The L^p-to-L^q Compactness of Commutators with $p > q$

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Abstract Let $1 < q < p < \infty$, $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, and T be a non-degenerate Calderón–Zygmund operator. We show that the commutator $[b, T]$ is compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if the symbol $b = a + c$ with $a \in L^r(\mathbb{R}^n)$ and c being any constant. Since both the corresponding Hardy–Littlewood maximal operator and the corresponding Calderón–Zygmund maximal operator are not bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we take the full advantage of the compact support of the approximation element in $C_c^\infty(\mathbb{R}^n)$, which seems to be redundant for many corresponding estimates when $p \le q$ but to be crucial when $p > q$. We also extend the results to the multilinear case.

1 Introduction

Let T denote a Calderón–Zygmund operator and $\mathrm{BMO}(\mathbb{R}^n)$ be the space of functions with bounded mean oscillation introduced by John and Nirenberg [21]. In the celebrated article of Coifman et al. [10], they proved that the commutator

$$[b, T](f) := bT(f) - T(bf)$$

is bounded on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ if and only if the symbol b belongs to $\mathrm{BMO}(\mathbb{R}^n)$. Later, Uchiyama [35] showed that $[b, T]$ is compact on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$ if and only if the symbol b belongs to $\mathrm{CMO}(\mathbb{R}^n)$, namely, the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}(\mathbb{R}^n)$. Here and hereafter, $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support. Since then, the commutators, generated by Calderón–Zygmund operators and $\mathrm{BMO}(\mathbb{R}^n)$ functions, have been widely studied in various branches of mathematics; see, for instance, [3, 4, 13, 14, 28, 29, 30].

Similar characterizations of the boundedness and the compactness also hold true in the fractional setting. To be precise, let $1 < p < q < \infty$ and b belong to the fractional variant of $\mathrm{BMO}(\mathbb{R}^n)$, that is,

$$\|b\|_{\mathrm{BMO}_\alpha(\mathbb{R}^n)} := \sup_{\text{cube } Q \subset \mathbb{R}^n} \frac{1}{|Q|^{1+\alpha/n}} \int_Q \left|b(x) - b_Q\right| dx < \infty$$

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with $\alpha \in (0,1)$ and

$$b_Q := \frac{1}{|Q|}\int_Q b(y)\,dy.$$

Then $\mathrm{BMO}_\alpha(\mathbb{R}^n)$ when $\alpha = 0$ coincides with $\mathrm{BMO}(\mathbb{R}^n)$, and one can derive the boundedness of commutators $[b,T]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with the help of fractional integrals; see, for instance, [16, Theorem 1.0.1]. The corresponding characterization of the compactness also holds true, for any $\alpha \in [0,1)$, via choosing b in the closure of $C_c^\infty(\mathbb{R}^n)$ in $\mathrm{BMO}_\alpha(\mathbb{R}^n)$; see, for instance, [15, Theorem 1.8]. The case $\alpha = 1$ requires a different formulation in terms of a suitably defined space $\mathrm{CMO}_\alpha(\mathbb{R}^n)$ which agrees with the aforementioned closure for $\alpha \in [0,1)$, but not for $\alpha = 1$; see [15, Theorems 1.7 and 1.8]. When considering the boundedness and the compactness of commutators, we always use some classical operators, such as (fractional) maximal operators and fractional integrals, which map $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $p \le q$. However, for the case $p > q$, these operators are no longer bounded. This may be a possible reason why the range $p > q$ has attracted little attention for a long time.

Recently, the first author [16] of this article completely settled the case $p > q$ on the boundedness of commutators $[b,T]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, and hence clarified the boundedness of commutators for all $p, q \in (1,\infty)$. This surprising characterization says that, for any "non-degenerate" Calderón–Zygmund operator T, the commutator $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1 < q < p < \infty$ if and only if $b \in L^r(\mathbb{R}^n)$ modulo constants, where $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. As an application, following an approach proposed by Lindberg [26], the first author [16] of this article further showed that every $f \in L^p(\mathbb{R}^n)$ can be represented as a convergent series of normalized Jacobians $Ju = \nabla u$ of $u \in \dot{W}^{1,np}(\mathbb{R}^n)^n$, which extends a famous result of Coifman et al. [9] and supports a conjecture of Iwaniec [20] about the solvability of the prescribed Jacobian equation $Ju = f \in L^p(\mathbb{R}^n)$.

In another direction, a quite general approach to the compactness of linear operators on the weighted space $L^p_\omega(\mathbb{R}^n)$ has been established very recently by the first author of this article and Lappas [17, 18], There are also recent results about the compactness of Calderón–Zygmund commutators in the two weight setting, namely, from $L^p_\sigma(\mathbb{R}^n)$ to $L^q_\omega(\mathbb{R}^n)$ with different weights; we refer the reader to Lacey and Li [23] for case $p = q$ and to Oikari, Sinko and one of us [19] for $p < q$. But the compactness of $[b,T]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $p > q$ is still unknown so far, even in the unweighted case.

The main purpose of this article is to investigate the compactness of commutators $[b,T]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any given $1 < q < p < \infty$. Besides natural interest in its own right, this question is motivated by the needs of Lindberg's program to the mentioned Jacobian conjecture of Iwaniec, as further developed in [27]. Indeed, we show in Section 2 that the boundedness of $[b,T]$ is *equivalent* to the compactness of $[b,T]$ in this range; see Theorem 2.11 below. This equivalence is essentially based on the density of $C_c^\infty(\mathbb{R}^n)$ in $L^r(\mathbb{R}^n)$. Since the implication "compact $\Longrightarrow$ bounded" is obvious, the main contribution of this article is the reverse implication "bounded $\Longrightarrow$ compact" obtained in Theorems 2.5 and 2.9 below. To this end, we apply the criterion of the compactness in Lebesgue spaces, namely, the Fréchet–Kolmogorov theorem; see Lemma 2.1 below. Since both the corresponding Hardy–Littlewood maximal operator and the corresponding Calderón–Zygmund maximal operator are not bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, we take the full advantage of the compact support of the approximating element in $C_c^\infty(\mathbb{R}^n)$, which seems to be redundant for many corresponding estimates when $p \le q$ but to be crucial when $p > q$. Moreover, we also discuss the corresponding multilinear and iterated results in Section 3. Apart from using the median method and the expectation of random signs, we also apply a characterization of $\dot{L}^r(\mathbb{R}^n)$ via sparse collections of cubes; see Proposition 3.4 below.

Throughout this article, we denote by C and $\widetilde{C}$ positive constants that are independent of the main parameters under consideration, but they may vary from line to line. Moreover, we use $C_{(\gamma,\,\beta,\,\ldots)}$ to denote a positive constant depending on the indicated parameters γ, β, ... Constants with subscripts, such as C_0 and A_1, do not change in different occurrences. Moreover, the symbol $f \lesssim g$ represents that $f \le Cg$ for some positive constant C. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \le h$. For any $p \in [1, \infty]$, let p' denote its *conjugate index*, that is, p' satisfies $1/p + 1/p' = 1$. We use $\mathbf{0}$ to denote the origin of $\mathbb{R}^n$ and, for any set $E \subset \mathbb{R}^n$, we use $\mathbf{1}_E$ to denote its characteristic function. Also, for any $\gamma \in (0, \infty]$, the *Lebesgue space* $L^\gamma(\mathbb{R}^n)$ is defined to be the set of all the Lebesgue measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_\gamma := \left[\int_{\mathbb{R}^n} |f(x)|^\gamma \, dx\right]^{1/\gamma}$$

with usual modification made when $\gamma = \infty$. For any $r \in (0, \infty]$, we use $L^r_{\rm loc}(\mathbb{R}^n)$ to denote the set of all the Lebesgue measurable functions f on $\mathbb{R}^n$ such that $\|f\mathbf{1}_B\|_r < \infty$ for any ball $B \subset \mathbb{R}^n$. Furthermore, for any $r \in (0, \infty]$, let

$$\dot{L}^r(\mathbb{R}^n) := \left\{b \in L^r_{\rm loc}(\mathbb{R}^n) : \ \|b\|_{\dot{L}^r(\mathbb{R}^n)} = \inf_c \|b - c\|_r < \infty\right\};$$

for any $b \in L^1_{\rm loc}(\mathbb{R}^n)$ and any bounded set $S \subset \mathbb{R}^n$ with $|S| > 0$, let

$$\langle b\rangle_S := \frac{1}{|S|}\int_S b(x)\, dx.$$

2 Main results and proofs

To obtain the compactness, we apply the following Fréchet–Kolmogorov theorem which can be found in, for instance, [37, p. 275] and [34]; see also [36, Theorem 1.1].

