Overpartitions and Bressoud's Conjecture, I

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Abstract. In 1980, Bressoud conjectured a combinatorial identity $A_j = B_j$ for $j = 0$ or 1, where the function A_j counts the number of partitions with certain congruence conditions and the function B_j counts the number of partitions with certain difference conditions. Bressoud's conjecture specializes to a wide variety of well-known theorems in the theory of partitions. Special cases of his conjecture have been subsequently proved by Bressoud, Andrews, Kim and Yee. Recently, Kim resolved Bressoud's conjecture for the case $j = 1$. In this paper, we introduce a new partition function $\overline{B}_j$ which can be viewed as an overpartition analogue of the partition function B_j introduced by Bressoud. By means of Gordon markings, we build bijections to obtain a relationship between $\overline{B}_1$ and B_0 and a relationship between $\overline{B}_0$ and B_1. Based on these former relationships, we further give overpartition analogues of many classical partition theorems including Euler's partition theorem, the Rogers-Ramanujan-Gordon identities, the Bressoud-Rogers-Ramanujan identities, the Andrews-Göllnitz-Gordon identities and the Bressoud-Göllnitz-Gordon identities.

Keywords: Bressoud's conjecture, Overpartitions, Euler's partition theorem, Rogers-Ramanujan identities, Göllnitz-Gordon identities, Bailey pairs, Gordon markings

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1 Introduction

Bressoud [10] proved and conjectured some partition identities involving the partition function B_j, which counts the number of partitions with certain difference conditions (see Definition 1.6). The main objective of this paper is to introduce a new partition function $\overline{B}_j$ which can be regarded as an overpartition analogue of the partition function B_j. We establish a relationship between $\overline{B}_1$ and B_0 and a relationship between $\overline{B}_0$ and B_1. Based on these two relationships, we obtain overpartition analogues of many classical partition theorems including Euler's partition theorem, the Rogers-Ramanujan-Gordon identities, the Bressoud-Rogers-Ramanujan identities, the Andrews-Göllnitz-Gordon identities and the Bressoud-Göllnitz-Gordon identities. It should be noted that the relationship between $\overline{B}_1$ and B_0 plays a crucial role in the proof of Bressoud's conjecture for $j=0$ in the subsequent paper [25].

Let us recall some common notation and terminologies on partitions from [5, Chapter 1]. A partition π of a positive integer n is a finite non-increasing sequence of positive integers $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ such that $\sum_{i=1}^{\ell}\pi_i=n$. An overpartition of n is a partition of n such that the first occurrence of a part can be overlined.

For example, there are five partitions of 4:

$$(4),(3,1),(2,2),(2,1,1),(1,1,1,1),$$

whereas there are fourteen overpartitions of 4:

$$(4),(\bar{4}),(3,1),(\bar{3},1),(3,\bar{1}),(\bar{3},\bar{1}),(2,2),(\bar{2},2),$$
$$(2,1,1),(\bar{2},1,1),(2,\bar{1},1),(\bar{2},\bar{1},1),(1,1,1,1),(\bar{1},1,1,1).$$

We impose the following order on the parts of an overpartition:

$$1<\bar{1}<2<\bar{2}<\cdots. \tag{1.1}$$

Let $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ be an ordinary partition (resp. an overpartition) with $\pi_1\geq\pi_2\geq\cdots\geq\pi_\ell\geq 1$. The number of parts of π is called the length of π, denoted $\ell(\pi)$. The weight of π is the sum of parts, denoted $|\pi|$.

In 1961, Gordon [22, p. 394] found an infinite family of combinatorial generalizations of the Rogers-Ramanujan identities, which has been known as the Rogers-Ramanujan-Gordon theorem.

Theorem 1.1 (Rogers-Ramanujan-Gordon)**.** *For $k\geq r\geq 1$, let $B_1(-;1,k,r;n)$ denote the number of partitions $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ of n, where $\pi_i\geq\pi_{i+k-1}+2$ for $1\leq i\leq\ell-k+1$, and at most $r-1$ of the π_i are equal to 1. For $k\geq r\geq 1$, let $A_1(-;1,k,r;n)$ denote the number of partitions of n into parts $\not\equiv 0,\pm r \pmod{2k+1}$. Then, for $k\geq r\geq 1$ and $n\geq 0$,*

$$A_1(-;1,k,r;n)=B_1(-;1,k,r;n).$$

An analytic proof of Theorem 1.1 was given by Andrews [3]. He discovered the following generating function version of Theorem 1.1, which has been called the Andrews-Gordon identity: For $k\geq r\geq 1$,

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q;q)_{N_{k-1}}}=\frac{(q^r,q^{2k-r+1},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}. \tag{1.2}$$

From now on, we assume that $|q|<1$ and adopt the standard notation [5]:

$$(a;q)_\infty=\prod_{i=0}^{\infty}(1-aq^i),\quad (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

and

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

In 1979, Bressoud [9] extended the Rogers-Ramanujan-Gordon theorem to even moduli, which has been called the Bressoud-Rogers-Ramanujan theorem.

Theorem 1.2 (Bressoud-Rogers-Ramanujan). *For $k > r \geq 1$, let $B_0(-; 1, k, r; n)$ denote the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n, where $\pi_i \geq \pi_{i+k-1} + 2$ for $1 \leq i \leq \ell - k + 1$, at most $r - 1$ of the π_i are equal to 1, and for $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + 1$, then*

$$\pi_i + \cdots + \pi_{i+k-2} \equiv r - 1 \pmod 2.$$

For $k > r \geq 1$, let $A_0(-; 1, k, r; n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm r \pmod{2k}$. Then, for $k > r \geq 1$ and $n \geq 0$,

$$A_0(-; 1, k, r; n) = B_0(-; 1, k, r; n).$$

Furthermore, Bressoud [10] obtained the following generating function version of Theorem 1.2: For $k > r \geq 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \cdots + N_{k-1}^2 + N_r + \cdots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}} = \frac{(q^r, q^{2k-r}, q^{2k}; q^{2k})_\infty}{(q; q)_\infty}. \tag{1.3}$$

Motivated by the Rogers-Ramanujan-Gordon identities, Andrews [2] found an infinite family of the combinatorial generalizations of the Göllnitz-Gordon identities, which has been referred to as the Andrews-Göllnitz-Gordon theorem.

Theorem 1.3 (Andrews-Göllnitz-Gordon). *For $k \geq r \geq 1$, let $B_1(1; 2, k, r; n)$ denote the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n such that no odd part is repeated, where $\pi_i \geq \pi_{i+k-1} + 2$ with strict inequality if π_i is even for $1 \leq i \leq \ell - k + 1$, and at most $r - 1$ of the π_i are less than or equal to 2. For $k \geq r \geq 1$, let $A_1(1; 2, k, r; n)$ denote the number of partitions of n into parts $\not\equiv 2 \pmod 4$ and $\not\equiv 0, \pm(2r - 1) \pmod{4k}$. Then, for $k \geq r \geq 1$ and $n \geq 0$,*

$$A_1(1; 2, k, r; n) = B_1(1; 2, k, r; n).$$

In 1980, Bressoud [10] extended the Andrews-Göllnitz-Gordon theorem to even moduli, which has been called the Bressoud-Göllnitz-Gordon theorem.

Theorem 1.4 (Bressoud-Göllnitz-Gordon). *For $k > r \geq 1$, let $B_0(1; 2, k, r; n)$ denote the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n such that no odd part is repeated, where $\pi_i \geq \pi_{i+k-1} + 2$ with strict inequality if π_i is even for $1 \leq i \leq \ell - k + 1$, at most $r - 1$ of the π_i are less than or equal to 2, and for $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + 2$ with strict inequality if π_i is odd, then*

$$\pi_i + \cdots + \pi_{i+k-2} \equiv r - 1 + V_\pi(\pi_i) \pmod 2,$$

where $V_\pi(t)$ denotes the number of odd parts not exceeding t in π. For $k > r \geq 1$, let $A_0(1;2,k,r;n)$ denote the number of partitions of n into parts not congruent to $2k-1 \pmod{4k-2}$ may be repeated, no part is congruent to $2 \pmod 4$, no part is multiple of $8k-4$, and no part is congruent to $\pm(2r-1) \pmod{4k-2}$. Then, for $k > r \geq 1$ and $n \geq 0$,

$$A_0(1;2,k,r;n) = B_0(1;2,k,r;n).$$

Bressoud [10] derived the following generating function versions of Theorem 1.3 and Theorem 1.4: For $j = 0$ or 1 and $(2k+j)/2 > r \geq 1$,

$$\begin{aligned}\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} &\frac{(-q^{1-2N_1};q^2)_{N_1} q^{2(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^2;q^2)_{N_1-N_2}\cdots(q^2;q^2)_{N_{k-2}-N_{k-1}}(q^{4-2j};q^{4-2j})_{N_{k-1}}} \\ &= \frac{(q^2;q^4)_\infty(q^{2r-1},q^{4k-2r-1+2j},q^{4k-2+2j};q^{4k-2+2j})_\infty}{(q;q)_\infty}.\end{aligned} \tag{1.4}$$

For $j = 0$ or 1, it is evident that the generating function of $A_j(1;2,k,r;n)$ defined in Theorem 1.3 and Theorem 1.4 equals the right-hand side of (1.4). Hence, the sum on the left-hand side of (1.4) can be considered as the generating function of $B_j(1;2,k,r;n)$ defined in Theorem 1.3 and Theorem 1.4. More precisely, when $j = 1$, the identity (1.4) can be viewed as the generating function version of Theorem 1.3, and when $j = 0$, the identity (1.4) can be seen as the generating function version of Theorem 1.4.

Bressoud obtained a far-reaching partition theorem utilizing an extension of Watson's q-analogue of Whipple's theorem (see [10, Theorem 1]). Throughout this paper, we assume that $\alpha_1, \alpha_2, \ldots, \alpha_\lambda$ and η are integers such that

$$0 < \alpha_1 < \cdots < \alpha_\lambda < \eta, \quad \text{and} \quad \alpha_i = \eta - \alpha_{\lambda+1-i} \quad \text{for} \quad 1 \leq i \leq \lambda. \tag{1.5}$$

When λ is odd, observing that $\eta = \alpha_{(\lambda+1)/2} + \alpha_{\lambda+1-(\lambda+1)/2} = 2\alpha_{(\lambda+1)/2}$, we see that η must be even in such case.

Theorem 1.5 (Bressoud)**.** *For $j = 0$ or 1 and $(2k+j)/2 > r \geq \lambda \geq 0$,*

$$\begin{aligned}\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} &\frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{(2-j)\eta};q^{(2-j)\eta})_{N_{k-1}}} \\ &\times \prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s} \prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty \\ &= \frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda};q^\eta)_\infty(q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2}+j)},q^{\eta(2k-\lambda+j)};q^{\eta(2k-\lambda+j)})_\infty}{(q^\eta;q^\eta)_\infty}.\end{aligned} \tag{1.6}$$

This theorem reduces to many infinite families of identities. For example, setting $\lambda = 0$, $\eta = 1$ and $j = 1$ or 0, we recover (1.2) and (1.3) respectively. Setting $\lambda = 1$, $\eta = 2$

and $\alpha_1 = 1$, we come to (1.4). To give a combinatorial interpretation of (1.6), Bressoud introduced two partition functions.

Definition 1.6 (Bressoud). *For $j = 0$ or 1 and $k \geq r \geq \lambda \geq 0$, define the partition function $B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ to be the number of partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n satisfying the following conditions*:

(1) *For $1 \leq i \leq \ell$, $\pi_i \equiv 0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$;*

(2) *Only multiples of η may be repeated;*

(3) *For $1 \leq i \leq \ell - k + 1$, $\pi_i \geq \pi_{i+k-1} + \eta$ with strict inequality if $\eta \mid \pi_i$;*

(4) *At most $r - 1$ of the π_i are less than or equal to η;*

(5) *For $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + \eta$ with strict inequality if $\eta \nmid \pi_i$, then*

$$[\pi_i/\eta] + \cdots + [\pi_{i+k-2}/\eta] \equiv r - 1 + V_\pi(\pi_i) \pmod{2-j},$$

where $V_\pi(t)$ denotes the number of parts not exceeding t which are not divisible by η in π and $[\ ]$ denotes the greatest integer function.

Definition 1.7 (Bressoud). *For $j = 0$ or 1 and $(2k + j)/2 > r \geq \lambda \geq 0$, define the partition function $A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ to be the number of partitions of n into parts congruent to $0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$ such that*

(1) *If λ is even, then only multiples of η may be repeated and no part is congruent to $0, \pm\eta(r - \lambda/2) \pmod{\eta(2k - \lambda + j)}$;*

(2) *If λ is odd and $j = 1$, then only multiples of $\eta/2$ may be repeated, no part is congruent to $\eta \pmod{2\eta}$, and no part is congruent to $0, \pm\eta(2r - \lambda)/2 \pmod{\eta(2k - \lambda + 1)}$;*

(3) *If λ is odd and $j = 0$, then only multiples of $\eta/2$ which are not congruent to $\eta(2k - \lambda)/2 \pmod{\eta(2k - \lambda)}$ may be repeated, no part is congruent to $\eta \pmod{2\eta}$, no part is congruent to $0 \pmod{2\eta(2k - \lambda)}$, and no part is congruent to $\pm\eta(2r - \lambda)/2 \pmod{\eta(2k - \lambda)}$.*

Bressoud [10] posed the following conjecture.

Conjecture 1.8 (Bressoud). *For $j = 0$ or 1, $(2k + j)/2 > r \geq \lambda \geq 0$ and $n \geq 0$,*

$$A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) = B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n).$$

This conjecture specializes to many infinite families of combinatorial identities. For example, setting $\lambda = 0$, $\eta = 1$ and $j = 1$ or 0, we find that it reduces to Theorem 1.1 and Theorem 1.2 respectively. For $\lambda = 1$, $\eta = 2$, $\alpha_1 = 1$ and $j = 1$ or 0, we see that it boils down to Theorem 1.3 and Theorem 1.4 respectively.

As remarked by Bressoud [10], it is not difficult to see that the generating function of $A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ is equal to the right-hand side of (1.6).

Theorem 1.9 (Bressoud)**.** *For $j = 0$ or 1 and $(2k+j)/2 > r \geq \lambda \geq 0$,*

$$\begin{aligned}&\sum_{n\geq 0} A_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n \\ &= \frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda};q^\eta)_\infty (q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2}+j)},q^{\eta(2k-\lambda+j)};q^{\eta(2k-\lambda+j)})_\infty}{(q^\eta;q^\eta)_\infty}.\end{aligned} \tag{1.7}$$

Nevertheless, it does not seem easy to prove that the left-hand side of (1.6) is indeed the generating function of $B_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$. In this regard, Bressoud [10] posed the following conjecture.

Conjecture 1.10 (Bressoud)**.** *For $j = 0$ or 1 and $(2k+j)/2 > r \geq \lambda \geq 0$,*

$$\begin{aligned}&\sum_{n\geq 0} B_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n \\ &= \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{(2-j)\eta};q^{(2-j)\eta})_{N_{k-1}}} \\ &\quad\times\prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s}\prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty.\end{aligned}$$

Andrews [4] proved Conjecture 1.8 for $\eta = \lambda + 1$ and $j = 1$. Kim and Yee [28] showed that the conjecture holds for $j = 1$ and $\lambda = 2$. In fact, they proved that Conjecture 1.10 is true for $j = 1$ and $\lambda = 2$ with the aid of Gordon markings introduced by Kurşungöz [29,30]. Recently, Kim [27] resolved Conjecture 1.8 for the case $j = 1$. To this end, she established the following theorem.

Theorem 1.11 (Bressoud-Kim)**.** *For $k \geq r \geq \lambda \geq 0$,*

$$\begin{aligned}&\sum_{n\geq 0} B_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n \\ &= \frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda};q^\eta)_\infty (q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2}+1)},q^{\eta(2k-\lambda+1)};q^{\eta(2k-\lambda+1)})_\infty}{(q^\eta;q^\eta)_\infty}.\end{aligned} \tag{1.8}$$

It is clear that Conjecture 1.8 for $j = 1$ is an immediate consequence of (1.7) and (1.8).

The main objective of this paper is to give overpartition analogues of the partition function B_j and the partition function A_j introduced by Bressoud and to establish overpartition analogues of some classical partition theorems. The overpartition analogues of classical partition theorems have caught much attention, see, for example, Chen, Sang and Shi [12–14], Choi, Kim and Lovejoy [15], Corteel and Lovejoy [16], Corteel, Lovejoy and Mallet [17], Corteel and Mallet [18], Dousse [19, 20], Goyal [23], He, Ji, Wang and

Zhao [24], He, Wang and Zhao [26], Kurşungöz [31], Lovejoy [32, 33, 35–37], Lovejoy and Mallet [38], Raghavendra and Padmavathamma [39], and Sang and Shi [41].

Lovejoy [32] established overpartition analogues of the Rogers-Ramanujan-Gordon theorem for the cases $i = 1$ and $i = k$, and the general case was obtained by Chen, Sang and Shi [13]. In Theorem 1.12 and for the rest of this paper, we adopt the following convention: For positive integers t and b, we define $t \pm b$ (resp. $\overline{t} \pm b$) as a non-overlined part (resp. an overlined part) of size $t \pm b$. The parts in an overpartition are ordered as in (1.1).

Theorem 1.12 (Chen-Sang-Shi)**.** *For $k \geq r \geq 1$, let $\overline{B}_1(-;1,k,r;n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n, where $\pi_i \geq \pi_{i+k-1} + 1$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - k + 1$, and at most $r - 1$ of the π_i are equal to 1. For $k > r \geq 1$, let $\overline{A}_1(-;1,k,r;n)$ denote the number of overpartitions of n such that non-overlined parts $\not\equiv 0, \pm r \pmod{2k}$, and for $k = r$, let $\overline{A}_1(-;1,k,k;n)$ denote the number of overpartitions of n into parts not divisible by k. Then, for $k \geq r \geq 1$ and $n \geq 0$,*

$$\overline{A}_1(-;1,k,r;n) = \overline{B}_1(-;1,k,r;n).$$

Chen, Sang and Shi [13] gave the following generating function version of Theorem 1.12: For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}(1+q^{-N_r})(-q^{1-N_1};q)_{N_1-1}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q;q)_{N_{k-1}}}$$
$$= \frac{(-q;q)_\infty (q^r, q^{2k-r}, q^{2k}; q^{2k})_\infty}{(q;q)_\infty}.$$

Corteel, Lovejoy and Mallet [17] established an overpartition analogue of the Bressoud-Rogers-Ramanujan theorem for the case $i = 1$, and the general case was obtained by Chen, Sang and Shi [14].

Theorem 1.13 (Chen-Sang-Shi)**.** *For $k \geq r \geq 1$, let $\overline{B}_0(-;1,k,r;n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n, where $\pi_i \geq \pi_{i+k-1} + 1$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - k + 1$, at most $r - 1$ of the π_i are equal to 1, and for $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + 1$ with strict inequality if π_i is overlined, then*

$$\pi_i + \cdots + \pi_{i+k-2} \equiv r - 1 + \overline{V}_\pi(\pi_i) \pmod{2}.$$

For $k \geq r \geq 1$, let $\overline{A}_0(-;1,k,r;n)$ denote the number of overpartitions of n such that non-overlined parts $\not\equiv 0, \pm r \pmod{2k-1}$. Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$\overline{A}_0(-;1,k,r;n) = \overline{B}_0(-;1,k,r;n).$$

In Theorem 1.13 and for the rest of this article, $\overline{V}_\pi(t)$ (resp. $\overline{V}_\pi(\overline{t})$), as used by Corteel, Lovejoy and Mallet [17], stands for the number of overlined parts not exceeding t (resp.

$\overline{t}$) in π. For example, for an overpartition $\pi=(\overline{7},7,6,\overline{5},5,\overline{2})$, we have $\overline{V}_\pi(5)=1$ and $\overline{V}_\pi(\overline{5})=2$.

The following generating function version of Theorem 1.13 was given by Sang and Shi [41]: For $k>r\geq 1$,

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1}}(1+q^{-N_r})(-q^{1-N_1};q)_{N_1-1}}{(q;q)_{N_1-N_2}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}$$
$$=\frac{(-q;q)_\infty(q^r,q^{2k-r-1},q^{2k-1};q^{2k-1})_\infty}{(q;q)_\infty}.$$

In this paper, we introduce two new partition functions $\overline{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ and $\overline{A}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ and build connections between $\overline{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ and $B_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$.

Definition 1.14. *For $j=0$ or 1 and $k\geq r\geq\lambda\geq 0$, define $\overline{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ to be the number of overpartitions $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ of n subject to the following conditions:*

(1) *For $1\leq i\leq\ell$, $\pi_i\equiv 0,\alpha_1,\ldots,\alpha_\lambda \pmod{\eta}$;*

(2) *Only multiples of η may be non-overlined;*

(3) *For $1\leq i\leq\ell-k+1$, $\pi_i\geq\pi_{i+k-1}+\eta$ with strict inequality if π_i is non-overlined;*

(4) *At most $r-1$ of the π_i are less than or equal to η;*

(5) *For $1\leq i\leq\ell-k+2$, if $\pi_i\leq\pi_{i+k-2}+\eta$ with strict inequality if π_i is overlined, then*

$$[\pi_i/\eta]+\cdots+[\pi_{i+k-2}/\eta]\equiv r-1+\overline{V}_\pi(\pi_i)\pmod{2-j}.$$

Definition 1.15. *For $j=0$ or 1 and $(2k-j)/2\geq r\geq\lambda\geq 0$, define the partition function $\overline{A}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ to be the number of overpartitions of n satisfying $\pi_i\equiv 0,\alpha_1,\ldots,\alpha_\lambda \pmod{\eta}$ such that*

(1) *If λ is even, then only multiplies of η may be non-overlined and there is no non-overlined part congruent to $0,\pm\eta(r-\lambda/2)\pmod{\eta(2k-\lambda+j-1)}$;*

(2) *If λ is odd and $j=1$, then only multiples of $\eta/2$ may be non-overlined, no non-overlined part is congruent to $\eta(2k-\lambda)/2\pmod{\eta(2k-\lambda)}$, no non-overlined part is congruent to $\eta\pmod{2\eta}$, no non-overlined part is congruent to $0\pmod{2\eta(2k-\lambda)}$, no non-overlined part is congruent to $\pm\eta(2r-\lambda)/2\pmod{\eta(2k-\lambda)}$, and no overlined part is congruent to $\eta/2\pmod{\eta}$ and not congruent to $\eta(2k-\lambda)/2\pmod{\eta(2k-\lambda)}$;*

(3) *If λ is odd and $j = 0$, then only multiples of $\eta/2$ may be non-overlined, no non-overlined part is congruent to $\eta \pmod{2\eta}$, no non-overlined part is congruent to $0, \pm\eta(2r-\lambda)/2 \pmod{\eta(2k-\lambda-1)}$, and no overlined part is congruent to $\eta/2 \pmod{\eta}$.*

Observe that for an overpartition π counted by $\overline{B}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ (resp. $\overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$) without overlined parts divisible by η, if we change the overlined parts in π to non-overlined parts, then we get an ordinary partition counted by $B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ (resp. $A_{1-j}(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1+j, r; n)$). Hence we say that $\overline{B}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ (resp. $\overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$) can be considered as an overpartition analogue of $B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ (resp. $A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$). In this case, $\overline{V}_\pi(t)$ reduces to the notation $V_\pi(t)$ introduced by Bressoud [10].

By means of Gordon markings, we build bijections to obtain the following relationships between $\overline{B}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ and $B_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$.

Theorem 1.16. *For $k \geq r \geq \lambda \geq 0$ and $k > \lambda$,*

$$\sum_{n\geq 0} \overline{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n = (-q^\eta; q^\eta)_\infty \sum_{n\geq 0} B_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n.$$

Theorem 1.17. *For $k > r \geq \lambda \geq 0$ and $k-1 > \lambda$,*

$$\sum_{n\geq 0} \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n = (-q^\eta; q^\eta)_\infty \sum_{n\geq 0} B_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r; n) q^n.$$

For $k-1 > \lambda$,

$$\sum_{n\geq 0} \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, k; n) q^n = (-q^\eta; q^\eta)_\infty \sum_{n\geq 0} B_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, k-1; n) q^n.$$

We also derive the generating function of $\overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$.

Theorem 1.18. *For $j = 0$ or 1 and $(2k-j)/2 \geq r \geq \lambda \geq 0$,*

$$\begin{aligned} &\sum_{n\geq 0} \overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) q^n \\ &= \frac{(-q^{\alpha_1}, \ldots, -q^{\alpha_\lambda}, -q^\eta; q^\eta)_\infty (q^{\eta(r-\frac{\lambda}{2})}, q^{\eta(2k-r-\frac{\lambda}{2}-1+j)}, q^{\eta(2k-\lambda-1+j)}; q^{\eta(2k-\lambda-1+j)})_\infty}{(q^\eta; q^\eta)_\infty}. \end{aligned} \tag{1.9}$$

By Theorem 1.2 and Theorem 1.16 with $\lambda = 0$ and $\eta = 1$, we find that for $k > r \geq 1$,

$$\sum_{n\geq 0} \overline{B}_1(-; 1, k, r; n) q^n = \frac{(-q; q)_\infty (q^r, q^{2k-r}, q^{2k}; q^{2k})_\infty}{(q; q)_\infty}.$$

Combining with Theorem 1.18 with $\lambda = 0$, $\eta = 1$ and $j = 1$, we can recover Theorem 1.12 for $k > r \geq 1$. By Theorem 1.4 and Theorem 1.16 with $\lambda = 1$ and $\eta = 2$, we find that for $k > r \geq 1$,

$$\sum_{n\geq 0} \overline{B}_1(1;2,k,r;n)q^n = \frac{(-q^2;q^2)_\infty(-q;q^2)_\infty(q^{2r-1},q^{4k-2r-1},q^{4k-2};q^{4k-2})_\infty}{(q^2;q^2)_\infty}.$$

Applying Theorem 1.18 with $\lambda = 1$, $\eta = 2$ and $j = 1$, we obtain a new overpartition analogue of the Andrews-Göllnitz-Gordon theorem.

Theorem 1.19. *For $k > r \geq 1$, let $\overline{B}_1(1;2,k,r;n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n such that (1) only even parts may be non-overlined; (2) $\pi_i \geq \pi_{i+k-1} + 2$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - k + 1$; (3) at most $r - 1$ of the π_i are less than or equal to 2.*

For $k > r \geq 1$, let $\overline{A}_1(1;2,k,r;n)$ denote the number of overpartitions of n such that (1) no non-overlined part is congruent to $2k-1 \pmod{4k-2}$; (2) no non-overlined part is congruent to $2 \pmod 4$; (3) no non-overlined part is congruent to $0 \pmod{8k-4}$; (4) no non-overlined part is congruent to $\pm(2r-1) \pmod{4k-2}$; (5) no overlined part is congruent to $1 \pmod 2$ and not congruent to $2k-1 \pmod{4k-2}$). Then, for $k > r \geq 1$ and $n \geq 0$,

$$\overline{A}_1(1;2,k,r;n) = \overline{B}_1(1;2,k,r;n).$$

The generating function version of Theorem 1.19 will be given in our subsequent paper [25]. It should be mentioned that Lovejoy [33] obtained an overpartition analogue of the Andrews-Göllnitz-Gordon theorem for $r = k$ and He, Ji, Wang and Zhao [24] found an overpartition analogue for the general case.

In view of Theorems 1.11, 1.17 and Theorem 1.18 for $j = 0$, we obtain the following overpartition analogue of Bressoud's Conjecture 1.8 for $j = 0$.

Theorem 1.20. *For $k \geq r \geq \lambda \geq 0$, $k - 1 > \lambda$ and $n \geq 0$, we have*

$$\overline{A}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n) = \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n).$$

The generating function version of Theorem 1.20 can be derived with the aid of Bailey pairs.

Theorem 1.21. *For $k \geq r > \lambda \geq 0$,*

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}(1+q^{-\eta N_r})(-q^{\eta-\eta N_{\lambda+1}};q^\eta)_{N_{\lambda+1}-1}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{2\eta};q^{2\eta})_{N_{k-1}}}$$
$$\times(-q^{\eta+\eta N_\lambda};q^\eta)_\infty \prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s} \prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty$$
$$= \frac{(-q^{\alpha_1},\ldots -q^{\alpha_\lambda},-q^\eta;q^\eta)_\infty(q^{(r-\frac{\lambda}{2})\eta},q^{(2k-r-\frac{\lambda}{2}-1)\eta},q^{(2k-\lambda-1)\eta};q^{(2k-\lambda-1)\eta})_\infty}{(q^\eta;q^\eta)_\infty}. \qquad (1.10)$$

Combining Theorem 1.18 for $j = 0$, Theorem 1.20 and Theorem 1.21, we obtain the following generating function of $\overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$, which can be regarded as the overpartition analogue of Bressoud's Conjecture 1.10 for $j = 0$.

