GAUGE-TRANSFORMED EXPONENTIAL INTEGRATOR FOR GENERALIZED KDV EQUATIONS WITH ROUGH DATA

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Abstract. In this paper, we propose a new exponential-type integrator for the gKdV equation under rough data. By introducing new frequency approximation techniques and a key gauge transform, the proposed scheme is explicit, stable and efficient in practice. The optimal convergence result of the scheme is established which says that it is first order accurate in H^γ-norm for the solution in $H^{\gamma+2}$ with any $\gamma \geq 0$, and the required regularity is lower than classical approaches. The results are confirmed by numerical experiments.

Keywords: gKdV equation, rough data, low-regularity method, first order accuracy, error estimates, exponential-type integrator

AMS Subject Classification: 65L05, 65L20, 65L70, 65M12, 65M15.

1. Introduction

In this paper, we consider the low-regularity integration approach to solve numerically the general KdV-type equation under the rough initial data on a torus:

$$\begin{cases} \partial_t u(t,x) + \partial_x^3 u(t,x) + \dfrac{1}{k}\partial_x\big(u(t,x)^k\big) = 0, \quad x \in \mathbb{T},\ t > 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $\mathbb{T} = (0, 2\pi)$, $u(t,x) : \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R}$ is the unknown and $u_0 \in H^s(\mathbb{T})$ is the given initial data with some $0 \leq s < \infty$. Here the power index $k > 0$ is an arbitrary given integer and based on its value, (1.1) is usually classified as: the (classical) KdV equation and the modified KdV equation (mKdV) for $k = 2$ and $k = 3$ respectively; the generalized KdV equation (gKdV) for all $k \geq 4$. As a classical mathematical model to describe the dynamics of waves on shallow water surface, the equation (1.1) has drawn extensive research interests from many different aspects.

Theoretically, the well-posedness of the Cauchy problem (1.1) has been intensively investigated, particularly for the initial data $u_0 \in H^s$ with different regularity. Let us give a brief review here. The global well-posedness theory for the classical KdV and the mKdV have been established respectively for $s \geq -1$ [20] and $s \geq \frac{1}{2}$ [6]. For the gKdV equation, the local well-posedness has been shown for $s \geq \frac{1}{2}$ firstly by Keel, Staffilani, Takaoka and Tao [7], and Bao and Wu [3] later obtained the global well-posedness in the quartic nonlinearity case of (1.1). Along the numerical aspect of (1.1), by assuming that the solution is smooth enough, different kinds of discretization techniques including the finite difference methods [8, 18], operator splittings [13, 14, 15, 37], finite elements [2, 5], spectral methods [9, 26, 35] and discontinues Galerkin methods [1, 23, 43], have been proposed and analyzed to solve the KdV-type model (1.1). However, when the solutions of the model are not smooth enough, e.g., the rough data case, the above numerical methods will lose their accuracy and then become less effective. In practice, the rough initial data may come from multiple reasons such as measurements or noise [4, 16].

To tackle the rough data case and increase the temporal accuracy, the so-called low-regularity integrator was firstly introduced by Ostermann and Schratz for nonlinear Schrödinger equations [30]. Compared with the traditional numerical discretizations, the low-regularity integrator is favoured for reaching the optimal convergence rate with the less regularity requirement of the solution. Ever since then, a trend has been started to design low-regularity integrators for important dispersive models [21, 22, 24, 25, 28, 29, 31, 32, 33, 39, 40], where the models possess no internal mechanism to smoothen the rough data. As one of the big three dispersive models, the KdV-type equations (1.1) have received special attentions. Hofmanovó and Schratz [12] considered the exponential-type integrator [11], which is based on the twisted variable $v = \mathrm{e}^{\partial_x^3 t}u$ and the Duhamel formula of (1.1) at $t_n = n\tau$ with $\tau > 0$ the time step:

$$v(t_n + \tau, x) = v(t_n, x) - \frac{1}{k}\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s,x)\big)^k ds.$$

By letting $v(t_n + s, x) \approx v(t_n, x)$ in the above, the key ingredient is to find the integration for s which in Fourier frequency space reads

$$\frac{1}{k}\int_0^\tau \sum_{l=l_1+l_2+\cdots+l_k} \mathrm{e}^{-i(t_n+s)(l^3-l_1^3-l_2^3-\cdots-l_k^3)}(il)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds.$$

When $k = 2$, owning to a key factorization formula $l^3 - l_1^3 - l_2^3 = 3ll_1l_2$ for the phase function, the above integral not only can be done exactly but also can be defined point-wisely in the physical space. As proved in [12], the resulting integrator is first-order accurate in H^1-norm for solving the classical KdV equation with initial data from H^3. The scheme therefore requires two additional bounded derivatives of the solution to reach the first order accuracy, which is indeed lower than the requirement of the splitting method [13] or the finite difference method [8]. Wu and Zhao extended the result to the second-order scheme [41] and proposed later an improvement [42] to further reduce the regularity requirement. These works are more or less based on the aforementioned factorization, which however is not true for $k \geq 3$. Fortunately for $k = 3$, i.e., the mKdV equation case, the celebrated Muira transform helps to derive the corresponding low-regularity integrator [38]. For $k \geq 4$, none of the current techniques are available anymore, and so a thoroughly new development is needed. On the other hand, it is known that with $k \geq 4$ in (1.1), the gKdV equation is non-integrable which differs its structure and dynamics significantly from the KdV and mKdV equations. Non-smooth dynamical phenomena like the self-similar blowups and the dispersive shock waves are of great interests in applications [19], and a low-regularity integrator would be a helpful addition towards efficient and accurate simulations.

We hence focus on the gKdV equation case in this work, i.e., $k \geq 4$ in (1.1), and we are going to propose a low-regularity integrator that is explicit and efficient. By rigorous analysis, we prove that the scheme provides the first order accuracy in H^γ for initial data from $H^{\gamma+2}$ for any $\gamma \geq 0$. Our key ideas include a new approximation technique for the phase function as

$$\mathrm{e}^{-i(t_n+s)(l^3-l_1^3-l_2^3-\cdots-l_k^3)} \approx \mathrm{e}^{-i(t_n+s)[l^3-l_1^3-(l_2^3+\cdots+l_k^3)]},$$

and a new vital *gauge transformation*

$$w(t,x) = u\Big(t, x + \frac{1}{2\pi}\int_0^t\int_{\mathbb{T}} u^{k-1}(\rho,x)dxd\rho\Big)$$

to eliminate the stability issue from the frequency approximations. We thus name the scheme as the *gauge-transformed exponential integrator (GTEI)*. The rest of the paper is organized as follows. In Section 2, we introduce some notations and tool lemmas that will be frequently used. The GTEI scheme is derived in Section 3 followed by its convergence theorem, and the proof is accomplished in Section 4. Numerical verifications are presented in Section 5.

2. Preliminary

In this section, we shall first give some notations and tool lemmas that will be used later for the derivation and analysis of the scheme. Then, we will write some direct exponential schemes for the gKdV equation as the benchmark of comparison.

2.1. **Notation and tool lemma.**

To discretize the time axis, we denote $\tau = \Delta t > 0$ as the time step and $t_n = n\tau$ for $n \in \mathbb{N}$ as the grid points. For two quantities A and B, $A \lesssim B$ or $B \gtrsim A$ denotes $A \leq CB$ for some constant $C > 0$ whose value may vary line by line but independent of τ, n.

The Fourier transform of a function $f(x)$ on $\mathbb{T}$ is defined by

$$\mathcal{F}(f) = \widehat{f}(l) = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{e}^{-ilx} f(x)\,dx, \quad \text{with the inversion} \quad f(x) = \sum_{l\in\mathbb{Z}} \mathrm{e}^{ilx}\widehat{f}(l).$$

Note that $\widehat{f}(0) = \frac{1}{2\pi}\int_{\mathbb{T}} f\,dx$, gives the average. The following classical properties hold:

$$\|f\|_{L^2(\mathbb{T})} = \sqrt{2\pi}\big\|\widehat{f}\big\|_{L^2((dl))} \quad \text{(Plancherel)}; \quad \langle f, g\rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}\,dx = 2\pi\sum_{l\in\mathbb{Z}}\widehat{f}(l)\overline{\widehat{g}(l)} \quad \text{(Parseval)};$$

$$\widehat{(fg)}(l) = \sum_{l_1\in\mathbb{Z}} \widehat{f}(l-l_1)\widehat{g}(l_1) \quad \text{(Convolution)}.$$

The Sobolev space $H^s(\mathbb{T})$ for $s \geq 0$ has the equivalent norm:

$$\big\|f\big\|_{H^s(\mathbb{T})} = \big\|J^s f\big\|_{L^2(\mathbb{T})} = \Big\|(1+l^2)^{\frac{s}{2}}\widehat{f}(l)\Big\|_{L^2((dl))},$$

where we denote the operator

$$J^s = (1-\partial_{xx})^{\frac{s}{2}}.$$

Moreover, we denote the inversion operator ∂_x^{-1} by

$$\widehat{\partial_x^{-1} f}(\xi) = \begin{cases} (il)^{-1}\widehat{f}(l), & \text{when } l \neq 0, \\ 0, & \text{when } l = 0. \end{cases}$$

For a space-time function $f(x,t)$, when there is no ambiguity we will omit the spatial variable x and denote $f(t) = f(x,t)$ for simplicity.

As a tool to overcome the absence of the algebraic property of H^s when $s \leq \frac{1}{2}$, we will need the Kato-Ponce inequality which was originally introduced in [17]. For our convergence analysis later, it is convenient to apply the following special form established in [41, 42]. Let us directly quote it here for brevity and refer the readers to [41, 42] for the proof.

Lemma 2.1. *(Kato-Ponce inequality) Let f, g be the Schwartz functions. Then for $s > 0$, $1 < p \leq \infty$, and $1 < p_1, p_2, p_3, p_4 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{p} = \frac{1}{p_3} + \frac{1}{p_4}$, the following inequality holds:*

$$\big\|J^s(fg)\big\|_{L^p} \leq C\Big(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|J^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}\Big),$$

where the constant $C > 0$ depends on $s, p, p_1, \cdots, p_4$. In particular, when $s > \frac{1}{p}$, then

$$\big\|J^s(fg)\big\|_{L^p} \leq C\|J^s f\|_{L^p}\|J^s g\|_{L^p},$$

where the constant $C > 0$ depends on s, p.

Remark 2.2. The estimate above was originally obtained in whole space case, but it also holds for the periodic functions, via replacing f by $f(x) - f(0)$ and the zero extension.

Lemma 2.3. *The following inequalities hold:*

(i) For any $\gamma \geq 0$, $\gamma_1 > \frac{1}{2}$, $f(x) \in H^\gamma(\mathbb{T})$, $g(x) \in H^{1+\gamma+\gamma_1}(\mathbb{T})$, then

$$\big\langle J^\gamma f, J^\gamma(\partial_x f \cdot g)\big\rangle \lesssim \|f\|_{H^\gamma}^2\|g\|_{H^{1+\gamma+\gamma_1}}.$$

(ii) For any $\gamma > \frac{1}{2}$, $f(x) \in H^\gamma(\mathbb{T})$, $g(x) \in H^{1+\gamma}(\mathbb{T})$, then

$$\big\langle J^\gamma f, J^\gamma(\partial_x f \cdot g)\big\rangle \lesssim \|f\|_{H^\gamma}^2\|g\|_{H^{1+\gamma}}.$$

2.2. **Direct exponential integrators.** To motivate our new scheme in the next section, we consider here some direct exponential integrators for the KdV equation (1.1) and point out their limitations.

In the framework of exponential-type integrators [11], we begin with the Duhamel formula of (1.1):

$$u(t_{n+1}) = \mathrm{e}^{-\tau\partial_x^3}u(t_n) - \int_0^\tau \frac{\mathrm{e}^{-(\tau-s)\partial_x^3}}{k}\partial_x \left(u(t_n+s)^k\right) ds. \tag{2.1}$$

The simplest approach is to directly use $u(t_n+s) \approx u(t_n)$ in the above integral, which apparently costs three spatial derivatives of the solution. Denote $u^n \approx u(t_n)$, we obtain the scheme that will be referred as *EI1*:

$$u^{n+1} = \mathrm{e}^{-\tau\partial_x^3}u^n - \frac{1-\mathrm{e}^{-\tau\partial_x^3}}{k\partial_x^2}\left(u^n\right)^k, \quad n \geq 0. \tag{2.2}$$

Concerning the operator ∂_x in the integrant of (2.1), EI1 in total needs **four** additional derivatives of the solution to guarantee its accuracy. The request of four derivatives of the solution is too much for rough data case. Therefore, one comes up with the filtered variable $\mathfrak{u}(t) := \mathrm{e}^{t\partial_x^3}u(t)$ and (2.1) becomes

$$\mathfrak{u}\left(t_{n+1}\right) = \mathfrak{u}\left(t_n\right) - \int_0^\tau \frac{\mathrm{e}^{(t_n+s)\partial_x^3}}{k}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\mathfrak{u}\left(t_n+s\right)\right)^k ds.$$

Now the approximation $\mathfrak{u}(t_n+s) \approx \mathfrak{u}(t_n)$ loses only one derivative, and so

$$\mathfrak{u}\left(t_{n+1}\right) \approx \mathfrak{u}\left(t_n\right) - \int_0^\tau \frac{\mathrm{e}^{(t_n+s)\partial_x^3}}{k}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\mathfrak{u}\left(t_n\right)\right)^k ds, \tag{2.3}$$

which costs in total two additional derivatives at this level. To close the scheme, one needs to further evaluate the above integration on the right-hand-side. Note that with the Fourier expansion of $u(t_n)$, this integral can always be done exactly in the frequency space, but that would involve k-th many convolutions. Certainly the direct computing of convolution is not welcome for practical implementations, and one therefore look for chances to apply the fast algorithms.