Lemma 2.1 (Fréchet–Kolmogorov). *Let $q \in (0, \infty)$. Then $\mathcal{F} \subset L^q(\mathbb{R}^n)$ is relatively compact in $L^q(\mathbb{R}^n)$ if and only if*

(i) *$\mathcal{F}$ is bounded, namely,*

$$\sup_{f\in\mathcal{F}} \|f\|_q < \infty;$$

(ii) *$\mathcal{F}$ uniformly vanishes at infinity, namely,*

$$\lim_{M\to\infty} \left\|f\mathbf{1}_{\{x\in\mathbb{R}:\ |x|\ge M\}}\right\|_q = 0 \quad \textit{uniformly for all } f \in \mathcal{F}.$$

(iii) *$\mathcal{F}$ is equicontinuous, namely,*

$$\lim_{\xi\to\mathbf{0}} \|f(\cdot + \xi) - f(\cdot)\|_q = 0 \quad \textit{uniformly for all } f \in \mathcal{F}.$$

In what follows, we consider a Calderón–Zygmund kernel K under the standard conditions

$$|K(x,y)| \le \frac{C_{(K)}}{|x-y|^n} \quad \text{for any } x,y\in\mathbb{R}^n \text{ and } x\neq y, \tag{2.1}$$

and, for any $x,\widetilde{x},y\in\mathbb{R}^n$ with $|x-\widetilde{x}|<\frac{1}{2}|x-y|$,

$$|K(x,y)-K(\widetilde{x},y)|+|K(y,x)-K(y,\widetilde{x})| \le \frac{C_{(K)}}{|x-y|^n}\omega\left(\frac{|x-\widetilde{x}|}{|x-y|}\right), \tag{2.2}$$

where $C_{(K)}$ denotes some positive constant depending only on K and the modulus of continuity $\omega:\ [0,1)\to[0,\infty)$ is increasing. Moreover, we say that ω satisfies the *Dini condition* if

$$\int_0^1 \frac{\omega(t)}{t}\,dt<\infty. \tag{2.3}$$

Notice that $\omega(t):=t^\alpha$ for any $t\in[0,\infty)$ and some $\alpha\in(0,1]$ satisfies the Dini condition. As usual, we assume that the Calderón–Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for any $p\in(1,\infty)$.

We also need the boundedness of the following two maximal operators on $L^p(\mathbb{R}^n)$. Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for any $f\in L^1_{\rm loc}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\mathcal{M}(f)(x):=\sup_{\text{ball } B\ni x}\frac{1}{|B|}\int_B |f(y)|\,dy;$$

and the *maximal truncated operator* T^* is defined by setting, for any $f\in C_{\rm c}^\infty(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$T^*(f)(x):=\sup_{\zeta\in(0,\infty)}\left|\int_{\{y\in\mathbb{R}^n:\ |x-y|>\zeta\}}K(x,y)f(y)\,dy\right|,$$

where the kernel K satisfies (2.1), (2.2), and (2.3). The following boundedness of both $\mathcal{M}$ and T^* on $L^p(\mathbb{R}^n)$ are well known; see, for instance, [12, Theorems 2.5 and 5.14].

Lemma 2.2. *Let $p\in(1,\infty)$. Then both $\mathcal{M}$ and T^* are bounded on $L^p(\mathbb{R}^n)$.*

To use the density argument of compact operators, we adapt the smooth truncated technique same as in [22, 8, 33]. Let $\eta\in(0,\infty)$ and T_η be the smooth truncated Calderón–Zygmund operator of T, that is, T_η is generated by the kernel

$$K_\eta(x,y):=K(x,y)\left[1-\varphi\left(\frac{|x-y|}{\eta}\right)\right] \quad \text{for any } x,y\in\mathbb{R}^n \tag{2.4}$$

with $\varphi\in C^\infty([0,\infty))$ satisfying both

$$0\le\varphi\le 1 \quad\text{and}\quad \varphi(x)=\begin{cases}1, & x\in[0,1/2],\\ 0, & x\in[1,\infty].\end{cases} \tag{2.5}$$

In what follows, for any $p,q\in(1,\infty)$, we use $\|T\|_{p\to q}$ to denote the *operator norm* of T from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. For any $p\in[1,\infty]$, we denote by p' its *conjugate exponent*, namely, $1/p+1/p'=1$. In addition, $C^1_{\rm c}(\mathbb{R}^n)$ denotes the set of all the differentiable functions b on $\mathbb{R}^n$ such that b has compact support and its gradient ∇b is continuous. Then we have the following approximation. In what follows, the symbol $\eta\to 0^+$ means $\eta\in(0,\infty)$ and $\eta\to 0$.

Lemma 2.3. *Let $1 < q < p < \infty$, $b \in C_c^1(\mathbb{R}^n)$, and T be a Calderón–Zygmund operator whose kernel satisfies* (2.1), (2.2), *and* (2.3). *Then, for any $\eta \in (0, \infty)$, $[b, T] - [b, T_\eta]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and*

$$\lim_{\eta \to 0^+} \left\| [b, T] - [b, T_\eta] \right\|_{p \to q} = 0.$$

Proof. Let all the symbols be the same as in the present lemma and let $\eta \in (0, \infty)$. Then $\operatorname{supp}(b) \subset B(\mathbf{0}, R_0)$ for some positive constant R_0, and hence, for any $y \in \mathbb{R}^n \setminus B(\mathbf{0}, R_0 + \eta)$ and any $x \in \mathbb{R}^n$ with $|x - y| \le \eta$, we have $b(x) = 0 = b(y)$ and hence

$$b(x) - b(y) = 0. \tag{2.6}$$

In what follows, without loss of generality, we may consider that both T and T_η for any $\eta \in (0, \infty)$ are well defined on any $f \in L^p(\mathbb{R}^n)$; otherwise, we need first to consider $f \in C_c^\infty(\mathbb{R}^n)$ and then to use a density argument. Thus, by (2.4), (2.5), (2.6), (2.1), and the mean value theorem, we conclude that, for any $\eta \in (0, \infty)$, $f \in L^p(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\left| [b, T](f)(x) - [b, T_\eta](f)(x) \right| \\
&\quad = \left| \lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x-y| < 1/\varepsilon} [b(x) - b(y)] K(x, y) \varphi\left(\frac{|x - y|}{\eta} \right) f(y)\, dy \right| \\
&\quad \le \int_{\mathbb{R}^n} |b(x) - b(y)|\, |K(x, y)|\, \mathbf{1}_{\{y \in \mathbb{R}^n :\ |x-y| \le \eta\}}(x, y)\, |f(y)|\, \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)}(y)\, dy \\
&\quad \lesssim \int_{\mathbb{R}^n} |b(x) - b(y)| \frac{\mathbf{1}_{\{y \in \mathbb{R}^n :\ |x-y| \le \eta\}}(x, y)}{|x - y|^n} |f(y)| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)}(y)\, dy \\
&\quad \lesssim \|\nabla b\|_\infty \sum_{k=0}^\infty \int_{\{y \in \mathbb{R}^n :\ 2^{-(k+1)}\eta < |x-y| \le 2^{-k}\eta\}} \frac{|f(y)| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)}(y)}{|x - y|^{n-1}}\, dy \\
&\quad \lesssim \|\nabla b\|_\infty \sum_{k=0}^\infty \frac{2^{-k}\eta}{[2^{-(k+1)}\eta]^n} \int_{\{y \in \mathbb{R}^n :\ |x-y| \le 2^{-k}\eta\}} |f(y)| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)}(y)\, dy \\
&\quad \lesssim \eta \|\nabla b\|_\infty \mathcal{M}(|f| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)})(x),
\end{aligned}$$

where $\|\nabla b\|_\infty$ denotes the essential supremum of $|\nabla b|$ on $\mathbb{R}^n$ and $\mathcal{M}$ the Hardy–Littlewood maximal operator. From this, Lemma 2.2, and the Hölder inequality, it follows that

$$\begin{aligned}
&\left\| [b, T](f) - [b, T_\eta](f) \right\|_q \\
&\quad \lesssim \eta \|\nabla b\|_\infty \left\| \mathcal{M}(|f| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)}) \right\|_q \lesssim \eta \|\nabla b\|_\infty \|\mathcal{M}\|_{q \to q} \left\| |f| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)} \right\|_q \\
&\quad \lesssim \eta \|\nabla b\|_\infty \|\mathcal{M}\|_{q \to q} \left\| \mathbf{1}_{B(\mathbf{0}, R_0 + \eta)} \right\|_r \|f\|_p \\
&\quad \sim \eta \|\nabla b\|_\infty \|\mathcal{M}\|_{q \to q} (R_0 + \eta)^{n/r} \|f\|_p
\end{aligned}$$

and hence $[b, T] - [b, T_\eta]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ and

$$\lim_{\eta \to 0^+} \left\| [b, T] - [b, T_\eta] \right\|_{p \to q} = 0.$$

This finishes the proof of Lemma 2.3. □

Remark 2.4. (i) Let all the symbols be the same as in Lemma 2.3. Then, by [16, Theorem 1.0.1] and $b \in C_c^1(\mathbb{R}^n) \subset L^r(\mathbb{R}^n)$ with $\frac{1}{r} := \frac{1}{p} - \frac{1}{q}$, we know that $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, which, together with Lemma 2.3, further implies that $[b, T_\eta]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any $\eta \in (0, \infty)$.

