Condition-based maintenance for a multi-component system in a dynamic operating environment

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Abstract

This paper develops a condition-based maintenance (CBM) model for a multi-component system operating under a dynamic environment. The degradation process of each component depends on both its intrinsic characteristic and the common operating environment. We model the environment evolution by a continuous-time Markov process, given which, the degradation increment of each component is described by a Poisson distribution. System reliability is firstly obtained, followed by a CBM policy to sustain system operation and ensure safety. In modelling the environmental effect on component degradation processes, two scenarios are considered. The first scenario considers renewable environment evolution while the second scenario on non-renewable environment evolution. The problem is casted into the Markov decision process (MDP) framework where the total expected discounted cost in the long-run horizon is utilized as the optimization objective to assess the policy. Structural properties of the optimal maintenance policy are investigated under mild conditions, which are further embedded into the value iteration algorithm to reduce the computational burden in calculating the maintenance cost. Applicability of the proposed model is illustrated through numerical examples.

Keywords: Dynamic environment; Condition-based maintenance;

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Multi-component systems; Markov decision process.

Notation

N	Number of components in the system
Φ	Set of components, $\Phi = \{1, 2, \cdots, N\}$
S_l	Set of environment states, $S_l = \{0, 1, \cdots, m\}$
$E_1 \backslash E_2$	Set-theoretic difference of sets E_1 and E_2
W_t	Environment state at time t
L_i	Failure state of component i, $i \in \Phi$
$\boldsymbol{x}$	vector of length N representing the degradation state of the system
x_i	the ith element in $\boldsymbol{x}$ denoting the degradation of component i
$\boldsymbol{S}$	Set of the system state
S_g	Set of system degradation states under which the system is functioning
$S_g^{(1)}$	Subset of S_g under which all components are functioning
$S_g^{(2)}$	Subset of S_g under which failures exist at the component level, $S_g = S_g^{(1)} \cup S_g^{(2)}$
$\lambda_i(t, w)$	Parameter of the Poisson distribution corresponding to component i in environment w, $i \in \Phi, w \in S_l$
$\boldsymbol{o}_t$	Realization of the system degradation at time t
$P_{wl}(t, \boldsymbol{x})$	Probability that the degradation increment is $\boldsymbol{x}$ with environment state l given the initial brand-new state and environment state l, $w, l \in S_l$.
$\boldsymbol{P}(t, \boldsymbol{x})$	Matrix form of $P_{wl}(t, \boldsymbol{x})$
$\boldsymbol{\Lambda}_i(t, \boldsymbol{x})$	Diagonal matrix with the (w, w)th entry $\lambda_i(t, w)$, $i \in \Phi, w \in S_l$
$\boldsymbol{\Lambda}(t, \boldsymbol{x})$	Diagonal matrix, $\boldsymbol{\Lambda}(t, \boldsymbol{x}) = \sum_{i \in \Phi} \boldsymbol{\Lambda}_i(t, \boldsymbol{x})$
$\mathrm{I}_{\{\cdot\}}$	Indicator function
$R(t; \boldsymbol{v})$	System reliability function at t given the initial brand new state and the environment state $\boldsymbol{v}$

$R(t; \boldsymbol{x}, \boldsymbol{v})$	System reliability function at t given the initial degradation state $\boldsymbol{x}$ and the environment state $\boldsymbol{v}$
c_I	Inspection cost each time
c_s	Maintenance set-up cost each time
c_{pi}	Preventive replacement cost of component i
c_{ri}	Corrective replacement cost of component i
c_d	Penalty of the system down-time per unitary time
c_f	System replacement cost
γ	Parameter associated with the discounted factor
$C_{ins}(\tau)$	Total expected inspection cost in the long-run horizon
$\kappa_\tau(\cdot)$	Transition kenel
$D_\tau(\boldsymbol{x}, w)$	Expected downtime during time period τ given the initial system degradation $\boldsymbol{x}$ and environmental state w
$U_\tau(\boldsymbol{x}, w)$	Expected cost until the next inspection epoch given the initial system degradation $\boldsymbol{x}$ and environmental state w
$V_\tau(\boldsymbol{x}, w)$	Value function when the environment is renewable, denoting minimum expected total discounted maintenance cost in the long-run horizon given the initial degradation state $\boldsymbol{x}$ and environmental state w
$\hat{V}_\tau(\boldsymbol{x}, w)$	Value function when the environment is non-renewable, denoting minimum expected total discounted maintenance cost in the long-run horizon given the initial degradation state $\boldsymbol{x}$ and environmental state w

1. Introduction

With the increasing demand on high quality and safety of systems/products, reliability evaluation and maintenance planning become increasingly more important in improving system safety and reducing operation budget in industrial field. Compared with the ideal laboratory environment, in practice, a system usually operates in a time-varying environment which exerts impacts on system functionality and performance. For instance, the lifetimes of components in an aircraft are related to the atmospheric operating condition such as pressure, temperature, mechanical vibration [1]. One can imagine that during flying and landing, the security requirements under the two scenarios are different. In the oil and gas industry, the flows of gas, oil and water contribute to the corrosion of throttle valves. In railway transportation, locomotive wheels have to be reprofiled occasionally to reduce the roughness and imbalance among the wheels [2]. The roughness and imbalance is associated with various environmental factors such as weather conditions, running speed, load, *etc.* It is therefore necessary to take the operating environment into consideration in assessing system reliability and developing maintenance strategies. Relevant studies have addressed the above issues in the literature [3, 4, 5]. However, existing works mainly focus on reliability assessment of single-component systems. Models concerning multi-component systems, especially multi-component degraded systems under dynamic environment are rather limited. In recent decades, conditional-based maintenance of multi-component degraded systems has received increasing attention [6, 7]. Most of the existing studies consider that maintenance decisions are based on the system, *i.e.*, the states of components. However, in practical operation, the health condition of a system depends not only on its manufacturing characteristics, but also on its working environment. For instance, the degradation of blades in offshore and onshore wind turbine are supposed to be different. Maintenance schedules should be adaptive to the working condition of the system. To the best of the authors' knowledge, little attention has been paid to the influence of the environment on the maintenance decision-making.

This paper will develop a generic model of degraded systems with heterogeneous components under a dynamic working environment. We intend to derive some system reliability measures and present how to make maintenance planning based on the observation: the state of system and the state of environment.

We consider the case that the system is periodically inspected, upon which, the degradation level of each component and their working environment state are fully and perfectly observed. A condition-based maintenance (CBM) model is constructed to prevent system failure: failed components are replaced, meanwhile, non-failed components can be preventively replaced, depending on their degradation levels. In terms of the environmental impact, this study considers two scenarios: the first one is that when the system is renewed, the working environment can be restored to its initial friendly state. As stated in [3], in the reliability analysis of the transmission system from a mining company, the wear metals iron was chosen as the covariate related to the system health. Whenever the system was replaced, this wear metals iron was removed and the environmental effect could be reset to the initial state. The other scenario is that the environment is independent of maintenance actions, such as the physical weather. The two scenarios will be referred to as renewable and non-renewable environment in the following contexts We formulate this problem into a Markov decision process (MDP) [8, 9] and the total expected discounted cost is utilized to assess the policy. Specifically, under some mild conditions, structural properties of the value function and the maintenance policy are derived. The main contributions of this paper are summarized as follows.

- We investigate the reliability measures and a CBM policy for a system with multiple heterogeneous degrading components where the system is generic and no specific system configuration is assumed.
- We discuss both renewable and non-renewable environment effects on the maintenance strategy and the corresponding maintenance cost.
- We provide exact system reliability measures and the impacts of the environment on the system reliability and maintenance policy are examined.
- Under mild conditions, the structural properties of the maintenance policy and the optimal maintenance cost are derived.

The rest of the paper is organized as follows. In Section 2, related literature review is given. In Section 3, we present the model assumptions, including description of the dynamic environment, system configuration and component characteristics. The system reliability and conditional reliability function given the component internal degradation and the dynamic working environment are presented. Section 4 focuses on the development of maintenance policy, and formulates the problem into the MDP framework. Structural properties of the optimal maintenance policy are investigated, which are further incorporated into a value iteration algorithm for calculation purpose. Numerical illustrations are presented to show the applicability of the proposed model in Section 5. Finally, our conclusions and future research directions are provided in Section 6.

2. Literature review

In the literature, most researchers take the model proposed in [10] as the first study concerning the environment impact on the system reliability where damages induced by random shocks depended on environment conditions. Since then, a bunch of studies have been developed [11, 12, 13, 14]. To model the effect of the dynamic environment on the system health condition, various approaches have been proposed. The proportional hazards rate model (PHM) and accelerated failure time model (AFTM) are extensively utilized where the environment effect is incorporated into the failure rate as a covariate [15, 16, 17, 18]. Markov chain has also received considerable attention in modelling the variation of operating condition, among which, a continuous time Markov chain (CTMC) is implemented to the scenarios that the transition epochs are arbitrary. Shen et al. [19] studied the availability and imperfect maintenance of a degrading system under a dynamic environment described by a CTMC. Under the constraints of system availability and operating times, the number of optimal imperfect actions were determined by minimizing the long term expected cost rate. Zhang et al. [5] incorporated the impact of the environmental influence on a degradation-threshold-shock model, where system reliability measures, expected total maintenance cost in the finite horizon and expected cost rate in the infinite horizon were derived. Discrete time Markov chain (DTMC) is utilized regarding cyclical transitions of the environment states. Zhao et al. [20] considered the conditional-based inspection and replacement policy of a non-monotone deteriorating system. The observable system degradation was the difference between two competing processes that depended on an environmental covariate modelled by a DTMC.

