\Big)' - \frac{d\varphi_2}{r}\Big(\frac{v_2}{\cos\theta}\Big)' \\
=&\frac{a}{\sin\theta}\big[(a-1)\cot\theta v_1 + cv_2\big]r^{-1}\varphi_1 + \frac{d}{\cos\theta}\big[bv_1 + (d-1)\tan\theta v_2\big]r^{-1}\varphi_2.
\end{aligned} \tag{79}
$$

Substituting (78) and (79) into (77), we obtain that

$$
u\cdot\nabla w = r^{-2}\big[-c(a+d)v_1v_2\varphi_1^2 + b(a+d)v_1v_2\varphi_2^2 + F_3\varphi_1\varphi_2\big], \tag{80}
$$

where

$$
F_3 = (a\cot\theta - d\tan\theta)(bv_1^2 + cv_2^2) + \Big(\frac{a^2-a}{\sin^2\theta} - \frac{d^2-d}{\cos^2\theta}\Big)v_1v_2. \tag{81}
$$

Next we compute Δw.

$$
\Delta w =\big(\partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2\big)w. \tag{82}
$$

By (78) and (72),

$$
\begin{aligned}
\partial_r^2 w =&\Big\{\big[a(a-1)(a-2)+abc\big]\frac{v_1}{\sin\theta} - cd(a+d-3)\frac{v_2}{\cos\theta}\Big\}r^{-3}\varphi_1 \\
&+\Big\{ab(a+d-3)\frac{v_1}{\sin\theta} - \big[d(d-1)(d-2)+bcd\big]\frac{v_2}{\cos\theta}\Big\}r^{-3}\varphi_2
\end{aligned} \tag{83}
$$

By (79) and (75),

$$
\begin{aligned}
\partial_\theta^2 w = &\Big\{\big[a(a-1)(a-2)\cot^2\theta - a(bc+a-1)\big]\frac{v_1}{\sin\theta} + ac\big[(a-2)\cot^2\theta - d\big]\frac{v_2}{\cos\theta}\Big\}r^{-1}\varphi_1 \\
&-\Big\{\big[d(d-1)(d-2)\tan^2\theta - d(bc+d-1)\big]\frac{v_2}{\cos\theta} + bd\big[(d-2)\tan^2\theta - a\big]\frac{v_1}{\sin\theta}\Big\}r^{-1}\varphi_2
\end{aligned} \tag{84}
$$

Substituting (78), (83) and (84) into (82), we obtain that

$$
\Delta w = r^{-3}(F_1\varphi_1 - F_2\varphi_2), \tag{85}
$$

where

$$
\begin{aligned}
F_1 =&a(a-1)(a-2)\frac{v_1}{\sin^3\theta} + c\big[a(a-2)\cot^2\theta - d(d+2a-2)\big]\frac{v_2}{\cos\theta}, \\
F_2 =&d(d-1)(d-2)\frac{v_2}{\cos^3\theta} + b\big[d(d-2)\tan^2\theta - a(a+2d-2)\big]\frac{v_1}{\sin\theta}.
\end{aligned} \tag{86}
$$

$\triangle w = u\cdot\nabla w$, (80) and (85) yield that

$$
F_1\varphi_1 - F_2\varphi_2 = (a+d)v_1v_2r\big(b\varphi_2^2 - c\varphi_1^2\big) + F_3r\varphi_1\varphi_2, \tag{87}
$$

where F_1, F_2, F_3 are given by (81) and (86), and v_1, v_2 satisfy (74).

Since φ_1, φ_2 satisfy equations (72), applying Lemma 2.3 for them, we obtain their expressions as shown in Lemma 2.3. If φ_1 and φ_2 are linearly dependent, one can reduce this problem to the one in Theorem 1.1, hence it suffices to consider that they are linearly independent.

Case (1): $b = 0$, $d = a$. At this time,

$$\begin{cases} \varphi_1 = C_1 r^a, \\ \varphi_2 = (cC_1 \ln r + C_2) r^a. \end{cases}$$

Since φ_1 and φ_2 are linearly independent, then $cC_1 \neq 0$. $\varphi_1, \varphi_2 \in C^0([0, \infty))$, then $a > 0$.