(ii) Observe that the compact support of $b \in C_c^1(\mathbb{R}^n)$ plays a key role in the proof of Lemma 2.3. In contrast to this, the corresponding approximation from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $p \le q$ only needs the smoothness of b, namely, $\|\nabla b\|_{L^\infty(\mathbb{R}^n)} < \infty$; see, for instance, [8, Lemma 7], [31, Lemma 3.1], and [32, Lemma 3.4].

Now, we state the first main theorem of this article.

Theorem 2.5. *Let* $1 < q < p < \infty$, $b \in L^r(\mathbb{R}^n)$ *with* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, *and* T *be a Calderón–Zygmund operator whose kernel satisfies* (2.1), (2.2), *and* (2.3). *Then the commutator* $[b, T]$ *is compact from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

Proof. Let all the symbols be the same as in the present theorem. To show the desired compactness, by the density of $C_c^\infty(\mathbb{R}^n)$ in $L^r(\mathbb{R}^n)$, Remark 2.4(i), Lemma 2.3, and [37, p. 278, Theorem(iii)], it suffices to prove that, for any given $b \in C_c^\infty(\mathbb{R}^n)$ and any $\eta \in (0, \infty)$ small enough, $[b, T_\eta]$ is compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. To this end, for any bounded subset $\mathcal{F} \subset L^p(\mathbb{R}^n)$, we show that the set $[b, T]\mathcal{F} := \{[b, T](f) : f \in \mathcal{F}\}$ satisfies (i), (ii), and (iii) of Lemma 2.1, and we proceed in order.

First, using both Lemma 2.3 and Remark 2.4(i), we find that $[b, T_\eta]\mathcal{F}$ satisfies the condition (i) of Lemma 2.1.

Next, from $b \in C_c^\infty(\mathbb{R}^n)$, it follows that $\operatorname{supp}(b) \subset B(\mathbf{0}, R_0)$ for some positive constant R_0. Let $M \in (2R_0, \infty)$. Then, for any $y \in B(\mathbf{0}, R_0)$ and $x \in \mathbb{R}^n$ with $|x| \in (M, \infty)$, we have $|x - y| \sim |x|$. Moreover, by this, (2.4), (2.1), and the Hölder inequality, we conclude that, for any $f \in \mathcal{F}$ and $x \in \mathbb{R}^n$ with $|x| \in (M, \infty)$,

$$\begin{aligned}\left|[b,\, T_\eta](f)(x)\right| &\lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^n} |f(y)|\, dy \lesssim \|b\|_\infty \int_{B(\mathbf{0},R_0)} \frac{|f(y)|}{|x|^n}\, dy \\ &\lesssim \frac{\|b\|_\infty}{|x|^n} \|f\|_p \left\|\mathbf{1}_{B(\mathbf{0},R_0)}\right\|_{p'} \lesssim \frac{\|b\|_\infty \|f\|_p R_0^{n/p'}}{|x|^n}\end{aligned}$$

and hence

$$\begin{aligned}&\left\|[b,\, T_\eta](f)\mathbf{1}_{\{x\in\mathbb{R}^n:\ |x|>M\}}\right\|_q \\ &\quad\lesssim \|b\|_\infty \|f\|_p R_0^{n/p'} \sum_{j=0}^{\infty} \left\| \frac{1}{|\cdot|^n} \mathbf{1}_{\{x\in\mathbb{R}^n:\ 2^j M<|x|\le 2^{j+1}M\}} \right\|_q \\ &\quad\lesssim \|b\|_\infty \|f\|_p R_0^{n/p'} \sum_{j=0}^{\infty} \frac{\|\mathbf{1}_{\{x\in\mathbb{R}^n:\ |x|\le 2^{j+1}M\}}\|_q}{(2^j M)^n} \\ &\quad\lesssim \|b\|_\infty \|f\|_p R_0^{n/p'} \sum_{j=0}^{\infty} \frac{(2^{j+1}M)^{n/q}}{(2^j M)^n} \sim \frac{\|b\|_\infty \|f\|_p R_0^{n/p'}}{M^{n/q'}}.\end{aligned}$$

Therefore, the condition (ii) of Lemma 2.1 holds true for $[b, T_\eta]\mathcal{F}$.

It remains to prove that $[b, T_\Omega^{(\eta)}]\mathcal{F}$ also satisfies the condition (iii) of Lemma 2.1. For any $f \in \mathcal{F}$, $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, and $x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
&[b, T_\eta](f)(x) - [b, T_\eta](f)(x+\xi) \\
&= \int_{\mathbb{R}^n} [b(x) - b(y)] K_\eta(x, y) f(y)\, dy \\
&\quad - \int_{\mathbb{R}^n} [b(x+\xi) - b(y)] K_\eta(x+\xi, y) f(y)\, dy \\
&= [b(x) - b(x+\xi)] \int_{\mathbb{R}^n} K_\eta(x, y) f(y)\, dy \\
&\quad + \int_{\mathbb{R}^n} [b(x+\xi) - b(y)] \left[K_\eta(x, y) - K_\eta(x+\xi, y)\right] f(y)\, dy \\
&= [b(x) - b(x+\xi)] \int_{\mathbb{R}^n} K_\eta(x, y) f(y)\, dy \\
&\quad + \int_{B(\mathbf{0}, R_0)} [b(x+\xi) - b(y)] \left[K_\eta(x, y) - K_\eta(x+\xi, y)\right] f(y)\, dy \\
&\quad + b(x+\xi) \int_{\mathbb{R}^n \setminus B(\mathbf{0}, R_0)} \left[K_\eta(x, y) - K_\eta(x+\xi, y)\right] f(y)\, dy \\
&=: L_1(x) + L_2(x) + L_3(x).
\end{aligned}
\tag{2.7}
$$

We first estimate L_1. For any $x \in \mathbb{R}^n$, using (2.1), we obtain

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n} K_\eta(x, y) f(y)\, dy\right| &\le \left|\int_{\{y\in\mathbb{R}^n:\ |x-y|>\eta/2\}} \left[K_\eta(x, y) - K(x, y)\right] f(y)\, dy\right| \\
&\quad + \left|\int_{\{y\in\mathbb{R}^n:\ |x-y|>\eta/2\}} K(x, y) f(y)\, dy\right| \\
&\lesssim \int_{\{y\in\mathbb{R}^n:\ \eta/2<|x-y|\le\eta\}} \frac{|f(y)|}{|x-y|^n}\, dy + T^*(f)(x) \\
&\lesssim \mathcal{M}(f)(x) + T^*(f)(x).
\end{aligned}
$$

From this, the Hölder inequality, and Lemma 2.2, it follows that

$$
\begin{aligned}
\|L_1\|_q &\le \|b(\cdot) - b(\cdot - \xi)\|_r \left\|\int_{\mathbb{R}^n} K_\eta(\cdot, y) f(y)\right\|_p \\
&\lesssim \|b(\cdot) - b(\cdot - \xi)\|_r \left(\|\mathcal{M}\|_{p\to p} + \left\|T^*\right\|_{p\to p}\right) \|f\|_p.
\end{aligned}
$$