In degradation modelling, for single-component systems, when the degradation state space is continuous, the Gamma process [21, 22, 23, 24] and Inverse Gaussian process [25, 26] have been widely implemented to model the monotonic degradation processes. Diffusion processes like Wiener process, Ornstein–Uhlenbeck process are preferable in describing non-monotonic degradations [27, 28, 29, 30, 31]. When the degradation state is discrete, in general, most of the studies assume that the degradation process exhibits Markov properties [32, 33, 34, 35]. For example, Markov chain and compound Poisson process have been widely used to model the degradation process of discrete states [see 36]. In fact, both the Gamma process and Inverse Gaussian process can be treated as limits of compound Poisson process [21, 25]. For degrading systems with multiple components, aside from modelling each component, dependency is also an important characteristic which should be taken into account: stochastic dependence, economic dependence, resource dependence, *etc.* are presented in detail in [8].

In recent decades, conditional-based maintenance of multi-component degraded systems has received increasing attention due to advanced sensors and the corresponding available data [6, 37, 29]. Poppe et al. [7] studied a hybrid CBM policy where two degradation threshold of the monitored component were determined to minimize the maintenance cost or the system downtime. Sun et al. [38] considered a K-out-of-N system where the degradation of each component was described by a Wiener process. They showed that the optimal maintenance policy was a multi-dimensional control limit policy. Similarly, Liu et al.[9] studied the optimal inspection and replacement policy of a two-component parallel system over a finite time horizon. The system was modelled by a bivariate Gamma process. Several studies have considered multi-component systems working under dynamic environment. For example, Zhang et al. [1] examined the reliability of a k-out-of-n system under a dynamic environment, where each component possessed binary states. The authors investigated system reliability measures such as the conditional reliability function, remaining useful life and asymptotic availability. Shi et al. [39] developed a condition-based maintenance policy of a degrading system where degradation and environmental observations were utilized to update the posterior distributions of the failure model parameters. They reported that cost-saving could be achieved when the environment information was incorporated in the decision making. Peng et al. [40] presented a summary of the state of the arts on the reliability modelling of complex systems with consideration of the effects of dynamic environments. Through an overview of the existing literature, it can be concluded that not much attention has been paid to modelling the influence of the dynamic working environment on multi-component systems. In addition, when considering maintenance interventions, most of the existing works assume that perfect maintenance can restore the system to the as-good-as-new state, meanwhile, the environment can be restored to its initial state too. However, this assumption can be violated in some situations. For instance, when the dynamic environment is seasons or weathers, it is independent of maintenance actions and is non-adjustable. Both the two scenarios should be taken into consideration.

3. Problem statement

3.1. System descriptions

In this study, we aim to develop a CBM model of a multi-component system operating in dynamic environment modelled by CTMC Consider a system consisting of N heterogeneous components. Denote the set of components as $\Phi = \{1, 2, \cdots, N\}$. At the start of operation, it is assumed that all the components are in the as-good-as-new state. The degradation level of component i is categorized into $L_i + 1$ different states, defined as $0, 1, \cdots, L_i$, where state 0 represents the as-good-as-new state and state L_i indicates the failure state. It is assumed that each component goes through a monotonic degradation process, where the degradation level is monotonically increasing if no maintenance action is executed. The degradation processes of components are assumed to be independent of each other given the working environment, *i.e.*, no interaction among the components is considered. The degradation state of the system is represented by $\boldsymbol{x} = (x_1, x_2, \cdots, x_N)$ where x_i is the ith entry of $\boldsymbol{x}$.

The system works in a dynamic environment which is modeled by a CTMC $\{W_t\}_{t\geq 0}$ with a finite state space $S_l = \{0, 1, ..., M\}$, infinitesimal generator $\boldsymbol{Q}$ and transition probability $\pi_{ij}(t)$, $i, j \in S_l$. Under environment w, the internal degradation increment of component j follows a Poisson distribution with parameter $\lambda_j(t, w)$ by time t, $j \in \Phi, t \geq 0$. Thus the degradation evolution of each component is characterized by its internal degradation and the working environment. The system state space is defined as $\boldsymbol{S} = \big\{(\boldsymbol{x}, w) : x_i \in \{0, 1, \cdots, L_i\}, w \in S_l\big\}$ where $\boldsymbol{x} = \{x_1, x_2, \cdots, x_N\}$ represents the system degradation state and w represents the environment state.

3.2. Descriptions of the maintenance process

Suppose that the failures are non-self-announcing at both the system and component level. The system is periodically inspected with interval τ. The inspection is perfect upon which the environment state and the state of each components can be fully observed. Upon inspection, the system is renewed to the as-good-as-new state if it fails. If the system can still operate but failure exists at the component level, then the failed components are replaced, at the same time, each non-failed component can be preventively replaced, or remain in its current state. Otherwise, if no component fails, then the decision-maker can choose to:

- do nothing and wait for the next decision epoch;
- preventively replace some degraded components.

The inspection and replacement times are negligible. The cost items involved in the maintenance policy include:

- Inspection cost: c_I,

- Maintenance set-up cost: c_s,
- The preventive maintenance cost of component i: c_{pi}, $i \in \Phi$,
- The corrective maintenance cost of component i: c_{ri}, $c_{pi} \leq c_{ri}$, $i \in \Phi$,
- The system replacement cost: c_f,
- Downtime penalty per unitary time due to system failure: c_d.

We intend to determine the optimal maintenance strategy, *i.e.*, the inspection interval and the corresponding maintenance policy that minimizes the long-run total discounted inspection and maintenance cost.

4. Optimal maintenance policy

This section develops a CBM policy for the system operating in dynamic environment, where the optimal policy is obtained in terms of the inspection interval and maintenance action for each system state. We first present the system reliability analysis, which is an essential part in measuring system failure probability and is also the stepping-stone for the subsequent maintenance analysis.

4.1. Reliability analysis

For a new system, denote $P_{wl}(t, \boldsymbol{x})$ as the probability that the system state is $(\boldsymbol{x}, l)$ by time t given its initial state $(\boldsymbol{0}, w)$ at time 0, $\boldsymbol{o}_t$ as the realization of the system degradation state at time t. Then $P_{wl}(t, \boldsymbol{x})$ can be expressed as

$$P_{wl}(t, \boldsymbol{x}) = P\big(\boldsymbol{o}_t = (x_1, x_2, \cdots, x_N), W_t = l \mid \boldsymbol{o}_0 = \boldsymbol{0}, W_0 = w\big). \quad (1)$$

Define $\boldsymbol{P}(t, \boldsymbol{x})$ as the matrix corresponding to $P_{wl}(t, \boldsymbol{x})$ which is a $|M+1| \times |M+1|$ matrix with the $(w, l)th$ element as $P_{wl}(t, \boldsymbol{x})$. Then the expression of $\boldsymbol{P}(t, \boldsymbol{x})$ can be derived with the following lemma.

Lemma 4.1. *Let $\boldsymbol{\Lambda}(t)$ be a diagonal matrix with the (i,i)th element given as $\Lambda_{ii}(t) = \sum_{l\in\Phi} \lambda_l(t,i)$. When $\boldsymbol{x} = \boldsymbol{0}$, meaning that no degradation increment occurs to each component, $\boldsymbol{P}(t,\boldsymbol{0})$ satisfies:*

$$\frac{\mathrm{d}\boldsymbol{P}(t,\boldsymbol{0})}{\mathrm{d}t} = \boldsymbol{P}(t,\boldsymbol{0})\big(\boldsymbol{Q} - \boldsymbol{\Lambda}(t)\big), \tag{2}$$

where $\boldsymbol{Q}$ is the infinitesimal generator of the CTMC. The initial value is $\boldsymbol{P}(0,\boldsymbol{0}) = \boldsymbol{I}$ which is an identity matrix. In particular, when $\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}$, meaning that the degradation increment of each component under given environment follows a homogeneous Poisson process, then $\boldsymbol{P}(t,\boldsymbol{0})$ can be further expressed by

$$\boldsymbol{P}(t,\boldsymbol{0}) = \exp\big((\boldsymbol{Q} - \boldsymbol{\Lambda})t\big). \tag{3}$$

The detailed proof is given in Appendix A.1.

To calculate $\boldsymbol{P}(t,\boldsymbol{x})$ in a general case, let us define $\boldsymbol{x}^{i-} = (x_1, x_2, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$ as the system degradation state where only the degradation of the ith component is 1 state less compared with $\boldsymbol{x}$, while the other component state are identical as $\boldsymbol{x}$. Then the following lemma can be concluded.