For $k = 2$, the convolution can be factorized thanks to an elegant algebraic structure: $(l_1+l_2)^3 - l_1^3 - l_2^3 = 3(l_1+l_2)l_1l_2$ [12], and so the integral in (2.3) can be done precisely in the physical space where the fast Fourier transform (FFT) can then be applied. In the case, several low-regularity integrators have been designed [41, 42]. For $k = 3$, although the algebraic structure for the three-frequency case fail, a low-regularity integrator has also been established [38] very recently under the help of Miura's transform.

However for $k \geq 4$, the previous techniques will all fail. There is no algebraic structure for factorizing nor magic Miura-type transform to reduce the power of nonlinearity. Further approximations to the integral in (2.3) is definitely needed, which is quite difficult because of the complex resonance structure from the high order nonlinearity. More precisely, one has to further approximate the operator $\mathrm{e}^{\pm(t_n+s)\partial_x^3}$ in (2.3), and this will certainly cost more derivatives. In such way, here we briefly write down an approximation for (2.3) in the same spirit as in [38]: replace the Airy operator with a Schrödinger operator [34], i.e., $\mathrm{e}^{s\partial_x^3} \approx \mathrm{e}^{is\partial_x^2}$, and

$$\begin{aligned}\mathfrak{u}(t_{n+1}) \approx& \mathfrak{u}(t_n) - \frac{1}{k}\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}\partial_x\left(\mathrm{e}^{-t_n\partial_x^3}\mathfrak{u}(t_n)\right)^k ds\\ =&\mathfrak{u}(t_n) + \frac{1}{k}\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3+i\tau\partial_x^2}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}\mathfrak{u}(t_n)\right)^k\right].\end{aligned}$$

By inverting the filtered variable, it leads to the scheme that will be referred as *EI2*:

$$u^{n+1} = \mathrm{e}^{-\tau\partial_x^3}u^n + \frac{1}{k}\mathrm{Re}\left[i\mathrm{e}^{-\tau\partial_x^3+i\tau\partial_x^2}\partial_x^{-1}\left(u^n\right)^k\right], \quad n \geq 0. \tag{2.4}$$

The EI2 is proved to give the first order accuracy with the cost of **three** spatial derivatives in [38] for the case $k = 3$. The result and proof can be straightforwardly extended to the case $k \geq 4$.

3. The new scheme and main theorem

In this section, we shall begin with the construction of our new exponential integrator for the gKdV equation (1.1) based on the gauge transform. So in the following, $k \geq 4$ is considered. Then, we shall give the main result of the paper on the convergence of the scheme.

3.1. **Gauge transformation.**

We introduce the *gauge transform* for the solution of (1.1) as

$$w(t,x) := u\Big(t, x + \frac{1}{2\pi}\int_0^t \int_{\mathbb{T}} u(\rho,x)^{k-1} dx d\rho\Big). \tag{3.1}$$

Note that $u(t,x)$ is a periodic function in x on $\mathbb{T}$ for every t, so $\int_{\mathbb{T}} (u(t,x))^{k-1}\, dx = \int_{\mathbb{T}} (w(t,x))^{k-1}\, dx$. The transformation is invertible. Indeed, we have the inverse transform as

$$u(t,x) = w\Big(t, x - \frac{1}{2\pi}\int_0^t \int_{\mathbb{T}} w(\rho,x)^{k-1} dx d\rho\Big). \tag{3.2}$$

Under the gauge transform (3.1), the equation (1.1) is transformed into the following equivalent form,

$$\partial_t w(t,x) + \partial_x^3 w(t,x) = -\partial_x w(t,x) \cdot w(t,x)^{k-1} + \partial_x w(t,x) \cdot \frac{1}{2\pi}\int_{\mathbb{T}} w(t,x)^{k-1} dx, \tag{3.3}$$

with the initial value $w(0,x) = u_0(x)$ unchanged.

The purpose of this transform is to eliminate the resonance term coming up from the approximation of the nonlinearity later, which is the key to avoid the stability issue. This will become clear in the detailed derivation below and the proof in the next section. Our strategy is to first construct the numerical solution to (3.3), and then obtain the one of (1.1) by the inverse gauge transformation.

3.2. **Derivation of the scheme.**

Based on (3.3), we construct the numerical solution in the framework of the exponential-type time integrator. The Duhamel formula of (3.3) at some t_n reads

$$\begin{aligned} w(t_{n+1}) =& \mathrm{e}^{-\tau\partial_x^3} w(t_n) - \int_0^\tau \mathrm{e}^{-(\tau-s)\partial_x^3}\Big[\partial_x w(t_n+s) \cdot w(t_n+s)^{k-1}\Big] ds \\ &+ \int_0^\tau \mathrm{e}^{-(\tau-s)\partial_x^3}\Big[\partial_x w(t_n+s) \cdot \frac{1}{2\pi}\int_{\mathbb{T}} w(t_n+s)^{k-1} dx\Big] ds. \end{aligned}$$

By introducing the twisted variable $v(t) := \mathrm{e}^{t\partial_x^3} w(t)$, the above formula can be rewritten as

$$\begin{aligned} v(t_{n+1}) =& v(t_n) - \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v(t_n+s)\Big) \cdot \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n+s)\Big)^{k-1}\Big] ds \\ &+ \int_0^\tau \partial_x v(t_n+s) \cdot \frac{1}{2\pi}\int_{\mathbb{T}} \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n+s)\Big)^{k-1} dx ds. \end{aligned} \tag{3.4}$$

Applying the approximation $v(t_n+s) \approx v(t_n)$, we have

$$v(t_{n+1}) = v(t_n) + I_1(t_n) + I_2(t_n) + R_1^n, \tag{3.5}$$

where we denote

$$I_1(t_n) := -\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v(t_n)\Big) \cdot \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n)\Big)^{k-1}\Big] ds, \tag{3.6a}$$

$$I_2(t_n) := \int_0^\tau \partial_x v(t_n) \cdot \frac{1}{2\pi}\int_{\mathbb{T}} \Big(\mathrm{e}^{-(t_n+s)\partial_x^3} v(t_n)\Big)^{k-1} dx ds, \tag{3.6b}$$

and

$$\begin{aligned} R_1^n := &-\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v(t_n+s)\Big)\cdot\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v\,(t_n+s)\Big)^{k-1} \\ &\qquad - \Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v(t_n)\Big)\cdot\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v\,(t_n)\Big)^{k-1}\Big]ds \\ &+\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v\,(t_n+s)\Big)\cdot\frac{1}{2\pi}\int_{\mathbb{T}}\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v\,(t_n+s)\Big)^{k-1}dx \\ &\qquad - \Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v\,(t_n)\Big)\cdot\frac{1}{2\pi}\int_{\mathbb{T}}\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v\,(t_n)\Big)^{k-1}dx\Big]ds. \end{aligned} \tag{3.7}$$

The term R_1^n would be the truncation term. We shall show later in the detailed analysis that $R_1^n = \mathcal{O}(\tau^2)$ and it depends on the second order derivative of the solution. Then to derive the scheme based on (3.5), it remains to evaluate $I_1(t_n)$ and $I_2(t_n)$. In order to get a practically efficient scheme as we discussed before, one should avoid convolutions in the Fourier frequency space. A closed form with point-wise definition in the physical space would be the ultimate goal and the key for approximating $I_1(t_n)$ and $I_2(t_n)$, because that would possibly allow us to apply fast algorithms. This will be further cleared in the derivation below.

Before we step into the detailed calculations, let us first briefly describe the main ideas and their motivations. By taking the Fourier transform on $I_1(t_n)$, we get

$$\widehat{I_1}(t_n,l) = -\int_0^\tau \sum_{l=l_1+l_2+\cdots+l_k} \mathrm{e}^{-i(t_n+s)(l^3-l_1^3-l_2^3-\cdots-l_k^3)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds, \tag{3.8}$$

where $l, l_1, l_2, \cdots, l_k \in \mathbb{Z}$. As mentioned, the phase function $l^3-l_1^3-l_2^3-\cdots-l_k^3$ is the obstacle if we integrate directly, since it cannot be factorized. To overcome this difficulty and meanwhile reduce the cost of derivatives, we need to find a new technique to approximate the exponential term

$$\mathrm{e}^{-is(l^3-l_1^3-l_2^3-\cdots-l_k^3)}.$$

With the EI2 (2.4) in mind which costs three derivatives for the first order accuracy, here we thus aim for a first-order scheme with the cost of two derivatives at most. We begin by rewriting

$$l^3-l_1^3-l_2^3-\cdots-l_k^3 = \alpha+\beta,$$

where

$$\alpha = l^3 - l_1^3 - (l_2+\cdots+l_k)^3, \quad \beta = (l_2+\cdots+l_k)^3 - l_2^3 - \cdots - l_k^3.$$

Then, we split the exponential term into:

$$\mathrm{e}^{-is(\alpha+\beta)} = \mathrm{e}^{-is\alpha} + (\mathrm{e}^{-is\beta}-1)\mathrm{e}^{-is\alpha},$$

and so (3.8) is split into

$$\begin{aligned} &\widehat{I_1}(t_n,l) \\ &= -\sum_{l=l_1+l_2+\cdots+l_k} \mathrm{e}^{-it_n(l^3-l_1^3-l_2^3-\cdots-l_k^3)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)\int_0^\tau \mathrm{e}^{-is\alpha}ds && \text{(3.9a)} \\ &\quad -\sum_{l=l_1+l_2+\cdots+l_k} \mathrm{e}^{-it_n(l^3-l_1^3-l_2^3-\cdots-l_k^3)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)\int_0^\tau (\mathrm{e}^{-is\beta}-1)\mathrm{e}^{-is\alpha}ds. && \text{(3.9b)} \end{aligned}$$

Noting $l = l_1+l_2+\cdots+l_k$, (3.9a) is factorable:

$$\alpha = 3ll_1(l_2+\cdots+l_k).$$

If $l \neq 0$, $l_1 \neq 0$, and $l_2+\cdots+l_k \neq 0$, we then find

$$\int_0^\tau \mathrm{e}^{-is\alpha}ds = \frac{\mathrm{e}^{-i\tau\alpha}-1}{-i\alpha} = \frac{\mathrm{e}^{-i\tau\alpha}-1}{-i3ll_1(l_2+\cdots+l_k)}, \tag{3.10}$$

which implies that (3.9a) can be exactly defined pointwisely in the physical space. The three special cases $l=0$ or $l_1=0$ or $l_2+\cdots+l_k=0$ will have to be considered separately for (3.8). Let us here quickly check the case $l_2+\cdots+l_k=0$ in (3.8), where we can find

$$\begin{aligned}&\int_0^\tau \sum_{\substack{l=l_1+\ldots+l_k\\ l_2+\cdots+l_k=0}} \mathrm{e}^{-i(t_n+s)(l^3-l_1^3-l_2^3-\cdots-l_k^3)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds\\ =&\sum_{\substack{l=l_1\\ l_2+\cdots+l_k=0}}(il_1)\widehat{v}(t_n,l_1)\int_0^\tau \mathrm{e}^{-i(t_n+s)(-l_2^3-\cdots-l_k^3)}\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds\\ =&\partial_x v(t_n)\int_0^\tau \frac{1}{2\pi}\int_{\mathbb{T}}\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n)\right)^{k-1}dxds.\end{aligned}$$

The above term matches precisely with the $I_2(t_n)$ that we introduced in (3.6b), and so it will be cancelled in (3.5) by $I_2(t_n)$. Note that the term $I_2(t_n)$ comes from (3.3) which is in fact induced by the gauge transformation. Now our motivation to introduce the transformation becomes clear. If one works directly on (1.1) without using the gauge transformation, the term $\partial_x v(t_n)$ in the above will present explicitly in the final scheme, which would cause unbalanced norms in the numerical solution from n to $n+1$ and consequently lead to instability. As for (3.9b), it can be treated directly as a truncation, since

$$\left|(\mathrm{e}^{-is\beta}-1)\mathrm{e}^{-is\alpha}\right|\le s|\beta| \quad \text{and} \quad |\beta|\lesssim \sum_{j\neq k\neq 1}|l_j|^2|l_k|.$$

The truncation error is therefore $\mathcal{O}(\tau^2)$ with the cost of only two spatial derivatives.

