LARGE GLOBAL SOLUTIONS FOR ENERGY-CRITICAL NONLINEAR SCHRÖDINGER EQUATION

RUOBING BAI, JIA SHEN, AND YIFEI WU

ABSTRACT. In this work, we consider the 3D defocusing energy-critical nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = |u|^4 u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3.$$

Applying the incoming and outgoing decomposition presented in the recent work [2], we prove that any radial function f with $\chi_{\le 1} f \in H^1$ and $\chi_{\ge 1} f \in H^{s_0}$ with $\frac{5}{6} < s_0 < 1$, there exists an outgoing component f_+ (or incoming component f_-) of f, such that when the initial data is f_+, then the corresponding solution is globally well-posed and scatters forward in time; when the initial data is f_-, then the corresponding solution is globally well-posed and scatters backward in time.

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the nonlinear Schrödinger equation (NLS) in 3 spatial dimensions (3D)

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^4 u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

with $\mu = \pm 1$. Here $u = u(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is a complex-valued function. The case $\mu = 1$ is referred to the defocusing case, and the case $\mu = -1$ is referred to the focusing case.

The solution satisfies the conservation of mass and energy, defined respectively by

$$M\big(u(t)\big) := \int_{\mathbb{R}^3} |u(t,x)|^2 dx = M(u_0), \tag{1.2}$$

and

$$E\big(u(t)\big) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t,x)|^2 dx + \frac{\mu}{6} \int_{\mathbb{R}^3} |u(t,x)|^6 dx = E(u_0). \tag{1.3}$$

The general form of the equation (1.1) is the following

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^p u, \qquad (t,x) \in \mathbb{R}^{1+d}, \\ u(0,x) = u_0(x). \end{cases} \tag{1.4}$$

The class of solutions to equation (1.4) is invariant under scaling

$$u(t,x) \to u_\lambda(t,x) = \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

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which maps the initial data

$$u(0) \to u_\lambda(0) := \lambda^{\frac{2}{p}} u_0(\lambda x).$$

Denote

$$s_c = \frac{d}{2} - \frac{2}{p},$$

then the scaling leaves $\dot{H}^{s_c}$ norm invariant, that is,

$$\|u(0)\|_{\dot{H}^{s_c}} = \|u_\lambda(0)\|_{\dot{H}^{s_c}}.$$

The well-posedness and scattering theory for the equation (1.4) has been widely studied. For the local well-posedness, Cazenave and Weissler [11] used the standard fixed point argument, and proved the equation (1.4) is locally well-posed in $H^s(\mathbb{R}^d)$ when $s \geq s_c$. Note that in the case of $s = s_c$ (critical regime), the time of existence depends on the profile of initial data rather than simply its norm. The fixed point argument can also be applied directly to prove the global well-posedness and scattering for the equation (1.4) with small initial data in $H^s(\mathbb{R}^d)$ when $s \geq s_c$.

Next, let us briefly review the large data global well-posedness and scattering theory for energy-critical NLS (1.4). Bourgain [6] firstly obtained such result for the 3D and 4D defocusing energy critical NLS with radial data in $\dot{H}^1(\mathbb{R}^3)$ by introducing the induction on energy method and spatial localized Morawetz estimate. Moreover, Grillakis [22] provided a different proof for the global well-posedness part of the result by Bourgain [6]. Tao [38] later generalized the results in [6, 22] to general dimensions with radially symmetric data. For non-radial problem, a major breakthrough was made by Colliander, Keel, Staffilani, Takaoka, and Tao in [12], where they obtained the related result for the 3D energy-critical defocusing NLS for general large data in $\dot{H}^1$. Then, the result was generalized by Ryckman and Visan [31] in dimension $d = 4$ and Visan [41] for higher dimensions. In the focusing case, Kenig and Merle [24] introduced the concentration compactness method, and obtained the global well-posedness and scattering in $\dot{H}^1(\mathbb{R}^d)$ $(d = 3, 4, 5)$ for the energy-critical NLS with radial initial data below the energy of ground state. Killip and Visan [26] later obtained the related result for dimensions $d \geq 5$ without the radial assumption. Then, Dodson [14] solved the 4D non-radial problem. Here, we only mention the papers for energy critical equations (1.4), and some other results for (1.4) can be found in [1, 3, 15–18, 25, 29, 32, 33] and the references therein.

Although the equation (1.4) is ill-posed in super-critical spaces, there are still some methods to study the well-posedness for a class of such data. The ill-posedness in some cases can be circumvented by an appropriate probabilistic method. Bourgain [4, 5] obtained the first almost sure local and global well-posedness results, which are based on the invariance of Gibbs measure associated to NLS on torus in one and two space dimensions. The random data approach has been further developed for the nonlinear dispersive equations, see for example [7–9, 13, 19, 27] and the references therein.

1.1. **Main result.** This paper aims to consider the global-wellposedness and scattering of the energy-critical NLS with rough and determined initial data. This is a continuing work of [2, 3], where the authors constructed the incoming and outgoing waves for the linear Schrödinger flow, and obtained global well-posedness and scattering with suitable scaling supercritical data in the energy subcritical case $p < \frac{4}{d-2}$ and supercritical case $p > \frac{4}{d-2}$, while the energy critical case when $p = \frac{4}{d-2}$ remains unsolved.

Next, we recall the the definitions of the incoming and outgoing components of functions for $d = 3, 4, 5$ introduced in [2].

Definition 1.1 (Deformed Fourier transformation). *Let $3 \leq d \leq 5$. Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and let $f \in \mathcal{S}(\mathbb{R}^d)$ with $|x|^\beta f \in L^1_{loc}(\mathbb{R}^d)$. Define*

$$\mathcal{F}f(\xi) = |\xi|^\alpha \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} |x|^\beta f(x) dx.$$

The inverse transform is defined by

Definition 1.2. *Let $3 \leq d \leq 5$. Let $f \in \mathcal{S}(\mathbb{R}^d)$, $|x|^\beta f \in L^1_{loc}(\mathbb{R}^d)$ and $|\xi|^{-\alpha}\mathcal{F}f \in L^1_{loc}(\mathbb{R}^d)$, then for any $x \in \mathbb{R}^d \setminus \{0\}$,*

$$f(x) = |x|^{-\beta} \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} |\xi|^{-\alpha} \mathcal{F}f d\xi.$$

Remark 1.3. If $f \in \mathcal{S}(\mathbb{R}^d)$ is radial, and $\beta > -d$, then it is easy to see that $|x|^\beta f \in L^1_{loc}(\mathbb{R}^d)$. Similarly, if $\alpha < d$, we can prove that $|\xi|^{-\alpha}\mathcal{F}f \in L^1_{loc}(\mathbb{R}^d)$.

We now give the radial version of the above deformed Fourier transform and its inverse transform:

$$\mathcal{F}f(\rho) = \rho^\alpha \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-2\pi i \rho r sin\theta} cos^{d-2}\theta r^{\beta+d-1} f(r) d\theta dr, \tag{1.5}$$

and

$$f(r) = r^{-\beta} \int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\pi i \rho r sin\theta} cos^{d-2}\theta \rho^{\beta+d-1} \mathcal{F}f(\rho) d\theta d\rho. \tag{1.6}$$

We denote

$$J(r) = \int_0^{\frac{\pi}{2}} e^{2\pi i r sin\theta} cos^{d-2}\theta d\theta,$$

and

$$K(r) = \chi_{\geq 2}(r)\Big[-\frac{1}{2\pi i r} + \frac{d-3}{(2\pi i r)^3}\Big], \quad d = 3, 4, 5.$$

Next, we define the incoming and outgoing components of functions.

Definition 1.4 (Incoming and outgoing components). *Let $3 \leq d \leq 5$. Let $\alpha < d$, $\beta > -d$, and the function $f \in L^1_{loc}(\mathbb{R}^d)$ is radial. Define the outgoing component of f as*

$$f_{out}(r) = r^{-\beta} \int_0^{+\infty} \big(J(\rho r) - K(\rho r)\big) \rho^{-\alpha+d-1} \mathcal{F}f(\rho) d\rho,$$

the incoming component of f as

$$f_{in}(r) = r^{-\beta} \int_0^{+\infty} \big(J(-\rho r) + K(\rho r)\big) \rho^{-\alpha+d-1} \mathcal{F}f(\rho) d\rho.$$

Remark 1.5. Throughout the present paper, we set the parameters in Definition 1.4 as

$$\alpha = 0, \text{ and } \beta = \frac{d-1}{2} - 2. \tag{1.7}$$

Moreover, from the process of the above definitions, we have $f_{in}(r) + f_{out}(r) = f(r)$.

The definition of incoming and outgoing wave can be compared to the in/out decomposition in scattering theory, which is inspired by the Enss method. In simple words, the in/out decomposition is to project some operator A on the positive or negative real axis. For example, A takes the form of $2A = x\cdot p + p\cdot x$ in Mourre's work [30], and $A = f(x)x\cdot p + p\cdot f(x)x$ in Sigal and Soffer's paper [35], where $p = -i\nabla_x$ and $f(x) \sim 1/|x|$ for $|x| \gtrsim 1$.

The new in/out decomposition for Schrödinger equations was introduced by Tao [37] in terms of the spherical waves for 3D case. Then, in [2], the authors give another definition for radial data through some kind of deformed Fourier transform, namely Definition 1.4. We refer the readers to the introduction in [2] for more complete explanation of the incoming/outgoing theory.

Note that the decomposition has singularity near the origin in the physical space. Then, we make the following modification to remove the singularity. In the sequel, we will assume that the initial data is smoother near the origin. Moreover, the low frequency part of the initial data is smooth naturally. Then, we make the incoming and outgoing decomposition of the remaining "real" rough part of the initial data, that is the high frequency and the away-from-origin parts.

Definition 1.6 (Modified incoming and outgoing components)**.** *Let $3 \le d \le 5$, and the radial function $f \in \mathcal{S}(\mathbb{R}^d)$. Define the modified outgoing component of f as*

$$f_+ = \frac{1}{2}P_{\le 1}f + \frac{1}{2}P_{\ge 1}\chi_{\le \varepsilon_0}f + (P_{\ge 1}\chi_{\ge \varepsilon_0}f)_{out};$$

the modified incoming component of f as

$$f_- = \frac{1}{2}P_{\le 1}f + \frac{1}{2}P_{\ge 1}\chi_{\le \varepsilon_0}f + (P_{\ge 1}\chi_{\ge \varepsilon_0}f)_{in}.$$

By the definition, we remark that $f = f_+ + f_-$. Moreover, if f is rough, then at least one of f_+ and f_- is rough.