Lemma 4.2. *For system state $\boldsymbol{x}$ with $x_i < L_i$, $\forall i \in \Phi$, $\boldsymbol{P}(t,\boldsymbol{x})$ satisfies*

$$\frac{\mathrm{d}\boldsymbol{P}(t,\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{P}(t,\boldsymbol{x})\big(\boldsymbol{Q} - \boldsymbol{\Lambda}(t)\big) + \sum_{i\in\Phi} \boldsymbol{P}(t,\boldsymbol{x}^{i-})\boldsymbol{\Lambda}_i(t), \tag{4}$$

where $\boldsymbol{Q}$ is the infinitesimal generator of the CTMC. $\boldsymbol{\Lambda}_i(t)$ is a diagonal matrix with the (j,j)th element given as $\lambda_i(t,j)$, $i \in \Phi$, $j \in S_l$, and $\boldsymbol{P}(t,\boldsymbol{x}^{i-}) = \boldsymbol{0}$ if $x_i = 0$. The initial value is $\boldsymbol{P}(0,\boldsymbol{x}) = \boldsymbol{0}$.

The detailed proof is provided in Appendix A.2. We have studied the scenario that no failure at component level occurs. For the scenarios that failure occurs at the component level, the following lemma can be used to derive $\boldsymbol{P}(t,\boldsymbol{x})$.

Lemma 4.3. *For a non-failed system state $\boldsymbol{x}$, where failures occur at the component level, i.e. $x_i = L_i$ for some $i \in \Phi$, then $\boldsymbol{P}(t,\boldsymbol{x})$ can be expressed as:*

$$\frac{\mathrm{d}\boldsymbol{P}(t,\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{P}(t,\boldsymbol{x})\big(\boldsymbol{Q} - \sum_{i\in\Phi, x_i\neq L_i} \boldsymbol{\Lambda}_i(t)\big) + \sum_{i\in\Phi} \boldsymbol{P}(t,\boldsymbol{x}^{i-})\boldsymbol{\Lambda}_i(t), \tag{5}$$

where $\boldsymbol{Q}$ *is the infinitesimal generator of the CTMC.* $\boldsymbol{\Lambda}_i(t)$ *is a diagonal matrix with the* (j,j)*th element given as* $\lambda_i(t,j)$*,* $i \in \Phi$*,* $j \in S_l$*, and* $\boldsymbol{P}(t, \boldsymbol{x}^{i-}) = \boldsymbol{0}$ *if* $x_i = 0$*.* $\boldsymbol{x}^{i-} = (x_1, x_2, \cdots, x_{i-1}, x_i - 1, x_{i+1}, \cdots, x_N)$*. The initial value is* $\boldsymbol{P}(0, \boldsymbol{x}) = \boldsymbol{0}$.

The proof is omitted here as it is similar to the proof of Lemma 4.2.

It is worth mentioning that the exact expression of $\boldsymbol{P}(t, \boldsymbol{x})$ is intractable due to the fact that $\boldsymbol{Q}$ and $\boldsymbol{\Lambda}(t)$ are not commute in general. We will turn to numerical methods to solve the above differential equations. In this work, we utilize the product-integration method to solve these equations. Details are given in Appendix A.3. A special case is that if the state of each component is binary, meaning that $L_i = 1$, $\forall i \in \Phi$, then the problem and associated solutions can be found in [1].

The system reliability can be obtained based on the above lemmas. Denote S_g as the set of the system degradation states under which the system is in the functioning state. For instance, consider a two-component system with $L_1 = 1$, and $L_2 = 2$. If it is a series system, then $S_g = \{(0,0), (0,1)\}$. If it is a parallel system, then $S_g = \{(0,0), (0,1), (0,2), (1,0), (1,1)\}$. If the system works if and only if component 1 works, then $S_g = \{(0,0), (0,1), (0,2)\}$.

The following theorem presents the system reliability and conditional reliability functions.

Theorem 4.1. *Let* $\boldsymbol{v} = (v_1, v_2, \cdots, v_M)$ *represent the initial environment state vector with* v_i *representing the probability the the initial environment state is* i*,* $i \in S_l$*. The system reliability function* $R(t; \boldsymbol{v})$ *can be given as*

$$R(t; \boldsymbol{v}) = \boldsymbol{v} \sum_{\boldsymbol{y} \in S_g} \boldsymbol{P}(t, \boldsymbol{y}) \boldsymbol{e}. \tag{6}$$

Given the initial degradation state level $\boldsymbol{x}$ *with* $x_i < L_i$*,* $\forall i \in \Phi$*, the conditional reliability function* $R(t; \boldsymbol{x})$ *can be derived as follows.*

$$R(t; \boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{v} \sum_{\boldsymbol{z} \in S_g} \boldsymbol{P}(t, \boldsymbol{z} - \boldsymbol{x}) \boldsymbol{e}, \tag{7}$$

where $\boldsymbol{e}$ is a column vector where each entry is 1, $\boldsymbol{P}(t, \boldsymbol{y})$ has been presented in Lemmas 4.1-4.3. $\boldsymbol{P}(t, \boldsymbol{z} - \boldsymbol{x}) = \boldsymbol{0}$ if $z_i < x_i$ for some $i \in \Phi$.

4.2. Maintenance analysis

For simplicity in maintenance analysis, it is assumed that the Poisson distribution parameter is independent of the time t, *i.e.*, for all $i \in \Phi$, $\lambda_i(t, w) = \lambda_i(w)$, which is independent of t and related only to the working environment in the following analysis. In this study, we consider two scenarios to model the effect of maintenance actions on the environment. The first one is that when the system is renewed, the working environment will also be restored to its initial friendly state. The other scenario is that the environment is independent of maintenance actions, such as the physical weather. Both of the two cases are studies in the subsequent maintenance modeling. We utilize the expected discounted total cost in the long-run horizon to assess the maintenance performance. Define $\exp(-\gamma t)$ as the discounted factor, where $\gamma \geq 0$. The long-run discount total inspection cost is given as:

$$C_{ins}(\tau) = c_I \sum_{k=0}^{\infty} \exp(-\gamma k \tau) = \frac{c_I}{1 - \exp(-\gamma \tau)}. \quad (8)$$

Since inspection is implemented at every τ intervals, the total discounted cost can be obtained by summating the inspection component $C_{ins}(\tau)$ and the maintenance cost at each inspection epoch. In the following, considering renewable and non-renewable environmental affects upon maintenance respectively, we will formulate the maintenance problem into a Markov decision process (MDP) framework and value iteration algorithm will be used to calculate the maintenance cost.

4.2.1. Maintenance policy for renewable environment state

If the environment state is renewable, when the system is replaced to the as-good-as-new state, the environment is restored to its initial friendly state. Define S_d as the set of the system degradation vectors, $S_g^{(1)}$ as the set of the system degradation states where all components are in their non-failed states, and $S_g^{(2)}$

as the set of the system degradation states where the system is functioning but failures at the component level exist. Then $S_g = S_g^{(1)} \cup S_g^{(2)}$ represents the set of the system degradation states denoting the system is functioning. $S_d \backslash S_g$ represents the system degradation states when the system fails.

Denote $V_\tau(\boldsymbol{x}, w)$ as the expected total discounted maintenance cost given the initial system degradation state $\boldsymbol{x}$ and the environment state w. Hence, the optimal inspection interval can be searched via

$$\tau^* = \arg\min_\tau [C_{ins}(\tau) + V_\tau(\mathbf{0}, 0)].$$

The Bellman equation of $V_\tau(\boldsymbol{x}, w)$ can be expressed as

$$V_\tau(\boldsymbol{x}, w) = \begin{cases} c_s + c_f + V_\tau(\mathbf{0}, 0), & \text{if} \quad \boldsymbol{x} \in S_d \backslash S_g, \\ \min\Big\{ c_s + C(\boldsymbol{x}, \boldsymbol{y}) + V_\tau(\boldsymbol{y}, w\mathrm{I}_{\{\boldsymbol{y} \neq \mathbf{0}\}}), \forall \boldsymbol{y} \in M(\boldsymbol{x}) \Big\}, & \text{if} \quad \boldsymbol{x} \in S_g^{(2)}, \\ \min\Big\{ D_\tau(\boldsymbol{x}, w) + e^{-\gamma\tau} U_\tau(\boldsymbol{x}, w), & \\ \qquad c_s + C(\boldsymbol{x}, \boldsymbol{y}) + V_\tau(\boldsymbol{y}, w\mathrm{I}_{\{\boldsymbol{y} \neq \mathbf{0}\}}), \forall \boldsymbol{y} \in M(\boldsymbol{x}) \Big\}, & \text{if} \quad \boldsymbol{x} \in S_g^{(1)}, \end{cases} \tag{9}$$

where c_I is the inspection cost, c_s is the set-up cost, c_f is the system replacement cost. $C(\boldsymbol{x}, \boldsymbol{y})$ is the maintenance cost such that the degradation vector transfers from $\boldsymbol{x}$ to $\boldsymbol{y}$, expressed as

$$C(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i \in s_r(\boldsymbol{x})} c_{ri} + \sum_{j \in \Phi \backslash s_r(\boldsymbol{x})} c_{pj} \mathrm{I}_{\{y_j = 0\}}, \tag{10}$$

where y_j is the jth element in $\boldsymbol{y}$. $c_{p\cdot}$ and $c_{r\cdot}$ are respectively the preventive and corrective maintenance cost of the considered component. $\mathrm{I}_{\{\cdot\}}$ is the indicator function. $s_r(\boldsymbol{x})$ is the set of failed components given the degradation observation $\boldsymbol{x}$. $M(\boldsymbol{x})$ contains all the possible degradation states after the corresponding maintenance action given the degradation vector $\boldsymbol{x}$.