Now we are ready to present the derivation for the scheme. By the above discussion, we divide (3.8) into the following terms:

$$\widehat{I}_1(t_n,l)=-\int_0^\tau \sum_{l=l_1+l_2+\cdots+l_k=0} \mathrm{e}^{-i(t_n+s)(\alpha+\beta)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds \tag{3.11a}$$

$$-\int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1=0}} \mathrm{e}^{-i(t_n+s)(\alpha+\beta)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds \tag{3.11b}$$

$$-\int_0^\tau \sum_{l=l_1} \mathrm{e}^{-i(t_n+s)(\alpha+\beta)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds \tag{3.11c}$$

$$-\int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1\neq 0, l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-i(t_n+s)(\alpha+\beta)}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds. \tag{3.11d}$$

For (3.11a), by the symmetry within $l_1,\ldots,l_k$, we find

$$(3.11a)=-\frac{1}{k}\int_0^\tau \sum_{l=l_1+l_2+\cdots+l_k=0} \mathrm{e}^{-i(t_n+s)(\alpha+\beta)}(il)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds=0.$$

It is direct to have (3.11b) $=0$ and (3.11c) $=-\widehat{I}_2(t_n,l)$. Hence, we have

$$\widehat{I}_1(t_n,l)+\widehat{I}_2(t_n,l)=(3.11\mathrm{d}).$$

For (3.11d), it can be approximated as

$$(3.11\mathrm{d})=-\int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1\neq 0, l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-it_n(\alpha+\beta)}\mathrm{e}^{-is\alpha}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds+\widehat{R_2^n}(l), \tag{3.12}$$

where

$$\widehat{R_2^n}(l) := -\int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1\neq 0, l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-it_n(\alpha+\beta)}\big(\mathrm{e}^{-is\beta}-1\big)\mathrm{e}^{-is\alpha}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds. \tag{3.13}$$

Therefore, (3.12) combined with (3.10) yields

$$(3.11\mathrm{d}) = \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1\neq 0, l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-it_n(\alpha+\beta)}\frac{\mathrm{e}^{-i\tau\alpha}-1}{3l(l_2+\cdots+l_k)}\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)+\widehat{R_2^n}(l).$$

The restriction $l_1 \neq 0$ can be removed, since when $l_1 = 0$, we have $\alpha = 0$ and the corresponding term vanishes. Hence,

$$(3.11\mathrm{d}) = \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-it_n(\alpha+\beta)}\frac{\mathrm{e}^{-i\tau\alpha}-1}{3l(l_2+\cdots+l_k)}\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)+\widehat{R_2^n}(l).$$

The inverse Fourier transform of the above gives

$$\begin{aligned} I_1(t_n)+I_2(t_n) = &-\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_{n+1}\partial_x^3}v(t_n)\right)\cdot\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\big)^{k-1}\right)\right]\\ &+\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)\cdot\left(\partial_x^{-1}\big(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\big)^{k-1}\right)\right]+R_2^n. \end{aligned}\tag{3.14}$$

Plugging (3.14) into (3.5), we obtain

$$\begin{aligned} v(t_{n+1}) = &v(t_n)-\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_{n+1}\partial_x^3}v(t_n)\right)\cdot\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\big)^{k-1}\right)\right]\\ &+\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)\cdot\left(\partial_x^{-1}\big(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\big)^{k-1}\right)\right]+R_1^n+R_2^n. \end{aligned}\tag{3.15}$$

By inverting the twisted variable $w(t) = e^{-t\partial_x^3}v(t)$ and dropping R_1^n, R_2^n, we find the approximation:

$$w(t_{n+1}) \approx \Phi\big(w(t_n)\big),$$

with

$$\Phi(f) := \mathrm{e}^{-\tau\partial_x^3}f-\frac{1}{3}\partial_x^{-1}\left[\left(\mathrm{e}^{-\tau\partial_x^3}f\right)\cdot\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}f^{k-1}\right)\right]+\frac{1}{3}\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\left(f\cdot\partial_x^{-1}f^{k-1}\right). \tag{3.16}$$

Finally, we invert the gauge transformation. Approximating the time integral in (3.2) by a Riemann sum/left-rectangle rule, we obtain

$$u(t_n,x) \approx w\Big(t_n, x-\frac{\tau}{2\pi}\sum_{j=0}^{n-1}\int_{\mathbb{T}} w(t_j,x)^{k-1}dx\Big).$$

Accordingly, the detailed final form of the scheme is summarized as follows. Denote $u^n = u^n(x) \approx u(t_n,x)$ as the numerical solution of (1.1) for $n \geq 0$. The first-order *gauge-transformed exponential integrator (GTEI)* for solving the gKdV equation (1.1) reads: $w^0(x) = u_0(x)$, and for $n \geq 0$,

$$u^{n+1}(x) = w^{n+1}\Big(x-\frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}}(w^j(x))^{k-1}dx\Big), \quad w^{n+1}(x) = \Phi\big(w^n(x)\big). \tag{3.17}$$

Practically, the above GTEI can be implemented as follows. The propagator Φ in (3.16) in space can be discretized by the Fourier pseudo-spectral method [36]. With the help of the FFT, the computational cost from w^n to w^{n+1} is $\mathcal{O}(N\log N)$ with N the number of spatial grid points. If one is only interested in the value of $u(t,x)$ at a specific time t, GTEI can proceed completely in w^n till $t = t_n$ and then perform the shift in space which can be done by trigonometric interpolations. Such trigonometric interpolations can be accelerated via the non-uniform FFT [10] if the intermediate values of u within $[0,t]$ is frequently needed. Note that inside the space-shift in (3.17), one only needs the average value of $(w^j(x))^{k-1}$ in space, so there is no need to store the full function $w^j(x)$ for all j. The average of $(w^j(x))^{k-1}$ corresponds the zero-mode of its Fourier coefficient, or equivalently it can be obtained by a direct Riemann sum on the spatial grids. Concerning that the gKdV equations may generate finite-time blowups, e.g., [5, 27], so the scheme (3.17) in principle would work before the occurrence of blowups.

3.3. **Main convergence result.** For the proposed GTEI scheme, we have the following main result on its convergence.

Theorem 3.1. *Consider the gKdV equation (1.1) for any* $k \geq 2$. *Let* u^n *be the numerical solution obtained from the GTEI scheme* (3.17) *up to some fixed time* $T \in (0, T^*)$ *with* $T^* > 0$ *the maximum time for the existence of the solution. Under the assumption that* $u_0 \in H^{\gamma+2}(\mathbb{T})$ *for some* $\gamma \geq 0$, *there exist constants* $\tau_0, C > 0$ *such that for any* $0 < \tau \leq \tau_0$ *we have*

$$\|u(t_n, \cdot) - u^n\|_{H^\gamma} \leq C\tau, \quad n = 0, 1 \ldots, \frac{T}{\tau},$$

where the constants τ_0 *and* C *depend on* T, k *and* $\|u\|_{L^\infty((0,T);H^{\gamma+2})}$.

The above theorem tells that the proposed GTEI scheme is indeed first order accurate with the cost of only **two** derivatives as desired, and so its regularity requirement is lower than the direct EIs in Section 2.2. This fact would make it more accurate for solving the gKdV equations (1.1) under the rough data case, and we will further illustrate this by numerical experiments later. The next section will be devoted to the rigorous proof of Theorem 3.1.

4. The first-order convergence analysis

In this section, we give the rigorous proof of Theorem 3.1 for the convergence of the proposed GTEI scheme (3.17). To do so, it is essential to firstly consider the convergence result on the transformed equation (3.3). This is stated in the following proposition.

Proposition 4.1. *Under the same assumptions of Theorem 3.1, let* w^n *be the numerical solution from* (3.17) *for the gauge-transformed gKdV equation* (3.3) *up to the fixed* $T > 0$. *Then, there exist constants* $\tau_0, C > 0$ *such that for any* $0 < \tau \leq \tau_0$,

$$\|w(t_n, \cdot) - w^n\|_{H^\gamma} \leq C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}, \tag{4.1}$$

where the constants τ_0 *and* C *depend on* T, k *and* $\|w\|_{L^\infty((0,T);H^{2+\gamma})}$.

To establish (4.1), it is more convenient to work on the twisted variable $v(t) = \mathrm{e}^{t\partial_x^3} w(t)$, since our approximations in Section 3.2 are mainly made based on (3.4). Under the assumption of Theorem 3.1 or Proposition 4.1, note that we have $u, v, w \in L^\infty((0,T); H^{\gamma+2}(\mathbb{T}))$. We shall denote $L_t^\infty H_x^s = L^\infty((0,T); H^s(\mathbb{T}))$ for short. Defining $v^n := \mathrm{e}^{t_n\partial_x^3} w^n$, we have

$$\|w(t_n, \cdot) - w^n\|_{H^\gamma} = \|v(t_n, \cdot) - v^n\|_{H^\gamma},$$

and so it reduces to consider the difference between the numerical solution v^n and the exact solution $v(t_n)$ of (3.4). Noticing the calculations in (3.11)-(3.14), it is direct to check that

$$\begin{aligned}
&-\frac{1}{3}\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_{n+1}\partial_x^3}v(t_n)\right)\cdot\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^{k-1}\right)\right]\\
&+\frac{1}{3}\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)\cdot\left(\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^{k-1}\right)\right]\\
=&-\int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1\neq 0, l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-it_n(\alpha+\beta)}\mathrm{e}^{-is\alpha}(il_1)\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)\mathrm{e}^{ilx}ds \qquad (4.2)\\
=&-\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\left[\left(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v\right)\cdot\left(\mathrm{e}^{-s\partial_x^3}\left(\mathrm{e}^{-t_n\partial_x^3}v\right)^{k-1}\right)\right]ds+\frac{\tau}{2\pi}\partial_x v\cdot\int_{\mathbb{T}}\left(\mathrm{e}^{-t_n\partial_x^3}v\right)^{k-1}dx.
\end{aligned}$$

By (3.15), we then define the numerical propagator Φ^n for the twisted variable as

$$\Phi^n\big(v\big) :=v - \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\left[\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x v\Big)\cdot\Big(\mathrm{e}^{-s\partial_x^3}\Big(\mathrm{e}^{-t_n\partial_x^3}v\Big)^{k-1}\Big)\right]ds + \frac{\tau}{2\pi}\partial_x v\cdot\int_{\mathbb{T}}\Big(\mathrm{e}^{-t_n\partial_x^3}v\Big)^{k-1}dx, \tag{4.3}$$

and so we have

$$v\big(t_{n+1}\big) - \Phi^n\big(v(t_n)\big) = R_1^n + R_2^n. \tag{4.4}$$

We rewrite

$$v\big(t_{n+1}\big) - v^{n+1} = v\big(t_{n+1}\big) - \Phi^n\big(v(t_n)\big) - \big[\Phi^n\big(v^n\big) - \Phi^n\big(v(t_n)\big)\big], \tag{4.5}$$

and the proof of Proposition 4.1 then breaks down into the following two estimates:

- Local error: $\big\|v\big(t_{n+1}\big) - \Phi^n\big(v(t_n)\big)\big\|_{H^\gamma} \le C\tau^2$;
- Stability: $\big\|\Phi^n\big(v^n\big) - \Phi^n\big(v(t_n)\big)\big\|_{H^\gamma} \le (1+C\tau)\|v(t_n)-v^n\|_{H^\gamma}$.

4.1. **Local error.**

The local error estimate is established in the following proposition.

Proposition 4.2. *Let $\gamma \ge 0$ and $v \in L_t^\infty H_x^{2+\gamma}$, then for any $\gamma_0 \in [\gamma, 1+\gamma]$, there exists some constant $C = C(\|v\|_{L_t^\infty H_x^{2+\gamma}}) > 0$ such that*

$$\big\|v\big(t_{n+1}\big) - \Phi^n\big(v(t_n)\big)\big\|_{H^{\gamma_0}} \le C\tau^{1+\alpha}, \quad \textit{for } \alpha = 1+\gamma-\gamma_0.$$

By (4.4), the proof of Proposition 4.2 consists of the estimates for R_1^n and R_2^n given in the following two lemmas in a sequel.