Theorem 1.22. *For $k \geq r > \lambda \geq 0$ and $k - 1 > \lambda$,*

$$\sum_{n\geq 0} \overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)q^n$$

$$= \sum_{N_1\geq\cdots\geq N_{k-1}\geq 0} \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}(1 + q^{-\eta N_r})(-q^{\eta-\eta N_{\lambda+1}}; q^\eta)_{N_{\lambda+1}-1}}{(q^\eta; q^\eta)_{N_1-N_2} \cdots (q^\eta; q^\eta)_{N_{k-2}-N_{k-1}}(q^{2\eta}; q^{2\eta})_{N_{k-1}}}$$

$$\times (-q^{\eta+\eta N_\lambda}; q^\eta)_\infty \prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s}; q^\eta)_{N_s} \prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}}; q^\eta)_\infty.$$

Theorem 1.20 and Theorem 1.21 specialize to overpartition analogues of a number of classical partition theorems. Setting $\lambda = 0$, $\eta = 1$, $k = 3$ and $r = 2$, we obtain an overpartition analogue of Euler's partition theorem [21]. Recall that Euler's partition theorem states that for $n \geq 1$, the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

Theorem 1.23. *Let $\overline{B}_0(-; 1, 3, 2; n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n, where $\pi_i \geq \pi_{i+2} + 1$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - 2$, and for $1 \leq i \leq \ell - 1$, if $\pi_i \leq \pi_{i+1} + 1$ with strict inequality if π_i is overlined, then $\pi_i + \pi_{i+1} \equiv 1 + \overline{V}_\pi(\pi_i) \pmod 2$. Let $\overline{A}_0(-; 1, 3, 2; n)$ denote the number of overpartitions of n such that no non-overlined part is congruent to $\equiv 0, \pm 2 \pmod 5$. Then, for $n \geq 0$,*

$$\overline{A}_0(-; 1, 3, 2; n) = \overline{B}_0(-; 1, 3, 2; n).$$

The generating function version takes the form:

$$\sum_{N_1\geq N_2\geq 0} \frac{q^{N_1^2+N_2^2+N_2}(1 + q^{-N_2})(-q^{1-N_1}; q)_{N_1-1}}{(q; q)_{N_1-N_2}(q^2; q^2)_{N_2}} = \frac{(-q; q)_\infty(q^2, q^3, q^5; q^5)_\infty}{(q; q)_\infty}.$$

For an overpartition $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ counted by $\overline{B}_0(-; 1, 3, 2; n)$, if there are no overlined parts in π, then $\overline{V}_\pi(\pi_i) = 0$ for $1 \leq i \leq \ell$. This implies that $\pi_i + \pi_{i+1}$ is odd if $\pi_i \leq \pi_{i+1} + 1$. Hence we deduce that $\pi_i > \pi_{i+1}$ for $1 \leq i \leq \ell - 1$. Therefore, π is a partition into distinct parts. For this reason, Theorem 1.23 can be perceived as an overpartition analogue of Euler's partition theorem.

Putting $\lambda = 0$ and $\eta = 1$ in Theorem 1.20, we are led to the overpartition analogue of the Bressoud-Rogers-Ramanujan theorem due to Chen, Sang and Shi [14]. In a similar way, Theorem 1.21 yields the generating function version found by Sang and Shi [41]. Setting $\lambda = 1$ and $\eta = 2$ in Theorem 1.20, we find an overpartition analogue of the Bressoud-Göllnitz-Gordon theorem.

Theorem 1.24. *For $k > 2$ and $k \geq r \geq 1$, let $\overline{B}_0(1;2,k,r;n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of n such that only even parts may be non-overlined, $\pi_i \geq \pi_{i+k-1} + 2$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - k + 1$, at most $r-1$ of the π_i are less than or equal to 2, and for $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + 2$ with strict inequality if π_i is overlined, then*

$$[\pi_i/2] + \cdots + [\pi_{i+k-2}/2] \equiv r - 1 + \overline{V}_\pi(\pi_i) \pmod 2.$$

For $k > 2$ and $k \geq r \geq 1$, let $\overline{A}_0(1;2,k,r;n)$ denote the number of overpartitions of n such that no non-overlined part is congruent to $2 \pmod 4$, no non-overlined part is congruent to $0, \pm(2r-1) \pmod{4k-4}$, and no overlined part is congruent to $1 \pmod 2$. Then, for $k > 2$, $k \geq r \geq 1$ and $n \geq 0$,

$$\overline{A}_0(1;2,k,r;n) = \overline{B}_0(1;2,k,r;n).$$

He, Wang and Zhao [26] established an overpartition analogue of the Bressoud-Göllnitz-Gordon theorem. Putting $\lambda = 1$ and $\eta = 2$ in Theorem 1.21, we get the generating function version of Theorem 1.24: For $k \geq r > 1$,

$$\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{2(N_1^2 + \cdots + N_{k-1}^2 + N_r + \cdots + N_{k-1})}(1 + q^{-2N_r})}{(q^2;q^2)_{N_1 - N_2} \cdots (q^2;q^2)_{N_{k-2} - N_{k-1}} (q^4;q^4)_{N_{k-1}}}$$
$$\times (-q^{1-2N_1};q^2)_{N_1} (-q^{2-2N_2};q^2)_{N_2 - 1} (-q^{2+2N_1};q^2)_\infty$$
$$= \frac{(-q;q^2)_\infty (-q^2;q^2)_\infty (q^{2r-1}, q^{4k-2r-3}, q^{4k-4}; q^{4k-4})_\infty}{(q^2;q^2)_\infty}.$$

This paper is organized as follows. In Section 2, we present a proof of Theorem 1.18. In Section 3, we introduce the notions of Gordon marking, reverse Gordon marking, and $(k-1)$-bands of an overpartition counted by $\overline{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$. Furthermore, we give a criterion to determine whether an overpartition counted by $\overline{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ is counted by $\overline{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ as well. In Section 4, we define the forward move and the backward move based on the Gordon marking and the reverse Gordon marking of an overpartition counted by $\overline{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$. These operations allow us to provide a combinatorial proof of Theorem 1.16. Section 5 is devoted to a combinatorial proof of Theorem 1.17. In Section 6, we give a proof of Theorem 1.21 with the aid of Bailey pairs. In Section 7, we discuss possible directions for future work.

2 Proof of Theorem 1.18

As mentioned in the introduction, the function $\overline{A}_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ can be viewed as the overpartition analogue of $A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$ introduced by Bressoud [10]. Similar to the case for $A_j(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r; n)$, it is not difficult to establish the generating function of $\overline{A}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ stated as in Theorem 1.18. For completeness, we include a detailed derivation.

Proof of Theorem 1.18. Clearly, the right-hand side of (1.9) can be interpreted as the generating function of $\overline{A}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ when λ is even. It remains to show that the right-hand side of (1.9) is also the generating function of $\overline{A}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ when λ is odd. When λ is odd, it is clear from (1.5) that $\eta=\alpha_{(\lambda+1)/2}+\alpha_{\lambda+1-(\lambda+1)/2}=2\alpha_{(\lambda+1)/2}$. This implies that η must be even in this event.

When $j=1$, by definition, we have for $k>r\geq\lambda\geq 0$,

$$\begin{aligned}\sum_{n\geq 0}&\overline{A}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n\\ &=(-q^{\alpha_1},\ldots,-q^{\alpha_{(\lambda-1)/2}},-q^{\alpha_{(\lambda+3)/2}},\ldots,-q^{\alpha_\lambda},-q^{\eta};q^{\eta})_\infty(-q^{\eta(2k-\lambda)/2};q^{\eta(2k-\lambda)})_\infty\\ &\quad\times\frac{(q^{\eta(2r-\lambda)/2},q^{\eta(4k-2r-\lambda)/2},q^{\eta(2k-\lambda)/2};q^{\eta(2k-\lambda)})_\infty(q^{2\eta(2k-\lambda)};q^{2\eta(2k-\lambda)})_\infty(q^{\eta};q^{2\eta})_\infty}{(q^{\eta/2};q^{\eta/2})_\infty}.\end{aligned}\tag{2.1}$$

Since η is even, we find that

$$\frac{(q^{\eta};q^{2\eta})_\infty}{(q^{\eta/2};q^{\eta/2})_\infty}=\frac{(q^{\eta/2},-q^{\eta/2};q^{\eta})_\infty}{(q^{\eta/2},q^{\eta};q^{\eta})_\infty}=\frac{(-q^{\eta/2};q^{\eta})_\infty}{(q^{\eta};q^{\eta})_\infty},\tag{2.2}$$

and

$$(-q^{\eta(2k-\lambda)/2},q^{\eta(2k-\lambda)/2};q^{\eta(2k-\lambda)})_\infty(q^{2\eta(2k-\lambda)};q^{2\eta(2k-\lambda)})_\infty=(q^{\eta(2k-\lambda)};q^{\eta(2k-\lambda)})_\infty.\tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1), and noting that $\alpha_{(\lambda+1)/2}=\eta/2$, we obtain that for $k>r\geq\lambda\geq 0$,

$$\begin{aligned}\sum_{n\geq 0}&\overline{A}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n\\ &=\frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda},-q^{\eta};q^{\eta})_\infty(q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2})},q^{\eta(2k-\lambda)};q^{\eta(2k-\lambda)})_\infty}{(q^{\eta};q^{\eta})_\infty}.\end{aligned}\tag{2.4}$$

When $j=0$, by definition, we have for $k\geq r\geq\lambda\geq 0$,

$$\begin{aligned}\sum_{n\geq 0}&\overline{A}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n\\ &=(-q^{\alpha_1},\ldots,-q^{\alpha_{(\lambda-1)/2}},-q^{\alpha_{(\lambda+3)/2}},\ldots,-q^{\alpha_\lambda},-q^{\eta};q^{\eta})_\infty\\ &\quad\times\frac{(q^{\eta(2r-\lambda)/2},q^{\eta(4k-2r-\lambda-2)/2},q^{\eta(2k-\lambda-1)};q^{\eta(2k-\lambda-1)})_\infty(q^{\eta};q^{2\eta})_\infty}{(q^{\eta/2};q^{\eta/2})_\infty}.\end{aligned}\tag{2.5}$$

Substituting (2.2) into (2.5), we obtain that for $k \geq r \geq \lambda \geq 0$,

$$\begin{aligned}&\sum_{n\geq 0}\overline{A}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)q^n\\&\quad=\frac{(-q^{\alpha_1},\ldots,-q^{\alpha_\lambda},-q^{\eta};q^{\eta})_\infty(q^{\eta(r-\frac{\lambda}{2})},q^{\eta(2k-r-\frac{\lambda}{2}-1)},q^{\eta(2k-\lambda-1)};q^{\eta(2k-\lambda-1)})_\infty}{(q^{\eta};q^{\eta})_\infty}.\end{aligned} \tag{2.6}$$

Combining (2.4) and (2.6), we conclude that (1.9) holds when λ is odd. This completes the proof. ∎

3 The (reverse) Gordon marking and $(k-1)$-bands

The main objective of this section is to give a criterion to determine whether an overpartition counted by $\overline{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ is also counted by $\overline{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$. Let $j=0$ or 1 and let λ, k and r be integers such that $k\geq r\geq\lambda\geq 0$. Let $\overline{\mathcal{B}}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ denote the set of overpartitions counted by $\overline{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ for $n\geq 0$. Let $\mathcal{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ denote the set of partitions counted by $B_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r;n)$ for $n\geq 0$. As mentioned in the introduction, we could use $\mathcal{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ to denote the set of overpartitions in $\overline{\mathcal{B}}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ without overlined parts divisible by η. For $\pi\in\overline{\mathcal{B}}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$, we call π a $\overline{B}_j$-overpartition for short.

The Gordon marking of an ordinary partition was introduced by Kurşungöz in [29, 30]. Kim [27] introduced the Gordon marking of an ordinary partition in $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$, which generalizes the Gordon marking of an ordinary partition. The Gordon marking of an overpartition was defined by Chen, Sang and Shi [13]. Now we define the Gordon marking of a $\overline{B}_1$-overpartition. Bear in mind that the parts in an overpartition are ordered as follows:

$$1<\bar{1}<2<\bar{2}<\cdots.$$

For positive integers t and b, we define $t\pm b$ (resp. $\bar{t}\pm b$) as a non-overlined part of size $t\pm b$ (resp. an overlined part of size $t\pm b$).

Definition 3.1 (Gordon marking). *For $k\geq r\geq\lambda\geq 0$, let $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ be an overpartition satisfying* (1) *and* (2) *in Definition 1.14. Assign a positive integer to each part of π as follows: First, assign 1 to π_ℓ. Then, for each π_i, assign s to π_i, where s is the smallest positive integer that is not used to mark the parts π_m such that $m>i$ and $\pi_m\geq\pi_i-\eta$ with strict inequality if π_i is overlined. Denote the Gordon marking of π by $G(\pi)$.*

It can be seen that for each π_i, the mark of π_i is the smallest positive integer that is not used to mark the parts after π_i in $[\pi_i-\eta,\pi_i]$ (resp. $(\pi_i-\eta,\pi_i)$) if π_i is non-overlined (resp. overlined). Assume that π_ℓ is assigned with 1. Then the part π_m of π is in $[\pi_i-\eta,\pi_i]$ (resp. $(\pi_i-\eta,\pi_i)$) means that $\pi_i-\eta\leq\pi_m\leq\pi_i$ (resp. $\pi_i-\eta<\pi_m<\pi_i$). For notational convenience, we use $I(\pi_i-\eta,\pi_i)$ to denote the interval $[\pi_i-\eta,\pi_i]$ if π_i is non-overlined, or the interval $(\pi_i-\eta,\pi_i)$ if π_i is overlined.

For example, let π be an overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ given by

$$\begin{aligned}\pi=(&\overline{80},80,80,\overline{70},70,\overline{69},\overline{60},60,\overline{55},\overline{51},50,\overline{49},\overline{45},\overline{41},\overline{39},\overline{35},\\&\overline{29},\overline{20},20,20,\overline{11},\overline{10},\overline{9},\overline{5},\overline{1}).\end{aligned}\tag{3.1}$$

The Gordon marking of π is given by

$$\begin{aligned}G(\pi)=(&\overline{80}_1,80_4,80_2,\overline{70}_1,70_3,\overline{69}_2,\overline{60}_4,60_1,\overline{55}_2,\overline{51}_3,50_4,\overline{49}_1,\overline{45}_2,\overline{41}_3,\overline{39}_1,\overline{35}_2,\\&\overline{29}_1,\overline{20}_4,20_3,20_2,\overline{11}_1,\overline{10}_4,\overline{9}_3,\overline{5}_2,\overline{1}_1),\end{aligned}\tag{3.2}$$

where the subscript of each part represents the mark in the Gordon marking.

For $k\geq r\geq\lambda\geq 0$, let $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ be an overpartition satisfying (1) and (2) in Definition 1.14. If the condition (3) in Definition 1.14 is also fulfilled, that is, for $1\leq i\leq \ell-k+1$, $\pi_i\geq\pi_{i+k-1}+\eta$ with strict inequality if π_i is non-overlined, then for each π_i, the number of parts after π_i belonging to $I(\pi_i-\eta,\pi_i)$ is at most $k-2$, so the marks in the Gordon marking of π do not exceed $k-1$. For the overpartition π in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ defined in (3.1), by (3.2), we see that the largest mark in $G(\pi)$ is 4.

If we assign a mark to each part starting with the largest part instead, then the resulting marking will be called the reverse Gordon marking.

Definition 3.2 (Reverse Gordon marking). *For $k\geq r\geq\lambda\geq 0$, let $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ be an overpartition satisfying* (1) *and* (2) *in Definition 1.14. Assign a positive integer to each part of π as follows: First assign* 1 *to π_1. Then, for each π_i, assign s to π_i, where s is the smallest positive integer that is not used to mark the parts π_m such that $m<i$ and $\pi_m\leq\pi_i+\eta$ with strict inequality if π_i is overlined. Denote the reverse Gordon marking of π by $RG(\pi)$.*

Analogously, for each π_i, the mark of π_i is the smallest positive integer that is not used to mark the parts before π_i belonging to $I(\pi_i,\pi_i+\eta)$. Furthermore, for π in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$, the marks in the reverse Gordon marking of π do not exceed $k-1$.

For the overpartition π in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ defined in (3.1), the reverse Gordon marking of π reads

$$\begin{aligned}RG(\pi)=(&\overline{80}_1,80_2,80_3,\overline{70}_1,70_4,\overline{69}_2,\overline{60}_1,60_3,\overline{55}_2,\overline{51}_4,50_1,\overline{49}_3,\overline{45}_2,\overline{41}_4,\overline{39}_1,\overline{35}_2,\\&\overline{29}_1,\overline{20}_2,20_3,20_4,\overline{11}_1,\overline{10}_2,\overline{9}_3,\overline{5}_4,\overline{1}_1),\end{aligned}$$

from which we see that the largest mark in $RG(\pi)$ is 4.

We proceed to give a criterion to determine whether a $\overline{B}_1$-overpartition is also a $\overline{B}_0$-overpartition. Let $\pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$ be an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. If there are no $k-1$ consecutive parts $\pi_i,\pi_{i+1},\ldots,\pi_{i+k-2}$ in π such that

$$\pi_i\leq\pi_{i+k-2}+\eta \text{ with strict inequality if } \pi_i \text{ is overlined,}\tag{3.3}$$

then by Definition 1.14, we see that π is also in $\overline{\mathcal{B}}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. Assume that there exist $k-1$ consecutive parts π_i, $\pi_{i+1},\ldots,\pi_{i+k-2}$ in π satisfying (3.3). By definition, π is not in $\overline{\mathcal{B}}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ if

$$[\pi_i/\eta]+\cdots+[\pi_{i+k-2}/\eta]\equiv r+\overline{V}_\pi(\pi_i) \pmod 2.$$

It follows that π is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ if and only if for any $k-1$ consecutive parts π_i, $\pi_{i+1},\ldots,\pi_{i+k-2}$ in π satisfying (3.3), we have

$$[\pi_i/\eta]+\cdots+[\pi_{i+k-2}/\eta]\equiv r-1+\overline{V}_\pi(\pi_i) \pmod 2. \tag{3.4}$$

The above $k-1$ consecutive parts satisfying (3.3) will be called *a* $(k-1)$*-band of* π in this sense that the difference between the largest element and the smallest element in a $(k-1)$-band is at most η. For the $(k-1)$-band $\{\pi_{i+l}\}_{0\leq l\leq k-2}$, if $\{\pi_{i+l}\}_{0\leq l\leq k-2}$ satisfy the congruence condition (3.4), then we say that the $(k-1)$-band $\{\pi_{i+l}\}_{0\leq l\leq k-2}$ is even. Otherwise, we say that it is odd.

For example, let π be the overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ defined in (3.1), where $k=5$. There are twelve 4-bands in π. It can be checked that all of them are even.

$$\{80,80,\overline{70},70\},\{70,\overline{69},\overline{60},60\},\{\overline{60},60,\overline{55},\overline{51}\},\{60,\overline{55},\overline{51},50\},$$
$$\{\overline{55},\overline{51},50,\overline{49}\},\{\overline{51},50,\overline{49},\overline{45}\},\{50,\overline{49},\overline{45},\overline{41}\},\{\overline{29},\overline{20},20,20\},$$
$$\{\overline{20},20,20,\overline{11}\},\{20,20,\overline{11},\overline{10}\},\{\overline{11},\overline{10},\overline{9},\overline{5}\},\{\overline{10},\overline{9},\overline{5},\overline{1}\}.$$

For the overpartitions π in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$, we see that π is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ if and only if all $(k-1)$-bands of π are even. For $k\geq r\geq\lambda\geq 0$, let π be an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. For each $(k-1)$-band $\{\pi_{i+l}\}_{0\leq l\leq k-2}$ of π, it is easy to see that the marks of π_{i+l} are distinct in the Gordon marking and the reverse Gordon marking of π. Hence there exists one part in $\{\pi_{i+l}\}_{0\leq l\leq k-2}$ marked with $k-1$ in the Gordon marking and the reverse Gordon marking of π.

Next we show that we may restrict our attention to certain special $(k-1)$-bands to determine whether π is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. Such special $(k-1)$-bands will be classified into two kinds depending on the positions of the $(k-1)$-marked parts of the Gordon marking or the reverse Gordon marking in the $(k-1)$-bands.

The $(k-1)$*-bands of the first kind* will be concerned with the case in which the $(k-1)$-marked part in the Gordon marking is the largest element in the band. Assume that there are N parts marked with $k-1$ in $G(\pi)$, and denote these $(k-1)$-marked parts by $\tilde{g}_1(\pi)>\tilde{g}_2(\pi)>\cdots>\tilde{g}_N(\pi)$. For each $(k-1)$-marked part $\tilde{g}_p(\pi)$ in $G(\pi)$, the number of the $(k-1)$-bands of π including $\tilde{g}_p(\pi)$ is at most $k-1$. We claim that there is a $(k-1)$-band of π such that $\tilde{g}_p(\pi)$ is the largest element of this $(k-1)$-band. Assume that $\tilde{g}_p(\pi)$ is the s-th part π_s of π. By Definition 3.1, we deduce that there exist $k-2$ parts π_m such that $m>s$ and $\pi_m\geq\pi_s-\eta$ with strict inequality if π_s is overlined. This implies that $\pi_s,\pi_{s+1},\ldots,\pi_{s+k-2}$ in π satisfy (3.3), that is, $\{\pi_{s+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of π.

Furthermore, the $(k-1)$-marked part $\tilde{g}_p(\pi)$ is the largest element of this $(k-1)$-band. So the claim is proved. *Such a* $(k-1)$*-band is called the* $(k-1)$*-band induced by* $\tilde{g}_p(\pi)$, *denoted* $\{\tilde{g}_p(\pi)\}_{k-1}$. Obviously, $\{\tilde{g}_1(\pi)\}_{k-1}, \{\tilde{g}_2(\pi)\}_{k-1}, \ldots, \{\tilde{g}_N(\pi)\}_{k-1}$ of π are disjoint.

For example, for the overpartition π given in (3.1), there are five 4-marked parts in $G(\pi)$, namely, $\tilde{g}_1(\pi) = 80$, $\tilde{g}_2(\pi) = \overline{60}$, $\tilde{g}_3(\pi) = 50$, $\tilde{g}_4(\pi) = \overline{20}$ and $\tilde{g}_5(\pi) = \overline{10}$. The 4-bands induced by $\tilde{g}_1(\pi), \tilde{g}_2(\pi), \tilde{g}_3(\pi), \tilde{g}_4(\pi)$ and $\tilde{g}_5(\pi)$ are illustrated in $G(\pi)$ below:

$$
\begin{aligned}
G(\pi) = (&\overline{80}_1, \overbrace{80_4, 80_2, \overline{70}_1, 70_3}^{\{80\}_4}, \overline{69}_2, \overbrace{\overline{60}_4, 60_1, \overline{55}_2, \overline{51}_3}^{\{\overline{60}\}_4}, \overbrace{50_4, \overline{49}_1, \overline{45}_2, \overline{41}_3}^{\{50\}_4}, \overline{39}_1, \overline{35}_2,\\
&\overline{29}_1, \underbrace{\overline{20}_4, 20_3, 20_2, \overline{11}_1}_{\{\overline{20}\}_4}, \underbrace{\overline{10}_4, \overline{9}_3, \overline{5}_2, \overline{1}_1}_{\{\overline{10}\}_4}).
\end{aligned}
$$

We now consider *the* $(k-1)$*-bands of the second kind* under the condition that the $(k-1)$-marked part in the reverse Gordon marking is the smallest part in the band. Assume that there are M parts marked with $k-1$ in $RG(\pi)$, namely, $\tilde{r}_1(\pi) > \tilde{r}_2(\pi) > \cdots > \tilde{r}_M(\pi)$. By the same reasoning, we see that there is a $(k-1)$-band of π in which $\tilde{r}_p(\pi)$ is the smallest element. *Such a* $(k-1)$*-band is called the* $(k-1)$*-band induced by* $\tilde{r}_p(\pi)$, *denoted* $\{\tilde{r}_p(\pi)\}_{k-1}$. Clearly, $\{\tilde{r}_1(\pi)\}_{k-1}, \{\tilde{r}_2(\pi)\}_{k-1}, \ldots, \{\tilde{r}_M(\pi)\}_{k-1}$ of π are disjoint.

For example, for the overpartition π given in (3.1), there are five 4-marked parts in $RG(\pi)$, which are $\tilde{r}_1(\pi) = 70$, $\tilde{r}_2(\pi) = \overline{51}$, $\tilde{r}_3(\pi) = \overline{41}$, $\tilde{r}_4(\pi) = 20$ and $\tilde{r}_5(\pi) = \overline{5}$. The 4-bands induced by $\tilde{r}_1(\pi)$, $\tilde{r}_2(\pi)$, $\tilde{r}_3(\pi)$, $\tilde{r}_4(\pi)$ and $\tilde{r}_5(\pi)$ are displayed below:

$$
\begin{aligned}
RG(\pi) = (&\overline{80}_1, \overbrace{80_2, 80_3, \overline{70}_1, 70_4}^{\{70\}_4}, \overline{69}_2, \overbrace{\overline{60}_1, 60_3, \overline{55}_2, \overline{51}_4}^{\{\overline{51}\}_4}, \overbrace{50_1, \overline{49}_3, \overline{45}_2, \overline{41}_4}^{\{\overline{41}\}_4}, \overline{39}_1, \overline{35}_2,\\
&\underbrace{\overline{29}_1, \overline{20}_2, 20_3, 20_4}_{\{20\}_4}, \underbrace{\overline{11}_1, \overline{10}_2, \overline{9}_3, \overline{5}_4}_{\{\overline{5}\}_4}, \overline{1}_1).
\end{aligned}
$$

It remains to show that the $(k-1)$-bands of π induced by the $(k-1)$-marked parts in $G(\pi)$ or $RG(\pi)$ are enough to determine whether π is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. For this purpose, we need the following property relating $G(\pi)$ and $RG(\pi)$.

Proposition 3.3. *For* $k \geq r \geq \lambda \geq 0$*, let* π *be an overpartition in* $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$*. Assume that there are* N *parts marked with* $k-1$ *in the Gordon marking of* π*, say,* $\tilde{g}_1(\pi) > \tilde{g}_2(\pi) > \cdots > \tilde{g}_N(\pi)$*, and there are* M *parts marked with* $k-1$ *in the reverse Gordon marking of* π*, say,* $\tilde{r}_1(\pi) > \tilde{r}_2(\pi) > \cdots > \tilde{r}_M(\pi)$*. Then* $N = M$*. Moveover, for each* $1 \leq i \leq N$*, we have* $\tilde{g}_i(\pi) \in \{\tilde{r}_i(\pi)\}_{k-1}$ *and* $\tilde{r}_i(\pi) \in \{\tilde{g}_i(\pi)\}_{k-1}$*, where* $\{\tilde{g}_i(\pi)\}_{k-1}$ *(resp.* $\{\tilde{r}_i(\pi)\}_{k-1}$*) is the* $(k-1)$*-band of* π *induced by* $\tilde{g}_i(\pi)$ *(resp.* $\tilde{r}_i(\pi)$*).*

Proof. For $N = 0$, there are no $(k-1)$-marked parts in $G(\pi)$, and so there are no $(k-1)$-bands in π. This implies that there are no $(k-1)$-marked parts in $RG(\pi)$. It follows that $M = 0$. Conversely, if $M = 0$, then $N = 0$.

We next consider the case $M, N > 0$. We first prove that $M \geq N$. For each fixed $(k-1)$-marked part $\tilde{g}_i(\pi)$ in $G(\pi)$, where $1 \leq i \leq N$, assume that $\tilde{g}_i(\pi)$ is the g_i-th part of $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$, that is, $\tilde{g}_i(\pi) = \pi_{g_i}$. Since π_{g_i} is the largest element in the $(k-1)$-band induced by π_{g_i}, we find that the parts

$$\pi_{g_i} \geq \pi_{g_i+1} \geq \cdots \geq \pi_{g_i+k-2}$$

are in the $(k-1)$-band of π induced by π_{g_i}. Moreover, the marks of these parts in $RG(\pi)$ are distinct. It follows that there exists t_i such that $0 \leq t_i \leq k-2$ and $\pi_{g_i+t_i}$ is marked with $k-1$ in $RG(\pi)$. Since the $(k-1)$-bands $\{\tilde{g}_1(\pi)\}_{k-1}, \{\tilde{g}_2(\pi)\}_{k-1}, \ldots, \{\tilde{g}_N(\pi)\}_{k-1}$ of π are disjoint, the parts $\pi_{g_1+t_1}$, $\pi_{g_2+t_2}, \ldots$, $\pi_{g_N+t_N}$ are distinct, which are marked with $k-1$ in $RG(\pi)$. This means that $M \geq N$. A similar argument yields $N \geq M$. We conclude that $M = N$. Note that the above proof also indicates that $\tilde{r}_i(\pi) = \pi_{g_i+t_i}$, which implies that $\tilde{r}_i(\pi) \in \{\tilde{g}_i(\pi)\}_{k-1}$ for $1 \leq i \leq N$. Similarly, $\tilde{g}_i(\pi) \in \{\tilde{r}_i(\pi)\}_{k-1}$ for $1 \leq i \leq N$, and thus the proof is complete. ∎

For example, for the overpartition π given in (3.1), there are five 4-marked parts in $G(\pi)$, and in the meantime there are five 4-marked parts in $RG(\pi)$.

We are now in a position to present the main result of this section.

Theorem 3.4. *For $k \geq r \geq \lambda \geq 0$, $k-1 > \lambda$ and $N \geq 0$, let $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ be an overpartition in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ with N parts marked with $k-1$ in $G(\pi)$ (resp. $RG(\pi)$), say $\tilde{g}_1(\pi) > \tilde{g}_2(\pi) > \cdots > \tilde{g}_N(\pi)$ (resp. $\tilde{r}_1(\pi) > \tilde{r}_2(\pi) > \cdots > \tilde{r}_N(\pi)$). Then π is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ if and only if for all $1 \leq i \leq N$, $\{\tilde{g}_i(\pi)\}_{k-1}$ (resp. $\{\tilde{r}_i(\pi)\}_{k-1}$) are even. In particular, for $\lambda = k-1$, the assertion holds if there are no overlined parts divisible by η in π.*

For example, the overpartition π given in (3.1) is also an overpartition in $\overline{\mathcal{B}}_0(1, 5, 9; 10, 5, 4)$. To prove Theorem 3.4, we need the following lemma.

Lemma 3.5. *For $k \geq r \geq \lambda \geq 0$ and $k-1 > \lambda$, let $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ be an overpartition in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, and let $\{\pi_{c+l}\}_{0 \leq l \leq k-2}$ and $\{\pi_{d+l}\}_{0 \leq l \leq k-2}$ be two $(k-1)$-bands of π. If $\pi_c > \pi_d$ and $\pi_c \leq \pi_{d+k-2} + 2\eta$ with strict inequality if π_c is overlined, then $\{\pi_{c+l}\}_{0 \leq l \leq k-2}$ and $\{\pi_{d+l}\}_{0 \leq l \leq k-2}$ are of the same parity. In particular, for $\lambda = k-1$, the assertion holds if there are no overlined parts divisible by η in π.*

The above lemma enables us to establish the following proposition, which, together with Proposition 3.3, leads to Theorem 3.4.