The Bellman equation can be interpreted as follows. Given the degradation vector $\boldsymbol{x}$, the state of each component and the operating environment can be perfectly revealed. The decision can thus be made based on $\boldsymbol{x}$. If $\boldsymbol{x}$ indicates that the system enters into the failed state, then it is renewed with cost c_f after which the system returns to the as-good-as-new state $\mathbf{0}$ and the environment state is restored to the most friendly scenario. If the system is functioning but failure occurs at the component level, i.e. $x \in S_g^{(2)}$, then the failed ones should be replaced, preventive maintenance may also be implemented upon the functioning components. When no component fails upon inspection, the decision-maker can choose to do the preventive maintenance at component level or system level, or do nothing and wait for the next inspection epoch. Under the scenario that do nothing and wait for the next decision epoch, the cost consists of the expected discounted downtime cost until the next inspection $D_\tau(\boldsymbol{x}, w)$ and the expected value function of the next inspection epoch $U_\tau(\boldsymbol{x}, w)$. The expected discounted downtime can be calculated as follows.

$$D_\tau(\boldsymbol{x}, w) = c_d \int_0^\tau \frac{e^{-\gamma\tau} - e^{-\gamma t}}{\gamma} \mathrm{d}R(t; \boldsymbol{x}, \boldsymbol{v}(w)) \tag{11}$$

where c_d is the downtime cost per unitary time. $R(t; \boldsymbol{x}, \boldsymbol{v}(w))$ is the conditional reliability function which has been presented in Theorem 4.1. We utilize $\boldsymbol{v}(w)$ is the row vector that the wth element is 1 and others are 0, and τ is the inspection period.

To obtain $U_\tau(\boldsymbol{x}, w)$, the transition probabilities of the system states should be calculated first. Define $\kappa_\tau\big((\boldsymbol{x}, w), (\boldsymbol{y}, l)\big)$ as the probability that the system degradation state transfers to $\boldsymbol{y}$ from $\boldsymbol{x}$ and the environment transfers from w to l between two consecutive inspection epoch when the inspection interval is τ. Then for $y \in S_g$, it follows

$$\kappa_\tau\big((\boldsymbol{x}, w), (\boldsymbol{y}, l)\big) = \boldsymbol{v}(w) \sum_{\boldsymbol{z} \in S_g} \boldsymbol{P}(t, \boldsymbol{z} - \boldsymbol{x}) \boldsymbol{e}(l), \tag{12}$$

$\boldsymbol{P}$ can be derived from Lemmas 4.1-4.3. $\boldsymbol{e}(l)$ is the column vector where the lth element is 1 and others are 0. Based on the transition kernel, $U_\tau(\boldsymbol{x}, w)$ can be derived as follows.

$$U_\tau(\boldsymbol{x}, w) = \sum_{y \in S_g} \kappa_\tau\big((\boldsymbol{x}, w), (\boldsymbol{y}, l)\big) V_\tau(\boldsymbol{y}, l) + (1 - R(\tau, \boldsymbol{x}, v(w)))(c_s + c_f + V_\tau(\mathbf{0}, 0)), \tag{13}$$

where S_g is the set of the system degradation vectors indicating that the system is functioning, κ_τ is the transition probability given by equation (12), R is the reliability function given in Theorem 4.1, c_s is the maintenance set-up cost, and c_f is the system replacement cost.

Based on the above Bellman equation, some structural properties of V_τ can be obtained.

Lemma 4.4. *$V_\tau(\boldsymbol{x}, w)$ is a non-decreasing function in x_i, $\forall i \in \Phi$, where x_i is the ith element of $\boldsymbol{x}$.*

The proof is provided in Appendix B.1.

Lemma 4.4 indicates that the expected total maintenance cost in the long-run horizon increases with the initial degradation states. With the monotonicity property of the value function, we can have the following theorem that provides insights for the optimal maintenance actions.

Theorem 4.2. *For the system with state $(\boldsymbol{x}, w)$ and inspection interval τ, we have the following statements:*

- *Given $(\boldsymbol{x}, w)$, if component i is replaced in the optimal policy, then for systems with state $(\boldsymbol{x}^{i+}, w)$, component i should also replaced.*
- *Given $(\boldsymbol{x}, w)$, if component i has not been replaced in the optimal policy, then for systems with state $(\boldsymbol{x}^{i-}, w)$, component i should also remain unchanged.*

where $\boldsymbol{x}^{i-} = (x_1, x_2, \cdots, x_{i-1}, x_i^{(-)}, x_{i+1}, \cdots, x_N)$ is the system degradation state where only the degradation of the ith component is less comparing to $\boldsymbol{x}$, and $\boldsymbol{x}^{i+} = (x_1, x_2, \cdots, x_{i-1}, x_i^{+}, x_{i+1}, \cdots, x_N)$ is the system degradation state where only the degradation of the ith component is larger comparing to $\boldsymbol{x}$.

The proof of Theorem 4.2 is provided in Appendix B.2. A special case is that when all the components follow an identical Poisson distribution for the internal degradation process, i.e. $\lambda_i(w) = \lambda_j(w)$, $\forall i, j \in \Phi$, $w \in S_l$, the optimal maintenance policy can be further simplified. Obviously, when the component characteristics and the maintenance unitary costs are identical to all the components, it is unnecessary to distinguish the components. In this case, if components are to be replaced, it is always optimal to replace the components with the largest degradation states. In addition, the following corollary shows the properties of the optimal maintenance policy.

Corollary 4.1. *If all the components follow an identical Poisson distribution for the internal degradation process, i.e.,* $\lambda_i(w) = \lambda_j(w)$, $\forall i, j \in \Phi$, $w \in S_l$,*and the maintenance costs are identical with* $c_{pi} = c_p < c_r = c_{ri}$ *for any* $i \in \Phi$*. Given the environment state* w*, the inspection interval* τ*, and initial system degradation state* $\boldsymbol{x}$ *with corresponding ordered degradation* $\tilde{\boldsymbol{x}}$*, where* $\tilde{x}_i$ *is the* i*th largest element in* $\boldsymbol{x}$*, it follows*

- *There exist a boundary* $l_1^*(\tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_n)$ *between "Do nothing and wait for the next decision epoch" and "replace 1 component". When* $\tilde{x}_1 < l_1^*(\tilde{x}_2, \tilde{x}_3, \cdots, \tilde{x}_n)$*, "do nothing" is better, otherwise "replace 1 component" is better.*

- *For* $i = 2, 3, \cdots, n-1$*, there exist a boundary* $l_i(\tilde{x}_{i+1}, \tilde{x}_{i+2}, \cdots, \tilde{x}_n)$ *between "replace* $i-1$ *components " and "replace* i *components". When* $\tilde{x}_i < l_i(\tilde{x}_{i+1}, \tilde{x}_{i+2}, \cdots, \tilde{x}_n)$*, "replace* $i-1$ *components" is better, otherwise, "replace* i *components" is better.*

Corollary 4.1 can be readily obtained based on Theorem 4.2, and therefore the detailed proof is omitted to avoid repetition. However, it is difficult to develop theoretical properties with respect to the environment state and the inspection interval. We intend to evaluate the influence of the two factors numerically.

4.2.2. Maintenance policy for non-renewable environment states

In this section, we consider the case that the environment state is non-renewable, meaning that if the system is in the failed states, then it is renewed to state $\mathbf{0}$ as discussed in Section 4.2.1, while the environment state remains unchanged. Other assumptions are as identical as those in Section 4.2.1. Likewise, the Bellman equation of the value function $\hat{V}_\tau(\boldsymbol{x}, w)$ is given as follows.

$$\hat{V}_\tau(\boldsymbol{x}, w) = \begin{cases} c_s + c_f + \hat{V}_\tau(\mathbf{0}, w), & \text{if} \quad \boldsymbol{x} \in S_d \backslash S_g, \\ \min\Big\{ c_s + C(\boldsymbol{x}, \boldsymbol{y}) + \hat{V}_\tau(\boldsymbol{y}, w), \forall \boldsymbol{y} \in M(\boldsymbol{x}) \Big\}, & \text{if} \quad \boldsymbol{x} \in S_g^{(2)}, \\ \min\Big\{ D_\tau(\boldsymbol{x}, w) + e^{-\gamma\tau} U_\tau(\boldsymbol{x}, w), & \\ \qquad c_s + C(\boldsymbol{x}, \boldsymbol{y}) + \hat{V}_\tau(\boldsymbol{y}, w), \forall \boldsymbol{y} \in M(\boldsymbol{x}) \Big\}, & \text{if} \quad \boldsymbol{x} \in S_g^{(1)}, \end{cases} \tag{14}$$

where $C(\boldsymbol{x}, \boldsymbol{y})$ a has been given in equations (10), $M(\boldsymbol{x})$ contains all the possible degradation states after the corresponding maintenance action given the degradation vector $\boldsymbol{x}$. $s_r(\boldsymbol{x})$ is the set of failed components given the degradation observation $\boldsymbol{x}$. It can be observed that the only difference between equation (9) and equation (14) lies in the effect of maintenance actions on the environmental states. In Section 4.2.1, when the environment is renewable, environment state is renewed to $\mathbf{0}$ together with renewal of the system states. Otherwise, the environment remain unchanged. When the environment is non-renewable, environment state is always remain unchanged whatever the maintenance actions on the system (14).