Substitute $b = 0$, $d = a$ and the expressions of φ_1 and φ_2 into (87), then we obtain

$$(C_1 F_1 - C_2 F_2) - cC_1 F_2 \ln r = r^{a+1} \big[cC_1^2 F_3 \ln r - 2acC_1^2 v_1 v_2 + C_1 C_2 F_3 \big],$$

which implies

$$\begin{cases} C_1 F_1 - C_2 F_2 = 0; \\ cC_1 F_2 = 0; \\ cC_1^2 F_3 = 0; \\ -2acC_1^2 v_1 v_2 + C_1 C_2 F_3 = 0. \end{cases}$$

Since $cC_1 \neq 0$ and $a > 0$, then $F_1 = F_2 = F_3 = v_1 v_2 \equiv 0$. Applying Lemma 2.4, we have $v_2 \equiv 0$, which contradicts our assumption that $v_2 \not\equiv 0$. Therefore this case does not exist.

Case (2): $b = 0$, $d \neq a$. We have

$$\begin{cases} \varphi_1 = C_1 r^a, \\ \varphi_2 = \frac{c}{a-d} C_1 r^a + C_2 r^d. \end{cases}$$

Since φ_1 and φ_2 are linearly independent, then $C_1 C_2 \neq 0$. $\varphi_1, \varphi_2 \in C^0$, then $a \geq 0$ and $d \geq 0$. $d \neq a$ implies that $a + d > 0$.

In this case, equation (87) becomes

$$C_1 \Big(F_1 - \frac{cF_2}{a-d} \Big) r^a - C_2 F_2 r^d = cC_1^2 \Big[\frac{F_3}{a-d} - (a+d) v_1 v_2 \Big] r^{2a+1} + C_1 C_2 F_3 r^{a+d+1}. \quad (88)$$

$r^a, r^d, r^{2a+1}, r^{a+d+1}$ appear in the above formula. Obviously, r^a is different from the other three, so is r^{a+d+1}. Then the coefficients of r^a and r^{a+d+1} must be 0, which give that

$$\begin{cases} F_1 = \dfrac{c}{a-d} F_2; & (89) \\ F_3 = 0; & (90) \\ C_2 F_2 r^d = c(a+d) C_1^2 v_1 v_2 r^{2a+1}. & (91) \end{cases}$$

This situation will be divided into two subcases for further discussion.

Case (2.1): $c = 0$. The above equations are reduced to

$$F_1 = F_2 = F_3 = 0,$$

namely

$$\begin{cases} a(a-1)(a-2) \frac{v_1}{\sin^3 \theta} = 0; \\ d(d-1)(d-2) \frac{v_2}{\cos^3 \theta} = 0; \\ \Big(\frac{a^2 - a}{\sin^2 \theta} - \frac{d^2 - d}{\cos^2 \theta} \Big) v_1 v_2 = 0. \end{cases} \quad (92)$$

Since $v_1 \not\equiv 0$ and $v_2 \not\equiv 0$, then the first and second equation of (92) give that

$$a = 0, 1, 2; \qquad d = 0, 1, 2.$$

Notice that $c = 0$, then the first equation of (74) is reduced to

$$a \cos\theta v_1 - \sin\theta v_1' = 0.$$

Since $a = 0, 1, 2$ and $v_1 \in C^2$, we use a argument similar to the one in Lemma 2.2, then we obtain that

$$v_1 = C_3 \sin^a \theta, \qquad C_3 \neq 0, \tag{93}$$

Similarly, since $b = 0$, the second equation of (74) is reduced to

$$d \sin\theta v_2 + \cos\theta v_2' = 0,$$

Since $d = 0, 1, 2$ and $v_2 \in C^2$, we apply Lemma 2.2 and obtain that

$$v_2 = C_4 \cos^d \theta, \qquad C_4 \neq 0. \tag{94}$$

Having (93) and (94), one can reduce the third equation of (92) to

$$\begin{cases} a^2 - a = 0; \\ d^2 - d = 0. \end{cases}$$

Since $d \neq a$, the above equations have solutions

$$\begin{cases} a = 1 \\ d = 0 \end{cases}; \qquad \begin{cases} a = 0 \\ d = 1 \end{cases}.$$