The main observation in [2] is that the decomposition allows us to cut the linear flow $e^{it\Delta}f$ into $e^{it\Delta}f_+$ and $e^{it\Delta}f_-$ such that up to a smooth part, the former moves forward in time and the latter moves backward in time, and the speed depends on the frequency which is faster for rougher data. This leads to some smoothing effect for the linear flow. Then, the authors obtained positively (or negatively) global well-posedness when the initial data is f_+ (or f_-) for the defocusing energy-subcritical NLS, namely when $p < \frac{4}{d-2}$, with a class of initial data in some scaling supercritical Sobolev space. This framework provides another approach to study the well-posedness for NLS in the supercritical space. The advantage of the setting is that the initial data is deterministic, and may be verified in numerical application.

However, the analysis on nonlinearity in the energy-subcritical case can not be applied directly to the energy-critical case. In this paper, we further consider and extend the incoming/outgoing theory in [2] to the 3D energy-critical case.

The following is our main result.

Theorem 1.7. *Let $s_0 \in (\frac{5}{6}, 1)$, $\alpha = 0$, and $\beta = -1$. Suppose that f is a radial function and there exists $\varepsilon_0 > 0$ such that*

$$\chi_{\leq \varepsilon_0} f \in H^1(\mathbb{R}^3), \quad (1 - \chi_{\leq \varepsilon_0}) f \in H^{s_0}(\mathbb{R}^3).$$

Then the solution u to the equation (1.1) with the initial data

$$u_0 = f_+ \quad (or \quad u_0 = f_-)$$

is global forward (or backward) in time. Moreover, there exists $u^+ \in H^{s_0}(\mathbb{R}^3)$ (or $u^- \in H^{s_0}(\mathbb{R}^3)$), such that

$$\lim_{t \to +\infty} \|u(t) - e^{it\Delta} u^+\|_{H^1(\mathbb{R}^3)} \to 0 \quad (or \lim_{t \to -\infty} \|u(t) - e^{it\Delta} u^-\|_{H^1(\mathbb{R}^3)} \to 0).$$

We make several remarks regarding the above result.

Remark 1.8.
(1) We are able to construct the global solutions for 3D defocusing energy-critical NLS with a class of data in some supercritical space. In fact, if f is not in $H^1(\mathbb{R}^3)$, at least one of f_+ and f_- is not in $H^1(\mathbb{R}^3)$.
(2) Previously, in the energy supercritical case [3], the related result requires some further size restrictions on certain parts of initial data. Comparably, our result in Theorem 1.7 does not rely on any size restriction.
(3) Compared to the energy subcritical case studied in [2], the main new difficulty for our problem is that we are lack of the spacetime estimates of the nonlinear components.
(4) The key common point between our setting and the randomized data problem [9, 19, 27, 33, 34] is that the well-chosen data leads to some improved linear estimates. However, our case is more limited: we do not have any weighted estimate, and the smoothing effect is not good enough either.
(5) The improvement for linear flow is better for higher dimension. In fact, we can prove that away from the time origin, $\||\nabla|^{s+\frac{1}{2}} e^{it\Delta} f_\pm\|_{L_t^2 L_x^\infty} \lesssim \|f_\pm\|_{H_x^s}$, which has the same scaling as $\dot{H}_x^{s+(d+1)/2}$. In this paper, we only consider the 3D case such that the improvement is the weakest in the above sense.
(6) By rescaling, it suffices to prove Theorem 1.7 when $\varepsilon_0 = 1$.

Now, we discuss the new difficulties comparing to the previous results.

• *Lack of spacetime estimate.* The first key drawback of the energy critical case is that the spacetime estimate for the nonlinear remainder is limited. Recall that in the energy subcritical case [2], the nonlinear remainder possesses all the energy subcritical spacetime estimates, which covers all the integration exponents. This can be obtained using the facts that the equation is energy subcritical, the Morawetz estimate can give an intercritical estimate, and the energy of the nonlinear remainder is almost conserved.

However, in the energy critical case, the Morawetz estimate still gives an energy subcritical control. Since there is not any a priori energy supercritcal estimate, we cannot expect to obtain any energy critical global bound, and the only useful global spacetime estimate for the nonlinear part seems $L_{t,x}^8$. In this sense, the energy-critical problem is much harder than the subcritical one.

• *The linear improvement is weaker.* We also remark that the improved estimate is not good enough comparing to the previous randomized data problem. More precisely, various $L_t^q L_x^\infty$-estimates with $q \geq 1$ for the linear flow play the key role when controlling the energy increment.

We briefly recall such improved estimates in previous studies. First, the key ingredient in the 4D radial almost sure scattering result [19, 27] is that the linear flow enables some weighted estimate. In fact, in [20], the authors observed an radial Sobolev improvement for the functions with unit-scale frequency support. Then combining the local smoothing, for the Wiener type randomization f^ω and $0 \le \alpha < 1$,

$$\||x|^\alpha \nabla e^{it\Delta} f^\omega\|_{L_t^2 L_x^\infty(\mathbb{R}\times\mathbb{R}^4)} < +\infty, \text{ for almost every } \omega. \tag{1.8}$$

Second, for the 4D nonradial problem, the author in [34] combined the angular and unit-scale wave packet decomposition to obtain that for the randomized data f^ω,

$$\|\nabla e^{it\Delta} f^\omega\|_{L_t^1 L_x^\infty(\mathbb{R}\times\mathbb{R}^4)} < +\infty, \text{ for almost every } \omega, \tag{1.9}$$

which covers all the time integration in energy increment. Third, for the nonradial 3D and 4D almost sure scattering problem, the authors in [33] obtained some $L_t^2 L_x^\infty$-estimate with sufficient smoothing effect for modified Wiener randomization f^ω and $d = 3, 4$,

$$\|\Delta e^{it\Delta} f^\omega\|_{L_t^2 L_x^\infty(\mathbb{R}\times\mathbb{R}^d)} < +\infty, \text{ for almost every } \omega. \tag{1.10}$$

However, for the incoming and outgoing wave theory in [2], we only have for the forward in time part(similar estimate holds for backward case):

$$\|\nabla e^{it\Delta} f_+\|_{L_t^2 L_x^6([0,+\infty)\times\mathbb{R}^3)} < \infty. \tag{1.11}$$

This estimate works for the energy subcritical case due to more flexible nonlinear ones, but seems very difficult to finish the argument in critical problem.

Therefore, we use the fact that the linear flow becomes smoother away from the origin, and obtain that for any small $\delta > 0$,

$$\|\nabla e^{it\Delta} f_+\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} < \infty. \tag{1.12}$$

We remark that this improved estimate is still weaker than the randomized data problem, but we have an advantage in our case that if the initial data is radial, then the solution keeps this symmetry. Note that in [19, 27], after the Wiener randomization, the data and solution does not keep the radial property any more.

1.2. **The key ingredients in the proofs.** In this subsection, we describe the key ingredients of the proof for Theorem 1.7. The argument is inspired by the related scattering theory for the randomized supercritical data problem, see [9, 19, 27, 39], while our main innovation is to use weaker improved linear estimates to obtain the desired conclusions, as described above. Next, we discuss our method in more details.

- *A conditional perturbation theory.* We consider the perturbation equation

$$\begin{cases} i\partial_t w + \Delta w = |v+w|^4(v+w), \\ w(0,x) = w_0(x) \in H^1(\mathbb{R}^3). \end{cases}$$

Given the maximal lifespan $[0, T^*)$, under the hypothesis of

$$\text{(a): } v \in S(I); \quad \text{(b): } w \in L_t^\infty([0,T^*); H^1),$$

where $S([0,T^*))$ is some suitably defined spacetime norm at $\dot{H}^1$ level, see (3.2) below, we establish the spacetime estimate that

$$\|w\|_{S([0,T^*))} < +\infty.$$

Here the bound is independent of T^*. This is a general theory on the scattering of the solution to a perturbation equation, which is available in our case by splitting u into a linear part v and a solution w of a perturbation equation.

Since the equation is energy critical, the local existence time depends on the profile of initial data, rather than its norm. Therefore, to prove the general theory, we will adopt the perturbation theory to approximate w by the solution of the original energy-critical NLS. The key ingredient of perturbation theory is to construct a suitable auxiliary space $S(I)$ that can close the estimates for nonlinear interaction, and meanwhile it matches the smooth effect of the linear flow benefited from the incoming/outgoing decomposition.

- *Supercritical spacetime estimates of the linear flow.* In this part, we check the hypothesis (a) above. Noting that u_0 merely belongs to H^{s_0}, $\frac{5}{6} < s_0 < 1$, the $S(I)$-estimates for linear solution are supercritical. Hence, we need to obtain enough smoothing effect from the incoming and outgoing decomposition by the delicate phase-space analysis method in [2]. However, the estimates presented in [2] are not sufficient in our case. Thus, we prove some finer spacetime estimates. In particular, these estimates imply that the solution becomes better when the time is away from the origin, which is crucial in the proof of the a priori estimate.

- *A priori estimate.* In this step, we shall check the hypothesis (b) above, that is to obtain the a priori estimate of the solution w to the perturbation equation in H^1 while the initial data is only in H^{s_0}, $\frac{5}{6} < s_0 < 1$. We first make a decomposition of initial data, such that w_0 is in H^1_x. Then, the proof of the a priori estimate is based on the Morawetz estimates, energy estimates and bootstrap argument. In the last step, we can obtain the $\|\nabla v\|_{L^2_t L^\infty_x}$-estimate for the linear flow v, which is sufficient for controlling the energy increment. However, such estimate is only available when t is away from the origin. Therefore, we consider the short time and long time cases, separately.

As for the short time interval, the estimates for linear solution $e^{it\Delta}u_0$ is not smooth enough, but it still enables some $\dot{H}^1_x$-critical spacetime estimates while the initial data u_0 itself is below the energy regularity. Thus, applying the local theory, we expect that the original solution u also has some $\dot{H}^1_x$ level estimates, and then the energy increment can be bounded suitably. Finally, we remark that the lower bound $\frac{5}{6}$ of the regularity condition is to ensure that the local $\dot{H}^1_x$-critical estimates hold.

1.3. **Organization of the paper.** The rest of the paper is organized as follows. In Section 2, we give some basic notations and lemmas that will be used throughout this paper. In Section 3, we give a general theory about the existence of the solution to a perturbation equation under suitable a priori hypothesis. In Section 4, we give the framework for proof of Theorem 1.7. In Section 5, We obtain the uniform bounded of linear solution v in the auxiliary space $Y(I)$. In Section 6, we prove the a priori estimate of solution w to the perturbation equation in H^1.