Similar to Lemma 4.4, it can be readily proved that $\hat{V}(\boldsymbol{x}, w)$ is a non-decreasing function in x_i when other elements in $\boldsymbol{x}$ and w remain unchanged. Unfortunately, it is difficult to obtain the monotonicity properties of the optimal maintenance policy due to the non-renewal environmental affect. The properties of the maintenance policies will be examined numerically in the following section. To further illustrate the applicability of the proposed model, A detailed numerical examples are presented in Section 5. The value iteration algorithm is implemented to calculate the value function V_τ where details are illustrated in Algorithm 1. Similarly, $\hat{V}_\tau$ can be obtained numerically.

Algorithm 1: The value iteration algorithm

Input : Component set Φ, failure threshold L_i $i \in \Phi$, environment state set S_l, infinitesimal generator $\boldsymbol{Q}$, Poisson distribution parameters $\lambda_i(w)$, Sets $\boldsymbol{S}$, S_d, S_g, $S_g^{(1)}$, $S_g^{(2)}$, $M(\boldsymbol{x})$, unitary costs, inspection period τ, discount γ, discretize step δ, precision ϵ.

Compute $\boldsymbol{P}(i\delta, \boldsymbol{x})$ using Eqs. (A.7) and (A.9) and $D_\tau(\boldsymbol{x}, w)$;

Compute the transition probability κ_τ in Eq. (12) and $C(\boldsymbol{x}, \boldsymbol{y})$ in Eq. (10);

Set $V_\tau^{(0)}(\boldsymbol{x}, w) = 0$, $\forall(\boldsymbol{x}, w) \in \boldsymbol{S}$;

while $|V_\tau^{(m)}(\boldsymbol{x}, w) - V_\tau^{(m-1)}(\boldsymbol{x}, w)| > \epsilon$, *for some* $(\boldsymbol{x}, w) \in S_d$, **do**

  for $(\boldsymbol{x}, w) \in S_d$, **do**

    if $(\boldsymbol{x}, w) \in S_d \backslash S_g$, **then**

      calculate $V_\tau^{(m)}(\boldsymbol{x}, w) = c_s + c_f + V_\tau^{(m-1)}(\boldsymbol{0}, 0)$

    else if $(\boldsymbol{x}, w) \in S_g^{(2)}$, **then**

      $V_\tau^{(m)}(\boldsymbol{x}, w) = \min\big\{c_s + C(\boldsymbol{x}, \boldsymbol{y}) + V_\tau^{(m-1)}(\boldsymbol{y}, w\mathrm{I}_{\{\boldsymbol{y} \neq \boldsymbol{0}\}}), \forall \boldsymbol{y} \in M(\boldsymbol{x})\big\}$

    else

      calculate $U_\tau^{(m)}(\boldsymbol{x}, w) = \sum_{(\boldsymbol{y}, l) \in \boldsymbol{S}} \kappa(\boldsymbol{x}, w, \boldsymbol{y}, l) V_\tau^{(m-1)}(\boldsymbol{y}, l)$;

      calculate $V_\tau^{(m)}(\boldsymbol{x}, w) = \min\big\{D_\tau(\boldsymbol{x}, w) + e^{-\gamma\tau} U_\tau^{(m)}(\boldsymbol{x}, w), c_s + C(\boldsymbol{x}, \boldsymbol{y}) + V_\tau^{(m-1)}(\boldsymbol{y}, w\mathrm{I}_{\{\boldsymbol{y} \neq \boldsymbol{0}\}}), \forall \boldsymbol{y} \in M(\boldsymbol{x})\big\}$

    end

  end

end

Output: For each $(\boldsymbol{x}, w) \in S_d$, $V_\tau(\boldsymbol{x}, w) = V_\tau^{(m)}(\boldsymbol{x}, w)$, $\forall(\boldsymbol{x}, w) \in S_d$ and the corresponding maintenance policy.

5. A numerical example

So far, we have evaluated the system reliability and maintenance of a degrading system under a dynamic environment, where the system consists of multiple heterogeneous components. For many mechanical systems, they usually operate in a changing environment where the system degradation is highly related to the environment. For instance, the degradations of chock valves depend on the oil and gas flows in the subsea production system. The weather condition, pressure and humidity play important roles in aircraft security. This model can help the decision-maker to develop the inspection and maintenance policy, and to estimate the maintenance cost which can help to control operation and maintenance budget in practice. In this section, we present a numerical example to illustrate the applicability of the proposed model. The following results are obtained by utilizing Matlab2018b on a Windows 8 Core 64-Bits operating system.

5.1. Reliability analysis

Consider a 3-component system in series that operates in a dynamic environment. Assume that the environment states can be classified into there types, for instance, it can been classified as “friendly”, “mild” and “severe” according to its impact on the system. The sojourn time in the friendly state is exponentially distributed with mean 1/3. Whenever the environment change occurs, it will transfer to the “mild” state with probability 1/3 and to the “severe” state with probability 2/3. Similarly, the transition properties of the environment with respect to the other states can be defined. We have the following infinitesimal generator to represent the transition properties.

$$\boldsymbol{Q} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix}. \tag{15}$$

Assume that under a harsher environment state, the failure rate of each component becomes larger. The following matrix presents the failure rate of each component under each environment state:

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0.6 & 0.7 & 0.8 \\ 0.6 & 0.65 & 0.7 \\ 0.7 & 0.8 & 0.9 \end{bmatrix}, \tag{16}$$

where the (i,j)th entry is the parameter of the Poisson distribution that is utilized to describe the degradation increment of component j under environment state i. The failure threshold is $L_i = 2, i \in \Phi$. Given the initial environment state, by utilizing Theorem 4.1, the system reliability function can be obtained. Figure 1 shows the variation of the system reliability function with various initial

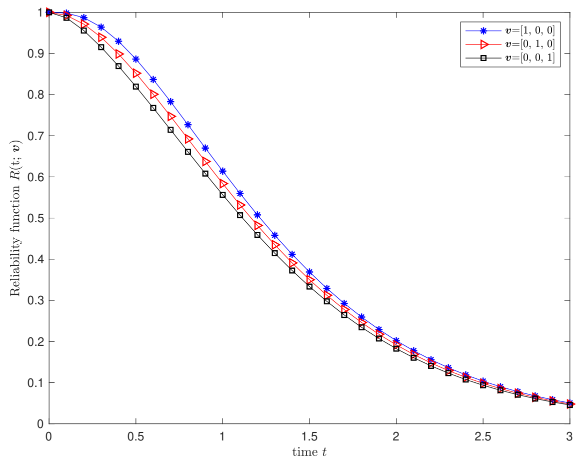

Figure 1: System reliability $R(t; \boldsymbol{v})$ with various initial environment states $\boldsymbol{v}$

environment states. It can be observed that the reliability shows a decreasing tendency under a harsher environmental state. As shown in $\boldsymbol{\Lambda}$, for each component, the parameter of the Poisson distribution increases with the environment state, meaning that it is more easier to enter into the deteriorating or failure state under a harsher initial environment.

Figure 2 shows the variation of the system reliability function with various initial system degradation states given the initial environmental state $\boldsymbol{v} = [1, 0, 0]$. Figure 3 shows the influence of the system reliability with respect to the system failure threshold $\boldsymbol{L}$. As expected, when the initial degradation level is higher or the failure threshold is smaller, the reliability function becomes smaller as the system inclines to enter into the failure state.