Proposition 3.6. *For $k \geq r \geq \lambda \geq 0$ and $k-1 > \lambda$, let $\tilde{g}_p(\pi)$ be a $(k-1)$-marked part in the Gordon marking of an overpartition π in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. Then the $(k-1)$-bands of π including $\tilde{g}_p(\pi)$ and the $(k-1)$-band induced by $\tilde{g}_p(\pi)$ are of the same parity. In particular, for $\lambda = k-1$, the assertion holds if there are no overlined parts divisible by η in π.*

For example, for the overpartition π given in (3.1), the 4-bands induced by $\tilde{g}_1(\pi)$, $\tilde{g}_2(\pi)$, $\tilde{g}_3(\pi)$, $\tilde{g}_4(\pi)$ and $\tilde{g}_5(\pi)$ are all even. Moreover, for any $1 \leq p \leq 5$, the $(k-1)$-bands including $\tilde{g}_p(\pi)$ and the $(k-1)$-band induced by $\tilde{g}_p(\pi)$ are of the same parity.

It should be mentioned that for $\lambda = k-1$, if there is an overlined part divisible by η in π, then a $(k-1)$-band of π including the $(k-1)$-marked part $\tilde{g}_p(\pi)$ may have a different parity from that of the $(k-1)$-band induced by $\tilde{g}_p(\pi)$. For example, let $\pi = (\overline{21}, \overline{20}, \overline{15}, \overline{11}, \overline{9}, \overline{5})$ be an overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,4,3)$, where $\lambda = 3$, $k = 4$, $\eta = 10$ and $r = 3$. The Gordon marking of π is

$$G(\pi) = (\overline{21}_3, \overline{20}_2, \overline{15}_1, \overline{11}_3, \overline{9}_2, \overline{5}_1),$$

from which we see that there are two 3-marked parts $\tilde{g}_1(\pi) = \overline{21}$ and $\tilde{g}_2(\pi) = \overline{11}$, and so we get the 3-band $\{\overline{11}, \overline{9}, \overline{5}\}$ induced by $\tilde{g}_2(\pi) = \overline{11}$ along with a 3-band $\{\overline{20}, \overline{15}, \overline{11}\}$ including $\tilde{g}_2(\pi) = \overline{11}$. Apparently, the 3-band $\{\overline{11}, \overline{9}, \overline{5}\}$ is even, since

$$[\overline{11}/10] + [\overline{9}/10] + [\overline{5}/10] = 1 \equiv r - 1 + \overline{V}_\pi(\overline{11}) \pmod 2,$$

and the 3-band $\{\overline{20}, \overline{15}, \overline{11}\}$ is odd, since

$$[\overline{20}/10] + [\overline{15}/10] + [\overline{11}/10] = 4 \equiv r + \overline{V}_\pi(\overline{20}) \pmod 2.$$

In the remainder of this section, we present the proofs of Lemma 3.5 and Proposition 3.6.

Proof of Lemma 3.5. To show that $\{\pi_{d+l}\}_{0\leq l\leq k-2}$ and $\{\pi_{c+l}\}_{0\leq l\leq k-2}$ are of the same parity, we write

$$[\pi_{d+k-2}/\eta] + \cdots + [\pi_d/\eta] \equiv a + \overline{V}_\pi(\pi_d) \pmod 2, \tag{3.5}$$

where $a = r-1$ or r. Trivially, $\{\pi_{d+l}\}_{0\leq l\leq k-2}$ is even when $a = r-1$, and $\{\pi_{d+l}\}_{0\leq l\leq k-2}$ is odd when $a = r$.

We intend to prove that

$$[\pi_{c+k-2}/\eta] + \cdots + [\pi_c/\eta] \equiv a + \overline{V}_\pi(\pi_c) \pmod 2, \tag{3.6}$$

where a is given as in (3.5). Since $\{\pi_{d+l}\}_{0\leq l\leq k-2}$ and $\{\pi_{c+l}\}_{0\leq l\leq k-2}$ are $(k-1)$-bands of π, we have

$$\begin{gathered} \pi_d \geq \pi_{d+1} \geq \cdots \geq \pi_{d+k-2}, \\ \text{where } \pi_d \leq \pi_{d+k-2} + \eta \text{ with strict inequality if } \pi_d \text{ is overlined, and} \end{gathered} \tag{3.7}$$

$$\begin{gathered} \pi_c \geq \pi_{c+1} \geq \cdots \geq \pi_{c+k-2}, \\ \text{where } \pi_c \leq \pi_{c+k-2} + \eta \text{ with strict inequality if } \pi_c \text{ is overlined.} \end{gathered} \tag{3.8}$$

Under the condition $\pi_c > \pi_d$, we have $c < d$. Assume that $d = c + t$ where $t \geq 1$. Given that $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, for $1 \leq i \leq \ell - k + 1$,

$$\pi_i \geq \pi_{i+k-1} + \eta \text{ with strict inequality if } \pi_i \text{ is non-overlined.} \tag{3.9}$$

It follows that there are at most $2k-2$ parts of π belonging to $I(\pi_c-2\eta,\pi_c)$. Therefore, by $\pi_{d+k-2}\geq \pi_c-2\eta$, we deduce that $1\leq t\leq k-1$, and so for $1\leq t\leq l\leq k-2$,

$$\pi_{c+l}=\pi_{d+l-t}. \tag{3.10}$$

Combining (3.7) and (3.8), we find that $\pi_{c+t}=\pi_d\leq \pi_{d+k-2}+\eta$ with strict inequality if π_d is overlined. Noting that $(d+k-2)-(c+t-1)=k-1$, using (3.9), we obtain that $\pi_{c+t-1}\geq \pi_{d+k-2}+\eta$ with strict inequality if π_{c+t-1} is non-overlined. It follows that $\pi_{c+t}<\pi_{c+t-1}$. The same argument yields $\pi_{d+k-1-t}<\pi_{d+k-2-t}$. To summarize, the overlapping structure of $\{\pi_{c+l}\}_{0\leq l\leq k-2}$ and $\{\pi_{d+l}\}_{0\leq l\leq k-2}$ can be described as follows, depending on two cases.

For $1\leq t<k-1$, we have

$$\begin{array}{ccccccccccccccc}
\pi_{d+k-2} & \leq & \cdots & \leq & \pi_{d+k-1-t} & < & \pi_{d+k-2-t} & \leq & \cdots & \leq & \pi_d & & & & \\
 & & & & & & \| & & & & \| & & & & \\
 & & & & & & \pi_{c+k-2} & \leq & \cdots & \leq & \pi_{c+t} & < & \pi_{c+t-1} & \leq \cdots \leq & \pi_c.
\end{array}$$

For $t=k-1$, we have

$$\pi_{d+k-2}\leq\cdots\leq\pi_d<\pi_{c+k-2}\leq\cdots\leq\pi_c.$$

We are now ready to prove (3.6). By (3.10), we have

$$\begin{aligned}
&[\pi_{c+k-2}/\eta]+\cdots+[\pi_{c+t}/\eta]+[\pi_{c+t-1}/\eta]+\cdots+[\pi_c/\eta]\\
&\quad=[\pi_{d+k-2-t}/\eta]+\cdots+[\pi_d/\eta]+[\pi_{c+t-1}/\eta]+\cdots+[\pi_c/\eta]\\
&\quad=[\pi_{d+k-2}/\eta]+\cdots+[\pi_d/\eta]\\
&\qquad\quad+[\pi_{c+t-1}/\eta]+\cdots+[\pi_c/\eta]-\left([\pi_{d+k-2}/\eta]+\cdots+[\pi_{d+k-1-t}/\eta]\right),
\end{aligned}$$

and by (3.5), we find that in order to show (3.6), it suffices to show that

$$\begin{aligned}
&[\pi_{c+t-1}/\eta]+\cdots+[\pi_c/\eta]-\left([\pi_{d+k-2}/\eta]+\cdots+[\pi_{d+k-1-t}/\eta]\right)\\
&\quad\equiv\overline{V}_\pi(\pi_c)-\overline{V}_\pi(\pi_d)\pmod 2.
\end{aligned}\tag{3.11}$$

We consider the following two cases.

Case 1: π_c is non-overlined. In this case, π_c is divisible by η, so we may write $\pi_c=(b+2)\eta$. In view of the condition that $\pi_{d+k-2}\geq\pi_c-2\eta=b\eta$, together with (3.9), we find that $\pi_{d+k-1-t}<\pi_c-\eta=(b+1)\eta$ and $\pi_{c+t-1}>(b+1)\eta$. Hence

$$b\eta\leq\pi_{d+k-2}\leq\cdots\leq\pi_{d+k-1-t}<(b+1)\eta,$$

and

$$(b+1)\eta<\pi_{c+t-1}\leq\cdots\leq\pi_c=(b+2)\eta.$$

This implies that for $k-1-t \leq l \leq k-2$, $[\pi_{d+l}/\eta] = b$, and for $0 \leq l \leq t-1$, $[\pi_{c+l}/\eta] = b+1$ if π_{c+l} is overlined, or $[\pi_{c+l}/\eta] = b+2$ if π_{c+l} is non-overlined. Consequently,

$$\begin{aligned}&[\pi_{c+t-1}/\eta] + \cdots + [\pi_c/\eta] - ([\pi_{d+k-2}/\eta] + \cdots + [\pi_{d+k-1-t}/\eta]) \\ &\equiv \overline{V}_\pi(\pi_c) - \overline{V}_\pi(\pi_d) \pmod 2,\end{aligned}$$

and so (3.11) is confirmed.

Case 2: π_c is overlined. Set $\alpha_0 = 0$. Then we may write $\pi_c = \overline{(b+1)\eta + \alpha_s}$, where $0 \leq s \leq \lambda$. Using the condition that $\pi_{d+k-2} > \pi_c - 2\eta = \overline{(b-1)\eta + \alpha_s}$ and the relation (3.9), we deduce that $\pi_{c+t-1} \geq \pi_{d+k-2} + \eta > \overline{b\eta + \alpha_s}$ and $\pi_{d+k-1-t} \leq \pi_c - \eta = \overline{b\eta + \alpha_s}$, so that

$$\overline{(b-1)\eta + \alpha_s} < \pi_{d+k-2} \leq \cdots \leq \pi_{d+k-1-t} \leq \overline{b\eta + \alpha_s}, \tag{3.12}$$

and

$$\overline{b\eta + \alpha_s} < \pi_{c+t-1} \leq \cdots \leq \pi_c = \overline{(b+1)\eta + \alpha_s}. \tag{3.13}$$

Assume that there are f_1 parts π_{d+l} in (3.12) satisfying $\overline{(b-1)\eta + \alpha_s} < \pi_{d+l} \leq \overline{(b-1)\eta + \alpha_\lambda}$. For such a part π_{d+l}, we have $[\pi_{d+l}/\eta] = b-1$. Assume that there are f_2 parts π_{d+l} in (3.12) satisfying $b\eta \leq \pi_{d+l} \leq \overline{b\eta + \alpha_s}$. For such a part π_{d+l}, we have $[\pi_{d+l}/\eta] = b$.

Assume that there are f_3 parts π_{c+l} in (3.13) satisfying $\overline{b\eta + \alpha_s} < \pi_{c+l} < (b+1)\eta$. In this case, we have $[\pi_{c+l}/\eta] = b$. Assume that there are f_4 parts π_{c+l} in (3.13) satisfying $\pi_{c+l} = (b+1)\eta$, which gives $[\pi_{c+l}/\eta] = b+1$. Assume that there are f_5 parts π_{c+l} in (3.13) satisfying $(b+1)\eta < \pi_{c+l} \leq \overline{(b+1)\eta + \alpha_s}$, which implies $[\pi_{c+l}/\eta] = b+1$. To sum up, we get

$$\begin{aligned}&[\pi_{c+t-1}/\eta] + \cdots + [\pi_c/\eta] - ([\pi_{d+k-2}/\eta] + \cdots + [\pi_{d+k-1-t}/\eta]) \\ &= bf_3 + (b+1)f_4 + (b+1)f_5 - (b-1)f_1 - bf_2,\end{aligned}$$

and

$$\overline{V}_\pi(\pi_c) - \overline{V}_\pi(\pi_d) = f_3 + f_5. \tag{3.14}$$

We proceed to show that $f_1 = f_3 + f_4$ and $f_2 = f_5$. By means of (3.9), we obtain $f_2 + k - t - 1 + f_3 + f_4 \leq k - 1$, that is, $f_2 + f_3 + f_4 \leq t$. Since $f_1 + f_2 = t$, we have

$$f_1 \geq f_3 + f_4. \tag{3.15}$$

To prove

$$f_1 \leq f_3 + f_4, \tag{3.16}$$

we consider three cases:

(1) If $t = k-1$, then $f_3 + f_4 + f_5 = t = k-1$. Using the condition that $k - 1 > \lambda$, we have

$$f_1 + f_5 \leq (\lambda - s) + (s + 1) = \lambda + 1 \leq k - 1.$$

This yields (3.16).

(2) If $1 \le t < k-1$ and $\pi_d < (b+1)\eta$, then $(b+1)\eta > \pi_d = \pi_{c+t} \ge \pi_{c+k-2} > \overline{b\eta + \alpha_s}$, and so we may write $\pi_d = \overline{b\eta + \alpha_g}$ with $g > s$. It follows that $\pi_{d+k-2} > \overline{(b-1)\eta + \alpha_g}$. Since

$$\overline{b\eta + \alpha_s} < \pi_{d+k-2-t} \le \cdots \le \pi_d = \overline{b\eta + \alpha_g},$$

we find that

$$k - t - 1 \le g - s. \tag{3.17}$$

Given the condition that $k - 1 > \lambda$, we obtain

$$f_1 + f_5 \le (\lambda - g) + (s + 1) = (\lambda + 1) - g + s \le k - 1 - g + s. \tag{3.18}$$

Combining (3.17) and (3.18) gives $f_1 + f_5 \le t$. Since $f_3 + f_4 + f_5 = t$, we arrive at (3.16).

(3) If $1 \le t < k-1$ and $\pi_d \ge (b+1)\eta$, then $\pi_{d+k-2} \ge \pi_d - \eta \ge b\eta$, and so

$$b\eta \le \pi_{d+k-2} \le \cdots \le \pi_{d+k-1-t} \le \overline{b\eta + \alpha_s}.$$

This implies that $f_1 = 0$, which leads to (3.16).

Returning to the special case there are no overlined parts divisible by η in π, we have $f_5 \le s$, and so (3.16) is also valid for $\lambda = k-1$. To sum up, (3.16) is justified for all cases. Combining with (3.15), we conclude that $f_1 = f_3 + f_4$.

It is now clear that $f_2 = f_5$ since $f_1 + f_2 = f_3 + f_4 + f_5 = t$ and $f_1 = f_3 + f_4$. Thus,

$$\begin{aligned}
&[\pi_{c+t-1}/\eta] + \cdots + [\pi_c/\eta] - ([\pi_{d+k-2}/\eta] + \cdots + [\pi_{d+k-1-t}/\eta]) \\
&\quad = bf_3 + (b+1)f_4 + (b+1)f_5 - (b-1)f_1 - bf_2 \\
&\quad = bf_3 + (b+1)f_4 + (b+1)f_5 - (b-1)(f_3 + f_4) - bf_5 \\
&\quad = f_3 + 2f_4 + f_5 \\
&\quad \equiv f_3 + f_5 \pmod 2.
\end{aligned} \tag{3.19}$$

Substituting (3.14) into (3.19), we reach (3.11), and this completes the proof. ∎

We conclude this section with a proof of Proposition 3.6 resorting to Lemma 3.5.

Proof of Proposition 3.6. Given a $(k-1)$-marked part $\tilde{g}_p(\pi)$ in $G(\pi)$, we like to show that a $(k-1)$-band of π including $\tilde{g}_p(\pi)$ has the same parity as that of the $(k-1)$-band of π induced by $\tilde{g}_p(\pi)$. Assume that $\tilde{g}_p(\pi)$ is the g_p-th part of $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, that is, $\pi_{g_p} = \tilde{g}_p(\pi)$, then $\{\pi_{g_p+l}\}_{0 \le l \le k-2}$ is the $(k-1)$-band induced by $\tilde{g}_p(\pi) = \pi_{g_p}$. Assume that $\{\pi_{c+l}\}_{0 \le l \le k-2}$ is a $(k-1)$-band of π including $\tilde{g}_p(\pi) = \pi_{g_p}$ and $g_p = c + t$, where $1 \le t \le k-2$. Since π is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, we have $\pi_{g_p-1} \ge \pi_{g_p+k-2} + \eta$ with strict inequality if π_{g_p+k-2} is non-overlined. By the definition of $(k-1)$-bands, we have $\pi_{g_p} \le \pi_{g_p+k-2} + \eta$ with strict inequality if π_{g_p+k-2} is overlined. Thus, $\pi_{g_p-1} > \pi_{g_p}$, and so $\pi_c \geq \pi_{c+t-1} = \pi_{g_p-1} > \pi_{g_p}$. The assumption that $\{\pi_{g_p+l}\}_{0\leq l\leq k-2}$ and $\{\pi_{c+l}\}_{0\leq l\leq k-2}$ are $(k-1)$-bands indicates

$$\pi_{g_p+k-2} \geq \pi_{g_p} - \eta = \pi_{c+t} - \eta \geq \pi_{c+k-2} - \eta \geq \pi_c - 2\eta,$$

with strict inequality if π_c is overlined. Thus, the conditions of Lemma 3.5 are satisfied, thereby $\{\tilde{g}_p(\pi)\}_{k-1}$ and $\{\pi_{c+l}\}_{0\leq l\leq k-2}$ are of the same parity. This completes the proof. ∎

4 Proof of Theorem 1.16

The main objective of this section is to give a combinatorial proof of Theorem 1.16. The relationship between $\overline{B}_1$ and B_0 stated in Theorem 1.16 plays a crucial role in the proof of Bressoud's conjecture for the case $j = 0$ in a subsequent paper [25].

Let $\mathcal{D}_\eta$ denote the set of partitions with distinct parts divisible by η. Theorem 1.16 is equivalent to the following combinatorial statement.

Theorem 4.1. *Let λ, k and r be integers such that $k \geq r \geq \lambda \geq 0$ and $k > \lambda$. There is a bijection Φ between $\mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, namely, for a pair $(\zeta, \mu) \in \mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, we have $\pi = \Phi(\zeta, \mu) \in \overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that $|\pi| = |\zeta| + |\mu|$.*

The bijection Φ is constructed via merging ζ and μ to produce an overpartition π in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. Recall that $\mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ is the set of overpartitions in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ without overlined parts divisible by η. Assume there are N parts marked with $k-1$ in $RG(\mu)$ and let $\zeta = (\eta\zeta_1, \ldots, \eta\zeta_c, \eta\zeta_{c+1} \ldots, \eta\zeta_{c+m})$ be a partition in $\mathcal{D}_\eta$ with $\zeta_1 > \cdots > \zeta_c > N \geq \zeta_{c+1} > \cdots > \zeta_{c+m} > 0$. In fact, the bijection Φ consists of two steps. The first step is to merge the parts $\eta\zeta_{c+1}, \eta\zeta_{c+2}, \ldots, \eta\zeta_{c+m}$ and μ. The second step is to merge the remaining parts $\eta\zeta_1, \eta\zeta_2, \ldots, \eta\zeta_c$ of ζ and μ to generate certain ovelined parts divisible by η. As will be seen, the overpartition ν obtained in the first step is in $\mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. In the meantime, there are no overlined parts divisible by η in ν. Eventually, the resulting overpartition π of the second step is in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$.

To describe the map Φ, we introduce the forward move and the backward move which are defined on the Gordon marking of a $\overline{B}_1$-overpartition and the reverse Gordon marking of a $\overline{B}_1$-overpartition. A precise description of the first merging operation will be given later based on the restricted forward move and the restricted backward move and an explanation of the second merging operation will be provided by means of the $(k-1)$-insertion operation and the $(k-1)$-separation operation.

4.1 The forward move and the backward move

Definition 4.2 (The forward move)**.** *For $k > \lambda$ and $N \geq 1$, let π be an overpartition satisfying* (1), (2) *and* (3) *in Definition 1.14. Assume that there are N parts marked with*

$k-1$ *in* $RG(\pi)$*, say* $\tilde{r}_1(\pi) > \tilde{r}_2(\pi) > \cdots > \tilde{r}_N(\pi)$*. For* $1 \le p \le N$*, the forward move* ϕ_p *is defined as follows: add* η *to each of* $\tilde{r}_1(\pi), \tilde{r}_2(\pi), \ldots, \tilde{r}_p(\pi)$ *and rearrange the parts in non-increasing order to obtain a new overpartition, denoted* $\phi_p(\pi)$*.*

For example, let π be the overpartition given in (3.1). Below is the reverse Gordon marking of π:

$$\begin{aligned} RG(\pi) = (&\overline{80}_1, \overbrace{\overline{80}_2, 80_3, \overline{70}_1, 70_4}^{\{70\}_4}, \overline{69}_2, \overbrace{\overline{60}_1, 60_3, \overline{55}_2, \overline{51}_4}^{\{\overline{51}\}_4}, \overbrace{\overline{50}_1, \overline{49}_3, \overline{45}_2, \overline{41}_4}^{\{\overline{41}\}_4}, \overline{39}_1, \overline{35}_2, \\ &\underbrace{\overline{29}_1, \overline{20}_2, 20_3, 20_4}_{\{20\}_4}, \underbrace{\overline{11}_1, \overline{10}_2, \overline{9}_3, \overline{5}_4}_{\{\overline{5}\}_4}, \overline{1}_1). \end{aligned} \tag{4.1}$$

There are five 4-marked parts in $RG(\pi)$. Applying the forward move ϕ_3 to π in (3.1), we obtain

$$\begin{aligned} \phi_3(\pi) = (&\overline{80}, 80, 80, 80, \overline{70}, \overline{69}, \overline{61}, \overline{60}, 60, \overline{55}, \overline{51}, 50, \overline{49}, \overline{45}, \overline{39}, \overline{35}, \\ &\overline{29}, \overline{20}, 20, 20, \overline{11}, \overline{10}, \overline{9}, \overline{5}, \overline{1}). \end{aligned}$$

The following proposition gives several properties of $\phi_p(\pi)$.

Proposition 4.3. *For* $k > \lambda$ *and* $N \ge 1$*, let* $\pi = (\pi_1, \ldots, \pi_\ell)$ *be an overpartition satisfying (1), (2) and (3) in Definition 1.14. Assume that there are* N *parts marked with* $k-1$ *in* $RG(\pi)$*, say* $\tilde{r}_1(\pi) > \tilde{r}_2(\pi) > \cdots > \tilde{r}_N(\pi)$*. For* $1 \le p \le N$*, let* $\omega = (\omega_1, \ldots, \omega_\ell) = \phi_p(\pi)$*. Then*

(1) *For* $1 \le i \le \ell$*,* $\omega_i \equiv 0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$*;*

(2) *Only multiples of* η *may be non-overlined in* ω*;*

(3) *There are at most* $k-1$ *marks in* $G(\omega)$ *and there are* N *parts marked with* $k-1$ *in* $G(\omega)$*, say* $\tilde{g}_1(\omega) > \tilde{g}_2(\omega) > \cdots > \tilde{g}_N(\omega)$*;*

(4) *Let* $\{\tilde{r}_i(\pi)\}_{k-1}$ *be the* $(k-1)$*-band of* π *induced by* $\tilde{r}_i(\pi)$*. Then* $\tilde{g}_i(\omega) = \tilde{r}_i(\pi) + \eta$ *for* $1 \le i \le p$*, and* $\tilde{g}_i(\omega) \in \{\tilde{r}_i(\pi)\}_{k-1}$ *for* $p < i \le N$*.*

For example, let π be the overpartition in $\overline{\mathcal{B}}_1(1, 5, 9; 10, 5, 4)$ given in (3.1), and let $\omega = \phi_3(\pi)$. Then the Gordon marking of ω is given by

$$\begin{aligned} G(\omega) = (&\overline{80}_1, \overbrace{80_4, 80_3, 80_2, \overline{70}_1}^{\{80\}_4}, \overline{69}_2, \overbrace{\overline{61}_4, \overline{60}_3, 60_1, \overline{55}_2}^{\{\overline{61}\}_4}, \overbrace{\overline{51}_4, 50_3, \overline{49}_1, \overline{45}_2}^{\{\overline{51}\}_4}, \overline{39}_1, \overline{35}_2, \\ &\overline{29}_1, \underbrace{\overline{20}_4, 20_3, 20_2, \overline{11}_1}_{\{\overline{20}\}_4}, \underbrace{\overline{10}_4, \overline{9}_3, \overline{5}_2, \overline{1}_1}_{\{\overline{10}\}_4}). \end{aligned} \tag{4.2}$$

It can be readily checked that ω satisfies the properties (1)-(4) in Proposition 4.3.

Proof of Proposition 4.3. To prove (1) and (2), it suffices to show that for $1 \leq i \leq p$, $\tilde{r}_i(\pi)+\eta$ cannot be repeated in ω if the part $\tilde{r}_i(\pi)$ is overlined. We now assume that $\tilde{r}_i(\pi)$ is overlined, so the new generated part $\tilde{r}_i(\pi)+\eta$ is also overlined. There are two cases.

Case 1: Assume that $\tilde{r}_i(\pi)+\eta$ is not a part of π. It is obvious that the generated overlined part $\tilde{r}_i(\pi)+\eta$ appears only once in ω.

Case 2: Assume that π contains an overlined part $\pi_t = \tilde{r}_i(\pi)+\eta$. We claim that π_t is marked with $k-1$ in $RG(\pi)$. Since π satisfies the condition (3) in Definition 1.14, the marks in $RG(\pi)$ do not exceed $k-1$. Assume that $\tilde{r}_i(\pi)$ is the r_i-th part of $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$, that is, $\pi_{r_i} = \tilde{r}_i(\pi)$. Then $\{\pi_{r_i-l}\}_{0\leq l\leq k-2}$ is the $(k-1)$-band induced by $\tilde{r}_i(\pi)$. It follows that the marks of the parts $\pi_{r_i-k+2}, \ldots, \pi_{r_i}$ are distinct in $RG(\pi)$ and π_{r_i} is marked with $k-1$. This implies that the marks of $\pi_{r_i-k+2}, \ldots, \pi_{r_i-1}$ in $RG(\pi)$ are distinct and less than $k-1$. Suppose to the contrary that the mark of π_t in $RG(\pi)$ is less than $k-1$. Consequently, there is a part π_{r_i-m} $(1 \leq m \leq k-2)$ in the $(k-1)$-band $\{\pi_{r_i-l}\}_{0\leq l\leq k-2}$ such that the mark of π_{r_i-m} is the same as the mark of π_t. Since $\pi_{r_i} < \pi_{r_i-m} < \pi_{r_i}+\eta$ and $\pi_t = \pi_{r_i}+\eta$, we obtain $\pi_{r_i-m} < \pi_t < \pi_{r_i-m}+\eta$, and so the marks of π_t and π_{r_i-m} are distinct. But this is impossible under the prior assumption. Therefore, the mark of π_t is $k-1$ in $RG(\pi)$, as claimed. In other words, we have $\tilde{r}_{i-1}(\pi) = \pi_t$. This enables us to employ the forward move to add η to $\tilde{r}_{i-1}(\pi)$. In the end, the generated overlined part $\tilde{r}_i(\pi)+\eta$ occurs only once in ω.

We now turn to the properties (3) and (4). Assume that $\tilde{r}_i(\pi)$ is the r_i-th part of $\pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$, that is, $\pi_{r_i} = \tilde{r}_i(\pi)$. In fact, the forward move consists of two steps. First, remove the $(k-1)$-marked parts $\pi_{r_1}, \pi_{r_2}, \ldots, \pi_{r_p}$ from π and denote the resulting overpartition by $\pi^{(1)}$. Since the largest mark in $RG(\pi)$ is $k-1$ and the parts removed from π are marked with $k-1$ in $RG(\pi)$, the marks of the remaining parts in $RG(\pi^{(1)})$ are the same as those in $RG(\pi)$. This implies that the marks of the parts of $\pi^{(1)}$ not less than π_{r_p} do not exceed $k-2$. Therefore, there are $N-p$ parts marked with $k-1$ in $RG(\pi^{(1)})$, denoted $\tilde{r}_1(\pi^{(1)}), \ldots, \tilde{r}_{N-p}(\pi^{(1)})$, for which $\tilde{r}_i(\pi^{(1)}) = \tilde{r}_{i+p}(\pi)$ and $\{\tilde{r}_i(\pi^{(1)})\}_{k-1} = \{\tilde{r}_{i+p}(\pi)\}_{k-1}$ for $1 \leq i \leq N-p$. In light of Proposition 3.3, we find that there are also $N-p$ parts marked with $k-1$ in $G(\pi^{(1)})$, denoted $\tilde{g}_1(\pi^{(1)}), \ldots, \tilde{g}_{N-p}(\pi^{(1)})$, for which $\tilde{g}_i(\pi^{(1)}) \in \{\tilde{r}_i(\pi^{(1)})\}_{k-1}$. Meanwhile, the marks of the parts not less than π_{r_p} in $G(\pi^{(1)})$ do not exceed $k-2$. So we deduce that $\tilde{g}_i(\pi^{(1)}) \in \{\tilde{r}_{i+p}(\pi)\}_{k-1}$ for $1 \leq i \leq N-p$.

The second step is to insert $\pi_{r_1}+\eta, \pi_{r_2}+\eta, \ldots, \pi_{r_p}+\eta$ into $\pi^{(1)}$ and to rearrange the parts in non-increasing order to obtain ω. We wish to show that for $1 \leq i \leq p$, $\pi_{r_i}+\eta$ is marked with $k-1$ in $G(\omega)$. We claim that $\pi_{r_i} > \pi_{r_i+1}$. For $1 \leq i \leq p$, since π_{r_i} is the $(k-1)$-marked part in $RG(\pi)$, we know that $\{\pi_{r_i-l}\}_{0\leq l\leq k-2}$ is the $(k-1)$-band of π induced by π_{r_i}, which ensures that $\pi_{r_i-k+2} \leq \pi_{r_i}+\eta$ with strict inequality if π_{r_i-k+2} is overlined. Under the assumption that π satisfies the condition (3) in Definition 1.14, we have $\pi_{r_i-k+2} \geq \pi_{r_i+1}+\eta$ with strict inequality if π_{r_i-k+2} is non-overlined. But $\pi_{r_i-k+2} \leq \pi_{r_i}+\eta$ with strict inequality if π_{r_i-k+2} is overlined, so we conclude that $\pi_{r_i} > \pi_{r_i+1}$, as claimed.