If $a = 1, d = 0$, then

$$\varphi_1 = C_1 r, \quad \varphi_2 = C_2, \quad v_1 = C_3 \sin\theta, \quad v_2 = C_4,$$

and

$$u = \begin{pmatrix} C_1 C_3 r \sin\theta \\ C_2 C_4 \end{pmatrix} = \begin{pmatrix} C_5 y \\ C_6 \end{pmatrix}, \quad p = -C_5 C_6 x + C_7. \tag{95}$$

If $a = 0, d = 1$, then

$$\varphi_1 = C_1, \quad \varphi_2 = C_2 r, \quad v_1 = C_3, \quad v_2 = C_4 \cos\theta,$$

and

$$u = \begin{pmatrix} C_1 C_3 \\ C_2 C_4 r \cos\theta \end{pmatrix} = \begin{pmatrix} C_5 \\ C_6 x \end{pmatrix}, \quad p = -C_5 C_6 y + C_7. \tag{96}$$

Case (2.2): $c \neq 0$. Consider equation (91).
If $d \neq 2a + 1$,

$$C_2 F_2 = c(a + d) C_1^2 v_1 v_2 \equiv 0,$$

namely

$$F_2 \equiv 0, \qquad v_1 v_2 \equiv 0.$$

According to Lemma 2.4, $v_2 \equiv 0$, which contradicts that $v_2 \not\equiv 0$. Therefore

$$d = 2a + 1.$$

With (86) and $b = 0$, we rewrite (89) as

$$a(a-1)(a-2)v_1 = \frac{c\sin\theta}{\cos^3\theta}\Big[\frac{d(d-1)(d-2)}{a-d}\sin^2\theta - a(a-2)\cos^4\theta + d(d+2a-2)\sin^2\theta\cos^2\theta\Big]v_2. \tag{97}$$

If $a(a-1)(a-2) = 0$, then

$$\Big[\frac{d(d-1)(d-2)}{a-d}\sin^2\theta - a(a-2)\cos^4\theta + d(d+2a-2)\sin^2\theta\cos^2\theta\Big]v_2 = 0.$$

Since $v_2 \not\equiv 0$, there holds

$$\begin{cases} d(d-1)(d-2) = 0; \\ a(a-2) = 0; \\ d(d+2a-2) = 0. \end{cases}$$

The above equations have solutions

$$\begin{cases} a = 0 \\ d = 0 \end{cases}; \qquad \begin{cases} a = 0 \\ d = 2 \end{cases}; \qquad \begin{cases} a = 2 \\ d = 0 \end{cases}.$$

These all contradict that $d = 2a + 1$. Therefore

$$a(a-1)(a-2) \neq 0.$$

This and (97) give that

$$v_1 = \frac{c}{a(a-1)(a-2)}\frac{\sin\theta}{\cos^3\theta}\Big[\frac{d(d-1)(d-2)}{a-d}\sin^2\theta - a(a-2)\cos^4\theta + d(d+2a-2)\sin^2\theta\cos^2\theta\Big]v_2. \tag{98}$$

Notice that when $b = 0$, equation (90) is

$$F_3 = c(a\cot\theta - d\tan\theta)v_2^2 + \Big(\frac{a^2-a}{\sin^2\theta} - \frac{d^2-d}{\cos^2\theta}\Big)v_1v_2 = 0.$$

Since $v_2 \not\equiv 0$, there exists an interval K_1, such that

$$v_2 \neq 0, \qquad \theta \in K_1.$$

Then

$$c(a\cot\theta - d\tan\theta)v_2 + \Big(\frac{a^2-a}{\sin^2\theta} - \frac{d^2-d}{\cos^2\theta}\Big)v_1 = 0, \qquad \theta \in K_1.$$