5.2. Illustration for maintenance models

Consider the system described in the above section for maintenance modelling. To further illustrate the effectiveness of the proposed model, the failure

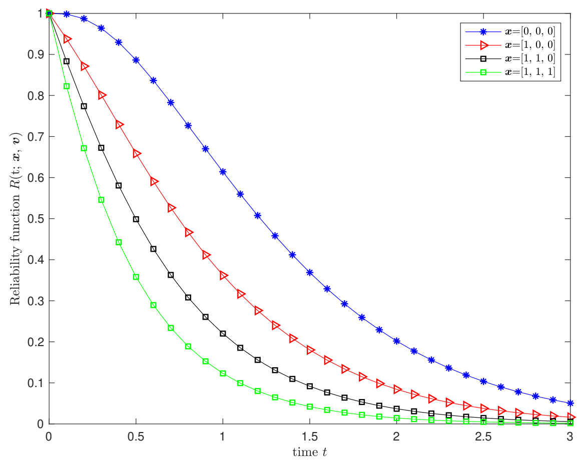

Figure 2: System reliability $R(t; \boldsymbol{x}, \boldsymbol{v})$ with various initial system degradation states $\boldsymbol{x}$

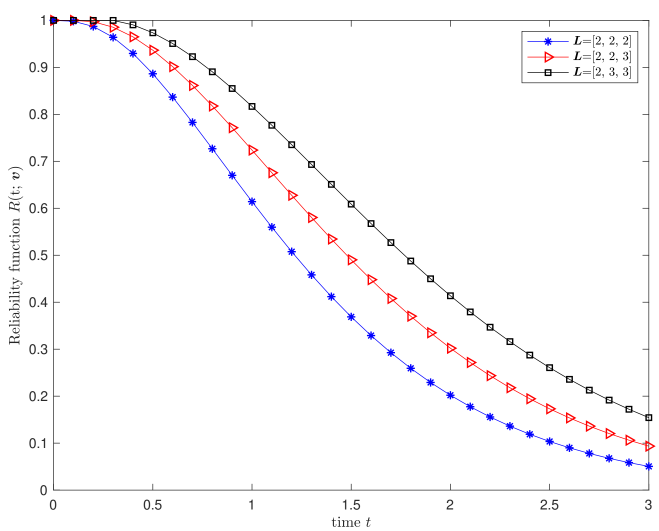

Figure 3: System reliability $R(t; \boldsymbol{v})$ with various failure thresholds $\boldsymbol{L}$

threshold of component i is changed to $L_i = 5$, $i \in \{1,2,3\}$. The parameter corresponding to the discount factor is $\gamma = 0.1$. Assume the unitary costs are $c_s = 1$, $c_I = 1.1$, $c_d = 10$, $c_f = 30$ for $i \in \{1,2,3\}$, $c_{pi} = 1 + i$, $c_{ri} = c_{pi} + 2$. The discretize step is 0.01 and the precision is $\epsilon = 10^{-6}$.

5.2.1. Maintenance under renewable and non-renewable environment

Figure 4 shows the variation of the total expected maintenance cost under various inspection intervals when the environment is renewable. It can be observed that the total expected cost is a unimodal function with respect to the inspection interval in the considered region. The optimum achieves at $\tau = 1$ and the corresponding cost is 54.18.

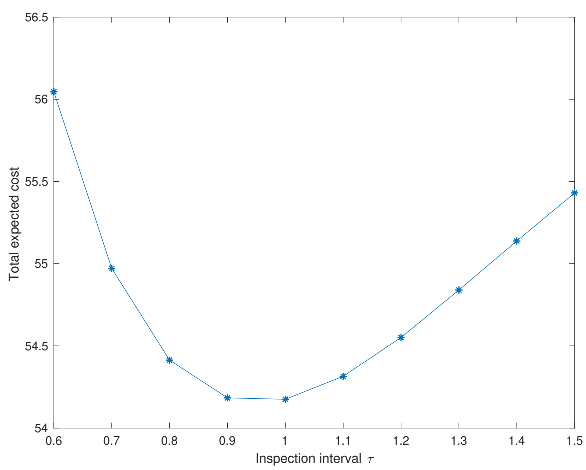

Figure 4: Total expected maintenance cost under various inspection intervals

Table 1 shows the variations of maintenance costs and the corresponding maintenance actions under renewable and non-renewable environment given $\tau = 1$. The value function V_τ corresponds to the scenario with renewable environment and $\hat{V}_\tau$ is associated with the non-renewable environment states. "DN" means do nothing and wait for the next decision epoch. "RE+component number" indicates the considered components are replaced. It can be observed that given w, V_τ and $\hat{V}_\tau$ increase with x_i, which verifies the statement of the monotonic property of the value function under the two scenarios. For each $\boldsymbol{x}$, with renewal environment states, it seems that the V_τ increases with the environment state. Part of the reason might be that with a lower environment state, the probability of having degradation increment is smaller as shown in $\boldsymbol{\Lambda}$ from Eq. (16). In addition, we can observe that "RE12" is the optimal maintenance policy for system with initial degradation $(2, 3, 0)$ and $(2, 4, 0)$, which coincides with Theorem 4.2. For each given $(\boldsymbol{x}, w)$, $\hat{V}_\tau(\boldsymbol{x}, w)$ is always larger than $V_\tau(\boldsymbol{x}, w)$. This is due to the fact that the system benefits more from the replacement when the environment is renewable.

In the subsequent analysis, we will focus on the scenario that the environment is renewable, while the analysis for non-renewable environment can be conducted in a similar way. To further investigate the impact of the environment, we consider the case that the components are homogenous with $\boldsymbol{\Lambda}$ given as

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.7 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 \end{bmatrix}, \tag{17}$$

while other parameters setting are remain unchanged. The optimal inspection interval is $\tau^* = 1.1$ and the associated total expected cost is 50.73. Focusing on the two scenarios where $x_3 = 0$ and $x_3 = 4$, the corresponding maintenance policies are presented in Table 2. The (i, j)th entry represents the maintenance policy given $x_1 = i - 1$ and $x_2 = j - 1$. It can be observed that for each degradation state $\boldsymbol{x}$, components are inclined to be replaced under environment $w = 3$ than under environment $w = 1$. An interesting observation is that it may be optimal to do maintenance with a lower initial system degradation but do nothing with a higher degradation. For instance, under $w = 3$, components 2 and 3 are replaced when the initial system degradation is $(0, 2, 4)$, while only component 3 is replaced when the initial system degradation is $(1, 2, 4)$.

Table 1: Variations of value functions and corresponding maintenance policies with $\boldsymbol{x}$ and w under renewable and non-renewable environments respectively

Environment state w	Degradation state $\boldsymbol{x}$	Value function $V_\tau(\boldsymbol{x}, w)$	Maintenance policy	Value function $\hat{V}_\tau(\boldsymbol{x}, w)$	Maintenance policy
0	(0, 0, 0)	42.6159	DN	44.0840	DN
	(0, 3, 0)	46.6159	RE2	48.0839	RE2
	(2, 3, 0)	48.6159	RE12	50.0839	RE12
	(2, 4, 0)	48.6159	RE12	50.0839	RE12
	(0, 2, 3)	50.4292	RE3	51.9016	RE3
	(1, 3, 3)	51.6190	RE23	53.0966	RE23
	(2, 3, 3)	52.6159	RE123	54.0839	RE123
1	(0, 0, 0)	44.0321	DN	45.5709	DN
	(0, 3, 0)	48.0320	RE2	49.5708	RE2
	(2, 3, 0)	50.0320	RE12	51.5708	RE12
	(2, 4, 0)	50.0320	RE12	51.5708	RE12
	(0, 2, 3)	51.7348	RE3	53.3133	RE3
	(1, 3, 3)	52.9907	RE23	54.5453	RE23
	(2, 3, 3)	54.0320	RE123	55.5708	RE123
2	(0, 0, 0)	45.4052	DN	47.1440	DN
	(0, 3, 0)	49.4051	RE2	51.1439	RE2
	(2, 3, 0)	51.4051	RE12	53.1439	RE12
	(2, 4, 0)	51.4051	RE12	53.1439	RE12
	(0, 2, 3)	52.9811	RE3	54.7935	RE3
	(1, 3, 3)	54.3187	RE23	56.0795	RE23
	(2, 3, 3)	55.4051	RE123	57.1439	RE123

Table 2: Variations of the expected discounted maintenance costs with $\boldsymbol{x}$ and w

	$w=1$					$w=3$				
$x_3=0$	DN	DN	DN	RE2	RE2	DN	DN	DN	RE2	RE2
	DN	DN	DN	RE2	RE2	DN	DN	DN	RE2	RE2
	DN	DN	DN	RE12	RE12	DN	DN	DN	RE12	RE12
	RE1	RE1	RE12	RE12	RE12	RE1	RE1	RE1	RE12	RE12
	RE1	RE1	RE12	RE12	RE12	RE1	RE1	RE1	RE12	RE12
$x_3=4$	RE3	RE3	RE23	RE23	RE23	RE3	RE3	RE3	RE23	RE23
	RE3	RE3	RE3	RE23	RE23	RE3	RE3	RE3	RE23	RE23
	RE13	RE13	RE123	RE123	RE123	RE13	RE13	RE13	RE123	RE123
	RE13	RE13	RE123	RE123	RE123	RE13	RE13	RE13	RE123	RE123
	RE13	RE13	RE123	RE123	RE123	RE13	RE13	RE13	RE123	RE123

5.2.2. Comparison with other maintenance policies

To further illustrate the advantages of the proposed maintenance policy, we compare it with two commonly used maintenance policies. The first one is the degradation-threshold policy, where a common threshold governs the maintenance of all components: each component is preventively replaced if its deterioration level exceeds the threshold X_p and the system is renewed upon failure. Table 3 shows the optimal total expected cost and the associated inspection interval with different maintenance thresholds. It is observed that $X_p = 2$ is the optimal threshold under which the optimal cost is 50.99, which is larger than 50.73 – the value obtained by our proposed model.

The second one is based on the working state of each component: upon inspection, replace the failed components and do nothing to the non-failed components. Hence, no PM actions is considered. Under this policy, the optimal inspection and maintenance cost is 64.87 with inspection period $\tau_2^* = 6.2$, which is equal to the first degradation-threshold policy where the PM threshold is 4. Given the initial environment $\boldsymbol{v} = [1, 0, 0]$, the system reliabilities are $R(\tau^*; \boldsymbol{v}) = 1$, $R(\tau_2^*; \boldsymbol{v}) = 0.14$. It is seen that the system is highly likely to be maintained at failure under the second maintenance policy. Our proposed model is more reliable and economical.