Lemma 4.3. *Under the same setup as in Proposition 4.2,*

$$\big\|R_1^n\big\|_{H^{\gamma_0}} \le C\tau^{1+\alpha}.$$

Proof. According to the definition of R_1^n in (3.7), we rewrite

$$R_1^n = -\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}\partial_x\big(v(t_n+s)-v(t_n)\big)\Big)\cdot\Big(\mathrm{e}^{-(t_n+s)\partial_x^3}v\left(t_n+s\right)\Big)^{k-1}\Big]ds + \text{ other similar terms}, \tag{4.6}$$

where "other similar terms" are the terms that contain the difference $v(t_n+s)-v(t_n)$ similar as in (4.6). They can be estimated by the same manner presented below and so we omit the details for them here for brevity. For (4.6), by Lemma 2.1 and Sobolev's inequality, we have that for any $\frac{1}{2} < \gamma_1 \le 1$,

$$\begin{aligned}
\|(4.6)\|_{H^{\gamma_0}} \lesssim& \int_0^\tau \Big(\big\|J^{\gamma_0}\partial_x\big(v(t_n+s)-v(t_n)\big)\big\|_{L^2}\big\|\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\big)^{k-1}\big\|_{L^\infty} \\
&+ \big\|\partial_x\big(v(t_n+s)-v(t_n)\big)\big\|_{L^2}\big\|J^{\gamma_0}\big(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\big)^{k-1}\big\|_{L^\infty}\Big)ds \\
\lesssim& \int_0^\tau \Big(\big\|\partial_x\big(v(t_n+s)-v(t_n)\big)\big\|_{H^{\gamma_0}}\|v\|_{L_t^\infty H_x^{\gamma_1}}^{k-1} + \big\|\partial_x\big(v(t_n+s)-v(t_n)\big)\big\|_{L^2}\|v\|_{L_t^\infty H_x^{\gamma_0+\gamma_1}}^{k-1}\Big)ds \\
\lesssim& \tau\left\|v(t_n+s)-v(t_n)\right\|_{H^{\gamma_0+1}}\|v\|_{L_t^\infty H_x^{\gamma_0+1}}^{k-1}.
\end{aligned}$$

Denote a frequency decomposition as $v = \sum_{|l|\le M}\widehat{v}\mathrm{e}^{ilx} + \sum_{|l|>M}\widehat{v}\mathrm{e}^{ilx} =: P_{\le M}v + P_{>M}v$ with some $M > 0$ to be determined. Then,

$$\|v(t_n+s)-v(t_n)\|_{H^{\gamma_0+1}} \le \big\|P_{\le M}\left(v(t_n+s)-v(t_n)\right)\big\|_{H^{\gamma_0+1}} + \big\|P_{>M}\left(v(t_n+s)-v(t_n)\right)\big\|_{H^{\gamma_0+1}}.$$

By the Bernstein inequality, we have

$$\begin{aligned}
\big\|P_{\le M}\left(v(t_n+s)-v(t_n)\right)\big\|_{H^{\gamma_0+1}} &\le M^{1-\alpha}\left\|v(t_n+s)-v(t_n)\right\|_{H^{\gamma_0+\alpha}}, \\
\big\|P_{>M}\left(v(t_n+s)-v(t_n)\right)\big\|_{H^{\gamma_0+1}} &\le M^{-\alpha}\left\|v(t_n+s)-v(t_n)\right\|_{H^{\gamma_0+1+\alpha}}.
\end{aligned}$$

On the one hand, by (3.3) the twisted variable $v = v(t)$ satisfies $\partial_t v = \partial_x v \cdot \frac{1}{2\pi}\int_{\mathbb{T}}(\mathrm{e}^{-t\partial_x^3}v)^{k-1}dx - \mathrm{e}^{t\partial_x^3}[(\partial_x \mathrm{e}^{-t\partial_x^3}v)\cdot(\mathrm{e}^{-t\partial_x^3}v)^{k-1}]$, and so by Lemma 2.1 we find

$$\|v(t_n+s)-v(t_n)\|_{H^{\gamma_0+\alpha}} \lesssim \tau \|\partial_t v\|_{L_t^\infty H_x^{\gamma_0+\alpha}} \lesssim \tau \|v\|^k_{L_t^\infty H_x^{\gamma_0+\alpha+1}}.$$

Then,

$$\big\|P_{\le M}\left(v(t_n+s)-v(t_n)\right)\big\|_{H^{\gamma_0+1}} \le M^{1-\alpha}\tau\|v\|^k_{L_t^\infty H_x^{\gamma_0+\alpha+1}}. \tag{4.7}$$

On the other hand, clearly $\|v(t_n+s)-v(t_n)\|_{H^{\gamma_0+1+\alpha}} \lesssim \|v\|_{L_t^\infty H_x^{\gamma_0+1+\alpha}}$. Therefore,

$$\big\|P_{>M}\left(v(t_n+s)-v(t_n)\right)\big\|_{H^{\gamma_0+1}} \le M^{-\alpha}\|v\|_{L_t^\infty H_x^{\gamma_0+1+\alpha}}. \tag{4.8}$$

Combining the estimates (4.7) and (4.8), we get

$$\|v(t_n+s)-v(t_n)\|_{H^{\gamma_0+1}} \le M^{1-\alpha}\tau\|v\|^k_{L_t^\infty H_x^{\gamma_0+\alpha+1}} + M^{-\alpha}\|v\|_{L_t^\infty H_x^{\gamma_0+1+\alpha}}.$$

Choosing $M=\tau^{-1}$ and noting $\gamma_0+1+\alpha = 2+\gamma$, we have

$$\|v(t_n+s)-v(t_n)\|_{H^{\gamma_0+1}} \le \tau^\alpha\|v\|^k_{L_t^\infty H_x^{\gamma+2}} + \tau^\alpha\|v\|_{L_t^\infty H_x^{\gamma+2}}.$$

This implies that $\|(4.6)\|_{H^{\gamma_0}} \le C\big(\|v\|_{L_t^\infty H_x^{2+\gamma}}\big)\tau^{1+\alpha}$ and the proof is completed. ∎

Lemma 4.4. *Under the same setup as in Proposition 4.2,*

$$\big\|R_2^n\big\|_{H^{\gamma_0}} \le C\tau^{1+\alpha}.$$

Proof. By Plancherel's identity, we have $\|R_2^n\|^2_{H^{\gamma_0}} = \sum\limits_l |l|^{2\gamma_0}\big|\widehat{R_2^n}(l)\big|^2$. By (3.13) and noting that

$$\left|\mathrm{e}^{is\beta}-1\right| \lesssim (s|\beta|)^\alpha \quad \text{for} \quad 0\le\alpha = 1+\gamma-\gamma_0 \le 1,$$

we find

$$\big|\widehat{R_2^n}(l)\big| \le \int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\neq 0\\ l_1\neq 0, l_2+\cdots+l_k\neq 0}} (s|\beta|)^\alpha |l_1||\widehat{v}(t_n,l_1)||\widehat{v}(t_n,l_2)|\cdots|\widehat{v}(t_n,l_k)|ds.$$

By symmetry, we may assume that $|l_2|\ge|l_3|\ge\cdots\ge|l_k|$. Then we get

$$|\beta| \lesssim |l_2|^2|l_3|,$$

which gives

$$\big|\widehat{R_2^n}(l)\big| \le \tau^{1+\alpha} \sum_{\substack{l=l_1+l_2+\cdots+l_k\\ |l_2|\ge|l_3|\ge\cdots\ge|l_k|}} |l_1||l_2|^{2\alpha}|l_3|^\alpha|\widehat{v}(t_n,l_1)||\widehat{v}(t_n,l_2)|\cdots|\widehat{v}(t_n,l_k)|.$$

To simplify the notations for presentations below, we denote $\tilde{v}(t,x) = \sum_{k\in\mathbb{Z}} \mathrm{e}^{ikx}|\widehat{v}(t,k)|$. Then, we have $\widehat{\tilde{v}}(t,k) = |\widehat{v}(t,k)|$ and

$$\|\tilde{v}\|^2_{H^{\gamma_0}} = 2\pi\sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma_0}\big|\widehat{\tilde{v}}_k(t)\big|^2 = 2\pi\sum_{k\in\mathbb{Z}}(1+k^2)^{\gamma_0}|\widehat{v}_k(t)|^2 = \|v\|^2_{H^{\gamma_0}}.$$

Hence, we may abuse the notations and assume directly that $\widehat{v}(t_n,l)\ge 0$ for any $l\in\mathbb{Z}$, otherwise one may replace v by $\tilde{v}$. According to this reduction, we further control $\widehat{R_2^n}(l)$ as

$$\big|\widehat{R_2^n}(l)\big| \lesssim \tau^{1+\alpha}\sum_{l=l_1+l_2+\cdots+l_k} |l_1||l_2|^{2\alpha}|l_3|^\alpha\widehat{v}(t_n,l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k).$$

Then by Plancherel's identity, we have

$$\big\|R_2^n\big\|_{H^{\gamma_0}} \lesssim \tau^{1+\alpha}\big\|J^{\gamma_0}\left(J^1 v\cdot J^{2\alpha}v\cdot J^\alpha v\cdot v^{k-3}\right)\big\|_{L^2}. \tag{4.9}$$

Now the estimate of (4.9) is split into the following two cases:

$$\text{Case 1: } 0\le\alpha\le\frac{1}{2}; \qquad \text{Case 2: } \frac{1}{2}<\alpha\le 1.$$

Case 1: $0 \le \alpha \le \frac{1}{2}$. In this case, to apply Lemma 2.1, we take the following parameters: let $0 < \epsilon \ll 1$,

$$p_1 = \begin{cases} 2, & \text{when } \alpha = 0, \\ \big(\frac{1}{2} - \epsilon\big)^{-1}, & \text{when } \alpha > 0; \end{cases} \qquad p_2 = \begin{cases} +\infty, & \text{when } \alpha = 0, \\ \epsilon^{-1}, & \text{when } \alpha > 0. \end{cases}$$

By Sobolev's inequality and choosing ϵ small enough, we have the following embedding inequalities:

$$\|f\|_{L^{p_1}} \lesssim \|f\|_{H^\alpha}, \qquad \|J^{2\alpha} f\|_{L^{p_2}} \lesssim \|f\|_{H^{1+\alpha}}. \tag{4.10}$$

Therefore, by (4.9), Lemma 2.1, (4.10) and Sobolev's inequality, we obtain

$$\begin{aligned} \big\|R_2^n\big\|_{H^{\gamma_0}} \lesssim & \tau^{1+\alpha}\big\|J^{1+\gamma_0} v(t_n)\big\|_{L^{p_1}}\big\|J^{2\alpha} v(t_n)\big\|_{L^{p_2}}\big\|J^{\alpha} v(t_n)\big\|_{L^\infty}\|v(t_n)\|_{L^\infty}^{k-3} \\ & + \tau^{1+\alpha}\big\|J^{1} v(t_n)\big\|_{L^{p_1}}\big\|J^{2\alpha+\gamma_0} v(t_n)\big\|_{L^{p_2}}\big\|J^{\alpha} v(t_n)\big\|_{L^\infty}\|v(t_n)\|_{L^\infty}^{k-3} \\ \lesssim & \tau^{1+\alpha}\|v\|_{L_t^\infty H_x^{1+\gamma_0+\alpha}}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^1}^{k-3} \\ & + \tau^{1+\alpha}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^{1+\gamma_0+\alpha}}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^1}^{k-3} \\ \lesssim & \tau^{1+\alpha}\|v\|_{L_t^\infty H_x^{2+\gamma}}^{k}. \end{aligned}$$

Case 2: $\frac{1}{2} < \alpha \le 1$. Noting that $2\alpha \le 1+\alpha, 1+\gamma_0+\alpha = 2+\gamma$ and $H^\alpha(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, by (4.9) we then have

$$\begin{aligned} \|R_2^n\|_{H^{\gamma_0}} \lesssim & \tau^{1+\alpha}\big\|J^{1+\gamma_0} v(t_n)\big\|_{L^\infty}\big\|J^{2\alpha} v(t_n)\big\|_{L^2}\big\|J^{\alpha} v(t_n)\big\|_{L^\infty}\|v(t_n)\|_{L^\infty}^{k-3} \\ & + \tau^{1+\alpha}\big\|J^{1} v(t_n)\big\|_{L^\infty}\big\|J^{2\alpha+\gamma_0} v(t_n)\big\|_{L^2}\big\|J^{\alpha} v(t_n)\big\|_{L^\infty}\|v(t_n)\|_{L^\infty}^{k-3} \\ \lesssim & \tau^{1+\alpha}\|v\|_{L_t^\infty H_x^{1+\gamma_0+\alpha}}\|v\|_{L_t^\infty H_x^{2\alpha}}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^1}^{k-3} \\ & + \tau^{1+\alpha}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^{\gamma_0+2\alpha}}\|v\|_{L_t^\infty H_x^{1+\alpha}}\|v\|_{L_t^\infty H_x^1}^{k-3} \\ \lesssim & \tau^{1+\alpha}\|v\|_{L_t^\infty H_x^{2+\gamma}}^{k}. \end{aligned}$$

This finishes the proof of the lemma. □

Proposition 4.2 is therefore proved by combining the results of Lemmas 4.3&4.4.

4.2. **Stability.** For short, we denote $e_n = v(t_n) - v^n$ as the error, and we have the following proposition stating the propagation of the error under two different norms.

Proposition 4.5. *Let $\gamma \ge 0, \gamma_0 \in (\frac{1}{2}, 1+\gamma)$ and $v \in L_t^\infty H_x^{2+\gamma}$, then*

$$\begin{aligned} \big\|\Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big)\big\|_{H^\gamma} \le & \big[1 + \tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^\gamma} + \tau^{\frac{2}{5}} Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^\gamma}^2, \\ \big\|\Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big)\big\|_{H^{\gamma_0}} \le & \big[1 + \tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^{\gamma_0}} + \tau^{-1}\|e_n\|_{H^\gamma}^3, \end{aligned}$$

where $Q(f) = \sum\limits_{l=0}^{2k-4} C_l f^l$ for some positive constants C_l depend only on k and $\|v\|_{L_t^\infty H_x^{2+\gamma}}$.