2. Preliminary

2.1. **Notations.** We write $X \lesssim Y$ or $Y \gtrsim X$ to denote the estimate $X \le CY$ for some constant $C > 0$. Throughout the whole paper, the letter C will denote different positive constants which are not important in our analysis and may vary line by line. If C depends upon some additional parameters, we will indicate this with subscripts; for example, $X \lesssim_a Y$

denotes the $X \le C(a)Y$ assertion for some $C(a)$ depending on a. The notation $a+$ denotes $a+\epsilon$ for some small ϵ. We use the following norms to denote the mixed spaces $L_t^q L_x^r$, that is

$$\|u\|_{L_t^q L_x^r} = \Big(\int \|u\|_{L_x^r}^q dt\Big)^{\frac{1}{q}}.$$

When $q=r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$. We use $\chi_{\le a}$ for $a \in \mathbb{R}^+$ to be the smooth function

$$\chi_{\le a}(x) = \begin{cases} 1, |x| \le a, \\ 0, |x| \ge \dfrac{11}{10}a. \end{cases}$$

Moreover, we denote $\chi_{\ge a} = 1 - \chi_{\le a}$ and $\chi_{a\le \cdot \le b} = \chi_{\le b} - \chi_{\le a}$. We denote $\chi_a = \chi_{\le 2a} - \chi_{\le a}$ for short.

For each number $N>0$, we define the Fourier multipliers $P_{\le N}, P_{>N}, P_N$ as

$$\widehat{P_{\le N} f}(\xi) := \chi_{\le N}(\xi)\hat{f}(\xi),$$
$$\widehat{P_{>N} f}(\xi) := \chi_{>N}(\xi)\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := \chi_N(\xi)\hat{f}(\xi),$$

and similarly $P_{\ge N}, P_{<N}$. These multipliers are usually used when N are *dyadic numbers* (that is, of the form 2^k for some integer k).

2.2. **Basic lemmas.** In this section, we state some useful lemmas which will be used in our later sections. Firstly, we recall the well-known Strichartz estimates.

Lemma 2.1. *(Strichartz's estimates, see [10, 21, 23, 36]) Let $I \subset \mathbb{R}$ be a time interval. For all admissible pairs $(q_j, r_j), j = 1, 2$, satisfying*

$$2 \le q_j, r_j \le \infty, \quad \frac{2}{q_j} + \frac{d}{r_j} = \frac{d}{2}, \quad \text{and} \quad (q, r, d) \neq (2, \infty, 2),$$

then the following statements hold:

$$\|e^{it\Delta} f\|_{L_t^{q_j} L_x^{r_j}(I\times\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}; \tag{2.1}$$

and

$$\Big\| \int_0^t e^{i(t-s)\Delta} F(s) ds \Big\|_{L_t^{q_1} L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q_2'} L_x^{r_2'}(I\times\mathbb{R}^d)}, \tag{2.2}$$

where $\frac{1}{q_2} + \frac{1}{q_2'} = \frac{1}{r_2} + \frac{1}{r_2'} = 1$.

We also need the following radial Sobolev inequality.

Lemma 2.2. *(Corollary A.3 in [40])Let α, p, q, s be the parameters which satisfy*

$$\alpha > -\frac{d}{q}, \quad \frac{1}{q} \le \frac{1}{p} \le \frac{1}{q} + s, \quad 1 \le p, q \le \infty, \quad 0 < s < d$$

with

$$\alpha + s = d(\frac{1}{p} - \frac{1}{q}).$$

Moreover, at most one of the equalities holds:

$$p = 1, \quad p = \infty, \quad q = 1, \quad q = \infty, \quad \frac{1}{p} = \frac{1}{q} + s.$$

Then for any radial function u,

$$\big\||x|^{\alpha}u\big\|_{L^q(\mathbb{R}^d)} \lesssim \||\nabla|^s u\|_{L^p(\mathbb{R}^d)}.$$

The following result is the Hardy inequality.

Lemma 2.3. *Let $1 < p < d$. Then,*

$$\big\||x|^{-1}u\big\|_{L^p_x(\mathbb{R}^d)} \lesssim \|\nabla u\|_{L^p_x(\mathbb{R}^d)}.$$

3. A general perturbation theory

In this section, we set up a general theory to give a sufficient condition for the existence of the solution to the following perturbation equation

$$\begin{cases} i\partial_t w + \Delta w = |w+v|^4(w+v), \\ w(0,x) = w_0(x). \end{cases} \tag{3.1}$$

Before giving the main results, we need some auxiliary spaces. We denote the space $S(I)$ with the corresponding norm as follows

$$\|u\|_{S(I)} := \|\nabla u\|_{L^2_t L^6_x(I\times\mathbb{R}^3)} + \|u\|_{L^8_t L^{12}_x(I\times\mathbb{R}^3)}. \tag{3.2}$$

In this section, our main result is as follows.

Proposition 3.1. *Let $0 \in I \subset \mathbb{R}^+$ and suppose that there exists a solution $w \in C(I; \dot{H}^1_x(\mathbb{R}^3))$ of (3.1). Assume that there exists a constant $C_0 > 0$, such that*

$$\|v\|_{S(\mathbb{R}^+)} \le C_0, \tag{3.3}$$

and there exists a constant $E_0 > 0$, such that

$$\sup_{t\in I}\|w(t)\|_{\dot{H}^1_x(\mathbb{R}^3)} \le E_0. \tag{3.4}$$

Then, there exists some $C = C(C_0, E_0) > 0$ (independent of I) such that

$$\|w\|_{S(I)} \le C(C_0, E_0).$$

To prove Proposition 3.1, we shall use the perturbation theory, which shows that the solution w of equation (3.1) can stay close to the solution $\widetilde{w}$ of the original energy critical equation:

$$\begin{cases} i\partial_t \widetilde{w} + \Delta \widetilde{w} = |\widetilde{w}|^4\widetilde{w}, \\ \widetilde{w}(0,x) = \widetilde{w}_0(x), \end{cases} \tag{3.5}$$

where $\widetilde{w} = \widetilde{w}(t,x) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is a complex-valued function. From the result in [12], we have that if $\widetilde{w}_0 \in \dot{H}^1_x(\mathbb{R}^3)$, then the equation (3.5) is globally well-posed and scatters, and $\widetilde{w}$ satisfies

$$\|\nabla\widetilde{w}\|_{S^0(\mathbb{R}^+)} + \|\widetilde{w}\|_{S(\mathbb{R}^+)} \le C(\|\widetilde{w}_0\|_{\dot{H}^1_x(\mathbb{R}^3)}). \tag{3.6}$$

Let $g(t,x) := w(t,x) - \widetilde{w}(t,x)$, then g satisfies the following equation:

$$\begin{cases} i\partial_t g + \Delta g = F(g+v, \widetilde{w}), \\ g(0,x) = w_0(x) - \widetilde{w}_0(x). \end{cases} \tag{3.7}$$

Where we denote

$$F(g+v,\widetilde{w}) = |g+v+\widetilde{w}|^4(g+v+\widetilde{w}) - |\widetilde{w}|^4\widetilde{w}.$$

Now, we give the estimate of the nonlinear term F.

Lemma 3.2. *Let $I \subset \mathbb{R}^+$ be some compact interval and $0 \in I$. Then*

$$\begin{aligned}\|\nabla F(g+v,\widetilde{w})\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} \lesssim &(\|g\|_{S(I)} + \|v\|_{S(I)})(\|g\|_{S(I)}^4 + \|v\|_{S(I)}^4 + \|\widetilde{w}\|_{S(I)}^4) \\ &+ \|\widetilde{w}\|_{S(I)}(\|g\|_{S(I)}^4 + \|v\|_{S(I)}^4).\end{aligned}$$

Proof. By the definition of $N^0(I)$ and Hölder's inequality, we have

$$\begin{aligned}\|\nabla u_1 u_2^4\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} &\lesssim \|\nabla u_1\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)} \|u_2\|_{L_t^8 L_x^{12}(I\times\mathbb{R}^3)}^4 \\ &\lesssim \|u_1\|_{S(I)}\|u_2\|_{S(I)}^4.\end{aligned}$$

Moreover, for the term $F(g+v,\widetilde{w})$, we have the pointwise estimates,

$$\begin{aligned}|\nabla F(g+v,\widetilde{w})| &= |\nabla(|g+v+\widetilde{w}|^4(g+v+\widetilde{w}) - |\widetilde{w}|^4\widetilde{w})| \\ &\lesssim (|\nabla g| + |\nabla v|)(|g|^4 + |v|^4 + |\widetilde{w}|^4) + |\nabla\widetilde{w}|(|g|^4 + |v|^4).\end{aligned}$$

Hence, this lemma follows by combining the above estimates. □

Then our perturbation result regarding to g is as follows.

Lemma 3.3. *Let $I \subset \mathbb{R}^+$, $0 \in I$ and $E_0 > 0$. Let $\widetilde{w} \in C(I; \dot{H}_x^1(\mathbb{R}^3))$ be the solution of (3.5) on I with*

$$\widetilde{w}_0 = w_0.$$

Then, there exists $\eta_1 = \eta_1(E_0)$ with the following properties. Assume that

$$\|v\|_{S(I)} \le \eta_1, \tag{3.8}$$

and

$$\|w_0\|_{\dot{H}_x^1(\mathbb{R}^3)} \le E_0, \tag{3.9}$$

then there exists a solution $g \in C(I; \dot{H}_x^1(\mathbb{R}^3))$ of (3.7) with initial data $g_0 = 0$ satisfying

$$\|g\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^3)} + \|g\|_{S(I)} \le C(E_0,\eta_1)\eta_1.$$

Proof. By the assumption of (3.9), and (3.6), we have

$$\|\widetilde{w}\|_{S(I)} \le C(E_0). \tag{3.10}$$

Fix some absolutely small $0 < \eta_2 \ll 1$, then we can split $I = \cup_{j=1}^J I_j$, $I_j = [t_{j-1}, t_j]$, $t_0 = 0$, such that

$$\frac{1}{2}\eta_2 \le \|\widetilde{w}\|_{S(I_j)} \le \eta_2. \tag{3.11}$$

Then $J = J(E_0,\eta_2)$ is finite. We also take some $\eta_1 \le \eta_2$ that will be decided later. The proof of this lemma will now be accomplished in two steps.