Table 3: Optimal maintenance costs and the corresponding inspection intervals

Optimal inspection & cost	$X_p = 1$	$X_p = 2$	$X_p = 3$	$X_p = 4$
Optimal inspection τ_1^*	2.1	1.0	0.8	6.2
Optimal cost	51.95	50.99	58.03	64.87

5.2.3. Sensitivity analysis

In the section, sensitivity analysis is presented to investigate the impact of parameter values on the optimal maintenance policy.

Impact of components' degradation. When $\mathbf{\Lambda}$ is changed to We change $\mathbf{\Lambda}$ to

$$\tilde{\mathbf{\Lambda}} = \begin{bmatrix} 0.7 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 \\ 0.9 & 0.9 & 0.9 \end{bmatrix}, \tag{18}$$

the optimum cost arrives at $\tau^* = 0.8$. and the associated total expected cost is 60.72. It can be observed that when the failure probability is higher, the maintenance cost is larger and the decision-maker prefers to inspect the system more frequently, with a smaller inspection interval τ.

Impact of environmental variations. To consider the impact of the working environment, let us consider the case where the infinitesimal generator is given as

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} -3 & 0.5 & 2.5 \\ 0.5 & -2 & 1.5 \\ 1 & 3 & -4 \end{bmatrix}. \tag{19}$$

The optimum cost arrives at $\tau^* = 1.1$ and the associated total expected cost is 51.62. which is larger than the value under $\boldsymbol{Q}$. This is due to the fact that with $\tilde{\boldsymbol{Q}}$, when environment state transition occurs, the environment is inclined to enter into a worse state, under which the probability of having degradation increments is larger, thus a larger cost is expected.

Impact of cost parameters. In the part, we examine the influence of the cost parameters on the optimal maintenance policy: the optimal inspection interval and maintenance cost. Tables 4 and 5 show the variation of the optimal maintenance cost and the associated inspection interval with different cost parameters. The ith element in $\boldsymbol{c}_p$ is c_{pi}, for $i = 1, 2, 3$. As expected, the maintenance cost increases with the cost parameters, and the optimal inspection interval increases with the inspection cost.

Table 4: Optimal maintenance costs and the corresponding inspection intervals with various $\boldsymbol{c}_p$ and c_s

Cost units	$\boldsymbol{c}_p = [1.5, 2.5, 3.5]$	$\boldsymbol{c}_p = [2.5, 3.5, 4.5]$	$c_s = 2$	$c_s = 0.5$
Optimal cost	48.38	52.95	54.25	48.83
Optimal inspection	1.1	1.2	1.3	1.1

Table 5: Optimal maintenance costs and the corresponding inspection intervals with different cost units

Cost units	$c_I = 1.0$	$c_I = 1.1$	$c_I = 1.2$	$c_I = 1.3$	$c_d = 12$	$c_f = 32$
Optimal cost	49.75	50.73	51.63	52.46	50.32	51.01
Optimal inspection	1.0	1.0	1.2	1.3	1.0	1.0

6. Conclusions

In this study, we have developed a condition-based maintenance model for a multi-unit system with heterogeneous components that operates in a dynamic environment. The environment evolution is modelled by a continuous-time Markov chain. Given the environment states, the degradation increment of each component is assumed to follow a specific Poisson distribution. The reliability and conditional reliability functions of the system are derived considering the internal degradation process and the environmental effects. Two scenarios considering whether the environment can be renewed or not are studied when developing maintenance policy. Structural properties of the maintenance policy given the inspection interval and the environment state are investigated in detail and further incorporated into the value iteration algorithm to solve the MDP model. We have also examined the influence of the working environment and other parameters settings on the optimal maintenance decisions. It has shown that under a harsher initial environment, the maintenance cost is expected to be larger. The proposed policy has also been compared with the classical CBM models with constant preventive thresholds, under which each component is replaced whenever its degradation level exceeds a predetermined value. It turns out that our policy is more cost-effective in presenting system failures. This work could be further extended in the following three aspects.

First, we have assumed that components are mutual independent given the working environment. It would be more realistic to consider dependencies among components such as the failure dependence, structural dependence, spare limitations, *etc*.

Secondly, we have assumed that both the environment state and the degradation state of each component are observable. When the system is partially observable, inspection and maintenance decisions based on imperfect information is more challenging and interesting.

Thirdly, we have supposed that the maintenance actions are perfect at the component level, *i.e.*, the state of each component is restored to the as-good-as-new state after maintenance. Imperfect maintenance can be an interesting and challenging topic in decision-makings for multi-component systems, at the cost of a heavier computational burden.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proofs for system reliability

Appendix A.1. The proof of Lemma 4.1

Proof: Define $\boldsymbol{o}_t = (x_1, x_2, \cdots, x_N)$ as the realization of the system degradation state at time t where x_i represents the degradation state of component i, $i \in \Phi$, $x_i \in \{0, 1, \cdots, L_i\}$. Let $P_{wl}(t+\Delta, \mathbf{0})$ be the probability that the system is in state $\mathbf{0}$ and the environment state is l by time $t+\Delta$ given the initial degradation state $\mathbf{0}$ and the environment state w. Then we have that

$$\begin{aligned} P_{wl}(t+\Delta, \mathbf{0}) &= \sum_{m \in S_l} P_{wm}(t, \mathbf{0}) \mathbb{P}(\boldsymbol{o}_{t+\Delta} = \mathbf{0}, W_{t+\Delta} = l \mid \boldsymbol{o}_t = \mathbf{0}, W_t = m) \quad \text{(A.1)} \\ &= \sum_{m \in S_l} P_{wm}(t, \mathbf{0}) \pi_{ml}(\Delta) \exp\big(- \sum_{i \in \Phi} \int_t^{t+\Delta} \lambda_i(\theta, m) \mathrm{d}\theta \big), \end{aligned}$$

where $\pi_{ml}(\Delta)$ is the transition probability from state m to l during time Δ, which satisfies

$$\pi_{ml}(\Delta) \approx \begin{cases} 1 + q_{mm}\Delta & \text{if } m = l, \\ q_{ml}\Delta & \text{if } m \neq l. \end{cases} \quad \text{(A.2)}$$

Hence, equation (A.1) can be further expressed as:

$$P_{wl}(t+\Delta, \mathbf{0}) = \big(P_{wl}(t, \mathbf{0}) + \sum_{m \in S_l} P_{wm}(t, \mathbf{0}) q_{ml} \Delta \big) \big(1 - \sum_{i \in \Phi} \lambda_i(t, m) \Delta \big),$$

which yields to

$$\frac{\mathrm{d}P_{wl}(t, \mathbf{0})}{\mathrm{d}t} = -P_{wl}(t, \mathbf{0}) \sum_{i \in \Phi} \lambda_i(t, m) + \sum_{m \in S_l} P_{wm}(t, \mathbf{0}) q_{ml}.$$

Writing in the matrix form, we have

$$\frac{\mathrm{d}\boldsymbol{P}(t+\Delta, \mathbf{0})}{\mathrm{d}t} = \boldsymbol{P}(t, \mathbf{0})(\boldsymbol{Q} - \boldsymbol{\Lambda}(t)), \quad \text{(A.3)}$$

where $\boldsymbol{Q}$ is the infinitesimal generator and $\boldsymbol{\Lambda}(t)$ is a diagonal matrix with the $(j,j)th$ element given by $\Lambda_{jj}(t) = \sum_{i \in \Phi} \lambda_j(t, i)$, $j \in S_l$. In general, the exact solution of equation (A.3) is intractable. A special case is that when $\boldsymbol{\Lambda}(t)$ is independent of t, i.e. $\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}$, meaning that the changement of the degradation under given environment state is constant, then $\boldsymbol{P}(t, \mathbf{0})$ can be given by

$$\boldsymbol{P}(t, \mathbf{0}) = \exp\big((\boldsymbol{Q} - \boldsymbol{\Lambda})t \big).$$

◇

Appendix A.2. The proof of Lemma 4.2

Proof: Let $P_{wl}(t+\Delta,\boldsymbol{x})$ be the probability that the system is in state $\boldsymbol{x}$ and the environment state is l by time $t+\Delta$ given the initial degradation state $\mathbf{0}$ and the environment state w. It is seen that

$$\begin{aligned}P_{wl}(t+\Delta,\boldsymbol{x}) \quad = \quad & \sum_{m\in S_l} P_{wm}(t,\boldsymbol{x})\pi_{ml}(\Delta)\exp\big(-\sum_{i\in\Phi}\int_t^{t+\Delta}\lambda_i(\theta,m)\mathrm{d}\theta\big),\\ & +\sum_{m\in S_l}\sum_{i\in\Phi}P_{wm}(t,\boldsymbol{x}^{i-})\pi_{ml}(\Delta)(1-\exp(-\lambda_i(t,m)\Delta))\\ & \times\exp\big(-\sum_{j\in\Phi\backslash i}\int_t^{t+\Delta}\lambda_j(\theta,m)\mathrm{d}\theta\big),\end{aligned}$$

which yields to

$$\frac{\mathrm{d}P_{wl}(t,\boldsymbol{x})}{\mathrm{d}t} = -P_{wl}(t,\boldsymbol{x})\sum_{i\in\Phi}\lambda_i(t,l) + \sum_{m\in S_l}P_{wm}(t,\boldsymbol{x})q_{ml} + \sum_{i\in\Phi}P_{wl}(t,\boldsymbol{x}^{i-})\lambda_i(t,l) \tag{A.4}$$