By this, the observation that b is uniformly continuous with compact support [or from the continuity of translations on $L^r(\mathbb{R}^n)$], and Lemma 2.2, we obtain

$$
\lim_{\xi\to\mathbf{0}} \|L_1\|_q = 0. \tag{2.8}
$$

Now, for any $x, y, \xi \in \mathbb{R}^n$ with $|x-y| < \eta/4$ and $|\xi| < \eta/8$, we have $|x-y|/\eta < 1/2$ and $|x+\xi-y|/\eta < 1/2$, which implies that $\varphi(|x-y|/\eta) = 1 = \varphi(|x+\xi-y|/\eta)$ and hence

$$K_\eta(x,y) = 0 = K_\eta(x+\xi, y). \tag{2.9}$$

Moreover, for any $x, y, \xi \in \mathbb{R}^n$ with $|x-y| \geq \eta/4$ and $|\xi| < \eta/8$, we have $|\xi| \leq |x-y|/2$ which, together with (2.2) and (2.1), further implies that

$$\begin{aligned}
&\left|K_\eta(x,y) - K_\eta(x+\xi,y)\right| \\
&\quad = \left|K(x,y)\left[1-\varphi\left(\frac{|x-y|}{\eta}\right)\right] - K_\eta(x+\xi,y)\left[1-\varphi\left(\frac{|x+\xi-y|}{\eta}\right)\right]\right| \\
&\quad \leq |K(x,y)-K(x+\xi,y)|\left|1-\varphi\left(\frac{|x-y|}{\eta}\right)\right| \\
&\quad\quad + |K(x+\xi,y)|\left|\varphi\left(\frac{|x-y|}{\eta}\right)-\varphi\left(\frac{|x+\xi-y|}{\eta}\right)\right| \\
&\quad \lesssim \frac{1}{|x-y|^n}\omega\left(\frac{|\xi|}{|x-y|}\right) \\
&\quad\quad + \frac{\|\varphi'\|_\infty}{|x+\xi-y|^n}\left|\frac{|x+\xi-y|}{\eta}-\frac{|x-y|}{\eta}\right|\mathbf{1}_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ \frac{1}{3}\eta\leq|x-y|\leq 2\eta\}}(x,y) \\
&\quad \lesssim \frac{1}{|x-y|^n}\omega\left(\frac{|\xi|}{|x-y|}\right) + \frac{1}{|x-y|^n}\frac{|\xi|}{\eta}\mathbf{1}_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ \frac{1}{3}\eta\leq|x-y|\leq 2\eta\}}(x,y) \\
&\quad \sim \frac{1}{|x-y|^n}\left[\omega\left(\frac{|\xi|}{|x-y|}\right)+\frac{|\xi|}{|x-y|}\right].
\end{aligned} \tag{2.10}$$

By both (2.9) and (2.10), we conclude that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|K_\eta(x,y)-K_\eta(x+\xi,y)\right||f(y)|\,dy \\
&\quad \lesssim |\xi|\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^{n+1}}\mathbf{1}_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ |x-y|\geq\eta/4\}}\,dy \\
&\quad\quad + \int_{\mathbb{R}^n}\frac{1}{|x-y|^n}\omega\left(\frac{|\xi|}{|x-y|}\right)|f(y)|\mathbf{1}_{\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ |x-y|\geq\eta/4\}}\,dy \\
&\quad \lesssim |\xi|\sum_{k=0}^\infty\left(2^k\eta\right)^{-(n+1)}\int_{\{y\in\mathbb{R}^n:\ 2^k\frac{\eta}{4}\leq|x-y|<2^{k+1}\frac{\eta}{4}\}}|f(y)|\,dy \\
&\quad\quad + \sum_{k=0}^\infty\left(2^k\eta\right)^{-n}\omega\left(\frac{|\xi|}{2^{k-2}\eta}\right)\int_{\{y\in\mathbb{R}^n:\ 2^k\frac{\eta}{4}\leq|x-y|<2^{k+1}\frac{\eta}{4}\}}|f(y)|\,dy \\
&\quad \lesssim \left[\frac{|\xi|}{\eta}+\int_0^{8|\xi|/\eta}\frac{\omega(s)}{s}\,ds\right]\mathcal{M}(f)(x).
\end{aligned}$$

From this, the Hölder inequality, and Lemma 2.2, we deduce that

$$\|L_2\|_q + \|L_3\|_q \tag{2.11}$$

$$
\begin{aligned}
&\le 2\|b\|_\infty \left\| \int_{\mathbb{R}^n} \left|K_\eta(\cdot, y) - K_\eta(\cdot + \xi, y)\right| |f(y)| \mathbf{1}_{B(\mathbf{0},R_0)}\, dy \right\|_q \\
&\quad + \|b\|_r \left\| \int_{\mathbb{R}^n} \left|K_\eta(\cdot, y) - K_\eta(\cdot + \xi, y)\right| |f(y)|\, dy \right\|_p \\
&\lesssim \left[\frac{|\xi|}{\eta} + \int_0^{8|\xi|/\eta} \frac{\omega(s)}{s}\, ds\right] \left[\|b\|_\infty \|\mathcal{M}(f\mathbf{1}_{B(\mathbf{0},R_0)})\|_q + \|b\|_r \|\mathcal{M}(f)\|_p\right] \\
&\lesssim \left[\frac{|\xi|}{\eta} + \int_0^{8|\xi|/\eta} \frac{\omega(s)}{s}\, ds\right] \left[\|b\|_\infty \|\mathcal{M}\|_{q\to q} R_0^{n/r} + \|b\|_r \|\mathcal{M}\|_{p\to p}\right] \|f\|_p.
\end{aligned}
$$

Combining (2.7), (2.8), (2.11), (2.3), and Lemma 2.2, we conclude that

$$
\lim_{\xi\to\mathbf{0}} \left\|[b, T_\eta](f)(\cdot + \xi) - [b, T_\eta](f)(\cdot)\right\|_q = 0
$$

uniformly for any $f \in \mathcal{F}$, which implies the condition (iii) of Lemma 2.1. Thus, $[b, T_\eta]$ is a compact operator for any given $b \in C_c^\infty(\mathbb{R}^n)$ and $\eta \in (0, \infty)$. This then finishes the proof of Theorem 2.5. □

Remark 2.6. One can avoid to use the compact support of b in (2.11) when $p = q$; see, for instance, [33].

Now, we consider a Calderón–Zygmund operator T_Ω with rough homogeneous kernel Ω. Such type of operators and their commutators have attracted a lot of attention; see, for instance, [5, 6, 7]. Precisely, let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and have mean value zero, namely, for any $\mu \in (0, \infty)$ and $x \in \mathbb{S}^{n-1}$,

$$
\Omega(\mu x) := \Omega(x) \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \Omega(x)\, d\sigma(x) = 0; \tag{2.12}
$$

here and hereafter, $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the *unit sphere* of $\mathbb{R}^n$ and $d\sigma$ the area measure on $\mathbb{S}^{n-1}$. Then, for any suitable function f and any $x \in \mathbb{R}^n$,

$$
\begin{aligned}
T_\Omega(f)(x) &:= \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy \\
&:= \lim_{\varepsilon\to 0^+} \int_{\varepsilon<|x-y|<1/\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy.
\end{aligned}
$$

It is well known that T_Ω is bounded on $L^p(\mathbb{R}^n)$ when $\Omega \in L^\nu(\mathbb{S}^{n-1})$ for some $\nu \in (1, \infty]$ and, moreover, for any $f \in L^p(\mathbb{R}^n)$,

$$
\|T_\Omega\|_p \lesssim \|\Omega\|_{L^\nu(\mathbb{S}^{n-1})} \|f\|_p \tag{2.13}
$$

with the implicit positive constant depending only on both n and ν; see, for instance, [12, p. 79, Theorem 4.2]. In particular, if $\Omega \in \mathrm{Lip}\,(\mathbb{S}^{n-1})$, namely, Ω satisfies the *Lipschitz condition*

$$
|\Omega(x) - \Omega(y)| \lesssim |x - y| \quad \text{for any } x, y \in \mathbb{S}^{n-1}
$$

with the implicit positive constant independent of both x and y, then the kernel $K(x, y) := \frac{\Omega(x-y)}{|x-y|^n}$ for any $x, y \in \mathbb{R}^n$ with $x \neq y$ satisfies (2.1), (2.2), and (2.3), which implies the following corollary.

Corollary 2.7. *Let* $\Omega \in \mathrm{Lip}\,(\mathbb{S}^{n-1})$ *satisfy* (2.12). *Then Theorem 2.5 holds true for* T_Ω.

For a rough Ω, we approximate it via $\mathrm{Lip}\,(\mathbb{S}^{n-1})$ and hence obtain the following approximation on $[b, T_\Omega]$.