Substituting (98) into the above equation, we obtain that in K_1,

$$a(a-1)(a-2)(a\cos^2\theta - d\sin^2\theta)\cos^4\theta + \big[(a^2-a)\cos^2\theta - (d^2-d)\sin^2\theta\big] \times \Big[\frac{d(d-1)(d-2)}{a-d}\sin^2\theta - a(a-2)\cos^4\theta + d(d+2a-2)\sin^2\theta\cos^2\theta\Big] = 0. \tag{99}$$

Notice that the left hand side of the above equation is a polynomial with respect to $\cos^2\theta$, and the constant term of this polynomial is $\frac{d^2(d-1)^2(d-2)}{d-a}$, then we have

$$\frac{d^2(d-1)^2(d-2)}{d-a}=0,$$

$$d=0,\ 1,\ 2.$$

Recall that $d=2a+1$, $a\geq 0$ and $a(a-1)(a-2)\neq 0$, then

$$d=2,\qquad a=\frac{1}{2}.$$

Substituting this into (99), we obtain that $-\frac{11}{16}\cos^2\theta=\sin^2\theta$, which is impossible. Therefore, Case (2.2) does not exist.

Case (3): $b\neq 0$, $\delta>0$. Now we have

$$\begin{cases}\varphi_1=C_1r^m+C_2r^n,\\ \varphi_2=\frac{m-a}{b}C_1r^m+\frac{n-a}{b}C_2r^n,\end{cases}$$

where m,n are two different real roots of equation (14) and $m>n$.

Since φ_1 and φ_2 are linearly independent, then $C_1C_2\neq 0$. $\varphi_1,\varphi_2\in C^0$, then $m>n\geq 0$.

By Vieta's theorem,

$$\begin{aligned}a+d=&m+n>0,\\ ad-bc=&mn.\end{aligned}\tag{100}$$

In this case, equation (87) becomes

$$\begin{aligned}&C_1\big[bF_1-(m-a)F_2\big]r^m+C_2\big[bF_1-(n-a)F_2\big]r^n\\ =&C_1^2\Big\{(a+d)\big[(m-a)^2-bc\big]v_1v_2+(m-a)F_3\Big\}r^{2m+1}\\ &+C_2^2\Big\{(a+d)\big[(n-a)^2-bc\big]v_1v_2+(n-a)F_3\Big\}r^{2n+1}\\ &+C_1C_2\Big\{2(a+d)\big[(m-a)(n-a)-bc\big]v_1v_2+(m+n-2a)F_3\Big\}r^{m+n+1}.\end{aligned}\tag{101}$$

Note that $r^m,r^n,r^{2m+1},r^{2n+1},r^{m+n+1}$ appear in the above equation. Since $m>n$,

$$2m+1>m+n+1>2n+1,m>n,$$

the coefficients of r^{2m+1}, r^{m+n+1} and r^n must be 0, which implies that

$$\begin{cases}(a+d)\big[(m-a)^2-bc\big]v_1v_2+(m-a)F_3=0;\\ 2(a+d)\big[(m-a)(n-a)-bc\big]v_1v_2+(m+n-2a)F_3=0;\\ bF_1=(n-a)F_2;\\ C_1\big[bF_1-(m-a)F_2\big]r^m=C_2^2\Big\{(a+d)\big[(n-a)^2-bc\big]v_1v_2+(n-a)F_3\Big\}r^{2n+1}.\end{cases}\tag{102}$$

Notice that m,n are roots of equation (14). This and (100) give that

$$\begin{aligned}(m-a)^2-bc=&(d-a)(m-a);\\ (n-a)^2-bc=&(d-a)(n-a);\\ (m-a)(n-a)=&-bc;\\ m+n=&a+d.\end{aligned}\tag{103}$$

Substitute the third equation of (102) into the forth one and apply (103), then one can rewrite (102) as

$$\begin{cases} (a+d)(d-a)(m-a)v_1v_2 + (m-a)F_3 = 0; \\ 4bc(a+d)v_1v_2 + (a-d)F_3 = 0; \\ F_1 = \frac{n-a}{b}F_2; \\ \big[(a+d)(d-a)(n-a)v_1v_2 + (n-a)F_3\big]r^{2n+1} = (n-m)C_1C_2^{-2}F_2r^m. \end{cases} \tag{104}$$