It can be expressed in the matrix form as follows:

$$\frac{\mathrm{d}\boldsymbol{P}(t,\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{P}(t,\boldsymbol{x})\big(\boldsymbol{Q}-\boldsymbol{\Lambda}(t)\big)+\sum_{i\in\Phi}\boldsymbol{P}(t,\boldsymbol{x}^{i-})\boldsymbol{\Lambda}_i(t), \tag{A.5}$$

where $\boldsymbol{Q}$ is the infinitesimal generator of the CTMC. $\boldsymbol{\Lambda}_i(t)$ is a diagonal matrix with the (j,j)th element given as $\lambda_i(t,j)$, $i\in\Phi$, $j\in S_l$, and $\boldsymbol{P}(t,\boldsymbol{x}^{i-})=\mathbf{0}$ if $x_i=0$. The initial value is $\boldsymbol{P}(t,\boldsymbol{x})=\mathbf{0}$. $\diamond$

Appendix A.3. Calculations of reliability functions

To calculate the system reliability, In this study, we will solve this problem by the product-integration method ([1]). To calculate $\boldsymbol{P}(t,\mathbf{0})$, as

$$\frac{\mathrm{d}\boldsymbol{P}(t,\mathbf{0})}{\mathrm{d}t} = \boldsymbol{P}(t)(\boldsymbol{Q}-\boldsymbol{\Lambda}(t)), \tag{A.6}$$

then $\boldsymbol{P}(t,\mathbf{0})$ can be numerically obtained as follows.

$$\boldsymbol{P}(t,\mathbf{0}) = \sum_{i=0}^{t/\delta}\big(\boldsymbol{I}+\boldsymbol{M}(i\delta)\delta\big). \tag{A.7}$$

where $\boldsymbol{M}(i\delta)=(\boldsymbol{Q}-\boldsymbol{\Lambda}(i\delta))$, $\boldsymbol{I}$ is the identity matrix, δ is the discretized step size.

To calculate $\boldsymbol{P}(t,\boldsymbol{x})$ where $x_i < L_i, \forall i \in \Phi$. As shown in Lemma 4.2, $\boldsymbol{P}(t,\boldsymbol{x})$ satisfies

$$\frac{\mathrm{d}\boldsymbol{P}(t,\boldsymbol{x})}{\mathrm{d}t} = \boldsymbol{P}(t,\boldsymbol{x})\big(\boldsymbol{Q} - \boldsymbol{\Lambda}(t)\big) + \sum_{i\in\Phi}\boldsymbol{P}(t,\boldsymbol{x}^{i-})\boldsymbol{\Lambda}_i(t), \tag{A.8}$$

Then $\boldsymbol{P}(t,\boldsymbol{x})$ can be obtained numerically as follows.

$$\boldsymbol{P}(t,\boldsymbol{x}) = \int_0^t \boldsymbol{N}(\theta)\Pi_{i=x/\delta}^{t/\delta}(\boldsymbol{I} + \boldsymbol{M}(i\delta)\delta)\mathrm{d}\theta, \tag{A.9}$$

where $\boldsymbol{M}(i\delta) = (\boldsymbol{Q} - \boldsymbol{\Lambda}(i\delta))$, $\boldsymbol{N}(\theta) = \sum_{i\in\Phi}\boldsymbol{P}(t,\boldsymbol{x}^{i-})\boldsymbol{\Lambda}_i(\theta)$, δ is the discretized step size, $\boldsymbol{I}$ is the identity matrix.

Similarly, $\boldsymbol{P}(t,\boldsymbol{x})$ in Lemma 4.3 can be calculated. Hence the system reliability function can be obtained.

Appendix B. Proofs for maintenance modelling

Appendix B.1. The proof of Lemma 4.4

Before the proof, we first introduce the definition of stochastic orders.

Definition. Random variable X_1 is said to be ***stochastically larger*** than X_2 if $\mathbb{P}(X_1 > t) \geq \mathbb{P}(X_2 > t)$, written as $X_1 \succ_{st} X_2$.

A property of this stochastic order is that, if $X_1 \succ_{st} X_2$, then for all increasing functions g, the corresponding expectations satisfy $\mathbb{E}(g(X_1)) \geq \mathbb{E}(g(X_2))$. The above property will be utilized in the following proof.

Proof: We will prove the monotonicity of V by mathematical induction. At the first iteration, set $V_\tau(\boldsymbol{x}, w) = 0$ for all system degradation state $\boldsymbol{x}$ and environment state w.

Assume that $V_\tau(\boldsymbol{x}, w)$ is a non-decreasing function in x_i at the nth iteration where x_i is the degradation state of component $i, i \in \Phi$. For simplicity, we drop off "τ" and use $V^{(n)}$ to represent V at the nth iteration. Then at the $n+1$th iteration, if $\boldsymbol{x} \notin S_g$, from equation (9), $V^{(n+1)}(\boldsymbol{x}, w) = c_s + c_f + V^{(n)}(\boldsymbol{0}, 0)$ which is constant and satisfy the non-decreasing property. If $\boldsymbol{x} \in S_g^{(2)}$, for all $y \in M(\boldsymbol{x})$, $C(\boldsymbol{x},\boldsymbol{y}) + V^{(n)}(\boldsymbol{y}, w)$ either remains constant or is a non-decreasing function of x_i, which yields the non-decreasing property of $V^{(n+1)}(\boldsymbol{x}, w)$ due to that operator "min" keeps the monotonicity.

If $\boldsymbol{x} \in S_g^{(1)}$, we need to prove the monotonicity of $D(\boldsymbol{x}, w)$ and $U^{(n)}(\boldsymbol{x}, w)$. According to the expression of the system reliability function in equation (7), for initial system degradation states $\boldsymbol{x}$ and $\boldsymbol{x}^{(i-)}$, clearly that $R(t; \boldsymbol{x}, \boldsymbol{v}) \leq R(t; \boldsymbol{x}^{(i-)}, \boldsymbol{v})$ for any initial environment state vector $\boldsymbol{v}$. Looking back at equation (11), we can conclude that $D(\boldsymbol{x}, w)$ is non-decreasing in x_i, $i \in \Phi$. In addition, $U^{(n)}(\boldsymbol{x}, w)$ is the expectation of the value function given the system degradation state $\boldsymbol{x}$ and the environment state w. Obvious that $\boldsymbol{y} \mid \boldsymbol{x} \succ_{st} \boldsymbol{y} \mid \boldsymbol{x}^{(i-)}$ and as $V^{(n)}$ is a non-decreasing function in x_i, thus $\mathbb{E}\big(\boldsymbol{y} \mid \boldsymbol{x}\big) \geq \mathbb{E}\big(\boldsymbol{y} \mid \boldsymbol{x}^{(i-)}\big)$ which means that $U^{(n)}(\boldsymbol{x}, w)$ is non-decreasing in x_i. Thus we can conclude that $V^{(n+1)}(\boldsymbol{x}, w)$ is non-decreasing in x_i when $\boldsymbol{x} \in S_g^{(1)}$. Hence, $V(\boldsymbol{x}, w)$ is non-decreasing in x_i, $i \in \Phi$. $\diamond$

Appendix B.2. The proof of Theorem 4.2

Proof: Define $\mathcal{A}_i(\boldsymbol{x}, w)$ as the action set in which component i is not replaced under each action. Define $V_\tau(a; \boldsymbol{x}, w)$ as the value function that action a is chosen in the decision-making given $(\boldsymbol{x}, w)$. Let a_i be the optimal action given $(\boldsymbol{x}, w)$ and assume that component i is replaced under a_i, it means that $\forall b \in \mathcal{A}_i(\boldsymbol{x}, w)$,

$$V_\tau(a_i; \boldsymbol{x}, w) \leq V_\tau(b; \boldsymbol{x}, w).$$

Given $(\boldsymbol{x}^{(i+)}, w)$, it is obvious that $\mathcal{A}_i(\boldsymbol{x}^{(i+)}, w) \subseteq \mathcal{A}_i(\boldsymbol{x}, w)$and $V_\tau(a_i; \boldsymbol{x}^{(i+)}, w) = V_\tau(a_i; \boldsymbol{x}, w)$. Due to the monotonicity of V_τ, $\forall c \in \mathcal{A}_i(\boldsymbol{x}^{(i+)}, w)$,

$$V_\tau(c; \boldsymbol{x}, w) \leq V_\tau(c; \boldsymbol{x}^{(i+)}, w),$$

which yields that $V_\tau(a_i; \boldsymbol{x}^{(i+)}, w) \leq V_\tau(c; \boldsymbol{x}^{(i+)}, w)$, indicating policy a_i is better than policies where component i is not replaced. The first statement in Theorem 4.2 is thus verified. Similarly, we can prove the second statement, which is omitted here. $\diamond$

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