Lemma 2.8. *Let* $1 < q < p < \infty$, $b \in L^r(\mathbb{R}^n)$ *with* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$, *and* T_Ω *be a Calderón–Zygmund operator with* Ω *satisfying* (2.12) *and* $\Omega \in L^v(\mathbb{S}^{n-1})$ *for some* $v \in (1, \infty)$. *Then, for any* $f \in L^p(\mathbb{R}^n)$,

$$\|[b, T_\Omega](f)\|_q \le \|b\|_r \|\Omega\|_{L^v(\mathbb{S}^{n-1})} \|f\|_p$$

and hence

$$\inf_{\widetilde{\Omega} \in \mathrm{Lip}\,(\mathbb{S}^{n-1})} \left\|[b, T_\Omega] - [b, T_{\widetilde{\Omega}}]\right\|_{p \to q} = 0.$$

Proof. Let all the symbols be the same as in the present lemma. Then, by the Hölder inequality and (2.13), we conclude that

$$\begin{aligned}\|[b, T_\Omega](f)\|_q &\le \|bT_\Omega(f)\|_q + \|T_\Omega(bf)\|_r \lesssim \|b\|_r \|T_\Omega(f)\|_p + \|\Omega\|_{L^v(\mathbb{S}^{n-1})} \|bf\|_q \\ &\lesssim \|b\|_r \|\Omega\|_{L^v(\mathbb{S}^{n-1})} \|f\|_p.\end{aligned}$$

Thus, for any $\widetilde{\Omega} \in \mathrm{Lip}\,(\mathbb{S}^{n-1})$,

$$\left\|[b, T_\Omega] - [b, T_{\widetilde{\Omega}}]\right\|_{p \to q} \lesssim \|b\|_r \left\|\Omega - \widetilde{\Omega}\right\|_{L^v(\mathbb{S}^{n-1})},$$

which, together with the density of $\mathrm{Lip}\,(\mathbb{S}^{n-1})$ in $L^v(\mathbb{S}^{n-1})$, then completes the proof of Lemma 2.8. □

The second main theorem of this article concerns the L^p-to-L^q compactness for commutators of rough homogeneous Calderón–Zygmund operators.

Theorem 2.9. *Let* $1 < q < p < \infty$, $b \in L^r(\mathbb{R}^n)$ *with* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *and* T_Ω *be a Calderón–Zygmund operator with rough homogeneous* Ω *satisfying* (2.12) *and* $\Omega \in L^v(\mathbb{S}^{n-1})$ *for some* $v \in (1, \infty]$. *Then the commutator* $[b, T_\Omega]$ *is compact from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

To prove Theorem 2.9, it suffices to use Corollary 2.7, Lemma 2.8, the density argument of compact operators (see the proof of Theorem 2.5), and the fact $L^\infty(\mathbb{S}^{n-1}) \subset L^v(\mathbb{S}^{n-1})$ for any $v \in (1, \infty)$; we omit the details.

Remark 2.10. It is fairly obvious to find that both Theorems 2.5 and 2.9 still hold true if we replace $b \in L^r(\mathbb{R}^n)$ by $b = a + c$ with $a \in L^r(\mathbb{R}^n)$ and c being any constant; we omit the details.

As in [16, Definition 2.1.1], K is called a *non-degenerate Calderón–Zygmund kernel* if K satisfies (at least) one of the following two conditions:

(i) K is a Calderón–Zygmund kernel satisfying (2.1), (2.2), $\omega(t) \to 0$ as $t \to 0^+$ and there exists a $c_0 \in (0, \infty)$ such that, for any $y \in \mathbb{R}^n$ and $r \in (0, \infty)$, there exists an $x \in \mathbb{R}^n \setminus B(y, r)$ satisfying

$$|K(x, y)| \ge \frac{1}{c_0 r^n};$$

(ii) K is a homogeneous Calderón–Zygmund kernel with $\Omega \in L^1(\mathbb{S}^{n-1}) \setminus \{0\}$. In particular, there exists a Lebesgue point $\theta_0 \in \mathbb{S}^{n-1}$ of Ω such that $\Omega(\theta_0) \neq 0$.

Combining Theorems 2.5 and 2.9, Remark 2.10, and [16, Theorem 1.0.1], we immediately obtain the following conclusion on non-degenerate Calderón–Zygmund operators. Recall that, for any $r \in (0, \infty]$,

$$\dot{L}^r(\mathbb{R}^n) := \left\{ b \in L^r_{\mathrm{loc}}(\mathbb{R}^n) : \ \|b\|_{\dot{L}^r(\mathbb{R}^n)} = \inf_c \|b - c\|_r < \infty \right\}.$$

Theorem 2.11. *Let $1 < q < p < \infty$ and T be a non-degenerate Calderón–Zygmund operator whose kernel K either*

satisfies (2.1), (2.2), *and* (2.3)

or $K(x, y) := \frac{\Omega(x-y)}{|x-y|^n}$ for any $x, y \in \mathbb{R}^n$ and $x \neq y$ with

$$\Omega \text{ satisfying } (2.12) \text{ and } \Omega \in L^v(\mathbb{S}^{n-1}) \text{ for some } v \in (1, \infty].$$

Then the following three statements are mutually equivalent:

(i) *$[b, T]$ is compact from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;*

(ii) *$[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$;*

(iii) *$b \in \dot{L}^r(\mathbb{R}^n)$ with $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$.*

Proof. The implication (i) $\Longrightarrow$ (ii) directly follows from the definition of compact operators, and the implication (ii) $\Longrightarrow$ (iii) is a part of [16, Theorem 1.0.1]. Moreover, using Theorems 2.5 and 2.9 and Remark 2.10, we obtain the implication (iii) $\Longrightarrow$ (i). This then finishes the proof of Theorem 2.11. □

3 Multilinear case

In this section, we briefly discuss how to extend the previous results to the multilinear setting. Let us begin with recalling some definitions and notation. Throughout this section, we fix an $m \in \mathbb{N}$ with $m \geq 2$. Let

$$\Delta := \left\{ (x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1} : \ x = y_1 = \cdots = y_m \right\}$$

be the diagonal in $(\mathbb{R}^n)^{m+1}$. A function $K : (\mathbb{R}^n)^{m+1} \setminus \Delta \to \mathbb{C}$ is called a *multilinear Calderón–Zygmund kernel* if there exists a positive constant C such that

$$|K(x, y_1, \ldots, y_m)| \leq \frac{C}{(\sum_{i=1}^m |x - y_i|)^{mn}} \quad \text{for any } (x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1} \setminus \Delta, \tag{3.1}$$

and, for any $(x, y_1, \ldots, y_m) \in (\mathbb{R}^n)^{m+1} \setminus \Delta$ and $h \in \mathbb{R}$ with $|h| \leq \frac{1}{2} \max_{i \in \{1,\ldots,m\}} |x - y_i|$,

$$|K(x + h, y_1, \ldots, y_m) - K(x, y_1, \ldots, y_m)| \tag{3.2}$$

$$+\sum_{i=1}^{m}|K(x,y_1,\ldots,y_i+h,\ldots,y_m)-K(x,y_1,\ldots,y_i,\ldots,y_m)|$$
$$\leq \frac{C}{(\sum_{i=1}^{m}|x-y_i|)^{mn}}\omega\left(\frac{|h|}{\sum_{i=1}^{m}|x-y_i|}\right),$$

where $\omega: [0,1)\to[0,\infty)$ is an increasing function with $\omega(0)=0$ and satisfies the Dini condition (2.3). Then T is called an *m-linear Calderón–Zygmund operator* if T is initially bounded from $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $q_i\in(1,\infty]$ for any $i\in\{1,\ldots,m\}$ and $\frac{1}{q}:=\sum_{i=1}^{m}\frac{1}{q_i}\in(0,\infty)$, and there exists a multilinear Calderón–Zygmund kernel K such that, for any $f_1,\ldots,f_m\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ and any $x\notin\bigcap_{i=1}^{m}\operatorname{supp} f_i$,

$$T(f_1,\ldots,f_m)(x)=\int_{\mathbb{R}^{mn}}K(x,y_1,\ldots,y_m)\prod_{i=1}^{m}f_i(y_i)\,dy_1\cdots dy_m, \tag{3.3}$$

where, for any $i\in\{1,\ldots,m\}$, $\operatorname{supp} f_i$ denotes the *support* of f_i, namely, the closure in $\mathbb{R}^n$ of the set $\{x\in\mathbb{R}^n:\ f_i(x)\neq 0\}$. Moreover, we say T is a *non-degenerate multilinear Calderón–Zygmund operator* if there exists a function K such that (3.1), (3.2), and (3.3) hold true with $\omega(0)\to 0$ when $t\to 0^+$ and, in addition, there exists a positive constant c_0 such that, for any $y\in\mathbb{R}^n$ and $r\in(0,\infty)$, there exists an $x\in\mathbb{R}^n\setminus B(y,r)$ satisfying

$$|K(x,y,\ldots,y)|\geq\frac{1}{c_0 r^{mn}}. \tag{3.4}$$

For any $i\in\{1,\ldots,m\}$, the *multilinear Calderón–Zygmund commutator* $[b,T]_i$ is defined by setting, for any suitable functions $\{f_i\}_{i=1}^{m}$ and any $x\in\mathbb{R}^n$,

$$[b,T]_i(f_1,\ldots,f_m)(x):=b(x)T(f_1,\ldots,f_m)(x)-T(f_1,\ldots,bf_i,\ldots,f_m)(x).$$