Proof. By (4.3) and recalling that $w(t_n) = \mathrm{e}^{-t_n\partial_x^3} v(t_n)$, $w^n = \mathrm{e}^{-t_n\partial_x^3} v^n$, we have

$$\begin{aligned} \Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big) = & e_n - \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\mathrm{e}^{-s\partial_x^3}\partial_x w(t_n) \cdot \mathrm{e}^{-s\partial_x^3} w(t_n)^{k-1}\Big] ds \\ & + \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big[\mathrm{e}^{-s\partial_x^3}\partial_x w^n \cdot \mathrm{e}^{-s\partial_x^3}\big(w^n\big)^{k-1}\Big] ds \\ & + \frac{\tau}{2\pi}\partial_x v(t_n)\int_{\mathbb{T}} w(t_n)^{k-1} dx - \frac{\tau}{2\pi}\partial_x v^n \int_{\mathbb{T}} \big(w^n\big)^{k-1} dx. \end{aligned}$$

Denote $\tilde{e}_n = w(t_n) - w^n$, and then $\tilde{e}_n = \mathrm{e}^{-t_n\partial_x^3} e_n$. By merging suitably the terms, we further obtain

$$
\begin{aligned}
&\Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big) \\
=& e_n - \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\partial_x w(t_n)\cdot \mathrm{e}^{-s\partial_x^3}\Big[C_1\tilde{e}_n w(t_n)^{k-2} + C_2\tilde{e}_n^2 w(t_n)^{k-3} + \cdots + C_{k-1}\tilde{e}_n^{k-1}\Big]\Big\}ds \\
&+ \frac{\tau}{2\pi}\partial_x v(t_n)\cdot\int_{\mathbb{T}}\Big[C_1\tilde{e}_n w(t_n)^{k-2} + C_2\tilde{e}_n^2 w(t_n)^{k-3} + \cdots + C_{k-1}\tilde{e}_n^{k-1}\Big]dx \\
&- \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\partial_x \tilde{e}_n\cdot \mathrm{e}^{-s\partial_x^3}\Big[\tilde{C}_1 w(t_n)^{k-1} + \tilde{C}_2\tilde{e}_n w(t_n)^{k-2} + \cdots + \tilde{C}_k\tilde{e}_n^{k-1}\Big]\Big\}ds \\
&+ \frac{\tau}{2\pi}\partial_x e_n\cdot\int_{\mathbb{T}}\Big[\tilde{C}_1 w(t_n)^{k-1} + \tilde{C}_2\tilde{e}_n w(t_n)^{k-2} + \cdots + \tilde{C}_k\tilde{e}_n^{k-1}\Big]dx,
\end{aligned}
$$

for some positive constants $C_j, j = 1,\cdots,k-1$ and $\tilde{C}_j, j = 1,\cdots,k$. Moreover, we denote

$$
\begin{aligned}
S_0 :=& -\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\partial_x w(t_n)\cdot \mathrm{e}^{-s\partial_x^3}\Big[C_1\tilde{e}_n w(t_n)^{k-2} + C_2\tilde{e}_n^2 w(t_n)^{k-3} + \cdots + C_{k-1}\tilde{e}_n^{k-1}\Big]\Big\}ds \\
&+ \frac{\tau}{2\pi}\partial_x v(t_n)\cdot\int_{\mathbb{T}}\Big[C_1\tilde{e}_n w(t_n)^{k-2} + C_2\tilde{e}_n^2 w(t_n)^{k-3} + \cdots + C_{k-1}\tilde{e}_n^{k-1}\Big]dx,
\end{aligned}
$$

and

$$
S_j := \tilde{C}_j\frac{\tau}{2\pi}\partial_x e_n\cdot\int_{\mathbb{T}}\tilde{e}_n^{j-1} w(t_n)^{k-j}dx - \tilde{C}_j\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\partial_x\tilde{e}_n\cdot \mathrm{e}^{-s\partial_x^3}\Big[\tilde{e}_n^{j-1} w(t_n)^{k-j}\Big]\Big\}ds, \tag{4.11}
$$

for $j = 1,\ldots,k$. Then, we may write

$$
\Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big) = e_n + S_0 + S_1 + \cdots + S_k,
$$

and for $\kappa = \gamma$ or γ_0, we have

$$
\big\|\Phi^n\big(v(t_n)\big) - \Phi^n\big(v^n\big)\big\|_{H^\kappa}^2 \le \|e_n\|_{H^\kappa}^2 + 2\sum_{j=0}^k\big|\big\langle J^\kappa e_n, J^\kappa S_j\big\rangle\big| + (k+1)\sum_{j=0}^k\big\|S_j\big\|_{H^\kappa}^2. \tag{4.12}
$$

The estimates for the right-hand-side of (4.12) are separated into the following three lemmas.

Lemma 4.6. *For the part of S_0 in* (4.12)*, with $\kappa = \gamma$ or γ_0 we have*

$$
\big|\big\langle J^\kappa e_n, J^\kappa S_0\big\rangle\big| + \|S_0\|_{H^\kappa}^2 \le \tau Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^\kappa}^2.
$$

Proof. By Lemma 2.1, Sobolev's and Cauchy-Schwarz's inequalities, we obtain that for any $\gamma \ge 0$,

$$
\begin{aligned}
\|S_0\|_{H^\gamma} \lesssim& \tau\|\partial_x w(t_n)\|_{H^{\gamma+\gamma_1}}\|\tilde{e}_n\|_{L^2}\left(\|w(t_n)\|_{H^{\gamma_1}}^{k-2} + \|\tilde{e}_n\|_{H^{\gamma_1}}^{k-2}\right) \\
&+ \tau\|\partial_x w(t_n)\|_{H^{\gamma_1}}\|\tilde{e}_n\|_{H^\gamma}\left(\|w(t_n)\|_{H^{\gamma_1}}^{k-2} + \|\tilde{e}_n\|_{H^{\gamma_1}}^{k-2}\right) \\
&+ \tau\|\partial_x w(t_n)\|_{H^{\gamma_1}}\|\tilde{e}_n\|_{L^2}\|w(t_n)\|_{H^{\gamma+\gamma_1}}\left(\|w(t_n)\|_{H^{\gamma_1}}^{k-3} + \|\tilde{e}_n\|_{H^{\gamma_1}}^{k-2}\right).
\end{aligned}
$$

Here $\gamma_1 > \frac{1}{2}$ is an arbitrary constant. By choosing $\gamma_1 = \min\{\gamma_0, 1\}$ and noting $\|\tilde{e}_n\|_{H^\gamma} = \|e_n\|_{H^\gamma}$, we get

$$
\|S_0\|_{H^\gamma} \le C\tau\|e_n\|_{H^\gamma}\left(1 + \|e_n\|_{H^{\gamma_0}}^{k-2}\right), \tag{4.13}
$$

where the constant C depends only on k and $\|w\|_{L_t^\infty H_x^{2+\gamma}}$. Similarly, since $\gamma_0 > \frac{1}{2}$ and noting that $\gamma_0 < 1 + \gamma$, we have

$$
\begin{aligned}
\|S_0\|_{H^{\gamma_0}} \lesssim& \tau\|\partial_x w(t_n)\|_{H^{\gamma_0}}\|\tilde{e}_n\|_{H^{\gamma_0}}\left(\|w(t_n)\|_{H^{\gamma_0}}^{k-2} + \|\tilde{e}_n\|_{H^{\gamma_0}}^{k-2}\right) \\
\le& C\tau\|e_n\|_{H^{\gamma_0}}\left(1 + \|e_n\|_{H^{\gamma_0}}^{k-2}\right).
\end{aligned} \tag{4.14}
$$

The estimates (4.13) and (4.14) together show that $\|S_0\|_{H^\kappa} \le C\tau\|e_n\|_{H^\kappa}\left(1 + \|e_n\|_{H^{\gamma_0}}^{k-2}\right)$ for $\kappa = \gamma$ or γ_0. Then, we find $\|S_0\|_{H^\kappa}^2 \le C\tau^2\|e_n\|_{H^\kappa}^2\left(1 + \|e_n\|_{H^{\gamma_0}}^{2k-4}\right)$ and

$$
\big|\big\langle J^\kappa e_n, J^\kappa S_0\big\rangle\big| \le \|e_n\|_{H^\kappa}\|S_0\|_{H^\kappa} \le C\tau\|e_n\|_{H^\kappa}^2\left(1 + \|e_n\|_{H^{\gamma_0}}^{k-2}\right).
$$

This proves the lemma by noting the definition of $Q(\cdot)$ in Proposition 4.5. □

Lemma 4.7. *For the part of S_1 in* (4.12)*, with $\kappa=\gamma$ or γ_0, it holds that*

$$\left|\left\langle J^\kappa e_n, J^\kappa S_1\right\rangle\right| + \|S_1\|_{H^\kappa}^2 \le C\tau\|e_n\|_{H^\kappa}^2.$$

Proof. We shall estimate the two quantities $\left\langle J^\kappa e_n, J^\kappa S_1\right\rangle$ and $\left\|S_1\right\|_{H^\kappa}^2$ in a sequel. Firstly, for $\left\langle J^\kappa e_n, J^\kappa S_1\right\rangle$, we have

$$\begin{aligned}\left|\left\langle J^\kappa e_n, J^\kappa S_1\right\rangle\right| \le&\tilde{C}_1\int_0^\tau \left|\left\langle J^\kappa e_n, J^\kappa \mathrm{e}^{(t_n+s)\partial_x^3}\left\{\mathrm{e}^{-s\partial_x^3}\partial_x\tilde{e}_n\cdot\left[\mathrm{e}^{-s\partial_x^3}w(t_n)^{k-1}\right]\right\}\right\rangle\right|ds\\ &+\tilde{C}_1\frac{\tau}{2\pi}\left|\int_{\mathbb{T}} w(t_n)^{k-1}dx\left\langle J^\kappa e_n, J^\kappa\partial_x e_n\right\rangle\right|\\ =&\tilde{C}_1\int_0^\tau\left|\left\langle \mathrm{e}^{-s\partial_x^3}J^\kappa\tilde{e}_n, J^\kappa\left\{\mathrm{e}^{-s\partial_x^3}\partial_x\tilde{e}_n\cdot\left[\mathrm{e}^{-s\partial_x^3}w(t_n)^{k-1}\right]\right\}\right\rangle\right|ds,\end{aligned}$$

where we have used the fact $\langle J^\kappa e_n, J^\kappa \partial_x e_n\rangle = 0$. Then from Lemma 2.3 (i) with any $\gamma_1>\frac{1}{2}$, we have that for $\kappa=\gamma$,

$$\left|\left\langle J^\gamma e_n, J^\gamma S_1\right\rangle\right| \lesssim \tau\|e_n\|_{H^\gamma}^2\|w(t_n)\|_{H^{\gamma+\gamma_1+1}}^{k-1} \le C\tau\|e_n\|_{H^\gamma}^2, \tag{4.15}$$

where we have used the relationship $\gamma+\gamma_1+1<2+\gamma$, and the constant $C=C(\|v(t_n)\|_{H^{2+\gamma}})$. On the other hand, by Lemma 2.3 (ii), since $\gamma_0>\frac{1}{2}$ we have that for $\kappa=\gamma_0$,

$$\left|\left\langle J^{\gamma_0} e_n, J^{\gamma_0} S_1\right\rangle\right| \lesssim \tau\|e_n\|_{H^{\gamma_0}}^2\|w(t_n)\|_{H^{\gamma_0+1}}^{k-1} \le C\tau\|e_n\|_{H^{\gamma_0}}^2. \tag{4.16}$$

The estimates (4.15) and (4.16) together give

$$\left|\left\langle J^\kappa e_n, J^\kappa S_1\right\rangle\right| \le C\tau\|e_n\|_{H^\kappa}^2. \tag{4.17}$$

Next, we estimate $\left\|S_1\right\|_{H^\kappa}^2$. By similar calculations as in (4.2), the Fourier transform of S_1 reads

$$\widehat{S}_1(l) = -\tilde{C}_1\int_0^\tau \sum_{\substack{l=l_1+l_2+\cdots+l_k\\ l_2+\cdots+l_k\neq 0}} \mathrm{e}^{-it_n(\alpha+\beta)}\mathrm{e}^{-is\alpha} il_1\widehat{e}_n(l_1)\widehat{v}(t_n,l_2)\cdots\widehat{v}(t_n,l_k)ds,$$

and it is straightforward to find out that

$$\begin{aligned}S_1=&-\frac{1}{3}\tilde{C}_1\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-\tau\partial_x^3}\tilde{e}_n\right)\cdot\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}w(t_n)^{k-1}\right)\right]+\frac{1}{3}\tilde{C}_1\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left[\tilde{e}_n\cdot\partial_x^{-1}w(t_n)^{k-1}\right]\\ &-\frac{\tau}{2\pi}\tilde{C}_1\int_{\mathbb{T}} e_n\cdot\left[\mathrm{e}^{t_n\partial_x^3}\partial_x w(t_n)^{k-1}\right]dx.\end{aligned}$$

By Lemma 2.1, Sobolev's and Hölder's inequalities, we obtain

$$\begin{aligned}\left\|J^\kappa\partial_x S_1\right\|_{L^2}\lesssim&\left\|J^\kappa\left[\left(\mathrm{e}^{-\tau\partial_x^3}\tilde{e}_n\right)\cdot\left(\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}w(t_n)^{k-1}\right)\right]\right\|_{L^2}+\left\|J^\kappa\left[\tilde{e}_n\cdot\partial_x^{-1}w(t_n)^{k-1}\right]\right\|_{L^2}\\ &+\tau\int_{\mathbb{T}}\left|e_n\cdot\left[\mathrm{e}^{t_n\partial_x^3}\partial_x w(t_n)^{k-1}\right]\right|dx\\ \lesssim&\|e_n\|_{H^\kappa}\|w(t_n)\|_{H^{\kappa+1}}^{k-1}+\tau\|e_n\|_{H^\kappa}\|w(t_n)\|_{H^{\kappa+1}}^{k-1}.\end{aligned} \tag{4.18}$$