We continue to prove that $\pi_{r_i}+\eta$ is marked with $k-1$ in $G(\omega)$. Based on the assumption that π satisfies the condition (3) in Definition 1.14, we obtain that $\pi_{r_i-k+1} \geq \pi_{r_i} + \eta$ with strict inequality if π_{r_i-k+1} is non-overlined, which implies that for $1 \leq i \leq p$,

$$\pi_{r_i} + \eta, \text{ and } \pi_{r_i-k+2}, \ldots, \pi_{r_i-1}$$

are parts of ω. Noting that $\pi_{r_i} > \pi_{r_i+1}$ and $\pi_{r_i-k+1} \geq \pi_{r_i} + \eta$ with strict inequality if π_{r_i} is non-overlined, it is clear that the mark of the part $\pi_{r_i} + \eta$ in $G(\omega)$ is the smallest positive integer that is not used to mark $\pi_{r_i-k+2}, \ldots, \pi_{r_i-1}$. Recalling that the marks of the parts of $\pi^{(1)}$ not less than π_{r_p} in $G(\pi^{(1)})$ do not exceed $k-2$, the marks of the parts $\pi_{r_i-k+2}, \pi_{r_i-k+3}, \ldots, \pi_{r_i-1}$ in $G(\omega)$ are less than $k-1$ for $1 \leq i \leq p$. Thus, the mark of $\pi_{r_i} + \eta$ in $G(\omega)$ is $k-1$. Meanwhile, the marks of remaining parts in $G(\omega)$ are the same as in $G(\pi^{(1)})$. Therefore, we reach the conclusion that there are N parts marked with $k-1$ in $G(\omega)$, and so the properties (3) and (4) are verified. This completes the proof. ∎

For example, for the overpartition π in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ with the reverse Gordon marking given in (4.1), there are five 4-marked parts in $RG(\pi)$. Then $\omega = \phi_3(\pi)$ can be constructed via two steps: The first step is to remove $\tilde{r}_1(\pi) = 70$, $\tilde{r}_2(\pi) = \overline{51}$, and $\tilde{r}_3(\pi) = \overline{41}$ from π to get $\pi^{(1)}$, whose reverse Gordon marking reads

$$\begin{aligned} RG(\pi^{(1)}) = (&\overline{80}_1, 80_2, 80_3, \overline{70}_1, \overline{69}_2, \overline{60}_1, 60_3, \overline{55}_2, 50_1, \overline{49}_3, \overline{45}_2, \overline{39}_1, \overline{35}_2, \\ &\underbrace{\overline{29}_1, \overline{20}_2, 20_3, 20_4}_{\{20\}_4}, \underbrace{\overline{11}_1, \overline{10}_2, \overline{9}_3, \overline{5}_4}_{\{\overline{5}\}_4}, \overline{1}_1). \end{aligned}$$

It can be checked that the marks of parts in $RG(\pi^{(1)})$ are the same as those in $RG(\pi)$. On the other hand, below is the Gordon marking of $\pi^{(1)}$

$$\begin{aligned} G(\pi^{(1)}) = (&\overline{80}_1, 80_3, 80_2, \overline{70}_1, \overline{69}_2, \overline{60}_3, 60_1, \overline{55}_2, 50_3, \overline{49}_1, \overline{45}_2, \overline{39}_1, \overline{35}_2, \\ &\overline{29}_1, \overline{20}_4, 20_3, 20_2, \overline{11}_1, \overline{10}_4, \overline{9}_3, \overline{5}_2, \overline{1}_1). \end{aligned}$$

Evidently, the 4-marked parts $\overline{20}_4$ and $\overline{10}_4$ in $G(\pi^{(1)})$ are in the 4-bands $\{\overline{29}, \overline{20}, 20, 20\}$ and $\{\overline{11}, \overline{10}, \overline{9}, \overline{5}\}$ of $\pi^{(1)}$, respectively. In the second step, we insert $\tilde{r}_1(\pi) + 10 = 80$, $\tilde{r}_2(\pi) + 10 = \overline{61}$, and $\tilde{r}_3(\pi) + 10 = \overline{51}$ into $\pi^{(1)}$ to get ω, whose Gordon marking is displayed in (4.2). As anticipated, the marks of $\tilde{r}_1(\pi) + 10 = 80$, $\tilde{r}_2(\pi) + 10 = \overline{61}$, and $\tilde{r}_3(\pi) + 10 = \overline{51}$ in $G(\omega)$ are 4. Meanwhile the marks of remaining parts in $G(\omega)$ are the same as in $G(\pi^{(1)})$. Therefore, ω satisfies the properties (1)-(4) in Proposition 4.3.

In parallel to the forward move, we now turn to the definition of the backward move relying on the Gordon marking of a $\overline{B}_1$-overpartition.

Definition 4.4 (The backward move). *For $k > \lambda$ and $N \geq p \geq 1$, let ω be an overpartition satisfying* (1), (2) *and* (3) *in Definition 1.14. Assume that there are N parts marked with $k-1$ in $G(\omega)$, denoted $\tilde{g}_1(\omega) > \tilde{g}_2(\omega) > \cdots > \tilde{g}_N(\omega)$, for which $\tilde{g}_p(\omega) \geq \overline{\eta + \alpha_1}$. The backward move ψ_p is defined as follows: subtract η from each of $\tilde{g}_1(\omega)$, $\tilde{g}_2(\omega), \ldots, \tilde{g}_p(\omega)$ and rearrange the parts in non-increasing order to obtain a new overpartition, denoted $\psi_p(\omega)$.*

For example, for the overpartition $\omega = \phi_3(\pi)$ with five 4-marked parts in $G(\omega)$ as in (4.2), the backward move ψ_3 transforms ω back to π in (4.1).

The backward move ψ_p possesses the following properties with respect to certain overpartitions satisfying (1), (2) and (3) in Definition 1.14.

Proposition 4.5. *For $k > \lambda$ and $N \geq p \geq 1$, let ω be an overpartition satisfying* (1), (2) *and* (3) *in Definition 1.14. Assume that there are N parts marked with $k-1$ in $G(\omega)$, denoted $\tilde{g}_1(\omega) > \tilde{g}_2(\omega) > \cdots > \tilde{g}_N(\omega)$, for which $\tilde{g}_p(\omega) \geq \overline{\eta + \alpha_1}$, and assume that $\tilde{g}_p(\omega)$ is a part in any $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega) - 2\eta, \tilde{g}_p(\omega))$. Let $\pi = (\pi_1, \ldots, \pi_\ell) = \psi_p(\omega)$. Then*

(1) *For $1 \leq i \leq \ell$, $\pi_i \equiv 0, \alpha_1, \ldots, \alpha_\lambda \pmod{\eta}$;*

(2) *Only multiples of η may be non-overlined in π;*

(3) *There are at most $k-1$ marks in $RG(\pi)$ and there are N parts marked with $k-1$ in $RG(\pi)$, say $\tilde{r}_1(\pi) > \cdots > \tilde{r}_N(\pi)$;*

(4) *Let $\{\tilde{g}_i(\omega)\}_{k-1}$ be the $(k-1)$-band of ω induced by $\tilde{g}_i(\omega)$. Then $\tilde{r}_i(\pi) = \tilde{g}_i(\omega) - \eta$ for $1 \leq i \leq p$, and $\tilde{r}_i(\pi) \in \{\tilde{g}_i(\omega)\}_{k-1}$ for $p < i \leq N$.*

Proof. To prove (1) and (2), it suffices to show that for $1 \leq i \leq p$, the part $\tilde{g}_i(\omega) - \eta$ occurs exactly once in π if $\tilde{g}_i(\omega)$ is overlined. We now assume that $\tilde{g}_i(\omega)$ is overlined, so that the generated part $\tilde{g}_i(\omega) - \eta$ is also overlined. There are two cases:

Case 1: Assume that the part $\tilde{g}_i(\omega) - \eta$ does not appear in ω. In this case, it is obvious that the generated part $\tilde{g}_i(\omega) - \eta$ occurs exactly once in π.

Case 2: Assume that ω contains the overlined part $\omega_{t_i} = \tilde{g}_i(\omega) - \eta$, where $1 \leq i \leq p$. Using the same argument as in the proof of Proposition 4.3, it can be shown that ω_{t_i} is marked with $k-1$ in $G(\omega)$, where $1 \leq i \leq p$.

We proceed to show that if ω satisfies the condition that $\tilde{g}_p(\omega)$ is a part in any $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega) - 2\eta, \tilde{g}_p(\omega))$, then ω does not contain the overlined part $\tilde{g}_p(\omega) - \eta$. Suppose to the contrary that ω contains the overlined part $\tilde{g}_p(\omega) - \eta$. Since $\tilde{g}_p(\omega) - \eta$ is marked with $k-1$ in $G(\omega)$, we have

$$\tilde{g}_{p+1}(\omega) = \tilde{g}_p(\omega) - \eta, \tag{4.3}$$

where $\tilde{g}_{p+1}(\omega)$ is also overlined. Assume that $\tilde{g}_{p+1}(\omega)$ is the g_{p+1}-th part of $\omega = (\omega_1, \omega_2, \ldots, \omega_\ell)$, that is, $\omega_{g_{p+1}} = \tilde{g}_{p+1}(\omega)$. Then $\{\omega_{g_{p+1}+l}\}_{0 \leq l \leq k-2}$ is the $(k-1)$-band induced by $\omega_{g_{p+1}}$, which, together with (4.3), leads to

$$\omega_{g_{p+1}+k-2} > \omega_{g_{p+1}} - \eta = \tilde{g}_p(\omega) - 2\eta.$$

By (4.3), we have

$$\omega_{g_{p+1}} = \tilde{g}_p(\omega) - \eta < \tilde{g}_p(\omega),$$

from which $\{\omega_{g_{p+1}+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega)-2\eta,\tilde{g}_p(\omega))$. But $\tilde{g}_p(\omega)$ is not a part in $\{\omega_{g_{p+1}+l}\}_{0\leq l\leq k-2}$, which contradicts the condition that $\tilde{g}_p(\omega)$ is a part in any $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega)-2\eta,\tilde{g}_p(\omega))$. This means that ω does not contain the overlined part $\tilde{g}_p(\omega)-\eta$. Using the fact that $\tilde{g}_i(\omega)-\eta$ is marked with $k-1$ in $G(\omega)$, we have $\tilde{g}_{i+1}(\omega)=\tilde{g}_i(\omega)-\eta$ where $1\leq i<p$. Applying the backward move to ω, we get the overpartition π in which the part $\tilde{g}_i(\omega)-\eta$ appears exactly once. So we have verified the properties (1) and (2).

We now turn to the properties (3) and (4). Similarly, the backward move consists of two steps. First, remove the $(k-1)$-marked parts $\tilde{g}_1(\omega),\ldots,\tilde{g}_p(\omega)$ from ω and denote the resulting overpartition by $\omega^{(1)}$. Along the same lines of reasoning as in the proof of Proposition 4.3, we deduce that the marks of the remaining parts in $RG(\omega)$ are the same as those in $RG(\omega^{(1)})$. This implies that there are $N-p$ parts marked with $k-1$ in $RG(\omega^{(1)})$, denoted $\tilde{r}_1(\omega^{(1)})>\cdots>\tilde{r}_{N-p}(\omega^{(1)})$, for which $\tilde{r}_i(\omega^{(1)})\in\{\tilde{g}_{i+p}(\omega)\}_{k-1}$. We proceed to demonstrate that $\tilde{r}_1(\omega^{(1)})\leq\tilde{g}_p(\omega)-2\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is non-overlined. Suppose to the contrary that $\tilde{r}_1(\omega^{(1)})\geq\tilde{g}_p(\omega)-2\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is overlined. Since $\tilde{r}_1(\omega^{(1)})\in\{\tilde{g}_{p+1}(\omega)\}_{k-1}$, we have $\tilde{r}_1(\omega^{(1)})\leq\tilde{g}_{p+1}(\omega)$. Note that $\tilde{g}_{p+1}(\omega)\leq\tilde{g}_p(\omega)-\eta$ with strict inequality if $\tilde{g}_{p+1}(\omega)$ is non-overlined. We obtain that $\tilde{r}_1(\omega^{(1)})\leq\tilde{g}_p(\omega)-\eta$ with strict inequality if $\tilde{r}_1(\omega^{(1)})$ is non-overlined. Assume that $\tilde{r}_1(\omega^{(1)})$ is the r_1-th part of $\omega^{(1)}=(\omega^{(1)}_1,\omega^{(1)}_2,\ldots,\omega^{(1)}_\ell)$, that is, $\omega^{(1)}_{r_1}=\tilde{r}_1(\omega^{(1)})$. Then $\{\omega^{(1)}_{r_1-l}\}_{0\leq l\leq k-2}$ is the $(k-1)$-band induced by $\omega^{(1)}_{r_1}$, which implies that $\omega^{(1)}_{r_1-k+2}\leq\omega^{(1)}_{r_1}+\eta$ with strict inequality if $\omega^{(1)}_{r_1}$ is overlined. But, $\tilde{r}_1(\omega^{(1)})\leq\tilde{g}_p(\omega)-\eta$ with strict inequality if $\tilde{r}_1(\omega^{(1)})$ is non-overlined, it follows that $\omega^{(1)}_{r_1-k+2}<\tilde{g}_p(\omega)$. Therefore, we conclude that $\{\omega^{(1)}_{r_1-l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of $\omega^{(1)}$ belonging to $I(\tilde{g}_p(\omega)-2\eta,\tilde{g}_p(\omega))$. By the construction of $\omega^{(1)}$, we see that $\{\omega^{(1)}_{r_1-l}\}_{0\leq l\leq k-2}$ is also a $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega)-2\eta,\tilde{g}_p(\omega))$. However, $\tilde{g}_p(\omega)$ is not in $\{\omega^{(1)}_{r_1-l}\}_{0\leq l\leq k-2}$, which contradicts the condition that $\tilde{g}_p(\omega)$ is a part in any $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega)-2\eta,\tilde{g}_p(\omega))$. Hence $\tilde{r}_1(\omega^{(1)})\leq\tilde{g}_p(\omega)-2\eta$ with strict inequality if $\tilde{r}_1(\omega^{(1)})$ is non-overlined.

The second step is to insert $\tilde{g}_1(\omega)-\eta,\ldots,\tilde{g}_p(\omega)-\eta$ into $\omega^{(1)}$ and to rearrange the parts in non-increasing order to obtain π. It can be shown that the mark of $\tilde{g}_i(\omega)-\eta$ in $RG(\pi)$ is equal to $k-1$ for $1\leq i\leq p$. Furthermore, the remaining parts not less than $\tilde{g}_p(\omega)-\eta$ in $RG(\pi)$ are the same as in $RG(\omega^{(1)})$. We need to show that the marks of remaining parts less than $\tilde{g}_p(\omega)-\eta$ in $RG(\pi)$ are the same as in $RG(\omega^{(1)})$. We first verify that the marks of the parts π_i of π such that $\tilde{g}_p(\omega)-2\eta\leq\pi_i<\tilde{g}_p(\omega)-\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is overlined in $RG(\pi)$ are the same as those in $RG(\omega^{(1)})$. Since $\tilde{r}_1(\omega^{(1)})\leq\tilde{g}_p(\omega)-2\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is non-overlined, the marks of the parts π_i of π such that $\tilde{g}_p(\omega)-2\eta\leq\pi_i<\tilde{g}_p(\omega)-\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is overlined are less than $k-1$ in $RG(\omega^{(1)})$. But the mark of $\tilde{g}_p(\omega)-\eta$ is $k-1$, we infer that the marks of the parts π_i of π such that $\tilde{g}_p(\omega)-2\eta\leq\pi_i<\tilde{g}_p(\omega)-\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is overlined in $RG(\pi)$ are the same as those in $RG(\omega^{(1)})$. Thus, the marks of the parts π_i of π such that $\pi_i\leq\tilde{g}_p(\omega)-2\eta$ with strict inequality if $\tilde{g}_p(\omega)$ is non-overlined in $RG(\pi)$ are the same as in $RG(\omega^{(1)})$. Therefore, the marks of remaining parts less than $\tilde{g}_p(\omega)-\eta$ in $RG(\pi)$ are the same as in $RG(\omega^{(1)})$. It follows that there are N parts marked with $k-1$ in $RG(\pi)$, and so the properties (3) and (4) are verified. This completes the proof. ∎

We now furnish an example to illustrate Proposition 4.5. Let ω be the overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ with the Gordon marking given by (4.2). There are five 4-marked parts in $G(\omega)$, namely, $\tilde{g}_1(\omega)=80$, $\tilde{g}_2(\omega)=\overline{61}$, $\tilde{g}_3(\omega)=\overline{51}$, $\tilde{g}_4(\omega)=\overline{20}$ and $\tilde{g}_5(\omega)=\overline{10}$. It can be checked that there are no 4-bands of ω belonging to $I(\tilde{g}_3(\omega)-2\eta,\tilde{g}_3(\omega))$. The overpartition $\pi=\psi_3(\omega)$ can be constructed as follows: First, remove $\tilde{g}_1(\omega)=80$, $\tilde{g}_2(\omega)=\overline{61}$, and $\tilde{g}_3(\omega)=\overline{51}$ from ω to get $\omega^{(1)}$. We have

$$G(\omega^{(1)})=(\overline{80}_1,80_3,80_2,\overline{70}_1,\overline{69}_2,\overline{60}_3,60_1,\overline{55}_2,50_3,\overline{49}_1,\overline{45}_2,\overline{39}_1,\overline{35}_2,$$

$$\overline{29}_1,\underbrace{\overline{20}_4,20_3,20_2,\overline{11}_1}_{\{\overline{20}\}_4},\underbrace{\overline{10}_4,\overline{9}_3,\overline{5}_2,\overline{1}_1}_{\{\overline{10}\}_4}).$$

It can be checked that the marks of parts in $G(\omega^{(1)})$ are the same as those in $G(\omega)$. The reverse Gordon marking of $\omega^{(1)}$ is given by

$$RG(\omega^{(1)})=(\overline{80}_1,80_2,80_3,\overline{70}_1,\overline{69}_2,\overline{60}_1,60_3,\overline{55}_2,50_1,\overline{49}_3,\overline{45}_2,\overline{39}_1,\overline{35}_2,$$

$$\overline{29}_1,\overline{20}_2,20_3,20_4,\overline{11}_1,\overline{10}_2,\overline{9}_3,\overline{5}_4,\overline{1}_1),$$

from which we see that the 4-marked parts 20 and $\overline{5}$ in $RG(\omega^{(1)})$ are also in $\{\overline{20}\}_4$ and $\{\overline{10}\}_4$ of $\omega^{(1)}$, respectively. Then π can be obtained by inserting $\tilde{g}_1(\omega)-10=70$, $\tilde{g}_2(\omega)-10=\overline{51}$, and $\tilde{g}_3(\omega)-10=\overline{41}$ into $\omega^{(1)}$. Below is the reverse Gordon marking of π

$$RG(\pi)=(\overline{80}_1,80_2,80_3,\overline{70}_1,70_4,\overline{69}_2,\overline{60}_1,60_3,\overline{55}_2,\overline{51}_4,50_1,\overline{49}_3,\overline{45}_2,\overline{41}_4,\overline{39}_1,\overline{35}_2,$$

$$\overline{29}_1,\overline{20}_2,20_3,20_4,\overline{11}_1,\overline{10}_2,\overline{9}_3,\overline{5}_4,\overline{1}_1).$$

Notice that $\tilde{g}_1(\omega)-10=70$, $\tilde{g}_2(\omega)-10=\overline{51}$, and $\tilde{g}_3(\omega)-10=\overline{41}$ are marked with 4 in $RG(\pi)$. The marks of the remaining parts in $RG(\pi)$ are the same as those in $RG(\omega^{(1)})$.

We remark that the condition in Proposition 4.5 that $\tilde{g}_p(\omega)$ is a part in any $(k-1)$-band of ω belonging to $I(\tilde{g}_p(\omega)-2\eta,\tilde{g}_p(\omega))$ is necessary. For example, let ω be an overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ having the Gordon marking

$$G(\omega)=(\overline{80}_1,80_4,80_3,80_2,\overline{70}_1,\overline{69}_2,\overline{61}_4,\overline{60}_3,60_1,\overline{55}_2,\overline{51}_4,\overline{50}_3,\overline{49}_1,\overline{45}_2,\overline{40}_4,40_3,\overline{39}_1,\overline{35}_2,$$

$$\overline{29}_1,\overline{20}_4,20_3,20_2,\overline{11}_1,\overline{10}_4,\overline{9}_3,\overline{5}_2,\overline{1}_1).$$

There are six 4-marked parts in $G(\omega)$, namely, $\tilde{g}_1(\omega)=80$, $\tilde{g}_2(\omega)=\overline{61}$, $\tilde{g}_3(\omega)=\overline{51}$, $\tilde{g}_4(\omega)=\overline{40}$, $\tilde{g}_5(\omega)=\overline{20}$ and $\tilde{g}_6(\omega)=\overline{10}$. Furthermore, ω has three 4-bands $\{\overline{49},\overline{45},\overline{40},40\}$, $\{\overline{45},\overline{40},40,\overline{39}\}$ and $\{\overline{40},40,\overline{39},\overline{35}\}$ in the interval $(\overline{31},\overline{51})$.

The overpartition $\pi=\psi_3(\omega)$ can be obtained by subtracting $\eta=10$ from each of $\tilde{g}_1(\omega)=80$, $\tilde{g}_2(\omega)=\overline{61}$, and $\tilde{g}_3(\omega)=\overline{51}$, and so we get

$$RG(\pi)=(\overline{80}_1,80_2,80_3,\overline{70}_1,70_4,\overline{69}_2,\overline{60}_1,60_3,\overline{55}_2,\overline{51}_4,\overline{50}_1,\overline{49}_3,\overline{45}_2,\overline{41}_4,\overline{40}_1,40_5,\overline{39}_3,\overline{35}_2,$$

$$\overline{29}_1,\overline{20}_2,20_3,20_4,\overline{11}_1,\overline{10}_2,\overline{9}_3,\overline{5}_4,\overline{1}_1).$$

Because of the occurrence of the 5-marked part 40 in $RG(\pi)$, the property (3) in Proposition 4.5 is violated.

4.2 The restricted moves

To describe the first step of the bijection Φ in Theorem 4.1, we will restrict the $(k-1)$-forward move and the $(k-1)$-backward move to two subsets of $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. We assume that λ, k and r are integers such that $k\geq r\geq\lambda\geq 0$ and $k>\lambda$. Recall that $\mathcal{B}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ denotes the set of overpartitions in $\overline{\mathcal{B}}_j(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ without overlined parts divisible by η. We will be concerned with the following two subsets of $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$.

- For $N\geq p\geq 1$, let $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$ denote the set of overpartitions γ in $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ such that there are N parts marked with $k-1$ in $RG(\gamma)$, denoted $\tilde{r}_1(\gamma)>\tilde{r}_2(\gamma)>\cdots>\tilde{r}_N(\gamma)$, and for all $1\leq i\leq p$, the parity of $\{\tilde{r}_i(\gamma)\}_{k-1}$ is the same as that of $\{\tilde{r}_{p+1}(\gamma)\}_{k-1}$.
- For $N\geq p\geq 1$, let $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$ denote the set of overpartitions γ in $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ such that there are N parts marked with $k-1$ in $RG(\gamma)$, denoted $\tilde{r}_1(\gamma)>\tilde{r}_2(\gamma)>\cdots>\tilde{r}_N(\gamma)$, and for all $1\leq i\leq p$, the parity of $\{\tilde{r}_i(\gamma)\}_{k-1}$ is opposite from the parity of $\{\tilde{r}_{p+1}(\gamma)\}_{k-1}$.

Notice that there are no $(k-1)$-bands $\{\tilde{r}_{N+1}(\gamma)\}_{k-1}$ in γ. In this case, we define the parity of the empty band to be even, and so $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,N)$ is a subset of $\mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$.

The following theorem shows that the forward move ϕ_p gives rise to a bijection between $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$ and $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$.

Theorem 4.6. *For $N\geq p\geq 1$, the forward move ϕ_p is a bijection between $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$ and $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. Moreover, for $\gamma\in\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$, let $\vartheta=\phi_p(\gamma)$, we have $|\vartheta|=|\gamma|+p\eta$.*

For example, let γ be the overpartition in $\mathcal{B}_1(1,5,9;10,5,4)$, whose reverse Gordon marking reads

$$RG(\gamma)=(\overline{81}_1,\overbrace{80_2,80_3,\overline{71}_1,70_4}^{\{70\}_4},\overline{69}_2,\overbrace{\overline{61}_1,60_3,\overline{59}_2,\overline{55}_4}^{\{\overline{55}\}_4},\overbrace{50_1,\overline{49}_2,\overline{45}_3,\overline{41}_4}^{\{\overline{41}\}_4},$$
$$\overline{39}_1,\overline{35}_2,\underbrace{\overline{29}_1,\overline{21}_2,20_3,20_4}_{\{20\}_4},\underbrace{\overline{11}_1,10_2,\overline{9}_3,\overline{5}_4}_{\{\overline{5}\}_4}).$$

There are five 4-marked parts in $RG(\gamma)$. Moreover, it can be checked that the 4-bands induced by $\tilde{r}_1(\gamma)=70$, $\tilde{r}_2(\gamma)=\overline{55}$, $\tilde{r}_3(\gamma)=\overline{41}$, $\tilde{r}_4(\gamma)=20$ are all even. Therefore, γ is an overpartition in $\mathcal{B}_e(1,5,9;10,5,4|5,3)$. Let $\vartheta=\phi_3(\gamma)$. Then the reverse Gordon marking of ϑ is given by

$$\begin{aligned}RG(\vartheta)=(&\overbrace{\overline{81}_1,80_2,80_3,80_4}^{\{80\}_4},\overline{71}_1,\overbrace{\overline{69}_2,\overline{65}_3,\overline{61}_1,60_4}^{\{60\}_4},\overline{59}_2,\overbrace{\overline{51}_1,50_3,\overline{49}_2,\overline{45}_4}^{\{\overline{45}\}_4},\\&\overline{39}_1,\overline{35}_2,\underbrace{\overline{29}_1,\overline{21}_2,20_3,20_4}_{\{20\}_4},\underbrace{\overline{11}_1,10_2,\overline{9}_3,\overline{5}_4}_{\{\overline{5}\}_4}).\end{aligned}$$

There are five 4-marked parts in $RG(\vartheta)$ and the 4-bands induced by $\tilde{r}_1(\vartheta)=80$, $\tilde{r}_2(\vartheta)=60$ and $\tilde{r}_3(\vartheta)=\overline{45}$ are odd, whereas the 4-band induced by $\tilde{r}_4(\vartheta)=20$ is even. This indicates that ϑ is an overpartition in $\mathcal{B}_d(1,5,9;10,5,4|5,3)$. Clearly, we have $|\vartheta|=|\gamma|+30$.

To prove Theorem 4.6, we establish two lemmas. From now on, we shall use $f_{\leq\eta}(\gamma)$ to denote the number of parts less than or equal to η in an overpartition γ.

Lemma 4.7. *For $N\geq p\geq 1$, let γ be an overpartition in $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$ and let $\vartheta=\phi_p(\gamma)$. Then ϑ is an overpartition in $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. Moreover, $|\vartheta|=|\gamma|+p\eta$.*

Proof. Clearly, γ is an overpartition in $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ with N parts marked with $k-1$ in $RG(\gamma)$. In view of Proposition 4.3, we find that $\vartheta=\phi_p(\gamma)$ satisfies (1), (2) and (3) in Definition 1.14. Furthermore, there are N parts marked with $k-1$ in $RG(\vartheta)$. Thus, to prove that ϑ belongs to $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$, it suffices to verify that the following conditions hold:

(A) $f_{\leq\eta}(\vartheta)\leq r-1$;

(B) For $1\leq i\leq p$, $\{\tilde{r}_i(\vartheta)\}_{k-1}$ and $\{\tilde{r}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities.

Condition (A). It is readily seen that $f_{\leq\eta}(\vartheta)$ equals either $f_{\leq\eta}(\gamma)$ or $f_{\leq\eta}(\gamma)-1$. Under the condition $f_{\leq\eta}(\gamma)\leq r-1$, we get $f_{\leq\eta}(\vartheta)\leq r-1$.

Condition (B). By the property (3) in Proposition 4.3, there are N parts marked with $k-1$ in $G(\vartheta)$, denoted $\tilde{g}_1(\vartheta)>\tilde{g}_2(\vartheta)>\cdots>\tilde{g}_N(\vartheta)$. It follows from Proposition 3.3 that there are also N parts marked with $k-1$ in $RG(\vartheta)$, denoted $\tilde{r}_1(\vartheta)>\tilde{r}_2(\vartheta)>\cdots>\tilde{r}_N(\vartheta)$ such that $\tilde{g}_i(\vartheta)\in\{\tilde{r}_i(\vartheta)\}_{k-1}$ for $1\leq i\leq N$. This implies that $\{\tilde{r}_i(\vartheta)\}_{k-1}$ is a $(k-1)$-band including $\tilde{g}_i(\vartheta)$. Utilizing Proposition 3.6, we obtain that for each $1\leq i\leq N$, $\{\tilde{r}_i(\vartheta)\}_{k-1}$ and $\{\tilde{g}_i(\vartheta)\}_{k-1}$ have the same parity. Therefore, to prove that $\{\tilde{r}_i(\vartheta)\}_{k-1}$ and $\{\tilde{r}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities for $1\leq i\leq p$, we are obliged to show that $\{\tilde{g}_i(\vartheta)\}_{k-1}$ and $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities for $1\leq i\leq p$.

For $1\leq i\leq N$, let

$$\tilde{r}_{i,1}(\gamma)\geq\cdots\geq\tilde{r}_{i,k-2}(\gamma)\geq\tilde{r}_i(\gamma)$$

be the parts in the $(k-1)$-band of γ induced by $\tilde{r}_i(\gamma)$, and let

$$\tilde{g}_i(\vartheta)\geq\tilde{g}_{i,2}(\vartheta)\geq\cdots\geq\tilde{g}_{i,k-1}(\vartheta)$$

be the parts in the $(k-1)$-band of ϑ induced by $\tilde{g}_i(\vartheta)$. Write

$$[\tilde{r}_{i,1}(\gamma)/\eta] + \cdots + [\tilde{r}_{i,k-2}(\gamma)/\eta] + [\tilde{r}_i(\gamma)/\eta] \equiv a_i(\gamma) + \overline{V}_\gamma(\tilde{r}_{i,1}(\gamma)) \pmod 2,$$

and

$$[\tilde{g}_i(\vartheta)/\eta] + [\tilde{g}_{i,2}(\vartheta)/\eta] + \cdots + [\tilde{g}_{i,k-1}(\vartheta)/\eta] \equiv a_i(\vartheta) + \overline{V}_\vartheta(\tilde{g}_i(\vartheta)) \pmod 2,$$

where $a_i(\gamma)$ (resp. $a_i(\vartheta)$) either equals $r-1$ or r for $1 \leq i \leq N$ and $a_{N+1}(\gamma)$ (resp. $a_{N+1}(\vartheta)) = r-1$ with the convention that the empty band is even.