Without loss of generality, we may only consider the case $i=1$, namely, the commutator $[b,T]_1$. First of all, we establish the boundedness of multilinear Calderón–Zygmund commutators. In what follows, for any $\gamma,\gamma_1,\ldots,\gamma_m\in(0,\infty]$, we use $\|T\|_{(\gamma_1,\ldots,\gamma_m)\to\gamma}$ to denote the *operator norm* of T from $L^{\gamma_1}(\mathbb{R}^n)\times\cdots\times L^{\gamma_m}(\mathbb{R}^n)$ to $L^{\gamma}(\mathbb{R}^n)$.

Proposition 3.1. *Let $0<q<p<\infty$, $p_i\in(1,\infty)$ for any $i\in\{1,\ldots,m\}$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $b\in\dot{L}^r(\mathbb{R}^n)$ with $\frac{1}{r}:=\frac{1}{q}-\frac{1}{p}<\frac{1}{p_1'}$, and T be an m-linear Calderón–Zygmund operator, where $\frac{1}{p_1}+\frac{1}{p_1'}=1$. Then the commutator $[b,T]_1$ is bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Proof. Let all the symbols be the same as in the present proposition. Then, by the Hölder inequality, the boundedness of T from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (see, for instance, [11, 25]), and the observation $1/(\frac{1}{p_1}+\frac{1}{r})=\frac{p_1 r}{p_1+r}>1$, together with the boundedness of T from $L^{\frac{p_1 r}{p_1+r}}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ (see, for instance, [11, 25] again), we conclude that, for any $(f_1,\ldots,f_m)\in L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$,

$$\|[b,T]_1(f_1,\ldots,f_m)\|_q$$
$$\leq\|bT(f_1,\ldots,f_m)\|_q+\|T(bf_1,f_2,\ldots,f_m)\|_q$$

$$\leq \|b\|_r \|T(f_1, \ldots, f_m)\|_p + \|T\|_{(\frac{p_1 r}{p_1+r}, p_2, \ldots, p_m) \to q} \|bf_1\|_{\frac{p_1 r}{p_1+r}} \prod_{i=2}^{m} \|f_i\|_{p_i}$$

$$\leq \left[\|T\|_{(p_1, \ldots, p_m) \to p} + \|T\|_{(\frac{p_1 r}{p_1+r}, p_2, \ldots, p_m) \to q} \right] \|b\|_r \prod_{i=1}^{m} \|f_i\|_{p_i},$$

which completes the proof of Proposition 3.1. □

Next, we consider the compactness of $[b, T]_1$ and likewise introduce the smooth truncated m-linear Calderón–Zygmund kernel: for any $\eta \in (0, \infty)$, $x \in \mathbb{R}^n$, and $y_i \in \mathbb{R}^n$ with $i \in \{1, \ldots, m\}$,

$$K_\eta(x, y_1, \ldots, y_m) := K(x, y_1, \ldots, y_m) \left[1 - \varphi\left(\frac{\max_{1 \leq i \leq m} |x - y_i|}{\eta} \right) \right]$$

with the same φ as in (2.5). Let $b \in C_c^1(\mathbb{R}^n)$ and, for any $i \in \{1, \ldots, m\}$, $f_i \in L^{p_i}(\mathbb{R}^n)$ with $p_i \in (1, \infty)$ and $\frac{1}{p} := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. We may assume that $\operatorname{supp}(b) \subset B(\mathbf{0}, R_0)$ for some positive constant R_0. Then, by similar arguments to the proof of Lemma 2.3 above (see also [2, Lemma 2.1]), we have, for any $\eta \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} &\left| [b, T]_1(f_1, \ldots, f_m)(x) - [b, T_\eta]_1(f_1, \ldots, f_m)(x) \right| \\ &\quad \lesssim \eta \|\nabla b\|_\infty \mathcal{M}(f_1 \mathbf{1}_{B(\mathbf{0}, R_0+\eta)}, f_2, \ldots, f_m)(x), \end{aligned}$$

where the implicit positive constant is independent of η, b, $\{f_i\}_{i=1}^m$, and x, and where $\mathcal{M}$ stands for the *m-linear Hardy–Littlewood maximal operator* defined by setting, for any $g_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $i \in \{1, \ldots, m\}$ and for any $x \in \mathbb{R}^n$,

$$\mathcal{M}(g_1, \ldots, g_m)(x) := \sup_{\text{ball } B \ni x} \prod_{i=1}^{m} \frac{1}{|B|} \int_B |g_i(y_i)| \, dy_i.$$

Then this immediately implies the multilinear analogy of Lemma 2.3. To obtain the multilinear analogy of Theorem 2.5, note that, for any $x \in \mathbb{R}^n$ with $|x| > M > 2R_0$, we have

$$\begin{aligned} &\left| [b, T_\eta]_1(f_1, \ldots, f_m)(x) \right| \\ &\quad \lesssim \|b\|_\infty \int_{\mathbb{R}^{mn}} \frac{(|f_1| \mathbf{1}_{B(\mathbf{0}, R_0)})(y_1) \prod_{i=2}^m |f_i(y_i)|}{(\sum_{i=1}^m |x - y_i|)^{mn}} \, dy_1 \cdots dy_m \\ &\quad \lesssim \|b\|_\infty \int_{\mathbb{R}^n} \frac{(|f_1| \mathbf{1}_{B(\mathbf{0}, R_0)})(y_1)}{|x - y_1|^n} \, dy_1 \prod_{i=2}^{m} \mathcal{M} f_i(x) \lesssim \frac{\|b\|_\infty \|f_1\|_{p_1} R_0^{n/p_1'}}{|x|^n} \prod_{i=2}^{m} \mathcal{M} f_i(x), \end{aligned}$$

where the implicit positive constant is independent of η, b, $\{f_i\}_{i=1}^m$, and x. From this, $\frac{1}{q} := \frac{1}{p} + \frac{1}{r} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{r}$, and the Hölder inequality, it follows that

$$\begin{aligned} &\left\| [b, T_\eta]_1(f_1, \ldots, f_m) \mathbf{1}_{\{x \in \mathbb{R}^n : \, |x| > M\}} \right\|_q \\ &\quad \lesssim \|b\|_\infty R_0^{n/p_1'} \left\| |x|^{-n} \mathbf{1}_{\{x \in \mathbb{R}^n : \, |x| > M\}} \right\|_{q_1} \prod_{i=1}^{m} \|f_i\|_{p_i} \end{aligned}$$

$$\sim \|b\|_\infty R_0^{n/p_1'} M^{-\frac{n}{q_1'}} \prod_{i=1}^m \|f_i\|_{p_i},$$

where $\frac{1}{q_1} := \frac{1}{p_1} + \frac{1}{r} < 1$ and the implicit positive constant is independent of η, b, $\{f_i\}_{i=1}^m$, and x. This verifies the condition (ii) of Lemma 2.1. The remaining arguments are quite similar to the proof of Theorem 2.5. So, by omitting some details, we obtain the following result.

Theorem 3.2. *Let $0 < q < p < \infty$, $p_i \in (1, \infty)$ for any $i \in \{1, \dots, m\}$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $b \in \dot{L}^r(\mathbb{R}^n)$ with $\frac{1}{r} := \frac{1}{q} - \frac{1}{p} < \frac{1}{p_1'}$, and T be an m-linear Calderón–Zygmund operator, where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$. Then the commutator $[b, T]_1$ is compact from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

One may be curious about whether or not the opposite direction holds true. Namely, if we know that T is an m-linear Calderón–Zygmund operator associated with some non-degenerate kernel and that the commutator $[b, T]_1$ is compact from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, can we show that $b = a + c$ for some $a \in L^r$ and some constant c? We give an affirmative answer to this question when $q \in (1, \infty)$ and b is real-valued as follows.