Consider the first and second equation of (104), where the determinant of the coefficients

$$\begin{aligned} \mathbf{D}_2 :=& \begin{vmatrix} (a+d)(d-a)(m-a) & m-a \\ 4bc(a+d) & a-d \end{vmatrix} \\ =&(a+d)(a-m)\big[(d-a)^2 + 4bc\big] = (a+d)(a-m)\delta. \end{aligned}$$

If $\mathbf{D}_2 \neq 0$, by Cramer's Rule,

$$v_1v_2 \equiv 0, \quad F_3 \equiv 0.$$

According to Lemma 2.4,

$$v_1 \equiv 0,$$

which contradicts that $v_1 \not\equiv 0$. Then we must have

$$\mathbf{D}_2 = 0.$$

Notice that $a+d > 0$ and $\delta > 0$, then

$$m = a.$$

Therefore, $n = a+d-m = d$, $bc = ad - mn = 0$. Since $b \neq 0$, then $c = 0$. In summary,

$$m = a, \quad n = d, \quad c = 0, \tag{105}$$

and $a > d \geq 0$.

With (105), we can reduce equations (104) to

$$\begin{cases} F_1 = \dfrac{d-a}{b}F_2; & (106) \\ F_3 = 0; & (107) \\ (a+d)(d-a)v_1v_2r^{2d+1} = C_1C_2^{-2}F_2r^a. & (108) \end{cases}$$

The following argument for this case is very similar to that for Case (2.2). For completeness, let us briefly describe the proof.

If $a \neq 2d+1$, equation (108) is equivalent to

$$(a+d)(d-a)v_1v_2 = C_1C_2^{-2}F_2 \equiv 0,$$

and then $v_1v_2 \equiv 0$, which contradicts that $v_1 \not\equiv 0$. Therefore

$$a = 2d+1.$$

With (86) and $c = 0$, we rewrite (106) as

$$d(d-1)(d-2)v_2 = \frac{b\cos\theta}{\sin^3\theta}\Big[\frac{a(a-1)(a-2)}{d-a}\cos^2\theta - d(d-2)\sin^4\theta + a(a+2d-2)\sin^2\theta\cos^2\theta\Big]v_1. \tag{109}$$

If $d(d-1)(d-2) = 0$, then

$$\Big[\frac{a(a-1)(a-2)}{d-a}\cos^2\theta - d(d-2)\sin^4\theta + a(a+2d-2)\sin^2\theta\cos^2\theta\Big]v_1 = 0.$$

Since $v_1 \not\equiv 0$, we must have

$$\begin{cases} a(a-1)(a-2) = 0; \\ d(d-2) = 0; \\ a(a+2d-2) = 0. \end{cases}$$

The above equations have solutions

$$\begin{cases} d = 0 \\ a = 0 \end{cases}; \qquad \begin{cases} d = 0 \\ a = 2 \end{cases}; \qquad \begin{cases} d = 2 \\ a = 0 \end{cases}.$$

These all contradict that $a = 2d + 1$. Therefore

$$d(d-1)(d-2) \neq 0.$$

This and (109) give that

$$v_2 = \frac{b}{d(d-1)(d-2)}\frac{\cos\theta}{\sin^3\theta}\Big[\frac{a(a-1)(a-2)}{d-a}\cos^2\theta - d(d-2)\sin^4\theta + a(a+2d-2)\sin^2\theta\cos^2\theta\Big]v_1. \tag{110}$$

Notice that when $c = 0$, equation (107) is

$$F_3 = b(a\cot\theta - d\tan\theta)v_1^2 + \Big(\frac{a^2-a}{\sin^2\theta} - \frac{d^2-d}{\cos^2\theta}\Big)v_1 v_2 = 0.$$