Since γ belongs to $\mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, p)$, where $1 \leq p \leq N$, we have for $1 \leq i \leq p$,

$$a_i(\gamma) = a_{p+1}(\gamma). \tag{4.4}$$

We proceed to show that $\{\tilde{g}_i(\vartheta)\}_{k-1}$ and $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities for $1 \leq i \leq p$, or equivalently, for $1 \leq i \leq p$,

$$a_i(\vartheta) \neq a_{p+1}(\vartheta). \tag{4.5}$$

The proof of Proposition 4.3 justifies the following relation for $1 \leq i \leq p$,

$$\begin{array}{ccccccc} \tilde{r}_i(\gamma)+\eta & \geq & \tilde{r}_{i,1}(\gamma) & \geq & \cdots & \geq & \tilde{r}_{i,k-2}(\gamma) \\ \shortparallel & & \shortparallel & & & & \shortparallel \\ \tilde{g}_i(\vartheta) & \geq & \tilde{g}_{i,2}(\vartheta) & \geq & \cdots & \geq & \tilde{g}_{i,k-1}(\vartheta). \end{array} \tag{4.6}$$

We claim that for $1 \leq i \leq p$,

$$\overline{V}_\vartheta(\tilde{g}_i(\vartheta)) = \overline{V}_\gamma(\tilde{r}_{i,1}(\gamma)). \tag{4.7}$$

Recall that $\overline{V}_\pi(t)$ (resp. $\overline{V}_\pi(\bar{t})$) stands for the number of overlined parts not exceeding t (resp. $\bar{t}$) in π.

Owing to the relation (4.6), we deduce that for $1 \leq i \leq p$,

$$\overline{V}_\gamma(\tilde{r}_{i,1}(\gamma)) - \overline{V}_\vartheta(\tilde{g}_{i,2}(\vartheta)) = \begin{cases} 1, & \text{if } \tilde{r}_i(\gamma) \not\equiv 0 \bmod \eta, \\ 0, & \text{otherwise,} \end{cases} \tag{4.8}$$

and

$$\overline{V}_\vartheta(\tilde{g}_i(\vartheta)) - \overline{V}_\vartheta(\tilde{g}_{i,2}(\vartheta)) = \begin{cases} 1, & \text{if } \tilde{g}_i(\vartheta) \not\equiv 0 \bmod \eta, \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

By definition, $\tilde{g}_i(\vartheta) = \tilde{r}_i(\gamma) + \eta$, and so $\tilde{g}_i(\vartheta)$ is divisible by η if and only if $\tilde{r}_i(\gamma)$ is divisible by η. Therefore, combining (4.8) and (4.9) gives (4.7), and hence the claim is proved.

Invoking the relation (4.6), we find that for $1 \leq i \leq p$,

$$\begin{aligned}
&[\tilde{g}_i(\vartheta)/\eta] + [\tilde{g}_{i,2}(\vartheta)/\eta] + \cdots + [\tilde{g}_{i,k-1}(\vartheta)/\eta] \\
&\quad = [(\tilde{r}_i(\gamma) + \eta)/\eta] + [\tilde{r}_{i,1}(\gamma)/\eta] + \cdots + [\tilde{r}_{i,k-2}(\gamma)/\eta] \\
&\quad = [\tilde{r}_i(\gamma)/\eta] + [\tilde{r}_{i,1}(\gamma)/\eta] + \cdots + [\tilde{r}_{i,k-2}(\gamma)/\eta] + 1 \\
&\quad \equiv a_i(\gamma) + \overline{V}_\gamma(\tilde{r}_{i,1}(\gamma)) + 1 \pmod 2.
\end{aligned}$$

It follows from (4.7) that $a_i(\vartheta) \equiv a_i(\gamma) + 1 \pmod 2$ for $1 \leq i \leq p$. In view of (4.4), we obtain that for $1 \leq i \leq p$,

$$a_i(\vartheta) \equiv a_{p+1}(\gamma) + 1 \pmod 2. \tag{4.10}$$

We next show that

$$a_{p+1}(\gamma) = a_{p+1}(\vartheta). \tag{4.11}$$

From Proposition 3.6, we know that the parity of $\{\tilde{r}_{p+1}(\vartheta)\}_{k-1}$ is the same as that of $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$. On the other hand, the construction of the forward move ϕ_p indicates that the parity of $\{\tilde{r}_{p+1}(\gamma)\}_{k-1}$ is the same as that of $\{\tilde{r}_{p+1}(\vartheta)\}_{k-1}$, and so the parity of $\{\tilde{r}_{p+1}(\gamma)\}_{k-1}$ agrees with that of $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$. Thereby, we get (4.11). Combining (4.10) and (4.11) gives (4.5). It follows that $\{\tilde{g}_i(\vartheta)\}_{k-1}$ and $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities for $1 \leq i \leq p$, and so $\{\tilde{r}_i(\vartheta)\}_{k-1}$ and $\{\tilde{r}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities for $1 \leq i \leq p$. Hence the condition (B) is satisfied.

We have shown that $\vartheta \in \mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, p)$. It is routine to verify that $|\vartheta| = |\gamma| + p\eta$, and thus the proof is complete. ∎

Lemma 4.8. *For $N \geq p \geq 1$, let ϑ be an overpartition in $\mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, p)$ and let $\gamma = \psi_p(\vartheta)$. Then γ is an overpartition in $\mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, p)$. Furthermore, $|\gamma| = |\vartheta| - p\eta$.*

Proof. In order to show that γ is an overpartition in $\mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, p)$, we need to prove that γ satisfies (1), (2) and (3) in Definition 1.14 and there are N parts marked with $k-1$ in $RG(\gamma)$, denoted $\tilde{r}_1(\gamma) > \tilde{r}_2(\gamma) > \cdots > \tilde{r}_N(\gamma)$. Moreover, the following conditions are also required:

(A) $f_{\leq \eta}(\gamma) \leq r - 1$;

(B) the parity of $\{\tilde{r}_i(\gamma)\}_{k-1}$ is the same as that of $\{\tilde{r}_{p+1}(\gamma)\}_{k-1}$ for $1 \leq i \leq p$.

Now we consider (1), (2) and (3) in Definition 1.14. Assume that $\tilde{g}_1(\vartheta) > \tilde{g}_2(\vartheta) > \cdots > \tilde{g}_N(\vartheta)$ are the $(k-1)$-marked parts in the Gordon marking of $\vartheta \in \mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, p)$. By Proposition 4.5, it is necessary to prove that

(C) $\tilde{g}_p(\vartheta) \geq \overline{\eta + \alpha_1}$;

(D) $\tilde{g}_p(\vartheta)$ is a part in any $(k-1)$-band of ϑ belonging to $I(\tilde{g}_p(\vartheta)-2\eta, \tilde{g}_p(\vartheta))$.

Condition (C). Given that $\vartheta \in \mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$, combining Proposition 3.3 and Proposition 3.6, we realize that $\{\tilde{g}_i(\vartheta)\}_{k-1}$ and $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ have opposite parities for $1 \le i \le p$. Suppose to the contrary that $\tilde{g}_p(\vartheta) < \overline{\eta+\alpha_1}$, which means that $\tilde{g}_p(\vartheta) \le \eta$. In this case, we have $p=N$. Observing that $\tilde{g}_N(\vartheta)$ is marked with $k-1$ in $G(\vartheta)$, so we get $f_{\le\eta}(\vartheta) = k-1$, that is, $r=k$. Assume that

$$\tilde{g}_N(\vartheta) \ge \tilde{g}_{N,2}(\vartheta) \ge \cdots \ge \tilde{g}_{N,k-1}(\vartheta)$$

are the parts in the $(k-1)$-band of ϑ induced by $\tilde{g}_N(\vartheta)$. Under the condition that $\{\tilde{g}_N(\vartheta)\}_{k-1}$ and $\{\tilde{g}_{N+1}(\vartheta)\}_{k-1}$ have opposite parities and the convention that the empty band is even, we deduce that $\{\tilde{g}_N(\vartheta)\}_{k-1}$ is odd, that is,

$$[\tilde{g}_N(\vartheta)/\eta] + [\tilde{g}_{N,2}(\vartheta)/\eta] + \cdots + [\tilde{g}_{N,k-1}(\vartheta)/\eta] \equiv r + \overline{V}_\vartheta(\tilde{g}_N(\vartheta)) \pmod 2. \tag{4.12}$$

On the other hand, since $\tilde{g}_N(\vartheta) \le \eta$, we obtain that

$$\begin{aligned} &[\tilde{g}_N(\vartheta)/\eta] + [\tilde{g}_{N,2}(\vartheta)/\eta] + \cdots + [\tilde{g}_{N,k-1}(\vartheta)/\eta] \\ &= f_{\le\eta}(\vartheta) - f_{<\eta}(\vartheta) = k-1-f_{<\eta}(\vartheta), \end{aligned} \tag{4.13}$$

where $f_{<\eta}(\vartheta)$ denotes the number of parts of ϑ less than η. Recall that $\overline{V}_\vartheta(\tilde{g}_N(\vartheta))$ counts the number of overlined parts of ϑ not exceeding $\tilde{g}_N(\vartheta)$. Again, under the assumption $\tilde{g}_N(\vartheta) \le \eta$, we have $\overline{V}_\vartheta(\tilde{g}_N(\vartheta)) = f_{<\eta}(\vartheta)$. Since $r=k$, (4.13) can be written as

$$[\tilde{g}_N(\vartheta)/\eta] + [\tilde{g}_{N,2}(\vartheta)/\eta] + \cdots + [\tilde{g}_{N,k-1}(\vartheta)/\eta] = r-1-\overline{V}_\vartheta(\tilde{g}_N(\vartheta)),$$

which contradicts (4.12). Hence $\tilde{g}_p(\vartheta) \ge \overline{\eta+\alpha_1}$.

Condition (D). Suppose that there is a $(k-1)$-band belonging to $I(\tilde{g}_p(\vartheta)-2\eta, \tilde{g}_p(\vartheta))$ which does not contain $\tilde{g}_p(\vartheta)$ as a part, and let

$$\vartheta_m \ge \vartheta_{m+1} \ge \cdots \ge \vartheta_{m+k-2}$$

be the parts in this $(k-1)$-band, that is, $\vartheta_m \le \vartheta_{m+k-2}+\eta$ with strict inequality if ϑ_m is overlined, $\vartheta_m < \tilde{g}_p(\vartheta)$ and $\vartheta_{m+k-2} \ge \tilde{g}_p(\vartheta)-2\eta$ with strict inequality if $\tilde{g}_p(\vartheta)$ is overlined. In view of Lemma 3.5, we deduce that $\{\tilde{g}_p(\vartheta)\}_{k-1}$ and $\{\vartheta_{m+l}\}_{0\le l\le k-2}$ are of the same parity.

Now, since $\{\vartheta_{m+l}\}_{0\le l\le k-2}$ is a $(k-1)$-band of ϑ, there is a part, say ϑ_{m+t} $(0 \le t \le k-2)$, marked with $k-1$ in $G(\vartheta)$. But $\vartheta_m < \tilde{g}_p(\vartheta)$, so we get $\vartheta_{m+t} = \tilde{g}_{p+1}(\vartheta)$. This implies that $\{\vartheta_{m+l}\}_{0\le l\le k-2}$ is a $(k-1)$-band of ϑ including $\tilde{g}_{p+1}(\vartheta)$. By Proposition 3.6, we deduce that $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ and $\{\vartheta_{m+l}\}_{0\le l\le k-2}$ are of the same parity. It follows that the parity of $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ is the same as that of $\{\tilde{g}_p(\vartheta)\}_{k-1}$, which contradicts the condition that $\vartheta \in \mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$, that is, the parity of $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ is opposite from the parity of $\{\tilde{g}_p(\vartheta)\}_{k-1}$. Therefore, $\tilde{g}_p(\vartheta)$ is a part in any $(k-1)$-band of ϑ belonging to $I(\tilde{g}_p(\vartheta)-2\eta, \tilde{g}_p(\vartheta))$.

Up to now, we have shown that ϑ satisfies the conditions (C) and (D). In view of Proposition 4.5, we see that γ satisfies (1), (2) and (3) in Definition 1.14 and there are N parts marked with $k-1$ in $RG(\gamma)$. We still have to show that γ satisfies the conditions (A) and (B).

Condition (A). By the condition (C), we have $\tilde{g}_p(\vartheta) \geq \overline{\eta+\alpha_1}$. We now consider two cases: (1) If $\tilde{g}_p(\vartheta) > 2\eta$, then $f_{\leq\eta}(\gamma) = f_{\leq\eta}(\vartheta) \leq r-1$. (2) If $\overline{\eta+\alpha_1} \leq \tilde{g}_p(\vartheta) \leq 2\eta$, then $f_{\leq\eta}(\gamma) = f_{\leq\eta}(\vartheta)+1$. In this case, we claim that $f_{\leq\eta}(\vartheta) < r-1$. Suppose to the contrary that $f_{\leq\eta}(\vartheta) = r-1$. Assume that

$$\tilde{g}_p(\vartheta) \geq \tilde{g}_{p,2}(\vartheta) \geq \cdots \geq \tilde{g}_{p,k-1}(\vartheta)$$

are the parts in the $(k-1)$-band of ϑ induced by $\tilde{g}_p(\vartheta)$. Then

$$\begin{aligned}
&[\tilde{g}_p(\vartheta)/\eta] + [\tilde{g}_{p,2}(\vartheta)/\eta] + \cdots + [\tilde{g}_{p,k-1}(\vartheta)/\eta] \\
&\quad \equiv \overline{V}_\vartheta(\tilde{g}_p(\vartheta)) - f_{<\eta}(\vartheta) + f_\eta(\vartheta) \\
&\quad \equiv f_{\leq\eta}(\vartheta) + \overline{V}_\vartheta(\tilde{g}_p(\vartheta)) \pmod 2,
\end{aligned}$$

which implies that $\{\tilde{g}_p(\vartheta)\}_{k-1}$ is even since $f_{\leq\eta}(\vartheta) = r-1$. Given that $\vartheta \in \mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta, k, r|N, p)$, we see that the parity of $\{\tilde{g}_p(\vartheta)\}_{k-1}$ is opposite from the parity of $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$. It follows that $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ is odd, and so $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$ is nonempty. On the other hand, since $\tilde{g}_{p+1}(\vartheta) \leq \tilde{g}_p(\vartheta) - \eta \leq \eta$, it is ensured by Lemma 3.5 that the parity of $\{\tilde{g}_p(\vartheta)\}_{k-1}$ is the same as that of $\{\tilde{g}_{p+1}(\vartheta)\}_{k-1}$, which leads to a contradiction. Hence $f_{\leq\eta}(\vartheta) < r-1$ when $\overline{\eta+\alpha_1} \leq \tilde{g}_p(\vartheta) \leq 2\eta$, and so $f_{\leq\eta}(\gamma) \leq r-1$.

Condition (B). Utilizing the property (4) in Proposition 4.5, we find that for $1 \leq i \leq p$, $\tilde{r}_i(\gamma) = \tilde{g}_i(\vartheta) - \eta$. The reasoning in the proof of Lemma 4.7 can be adapted to deduce that the parity of $\{\tilde{r}_i(\gamma)\}_{k-1}$ is the same as that of $\{\tilde{r}_{p+1}(\gamma)\}_{k-1}$ for $1 \leq i \leq p$.

Thus we conclude that γ is an overpartition in $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. It is manifest from the construction of ψ_p that $|\gamma| = |\vartheta| - p\eta$. This completes the proof. ∎

Proof of Theorem 4.6. Let $\gamma \in \mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. Utilizing Lemma 4.7, we find that $\phi_p(\gamma)$ belongs to $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. In view of the property (4) in Proposition 4.3, we deduce that $\psi_p(\phi_p(\gamma)) = \gamma$.

Analogously, let $\vartheta \in \mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. Invoking Lemma 4.8, we get $\psi_p(\vartheta) \in \mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. By virtue of the property (4) in Proposition 4.5, we obtain that $\phi_p(\psi_p(\vartheta)) = \vartheta$.

Thus, we arrive at the assertion that the forward move ϕ_p is a bijection between $\mathcal{B}_e(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$ and $\mathcal{B}_d(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,p)$. This completes the proof. ∎

4.3 The $(k-1)$-insertion and the $(k-1)$-separation

As mentioned before, a merging operation is needed in the construction of the bijection Φ between $\mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ and $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. The main objective of this subsection is to present a description of this merging operation in terms of the $(k-1)$-insertion operation and the $(k-1)$-separation operation. To be more specific, the merging operation is meant to take the parts divisible by η and the parts of the overpartition in $\mathcal{B}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ to generate certain overlined parts divisible by η. As a result, we get an overpartition π in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. To this end, we shall prepare two subsets of $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. Assume that $a=\eta$ or α_i for some $1\leq i\leq\lambda$.

- For $s\geq N\geq 0$, let $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ denote the set of overpartitions τ in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ satisfying
  (1) There are N parts marked with $k-1$ in $RG(\tau)$, denoted $\tilde{r}_1(\tau)>\tilde{r}_2(\tau)>\cdots>\tilde{r}_N(\tau)$;
  (2) Assume that p is the smallest integer satisfying $\tilde{r}_{p+1}(\tau)+\eta\leq\overline{(s-p)\eta+a}$ with the convention that $\tilde{r}_{N+1}(\tau)=-\infty$. Then the largest overlined part $\equiv a \pmod{\eta}$ in τ is less than $\overline{(s-p)\eta+a}$;
  (3) If $f_{\leq\eta}(\tau)=r-1$, $s=N\geq 1$ and $a\neq\eta$, then $\tilde{r}_N(\tau)\leq\eta$;
  (4) If $s=N=0$ and $a\neq\eta$, then $f_{\leq\eta}(\tau)<r-1$.

- For $s\geq N\geq 0$, let $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ denote the set of overpartitions σ in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ subject to the following conditions:
  (1) There exists an overlined part $\equiv a \pmod{\eta}$ in σ, and assume that the largest overlined part $\equiv a \pmod{\eta}$ in σ is $\overline{t\eta+a}$;
  (2) Let $\hat{\sigma}$ be the overpartition obtained by removing $\overline{t\eta+a}$ from σ. Then there are N parts marked with $k-1$ in $G(\hat{\sigma})$, denoted $\tilde{g}_1(\hat{\sigma})>\tilde{g}_2(\hat{\sigma})>\cdots>\tilde{g}_N(\hat{\sigma})$;
  (3) Assume that p is the smallest integer such that $\tilde{g}_{p+1}(\hat{\sigma})<\overline{t\eta+a}$ with the convention that $\tilde{g}_{N+1}(\hat{\sigma})=-\infty$. Then $s=p+t$.

For example, let $N=5$, $s=6$ and $a=10$ and let τ be the overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$ with the reverse Gordon marking

$$\begin{aligned}RG(\tau)=(&\overline{85}_1,80_2,80_3,\overline{75}_1,70_4,\overline{69}_2,\overline{61}_1,60_3,\overline{59}_2,\overline{55}_4,50_1,\overline{49}_2,\overline{45}_3,\overline{41}_4,\\&\overline{39}_1,\overline{35}_2,\overline{29}_1,\overline{20}_2,20_3,20_4,\overline{11}_1,\overline{10}_2,\overline{9}_3,\overline{5}_4,\overline{1}_1).\end{aligned}\tag{4.14}$$

There are five 4-marked parts $\tilde{r}_1(\tau)=70$, $\tilde{r}_2(\tau)=\overline{55}$, $\tilde{r}_3(\tau)=\overline{41}$, $\tilde{r}_4(\tau)=20$ and $\tilde{r}_5(\tau)=\overline{5}$ in $RG(\tau)$. Then $p=3$ is the smallest integer such that $30=\tilde{r}_{p+1}(\tau)+\eta\leq\overline{(s-p)\eta+a}=\overline{40}$. Meanwhile, the largest overlined part divisible by 10 in τ is $\overline{20}$, which is less than $\overline{(s-p)\eta+a}=\overline{40}$. So τ is an overpartition in $\overline{\mathcal{B}}^{10}_<(1,5,9;10,5,4|5,6)$.

Next example is concerned with determining whether an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ belongs to $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$. Let $N=5$, $s=6$, $a=10$ and let

$$\begin{aligned}\sigma=(&\overline{85},80,80,80,\overline{75},\overline{69},\overline{65},\overline{61},60,\overline{59},\overline{51},50,\overline{49},\overline{45},\overline{40},\\&\overline{39},\overline{35},\overline{29},\overline{20},20,20,\overline{11},\overline{10},\overline{9},\overline{5},\overline{1})\end{aligned}\tag{4.15}$$

be an overpartition in $\overline{\mathcal{B}}_1(1,5,9;10,5,4)$. The largest overlined part divisible by 10 of σ is $\overline{40}$, and so $t=3$. Removing $\overline{40}$ from σ, we get $\hat{\sigma}$ with the Gordon marking

$$\begin{aligned}G(\hat{\sigma})=(&\overline{85}_2,80_4,80_3,80_1,\overline{75}_2,\overline{69}_1,\overline{65}_4,\overline{61}_3,60_2,\overline{59}_1,\overline{51}_4,50_3,\overline{49}_1,\overline{45}_2,\\&\overline{39}_1,\overline{35}_2,\overline{29}_1,\overline{20}_4,20_3,20_2,\overline{11}_1,\overline{10}_4,\overline{9}_3,\overline{5}_2,\overline{1}_1).\end{aligned}\tag{4.16}$$

There are five 4-marked parts $\tilde{g}_1(\hat{\sigma})=80$, $\tilde{g}_2(\hat{\sigma})=\overline{65}$, $\tilde{g}_3(\hat{\sigma})=\overline{51}$, $\tilde{g}_4(\hat{\sigma})=\overline{20}$ and $\tilde{g}_5(\hat{\sigma})=\overline{10}$ in $G(\hat{\sigma})$ and $p=3$ is the smallest integer such that $\overline{20}=\tilde{g}_{p+1}(\hat{\sigma})<\overline{40}$. Indeed, $p+t=s$ holds. Thus, we conclude that σ is an overpartition in $\overline{\mathcal{B}}^{10}_=(1,5,9;10,5,4|5,6)$.

We next give the definition of the $(k-1)$-insertion operation, which serves as a bijection between $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ and $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$.

Definition 4.9 (The $(k-1)$-insertion)**.** *For $s\geq N\geq 0$, let τ be an overpartition in $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ with N parts marked with $k-1$ in $RG(\tau)$, denoted $\tilde{r}_1(\tau)>\cdots>\tilde{r}_N(\tau)$. Assume that p is the smallest integer such that $0\leq p\leq N$ and $\overline{(s-p)\eta+a}\geq\tilde{r}_{p+1}(\tau)+\eta$. The $(k-1)$-insertion $I^a_s\colon\tau\to\sigma$ is defined as follows: first apply the forward move ϕ_p to τ to get $\tau'=\phi_p(\tau)$, then insert $\overline{(s-p)\eta+a}$ into τ' as an overlined part of σ.*

It should be understood that when $p=0$, the forward move ϕ_p is considered as the identity map, that is, $\phi_p(\tau)=\tau$. In this paper, we adopt the $(k-1)$-insertion with $a=\eta$. The case $a=\alpha_1$ will be used in our second paper [25].

For example, take the overpartition τ in $\overline{\mathcal{B}}^{10}_<(1,5,9;10,5,4|5,6)$ whose reverse Gordon marking is given in (4.14). In this case, $p=3$ is the smallest integer such that $\overline{(s-p)\eta+a}=\overline{40}\geq 30=\tilde{r}_{p+1}(\tau)+\eta$, where $s=6$ and $a=10$. Applying the forward move ϕ_3 to τ, we get

$$\begin{aligned}\tau'=(&\overline{85},80,80,80,\overline{75},\overline{69},\overline{65},\overline{61},60,\overline{59},\overline{51},50,\overline{49},\overline{45},\\&\overline{39},\overline{35},\overline{29},\overline{20},20,20,\overline{11},\overline{10},\overline{9},\overline{5},\overline{1}),\end{aligned}$$

whose Gordon marking agrees with the one in (4.16). Inserting $\overline{(s-p)\eta+a}=\overline{40}$ into τ', we obtain $\sigma=I^a_s(\tau)$ as in (4.15), which belongs to $\overline{\mathcal{B}}^{10}_=(1,5,9;10,5,4|5,6)$. Clearly, $|\sigma|=|\tau|+70$.

Theorem 4.10. *For $s\geq N\geq 0$, the $(k-1)$-insertion I^a_s is a bijection between $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ and $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$. Moreover, for $\tau\in\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$, let $\sigma=I^a_s(\tau)$, we have $|\sigma|=|\tau|+s\eta+a$.*

The proof of the above theorem consists of three parts. Lemma 4.11 shows that the $(k-1)$-insertion is a map from $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ to $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$. Lemma 4.13 provides a map (that is, the $(k-1)$-separation) from $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ to $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$. Then we will finish the proof of Theorem 4.10 by confirming that the $(k-1)$-insertion and the $(k-1)$-separation are inverses of each other.

Lemma 4.11. *For $s \geq N \geq 0$, let τ be an overpartition in $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ and let $\sigma = I^a_s(\tau)$. Then σ is an overpartition in $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$. Moreover, $|\sigma| = |\tau| + s\eta + a$.*

Proof. To prove that σ belongs to $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$, we must verify the following conditions:

(A) There exists an overlined part $\equiv a \pmod{\eta}$ in σ, and assume that the largest overlined part $\equiv a \pmod{\eta}$ in σ is $\overline{t\eta+a}$;

(B) Let $\hat{\sigma}$ be the overpartition obtained by removing $\overline{t\eta+a}$ from σ. Then there are N parts marked with $k-1$ in $G(\hat{\sigma})$, denoted $\tilde{g}_1(\hat{\sigma}) > \tilde{g}_2(\hat{\sigma}) > \cdots > \tilde{g}_N(\hat{\sigma})$;

(C) Let p be the smallest integer such that $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. Then we have $p+t=s$;

(D) $f_{\leq\eta}(\sigma) \leq r-1$;

(E) The marks in $G(\sigma)$ do not exceed $k-1$.

Condition (A). Let $\tilde{r}_1(\tau) > \cdots > \tilde{r}_N(\tau)$ be the $(k-1)$-marked parts in $RG(\tau)$. Assume that p is the smallest integer such that $0 \leq p \leq N$ and

$$\overline{(s-p)\eta+a} \geq \tilde{r}_{p+1}(\tau)+\eta. \tag{4.17}$$

By the choice of p, we find that for $p \geq 1$ and $1 \leq i \leq p$,

$$\overline{(s-i+1)\eta+a} < \tilde{r}_i(\tau)+\eta. \tag{4.18}$$

Since $\tau \in \overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$, the largest overlined part $\equiv a \pmod{\eta}$ in τ is less than $\overline{(s-p)\eta+a}$. By the construction of I^a_s, together with (4.18), we deduce that the largest overlined part $\equiv a \pmod{\eta}$ in σ is $\overline{(s-p)\eta+a}$, that is, $t=s-p$.

Condition (B). Since $\hat{\sigma}$ is the overpartition obtained by removing $\overline{t\eta+a}$ from σ, by the construction of I^a_s, we find that $\hat{\sigma} = \phi_p(\tau)$. In view of Proposition 4.3, we know that there are N parts marked with $k-1$ in $G(\hat{\sigma})$, denoted $\tilde{g}_1(\hat{\sigma}) > \tilde{g}_2(\hat{\sigma}) > \cdots > \tilde{g}_N(\hat{\sigma})$.

Condition (C). From the proof for the condition (A), we observe that the largest overlined part $\equiv a \pmod{\eta}$ in σ is $\overline{(s-p)\eta+a}$, that is, $t=s-p$. We attempt to show that p is the smallest integer such that $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. By Proposition 4.3, we get

$$\tilde{g}_i(\hat{\sigma}) = \tilde{r}_i(\tau)+\eta \text{ for } 1 \leq i \leq p, \quad \text{and} \quad \tilde{r}_i(\tau) \leq \tilde{g}_i(\hat{\sigma}) \leq \tilde{r}_{i,1}(\tau) \text{ for } p < i \leq N,$$

where $\tilde{r}_{i,1}(\tau) \geq \cdots \geq \tilde{r}_{i,k-2}(\tau) \geq \tilde{r}_i(\tau)$ are the parts in the $(k-1)$-band of τ induced by $\tilde{r}_i(\tau)$. It follows that for $1 \leq i \leq p$,

$$\tilde{g}_i(\hat{\sigma}) = \tilde{r}_i(\tau) + \eta > \overline{(s-i+1)\eta + a}, \tag{4.19}$$

and

$$\tilde{g}_{p+1}(\hat{\sigma}) \leq \tilde{r}_{p+1,1}(\tau) \leq \tilde{r}_{p+1}(\tau) + \eta,$$

with strict inequality if $\tilde{r}_{p+1}(\tau)$ is overlined. Consequently, in view of (4.17), we deduce that $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta + a}$. But, by (4.19), we find that $\tilde{g}_i(\hat{\sigma}) > \overline{t\eta + a}$ for $1 \leq i \leq p$, from which it follows that p is the smallest integer such that $\overline{t\eta + a} > \tilde{g}_{p+1}(\hat{\sigma})$.

Condition (D). By the construction of $\hat{\sigma}$, we know that $f_{\leq \eta}(\hat{\sigma}) \leq r-1$. To show that $f_{\leq \eta}(\sigma) \leq r-1$, we consider the following two cases:

Case 1: If $\overline{(s-p)\eta + a} > \eta$, then $f_{\leq \eta}(\sigma) = f_{\leq \eta}(\hat{\sigma}) \leq r-1$.

Case 2: If $\overline{(s-p)\eta + a} \leq \eta$, then $p = s$ and $a \neq \eta$. Moreover, because of the choice of p, we further have $s = p = N$. We now encounter two subcases.

Subcase 2.1: If $f_{\leq \eta}(\tau) < r-1$, then $f_{\leq \eta}(\sigma) = f_{\leq \eta}(\hat{\sigma}) + 1 \leq f_{\leq \eta}(\tau) + 1 \leq r-1$.