Step 1. In this step, we are going to prove that under the assumption that for any $j \ge 1$,

$$\|g(t_{j-1})\|_{\dot{H}_x^1(\mathbb{R}^3)} \le \eta_2, \tag{3.12}$$

then there exists some suitable constant $B_0 > 1$ independent of j, η_1, and η_2, such that

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_j\times\mathbb{R}^3)} + \|g\|_{S(I_j)} \le B_0(\|g(t_{j-1})\|_{\dot{H}_x^1(\mathbb{R}^3)} + \eta_1). \tag{3.13}$$

To prove (3.13), first recall the Duhamel formula to the equation (3.7),

$$g(t) = e^{i(t-t_{j-1})\Delta} g(t_{j-1}) - i \int_{t_{j-1}}^{t} e^{i(t-s)\Delta} F(g+v, \widetilde{w})(s) ds. \tag{3.14}$$

Using Duhamel formula (3.14), Lemma 2.1, and Sobolev inequality, we have

$$\begin{aligned} \|g\|_{L_t^\infty \dot{H}_x^1(I_j\times\mathbb{R}^3)} + \|g\|_{S(I_j)} \lesssim & \|g(t_{j-1})\|_{\dot{H}_x^1(\mathbb{R}^3)} + \Big\| \int_{t_{j-1}}^{t} e^{i(t-s)\Delta} \nabla F(g+v,\widetilde{w})(s)ds \Big\|_{L_t^\infty L_x^2 \cap L_t^8 L_x^{\frac{12}{5}}(I_j\times\mathbb{R}^3)} \\ \lesssim & \|g(t_{j-1})\|_{\dot{H}_x^1(\mathbb{R}^3)} + \big\|\nabla F(g+v,\widetilde{w})\big\|_{L_t^1 L_x^2(I_j\times\mathbb{R}^3)}. \end{aligned} \tag{3.15}$$

Moreover, by the Lemma 3.2, (3.8), and (3.11), we have

$$\begin{aligned} \big\|\nabla F(g+v,\widetilde{w})\big\|_{L_t^1 L_x^2(I_j\times\mathbb{R}^3)} \lesssim & (\|g\|_{S(I_j)} + \eta_1)(\|g\|_{S(I_j)}^4 + \eta_1^4 + \eta_2^4) + \eta_2(\|g\|_{S(I_j)}^4 + \eta_1^4) \\ \lesssim & (\eta_2^4 + \|g\|_{S(I_j)}^4)\|g\|_{S(I_j)} + \eta_1\eta_2^4. \end{aligned}$$

Further by (3.15) implies that

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_j\times\mathbb{R}^3)} + \|g\|_{S(I_j)} \lesssim \|g(t_{j-1})\|_{\dot{H}_x^1(\mathbb{R}^3)} + (\eta_1^4 + \|g\|_{S(I_j)}^4)\|g\|_{S(I_j)} + \eta_1. \tag{3.16}$$

Noting the assumptions of smallness conditions (3.12), by (3.16) and the bootstrap method, we can obtain

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_j\times\mathbb{R}^3)} + \|g\|_{S(I_j)} \lesssim \|g(t_{j-1})\|_{\dot{H}_x^1(\mathbb{R}^3)} + \eta_1. \tag{3.17}$$

Hence, we can choose suitable constant $B_0 > 1$, such that (3.13) holds.

Step 2. Next, we shall to get the desired results through induction. To start with, we take the parameter η_1 such that

$$J B_0^J \eta_1 \leq \eta_2. \tag{3.18}$$

Therefore, η_1 depends only on E_0 and the absolute small constant η_2.

First, we consider the subinterval $I_1 = [0, t_1]$. In this case, $g_0 = w_0 - \widetilde{w}_0 = 0$. Thus, we have

$$\|g_0\|_{\dot{H}^1(\mathbb{R}^3)} = 0.$$

By the first step, we get the existence of g on I_1, and

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_1\times\mathbb{R}^3)} + \|g\|_{S(I_1)} \leq B_0\eta_1. \tag{3.19}$$

Secondly, we consider the subinterval $I_2 = [t_1, t_2]$. In this case, by (3.19) and (3.18),

$$\|g(t_1)\|_{\dot{H}_x^1(\mathbb{R}^3)} \leq B_0\eta_1 \leq \eta_2.$$

The above estimate satisfies the assumption (3.12). Thus, by *Step* 1, we have

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_2\times\mathbb{R}^3)} + \|g\|_{S(I_2)} \leq B_0(B_0\eta_1 + \eta_1) \leq 2B_0^2\eta_1. \tag{3.20}$$

Now we start the induction procedure from I_2. We aim to prove that for any $j = 1, 2, \ldots, J$,

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_j\times\mathbb{R}^3)} + \|g\|_{S(I_j)} \leq jB_0^j\eta_1. \tag{3.21}$$

For $j = 2$ case, we have that the above estimate holds. Next, for $j = k$ case, we suppose the above estimate (3.21) holds, that is

$$\|g\|_{L_t^\infty \dot{H}_x^1(I_k\times\mathbb{R}^3)} + \|g\|_{S(I_k)} \leq kB_0^k\eta_1. \tag{3.22}$$

Hence, it suffices to prove the $j = k+1$ case. Using (3.22) and (3.18), we obtain

$$\|g(t_k)\|_{\dot H^1_x(\mathbb{R}^3)} \le kB_0^k\eta_1 \le JB_0^J\eta_1 \le \eta_2. \tag{3.23}$$

By *Step* 1, we obtain the existence of g on I_{k+1}, and

$$\|g\|_{L_t^\infty \dot H^1_x(I_{k+1}\times\mathbb{R}^3)} + \|g\|_{S(I_{k+1})} \le B_0(kB_0^k\eta_1+\eta_1) \le (k+1)B_0^{k+1}\eta_1. \tag{3.24}$$

Hence, by induction, we have (3.21) holds for any $j = 1,2,\dots,J$.

Then, we have the existence of g on the whole interval I, and for any j,

$$\|g\|_{L_t^\infty \dot H^1_x(I_j\times\mathbb{R}^3)} + \|g\|_{S(I_j)} \le jB_0^j\eta_1 \le \eta_2.$$

Summing this over all subintervals I_j, we complete the proof of this lemma. □

Finally, we are in a position to prove Proposition 3.1. Although the perturbation theory is a local result, we can apply it by iteration to proving that the global spacetime norms of w is uniformly bounded.

Proof of Proposition 3.1. For any fixed $t_0 \in I$, we consider the following equation

$$\begin{cases} i\partial_t\widetilde w + \Delta\widetilde w = |\widetilde w|^4\widetilde w, \\ \widetilde w(t_0,x) = w(t_0,x). \end{cases} \tag{3.25}$$

By (3.6), we have that there exists a global solution $\widetilde w(t,x) = \widetilde w^{(t_0)}(t,x)$ of the equation (3.25) with

$$\|\widetilde w^{(t_0)}\|_{S(\mathbb{R}^+)} \le C(\|w(t_0)\|_{\dot H^1_x(\mathbb{R}^3)}). \tag{3.26}$$

Now, let $\eta_1 = \eta_1(E_0)$ be defined as in Lemma 3.3. By the assumption (3.3), we can split $I = \cup_{l=1}^L \widetilde I_l$, $\widetilde I_l = [\tau_{l-1},\tau_l]$, $\tau_0 = 0$, such that

$$\frac12\eta_1 \le \|v\|_{S(\widetilde I_l)} \le \eta_1.$$

Then $L(C_0,\eta_1)$ is finite. We consider subinterval $\widetilde I_1$ firstly. By (3.4), we can take $t_0 = \tau_0 = 0$ for the equation (3.25), and clearly have

$$\|\widetilde w(0)\|_{\dot H^1_x(\mathbb{R}^3)} = \|w(0)\|_{\dot H^1_x(\mathbb{R}^3)} \le E_0.$$

Then, using Lemma 3.3 on $\widetilde I_1$, we obtain the existence of $w \in C\big(\widetilde I_1; \dot H^1_x(\mathbb{R}^3)\big)$.

Next, we use the a priori energy estimate assumption for w, namely (3.4) to extending the perturbation argument. Now, for $\widetilde I_2$, we can take $t_0 = \tau_1$ for the equation (3.25). Using (3.4),

$$\sup_{t\in[0,\tau_1]} \|w(t)\|_{\dot H^1_x(\mathbb{R}^3)} \le E_0.$$

Particularly,

$$\|\widetilde w(\tau_1)\|_{\dot H^1_x(\mathbb{R}^3)} = \|w(\tau_1)\|_{\dot H^1_x(\mathbb{R}^3)} \le E_0.$$

Then, we can apply Lemma 3.3 on $\widetilde I_2$ after translation in t from the starting point τ_1, and obtained the well-posedness on $\widetilde I_2$.

Inductively, under the assumptions of Proposition 3.1, we can obtain the existence of $w \in C\big(\widetilde{I}_l; \dot{H}^1_x(\mathbb{R}^3)\big)$ for $l = 1, 2, \dots, L$. Moreover, from Lemma 3.3 and (3.6), we have

$$\|w\|_{S(I)} \leq C(C_0, E_0). \tag{3.27}$$

Hence, we finish the proof of the proposition. □

4. Framework of the proof

4.1. **Linear and nonlinear decomposition.**

Now, we turn to the proof of Theorem 1.7. We only consider the energy critical NLS (1.1) with the initial data $u_0 = f_+$, since it can be treated in the same way when the initial data $u_0 = f_-$. We first recall Definition 1.6 that

$$f_+ = \frac{1}{2} P_{\leq 1} f + \frac{1}{2} P_{\geq 1} \chi_{\leq 1} f + (P_{\geq 1} \chi_{\geq 1} f)_{out}.$$

Fixing $\delta_0 > 0$, we take $N = N(\delta_0) > 0$, such that

$$\|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)} \leq \delta_0. \tag{4.1}$$

Denote $v_0 := (P_{\geq N} \chi_{\geq 1} f)_{out}$, and $w_0 := \frac{1}{2} P_{\leq 1} f + \frac{1}{2} P_{\geq 1} \chi_{\leq 1} f + (P_{1 \leq \cdot \leq N} \chi_{\geq 1} f)_{out}$. Then we split the solution u of (1.1) as $u = v + w$, where

$$v = e^{it\Delta} v_0,$$

and w satisfies the equation

$$\begin{cases} i\partial_t w + \Delta w = |w + v|^4 (w + v), \\ w(0, x) = w_0(x). \end{cases} \tag{4.2}$$

Now we need the following two hypotheses: assume that there exist constants $C_0, E_0 > 0$,

(H1)

$$\|v\|_{S(\mathbb{R}^+)} \leq C_0,$$

(H2) For any $0 \in I \subset \mathbb{R}^+$, if $w \in C(I; \dot{H}^1_x(\mathbb{R}^3))$, then

$$\sup_{t \in I} \|w(t)\|_{\dot{H}^1_x(\mathbb{R}^3)} \leq E_0.$$

Then, under the above two hypotheses, the general theory in the third section is available for our case, we can obtain that the solution w to the above perturbation equation is global in $\dot{H}^1$. Hence, it suffices to verify the above two hypotheses (H1) and (H2).

4.2. **Proof of Theorem 1.7 under the hypotheses (H1) and (H2).**

We are now in a position to give the proof of Theorem 1.7.