Since $v_1 \not\equiv 0$, there exists an interval K_2, such that

$$v_1 \neq 0, \qquad \theta \in K_2.$$

Then

$$b(a\cot\theta - d\tan\theta)v_1 + \Big(\frac{a^2-a}{\sin^2\theta} - \frac{d^2-d}{\cos^2\theta}\Big)v_2 = 0, \qquad \theta \in K_2.$$

Substituting (110) into the above equation, we obtain that in K_2,

$$d(d-1)(d-2)(a\cos^2\theta - d\sin^2\theta)\sin^4\theta + \big[(a^2-a)\cos^2\theta - (d^2-d)\sin^2\theta\big] \times \Big[\frac{a(a-1)(a-2)}{d-a}\cos^2\theta - d(d-2)\sin^4\theta + a(a+2d-2)\sin^2\theta\cos^2\theta\Big] = 0.$$

Notice that the left hand side of the above equation is a polynomial with respect to $\sin^2\theta$, and the constant term of this polynomial is $\frac{a^2(a-1)^2(a-2)}{d-a}$, then we have

$$\frac{a^2(a-1)^2(a-2)}{d-a} = 0,$$

and $a = 0,\ 1,\ 2$. Recall that $a = 2d + 1$, $d \geq 0$ and $d(d-1)(d-2) \neq 0$, then

$$a = 2, \qquad d = \frac{1}{2}.$$

This is also impossible by similar arguments as Case (2.2). Therefore, Case (3) does not exist.

Case (4) and (5): $b \neq 0$, $\delta \leq 0$. In these two cases, write

$$\begin{cases} \varphi_1 = (C_1 f_1 + C_2 f_2) r^k, \\ \varphi_2 = (B_1 f_1 + B_2 f_2) r^k. \end{cases} \tag{111}$$

In Case (4),

$$\begin{cases} k = l, \\ f_1 = \ln r, & f_2 = 1, \\ B_1 = \frac{l-a}{b} C_1, & B_2 = \frac{C_1 + (l-a) C_2}{b}. \end{cases} \tag{112}$$

where l is the unique real root of equation (14). Since φ_1 and φ_2 are linearly independent, then $C_1 \neq 0$. $\varphi_1, \varphi_2 \in C^0$ at $r = 0$, then $k = l > 0$.

In Case (5),

$$\begin{cases} k = \lambda, \\ f_1 = \cos(\mu \ln r), & f_2 = \sin(\mu \ln r), \\ B_1 = \frac{(\lambda - a) C_1 + \mu C_2}{b}, & B_2 = \frac{(\lambda - a) C_2 - \mu C_1}{b}. \end{cases} \tag{113}$$

where $\lambda \pm \mu i$ ($\mu \neq 0$) are the complex roots of equation (14). Since φ_1 and φ_2 are linearly independent, then $C_1^2 + C_2^2 \neq 0$. $\varphi_1, \varphi_2 \in C^0$ at $r = 0$, then $k = \lambda > 0$.

Substitute (111) into (87) and denote $G = (a + d) v_1 v_2$, then we obtain that

$$\begin{aligned} &(C_1 F_1 - B_1 F_2) f_1 + (C_2 F_1 - B_2 F_2) f_2 \\ = {}& r^{k+1} \Big\{ f_1^2 \big[(b B_1^2 - c C_1^2) G + C_1 B_1 F_3 \big] + f_2^2 \big[(b B_2^2 - c C_2^2) G + C_2 B_2 F_3 \big] \\ &+ f_1 f_2 \big[2 (b B_1 B_2 - c C_1 C_2) G + (C_1 B_2 + C_2 B_1) F_3 \big] \Big\}. \end{aligned} \tag{114}$$

Since $f_1, f_2, r^{k+1} f_1^2, r^{k+1} f_2^2$ and $r^{k+1} f_1 f_2$ are linearly independent, the above equations are equivalent to

$$\begin{cases} C_1 F_1 - B_1 F_2 = 0; \\ C_2 F_1 - B_2 F_2 = 0; \\ (b B_1^2 - c C_1^2) G + C_1 B_1 F_3 = 0; \\ (b B_2^2 - c C_2^2) G + C_2 B_2 F_3 = 0; \\ 2 (b B_1 B_2 - c C_1 C_2) G + (C_1 B_2 + C_2 B_1) F_3 = 0. \end{cases} \tag{115}$$