Subcase 2.2: If $f_{\leq \eta}(\tau) = r-1$, then $N \geq 1$. Based on the fact that $s = p = N \geq 1$ and the condition (3) in the definition of $\overline{\mathcal{B}}^{a}_{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$, we see that $\tilde{r}_N(\tau) \leq \eta$. In the case $p = N$, we may employ the forward move to add η to $\tilde{r}_N(\tau)$ in τ to get $f_{\leq \eta}(\hat{\sigma}) = f_{\leq \eta}(\tau) - 1$. Hence $f_{\leq \eta}(\sigma) = f_{\leq \eta}(\hat{\sigma}) + 1 = f_{\leq \eta}(\tau) = r-1$.

Condition (E). Recall that the marks in $G(\hat{\sigma})$ do not exceed $k-1$ and σ is obtained by inserting $\overline{(s-p)\eta + a}$ into $\hat{\sigma}$. To show that the marks in $G(\sigma)$ do not exceed $k-1$, it is enough to prove that there are no $(k-1)$-bands of $\hat{\sigma}$ in $(\overline{(s-p-1)\eta + a}, \overline{(s-p+1)\eta + a})$. Suppose to the contrary that there exists a $(k-1)$-band $\{\hat{\sigma}_{m+l}\}_{0 \leq l \leq k-2}$ of $\hat{\sigma}$ in $(\overline{(s-p-1)\eta + a}, \overline{(s-p+1)\eta + a})$, namely,

$$\overline{(s-p+1)\eta + a} > \hat{\sigma}_m \geq \hat{\sigma}_{m+1} \geq \cdots \geq \hat{\sigma}_{m+k-2} > \overline{(s-p-1)\eta + a}.$$

From the construction of the $(k-1)$-insertion, we find that $\{\hat{\sigma}_{m+l}\}_{0 \leq l \leq k-2}$ is also a $(k-1)$-band of τ. Hence there is a $(k-1)$-marked part $\hat{\sigma}_{m+t}$ $(0 \leq t \leq k-2)$ in $RG(\tau)$.

Case 1: $\hat{\sigma}_m < \tilde{r}_p(\tau)$. In this case, $\tilde{r}_{p+1}(\tau) \geq \hat{\sigma}_{m+t} > \overline{(s-p-1)\eta + a}$, which contradicts (4.17).

Case 2: $\hat{\sigma}_m > \tilde{r}_p(\tau)$. Setting $i = p$ in (4.18) gives $\overline{(s-p+1)\eta + a} < \tilde{r}_p(\tau) + \eta$, whence $\hat{\sigma}_m < \overline{(s-p+1)\eta + a} < \tilde{r}_p(\tau) + \eta$. Consequently, $\tilde{r}_p(\tau)$ is a part of τ in $(\hat{\sigma}_m - \eta, \hat{\sigma}_m)$, that is, $\hat{\sigma}_m - \eta < \tilde{r}_p(\tau) < \hat{\sigma}_m$. Since $\{\hat{\sigma}_{m+l}\}_{0 \leq l \leq k-2}$ is a $(k-1)$-band of $\hat{\sigma}$, there are exactly $k-2$ parts of $\hat{\sigma}$ after $\hat{\sigma}_m$ belonging to $I(\hat{\sigma}_m - \eta, \hat{\sigma}_m)$. Recalling that $\tilde{r}_p(\tau)$ does not appear in $\hat{\sigma}$, we infer that there are exactly $k-1$ parts of τ after $\hat{\sigma}_m$ belonging to $I(\hat{\sigma}_m - \eta, \hat{\sigma}_m)$. This implies that there is one part belonging to $I(\hat{\sigma}_m - \eta, \hat{\sigma}_m)$ marked with k in $RG(\tau)$, which is again a contradiction since the marks in $RG(\tau)$ are supposed not to exceed $k-1$.

Therefore, there are no $(k-1)$-bands of $\hat{\sigma}$ in $(\overline{(s-p-1)\eta+a}, \overline{(s-p+1)\eta+a})$. That is to say, the marks in $G(\sigma)$ do not exceed $k-1$ after inserting $\overline{(s-p)\eta+a}$ into $\hat{\sigma}$, and so the condition (E) is verified.

Thus, we have shown that σ is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. Clearly, $|\sigma| = |\tau| + s\eta + a$. This completes the proof. ∎

We now define the $(k-1)$-separation, which plays the role of the inverse map of the $(k-1)$-insertion.

Definition 4.12 (The $(k-1)$-separation)**.** *For $s \geq N \geq 0$, let σ be an overpartition in $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$ with the largest overlined part $\equiv a \pmod{\eta}$ being $\overline{t\eta+a}$. The $(k-1)$-separation $J^a_s \colon \sigma \to \tau$ is defined as follows: First remove $\overline{t\eta+a}$ from σ to produce $\hat{\sigma}$, and then apply the backward move ψ_{s-t} to $\hat{\sigma}$ to obtain τ.*

The following lemma states that the $(k-1)$-separation has the specified image set.

Lemma 4.13. *For $s \geq N \geq 0$, let σ be an overpartition in $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$, and let $\tau = J^a_s(\sigma)$. Then τ is an overpartition in $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. Moreover, $|\tau| = |\sigma| - s\eta - a$.*

Proof. To prove that τ belongs to $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$, we need to check the following conditions:

(A) There are N parts marked with $k-1$ in $RG(\tau)$, denoted $\tilde{r}_1(\tau) > \tilde{r}_2(\tau) > \cdots > \tilde{r}_N(\tau)$;

(B) Assume that p is the smallest integer such that $\tilde{r}_{p+1}(\tau) + \eta \leq \overline{(s-p)\eta+a}$. Then the largest overlined part $\equiv a \pmod{\eta}$ in τ is less than $\overline{(s-p)\eta+a}$;

(C) $f_{\leq\eta}(\tau) \leq r-1$;

(D) If $s = N = 0$ and $a \neq \eta$, then $f_{\leq\eta}(\tau) < r-1$;

(E) If $f_{\leq\eta}(\tau) = r-1$, $s = N \geq 1$ and $a \neq \eta$, then $\tilde{r}_N(\tau) \leq \eta$.

Condition (A). Assume that the largest overlined part $\equiv a \pmod{\eta}$ in σ is $\overline{t\eta+a}$. Let $\hat{\sigma}$ be the overpartition obtained by removing $\overline{t\eta+a}$ from σ. By definition, there are N parts marked with $k-1$ in $G(\hat{\sigma})$, denoted $\tilde{g}_1(\hat{\sigma}) > \tilde{g}_2(\hat{\sigma}) > \cdots > \tilde{g}_N(\hat{\sigma})$. Assume that p is the smallest integer such that $\overline{t\eta+a} > \tilde{g}_{p+1}(\hat{\sigma})$. Since σ is an overpartition in $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$, we have $p + t = s$. To show that the condition (A) holds, in view of Proposition 4.5, it suffices to verify the following statements:

(A1) $\tilde{g}_p(\hat{\sigma}) \geq \overline{\eta+\alpha_1}$;

(A2) $\tilde{g}_p(\hat{\sigma})$ is a part in any $(k-1)$-band of $\hat{\sigma}$ belonging to $I(\tilde{g}_p(\hat{\sigma}) - 2\eta, \tilde{g}_p(\hat{\sigma}))$.

Condition (A1). Since $\overline{t\eta + a}$ does not appear in $\hat{\sigma}$, the minimality of p implies that $\overline{t\eta + a} < \tilde{g}_p(\hat{\sigma})$. Under the condition that the marks in $G(\sigma)$ do not exceed $k-1$, it is obvious that there are no $(k-1)$-bands of $\hat{\sigma}$ in $(\overline{(t-1)\eta + a}, \overline{(t+1)\eta + a})$. Let

$$\tilde{g}_p(\hat{\sigma}) \geq \tilde{g}_{p,2}(\hat{\sigma}) \geq \cdots \geq \tilde{g}_{p,k-2}(\hat{\sigma})$$

be the parts in the $(k-1)$-band of $\hat{\sigma}$ induced by $\tilde{g}_p(\hat{\sigma})$. Then

$$\tilde{g}_{p,k-2}(\hat{\sigma}) \geq \tilde{g}_p(\hat{\sigma}) - \eta > \overline{(t-1)\eta + a}.$$

Consequently, $\tilde{g}_p(\hat{\sigma}) \geq \overline{(t+1)\eta + a}$ since there are no $(k-1)$-bands of $\hat{\sigma}$ in $(\overline{(t-1)\eta + a}, \overline{(t+1)\eta + a})$. Using the fact that the largest overlined part $\equiv a \pmod{\eta}$ in $\hat{\sigma}$ is less than $\overline{t\eta + a}$, we are led to the strict inequality

$$\tilde{g}_p(\hat{\sigma}) > \overline{(t+1)\eta + a}, \tag{4.20}$$

which yields (A1).

Condition (A2). Suppose to the contrary that there is a $(k-1)$-band of $\hat{\sigma}$ belonging to $I(\tilde{g}_p(\hat{\sigma}) - 2\eta, \tilde{g}_p(\hat{\sigma}))$ that does not contain $\tilde{g}_p(\hat{\sigma})$ as a part, and let

$$\hat{\sigma}_m \geq \cdots \geq \hat{\sigma}_{m+k-2}$$

be the parts in this $(k-1)$-band. We assume that $\hat{\sigma}_{m+l}$ $(0 \leq l \leq k-2)$ is a part in this $(k-1)$-band marked with $k-1$ in $G(\hat{\sigma})$. Evidently, $\hat{\sigma}_{m+l} \leq \hat{\sigma}_m < \tilde{g}_p(\hat{\sigma})$. We claim that $\hat{\sigma}_{m+l} > \tilde{g}_{p+1}(\hat{\sigma})$. According to (4.20), we get

$$\overline{(t-1)\eta + a} < \tilde{g}_p(\hat{\sigma}) - 2\eta \leq \hat{\sigma}_{m+k-2} \leq \cdots \leq \hat{\sigma}_m. \tag{4.21}$$

As mentioned before, there are no $(k-1)$-bands of $\hat{\sigma}$ in $(\overline{(t-1)\eta + a}, \overline{(t+1)\eta + a})$, and so it follows from (4.21) that $\hat{\sigma}_m \geq \overline{(t+1)\eta + a}$. In fact, we attain the strict inequality $\hat{\sigma}_m > \overline{(t+1)\eta + a}$ owing to the fact that the largest overlined part $\equiv a \pmod{\eta}$ in $\hat{\sigma}$ is less than $\overline{t\eta + a}$. The assumption that $\{\hat{\sigma}_{m+l}\}_{0 \leq l \leq k-2}$ is a $(k-1)$-band ensures that $\hat{\sigma}_{m+k-2} \geq \hat{\sigma}_m - \eta$. Noting that $\hat{\sigma}_m > \overline{(t+1)\eta + a}$, we obtain that $\hat{\sigma}_{m+k-2} > \overline{t\eta + a}$. But $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta + a}$, we arrive at

$$\hat{\sigma}_{m+l} \geq \hat{\sigma}_{m+k-2} > \overline{t\eta + a} > \tilde{g}_{p+1}(\hat{\sigma}),$$

as claimed. Thus, we conclude that $\tilde{g}_{p+1}(\hat{\sigma}) < \hat{\sigma}_{m+l} < \tilde{g}_p(\hat{\sigma})$. However, $\hat{\sigma}_{m+l}$ is marked with $k-1$ in $G(\hat{\sigma})$, which leads to a contradiction since there are no $(k-1)$-marked parts in $G(\hat{\sigma})$ between $\tilde{g}_p(\hat{\sigma})$ and $\tilde{g}_{p+1}(\hat{\sigma})$. This confirms the condition (A2).

With the conditions (A1) and (A2) in hand, Proposition 4.5 guarantees that there are N parts marked with $k-1$ in $RG(\tau)$. In addition, it gives that

$$\tilde{r}_i(\tau) = \tilde{g}_i(\hat{\sigma}) - \eta \text{ for } 1 \leq i \leq p, \quad \text{and} \quad \tilde{g}_{i,k-1}(\hat{\sigma}) \leq \tilde{r}_i(\tau) \leq \tilde{g}_i(\hat{\sigma}) \text{ for } p < i \leq N. \tag{4.22}$$

Thus, we have proved that τ satisfies the condition (A).

Condition (B). We aim to show that p is also the smallest integer such that $\tilde{r}_{p+1}(\tau)+\eta \leq \overline{(s-p)\eta+a}$, where p is defined to be the smallest integer such that $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. Applying (4.22) with $i=p+1$ yields that $\tilde{r}_{p+1}(\tau) \leq \tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. Let

$$\tilde{r}_{p+1,1}(\tau) \geq \cdots \geq \tilde{r}_{p+1,k-2}(\tau) \geq \tilde{r}_{p+1}(\tau)$$

be the parts in the $(k-1)$-band of τ induced by $\tilde{r}_{p+1}(\tau)$. Then

$$\tilde{r}_{p+1,1}(\tau) \leq \tilde{r}_{p+1}(\tau)+\eta < \overline{(t+1)\eta+a}.$$

Since $\{\tilde{r}_{p+1}(\tau)\}_{k-1}$ is also a $(k-1)$-band of $\hat{\sigma}$ and there are no $(k-1)$-bands of $\hat{\sigma}$ in $(\overline{(t-1)\eta+a}, \overline{(t+1)\eta+a})$, we deduce that

$$\tilde{r}_{p+1}(\tau) \leq \overline{(t-1)\eta+a} = \overline{(s-p-1)\eta+a}. \tag{4.23}$$

Combining (4.20) and (4.22), we find that

$$\tilde{r}_{p}(\tau) = \tilde{g}_{p}(\hat{\sigma})-\eta > \overline{t\eta+a} = \overline{(s-p)\eta+a}.$$

Hence for $1 \leq i < p$,

$$\tilde{r}_{i}(\tau) \geq \tilde{r}_{i+1}(\tau)+\eta \geq \cdots \geq \tilde{r}_{p}(\tau)+(p-i)\eta > \overline{(s-i)\eta+a}. \tag{4.24}$$

By inspection of (4.23) and (4.24), we conclude that p is the smallest integer such that $\tilde{r}_{p+1}(\tau)+\eta \leq \overline{(s-p)\eta+a}$. On the other hand, by the definition of J_s^a, we obtain that the largest overlined part $\equiv a \pmod{\eta}$ in τ is less than $\overline{(s-p)\eta+a}$, and so the condition (B) is justified.

Condition (C). To show that $f_{\leq\eta}(\tau) \leq r-1$, we consider three cases:

Case c1: $p=0$. In this case, $\tau=\hat{\sigma}$, so $f_{\leq\eta}(\tau) \leq f_{\leq\eta}(\hat{\sigma}) \leq r-1$.

Case c2: $p \geq 1$ and $\tilde{g}_p(\hat{\sigma})-\eta > \eta$. In this case, $f_{\leq\eta}(\tau) = f_{\leq\eta}(\hat{\sigma}) \leq f_{\leq\eta}(\sigma) \leq r-1$.

Case c3: $p \geq 1$ and $\tilde{g}_p(\hat{\sigma})-\eta \leq \eta$. In this case, $f_{\leq\eta}(\tau) = f_{\leq\eta}(\hat{\sigma})+1$. It follows from (4.20) that $\eta \geq \tilde{g}_p(\hat{\sigma})-\eta > \overline{t\eta+a}$, and so $f_{\leq\eta}(\hat{\sigma}) = f_{\leq\eta}(\sigma)-1$. Hence $f_{\leq\eta}(\tau) = f_{\leq\eta}(\hat{\sigma})+1 = f_{\leq\eta}(\sigma) \leq r-1$.

Condition (D). If $s=N=0$ and $a \neq \eta$, then τ is obtained by removing $\overline{a}$ from σ. This implies that $f_{\leq\eta}(\tau) = f_{\leq\eta}(\sigma)-1 < r-1$.

Condition (E). There are two cases.

Case e1: If $\tilde{g}_p(\hat{\sigma})-\eta \leq \eta$, then $p=N$ and $\tilde{r}_N(\tau) = \tilde{g}_N(\hat{\sigma})-\eta \leq \eta$.

Case e2: If $\tilde{g}_p(\hat{\sigma})-\eta > \eta$, then $f_{\leq\eta}(\hat{\sigma}) = f_{\leq\eta}(\tau)$. The condition $f_{\leq\eta}(\tau)=r-1$ implies that $f_{\leq\eta}(\hat{\sigma})=r-1$. We claim that $p<N$ in this case. Suppose to the contrary that $p=N$. If so, we have $t=s-p=N-p=0$. This implies that $\hat{\sigma}$ is obtained by removing

$\overline{a}$ from σ, and thus $f_{\leq\eta}(\hat{\sigma}) = f_{\leq\eta}(\sigma) - 1 < r - 1$, which contradicts $f_{\leq\eta}(\hat{\sigma}) = r - 1$. Hence we have $p < N$. In light of (4.23), we obtain that

$$\tilde{r}_N(\pi) \leq \tilde{r}_{N-1}(\pi) - \eta \leq \cdots \leq \tilde{r}_{p+1}(\pi) - (N-p-1)\eta \leq \overline{(s-N)\eta + a} = \overline{a} < \eta.$$

Therefore, we have proved that the condition (E) is fulfilled.

So far we have accomplished the task of showing that τ is an overpartition in $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. Evidently, $|\tau| = |\sigma| - s\eta - a$. This completes the proof. ∎

We are now ready to give a proof of Theorem 4.10 based on Lemma 4.11 and Lemma 4.13.

Proof of Theorem 4.10. Let $\tau \in \overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. Utilizing Lemma 4.11, we find that $I^a_s(\tau)$ belongs to $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. Appealing to the condition (C) in the proof of Lemma 4.11 and the property (4) in Proposition 4.3, we deduce that $J^a_s(I^a_s(\tau)) = \tau$.

Conversely, let $\gamma \in \overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. Invoking Lemma 4.13, we know that $J^a_s(\gamma) \in \overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. By virtue of the condition (B) in the proof of Lemma 4.13 and the property (4) in Proposition 4.5, we obtain that $I^a_s(J^a_s(\gamma)) = \gamma$.

Therefore, the map I^a_s is a bijection between $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$ and $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. This completes the proof. ∎

The following theorem gives a criterion to determine whether an overpartition in $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$ is also an overpartition in $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N', s')$, which involves the successive application of the $(k-1)$-insertion operations.

Theorem 4.14. *For $s \geq N \geq 0$, let σ be an overpartition in $\overline{\mathcal{B}}^a_=(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, s)$. Assume that there are N' parts marked with $k-1$ in the reverse Gordon marking of σ. Then σ is also an overpartition in $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N', s')$ if and only if $s' > s$.*

Proof. We first show that if $s' > s$, then σ is in $\overline{\mathcal{B}}^a_<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N', s')$. Let $\tilde{r}_1(\sigma) > \cdots > \tilde{r}_{N'}(\sigma)$ be the $(k-1)$-marked parts in $RG(\sigma)$. We are required to prove that σ satisfies the following conditions:

(A) If p' is the smallest integer such that

$$\overline{(s'-p')\eta + a} \geq \tilde{r}_{p'+1}(\sigma) + \eta,$$

then the largest overlined part $\equiv a \pmod{\eta}$ in σ is less than $\overline{(s'-p')\eta + a}$;

(B) If $f_{\leq\eta}(\sigma) = r - 1$, $s' = N' \geq 1$ and $a \neq \eta$, then $\tilde{r}_{N'}(\sigma) \leq \eta$.

Condition (A). Assume that $\overline{t\eta + a}$ is the largest overlined part $\equiv a \pmod{\eta}$ in σ. Let $\hat{\sigma}$ be the overpartition obtained from σ by removing $\overline{t\eta + a}$. By definition, there are N parts marked with $k-1$ in $G(\hat{\sigma})$, denoted $\tilde{g}_1(\hat{\sigma}) > \cdots > \tilde{g}_N(\hat{\sigma})$. Let p be the smallest integer such that $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. Since $\sigma \in \overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$, we have $p = s-t$. Using Proposition 3.3, we find that there are also N parts marked with $k-1$ in $RG(\hat{\sigma})$, denoted $\tilde{r}_1(\hat{\sigma}) > \cdots > \tilde{r}_N(\hat{\sigma})$ and that $\tilde{r}_i(\hat{\sigma}) \leq \tilde{g}_i(\hat{\sigma})$ for $1 \leq i \leq N$. In particular, $\tilde{r}_{p+1}(\hat{\sigma}) \leq \tilde{g}_{p+1}(\hat{\sigma})$. But $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$, so we get $\tilde{r}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$.

We now attempt to show that $\tilde{r}_{p+1}(\sigma) \leq \overline{t\eta+a}$. Suppose to the contrary that $\tilde{r}_{p+1}(\sigma) > \overline{t\eta+a}$. Since $\hat{\sigma}$ is the overpartition obtained from σ by removing $\overline{t\eta+a}$, we find that $\tilde{r}_{p+1}(\hat{\sigma}) = \tilde{r}_{p+1}(\sigma)$, which implies $\tilde{r}_{p+1}(\hat{\sigma}) > \overline{t\eta+a}$, contradicting the preceding assertion that $\tilde{r}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. This proves $\tilde{r}_{p+1}(\sigma) \leq \overline{t\eta+a}$.

Examining the construction of $\hat{\sigma}$, we notice that N' equals either N or $N+1$. Under the condition that $s' > s$, we get $s' \geq s+1 \geq N+1 \geq N'$. Let p' be the smallest integer such that

$$\overline{(s'-p')\eta+a} \geq \tilde{r}_{p'+1}(\sigma)+\eta.$$

Since $s'-p > s-p = t$ and $\tilde{r}_{p+1}(\sigma) \leq \overline{t\eta+a}$, we find that

$$\overline{(s'-p)\eta+a} \geq \overline{(t+1)\eta+a} = \overline{t\eta+a}+\eta \geq \tilde{r}_{p+1}(\sigma)+\eta.$$

Hence the choice of p' implies that

$$p' \leq p \leq N \leq N'.$$

This leads to

$$\overline{(s'-p')\eta+a} > \overline{(s-p)\eta+a} = \overline{t\eta+a}.$$

This proves the condition (A) because $\overline{t\eta+a}$ is the largest overlined part $\equiv a \pmod{\eta}$ of σ.

Condition (B). As we know, $\tilde{r}_{p+1}(\sigma) \leq \overline{t\eta+a}$, so that

$$\overline{t\eta+a} \geq \tilde{r}_{p+1}(\sigma) \geq \tilde{r}_{p+2}(\sigma)+\eta \geq \cdots \geq \tilde{r}_{N+1}(\sigma)+(N-p)\eta.$$

Observing that $s' \geq s+1 \geq N+1 \geq N'$, we find that $N' = N+1 \geq 1$ and $s = N$ when $s' = N'$. It follows that $\tilde{r}_{N'}(\sigma) = \tilde{r}_{N+1}(\sigma) \leq \overline{t\eta+a}-(N-p)\eta = \overline{a} < \eta$ because $p+t = s = N$. Thus, we have proved the condition (B) is valid.

This completes the proof of the sufficiency. Conversely, assume that σ is in both $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$ and $\overline{\mathcal{B}}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N',s')$, we intend to show that $s' > s$.

Given that σ belongs to $\overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$, we may assume that $\overline{t\eta+a}$ is the largest overlined part $\equiv a \pmod{\eta}$ in σ. Let $\hat{\sigma}$ be the overpartition obtained from σ by removing $\overline{t\eta+a}$. Then there are N parts marked with $k-1$ in $G(\hat{\sigma})$, denote $\tilde{g}_1(\hat{\sigma}) > \cdots > \tilde{g}_N(\hat{\sigma})$. Let p be the smallest integer such that $\tilde{g}_{p+1}(\hat{\sigma}) < \overline{t\eta+a}$. Since $\sigma \in \overline{\mathcal{B}}^a_=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N,s)$, we have $p = s-t$. By the reasoning in the proof of Lemma 4.13, we establish that

$$\tilde{g}_p(\hat{\sigma}) > \overline{(t+1)\eta+a}. \tag{4.25}$$

On the other hand, since σ is also in $\mathcal{B}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N',s')$, there are N' parts marked with $k-1$ in $RG(\sigma)$, denote $\tilde{r}_1(\sigma)>\cdots>\tilde{r}_{N'}(\sigma)$. Assume that p' is the smallest integer such that

$$\overline{(s'-p')\eta+a}\geq\tilde{r}_{p'+1}(\sigma)+\eta. \tag{4.26}$$

The condition that $\sigma\in\mathcal{B}^a_<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N',s')$ ensures that the largest overlined part $\equiv a\pmod\eta$ in σ is less than $\overline{(s'-p')\eta+a}$. But the largest overlined part $\equiv a\pmod\eta$ in σ is supposed to be $\overline{t\eta+a}$, so we get $s'-p'>t=s-p$.

Our final goal is to show that $s'>s$. Suppose to the contrary that $s'\leq s$. This implies that $p'<p$ since $s'-p'>s-p$. Let $\tilde{r}_1(\hat{\sigma})>\cdots>\tilde{r}_N(\hat{\sigma})$ be the $(k-1)$-marked parts in $RG(\hat{\sigma})$. In view of Proposition 3.3, we find that $\tilde{r}_p(\hat{\sigma})\geq\tilde{g}_p(\hat{\sigma})-\eta$. Comparison with (4.25) yields

$$\tilde{r}_p(\hat{\sigma})\geq\tilde{g}_p(\hat{\sigma})-\eta>\overline{t\eta+a},$$

so that

$$\tilde{r}_p(\sigma)=\tilde{r}_p(\hat{\sigma})>\overline{t\eta+a}. \tag{4.27}$$

Using the fact that $p'<p$, we obtain that

$$\tilde{r}_{p'+1}(\sigma)\geq\tilde{r}_{p'+2}(\sigma)+\eta\geq\cdots\geq\tilde{r}_p(\sigma)+(p-p'-1)\eta. \tag{4.28}$$

Substituting (4.27) into (4.28), we arrive at

$$\tilde{r}_{p'+1}(\sigma)>\overline{t\eta+a}+(p-p'-1)\eta=\overline{(s-p')\eta+a}-\eta\geq\overline{(s'-p')\eta+a}-\eta,$$

that is,

$$\tilde{r}_{p'+1}(\sigma)+\eta>\overline{(s'-p')\eta+a},$$

which is in contradiction to (4.26). Thus, we have shown $s'>s$. This completes the proof. ∎

4.4 Proof of Theorem 4.1

In this subsection, we will give a proof of Theorem 4.1 by successively applying the forward move and the $(k-1)$-insertion with $a=\eta$.

Proof of Theorem 4.1. Let μ be an overpartition in $\mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ and let ζ be a partition with distinct parts divisible by η. We wish to construct an overpartition $\pi=\Phi(\zeta,\mu)$ in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ such that $|\pi|=|\zeta|+|\mu|$. There are two cases:

Case 1: $\zeta=\emptyset$. Set $\pi=\mu$. Obviously, $\pi\in\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ and $|\pi|=|\zeta|+|\mu|$.

Case 2: $\zeta\neq\emptyset$. Assume that there are N parts marked with $k-1$ in $G(\mu)$, and set $\zeta=(\eta\zeta_1,\ldots,\eta\zeta_c,\eta\zeta_{c+1}\ldots,\eta\zeta_{c+m})$, where $\zeta_1>\cdots>\zeta_c>N\geq\zeta_{c+1}>\cdots>\zeta_{c+m}>0$. We first merge $\eta\zeta_{c+1},\ldots,\eta\zeta_{c+m}$ and μ by successively applying the forward move. Then, we will merge $\eta\zeta_1,\ldots,\eta\zeta_c$ and μ by applying the $(k-1)$-insertion with $a=\eta$ to generate c overlined parts divisible by η.

Step 1. Let $\tilde{g}_1(\mu) > \tilde{g}_2(\mu) > \cdots > \tilde{g}_N(\mu)$ be the $(k-1)$-marked parts in $G(\mu)$. Note that $\mu \in \mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, we see that $\{\tilde{g}_i(\mu)\}_{k-1}$ are even for $1 \leq i \leq N$. We first merge $\eta\zeta_{c+1}, \ldots, \eta\zeta_{c+m}$ into μ by successively applying the forward move. Denote the intermediate overpartitions by $\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(m)}$ with $\mu^{(0)} = \mu$.

Since $\zeta_{c+1} \leq N$, we find that μ is an overpartition in $\mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, \zeta_{c+1})$. Set $b = 0$ and repeat the following procedure until $b = m$:

(A) Merge $\eta\zeta_{c+b+1}$ into $\mu^{(b)}$. Apply the forward move $\phi_{\zeta_{c+b+1}}$ to $\mu^{(b)}$ to obtain $\mu^{(b+1)}$, that is,

$$\mu^{(b+1)} = \phi_{\zeta_{c+b+1}}(\mu^{(b)}).$$

Since

$$\mu^{(b)} \in \mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, \zeta_{c+b+1}),$$

in view of Lemma 4.7, we deduce that

$$\mu^{(b+1)} \in \mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, \zeta_{c+b+1}),$$

and

$$|\mu^{(b+1)}| = |\mu^{(b)}| + \eta\zeta_{c+b+1}.$$

(B) Replace b by $b+1$. If $b = m$, then we are done. If $b < m$, then we have

$$\mu^{(b)} \in \mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, \zeta_{c+b+1}),$$

since $\zeta_{c+b+1} < \zeta_{c+b} \leq N$. Go back to (A).

Eventually, the above procedure yields $\mu^{(m)} \in \mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, \zeta_{c+m})$ such that

$$|\mu^{(m)}| = |\mu^{(0)}| + \eta\zeta_{c+1} + \cdots + \eta\zeta_{c+m}. \tag{4.29}$$

Step 2. We continue to merge $\eta\zeta_c, \ldots, \eta\zeta_1$ into $\mu^{(m)}$ by successively applying the $(k-1)$-insertion with $a = \eta$. Denote the intermediate overpartitions by $\mu^{(m)}, \mu^{(m+1)}, \ldots, \mu^{(m+c)}$ and set $\pi = \mu^{(m+c)}$. Assume that there are $N(\mu^{(i)})$ parts marked with $k-1$ in $RG(\mu^{(i)})$, where $m \leq i \leq m+c$ and $N(\mu^{(m)}) = N$.