Proof. First of all, by Proposition 3.1, under the hypotheses (H1) and (H2), we obtain the global existence of w in the forward time and

$$\|w\|_{S(\mathbb{R}^+)} \leq C(C_0, E_0). \tag{4.3}$$

Next, we prove the scattering statement in Theorem 1.7. Set

$$u^+ = f_+ - i \int_0^{+\infty} e^{-is\Delta} (|u|^4 u)(s) ds.$$

Then we have

$$u(t) - e^{it\Delta}u^+ = i\int_t^{+\infty} e^{i(t-s)\Delta}(|u|^4u)(s)ds.$$

Now, we claim that $u^+ - f_+ \in H^1$ and

$$\|u(t) - e^{it\Delta}u^+\|_{H^1_x(\mathbb{R}^3)} \to 0, \quad as \quad t \to +\infty.$$

Indeed, it is reduced to prove that

$$\Big\| \int_t^{+\infty} e^{i(t-s)\Delta}(|u|^4u)(s)ds\Big\|_{H^1_x(\mathbb{R}^3)} \to 0, \quad as \quad t \to +\infty. \tag{4.4}$$

First, by Strichartz's estimate (2.2) and Hölder's inequality, we have

$$\begin{aligned}\Big\| \int_t^{+\infty} e^{i(t-s)\Delta}(|u|^4u)(s)ds\Big\|_{\dot H^1_x(\mathbb{R}^3)} &\lesssim \|\nabla(|u|^4u)\|_{L^1_tL^2_x([t,+\infty)\times\mathbb{R}^3)}\\ &\lesssim \|\nabla u\|_{L^2_tL^6_x([t,+\infty)\times\mathbb{R}^3)}\|u\|^4_{L^8_tL^{12}_x([t,+\infty)\times\mathbb{R}^3)}. \end{aligned}\tag{4.5}$$

Moreover, by (H1) and (4.3), we have

$$\begin{aligned}\|\nabla u\|_{L^2_tL^6_x([t,+\infty)\times\mathbb{R}^3)} + \|u\|_{L^8_tL^{12}_x([t,+\infty)\times\mathbb{R}^3)} &\lesssim \|\nabla w\|_{L^2_tL^6_x([t,+\infty)\times\mathbb{R}^3)} + \|w\|_{L^8_tL^{12}_x([t,+\infty)\times\mathbb{R}^3)} + \|v\|_{S([t,+\infty))}\\ &\le C(C_0, E_0). \end{aligned}\tag{4.6}$$

Combining (4.5) with (4.6), we get

$$\lim_{t\to+\infty}\Big\| \int_t^{+\infty} e^{i(t-s)\Delta}(|u|^4u)(s)ds\Big\|_{\dot H^1_x(\mathbb{R}^3)} = 0.$$

Next, similarly as above, we have

$$\begin{aligned}\Big\| \int_t^{+\infty} e^{i(t-s)\Delta}(|u|^4u)(s)ds\Big\|_{L^2_x(\mathbb{R}^3)} &\lesssim \big\||u|^4u\big\|_{L^2_tL^{\frac65}_x([t,+\infty)\times\mathbb{R}^3)}\\ &\lesssim \|u\|^4_{L^8_tL^{12}_x([t,+\infty)\times\mathbb{R}^3)}\|u\|_{L^\infty_tL^2_x([t,+\infty)\times\mathbb{R}^3)}\\ &\to 0, \quad as \quad t\to+\infty.\end{aligned}$$

Hence, we obtain (4.4). This proves the scattering statement and thus finishes the proof of Theorem 1.7. □

5. Linear estimates

In this section, we give the proof of the validity of hypothesis (H1). Before this, we recall that $v = e^{it\Delta}(P_{\ge N}\chi_{\ge1}f)_{out}$, and $\chi_{\ge1}f \in H^{s_0}(\mathbb{R}^3)$.

Proposition 5.1. *Let $s_0 \in (\frac56, 1)$. Then there exists a constant $C_0 > 0$, such that*

$$\|v\|_{S(\mathbb{R}^+)} \le C_0.$$

First of all, we need the frequency restricted outgoing component of f as follows, for any fixed integer k

$$f_{out,k}(r) = r\int_0^{+\infty} \big(J(\rho r) - K(\rho r)\big)\chi_{2^k}(\rho)\rho^2\mathcal{F}f(\rho)d\rho;$$

correspondingly, the frequency restricted incoming component of f as follows,

$$f_{in,k}(r) = r\int_0^{+\infty} \big(J(-\rho r) + K(\rho r)\big)\chi_{2^k}(\rho)\rho^2 \mathcal{F}f(\rho)d\rho,$$

where $\mathcal{F}f$ is the deformed Fourier transformation in Definition 1.1 with $\alpha = 0$ and $\beta = -1$. Here we remark that $f_{out,k} + f_{in,k} = |x|P_{2^k}(|x|^{-1}f)$.

Moreover, by the definitions of $f_{out/in}$ and $f_{out/in,k}$,

$$f_{out/in}(r) = \sum_{k=-\infty}^{+\infty} f_{out/in,k}(r).$$

Correspondingly, for $k_0 \in \mathbb{Z}$, we denote

$$f_{out/in,\geq k_0}(r) = \sum_{k=k_0}^{k=+\infty} f_{out/in,k}(r); \quad f_{out/in,\leq k_0}(r) = \sum_{k=-\infty}^{k=k_0} f_{out/in,k}(r).$$

5.1. **Known results from [2].** Next, we recall some useful lemmas, see Proposition 3.11, Lemma 3.12, Proposition 4.1 and Proposition 4.3 in [2] for the proof.

Lemma 5.2. *Suppose that $f \in L^2(\mathbb{R}^3)$, then*

$$\|f_{out/in}\|_{L^2(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)}.$$

The next lemma shows that if a function f has high frequency $f = P_{2^k}f$, then its incoming/outgoing component will have almost the same frequency plus a smooth perturbation.

Lemma 5.3. *Let $k \geq 0$ be an integer. Suppose that $f \in L^2(\mathbb{R}^3)$, then*

$$(P_{2^k}(\chi_{\geq 1}f))_{out/in} = (P_{2^k}(\chi_{\geq 1}f))_{out/in,k-1\leq\cdot\leq k+1} + h_k,$$

where h_k satisfies the following estimate,

$$\|h_k\|_{H^2(\mathbb{R}^3)} \lesssim 2^{-10k}\|P_{2^k}(\chi_{\geq 1}f)\|_{L^2(\mathbb{R}^3)}.$$

Moreover,

$$\|\chi_{\leq \frac14}(P_{2^k}(\chi_{\geq 1}f))_{out/in,k-1\leq\cdot\leq k+1}\|_{H^2(\mathbb{R}^3)} \lesssim 2^{-2k}\|P_{2^k}(\chi_{\geq 1}f)\|_{L^2(\mathbb{R}^3)}.$$

The following result is the incoming/outgoing linear flow's estimate related to the inside region.

Lemma 5.4. *Let $k \geq 0$ be an integer. Then there exists $\delta > 0$, such that for any triple (γ, q, r) satisfying that*

$$q \geq 2, \quad r > 2, \quad 0 \leq \gamma \leq 1, \quad \frac{2}{q} + \frac{5}{r} < \frac{5}{2},$$

the following estimates holds,

$$\begin{aligned}&\big\||\nabla|^\gamma[\chi_{\leq \delta(1+2^k t)}e^{it\Delta}(\chi_{\geq \frac14}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1})]\big\|_{L^q_t L^r_x(\mathbb{R}^+\times\mathbb{R}^3)}\\ &\lesssim 2^{-(2-(\gamma-\frac{2}{q}-\frac{3}{r}))k}\|P_{2^k}(\chi_{\geq 1}f)\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

The same estimate holds when $e^{it\Delta}$ and $_{out}$ are replaced by $e^{-it\Delta}$ and $_{in}$, respectively.

The following result is the incoming/outgoing linear flow's estimate related to the outside region.

Lemma 5.5. *Let $k \geq 0$ be an integer. Moreover, let r, γ_1, γ_2, s be parameters satisfying*

$$r > 2, \quad \gamma_1 \geq 0, \quad \gamma_2 \geq 0, \quad s + \frac{1}{r} \geq \frac{1}{2}, \quad \gamma_1 + s = \frac{3}{2} - \frac{3}{r}.$$

Then for any $t > 0$,

$$\begin{aligned} &\big\| |\nabla|^{\gamma_2} [\chi_{\geq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1})] \big\|_{L_x^r(\mathbb{R}^3)} \\ &\quad \lesssim (1+2^k t)^{-\gamma_1} 2^{(\gamma_2+s+)k} \| P_{2^k}(\chi_{\geq 1} f)\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

The same estimate holds when $e^{it\Delta}$ and $_{out}$ are replaced by $e^{-it\Delta}$ and $_{in}$, respectively.

5.2. **Further estimates.** Now, we can obtain the following space-time estimates based on the above lemma.

Corollary 5.6. *Let (γ, q, r) be a triple satisfying*

$$\gamma \geq 0, \quad q \geq 1, \quad r > 2, \quad \frac{1}{q} < 1 - \frac{2}{r},$$

then for any $\delta > 0$, the following estimates hold,

$$\begin{aligned} &\big\| |\nabla|^{\gamma} [\chi_{\geq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1})] \big\|_{L_t^q L_x^r(\mathbb{R}^+ \times \mathbb{R}^3)} \\ &\quad \lesssim 2^{(-\frac{1}{q} - \frac{1}{r} + \frac{1}{2} + \gamma +)k} \| P_{2^k}(\chi_{\geq 1} f)\|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\big\| |\nabla|^{\gamma} [\chi_{\geq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1})] \big\|_{L_t^2 L_x^\infty([\delta,+\infty) \times \mathbb{R}^3)} \\ &\quad \lesssim_\delta 2^{(-\frac{1}{2} + \gamma +)k} \| P_{2^k}(\chi_{\geq 1} f)\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{5.2}$$

The same estimate holds when $e^{it\Delta}$ and $_{out}$ are replaced by $e^{-it\Delta}$ and $_{in}$, respectively.

Proof. (5.1) was proved by Corollary 4.4 in [2]. In addition, we also need the result (5.2).

From Lemma 5.5, we have

$$\begin{aligned} &\big\| |\nabla|^{\gamma} [\chi_{\geq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1})] \big\|_{L_t^2 L_x^\infty([\delta,+\infty) \times \mathbb{R}^3)} \\ &\quad \lesssim 2^{(\gamma + \frac{1}{2} +)k} \| (1+2^k t)^{-1} \|_{L_t^2([\delta,+\infty))} \| P_{2^k}(\chi_{\geq 1} f)\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Furthermore, we have

$$\| (1+2^k t)^{-1} \|_{L_t^2([\delta,+\infty))} \lesssim \delta^{-\frac{1}{2}} 2^{-k}.$$

Combining the above two estimates, we prove the corollary. □

Remark 5.7. Although the implicit constant depends on δ, we can also treat the parameter as a constant. Thus, in the following, we will omit the dependence on δ for short.