Theorem 3.3. *Let $1 < q < p < \infty$, $p_i \in (1, \infty)$ for any $i \in \{1, \dots, m\}$, and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let T be an m-linear non-degenerate Calderón–Zygmund operator and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be real-valued. If the commutator $[b, T]_1$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $b \in \dot{L}^r(\mathbb{R}^n)$ with $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$.*

To prove the above theorem, we need the following characterization of $\dot{L}^r(\mathbb{R}^n)$, whose proof can be found at [1, Proposition 3.2]. In what follows, a collection $\mathscr{S}$ of cubes is said to be *sparse* if there exists a family of pairwise disjoint subsets, $\{E(S)\}_{S \in \mathscr{S}}$, such that, for any $S \in \mathscr{S}$,

$$E(S) \subset S \text{ and } |E(S)| \geq \frac{1}{2}|S|;$$

for any locally integrable function b and any bounded measurable set S with $|S| > 0$, let

$$\langle b \rangle_S := \frac{1}{|S|} \int_S b(x)\, dx.$$

Proposition 3.4. *Let $r \in (1, \infty)$ and $b \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\|b\|_{\dot{L}^r(\mathbb{R}^n)} \sim \sup \left\{ \sum_{S \in \mathscr{S}} \lambda_S \int_S |b(x) - \langle b \rangle_S|\, dx : \mathscr{S} \text{ is sparse, } \sum_{S \in \mathscr{S}} |S| \lambda_S^{r'} \leq 1 \right\}$$

with the positive equivalence constants independent of b.

Now, we are ready to prove Theorem 3.3, via following the median method used in [16, 24]. In what follows, for any locally integrable functions f and g, let $\langle f, g \rangle := \int_{\mathbb{R}^n} |f(x)g(x)|\, dx$.

Proof of Theorem 3.3. Let all the symbols be the same as in the present theorem. By (3.4), we find that, for any cube Q, there exists a cube $\widetilde{Q}$ such that $\ell(Q) = \ell(\widetilde{Q})$ and $\operatorname{dist}(Q, \widetilde{Q}) \geq C_0 \ell(Q)$, and there exists a $\sigma_Q \in \mathbb{C}$ with $|\sigma_Q| = 1$ and

$$\Re\left(\sigma_Q K(x, y_1, \dots, y_m)\right) \sim |Q|^{-m}, \quad \forall\, x \in \widetilde{Q},\ \forall\, y_1, \dots, y_m \in Q; \tag{3.5}$$

see, for instance, [16, Proposition 2.2.1 and Remark 4.1.2]. Here and hereafter, we use $\Re(z)$ to denote the *real part* of any $z \in \mathbb{C}$. Now, we apply Proposition 3.4 and fix a sparse family $\mathscr{S}$. For any cube $S \in \mathscr{S}$, let α_S be the *median* of b on $\widetilde{S}$, namely,

$$\min\left(|\widetilde{S} \cap \{b \le \alpha_S\}|,\ |\widetilde{S} \cap \{b \ge \alpha_S\}|\right) \ge \frac{1}{2}|\widetilde{S}| = \frac{1}{2}|S|.$$

By this and (3.5), we obtain, for any $x \in \widetilde{S} \cap \{b \ge \alpha_S\}$,

$$\begin{aligned}
&\int_S [b(y_1) - \alpha_S]_-\, dy_1 \\
&\quad = \int_{S\cap\{b\le\alpha_S\}} [\alpha_S - b(y_1)]\, dy_1 \\
&\quad \lesssim |S|\Re\left(\sigma_S \int_{S^{m-1}} \int_{S\cap\{y_1: b(y_1)\le\alpha_S\}} [b(x) - b(y_1)]K(x, y_1, \cdots, y_m)\, dy_1 \cdots dy_m\right) \\
&\quad = |S|\Re\left(\sigma_S [b, T]_1(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)(x)\right).
\end{aligned}$$

Likewise, for any $x \in \widetilde{S} \cap \{b \le \alpha_S\}$, we have

$$\int_S [b(y_1) - \alpha_S]_+\, dy_1 \lesssim -|S|\Re\left(\sigma_S [b, T]_1(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)(x)\right).$$

Thus, for any $\{\lambda_S\}_{S\in\mathscr{S}} \subset [0, \infty)$, we find that

$$\begin{aligned}
&\sum_{S\in\mathscr{S}} \lambda_S \int_S |b(x) - \langle b\rangle_S|\, dx \\
&\quad \le 2 \sum_{S\in\mathscr{S}} \lambda_S \int_S |b(x) - \alpha_S|\, dx \\
&\quad \lesssim \sum_{S\in\mathscr{S}} \lambda_S |S| \Re\left(\sigma_S \left\langle [b, T]_1(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)\right\rangle_{\widetilde{S}\cap\{b\ge\alpha_S\}}\right) \\
&\qquad - \sum_{S\in\mathscr{S}} \lambda_S |S| \Re\left(\sigma_S \left\langle [b, T]_1(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)\right\rangle_{\widetilde{S}\cap\{b\le\alpha_S\}}\right) \\
&\quad \lesssim \sum_{S\in\mathscr{S}} \lambda_S |S| \left\langle \left|[b, T]_1(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)\right|\right\rangle_{\widetilde{S}} \\
&\qquad + \sum_{S\in\mathscr{S}} \lambda_S |S| \left\langle \left|[b, T]_1(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)\right|\right\rangle_{\widetilde{S}}.
\end{aligned}$$

We next only focus on estimating the first term in the last step since the other one is similar. Moreover, by the monotone convergence theorem, we may assume that $\mathscr{S}$ contains only finitely many elements. Now, let g_S be the function such that

$$\left|[b, T]_1(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S)\right| = [b, T]_1(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) g_S.$$

For each $j \in \{1, \ldots, m\}$, let $\{\varepsilon_S^{(j)}\}_{S\in\mathscr{S}}$ be a collection of independent random signs, and we denote by $\mathbb{E}^{(j)}$ the corresponding *expectation*, namely, for any function f defined on two random signs,

$$\mathbb{E}^{(j)} f(\varepsilon_1, \varepsilon_2) := \sum_{\varepsilon_1=\pm1,\ \varepsilon_2=\pm1} \frac{f(\varepsilon_1, \varepsilon_2)}{4},$$

where $\varepsilon_1, \varepsilon_2 \in \{\varepsilon_S^{(j)}\}_{S\in\mathscr{S}}$. Write $\mathbb{E} := \mathbb{E}^{(1)} \cdots \mathbb{E}^{(m)}$ and observe that

$$1 = r'\left(\frac{1}{q'} + \frac{1}{p}\right) = \frac{r'}{q'} + \frac{r'}{p_1} + \cdots + \frac{r'}{p_m},$$

where $\frac{1}{r} + \frac{1}{r'} = 1 = \frac{1}{q} + \frac{1}{q'}$. From this, the linearity of $[b, T]_1(\cdot)$, the Hölder inequality, Proposition 3.1, [16, Lemma 2.5.4], and $\sum_{S\in\mathscr{S}} |S|\lambda_S^{r'} \le 1$, it follows that