Then, based on (4.11) and using integration-by-parts, we find

$$\begin{aligned}\left\|S_1\right\|_{H^\kappa}^2=\left\langle J^\kappa S_1, J^\kappa S_1\right\rangle=&-\tilde{C}_1\int_0^\tau\left\langle J^\kappa S_1, J^\kappa\mathrm{e}^{(t_n+s)\partial_x^3}\left\{\mathrm{e}^{-s\partial_x^3}\partial_x\tilde{e}_n\cdot\left[\mathrm{e}^{-s\partial_x^3}w(t_n)^{k-1}\right]\right\}\right\rangle ds\\ &+\tilde{C}_1\frac{\tau}{2\pi}\int_{\mathbb{T}} w(t_n)^{k-1}dx\cdot\left\langle J^\kappa S_1, J^\kappa\partial_x e_n\right\rangle\\ =&\tilde{C}_1\int_0^\tau\left\langle J^\kappa\partial_x S_1, J^\kappa\mathrm{e}^{(t_n+s)\partial_x^3}\left\{\mathrm{e}^{-s\partial_x^3}\tilde{e}_n\cdot\left[\mathrm{e}^{-s\partial_x^3}w(t_n)^{k-1}\right]\right\}\right\rangle ds\\ &+\tilde{C}_1\int_0^\tau\left\langle J^\kappa S_1, J^\kappa\mathrm{e}^{(t_n+s)\partial_x^3}\left\{\mathrm{e}^{-s\partial_x^3}\tilde{e}_n\cdot\left[\mathrm{e}^{-s\partial_x^3}\partial_x w(t_n)^{k-1}\right]\right\}\right\rangle ds\\ &-\tilde{C}_1\frac{\tau}{2\pi}\int_{\mathbb{T}} w(t_n)^{k-1}dx\cdot\left\langle J^\kappa\partial_x S_1, J^\kappa e_n\right\rangle.\end{aligned}$$

By Cauchy-Schwarz's inequality, we get

$$\begin{aligned}\big\|S_1\big\|_{H^\kappa}^2 \lesssim& \int_0^\tau \big\|J^\kappa\partial_x S_1\big\|_{L^2}\Big\|J^\kappa \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\tilde e_n\cdot\Big[\mathrm{e}^{-s\partial_x^3}w(t_n)^{k-1}\Big]\Big\}\Big\|_{L^2}\,ds\\ &+\int_0^\tau \big\|J^\kappa S_1\big\|_{L^2}\Big\|J^\kappa \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\tilde e_n\cdot\Big[\mathrm{e}^{-s\partial_x^3}\partial_x w(t_n)^{k-1}\Big]\Big\}\Big\|_{L^2}\,ds\\ &+\tau\big\|w(t_n)\big\|_{L^{k-1}}^{k-1}\big\|J^\kappa\partial_x S_1\big\|_{L^2}\|e_n\|_{H^\kappa}.\end{aligned}$$

Hence, from (4.18), Lemma 2.1 and Sobolev's inequality, we get that for any $\gamma\ge 0$, $\frac12<\gamma_1\le 1$,

$$\begin{aligned}\big\|S_1\big\|_{H^\gamma}^2 \lesssim&\tau\big\|J^\gamma\partial_x S_1\big\|_{L^2}\big\|e_n\big\|_{H^\gamma}\big\|w(t_n)\big\|_{H^{\gamma+\gamma_1}}+\tau\big\|J^\gamma S_1\big\|_{L^2}\big\|e_n\big\|_{H^\gamma}\big\|w(t_n)\big\|_{H^{\gamma+\gamma_1+1}}\\ &+\tau\big\|w(t_n)\big\|_{H^{\gamma_1}}^{k-1}\big\|J^\gamma\partial_x S_1\big\|_{L^2}\|e_n\|_{H^\gamma}\\ \lesssim&\tau\|e_n\|_{H^\gamma}^2\big\|w(t_n)\big\|_{H^{2+\gamma}}^{2k-2}.\end{aligned}$$

Similarly, we get that for any $\frac12<\gamma_0<1+\gamma$,

$$\begin{aligned}\big\|S_1\big\|_{H^{\gamma_0}}^2 \lesssim&\tau\big\|J^{\gamma_0}\partial_x S_1\big\|_{L^2}\big\|e_n\big\|_{H^{\gamma_0}}\big\|w(t_n)\big\|_{H^{\gamma_0}}+\tau\big\|J^{\gamma_0} S_1\big\|_{L^2}\big\|e_n\big\|_{H^{\gamma_0}}\big\|w(t_n)\big\|_{H^{\gamma_0+1}}\\ &+\tau\big\|w(t_n)\big\|_{H^{\gamma_0}}^{k-1}\big\|J^{\gamma_0}\partial_x S_1\big\|_{L^2}\|e_n\|_{H^{\gamma_0}}\\ \lesssim&\tau\|e_n\|_{H^{\gamma_0}}^2\big\|w(t_n)\big\|_{H^{2+\gamma}}^{2k-2}.\end{aligned}$$

Therefore, it infers that for $\kappa=\gamma$ or γ_0,

$$\|S_1\|_{H^\kappa}^2\le C\tau\|e_n\|_{H^\kappa}^2.$$

This combining with (4.17), yield the desired estimate of the lemma. □

Lemma 4.8. *For the part of S_j in* (4.12) *with $j\ge 2$, we have the following two estimates*

$$\big|\big\langle J^{\gamma_0}e_n,J^{\gamma_0}S_j\big\rangle\big|+\|S_j\|_{H^{\gamma_0}}^2\le Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^{\gamma_0}}\big\|e_n\big\|_{H^\gamma}^2+\tau Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^{\gamma_0}}^2, \tag{4.19a}$$

$$\big|\big\langle J^{\gamma}e_n,J^{\gamma}S_j\big\rangle\big|+\|S_j\|_{H^{\gamma}}^2\le \tau^{\frac25}Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^3. \tag{4.19b}$$

Proof. H^{γ_0}**-estimate for** (4.19a)**.** Similarly as for S_1, we can find for S_j with $j\ge2$ in (4.11),

$$\begin{aligned}S_j=&-\frac13\tilde C_j\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\Big[\mathrm{e}^{-\tau\partial_x^3}\tilde e_n\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\tilde e_n^{j-1}w(t_n)^{k-j}\big)\Big]\\ &+\frac13\tilde C_j\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\big[\tilde e_n\cdot\partial_x^{-1}\big(\tilde e_n^{j-1}w(t_n)^{k-j}\big)\big]-\frac{\tau}{2\pi}\tilde C_j\int_{\mathbb T}e_n\cdot\mathrm{e}^{t_n\partial_x^3}\partial_x\big[\tilde e_n^{j-1}w(t_n)^{k-j}\big]dx.\end{aligned}$$

For the last term in the above, by integration-by-parts we have the identity:

$$\int_{\mathbb T}e_n\cdot\mathrm{e}^{t_n\partial_x^3}\partial_x\big[\tilde e_n^{j-1}w(t_n)^{k-j}\big]dx=\int_{\mathbb T}\tilde e_n\cdot\partial_x\big[\tilde e_n^{j-1}w(t_n)^{k-j}\big]dx=\frac1j\int_{\mathbb T}\tilde e_n^{j}\cdot\partial_x\big[w(t_n)^{k-j}\big]dx.$$

Therefore, we further get that

$$\begin{aligned}S_j=&-\frac13\tilde C_j\mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\Big[\mathrm{e}^{-\tau\partial_x^3}\tilde e_n\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\tilde e_n^{j-1}w(t_n)^{k-j}\big)\Big]\\ &+\frac13\tilde C_j\mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\big[\tilde e_n\cdot\partial_x^{-1}\big(\tilde e_n^{j-1}w(t_n)^{k-j}\big)\big]-\frac{\tau}{2\pi}\frac{\tilde C_j}{j}\int_{\mathbb T}\tilde e_n^{j}\cdot\partial_x\big[w(t_n)^{k-j}\big]dx.\end{aligned}\tag{4.20}$$

Noting that $\frac12<\gamma_0<1+\gamma$, by Lemma 2.1 and Sobolev's inequality, we get

$$\begin{aligned}\big\|S_j\big\|_{H^{\gamma_0}}\lesssim&\Big\|\mathrm{e}^{-\tau\partial_x^3}\tilde e_n\cdot\mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big[\tilde e_n^{j-1}w(t_n)^{k-j}\big]\Big\|_{H^\gamma}+\Big\|\tilde e_n\cdot\partial_x^{-1}\big[\tilde e_n^{j-1}w(t_n)^{k-j}\big]\Big\|_{H^\gamma}\\ &+\tau\Big|\int_{\mathbb T}\tilde e_n^{j}\cdot\partial_x\big[w(t_n)^{k-j}\big]dx\Big|\\ \lesssim&\|e_n\|_{H^\gamma}^2\|e_n\|_{H^{\gamma_0}}^{j-2}\big\|w(t_n)\big\|_{H^{\gamma_0}}^{k-j}+\tau\|e_n\|_{L^2}^2\|e_n\|_{H^{\gamma_0}}^{j-2}\big\|w(t_n)\big\|_{H^{\gamma_0+1}}^{k-j}\\ \le&Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2+\tau Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^{\gamma_0}}^2.\end{aligned}$$

Then (4.19a) follows from the estimate above and the Hölder inequality.

H^γ-estimate for (4.19b)**.** With some $M \geq 1$ to be determined, we consider a frequency decomposition to (4.11) as:

$$S_j = -\tilde{C}_j \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n \cdot \mathrm{e}^{-s\partial_x^3}\Big[\tilde{e}_n^{j-1}w(t_n)^{k-j}\Big]\Big\}ds \tag{4.21a}$$

$$-\tilde{C}_j \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\Big\{\mathrm{e}^{-s\partial_x^3}\partial_x P_{>M}\tilde{e}_n \cdot \mathrm{e}^{-s\partial_x^3}\Big[\tilde{e}_n^{j-1}w(t_n)^{k-j}\Big]\Big\}ds + \frac{\tilde{C}_j\tau}{2\pi}\partial_x e_n \cdot \int_{\mathbb{T}} \tilde{e}_n^{j-1}w(t_n)^{k-j}dx, \tag{4.21b}$$

and then we estimate the above two parts as follows. For (4.21a), by Lemma 2.1, we have that for any $\gamma \geq 0$,

$$\begin{aligned}
\|(4.21\text{a})\|_{H^\gamma} \lesssim& \int_0^\tau \Big\|\mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n \cdot \mathrm{e}^{-s\partial_x^3}\Big[\tilde{e}_n^{j-1}w(t_n)^{k-j}\Big]\Big\|_{H^\gamma}\, ds\\
\lesssim& \int_0^\tau \Big\|J^\gamma \mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n\Big\|_{L^\infty}\Big\|\mathrm{e}^{-s\partial_x^3}\Big[\tilde{e}_n^{j-1}w(t_n)^{k-j}\Big]\Big\|_{L^2}\, ds\\
&+ \int_0^\tau \Big\|\mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n\Big\|_{L^\infty}\Big\|J^\gamma \mathrm{e}^{-s\partial_x^3}\Big[\tilde{e}_n^{j-1}w(t_n)^{k-j}\Big]\Big\|_{L^2}\, ds.
\end{aligned}$$

Note that by Bernstein's inequality, we have

$$\Big\|J^\gamma \mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n\Big\|_{L^\infty} \lesssim M^{\frac12}\Big\|J^\gamma \mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n\Big\|_{L^2} \lesssim M^{\frac32}\|e_n\|_{H^\gamma},$$

and similarly,

$$\Big\|\mathrm{e}^{-s\partial_x^3}\partial_x P_{\leq M}\tilde{e}_n\Big\|_{L^\infty} \lesssim M^{\frac32}\|e_n\|_{L^2}.$$

Using these two estimates, Lemma 2.1 and Sobolev's inequality, we further obtain that for any $\gamma \geq 0$, $\gamma_1 > \frac12$,

$$\|(4.21\text{a})\|_{H^\gamma} \lesssim \tau M^{\frac32}\big\|e_n\big\|_{L^2}\|e_n\|_{H^\gamma}\big\|e_n\big\|_{H^{\gamma_1}}^{j-2}\big\|w(t_n)\big\|_{H^{\gamma+\gamma_1}}^{k-j}.$$

Choosing $\gamma_1 = \min\{\gamma_0, 1\}$, then we have

$$\|(4.21\text{a})\|_{H^\gamma} \leq \tau M^{\frac32}Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2. \tag{4.22}$$

For (4.21b), similar as (4.20), we rewrite it as

$$\begin{aligned}
(4.21\text{b}) =& -\frac13\tilde{C}_j \mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\Big[\mathrm{e}^{-\tau\partial_x^3}P_{>M}\tilde{e}_n \cdot \mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\Big] && (4.23\text{a})\\
&+ \frac13\tilde{C}_j \mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\big[P_{>M}\tilde{e}_n \cdot \partial_x^{-1}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\big] && (4.23\text{b})\\
&+ \tilde{C}_j\frac{\tau}{2\pi}\partial_x P_{\leq M}e_n \cdot \int_{\mathbb{T}}\tilde{e}_n^{j-1}w(t_n)^{k-j}\,dx && (4.23\text{c})\\
&- \frac{\tau}{2\pi}\tilde{C}_j\int_{\mathbb{T}} P_{\leq M}\partial_x e_n \cdot \mathrm{e}^{t_n\partial_x^3}\big[\tilde{e}_n^{j-1}w(t_n)^{k-j}\big]dx && (4.23\text{d})\\
&- \frac{\tau}{2\pi}\frac{\tilde{C}_j}{j}\int_{\mathbb{T}}\tilde{e}_n^j \cdot \partial_x\big[w(t_n)^{k-j}\big]dx, && (4.23\text{e})
\end{aligned}$$

and we estimate these five terms one by one.