Next, we gather some space-time norms that will be used below. Define the $Y(I)$ space by its norm

$$\begin{aligned} \|v\|_{Y(I)} :=& N^{s_0 - \frac{5}{6} -} \|\nabla v\|_{L_t^2 L_x^6(I \times \mathbb{R}^3)} + N^{s_0 - \frac{7}{24} -} \|v\|_{L_t^8 L_x^{12}(I \times \mathbb{R}^3)} \\ &+ N^{s_0 - \frac{1}{3} -} \|v\|_{L_t^\infty L_x^6(I \times \mathbb{R}^3)} + N^{s_0 -} \|v\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^3)}. \end{aligned}$$

Then, by the above lemmas and corollary, we have

Lemma 5.8. *Let $N \geq 1$, $\frac{5}{6} < s_0 < 1$, then the following estimates hold,*

$$\|v\|_{Y(\mathbb{R}^+)} \lesssim \|P_{\geq N}\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Moreover, for any $\delta > 0$,

$$\|\nabla v\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{1}{2}+}\|P_{\geq N}\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Proof. The estimates above v on $\mathbb{R}^+$ was proved by Proposition 4.5 in [2], we only sketch the proof for completeness, and prove the spacetime estimate of v on $[\delta, +\infty)$. Let $N = 2^{k_0}$ for some $k_0 \in \mathbb{N}$. By Lemma 5.3, we write

$$\begin{aligned}
v =& e^{it\Delta}(P_{\geq N}\chi_{\geq 1}f)_{out} \\
=& \sum_{k=k_0}^{\infty} e^{it\Delta}(P_{2^k}(\chi_{\geq 1}f))_{out} \\
=& \sum_{k=k_0}^{\infty} e^{it\Delta}(\chi_{\leq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1}) + \sum_{k=k_0}^{\infty} e^{it\Delta}h_k \\
&+ \sum_{k=k_0}^{\infty} e^{it\Delta}(\chi_{\geq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1}) \\
=& \sum_{k=k_0}^{\infty} e^{it\Delta}(\chi_{\leq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1}) + \sum_{k=k_0}^{\infty} e^{it\Delta}h_k \\
&+ \sum_{k=k_0}^{\infty} \chi_{\leq \delta(1+2^k t)} e^{it\Delta}(\chi_{\geq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1}) \\
&+ \sum_{k=k_0}^{\infty} \chi_{\geq \delta(1+2^k t)} e^{it\Delta}(\chi_{\geq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1}).
\end{aligned}$$

Then by Lemma 2.1 and Lemma 5.3, we obtain

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \left\|e^{it\Delta}|\nabla|(\chi_{\leq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1})\right\|_{L_t^2 L_x^6(\mathbb{R}^+\times\mathbb{R}^3)} \\
\lesssim& \sum_{k=k_0}^{\infty} \|\chi_{\leq \frac{1}{4}}(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1}\|_{H^2(\mathbb{R}^3)} \\
\lesssim& \sum_{k=k_0}^{\infty} 2^{-2k}\|P_{2^k}(\chi_{\geq 1}f)\|_{L^2(\mathbb{R}^3)} \\
\lesssim& N^{-2-s_0}\|P_{\geq N}\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}.
\end{aligned} \tag{5.3}$$

In the same way as above, we can also obtain

$$\sum_{k=k_0}^{\infty} \|e^{it\Delta}|\nabla|h_k\|_{L_t^2 L_x^6(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim N^{-2-s_0}\|P_{\geq N}\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}. \tag{5.4}$$

Next, by Lemma 5.4, for any $s_0 \geq 0$, we have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \big\| |\nabla| \big[\chi_{\leq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1}) \big] \big\|_{L_t^2 L_x^6(\mathbb{R}^+\times\mathbb{R}^3)} \\
&\quad \lesssim \sum_{k=k_0}^{\infty} 2^{-(2-(1-\frac{2}{2}-\frac{3}{6}))k} \|P_{2^k}(\chi_{\geq 1} f)\|_{L^2(\mathbb{R}^3)} \\
&\quad \lesssim N^{-s_0-\frac{5}{2}} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}. \qquad (5.5)
\end{aligned}$$

Finally, from Corollary 5.6, noting that $\frac{5}{6} < s_0 < 1$, we obtain that

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \big\| |\nabla| \big[\chi_{\geq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1}) \big] \big\|_{L_t^2 L_x^6(\mathbb{R}^+\times\mathbb{R}^3)} \\
&\quad \lesssim \sum_{k=k_0}^{\infty} 2^{(-\frac{1}{2}-\frac{1}{6}+\frac{1}{2}+1+)k} \|P_{2^k}(\chi_{\geq 1} f)\|_{L^2(\mathbb{R}^3)} \\
&\quad \lesssim N^{-(s_0-\frac{5}{6})+} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}. \qquad (5.6)
\end{aligned}$$

Then collecting the estimates (5.3)-(5.6), we get

$$\|\nabla v\|_{L_t^2 L_x^6(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{5}{6}+} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Similarly as above, by Lemma 2.1, Lemma 5.3, Lemma 5.4, Corollary 5.6, and Sobolev inequality, we can obtain

$$\|v\|_{L_t^8 L_x^{12}(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{7}{24}+} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)},$$

$$\|v\|_{L_t^\infty L_x^6(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{1}{3}+} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)},$$

and

$$\|v\|_{L_t^2 L_x^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \lesssim N^{-s_0+} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Next, we will estimate the term $\|\nabla v\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)}$, the proof is similar as above. By Lemma 2.1, Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \big\| e^{it\Delta} |\nabla| (\chi_{\leq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1}) \big\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} \\
&\quad + \sum_{k=k_0}^{\infty} \big\| |\nabla| \big[\chi_{\leq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1}) \big] \big\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} \\
&\quad + \sum_{k=k_0}^{\infty} \|e^{it\Delta} |\nabla| h_k\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} \\
&\quad \lesssim N^{-2-s_0} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}.
\end{aligned}$$

Furthermore, by Corollary 5.6, we obtain

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \big\| |\nabla| \big[\chi_{\geq \delta(1+2^k t)} e^{it\Delta} (\chi_{\geq \frac{1}{4}} (P_{2^k}(\chi_{\geq 1} f))_{out,k-1\leq \cdot \leq k+1}) \big] \big\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} \\
&\quad \lesssim N^{-s_0+\frac{1}{2}+} \|P_{\geq N} \chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}.
\end{aligned}$$

Hence, by the above two estimates, we obtain

$$\|\nabla v\|_{L_t^2 L_x^\infty([\delta,+\infty)\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{1}{2}+}\|P_{\geq N}\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Thus, we get the desired estimates and complete the proof of the lemma. □

Hence, by Lemma 5.8, we can obtain Proposition 5.1 by choosing a suitable constant $C_0 > 0$.

6. A priori estimate

In this section, we give the proof of the validity of a priori assumptions (H2). To this end, we define the working space $X_N(I)$ for $I \subset \mathbb{R}^+$ by its norm

$$\|h\|_{X_N(I)} = N^{3(s_0-1)}\|h\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^3)} + N^{\frac{9}{8}(s_0-1)}\|h\|_{L_{t,x}^8(I\times\mathbb{R}^3)}.$$

Then we have

$$\begin{aligned}\|h\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^3)} &\leq N^{3(1-s_0)}\|h\|_{X_N(I)},\\ \|h\|_{L_{t,x}^8(I\times\mathbb{R}^3)} &\leq N^{\frac{9}{8}(1-s_0)}\|h\|_{X_N(I)}.\end{aligned} \tag{6.1}$$

The following is the main result in this section. Before this, we recall that w solves the following equation

$$\begin{cases} i\partial_t w + \Delta w = |w+v|^4(w+v),\\ w(0,x) = w_0(x),\end{cases}$$

where $w_0 := \frac{1}{2}P_{\leq 1}f + \frac{1}{2}P_{\geq 1}\chi_{\leq 1}f + (P_{1\leq\cdot\leq N}\chi_{\geq 1}f)_{out}$.

Proposition 6.1. *Let $s_0 \in (\frac{5}{6}, 1)$. Let $w \in C(I;\dot H_x^1(\mathbb{R}^3))$ be the solution of the perturbation equation (3.1) with $w(0) = w_0$. Then there exists a constant $E_0 = E_0(N) > 0$, such that*

$$\sup_{t\in I}\|w(t)\|_{\dot H_x^1(\mathbb{R}^3)} \leq E_0.$$

To prove Proposition 6.1, we need some spacetime norms of w are uniformly bounded. First of all, we have the initial data w_0 is in $\dot H^1$. Indeed, by Bernstein's inequality, we have

$$\|P_{\leq 1}f + P_{\geq 1}\chi_{\leq 1}f\|_{\dot H^1(\mathbb{R}^3)} \lesssim \|\chi_{\leq 1}f\|_{H^1(\mathbb{R}^3)} + \|\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Moreover, by Lemmas 5.2 and 5.3, we have

$$\begin{aligned}\|(P_{1\leq\cdot\leq N}\chi_{\geq 1}f)_{out}\|_{\dot H^1(\mathbb{R}^3)} &\lesssim \sum_{k=0}^{k_0}\left(\|(P_{2^k}(\chi_{\geq 1}f))_{out,k-1\leq\cdot\leq k+1})\|_{\dot H^1(\mathbb{R}^3)} + \|h_k\|_{\dot H^1(\mathbb{R}^3)}\right)\\ &\lesssim \sum_{k=0}^{k_0}\left(2^k\|P_{2^k}(\chi_{\geq 1}f)\|_{L^2(\mathbb{R}^3)} + 2^{-10k}\|P_{2^k}(\chi_{\geq 1}f)\|_{L^2(\mathbb{R}^3)}\right)\\ &\lesssim N^{1-s_0}\|\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)} + \|\chi_{\geq 1}f\|_{H^{s_0}(\mathbb{R}^3)}\\ &\lesssim N^{1-s_0}.\end{aligned}$$

Hence, combining the above two estimates, we have

$$\|w_0\|_{\dot H^1(\mathbb{R}^3)} \lesssim N^{1-s_0}. \tag{6.2}$$

Next, we have that the $L_t^\infty L_x^2(I\times\mathbb{R}^3)$ norm of w is uniformly bounded. In fact, by the conservation of mass (1.2) and Lemma 5.2 , we obtain

$$\begin{aligned}\|w\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} &\lesssim \|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} + \|v\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)}\\ &\lesssim \|f_+\|_{L^2(\mathbb{R}^3)} + \|f\|_{H^{s_0}(\mathbb{R}^3)}\\ &\lesssim \|f\|_{H^{s_0}(\mathbb{R}^3)}.\end{aligned}\tag{6.3}$$

Now, we start with the Morawetz estimates.

6.1. **Morawetz estimates.** In this subsection, we consider the Morawetz-type estimate by Lin-Strauss [28].