Assume that p is the smallest integer such that $0 \leq p \leq N$ and $\overline{(\zeta_c - p)\eta} \geq \tilde{r}_{p+1}(\mu^{(m)}) + \eta$. Such an integer p exists because $\zeta_c > N$ and $\overline{(\zeta_c - N)\eta} > 0 \geq -\infty = \tilde{r}_{N+1}(\mu^{(m)}) + \eta$. Since $\mu^{(m)} \in \mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N, \zeta_{c+m})$, there are no overlined parts divisible by η in $\mu^{(m)}$. Hence the largest overlined part divisible by η in $\mu^{(m)}$ is less than $\overline{(\zeta_c - p)\eta}$. It follows that

$$\mu^{(m)} \in \overline{\mathcal{B}}^{\eta}_{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|N(\mu^{(m)}), \zeta_c - 1).$$

Merging $\eta\zeta_c, \ldots, \eta\zeta_1$ into $\mu^{(m)}$, the following procedure generates c overlined parts divisible by η. We start with setting $b = 0$.

(A) Merge $\eta\zeta_{c-b}$ into $\mu^{(m+b)}$ to generate an overlined part divisible by η. More precisely, applying the $(k-1)$-insertion $I^{\eta}_{\zeta_{c-b}-1}$ to $\mu^{(m+b)}$, we obtain

$$\mu^{(m+b+1)} = I^{\eta}_{\zeta_{c-b}-1}(\mu^{(m+b)}).$$

Since

$$\mu^{(m+b)} \in \overline{\mathcal{B}}^{\eta}_{<}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N(\mu^{(m+b)}),\zeta_{c-b}-1),$$

in view of Lemma 4.11, we find that

$$\mu^{(m+b+1)} \in \overline{\mathcal{B}}^{\eta}_{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N(\mu^{(m+b)}),\zeta_{c-b}-1),$$

and

$$|\mu^{(m+b+1)}| = |\mu^{(m+b)}| + \eta\zeta_{c-b}.$$

(B) Replace b by $b+1$. If $b=c$, then we are done. If $b<c$, since $\zeta_{c-b} > \zeta_{c-b+1}$, it follows from Theorem 4.14 that

$$\mu^{(m+b)} \in \overline{\mathcal{B}}^{\eta}_{<}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N(\mu^{(m+b)}),\zeta_{c-b}-1).$$

Go back to (A).

The above procedure generates an overpartition $\pi = \mu^{(m+c)} \in \overline{\mathcal{B}}^{\eta}_{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r |N(\mu^{(m+c-1)}),\zeta_1-1)$ such that

$$|\mu^{(m+c)}| = |\mu^{(m)}| + \eta\zeta_c + \cdots + \eta\zeta_1. \tag{4.30}$$

From the construction of the $(k-1)$-insertion with $a=\eta$, it can be seen that π is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ with c overlined parts divisible by η. Furthermore, combining (4.29) with (4.30), we find that $|\pi| = |\mu| + |\zeta|$. Therefore, Φ is a desired map from $\mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ to $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$.

To prove that Φ is a bijection, we shall define the inverse map Ψ of Φ from $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ to $\mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ by successively applying the $(k-1)$-separation with $a=\eta$ and the backward move. Let π be an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. We shall construct a pair of overpartitions (ζ,μ), that is, $\Psi(\pi)=(\zeta,\mu)$, such that $|\zeta|+|\mu| = |\pi|$, where $\zeta \in \mathcal{D}_\eta$ and $\mu \in \mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$.

There are two steps in the construction of (ζ,μ) from π. In the first step, we eliminate all overlined parts of π divisible by η by successively applying the $(k-1)$-separation with $a=\eta$. In the second step, we successively apply the backward move to the resulting overpartition in the first step so that all $(k-1)$-bands of the obtained overpartition are even.

Step 1. Assume that there are $c \geq 0$ overlined parts divisible by η in π. We eliminate the c overlined parts divisible by η from π by applying the $(k-1)$-separation with $a=\eta$. Denote the intermediate pairs by $(\zeta^{(0)},\pi^{(0)}),\ldots,(\zeta^{(c)},\pi^{(c)})$, with $(\zeta^{(0)},\pi^{(0)}) = (\emptyset,\pi)$. There are two cases:

Case 1: $c=0$. Then set $\zeta^{(c)}=\emptyset$ and $\pi^{(c)}=\pi$.

Case 2: $c\geq 1$. Assume that $\overline{\eta t_0}>\overline{\eta t_1}>\cdots>\overline{\eta t_{c-1}}$ are the overlined parts of π divisible by η. Set $b=0$ and carry out the following procedure.

(A) Let $\hat{\pi}^{(b)}$ be the overpartition obtained from $\pi^{(b)}$ by removing the overlined part $\overline{\eta t_b}$. Assume that $\tilde{g}_1(\hat{\pi}^{(b)})>\cdots>\tilde{g}_{N(\hat{\pi}^{(b)})}(\hat{\pi}^{(b)})$ are the $N(\hat{\pi}^{(b)})$ parts marked with $k-1$ in $G(\hat{\pi}^{(b)})$, and p_b is the smallest integer such that $\tilde{g}_{p_b+1}(\hat{\pi}^{(b)})<\overline{\eta t_b}$. Let $s^{(b)}=p_b+t_b$. By definition,

$$\pi^{(b)}\in\overline{\mathcal{B}}_{=}^{\eta}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N(\hat{\pi}^{(b)}),s^{(b)}-1).$$

Apply the $(k-1)$-separation $J^{\eta}_{s^{(b)}-1}$ to $\pi^{(b)}$ to get $\pi^{(b+1)}$, that is,

$$\pi^{(b+1)}=J^{\eta}_{s^{(b)}-1}(\pi^{(b)}).$$

By means of Lemma 4.13, we find that

$$\pi^{(b+1)}\in\overline{\mathcal{B}}_{<}^{\eta}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|N(\hat{\pi}^{(b)}),s^{(b)}-1),$$

and

$$\left|\pi^{(b+1)}\right|=|\pi^{(b)}|-\eta s^{(b)}.$$

Then insert $\eta s^{(b)}$ into $\zeta^{(b)}$ as a part to obtain $\zeta^{(b+1)}$.

(B) Replace b by $b+1$. If $b=c$, then we are done. Otherwise, go back to (A).

Observe that for $0\leq b\leq c$, there are $c-b$ overlined parts divisible by η in $\pi^{(b)}$. Theorem 4.14 reveals that for $0\leq b<c-1$,

$$s^{(b)}>s^{(b+1)}>N(\hat{\pi}^{(b+1)}).\tag{4.31}$$

Therefore, there are no overlined parts divisible by η in $\pi^{(c)}$ and $\zeta^{(c)}=(\eta s^{(0)},\ldots,\eta s^{(c-1)})$ is a partition with distinct parts divisible by η. Moreover, we have

$$|\pi|=|\pi^{(c)}|+|\zeta^{(c)}|.\tag{4.32}$$

Let us now move on to the second step.

Step 2. Applying the backward move successively to $\pi^{(c)}$, we are led to a pair of overpartitions $(\zeta,\mu)\in\mathcal{D}_\eta\times\mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ such that $|\pi^{(c)}|=|\mu|+|\zeta|$. Let N be the number of the $(k-1)$-marked parts in $G(\pi^{(c)})$ and let $\tilde{g}_1(\pi^{(c)})>\cdots>\tilde{g}_N(\pi^{(c)})$ be the $(k-1)$-marked parts in $G(\pi^{(c)})$. There are two cases:

Case 1: All the $(k-1)$-bands $\{\tilde{g}_i(\pi^{(c)})\}_{k-1}$ of $\pi^{(c)}$ are even. In view of Theorem 3.4, we have $\pi^{(c)}\in\mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. Set $\mu=\pi^{(c)}$ and $\zeta=\zeta^{(c)}$. Then $(\zeta,\mu)\in\mathcal{D}_\eta\times\mathcal{B}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$ and $|\pi|=|\mu|+|\zeta|$.

Case 2: There exists i such that $1\leq i\leq N$ and $\{\tilde{g}_i(\pi^{(c)})\}_{k-1}$ is odd.

In this case, we set $b=0$ and execute the following procedure. Denote the intermediate pairs by $(\zeta^{(c)},\pi^{(c)}),(\zeta^{(c+1)},\pi^{(c+1)})$, and so on.

(A) Let $\tilde{g}_1(\pi^{(c+b)}) > \cdots > \tilde{g}_N(\pi^{(c+b)})$ be the $(k-1)$-marked parts in $G(\pi^{(c+b)})$ and let $1 \leq p_{c+b} \leq N$ be the smallest integer such that $\{\tilde{g}_{p_{c+b}}(\pi^{(c+b)})\}_{k-1}$ and $\{\tilde{g}_{p_{c+b}+1}(\pi^{(c+b)})\}_{k-1}$ have opposite parities. By definition, we get

$$\pi^{(c+b)} \in \mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | N, p_{c+b}).$$

Apply the backward move $\psi_{p_{c+b}}$ to $\pi^{(c+b)}$ to get $\pi^{(c+b+1)}$, that is,

$$\pi^{(c+b+1)} = \psi_{p_{c+b}}(\pi^{(c+b)}).$$

By Lemma 4.8, we obtain that

$$\pi^{(c+b+1)} \in \mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | N, p_{c+b}),$$

and

$$|\pi^{(c+b+1)}| = |\pi^{(c+b)}| - \eta p_{c+b}. \tag{4.33}$$

Then insert ηp_{c+b} into $\zeta^{(c+b)}$ as a part to get a partition $\zeta^{(c+b+1)}$.

(B) Replace b by $b+1$. If all the $(k-1)$-bands $\{\tilde{g}_i(\pi^{(c+b)})\}_{k-1}$ of $\pi^{(c+b)}$ are even, then we are done. Otherwise, go back to (A).

We claim that during the above procedure, we have

$$N \geq p_{c+b+1} > p_{c+b}. \tag{4.34}$$

Given $b \geq 0$, since $\pi^{(c+b+1)} \in \mathcal{B}_e(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | N, p_{c+b})$, we know that p_{c+b} is the least integer such that $\{\tilde{g}_i(\pi^{(c+b+1)})\}_{k-1}$ have the same parity for $1 \leq i \leq p_{c+b}+1$. Whereas $\pi^{(c+b+1)}$ is in $\mathcal{B}_d(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | N, p_{c+b+1})$, so that p_{c+b+1} is the least integer such that $\{\tilde{g}_{p_{c+b+1}}(\pi^{(c+b+1)})\}_{k-1}$ and $\{\tilde{g}_{p_{c+b+1}+1}(\pi^{(c+b+1)})\}_{k-1}$ have opposite parities. Hence we obtain (4.34), and this proves the claim.

The relation (4.34) ensures that the above procedure terminates after at most N iterations. Assume that it terminates with $b = m$, that is, all the $(k-1)$-bands $\{\tilde{g}_i(\pi^{(c+m)})\}_{k-1}$ are even for $1 \leq i \leq N$. Set

$$\mu = \pi^{(c+m)} \quad \text{and} \quad \zeta = \zeta^{(c+m)} = (\eta s_0, \ldots, \eta s_{c-1}, \eta p_{c+m-1}, \ldots, \eta p_c).$$

Utilizing Theorem 3.4, we find that μ is an overpartition in $\mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. Observe that $N = N(\hat{\pi}^{(c-1)})$ when $c \geq 1$. In light of (4.31) and (4.34), we conclude that ζ is a partition with distinct parts divisible by η. Combining (4.32) and (4.33), we have $|\pi| = |\mu| + |\zeta|$. Therefore, Ψ is a map from $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ to $\mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$.

Combining Theorem 4.6 and Theorem 4.10, we obtain that $\Psi(\Phi(\zeta, \mu)) = (\zeta, \mu)$ for all $(\zeta, \mu) \in \mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and $\Phi(\Psi(\pi)) = \pi$ for all $\pi \in \overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. Hence Φ is a bijection between $\mathcal{D}_\eta \times \mathcal{B}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. This completes the proof. ∎

4.5 An example

We provide an example to illustrate the bijection Φ in Theorem 4.1. Let

$$\zeta = (100, 80, 50, 40, 20)$$

be a partition in $\mathcal{D}_{10}$, and let μ be an overpartition in $\mathcal{B}_0(3, 7; 10, 4, 3)$ with the reverse Gordon marking

$$\begin{aligned} RG(\mu) = (&\overbrace{\overline{87}_1, 80_2, 80_3}^{\{80\}_3}, \overline{67}_1, \overline{63}_2, \overbrace{\overline{57}_1, 50_2, 50_3}^{\{50\}_3}, \overline{43}_1, \overbrace{\overline{37}_2, \overline{33}_1, 30_3}^{\{30\}_3}, \\ &\underbrace{20_1, 20_2, \overline{13}_3}_{\{\overline{13}\}_3}, \overline{7}_1, \overline{3}_2). \end{aligned}$$

The overpartition $\pi = \Phi(\zeta, \mu)$ is obtained by successively applying the forward move and the 3-insertion with $a = 10$. Observe that there are four 3-marked parts in $RG(\mu)$, that is, $N = 4$. We first merge 40 and 20 of ζ into μ by successively applying the forward move and then merge 50, 80 and 100 of ζ into the resulting overpartition by successively applying the 3-insertion with $a = 10$.

Step 1. Merge 40 and 20 of ζ into μ by successively applying the forward move.

Note that $\{80\}_3$, $\{50\}_3$, $\{30\}_3$, $\{\overline{13}\}_3$ in $RG(\mu)$ are all even, so $\mu \in \mathcal{B}_e(3, 7; 10, 4, 3|4, 4)$.

- Set $\mu^{(0)} = \mu$, and merge 40 into $\mu^{(0)}$.

  Apply the forward move ϕ_4 to $\mu^{(0)}$ to get $\mu^{(1)}$, namely, add $\eta = 10$ to each of the 3-marked parts $80, 50, 30$ and $\overline{13}$ in $RG(\mu^{(0)})$ respectively and rearrange the parts in non-increasing order to obtain $\mu^{(1)} = (90, \overline{87}, 80, \overline{67}_1, \overline{63}, 60, \overline{57}, 50, \overline{43}, 40, \overline{37}, \overline{33}, \overline{23}, 20, 20, \overline{7}, \overline{3})$. The reverse Gordon marking of $\mu^{(1)}$ is given by

$$\begin{aligned} RG(\mu^{(1)}) = (&\overbrace{90_1, \overline{87}_2, 80_3}^{\{80\}_3}, \overbrace{\overline{67}_1, \overline{63}_2, 60_3}^{\{60\}_3}, \overline{57}_1, \overbrace{50_2, \overline{43}_1, 40_3}^{\{40\}_3}, \\ &\overline{37}_2, \overline{33}_1, \underbrace{\overline{23}_1, 20_2, 20_3}_{\{20\}_3}, \overline{7}_1, \overline{3}_2). \end{aligned}$$

  By Lemma 4.7, we deduce that $\mu^{(1)} \in \mathcal{B}_d(3, 7; 10, 4, 3|4, 4)$. Indeed, $\{80\}_3$, $\{60\}_3$, $\{40\}_3$, $\{20\}_3$ in $RG(\mu^{(1)})$ are odd. This implies that $\mu^{(1)} \in \mathcal{B}_e(3, 7; 10, 4, 3|4, 2)$.

- Merge 20 into $\mu^{(1)}$.

  Apply the forward move ϕ_2 to $\mu^{(1)}$ to obtain $\mu^{(2)}$, namely, add $\eta = 10$ to each of the 3-marked parts 80 and 60 in $RG(\mu^{(1)})$. We get

$$\begin{aligned} RG(\mu^{(2)}) = (&\overbrace{90_1, 90_2, \overline{87}_3}^{\{\overline{87}\}_3}, \overbrace{70_1, \overline{67}_2, \overline{63}_3}^{\{\overline{63}\}_3}, \overline{57}_1, \overbrace{50_2, \overline{43}_1, 40_3}^{\{40\}_3}, \\ &\overline{37}_2, \overline{33}_1, \underbrace{\overline{23}_1, 20_2, 20_3}_{\{20\}_3}, \overline{7}_1, \overline{3}_2). \end{aligned}$$

Again, it follows from Lemma 4.7 that $\mu^{(2)} \in \mathcal{B}_d(3,7;10,4,3|4,2)$. In fact, $\{\overline{87}\}_3$ and $\{\overline{63}\}_3$ in $RG(\mu^{(2)})$ are even, but $\{40\}_3$ in $RG(\mu^{(2)})$ is odd.

Step 2. Successively employ the 3-insertion with $a = 10$ to merge 50, 80 and 100 of ζ into $\mu^{(2)}$.

- We start with merging 50 into $\mu^{(2)}$, and set $s = 4$.

  There are four 3-marked parts in $RG(\mu^{(2)})$, which are $\tilde{r}_1(\mu^{(2)}) = \overline{87}$, $\tilde{r}_2(\mu^{(2)}) = \overline{63}$, $\tilde{r}_3(\mu^{(2)}) = 40$ and $\tilde{r}_4(\mu^{(2)}) = 20$. In this occasion, $p = 4$ is the smallest integer such that $\overline{(4-p)\cdot 10 + 10} = \overline{10} \geq \tilde{r}_{p+1}(\mu^{(2)}) + 10 = -\infty$ and there are no overlined parts divisible by 10 in $\mu^{(2)}$. Hence $\mu^{(2)} \in \overline{\mathcal{B}}_<^{10}(3,7;10,4,3|4,4)$.

  Apply the 3-insertion I_4^{10} to $\mu^{(2)}$ to get $\mu^{(3)}$. More precisely, add $\eta = 10$ to each of the 3-marked parts $\overline{87}$, $\overline{63}$, 40 and 20 in $RG(\mu^{(2)})$ and then insert $\overline{10}$ into the resulting overpartition as an overlined part. The resulting reverse Gordon marking reads

$$RG(\mu^{(3)}) = (\overline{97}_1, 90_2, 90_3, \overline{73}_1, 70_2, \overline{67}_3, \overline{57}_1, 50_2, 50_3, \overline{43}_1,$$
$$\overline{37}_2, \overline{33}_1, 30_3, \overline{23}_1, 20_2, \overline{10}_1, \overline{7}_2, \overline{3}_3).$$

  Utilizing Lemma 4.11 gives $\mu^{(3)} \in \overline{\mathcal{B}}_=^{10}(3,7;10,4,3|4,4)$.

- Merge 80 into $\mu^{(3)}$ and set $s = 7$.

  There are five 3-marked parts in $RG(\mu^{(3)})$, to wit, $\tilde{r}_1(\mu^{(3)}) = 90$, $\tilde{r}_2(\mu^{(3)}) = \overline{67}$, $\tilde{r}_3(\mu^{(3)}) = 50$, $\tilde{r}_4(\mu^{(3)}) = 30$ and $\tilde{r}_5(\mu^{(3)}) = \overline{3}$. Moreover, $p = 2$ is the smallest integer such that $\overline{(7-p)\cdot 10 + 10} = \overline{60} \geq \tilde{r}_{p+1}(\mu^{(3)}) + 10 = 60$. Given that $\mu^{(3)} \in \overline{\mathcal{B}}_=^{10}(3,7;10,4,3|4,4)$, Theorem 4.14 yields that $\mu^{(3)} \in \overline{\mathcal{B}}_<^{10}(3,7;10,4,3|5,7)$.

  Apply the 3-insertion I_7^{10} to $\mu^{(3)}$ to get $\mu^{(4)}$, that is, add $\eta = 10$ to each of the 3-marked parts 90 and $\overline{67}$ in $RG(\mu^{(3)})$ and then insert $\overline{60}$ into the resulting overpartition as an overlined part. We are led to

$$RG(\mu^{(4)}) = (100_1, \overline{97}_2, 90_3, \overline{77}_1, \overline{73}_2, 70_3, \overline{60}_1, \overline{57}_2, 50_1, 50_3, \overline{43}_2,$$
$$\overline{37}_1, \overline{33}_2, 30_3, \overline{23}_1, 20_2, \overline{10}_1, \overline{7}_2, \overline{3}_3).$$

  As asserted by Lemma 4.11, we have $\mu^{(4)} \in \overline{\mathcal{B}}_=^{10}(3,7;10,4,3|5,7)$.

- Finally, merge 100 into $\mu^{(4)}$, and set $s = 9$.

  There are five 3-marked parts in $RG(\mu^{(4)})$, namely, $\tilde{r}_1(\mu^{(4)}) = 90$, $\tilde{r}_2(\mu^{(4)}) = 70$, $\tilde{r}_3(\mu^{(4)}) = 50$, $\tilde{r}_4(\mu^{(4)}) = 30$ and $\tilde{r}_5(\mu^{(4)}) = \overline{3}$. Moreover, $p = 0$ is the smallest integer such that $\overline{(9-p)\cdot 10 + 10} = \overline{100} \geq \tilde{r}_1(\mu^{(4)}) + 10 = 100$. Knowing that $\mu^{(4)} \in \overline{\mathcal{B}}_=^{10}(3,7;10,4,3|5,7)$, Theorem 4.14 indicates that $\mu^{(4)} \in \overline{\mathcal{B}}_<^{10}(3,7;10,4,3|5,9)$.

Apply the 3-insertion I_9^{10} to $\mu^{(4)}$ to get $\mu^{(5)}$. In other words, insert $\overline{100}$ into $\mu^{(4)}$ as an overlined part to generate

$$\begin{aligned} RG(\mu^{(5)}) = (&\overline{100}_1, 100_2, \overline{97}_3, 90_1, \overline{77}_1, \overline{73}_2, 70_3, \overline{60}_1, \overline{57}_2, 50_1, 50_3, \overline{43}_2, \\ &\overline{37}_1, \overline{33}_2, 30_3, \overline{23}_1, 20_2, \overline{10}_1, \overline{7}_2, \overline{3}_3). \end{aligned} \tag{4.35}$$

Using Lemma 4.11 again, we conclude that $\mu^{(5)} \in \overline{\mathcal{B}}_{=}^{10}(3, 7; 10, 4, 3|5, 9)$.

Set $\pi = \mu^{(5)}$. Clearly, π is an overpartition in $\overline{\mathcal{B}}_1(3, 7; 10, 4, 3)$ such that $|\pi| = |\mu| + |\zeta|$.

Conversely, let π be an overpartition in $\overline{\mathcal{B}}_1(3, 7; 10, 4, 3)$ whose reverse Gordon marking is given by (4.35). The pair of overpartitions $\Psi(\pi) = (\zeta, \mu)$ can be recovered by successively applying the 3-separation with $a = 10$ and the backward move. There are three overlined parts in π divisible by 10, as identified by $\overline{100}$, $\overline{60}$ and $\overline{10}$.

Step 1. Eliminate $\overline{100}$, $\overline{60}$ and $\overline{10}$ from π by successively using the 3-separation with $a = 10$.

- Eliminate $\overline{100}$ from π, and set $t_0 = 10$.

  Set $\pi^{(0)} = \pi$ and $\zeta^{(0)} = \emptyset$. Let $\hat{\pi}^{(0)}$ be the overpartition obtained from $\pi^{(0)}$ by removing $\overline{100}$, which has the Gordon marking

  $$\begin{aligned} G(\hat{\pi}^{(0)}) = (&100_3, \overline{97}_2, 90_1, \overline{77}_3, \overline{73}_2, 70_1, \overline{60}_2, \overline{57}_1, 50_3, 50_2, \overline{43}_1, \overline{37}_2, \overline{33}_1 \\ &30_3, \overline{23}_2, 20_1, \overline{10}_3, \overline{7}_2, \overline{3}_1). \end{aligned} \tag{4.36}$$

  There are five 3-marked parts in $G(\hat{\pi}^{(0)})$, namely, $\tilde{g}_1(\hat{\pi}^{(0)}) = 100$, $\tilde{g}_2(\hat{\pi}^{(0)}) = \overline{77}$, $\tilde{g}_3(\hat{\pi}^{(0)}) = 50$, $\tilde{g}_4(\hat{\pi}^{(0)}) = 30$ and $\tilde{g}_5(\hat{\pi}^{(0)}) = \overline{10}$. Moreover, $p_0 = 0$ is the smallest integer such that $\overline{10 \cdot t_0} = \overline{100} > \tilde{g}_{p_0+1}(\hat{\pi}^{(0)}) = 100$. Set $s_0 = p_0 + t_0 = 10$. Then $\pi^{(0)} \in \overline{\mathcal{B}}_{=}^{10}(3, 7; 10, 4, 3|5, 9)$.

  Set $\zeta^{(1)} = (100)$. Apply the 3-separation J_9^{10} to $\pi^{(0)}$ to get $\pi^{(1)}$. In other words, $\pi^{(1)}$ is obtained from $\pi^{(0)}$ by removing $\overline{100}$, which means that $\pi^{(1)} = \hat{\pi}^{(0)}$ and the Gordon marking of $\pi^{(1)}$ is given by (4.36). Appealing to Lemma 4.13, we deduce that $\pi^{(1)} \in \overline{\mathcal{B}}_{<}^{10}(3, 7; 10, 4, 3|5, 9)$.

- Eliminate $\overline{60}$ from $\pi^{(1)}$, and set $t_1 = 6$.

  Let $\hat{\pi}^{(1)}$ be the overpartition obtained from $\pi^{(1)}$ by removing $\overline{60}$. We have

  $$\begin{aligned} G(\hat{\pi}^{(1)}) = (&100_3, \overline{97}_2, 90_1, \overline{77}_3, \overline{73}_2, 70_1, \overline{57}_1, 50_3, 50_2, \overline{43}_1, \overline{37}_2, \overline{33}_1 \\ &30_3, \overline{23}_2, 20_1, \overline{10}_3, \overline{7}_2, \overline{3}_1). \end{aligned}$$

  There are five 3-marked parts in $G(\hat{\pi}^{(1)})$, which are $\tilde{g}_1(\hat{\pi}^{(1)}) = 100$, $\tilde{g}_2(\hat{\pi}^{(1)}) = \overline{77}$, $\tilde{g}_3(\hat{\pi}^{(1)}) = 50$, $\tilde{g}_4(\hat{\pi}^{(1)}) = 30$ and $\tilde{g}_5(\hat{\pi}^{(1)}) = \overline{10}$. Now, $p_1 = 2$ is the smallest integer such that $\overline{10 \cdot t_1} = \overline{60} > \tilde{g}_{p_1+1}(\hat{\pi}^{(1)}) = 50$. Set $s_1 = p_1 + t_1 = 8$, and we get $\pi^{(1)} \in \overline{\mathcal{B}}_{=}^{10}(3,7;10,4,3|5,7)$. Clearly, $s_0 > s_1$, in agreement with Theorem 4.14.

Set $\zeta^{(2)} = (100, 80)$. Apply the 3-separation J_7^{10} to $\pi^{(1)}$ to get $\pi^{(2)}$, namely, remove $\overline{60}$ from $\pi^{(1)}$ to get $\hat{\pi}^{(1)}$, and then subtract $\eta = 10$ from each of the 3-marked parts 100 and $\overline{77}$ in $G(\hat{\pi}^{(1)})$ to get $\pi^{(2)}$. The Gordon marking of $\pi^{(2)}$ is given below:

$$G(\pi^{(2)}) = (\overline{97}_3, 90_2, 90_1, \overline{73}_3, 70_2, \overline{67}_1, \overline{57}_1, 50_3, 50_2, \overline{43}_1, \overline{37}_2, \overline{33}_1 \\ 30_3, \overline{23}_2, 20_1, \overline{10}_3, \overline{7}_2, \overline{3}_1).$$

We now have $\pi^{(2)} \in \overline{\mathcal{B}}_{<}^{10}(3,7;10,4,3|5,7)$, as expected by Lemma 4.13.

- Finally, eliminate $\overline{10}$ from $\pi^{(2)}$, and set $t_2 = 1$.

  Let $\hat{\pi}^{(2)}$ be the overpartition obtained from $\pi^{(2)}$ by removing $\overline{10}$, so that

$$G(\hat{\pi}^{(2)}) = (\overline{97}_3, 90_2, 90_1, \overline{73}_3, 70_2, \overline{67}_1, \overline{57}_1, 50_3, 50_2, \overline{43}_1, \overline{37}_2, \overline{33}_1 \\ 30_3, \overline{23}_2, 20_1, \overline{7}_2, \overline{3}_1).$$

  There are four 3-marked parts in $G(\hat{\pi}^{(2)})$, namely, $\tilde{g}_1(\hat{\pi}^{(2)}) = \overline{97}$, $\tilde{g}_2(\hat{\pi}^{(2)}) = \overline{73}$, $\tilde{g}_3(\hat{\pi}^{(2)}) = 50$ and $\tilde{g}_4(\hat{\pi}^{(2)}) = 30$. Meanwhile, $p_2 = 4$ is the smallest integer such that $\overline{10 \cdot t_2} = \overline{10} > \tilde{g}_{p_2+1}(\hat{\pi}^{(1)}) = -\infty$. Set $s_2 = t_2 + p_2 = 5$. Then $\pi^{(2)} \in \overline{\mathcal{B}}_{=}^{10}(3,7;10,4,3|4,4)$. In accordance with Theorem 4.14, we have $s_1 > s_2$.

  Set $\zeta^{(3)} = (100, 80, 50)$. Apply the 3-separation J_4^{10} to $\pi^{(2)}$ to get $\pi^{(3)}$. To wit, remove $\overline{10}$ from $\pi^{(2)}$ to get $\hat{\pi}^{(2)}$, then subtract $\eta = 10$ from each of the 3-marked parts $\overline{97}$, $\overline{73}$, 50 and 30 in $G(\hat{\pi}^{(2)})$ to obtain $\pi^{(3)}$. We get

$$G(\pi^{(3)}) = (\overbrace{90_3, 90_2, \overline{87}_1}^{\{90\}_3}, \overbrace{70_3, \overline{67}_1, \overline{63}_2}^{\{70\}_3}, \overline{57}_1, 50_2, \overline{43}_1, \overbrace{40_3, \overline{37}_2, \overline{33}_1}^{\{40\}_3}, \\ \underbrace{\overline{23}_3, 20_2, 20_1}_{\{\overline{23}\}_3}, \overline{7}_2, \overline{3}_1).$$

  Using Lemma 4.13, we have $\pi^{(3)} \in \overline{\mathcal{B}}_{=}^{10}(3,7;10,4,3|4,4)$.

There are no overlined parts divisible by 10 in $\pi^{(3)}$. The fact that $\zeta^{(3)} = (100, 80, 50)$ is a partition with distinct parts reflects the claim of Theorem 4.14.