$$\begin{aligned}
&\sum_{S\in\mathscr{S}} \lambda_S |S| \left\langle \left|[b,T]_1\left(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \ldots, \mathbf{1}_S\right)\right| \right\rangle_{\widetilde{S}} \\
&= \sum_{S\in\mathscr{S}} \left\langle [b,T]_1 \left(\lambda_S^{\frac{r'}{p_1}} \mathbf{1}_{S\cap\{b\le\alpha_S\}}, \lambda_S^{\frac{r'}{p_2}} \mathbf{1}_S, \ldots, \lambda_S^{\frac{r'}{p_m}} \mathbf{1}_S\right),\ \lambda_S^{\frac{r'}{q'}} g_S \mathbf{1}_{\widetilde{S}} \right\rangle \\
&= \mathbb{E}\Bigg\langle [b,T]_1 \Bigg(\sum_{S_1\in\mathscr{S}} \varepsilon_{S_1}^{(1)} \lambda_{S_1}^{\frac{r'}{p_1}} \mathbf{1}_{S_1\cap\{b\le\alpha_{S_1}\}}, \sum_{S_2\in\mathscr{S}} \varepsilon_{S_2}^{(1)} \varepsilon_{S_2}^{(2)} \lambda_{S_2}^{\frac{r'}{p_2}} \mathbf{1}_{S_2}, \ldots, \\
&\qquad \sum_{S_m\in\mathscr{S}} \varepsilon_{S_m}^{(m-1)} \varepsilon_{S_m}^{(m)} \lambda_{S_m}^{\frac{r'}{p_m}} \mathbf{1}_{S_m} \Bigg), \sum_{S_{m+1}\in\mathscr{S}} \varepsilon_{S_{m+1}}^{(m)} \lambda_{S_{m+1}}^{\frac{r'}{q'}} g_S \mathbf{1}_{\widetilde{S}_{m+1}} \Bigg\rangle \\
&\lesssim \mathbb{E} \left\| \sum_{S_1\in\mathscr{S}} \varepsilon_{S_1}^{(1)} \lambda_{S_1}^{\frac{r'}{p_1}} \mathbf{1}_{S_1\cap\{b\le\alpha_{S_1}\}} \right\|_{p_1} \cdots \left\| \sum_{S_m\in\mathscr{S}} \varepsilon_{S_m}^{(m-1)} \varepsilon_{S_m}^{(m)} \lambda_{S_m}^{\frac{r'}{p_m}} \mathbf{1}_{S_m} \right\|_{p_m} \\
&\qquad \cdot \left\| \sum_{S_{m+1}\in\mathscr{S}} \varepsilon_{S_{m+1}}^{(m)} \lambda_{S_{m+1}}^{\frac{r'}{q'}} g_S \mathbf{1}_{\widetilde{S}_{m+1}} \right\|_{q'} \\
&\lesssim \left\| \sum_{S_1\in\mathscr{S}} \lambda_{S_1}^{\frac{r'}{p_1}} \mathbf{1}_{S_1} \right\|_{p_1} \cdots \left\| \sum_{S_m\in\mathscr{S}} \lambda_{S_m}^{\frac{r'}{p_m}} \mathbf{1}_{S_m} \right\|_{p_m} \cdot \left\| \sum_{S_{m+1}\in\mathscr{S}} \lambda_{S_{m+1}}^{\frac{r'}{q'}} \mathbf{1}_{\widetilde{S}_{m+1}} \right\|_{q'} \\
&\sim \left(\sum_{S\in\mathscr{S}} \lambda_S^{r'} |S| \right)^{\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{q'}} \lesssim 1.
\end{aligned}$$

This, together with Proposition 3.4, then completes the proof of Theorem 3.3. □

The above multilinear results can also be extended to the iterated case. In what follows, for any $k \in \mathbb{N}$, let $T_{b,1}^{(k)} := [b, T_{b,1}^{(k-1)}]_1$ be the k-times iterated commutator of $[b, T]_1 =: T_{b,1}^{(1)}$. Combining [16, Theorem 4.0.1] and Theorems 3.2 and 3.3, we immediately have the following results; we omit the details here.

Theorem 3.5. *Let* $0 < q < p < \infty$, $p_i \in (1, \infty)$ *for any* $i \in \{1, \ldots, m\}$, *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Let* $k \in \mathbb{N}$ *and* $b \in \dot{L}^{kr}(\mathbb{R}^n)$ *with* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p} < \frac{1}{p_1'}$, *where* $\frac{1}{p_1} + \frac{1}{p_1'} = 1$. *Let* T *be an* m*-linear Calderón–Zygmund operator. Then the commutator* $T_{b,1}^{(k)}$ *is compact from* $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

Theorem 3.6. *Let* $1 < q < p < \infty$, $p_i \in (1, \infty)$ *for any* $i \in \{1, \ldots, m\}$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, *and* $k \in \mathbb{N}$. *Let* T *be an* m*-linear non-degenerate Calderón–Zygmund operator and* $b \in L^k_{\mathrm{loc}}(\mathbb{R}^n)$

be real-valued. If the commutator $T^{(k)}_{b,1}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $b \in \dot{L}^{kr}(\mathbb{R}^n)$ with $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$.

Proof. We sketch the proof of the present theorem. Let all the symbols be the same as in the present theorem. First of all, the same arguments as that used in the proof of Theorem 3.3 give us that

$$\int_S |b(x) - \langle b\rangle_S|^k\, dx \lesssim |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}} + |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}.$$

Then, for any $\{\lambda_S\}_{S\in\mathscr{S}} \subset [0,\infty)$ with $\sum_{S\in\mathscr{S}} |S| \lambda_S^{(kr)'} \le 1$, by both the Hölder inequality and the Riesz representation theorem of ℓ^r, we have

$$\begin{aligned}
&\sum_{S\in\mathscr{S}} \lambda_S \int_S |b(x) - \langle b\rangle_S|\, dx \\
&\quad\le \sum_{S\in\mathscr{S}} \lambda_S \left[\int_S |b(x) - \langle b\rangle_S|^k\, dx\right]^{1/k} |S|^{1/k'} \\
&\quad\lesssim \sum_{S\in\mathscr{S}} \lambda_S |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}^{1/k} \\
&\qquad+ \sum_{S\in\mathscr{S}} \lambda_S |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}^{1/k} \\
&\quad\lesssim \left[\sum_{S\in\mathscr{S}} |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}^{r}\right]^{1/(kr)} \left[\sum_{S\in\mathscr{S}} |S| \lambda_S^{(kr)'}\right]^{\frac{1}{(kr)'}} \\
&\qquad+ \left[\sum_{S\in\mathscr{S}} |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}^{r}\right]^{1/(kr)} \left[\sum_{S\in\mathscr{S}} |S| \lambda_S^{(kr)'}\right]^{\frac{1}{(kr)'}} \\
&\quad\lesssim \left\| \left\{ |S|^{\frac{1}{r}} \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}} \right\}_{S\in\mathscr{S}} \right\|_{\ell^r}^{1/k} \\
&\qquad+ \left\| \left\{ |S|^{\frac{1}{r}} \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}} \right\}_{S\in\mathscr{S}} \right\|_{\ell^r}^{1/k} \\
&\quad\sim \left[\sum_{S\in\mathscr{S}} \tau_S |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\le\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}\right]^{1/k} \\
&\qquad+ \left[\sum_{S\in\mathscr{S}} \tau_S |S| \left\langle \left| T^{(k)}_{b,1}(\mathbf{1}_{S\cap\{b\ge\alpha_S\}}, \mathbf{1}_S, \cdots, \mathbf{1}_S) \right| \right\rangle_{\widetilde{S}}\right]^{1/k},
\end{aligned}$$

where the non-negative sequence $\{\tau_S\}_{S\in\mathscr{S}}$ satisfies

$$\left\| \left\{ \tau_S |S|^{\frac{1}{r'}} \right\}_{S\in\mathscr{S}} \right\|_{\ell^{r'}} := \sum_{S\in\mathscr{S}} \tau_S^{r'} |S| \le 1.$$

Then the remaining arguments are the same as those used in the proof of Theorem 3.3; we omit the details. This finishes the proof of Theorem 3.6. □

Corollary 3.7. *Let* $1 < q < p < \infty$, $p_i \in (1, \infty)$ *for any* $i \in \{1, \ldots, m\}$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, *and* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. *Let* T *be an* m*-linear Calderón–Zygmund operator,* $b \in L^k_{\mathrm{loc}}(\mathbb{R}^n)$ *be real-valued, and* $k \in \mathbb{N}$. *Then the following three statements are mutually equivalent:*

(i) $T^{(k)}_{b,1}$ *is compact from* $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$*;*

(ii) $T^{(k)}_{b,1}$ *is bounded from* $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$*;*

(iii) $b \in \dot{L}^{kr}(\mathbb{R}^n)$.

Proof. The implication (i) $\Longrightarrow$ (ii) directly follows from the definition of compact operators. Moreover, using Theorem 3.6, we obtain the implication (ii) $\Longrightarrow$ (iii). Furthermore, Theorem 3.5 shows the implication (iii) $\Longrightarrow$ (i), which completes the proof of Corollary 3.7. We note that the assumption of Theorem 3.5 that $\frac{1}{r} := \frac{1}{q} - \frac{1}{p} < \frac{1}{p_1'}$ follows from the present assumptions; namely, we have $\frac{1}{q} < 1 = \frac{1}{p_1} + \frac{1}{p_1'} < \frac{1}{p} + \frac{1}{p_1'}$. This finishes the proof of Corollary 3.7. □

Remark 3.8. (i) By some routine modifications, we find that Proposition 3.1 and Theorems 3.2, 3.3, 3.5, and 3.6 hold true with $[b, T]_1$ replaced by $[b, T]_i$ for any $i \in \{2, \ldots, m\}$.

(ii) It is still *unclear* whether or not the assumptions that $q > 1$ and that b is real-valued in Theorems 3.3 and 3.6 are necessary.

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