Lemma 6.2. *Let $\frac{5}{6} < s_0 < 1$. Let $w \in C(I;\dot H_x^1(\mathbb{R}^3))$ be the solution of the perturbation equation (3.1) with $w(0)=w_0$. Suppose that $\|w\|_{X_N(I)} \geq 1$, then*

$$\int_I\int_{\mathbb{R}^3} \frac{|w(t,x)|^6}{|x|}dxdt \lesssim N^{3(1-s_0)}(\|w\|_{X_N(I)} + \delta_0\|w\|^5_{X_N(I)}).$$

Proof. Let

$$M(t) = \mathrm{Im}\int_{\mathbb{R}^3} \frac{x}{|x|}\cdot\nabla w(t,x)\overline{w}(t,x)dx.$$

By integration-by-parts, we have

$$\begin{aligned}M'(t) &= \mathrm{Im}\int_{\mathbb{R}^3}\frac{x}{|x|}\cdot(\nabla w_t\overline{w} + \nabla w\overline{w_t})dx\\ &= (-1)\mathrm{Im}\int_{\mathbb{R}^3}(\frac{2}{|x|}w_t\overline{w} + \frac{x}{|x|}\cdot\nabla\overline{w}w_t)dx + \mathrm{Im}\int_{\mathbb{R}^3}\frac{x}{|x|}\cdot\nabla w\overline{w_t}dx\\ &= 2\mathrm{Im}\int_{\mathbb{R}^3}(\frac{x}{|x|}\cdot\nabla w + \frac{1}{|x|}w)\overline{w_t}dx.\end{aligned}$$

By the equation (4.2)

$$\begin{aligned}\overline{w_t} &= -i\Delta\overline{w} + i|u|^4\overline{u}\\ &= -i\Delta\overline{w} + i|w|^4\overline{w} + i|u|^4\overline{u} - i|w|^4\overline{w}.\end{aligned}$$

From the above two equalities, we have

$$\begin{aligned}M'(t) =& 2\mathrm{Im}\int_{\mathbb{R}^3}(\frac{x}{|x|}\cdot\nabla w + \frac{1}{|x|}w)(-i\Delta\overline{w})dx\\ &+ 2\mathrm{Im}\int_{\mathbb{R}^3}(\frac{x}{|x|}\cdot\nabla w + \frac{1}{|x|}w)(i|w|^4\overline{w})dx\\ &+ 2\mathrm{Im}\int_{\mathbb{R}^3}(\frac{x}{|x|}\cdot\nabla w + \frac{1}{|x|}w)(i|u|^4\overline{u} - i|w|^4\overline{w})dx\\ :=& I_1 + I_2 + I_3.\end{aligned}\tag{6.4}$$

For I_1, by integration-by-parts, we have

$$I_1 = \int_{\mathbb{R}^3}\frac{1}{|x|}\big(|\nabla w|^2 - |\frac{x}{|x|}\cdot\nabla w|^2\big)dx + 2\pi|w(t,0)|^2 \geq 0.$$

For I_2, by integration-by-parts again, we have

$$I_2 = \frac{2}{3}\int_{\mathbb{R}^3} \frac{|w(t,x)|^6}{|x|}dx.$$

For I_3, by Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}|I_3| &\lesssim \Big|\mathrm{Im}\int_{\mathbb{R}^3}(\frac{x}{|x|}\cdot\nabla w + \frac{1}{|x|}w)(i|u|^4\overline{u} - i|w|^4\overline{w})dx\Big| \\ &\lesssim \|\nabla w\|_{L^2_x(\mathbb{R}^3)}\big\||u|^4u - |w|^4w\big\|_{L^2_x(\mathbb{R}^3)}.\end{aligned}$$

Hence, by the above three estimates and integrating in time in (6.4), we can obtain

$$\int_I\int_{\mathbb{R}^3}\frac{|w(t,x)|^6}{|x|}dxdt \lesssim \sup_{t\in I}M(t) + \|\nabla w\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)}\big\||u|^4u - |w|^4w\big\|_{L^1_tL^2_x(I\times\mathbb{R}^3)}. \tag{6.5}$$

For the first term, by (6.1), (6.3) and Hölder's inequality

$$\sup_{t\in I}M(t) \lesssim \|w\|_{L^\infty_t L^2_x(I\times\mathbb{R}^3)}\|w\|_{L^\infty_t \dot H^1_x(I\times\mathbb{R}^3)} \lesssim N^{3(1-s_0)}\|w\|_{X_N(I)}. \tag{6.6}$$

Next, for the term $\big\||u|^4u - |w|^4w\big\|_{L^1_tL^2_x(I\times\mathbb{R}^3)}$, noting that $u = w + v$, by Hölder's inequality we have

$$\begin{aligned}\big\||u|^4u - |w|^4w\big\|_{L^1_tL^2_x(I\times\mathbb{R}^3)} &\lesssim \big\||u+w|^4|u-w|\big\|_{L^1_tL^2_x(I\times\mathbb{R}^3)} \\ &\lesssim \big\|(|w|^4 + |v|^4)|v|\big\|_{L^1_tL^2_x(I\times\mathbb{R}^3)} \\ &\lesssim \|v\|^5_{L^5_tL^{10}_x(I\times\mathbb{R}^3)} + \|v\|_{L^2_tL^\infty_x(I\times\mathbb{R}^3)}\|w\|^4_{L^8_{t,x}(I\times\mathbb{R}^3)}.\end{aligned}$$

For the estimates about v, by Lemma 5.8, interpolation inequality and (4.1), we have

$$\|v\|_{L^2_tL^\infty_x(I\times\mathbb{R}^3)} \lesssim N^{-s_0+}\delta_0,$$

and

$$\|v\|_{L^5_tL^{10}_x(I\times\mathbb{R}^3)} \lesssim \|v\|^{\frac{2}{5}}_{L^2_tL^\infty_x(I\times\mathbb{R}^3)}\|v\|^{\frac{3}{5}}_{L^\infty_tL^6_x(I\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{1}{5}+}\delta_0.$$

Further, by (6.1) and the above three estimates, we obtain that

$$\begin{aligned}\big\||u|^4u - |w|^4w\big\|_{L^1_tL^2_x(I\times\mathbb{R}^3)} &\lesssim N^{-5s_0+1+}\delta_0^5 + N^{-s_0+}\delta_0 N^{\frac{9}{2}(1-s_0)}\|w\|^4_{X_N(I)} \\ &\lesssim N^{-\frac{11}{2}s_0+\frac{9}{2}+}\delta_0\|w\|^4_{X_N(I)} \\ &\lesssim \delta_0\|w\|^4_{X_N(I)}.\end{aligned} \tag{6.7}$$

Hence, by (6.1), (6.5), (6.6) and (6.7), we have

$$\int_I\int_{\mathbb{R}^3}\frac{|w(t,x)|^6}{|x|}dxdt \lesssim N^{3(1-s_0)}(\|w\|_{X_N(I)} + \delta_0\|w\|^5_{X_N(I)}).$$

This finishes the proof. □

From the above lemma we have the following result.

Corollary 6.3. *Under the same assumptions as in Lemma 6.2, then*

$$\|w\|_{L^8_{t,x}(I\times\mathbb{R}^3)} \lesssim N^{\frac{9}{8}(1-s_0)}\big(\|w\|^{\frac{3}{8}}_{X_N(I)} + \delta_0^{\frac{1}{8}}\|w\|^{\frac{7}{8}}_{X_N(I)}\big).$$

Proof. By Hölder's inequality,

$$\int_I\int_{\mathbb{R}^3}|w(t,x)|^8dxdt=\int_I\int_{\mathbb{R}^3}\frac{|w|^6}{|x|}|x||w|^2dxdt$$
$$\lesssim\big\||x|^{\frac12}w\big\|^2_{L^\infty_{t,x}(I\times\mathbb{R}^3)}\int_I\int_{\mathbb{R}^3}\frac{|w(t,x)|^6}{|x|}dxdt.$$

By Lemma 2.2 and (6.1), we have

$$\big\||x|^{\frac12}w\big\|_{L^\infty_{t,x}(I\times\mathbb{R}^3)}\lesssim\|w\|_{L^\infty_t\dot H^1_x(I\times\mathbb{R}^3)}\lesssim N^{3(1-s_0)}\|w\|_{X_N(I)}.$$

Hence, by Lemma 6.2 and the above estimates, we obtain

$$\int_I\int_{\mathbb{R}^3}|w(t,x)|^8dxdt\lesssim N^{9(1-s_0)}(\|w\|^3_{X_N(I)}+\delta_0\|w\|^7_{X_N(I)}),$$

which gives the desired estimate. □

6.2. **Energy estimate.**

Lemma 6.4. *Let $\frac56<s_0<1$. Let $w\in C(I;\dot H^1_x(\mathbb{R}^3))$ be the solution of the perturbation equation (3.1) with $w(0)=w_0$. Suppose that $\|w\|_{X_N(I)}\geq 1$, then*

$$\|w\|_{L^\infty_t\dot H^1_x(I\times\mathbb{R}^3)}\lesssim N^{3(1-s_0)}\big(1+\delta_0^{\frac12}\|w\|^{\frac52}_{X_N(I)}\big).$$

Proof. For simplicity, we denote $I=[0,T)$ and for any $t\in I$, let

$$\widetilde E(t):=\frac12\int_{\mathbb{R}^3}|\nabla w(t,x)|^2dx+\frac16\int_{\mathbb{R}^3}|u(t,x)|^6dx.$$

Noting that $u=w+v$ with v is a linear solution. Taking product with w_t on the equation (3.1) and integration-by-parts, we have

$$\frac{d}{dt}\widetilde E(t)=\mathrm{Re}\int_{\mathbb{R}^3}|u|^4u\overline{v_t}dx=\mathrm{Im}\int_{\mathbb{R}^3}|u|^4u\Delta\bar vdx.$$

Integrating the above equality in time from t_0 to t, we get

$$\widetilde E(t)=\widetilde E(t_0)+\mathrm{Im}\int_{t_0}^t\int_{\mathbb{R}^3}|u|^4u\Delta\bar v dxdt',\tag{6.8}$$

where t_0 will be determined later.