For (4.23a), according to the frequency restriction, there exists some constant $c = c(k) > 0$ such that

$$\begin{aligned}
(4.23\text{a}) =& -\frac13\tilde{C}_j \mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}P_{>cM}\Big[\mathrm{e}^{-\tau\partial_x^3}P_{>M}\tilde{e}_n \cdot \mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\Big]\\
&- \frac13\tilde{C}_j \mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\Big[\mathrm{e}^{-\tau\partial_x^3}P_{>M}\tilde{e}_n \cdot \mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}P_{>cM}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\Big].
\end{aligned}$$

Therefore, by Beinstein's inequality, Sobolev's inequality and Lemma 2.1, we have that for any $\gamma \geq 0$, $\gamma_1 > \frac{1}{2}$,

$$\begin{aligned}\big\|(4.23\text{a})\big\|_{H^\gamma} \lesssim & M^{-1}\left\|\mathrm{e}^{-\tau\partial_x^3}P_{>M}\tilde{e}_n\cdot \mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\right\|_{H^\gamma}\\ &+\left\|J^\gamma\left[\mathrm{e}^{-\tau\partial_x^3}P_{>M}\tilde{e}_n\cdot \mathrm{e}^{-\tau\partial_x^3}\partial_x^{-1}P_{>cM}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\right]\right\|_{L^1}\\ \lesssim & M^{-1}\big\|e_n\big\|_{H^\gamma}\big\|\tilde{e}_n^{j-1}w(t_n)^{k-j}\big\|_{H^\gamma}+\big\|e_n\big\|_{H^\gamma}\cdot M^{-1}\big\|P_{>cM}\big(\tilde{e}_n^{j-1}w(t_n)^{k-j}\big)\big\|_{H^\gamma}\\ \lesssim & M^{-1}\big\|e_n\big\|_{H^\gamma}^2\big\|e_n\big\|_{H^{\gamma_1}}^{j-2}\big\|w(t_n)\big\|_{H^{\gamma+\gamma_1}}^{k-j}\leq M^{-1}Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.\end{aligned}$$

Here $\gamma_1 = \min\{\gamma_0, 1\}$ is chosen in the last step. The term in (4.23b) can be treated similarly as (4.23a), and we have

$$\big\|(4.23\text{b})\big\|_{H^\gamma} \leq M^{-1}Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.$$

For (4.23c), using Lemma 2.1, Bernstein's and Sobolev's inequalities, we can find that for any $\gamma \geq 0$, $\frac{1}{2} < \gamma_0 < 1 + \gamma$,

$$\begin{aligned}\|(4.23\text{c})\|_{H^\gamma} \lesssim & \tau\left\|\partial_x P_{\leq M}e_n\cdot\int_{\mathbb{T}}\tilde{e}_n^{j-1}w(t_n)^{k-j}\,dx\right\|_{H^\gamma}\lesssim \tau\left\|\partial_x P_{\leq M}e_n\right\|_{H^\gamma}\left|\int_{\mathbb{T}}\tilde{e}_n^{j-1}w(t_n)^{k-j}\,dx\right|\\ \lesssim & \tau M\|e_n\|_{H^\gamma}\big\|e_n\big\|_{L^2}\big\|e_n\big\|_{H^{\gamma_0}}^{j-2}\big\|w(t_n)\big\|_{H^{\gamma_0}}^{k-j}\leq \tau MQ(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.\end{aligned}$$

For (4.23d), by Lemma 2.1 and Sobolev's inequality, we get that for any $\gamma \geq 0$, $\frac{1}{2} < \gamma_0 < 1 + \gamma$,

$$\begin{aligned}\|(4.23\text{d})\|_{H^\gamma} \lesssim & \tau\left|\int_{\mathbb{T}}P_{\leq M}\partial_x e_n\cdot \mathrm{e}^{t_n\partial_x^3}\left[\tilde{e}_n^{j-1}w(t_n)^{k-j}\right]dx\right|\lesssim \tau\left\|\partial_x P_{\leq M}e_n\right\|_{L^2}\left\|\tilde{e}_n^{j-1}w(t_n)^{k-j}\right\|_{L^2}\\ \lesssim & \tau M\big\|e_n\big\|_{L^2}^2\big\|e_n\big\|_{H^{\gamma_0}}^{j-2}\big\|w(t_n)\big\|_{H^{\gamma_0}}^{k-j}\leq \tau MQ(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.\end{aligned}$$

For (4.23e), by Hölder's and Sobolev's inequalities, we obtain that for any $\frac{1}{2} < \gamma_0 < 1 + \gamma$,

$$\|(4.23\text{e})\|_{H^\gamma} \lesssim \tau\big\|e_n\big\|_{L^2}^2\big\|e_n\big\|_{H^{\gamma_0}}^{j-2}\big\|w(t_n)\big\|_{H^{1+\gamma_0}}^{k-j}\leq \tau Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.$$

Then combining with the above five estimates for (4.23), we have

$$\big\|(4.21\text{b})\big\|_{H^\gamma} \leq \big(M^{-1}+\tau M\big)\,Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2. \tag{4.24}$$

With the estimates (4.22) and (4.24) together, we obtain

$$\big\|S_j\big\|_{H^\gamma} \leq \left(M^{-1}+\tau M^{\frac{3}{2}}\right)Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.$$

Choosing $M = \tau^{-\frac{2}{5}}$, it gives

$$\big\|S_j\big\|_{H^\gamma} \leq \tau^{\frac{2}{5}}Q(\|e_n\|_{H^{\gamma_0}})\big\|e_n\big\|_{H^\gamma}^2.$$

Again, (4.19b) follows from the estimate above and the Hölder inequality.

□

Now we continue to prove Proposition 4.5.

H^γ-estimate. By Lemmas 4.6-4.8, we conclude that

$$\big\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\big\|_{H^\gamma}^2 \leq \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^\gamma}^2+\tau^{\frac{2}{5}}Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^\gamma}^3,$$

where $Q(f) = \sum\limits_{l=0}^{2k-4} C_l f^l$ for some $C_l > 0$ depending only on k and $\|v\|_{L_t^\infty H_x^{2+\gamma}}$. Using the facts

$$\sqrt{A+B}-\sqrt{A}\leq \min\{B/\sqrt{A},\sqrt{B}\},\quad \text{for any } A, B > 0, \tag{4.25}$$

and $\sqrt{1+C\tau} \sim 1 + C\tau$ when τ is small enough, we get

$$\big\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\big\|_{H^\gamma} \leq \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^\gamma}+\tau^{\frac{2}{5}}Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^\gamma}^2.$$

H^{γ_0}-estimate. Similarly as the above H^γ-estimate, using Lemmas 4.6-4.8, we can get

$$\big\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\big\|_{H^{\gamma_0}}^2 \leq \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^{\gamma_0}}^2+Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^{\gamma_0}}\|e_n\|_{H^\gamma}^2.$$

By Hölder's inequality, we further obtain

$$\left\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\right\|_{H^{\gamma_0}}^2 \le \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^{\gamma_0}}^2+\tau^{-1}\|e_n\|_{H^\gamma}^4.$$

Then by (4.25), we have

$$\left\|\Phi^n\big(v(t_n)\big)-\Phi^n\big(v^n\big)\right\|_{H^{\gamma_0}} \le \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^{\gamma_0}}+\tau^{-1}\|e_n\|_{H^\gamma}^3.$$

This finishes the proof of Proposition 4.5. □

4.3. **Bootstrap and proof of Proposition 4.1.** Recall that $e_n=v^n-v(t_n)$. Then by (4.5),

$$\|e_{n+1}\|_{H^\kappa}\le \left\|\Phi^n\big(v^n\big)-\Phi^n\big(v(t_n)\big)\right\|_{H^\kappa}+\left\|v\big(t_{n+1}\big)-\Phi^n\big(v(t_n)\big)\right\|_{H^\kappa},$$

for $\kappa=\gamma\ge 0$ or $\kappa=\gamma_0$ with $\max\{\frac12,\gamma\}<\gamma_0<1+\gamma$. Therefore by Propositions 4.2&4.5, we get

$$\|e_{n+1}\|_{H^\gamma}\le \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^\gamma}+\tau^{\frac25}Q(\|e_n\|_{H^{\gamma_0}})\|e_n\|_{H^\gamma}^2+\tau^2, \tag{4.26a}$$

$$\|e_{n+1}\|_{H^{\gamma_0}}\le \big[1+\tau Q(\|e_n\|_{H^{\gamma_0}})\big]\|e_n\|_{H^{\gamma_0}}+\tau^{-1}\|e_n\|_{H^\gamma}^3+\tau^{1+\alpha}, \tag{4.26b}$$

for $\alpha=1+\gamma-\gamma_0$. Now we fix $\gamma_0=0.6+\gamma$ (thus $\alpha=0.4$) and let

$$X_n=\tau^{-1}\|e_n\|_{H^\gamma}+\tau^{-0.4}\|e_n\|_{H^{\gamma_0}},$$

then by (4.26), we can get

$$X_{n+1}\le \big[1+\tau Q\big(\tau^{0.4}X_n\big)\big]\,X_n+\tau^{\frac75}Q\big(\tau^{0.4}X_n\big)X_n^2+\tau^{2-0.4}X_n^3+C\tau.$$

Denote $Q_{\ge1}(f):=Q(f)-C_0=\sum\limits_{l=1}^{2k-4}C_lf^l$. Then choosing $\tau\le1$, the estimate above is simplified as

$$X_{n+1}\le \big[1+C_0\tau+Q_{\ge1}\big(\tau^{0.4}X_n\big)\tau\big]\,X_n+C\tau, \tag{4.27}$$

for some $C>0$ depending only on $\|v\|_{L_t^\infty H_x^{2+\gamma}}$.

Now we claim that for some $\tau_0>0$ satisfying

$$Q_{\ge1}\big(\tau_0^{0.4}C_0^{-1}C\mathrm{e}^{2C_0T}\big)\le C_0, \tag{4.28}$$

we have for any $0<\tau\le\tau_0$,

$$X_n\le C\tau\sum_{j=0}^n(1+2C_0\tau)^j,\quad\text{for all}\quad 0\le n\le\frac{T}{\tau}. \tag{4.29}$$

Noting that $X_0=0$, we use induction and assume that it holds till some $0\le n_0\le T/\tau-1$, i.e.,

$$X_n\le C\tau\sum_{j=0}^n(1+2C_0\tau)^j,\quad\text{for any}\quad 0\le n\le n_0.$$

Note that $n\le T/\tau$, we have the uniform bound of X_n as $X_n\le C_0^{-1}C\mathrm{e}^{2C_0T}$. Thus by (4.28), it implies that $Q_{\ge1}\big(\tau^{0.4}X_n\big)\le C_0$ for any $0\le n\le n_0$, and then by (4.27) we can find

$$\begin{aligned}X_{n_0+1}&\le \big[1+C_0\tau+Q_{\ge1}\big(\tau^{0.4}X_{n_0}\big)\tau\big]\,X_{n_0}+C\tau\le(1+2C_0\tau)\,X_{n_0}+C\tau\\&\le(1+2C_0\tau)\,C\tau\sum_{j=0}^{n_0}(1+2C_0\tau)^j+C\tau=C\tau\sum_{j=0}^{n_0+1}(1+2C_0\tau)^j.\end{aligned}$$

Therefore, the claim (4.29) holds, and we have $X_n\le C_0^{-1}C\mathrm{e}^{2C_0T}$ for any $0\le n\le T/\tau$. By the definition of X_n, this yields that for some $C>0$,

$$\|v^n-v(t_n)\|_{H^\gamma}\le C\tau,\quad\text{for any}\quad 0\le n\le\frac{T}{\tau}.$$

This completes the proof of Proposition 4.1. □

4.4. **Proof of Theorem 3.1.** Now, we can use Proposition 4.1 to give the proof of Theorem 3.1. Let $t = t_{n+1}$ in (3.2), and then by subtracting it from (3.17), we find

$$\begin{aligned}
&u^{n+1}(x) - u(t_{n+1}, x)\\
=&w^{n+1}\Big(x - \frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}}(w^j(x))^{k-1}dx\Big) - w\Big(t_{n+1}, x - \frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}}(w^j(x))^{k-1}dx\Big) \quad (4.30a)\\
&+ w\Big(t_{n+1}, x - \frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}}(w^j(x))^{k-1}dx\Big) - w\Big(t_{n+1}, x - \frac{1}{2\pi}\int_0^{t_{n+1}}\int_{\mathbb{T}} w(\rho, x)^{k-1}dxd\rho\Big). \quad (4.30b)
\end{aligned}$$