For Case (4), we substitute (112) into the last three equations of (115) and take into account that $C_1 \neq 0$, then we obtain that

$$\begin{cases} \big[(l-a)^2 - bc \big] G + (l-a) F_3 = 0; & (116) \\ \Big\{ C_1^2 + 2(l-a) C_1 C_2 + \big[(l-a)^2 - bc \big] C_2^2 \Big\} G + \big[C_1 C_2 + (l-a) C_2^2 \big] F_3 = 0; & (117) \\ \Big\{ 2(l-a) C_1 + 2 \big[(l-a)^2 - bc \big] C_2 \Big\} G + \big[C_1 + 2(l-a) C_2 \big] F_3 = 0. & (118) \end{cases}$$

$\frac{(117) - (116) \times C_2^2}{C_1}$ gives that

$$\big[C_1 + 2(l-a) C_2 \big] G + C_2 F_3 = 0. \tag{119}$$

$\frac{(118)-(116)\times 2C_2}{C_1}$ gives that

$$2(l-a)G+F_3=0. \tag{120}$$

$\frac{(119)-(120)\times C_2}{C_1}$ gives that

$$G=0,$$

namely $G=(a+d)v_1v_2=0$. Notice that $a+d=2l>0$, then $v_1v_2=0$. Since $b\neq 0$, we apply Lemma 2.4 again, then we have $v_1\equiv 0$, which contradicts that $v_1\not\equiv 0$. Therefore Case (4) does not exist.

For Case (5), we substitute (113) into the last three equations of (115), then we obtain that

$$\begin{cases} \Big\{\big[(\lambda-a)^2-bc\big]C_1^2+\mu^2C_2^2+2\mu(\lambda-a)C_1C_2\Big\}G+\big[(\lambda-a)C_1^2+\mu C_1C_2\big]F_3=0; & (121)\\ \Big\{\big[(\lambda-a)^2-bc\big]C_2^2+\mu^2C_1^2-2\mu(\lambda-a)C_1C_2\Big\}G+\big[(\lambda-a)C_2^2-\mu C_1C_2\big]F_3=0; & (122)\\ \Big\{2\mu(\lambda-a)\big(C_2^2-C_1^2\big)+2\big[(\lambda-a)^2-\mu^2-bc\big]C_1C_2\Big\}G & \\ \qquad\qquad +\big[\mu\big(C_2^2-C_1^2\big)+2(\lambda-a)C_1C_2\big]F_3=0. & (123) \end{cases}$$

We claim that

$$G=0.$$

Firstly, if $C_1=0$, then $C_2\neq 0$ due to $C_1^2+C_2^2\neq 0$, and (121) becomes $\mu^2C_2^2G=0$. Since $\mu\neq 0$, then $G=0$.
Secondly, if $C_2=0$, then $C_1\neq 0$ and (122) becomes $\mu^2C_1^2G=0$ and thus $G=0$.
Finally, if $C_1C_2\neq 0$, (121) $\times\frac{C_2}{C_1}+$ (122) $\times\frac{C_1}{C_2}-$ (123) gives that

$$\mu^2\frac{\big(C_1^2+C_2^2\big)^2}{C_1C_2}G=0,$$

then $G=0$. Thus the claim is proved.
Consequently, $G=(a+d)v_1v_2=0$. Notice that $a+d=2\lambda>0$, then $v_1v_2=0$, which contradicts that $v_1\not\equiv 0$. Therefore Case (5) does not exist.

Finally, when

$$u=\begin{pmatrix}\varphi_1(r)v_1(\theta)\\ \varphi_2(r)v_2(\theta)\end{pmatrix},$$

all solutions of equations (2) are (i), (ii), (iii), (iv) and (v) as shown in Theorem 1.1, (95) and (96).

The proof is complete. □

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