For the first term $\widetilde E(t_0)$ in (6.8), by Sobolev inequality, we have

$$\begin{aligned}\widetilde E(t_0)&\lesssim\|w(t_0)\|^2_{\dot H^1_x(\mathbb{R}^3)}+\|u(t_0)\|^6_{L^6_x(\mathbb{R}^3)}\\&\lesssim\|\nabla w\|^2_{L^\infty_tL^2_x([0,t_0]\times\mathbb{R}^3)}+\|\nabla w\|^6_{L^\infty_tL^2_x([0,t_0]\times\mathbb{R}^3)}+\|v\|^6_{L^\infty_tL^6_x([0,t_0]\times\mathbb{R}^3)}.\end{aligned}\tag{6.9}$$

Now, we estimate the term $\|\nabla w\|_{L^\infty_tL^2_x([0,t_0]\times\mathbb{R}^3)}$. First of all, by Lemma 2.1 and Lemma 5.8, we have

$$\begin{aligned}\|\nabla e^{it\Delta}u_0\|_{L^2_tL^6_x(I\times\mathbb{R}^3)}&\lesssim\|\nabla e^{it\Delta}(P_{\leq1}f+P_{\geq1}\chi_{\leq1}f)\|_{L^2_tL^6_x(I\times\mathbb{R}^3)}+\|\nabla e^{it\Delta}(P_{\geq1}\chi_{\geq1}f)_{out}\|_{L^2_tL^6_x(I\times\mathbb{R}^3)}\\&\lesssim\|P_{\leq1}f+P_{\geq1}\chi_{\leq1}f\|_{\dot H^1(\mathbb{R}^3)}+\|P_{\geq1}\chi_{\geq1}f\|_{H^{s_0}(\mathbb{R}^3)}\\&\lesssim\|\chi_{\leq1}f\|_{H^1(\mathbb{R}^3)}+\|\chi_{\geq1}f\|_{H^{s_0}(\mathbb{R}^3)}.\end{aligned}\tag{6.10}$$

Furthermore, by Lemma 5.8 and interpolation inequality

$$\|v\|_{L_t^6 L_x^{18}(I\times\mathbb{R}^3)} \lesssim \|\nabla v\|_{L_t^2 L_x^6(I\times\mathbb{R}^3)}^{\frac{1}{3}} \|v\|_{L_t^\infty L_x^6(I\times\mathbb{R}^3)}^{\frac{2}{3}} \lesssim \|\chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}.$$

In the same way as (6.10), we have

$$\|e^{it\Delta}u_0\|_{L_t^6 L_x^{18}(I\times\mathbb{R}^3)} \lesssim \|\chi_{\leq 1} f\|_{H^1(\mathbb{R}^3)} + \|\chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}.$$

Hence, given small constant $\eta_0 > 0$, by choosing $t_0 = t_0(u_0, \eta_0, \|\chi_{\leq 1} f\|_{H^1(\mathbb{R}^3)} + \|\chi_{\geq 1} f\|_{H^{s_0}(\mathbb{R}^3)}) > 0$ small enough, we can obtain

$$\|e^{it\Delta}u_0\|_{L_t^2 \dot W_x^{1,6}\cap L_t^6 L_x^{18}([0,t_0]\times\mathbb{R}^3)} \leq \eta_0.$$

Then using the standard fixed point argument, we can obtain

$$\|u\|_{L_t^2 \dot W_x^{1,6}\cap L_t^6 L_x^{18}([0,t_0]\times\mathbb{R}^3)} \leq \eta_0. \tag{6.11}$$

By (4.1), (6.11), Lemma 2.1 and Lemma 5.8, we have

$$\begin{aligned}
\|\nabla w\|_{L_t^\infty L_x^2([0,t_0]\times\mathbb{R}^3)} &\lesssim \|w_0\|_{\dot H_x^1(\mathbb{R}^3)} + \|\nabla(|u|^4 u)\|_{L_t^1 L_x^2([0,t_0]\times\mathbb{R}^3)} \\
&\lesssim N^{1-s_0} + \|\nabla u\|_{L_t^2 L_x^6([0,t_0]\times\mathbb{R}^3)} \|u\|^3_{L_t^6 L_x^{18}([0,t_0]\times\mathbb{R}^3)} \|u\|_{L_t^\infty L_x^6([0,t_0]\times\mathbb{R}^3)} \\
&\lesssim N^{1-s_0} + \eta_0^4 \big(\|\nabla w\|_{L_t^\infty L_x^2([0,t_0]\times\mathbb{R}^3)} + \|v\|_{L_t^\infty L_x^6([0,t_0]\times\mathbb{R}^3)}\big) \\
&\lesssim N^{1-s_0} + \eta_0^4 \|\nabla w\|_{L_t^\infty L_x^2([0,t_0]\times\mathbb{R}^3)} + \eta_0^4 N^{-s_0+\frac{1}{3}+}\delta_0 \\
&\lesssim N^{1-s_0} + \eta_0^4 \|\nabla w\|_{L_t^\infty L_x^2([0,t_0]\times\mathbb{R}^3)}.
\end{aligned}$$

Thus, we obtain

$$\|\nabla w\|_{L_t^\infty L_x^2([0,t_0]\times\mathbb{R}^3)} \lesssim N^{1-s_0}. \tag{6.12}$$

Hence, by (6.9), (6.12) and Lemma 5.8, we obtain

$$\begin{aligned}
\widetilde{E}(t_0) &\lesssim N^{2(1-s_0)} + N^{6(1-s_0)} + N^{-6s_0+2+}\delta_0^6 \\
&\lesssim N^{6(1-s_0)}.
\end{aligned} \tag{6.13}$$

Next, we consider the second term in (6.8), by integration-by-parts, we have

$$\begin{aligned}
\operatorname{Im} \int_{t_0}^{t} \int_{\mathbb{R}^3} |u|^4 u \Delta \bar v \, dx dt' &\lesssim \int_{t_0}^{t} \int_{\mathbb{R}^3} |u|^4 |\nabla u||\nabla v| \, dx dt' \\
&\lesssim \int_{t_0}^{T} \int_{\mathbb{R}^3} |u|^4 |\nabla w||\nabla v| \, dx dt' + \int_{t_0}^{T} \int_{\mathbb{R}^3} |u|^4 |\nabla v|^2 \, dx dt' \\
&:= I_1 + I_2.
\end{aligned} \tag{6.14}$$

We consider the term I_1 firstly. By (4.1), Lemma 5.8 and interpolation inequality, we have

$$\|\nabla v\|_{L_t^2 L_x^\infty([t_0,T]\times\mathbb{R}^3)} \lesssim N^{-s_0+\frac{1}{2}+}\delta_0, \tag{6.15}$$

and

$$\|v\|_{L_{t,x}^8([t_0,T]\times\mathbb{R}^3)} \lesssim \|v\|_{L_t^2 L_x^\infty(I\times\mathbb{R}^3)}^{\frac{1}{4}} \|v\|_{L_t^\infty L_x^6(I\times\mathbb{R}^3)}^{\frac{3}{4}} \lesssim N^{-s_0+\frac{1}{4}+}\delta_0. \tag{6.16}$$

Further, by using (6.1), (6.15) and (6.16), we obtain

$$\begin{aligned}
I_1 &\lesssim \|\nabla v\|_{L_t^2 L_x^\infty([t_0,T]\times\mathbb{R}^3)} \|\nabla w\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \|u\|^4_{L^8_{t,x}(I\times\mathbb{R}^3)} \\
&\lesssim \|\nabla v\|_{L_t^2 L_x^\infty([t_0,T]\times\mathbb{R}^3)} \|\nabla w\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \big(\|w\|^4_{L^8_{t,x}(I\times\mathbb{R}^3)} + \|v\|^4_{L^8_{t,x}(I\times\mathbb{R}^3)}\big) \\
&\lesssim N^{-s_0+\frac12+}\delta_0 \cdot N^{3(1-s_0)}\|w\|_{X_N(I)} \cdot \big(N^{\frac92(1-s_0)}\|w\|^4_{X_N(I)} + N^{-4s_0+1+}\delta_0^4\big) \\
&\lesssim N^{-s_0+\frac12+}\delta_0 \cdot N^{3(1-s_0)}\|w\|_{X_N(I)} \cdot N^{\frac92(1-s_0)}\|w\|^4_{X_N(I)} \\
&\lesssim N^{-\frac{17}{2}s_0+8+}\delta_0\|w\|^5_{X_N(I)} \\
&\lesssim N^{6(1-s_0)}\delta_0\|w\|^5_{X_N(I)}.
\end{aligned} \tag{6.17}$$

Next, we consider the term I_2. By (1.2), (4.1), (6.15) and Lemma 5.8, we obtain

$$\begin{aligned}
I_2 \lesssim& \|\nabla v\|^2_{L_t^2 L_x^\infty([t_0,T]\times\mathbb{R}^3)} \|u\|_{L_t^\infty L_x^2(I\times\mathbb{R}^3)} \|u\|^3_{L_t^\infty L_x^6(I\times\mathbb{R}^3)} \\
\lesssim& N^{-2s_0+1+}\delta_0^2 \cdot \big(N^{9(1-s_0)}\|w\|^3_{X_N(I)} + N^{-3s_0+1+}\delta_0^3\big) \\
\lesssim& N^{-11s_0+10+}\delta_0^2\|w\|^3_{X_N(I)} \\
\lesssim& N^{6(1-s_0)}\delta_0^2\|w\|^3_{X_N(I)}.
\end{aligned} \tag{6.18}$$

Hence, combining (6.8), (6.13), (6.14), (6.17) with (6.18), we can obtain

$$\begin{aligned}
\sup_{t\in I}\widetilde{E}(t) &\lesssim N^{6(1-s_0)}\big(1 + \delta_0\|w\|^5_{X_N(I)} + \delta_0^2\|w\|^3_{X_N(I)}\big) \\
&\lesssim N^{6(1-s_0)}\big(1 + \delta_0\|w\|^5_{X_N(I)}\big).
\end{aligned}$$

Further, from the definition of $\widetilde{E}(t)$, we have

$$\begin{aligned}
\|w\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^3)} &\lesssim \big(\sup_{t\in I}\widetilde{E}(t)\big)^{\frac12} \\
&\lesssim N^{3(1-s_0)}\big(1 + \delta_0^{\frac12}\|w\|^{\frac52}_{X_N(I)}\big).
\end{aligned}$$

This completes the proof of this lemma. □

Now, we aim to prove Proposition 6.1, which shows the assumption (H2) is valid.

Proof of Proposition 6.1. First, we show that for any I such that $0\in I\subset\mathbb{R}^+$,

$$\|w\|_{X_N(I)} \lesssim 1. \tag{6.19}$$

Indeed, if $\|w\|_{X_N(I)} \le 1$, then (6.19) already holds. Therefore, we can assume $\|w\|_{X_N(I)} \ge 1$. Using Corollary 6.3, Lemma 6.4, and Young's inequality, we obtain

$$\begin{aligned}
\|w\|_{X_N(I)} &= N^{-3(1-s_0)}\|w\|_{L_t^\infty \dot H_x^1(I\times\mathbb{R}^3)} + N^{-\frac98(1-s_0)}\|w\|_{L^8_{t,x}(I\times\mathbb{R}^3)} \\
&\le C(1 + \delta_0^{\frac12}\|w\|^{\frac52}_{X_N(I)} + \|w\|^{\frac38}_{X_N(I)} + \delta_0^{\frac18}\|w\|^{\frac78}_{X_N(I)}) \\
&\le C + C\delta_0^{\frac12}\|w\|^{\frac52}_{X_N(I)} + \frac14\|w\|_{X_N(I)} + \delta_0^{\frac17}\|w\|_{X_N(I)} \\
&\le C + C\delta_0^{\frac12}\|w\|^{\frac52}_{X_N(I)} + \frac12\|w\|_{X_N(I)}.
\end{aligned}$$

Then, we get

$$\|w\|_{X_N(I)} \lesssim 1 + \delta_0^{\frac12}\|w\|^{\frac52}_{X_N(I)}.$$

By the usual bootstrap argument, we obtain (6.19). Hence, we finish the proof of Proposition 6.1. □

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Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

Email address: `baimaths@hotmail.com`

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

Email address: `shenjia@tju.edu.cn`

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

Email address: `yerfmath@gmail.com`