Hence, we have

$$\|u^{n+1} - u\,(t_{n+1})\,\|_{H^\gamma} \le \|(4.30a)\|_{H^\gamma} + \|(4.30b)\|_{H^\gamma}. \tag{4.31}$$

By a shift of variable, it is direct to have $\|(4.30a)\|_{H^\gamma} \le \|w^{n+1} - w(t_{n+1})\|_{H^\gamma}$. From Proposition 4.1, there exit constants C, τ_0 depending on T, k and $\|w\|_{L^\infty((0,T);H^{2+\gamma}(\mathbb{T}))}$ such that

$$\|(4.30a)\|_{H^\gamma} \le C\tau, \quad 0 < \tau \le \tau_0. \tag{4.32}$$

For (4.30b), we further write it as

$$\begin{aligned}
&(4.30b)\\
=&w\Big(t_{n+1}, x - \frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}}(w^j(x))^{k-1}dx\Big) - w\Big(t_{n+1}, x - \frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}} w(t_j, x)^{k-1}dx\Big) \quad (4.33a)\\
&+ w\Big(t_{n+1}, x - \frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}} w(t_j, x)^{k-1}dx\Big) - w\Big(t_{n+1}, x - \frac{1}{2\pi}\int_0^{t_{n+1}}\int_{\mathbb{T}} w(\rho, x)^{k-1}dxd\rho\Big). \quad (4.33b)
\end{aligned}$$

Then,

$$\|(4.30b)\|_{H^\gamma} \le \|(4.33a)\|_{H^\gamma} + \|(4.33b)\|_{H^\gamma}. \tag{4.34}$$

For (4.33a), we have

$$\|(4.33a)\|_{H^\gamma} \lesssim \big\|\partial_x w(t_{n+1})\big\|_{H^\gamma}\,\frac{\tau}{2\pi}\sum_{j=0}^{n}\int_{\mathbb{T}}\Big|(w^j(x))^{k-1} - w(t_j, x)^{k-1}\Big|dx. \tag{4.35}$$

By Lemma 2.1, we have for $\gamma_0 \in (\frac{1}{2}, 1+\gamma)$,

$$\int_{\mathbb{T}}\big|(w^j(x))^{k-1} - w(t_j, x)^{k-1}\big|dx \lesssim \|w^j(x) - w(t_j)\|_{L^2}\left[\|w^j(x)\|_{H^{\gamma_0}}^{k-2} + \|w(t_j)\|_{H^{\gamma_0}}^{k-2}\right].$$

From (4.1), we have $\|w^n\|_{H^{\gamma_0}} \le C$, where the constant C depends on T, k and $\|w\|_{L^\infty((0,T);H^{2+\gamma}(\mathbb{T}))}$. This estimate combining with Proposition 4.1 yield that

$$\int_{\mathbb{T}}\big|(w^j(x))^{k-1} - w(t_j, x)^{k-1}\big|dx \le C\tau.$$

Inserting the above into (4.35), we obtain

$$\|(4.33a)\|_{H^\gamma} \le Cn\tau^2 \le C\tau. \tag{4.36}$$

For (4.33b), similarly as (4.33a), we can get

$$\|(4.33b)\|_{H^\gamma} \lesssim \big\|\partial_x w(t_{n+1})\big\|_{H^\gamma}\,\frac{1}{2\pi}\sum_{j=0}^{n}\Big|\int_{t_j}^{t_{j+1}}\int_{\mathbb{T}}\Big[w(t_j, x)^{k-1} - w(\rho, x)^{k-1}\Big]dxd\rho\Big|. \tag{4.37}$$

Moreover, noting (3.3) we have for $\rho \in [t_j, t_{j+1}]$,

$$\Big|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\big[w(t_j,x)^{k-1}-w(\rho,x)^{k-1}\big]dxd\rho\Big| \lesssim \Big|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\int_{t_j}^{\rho}\partial_t w(t)\cdot w(t)^{k-2}dtdxd\rho\Big|$$

$$\lesssim\Big|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\int_{t_j}^{\rho}\partial_x^3 w(t)\cdot w(t)^{k-2}dtdxd\rho\Big| \tag{4.38a}$$

$$+\Big|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\int_{t_j}^{\rho}\partial_x w(t,x)\cdot w(t)^{k-2}dtdxd\rho\Big| \tag{4.38b}$$

$$+\Big|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\int_{t_j}^{\rho}\partial_x w(t,x)\frac{1}{2\pi}\int_{\mathbb{T}}w(t,x)^{k-1}dx\cdot w(t)^{k-2}dtdxd\rho\Big|. \tag{4.38c}$$

For (4.38a), by integration-by-parts, Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}(4.38\text{a}) &\lesssim \Big|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\int_{t_j}^{\rho}\partial_x^2 w(t)\cdot\partial_x\big[w(t)^{k-2}\big]dtdxd\rho\Big|\\ &\lesssim \tau^2\big\|\partial_x^2 w(t)\big\|_{L^2}\big\|\partial_x\big[w(t)^{k-2}\big]\big\|_{L^2}\lesssim\tau^2\|w\|^{k-1}_{L^\infty((t_n,t_{n+1});H^{2+\gamma})}.\end{aligned}$$

For (4.38b), again by Hölder's inequality and Sobolev's inequality, we have

$$(4.38\text{b})\lesssim\tau^2\big\|\partial_x w(t)\big\|_{L^2}\big\|w(t)^{k-2}\big\|_{L^2}\lesssim\tau^2\|w\|^{k-1}_{L^\infty((t_n,t_{n+1});H^{2+\gamma})}.$$

Similarly for (4.38c), we have

$$(4.38\text{c})\lesssim\tau^2\|w\|^{k-1}_{L_t^\infty L_x^{k-1}}\|\partial_x w(t)\|_{L^2}\big\|w(t)^{k-2}\big\|_{L^2}\lesssim\tau^2\|w\|^{2k-2}_{L^\infty((t_n,t_{n+1});H^{2+\gamma})}.$$

Combining the three estimates, we obtain from (4.38) that

$$\left|\int_{t_j}^{t_{j+1}}\!\!\int_{\mathbb{T}}\Big[w(t_j,x)^{k-1}-w(\rho,x)^{k-1}\Big]dxd\rho\right|\le C\tau^2,$$

where C depends on T, k and $\|w\|_{L^\infty((0,T);H^{2+\gamma}(\mathbb{T}))}$. Putting this estimate into (4.37), we get

$$\|(4.33\text{b})\|_{H^\gamma}\le Cn\tau^2\le C\tau. \tag{4.39}$$

Combining (4.34), (4.36) and (4.39), we get

$$\|(4.30\text{b})\|_{H^\gamma}\le C\tau. \tag{4.40}$$

Finally, plugging (4.32) and (4.40) into (4.31), we obtain that for $0<\tau\le\tau_0$,

$$\|u^{n+1}-u\left(t_{n+1}\right)\|_{H^\gamma}\le C\tau,$$

where the constants τ_0, C depend on T, k and $\|u\|_{L^\infty((0,T);H^{2+\gamma}(\mathbb{T}))}$. This proves Theorem 3.1. □

5. Numerical results

In this section, we will carry out numerical experiments of the presented schemes. They will be devoted to verifying the convergence result in Theorem 3.1 and illustrating the advantage of GTEI (3.17) for solving the gKdV equation (1.1). The two direct schemes, i.e., EI1 (2.2) and EI2 (2.4), will be considered as the benchmark for comparisons.

To get an initial data with the desired regularity, we construct $u_0(x)$ for the gKdV equation (1.1) by the following strategy [30]. Choose $N>0$ as an even integer and discretize the spatial domain $\mathbb{T}=(0,2\pi)$ with grid points $x_j=j\frac{2\pi}{N}$ for $j=0,\dots,N$. Take a uniformly distributed random vector $\mathcal{U}^N=\text{rand}(N,1)\in[0,1]^N$. Then, we define

$$u_0(x):=\frac{|\partial_{x,N}|^{-s}\mathcal{U}^N}{\||\partial_{x,N}|^{-\theta}\mathcal{U}^N\|_{L^\infty}},\quad x\in\mathbb{T}, \tag{5.1}$$

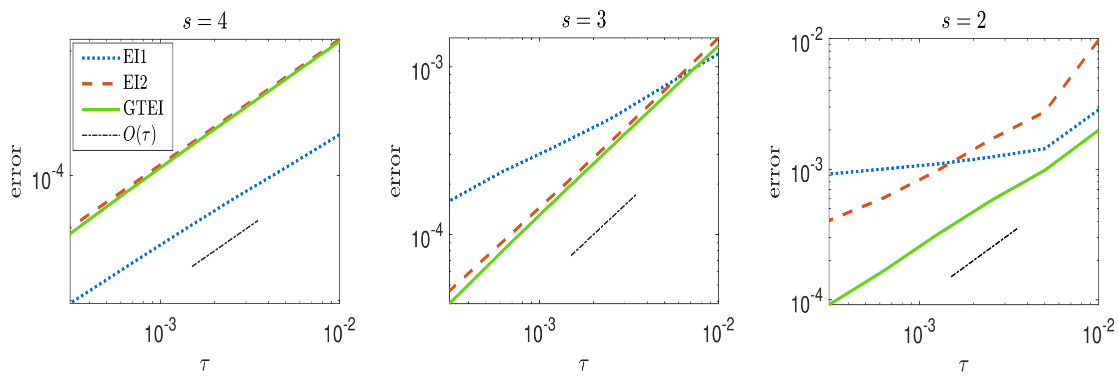

FIGURE 1. Temporal error $\|u - u^n\|_{L^2}/\|u\|_{L^2}$ of EI1, EI2, GTEI at $t_n = 1$ for (1.1) with $k = 4$ under $u_0 \in H^s$ for $s = 4$, $s = 3$, $s = 2$.

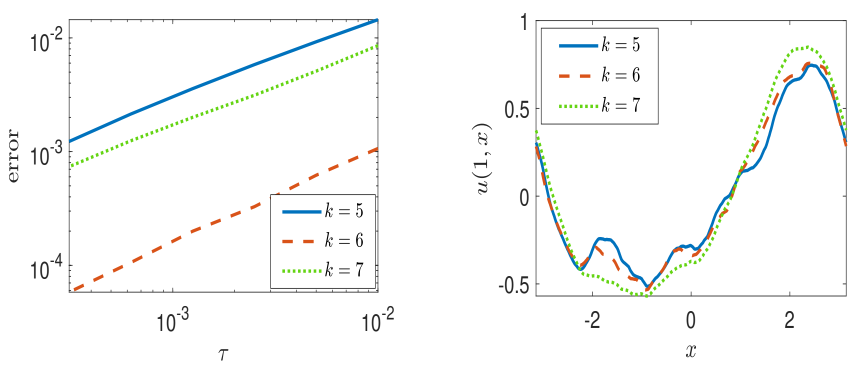

FIGURE 2. Temporal error $\|u - u^n\|_{L^2}/\|u\|_{L^2}$ of GTEI at $t_n = 1$ (left) and the solution profile $u(t = 1, x)$ (right) for (1.1) with $k = 5, 6, 7$ under $u_0 \in H^2$.

where the pseudo-differential operator $|\partial_{x,N}|^{-s}$ for $s \geq 0$ reads: for Fourier modes $l = -N/2, \ldots, N/2 - 1$,

$$\left(|\partial_{x,N}|^{-s}\right)_l = \begin{cases} |l|^{-s}, & \text{if } l \neq 0, \\ 0, & \text{if } l = 0. \end{cases}$$

Thus, we get $u_0 \in H^s(\mathbb{T})$ for any $s \geq 0$.

We implement the spatial discretizations of the numerical methods within discussions by the Fourier pseudo-spectral method [36] with a large number of grid points $N = 2^{14}$ in the torus domain $\mathbb{T}$. We present the error $u(t_n, x) - u^n(x)$ in the L^2 norm at the final time $t_n = T = 1$, where the 'exact' solution is obtained numerically by the EI2 scheme (2.4) with $\tau = 10^{-5}$. Firstly, we fix $k = 4$ in the gKdV equation (1.1). Figure 1 shows the errors of GTEI (3.17), EI2 (2.4) and EI1 (2.2) by using different time step τ for the initial data u_0 in H^4 or H^3 or H^2 as defined in (5.1). In Figure 2, we show the error of GTEI for solving (1.1) with several different k under u_0 in H^2.

The numerical results illustrate that the proposed GTEI scheme (3.17) is always first order accurate in the L^2-norm for solving (1.1) with $u_0 \in H^2$, which verifies the theoretical result. In contrast, the two direct EIs need the solution to be smoother to reach the first order convergence rate. Thus as a conclusion, the GTEI scheme (3.17) is more accurate for the gKdV equation (1.1) under rough data.

Acknowledgements

B. Li and Y. Wu are partially supported by NSFC 12171356. X. Zhao is supported by NSFC 12271413, 11901440 and the Natural Science Foundation of Hubei Province 2019CFA007.

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