Step 2. Successively apply the backward move to $\pi^{(3)}$ to derive a pair of overpartitions (ζ, μ) in $\mathcal{D}_\eta \times \mathcal{B}_0(3,7;10,4,3)$.

There are four 3-marked parts in $G(\pi^{(3)})$, namely, $\tilde{g}_1(\pi^{(3)}) = 90$, $\tilde{g}_2(\pi^{(3)}) = 70$, $\tilde{g}_3(\pi^{(3)}) = 40$ and $\tilde{g}_4(\pi^{(3)}) = \overline{23}$. Moreover, $\{90\}_3$ and $\{70\}_3$ are even and $\{40\}_3$ and $\{\overline{23}\}_3$ are odd, whereas $p_3 = 2$ is the smallest integer such that $\{\tilde{g}_{p_3}(\pi^{(3)})\}_3$ and $\{\tilde{g}_{p_3+1}(\pi^{(3)})\}_3$ have opposite parities. Hence $\pi^{(3)} \in \mathcal{B}_d(3,7;10,4,3|4,2)$.

- Set $\zeta^{(4)} = (100, 80, 50, 20)$. Apply the backward move ψ_2 to $\pi^{(3)}$ to produce $\pi^{(4)}$. Strictly speaking, subtract $\eta = 10$ from each of the 3-marked parts 90 and 70 in $G(\pi^{(3)})$ to get $\pi^{(4)}$. The Gordon marking of $\pi^{(4)}$ is

$$G(\pi^{(4)}) = (\overbrace{90_3, \overline{87}_2, 80_1}^{\{90\}_3}, \overline{67}_1, \overline{63}_2, \overbrace{60_3, \overline{57}_1, 50_2}^{\{60\}_3}, \overline{43}_1, \overbrace{40_3, \overline{37}_2, \overline{33}_1}^{\{40\}_3},$$
$$\underbrace{\overline{23}_3, 20_2, 20_1}_{\{\overline{23}\}_3}, \overline{7}_2, \overline{3}_1).$$

  In view of Lemma 4.8, we may say that $\pi^{(4)} \in \mathcal{B}_e(3, 7; 10, 4, 3|4, 2)$. To be more specific, $\{90\}_3$, $\{60\}_3$ $\{40\}_3$ and $\{\overline{23}\}_3$ are all odd. Hence $p_4 = 4$ is the smallest integer such that $\{\tilde{g}_{p_4}(\pi^{(4)})\}_3$ and $\{\tilde{g}_{p_4+1}(\pi^{(4)})\}_3$ have opposite parities. It follows that $\pi^{(4)} \in \mathcal{B}_d(3, 7; 10, 4, 3|4, 4)$. Obviously, $N \geq p_4 > p_3$.

- Set $\zeta^{(5)} = (100, 80, 50, 40, 20)$. Apply the backward move ψ_4 to $\pi^{(4)}$ to obtain $\pi^{(5)}$, namely, subtract $\eta = 10$ from each of the 3-marked parts 90, 60, 40 and $\overline{23}$ in $G(\pi^{(4)})$. We get

$$G(\pi^{(5)}) = (\overbrace{\overline{87}_3, 80_2, 80_1}^{\{\overline{87}\}_3}, \overline{67}_1, \overline{63}_2, \overline{57}_1, \overbrace{50_3, \overline{50}_2, \overline{43}_1}^{\{50\}_3}, \overbrace{\overline{37}_3, \overline{33}_2, 30_1}^{\{\overline{37}\}_3},$$
$$\underbrace{20_3, 20_2, \overline{13}_1}_{\{20\}_3}, \overline{7}_2, \overline{3}_1).$$

  By Lemma 4.8, we see that $\pi^{(5)} \in \mathcal{B}_e(3, 7; 10, 4, 3|4, 4)$. More precisely, $\{\overline{87}\}_3$, $\{50\}_3$, $\{\overline{37}\}_3$, $\{20\}_3$ in $G(\pi^{(5)})$ are even. Resorting to Theorem 3.4, we arrive at $\pi^{(5)} \in \mathcal{B}_0(3, 7; 10, 4, 3)$.

In conclusion, set $\zeta = \zeta^{(5)}$ and $\mu = \pi^{(5)}$. Then $(\zeta, \mu) \in \mathcal{D}_{10} \times \mathcal{B}_0(3, 7; 10, 4, 3)$ and $|\pi| = |\mu| + |\zeta|$.

5 Proof of Theorem 1.17

The goal of this section is to give a proof of Theorem 1.17, which can be restated in purely combinatorial terms. Here we use the common notation $\delta_{r,k} = 1$ if $r = k$, and $\delta_{r,k} = 0$ otherwise.

Theorem 5.1. *Let k, r and λ be integers such that $k \geq r \geq \lambda \geq 0$ and $k - 1 > \lambda$. There is a bijection Θ between $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and $\mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k - 1, r - \delta_{r,k})$, namely, for an overpartition $\nu \in \overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, we have $\Theta(\nu) = (\zeta, \omega) \in \mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k - 1, r - \delta_{r,k})$ such that $|\nu| = |\zeta| + |\omega|$ and $\ell(\nu) = \ell(\zeta) + \ell(\omega)$.*

Since there are no overlined parts divisible by η in ω and there are no $(k-1)$-marked parts in $RG(\omega)$, in order to obtain (ζ, ω), we need to remove all overlined parts divisible by η and certain non-overlined parts divisible by η from ν to generate ω, and use the removed parts to generate ζ. To this end, we shall define the $(k-1)$-reduction operation and the $(k-1)$-augmentation operation, which are the main ingredients in the construction of Θ.

5.1 The $(k-1)$-reduction and the $(k-1)$-augmentation

The definitions of the $(k-1)$-reduction and the $(k-1)$-augmentation are based on two subsets of $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. To describe these two subsets, we need to introduce the following notation. Define $ol(\nu)$ to be the largest overlined part divisible by η in ν with the convention that $ol(\nu) = \overline{0}$ if there are no overlined parts divisible by η in ν. Define $\tilde{r}_1(\nu)$ to be the largest $(k-1)$-marked part in $RG(\nu)$ with the convention that $\tilde{r}_1(\nu) = -\infty$ if there are no $(k-1)$-marked parts in $RG(\nu)$.

We now assume that k, r and λ are integers such that $k \geq r \geq \lambda \geq 0$ and $k-1 > \lambda$.

- For $t \geq 1$, let $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ denote the set of overpartitions ν in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that either $ol(\nu) = \overline{t\eta}$ and $\tilde{r}_1(\nu) \leq \overline{t\eta}$, or $ol(\nu) < \overline{t\eta}$ and $\overline{(t-1)\eta} < \tilde{r}_1(\nu) \leq t\eta$.
- For $t \geq 1$, let $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ denote the set of overpartitions ν in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that $ol(\nu) < \overline{t\eta}$ and $\tilde{r}_1(\nu) \leq \overline{(t-1)\eta}$.

With the above two subsets in hand, we are ready to give the definition of the $(k-1)$-reduction operation.

Definition 5.2 (The $(k-1)$-reduction)**.** *For $t \geq 1$, let ν be an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. Define the $(k-1)$-reduction $D_t \colon \nu \to \omega$ as follows: If $ol(\nu) = \overline{t\eta}$, then ω is obtained from ν by removing the overlined part $\overline{t\eta}$. Otherwise, ω is obtained from ν by removing a non-overlined part $t\eta$.*

The following proposition guarantees that the $(k-1)$-reduction is well defined.

Proposition 5.3. *For $t \geq 1$, let ν be an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that $ol(\nu) < \overline{t\eta}$ and $\overline{(t-1)\eta} < \tilde{r}_1(\nu) \leq t\eta$. Then ν contains a non-overlined part $t\eta$.*

Proof. Assume that $\tilde{r}_1(\nu)$ is the r_1-th part of $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$, that is, $\tilde{r}_1(\nu) = \nu_{r_1}$. Since ν_{r_1} is a $(k-1)$-marked part in $RG(\nu)$, there is a unique $(k-1)$-band of ν induced by ν_{r_1}. Assume that

$$\nu_{r_1-k+2} \geq \nu_{r_1-k+3} \geq \cdots \geq \nu_{r_1}$$

are the parts in the $(k-1)$-band induced by ν_{r_1}, where $\nu_{r_1-k+2} \leq \nu_{r_1} + \eta$ with strict inequality if ν_{r_1} is overlined. Under the condition $\overline{(t-1)\eta} < \nu_{r_1} \leq t\eta$, we deduce that $\nu_{r_1-k+2} \leq (t+1)\eta$, and so

$$(t+1)\eta \geq \nu_{r_1-k+2} \geq \nu_{r_1-k+3} \geq \cdots \geq \nu_{r_1} > \overline{(t-1)\eta}.$$

Moreover, we may assume that m is the smallest integer such that $r_1 - k + 2 \leq m \leq r_1$ and $\nu_m \leq t\eta$. This implies that $\nu_l \leq t\eta$ for $m \leq l \leq r_1$ and $\nu_l \geq \overline{t\eta}$ for $r_1 - k + 2 \leq l < m$. We claim that $\nu_m = t\eta$. Suppose to the contrary that $\nu_m < t\eta$. In this case, we have $\overline{(t-1)\eta} < \nu_{r_1} \leq \nu_m < t\eta$, so we can write $\nu_{r_1} = \overline{(t-1)\eta + \alpha_i}$, where $1 \leq i \leq \lambda$. Then we have

$$\overline{(t-1)\eta + \alpha_i} = \nu_{r_1} \leq \cdots \leq \nu_m \leq \overline{(t-1)\eta + \alpha_\lambda}, \tag{5.1}$$

and $\nu_{r_1-k+2} < \nu_{r_1} + \eta = \overline{t\eta + \alpha_i}$. The condition $ol(\nu) < \overline{t\eta}$ implies that $\nu_l > \overline{t\eta}$ for $r_1 - k + 2 \leq l < m$. Hence we have

$$\overline{t\eta} < \nu_{m-1} \leq \cdots \leq \nu_{r_1-k+2} < \overline{t\eta + \alpha_i}. \tag{5.2}$$

Combining (5.1) and (5.2), we deduce that $k - 1 \leq (\lambda - i + 1) + (i - 1) = \lambda$, which contradicts the assumption that $k - 1 > \lambda$. Hence $\nu_m = t\eta$. This completes the proof. ∎

The following theorem says that the $(k-1)$-reduction operation is indeed a bijection.

Theorem 5.4. *For $t \geq 1$, the $(k-1)$-reduction D_t is a bijection between $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ and $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. Moreover, for $\nu \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ and $\omega = D_t(\nu)$, we have $|\omega| = |\nu| - t\eta$ and $\ell(\omega) = \ell(\nu) - 1$.*

The proof of Theorem 5.4 consists of three parts. In Lemma 5.5, we show that the $(k-1)$-reduction is a map from $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ to $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. Lemma 5.7 exhibits the $(k-1)$-augmentation map from $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ to $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. Then we show that the $(k-1)$-reduction and the $(k-1)$-augmentation are inverses of each other.

Lemma 5.5. *For $t \geq 1$, let ν be an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ and let $\omega = D_t(\nu)$. Then ω is overpartition in $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. Furthermore, $|\omega| = |\nu| - t\eta$ and $\ell(\omega) = \ell(\nu) - 1$.*

Proof. By definition, we wish to show that ω satisfies the following conditions:

(A) ω is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$;

(B) $ol(\omega) < \overline{t\eta}$ and $\tilde{r}_1(\omega) \leq \overline{(t-1)\eta}$;

(C) All the $(k-1)$-bands of ω induced by the $(k-1)$-marked parts in $RG(\omega)$ are even.

Condition (A). Given the precondition $\nu \in \overline{\mathcal{B}}_0^=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$, it is immediate from the construction of ω that it satisfies (1)-(4) in the definition of $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$. That is to say, ω is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$.

Condition (B). Since $ol(\nu) \leq \overline{t\eta}$, $\tilde{r}_1(\nu) \leq \overline{t\eta}$ and ω is obtained from ν by removing an overlined part $\overline{t\eta}$ or a non-overlined part $t\eta$, we obtain that $ol(\omega) < \overline{t\eta}$ and $\tilde{r}_1(\omega) \leq t\eta$.

We further show that $\tilde{r}_1(\omega) \leq \overline{(t-1)\eta}$. Suppose to the contrary that $\tilde{r}_1(\omega) > \overline{(t-1)\eta}$. In this case, we have $\overline{(t-1)\eta} < \tilde{r}_1(\omega) \leq t\eta$. Since $\tilde{r}_1(\omega)$ is the largest $(k-1)$-marked part in $RG(\omega)$, there are exactly $k-2$ parts of ω appearing before $\tilde{r}_1(\omega)$ in the interval $I(\tilde{r}_1(\omega),\tilde{r}_1(\omega)+\eta)$. The assumption $\overline{(t-1)\eta} < \tilde{r}_1(\omega) \leq t\eta$ implies that $\overline{t\eta} < \tilde{r}_1(\omega) + \eta \leq (t+1)\eta$. Hence the removed part of ν (that is, $t\eta$ or $\overline{t\eta}$) is also in the interval $I(\tilde{r}_1(\omega),\tilde{r}_1(\omega)+\eta)$. It follows that there are exactly $k-1$ parts of ν appearing before $\tilde{r}_1(\omega)$ in the interval $I(\tilde{r}_1(\omega),\tilde{r}_1(\omega)+\eta)$. This means that there exists a part of ν marked with k in $RG(\nu)$, which is impossible because $\nu \in \overline{\mathcal{B}}_0^=(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$.

Condition (C). Given a $(k-1)$-marked part ω_i in $RG(\omega)$, assume that $\{\omega_{i-l}\}_{0\leq l\leq k-2}$ is the $(k-1)$-band induced by the $(k-1)$-marked part ω_i. We aim to show that $\{\omega_{i-l}\}_{0\leq l\leq k-2}$ is even in ω. Using the condition (B), we know that $\tilde{r}_1(\omega) \leq \overline{(t-1)\eta}$, and so $\omega_i \leq \overline{(t-1)\eta}$. The assumption that $\{\omega_{i-l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band yields $\omega_{i-k+2} < \overline{t\eta}$, more precisely,

$$\overline{t\eta} > \omega_{i-k+2} \geq \omega_{i-k+3} \geq \cdots \geq \omega_i. \tag{5.3}$$

It follows that $\{\omega_{i-l}\}_{0\leq l\leq k-2}$ is also a $(k-1)$-band in ν. Since $\nu \in \overline{\mathcal{B}}_0(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$, we find that the $(k-1)$-band $\{\omega_{i-l}\}_{0\leq l\leq k-2}$ is even in ν, that is,

$$[\omega_{i-k+2}/\eta] + \cdots + [\omega_i/\eta] \equiv r-1+\overline{V}_\nu(\omega_{i-k+2}) \pmod 2. \tag{5.4}$$

Noting that ω is obtained from ν by removing an overlined part $\overline{t\eta}$ or a non-overlined part $t\eta$, by (5.3), we get $\overline{V}_\nu(\omega_{i-k+2}) = \overline{V}_\omega(\omega_{i-k+2})$. Therefore, it is immediate from (5.4) that $\{\omega_{i-l}\}_{0\leq l\leq k-2}$ is also even in ω, and so the condition (C) is justified.

In conclusion, we have shown that ω is an overpartition in $\overline{\mathcal{B}}_0^<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. Clearly, $|\omega| = |\nu| - t\eta$ and $\ell(\omega) = \ell(\nu) - 1$. This completes the proof. ∎

We now turn to the $(k-1)$-augmentation operation, which will be shown to be the inverse map of the $(k-1)$-reduction operation.

Definition 5.6 (The $(k-1)$-augmentation)**.** *For $t \geq 1$, let ω be an overpartition in $\overline{\mathcal{B}}_0^<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. We say that ω satisfies the condition U if there exist $k-2$ consecutive parts $\omega_i,\ldots,\omega_{i+k-3}$ of ω such that*

(1) $(t+1)\eta \geq \omega_i \geq \cdots \geq \omega_{i+k-3} > \overline{(t-1)\eta}$;

(2) $\omega_i \leq \omega_{i+k-3} + \eta$ *with strict inequality if* ω_i *is overlined*;

(3) $[\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] \equiv t+r-1+\overline{V}_\omega(\omega_i) \pmod 2$.

The $(k-1)$-augmentation $C_t\colon \omega \to \nu$ is defined as follows: If ω satisfies the condition U, then ν is obtained by inserting $t\eta$ into ω as a non-overlined part. Otherwise, we say that ω satisfies the condition O and ν is obtained by inserting $\overline{t\eta}$ into ω as an overlined part.

The following lemma says that the $(k-1)$-augmentation is a map from $\overline{\mathcal{B}}_0^{<}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$ to $\overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$.

Lemma 5.7. *For $t \geq 1$, let ω be an overpartition in $\overline{\mathcal{B}}_0^{<}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$ and let $\nu = C_t(\omega)$. Then ν is an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$ such that $|\nu| = |\omega| + t\eta$ and $\ell(\nu) = \ell(\omega)+1$.*

Proof. To prove that ν is an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$, we need to verify that ν satisfies the following conditions:

(A) $f_{\leq\eta}(\nu) \leq r-1$;

(B) For $1 \leq i \leq \ell(\nu)-k+1$, $\nu_i \geq \nu_{i+k-1}+\eta$ with strict inequality if ν_i is non-overlined;

(C) $ol(\nu) = \overline{t\eta}$ and $\tilde{r}_1(\nu) \leq \overline{t\eta}$, or $ol(\nu) < \overline{t\eta}$ and $\overline{(t-1)\eta} < \tilde{r}_1(\nu) \leq t\eta$;

(D) All $(k-1)$-bands of ν are even.

Condition (A). It is clear that $f_{\leq\eta}(\omega) \leq r-1$ since $\omega \in \overline{\mathcal{B}}_0^{<}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. To prove that $f_{\leq\eta}(\nu) \leq r-1$, we consider three cases:

Case 1. $t \geq 2$. In this case, $f_{\leq\eta}(\nu) = f_{\leq\eta}(\omega) \leq r-1$.

Case 2. $t = 1$ and $f_{\leq\eta}(\omega) < r-1$. We have $f_{\leq\eta}(\nu) \leq f_{\leq\eta}(\omega)+1 \leq r-1$.

Case 3. $t = 1$ and $f_{\leq\eta}(\omega) = r-1$. We claim that ω satisfies the condition O. Assume that

$$2\eta \geq \omega_i \geq \cdots \geq \omega_{i+k-3} > 0 \tag{5.5}$$

are the $k-2$ consecutive parts of ω such that $\omega_i \leq \omega_{i+k-3}+\eta$ with strict inequality whenever ω_i is overlined. Since $\tilde{r}_1(\omega) \leq \overline{(t-1)\eta} = \overline{0}$, there are no $(k-1)$-marked parts in $RG(\omega)$, that is, there are no $(k-1)$-bands of ω, which implies that $f_{\leq\eta}(\omega) \leq k-2$. Therefore, all parts of ω not exceeding η are after ω_{i-1}. Hence, by (5.5), we obtain that

$$\begin{aligned}
&[\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] \\
&\qquad \equiv f_{=\eta}(\omega) + (\overline{V}_\omega(\omega_i) - f_{<\eta}(\omega)) \\
&\qquad \equiv f_{\leq\eta}(\omega) + \overline{V}_\omega(\omega_i) \\
&\qquad = r-1+\overline{V}_\omega(\omega_i) \pmod 2.
\end{aligned}$$

So the claim is confirmed, and hence ν is obtained by inserting $\overline{\eta}$ into ω as an overlined part, from which we get $f_{\leq\eta}(\nu) = f_{\leq\eta}(\omega) = r-1$.

Condition (B). Suppose to the contrary that there exists $1 \leq c \leq \ell(\nu) - k + 1$ such that

$$\nu_c \leq \nu_{c+k-1} + \eta \text{ with strict inequality if } \nu_c \text{ is overlined.} \tag{5.6}$$

Assume that the m-th part of ν is the inserted part of the $(k-1)$-augmentation operation, that is, $\nu_m = \overline{t\eta}$ or $t\eta$. Since $\omega \in \overline{\mathcal{B}}_0^<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$, we have $\nu_c \geq \nu_m \geq \nu_{c+k-1}$. Comparing with (5.6), we get $\nu_c \leq \nu_m + \eta = (t+1)\eta$ and $\nu_{c+k-1} \geq \nu_m - \eta = (t-1)\eta$ with strict inequality if ν_m is overlined. Thus, we arrive at

$$(t+1)\eta \geq \nu_c \geq \nu_{c+1} \geq \cdots \geq \nu_{c+k-1} \geq (t-1)\eta.$$

The condition $\tilde{r}_1(\omega) \leq \overline{(t-1)\eta}$ implies that there are no $(k-1)$-bands of ω in $(\overline{(t-1)\eta}, (t+1)\eta]$. It follows that $\nu_{c+k-1} = \overline{(t-1)\eta}$ or $(t-1)\eta$, and so $\nu_c \leq t\eta$. But $\nu_c \geq \nu_m \geq t\eta$, we obtain that

$$\nu_c = \nu_m = t\eta. \tag{5.7}$$

Hence, $\omega_{c+i-1} = \nu_{c+i}$, where $1 \leq i \leq k-1$. More precisely,

$$t\eta \geq \omega_c \geq \omega_{c+1} \geq \cdots \geq \omega_{c+k-2} \geq (t-1)\eta,$$

and

$$\omega_{c+k-2} = \nu_{c+k-1} = \overline{(t-1)\eta} \text{ or } (t-1)\eta. \tag{5.8}$$

Therefore, $\{\omega_{c+l}\}_{0 \leq l \leq k-2}$ is a $(k-1)$-band of ω in $[(t-1)\eta, t\eta]$. Using the condition $\omega \in \overline{\mathcal{B}}_0^<(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$, we find that

$$[\omega_c/\eta] + \cdots + [\omega_{c+k-2}/\eta] \equiv r - 1 + \overline{V}_\omega(\omega_c). \tag{5.9}$$

It is clear from (5.7) that ν is obtained by inserting a non-overlined part $t\eta$ into ω. So ω satisfies the condition U, which means that there exist $k-2$ consecutive parts in $(\overline{(t-1)\eta}, (t+1)\eta]$, say

$$(t+1)\eta \geq \omega_i \geq \cdots \geq \omega_{i+k-3} > \overline{(t-1)\eta},$$

satisfying $\omega_i \leq \omega_{i+k-3} + \eta$ with strict inequality if ω_i is overlined, and

$$[\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] \equiv t + r - 1 + \overline{V}_\omega(\omega_i) \pmod 2. \tag{5.10}$$

Now, (5.8) yields that $\omega_{i+k-3} > \overline{(t-1)\eta} \geq \omega_{c+k-2}$, which implies that $i \leq c$. Set $c = i + t$, where $t \geq 0$. Then we have

$$(t+1)\eta \geq \omega_i \geq \cdots \geq \omega_{i+t-1} > \overline{t\eta}, \tag{5.11}$$

and

$$t\eta > \omega_{c+k-t-2} \geq \cdots \geq \omega_{c+k-3} \geq (t-1)\eta. \tag{5.12}$$

Combining (5.11) and (5.12), we obtain that

$$[\omega_i/\eta] + \cdots + [\omega_{i+t-1}/\eta] \equiv [\omega_{c+k-t-2}/\eta] + \cdots + [\omega_{c+k-3}/\eta] + \overline{V}_\omega(\omega_i) - \overline{V}_\omega(\omega_c) \pmod 2,$$

which can be rewritten as

$$[\omega_i/\eta]+\cdots+[\omega_{i+k-3}/\eta]\equiv[\omega_c/\eta]+\cdots+[\omega_{c+k-3}/\eta]+\overline{V}_\omega(\omega_i)-\overline{V}_\omega(\omega_c) \pmod 2. \quad (5.13)$$

Substituting (5.8) and (5.9) into (5.13) gives

$$\begin{aligned}&[\omega_i/\eta]+\cdots+[\omega_{i+k-3}/\eta]\\ &\quad=[\omega_c/\eta]+\cdots+[\omega_{c+k-3}/\eta]+[\omega_{c+k-2}/\eta]-(t-1)+\overline{V}_\omega(\omega_i)-\overline{V}_\omega(\omega_c)\\ &\quad\equiv r-1+\overline{V}_\omega(\omega_c)-(t-1)+\overline{V}_\omega(\omega_i)-\overline{V}_\omega(\omega_c)\\ &\quad\equiv t+r+\overline{V}_\omega(\omega_i) \pmod 2,\end{aligned}$$

which contradicts (5.10). Hence the condition (B) holds. Together with the condition (A), we conclude that ν is an overpartition in $\overline{\mathcal{B}}_1(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$.

Condition (C). We consider the following two cases.

Case 1. ω satisfies the condition O in Definition 5.6. In this circumstance, ν is obtained from ω by inserting $\overline{t\eta}$ as an overlined part. Obviously, $ol(\nu)=\overline{t\eta}$ and $\tilde{r}_1(\nu)\leq\overline{t\eta}$.

Case 2. ω satisfies the condition U in Definition 5.6. If so, ν is obtained from ω by inserting $t\eta$ as a non-overlined part. Under the condition that $ol(\omega)<\overline{t\eta}$ and $\tilde{r}_1(\omega)\leq\overline{(t-1)\eta}$, we deduce that $ol(\nu)=ol(\omega)<\overline{t\eta}$ and $\tilde{r}_1(\nu)\leq t\eta$.

To prove that $\tilde{r}_1(\nu)>\overline{(t-1)\eta}$, it suffices to show that there is a $(k-1)$-band in $(\overline{(t-1)\eta},(t+1)\eta]$. With the assumption that ω satisfies the condition U, we know that there exist $k-2$ consecutive parts of ω, say

$$(t+1)\eta\geq\omega_s\geq\cdots\geq\omega_{s+k-3}>\overline{(t-1)\eta},$$

satisfying $\omega_s\leq\omega_{s+k-3}+\eta$ with strict inequality if ω_s is overlined, and

$$[\omega_s/\eta]+\cdots+[\omega_{s+k-3}/\eta]\equiv t+r-1+\overline{V}_\omega(\omega_s) \pmod 2. \quad (5.14)$$

Since $\tilde{r}_1(\omega)\leq\overline{(t-1)\eta}$, there are no $(k-1)$-bands of ω in $(\overline{(t-1)\eta},(t+1)\eta]$. It follows that $\omega_{s-1}>\overline{t\eta}$ and $\omega_{s+k-2}<t\eta$. Assume that c is the smallest integer such that $\nu_c\leq t\eta$. Then we have $s\leq c\leq s+k-2$, $\nu_l=\omega_l$ for $s\leq l\leq c-1$, and $\nu_{l+1}=\omega_l$ for $c\leq l\leq s+k-3$, namely,

$$(t+1)\eta\geq\nu_s\geq\cdots\geq\nu_{c-1}>t\eta\geq\nu_{c+1}\geq\cdots\geq\nu_{s+k-2}>\overline{(t-1)\eta} \quad (5.15)$$

are the $k-1$ parts of ν such that $\nu_s\leq\nu_{s+k-2}+\eta$ with strict inequality if ν_s is overlined. Hence $\{\nu_{s+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of ν. So we arrive at $\tilde{r}_1(\nu)>\overline{(t-1)\eta}$, and this proves that the condition (C) is valid.

Condition (D). There are two cases.

Case 1. ω satisfies the condition O in Definition 5.6. Then ν is obtained from ω by inserting $\overline{t\eta}$ as an overlined part. From the condition (C), we know that $\tilde{r}_1(\nu)\leq\overline{t\eta}$ in this event. Assume that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of ν. We aim to show that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is even in ν. Since $\tilde{r}_1(\nu) \leq \overline{t\eta}$ and there is a part in $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ marked with $k-1$ in $RG(\nu)$, we get $\nu_{i+k-2} \leq \overline{t\eta}$. There are two subcases.

Subcase 1.1. $\nu_{i+k-2} \leq \overline{(t-1)\eta}$. In this case, the assumption that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band implies that $\nu_i \leq \nu_{i+k-2}$ with strict inequality if ν_i is overlined, and so $\nu_i < \overline{t\eta}$. Recall that ν is obtained from ω by inserting $\overline{t\eta}$ as an overlined part, we find that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is also a $(k-1)$-band of ω and $\overline{V}_\omega(\nu_i) = \overline{V}_\nu(\nu_i)$. Since $\omega \in \overline{\mathcal{B}}_0^<(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r)$, we see that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is even in ω, and so $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is also even in ν.

Subcase 1.2. $\overline{(t-1)\eta} < \nu_{i+k-2} \leq \overline{t\eta}$. In this case, using the same reasoning as in the Subcase 1.1, we obtain that $\nu_i \leq (t+1)\eta$, so that

$$(t+1)\eta \geq \nu_i \geq \nu_{i+1} \geq \cdots \geq \nu_{i+k-2} > \overline{(t-1)\eta}.$$

Utilizing the condition (B), we deduce that $\nu_{i-1} > \overline{t\eta}$ and $\nu_{i+k-1} < t\eta$. Hence the inserted part $\overline{t\eta}$ in ν belongs to $\{\nu_{i+l}\}_{0\leq l\leq k-2}$, so we may write $\nu_m = \overline{t\eta}$, where $i \leq m \leq i+k-2$. By the construction of ν, we see that $\omega_l = \nu_l$ for $i \leq l \leq m-1$, and $\omega_l = \nu_{l+1}$ for $m \leq l \leq i+k-3$. This implies that

$$(t+1)\eta \geq \omega_i \geq \cdots \geq \omega_{i+k-3} > \overline{(t-1)\eta}$$

are the $k-2$ consecutive parts of ω such that $\omega_i \leq \omega_{i+k-3} + \eta$ with strict inequality provided ω_i is overlined, and $\overline{V}_\nu(\nu_i) = \overline{V}_\omega(\omega_i) + 1$. Under the assumption that ω satisfies the condition O, we find that

$$[\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] \equiv t + r + \overline{V}_\omega(\omega_i) \pmod 2.$$

Therefore,

$$\begin{aligned}
&[\nu_i/\eta] + \cdots + [\nu_m/\eta] + \cdots + [\nu_{i+k-2}/\eta] \\
&\quad = [\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] + t \\
&\quad \equiv t + r + \overline{V}_\omega(\omega_i) + t \\
&\quad \equiv r - 1 + \overline{V}_\nu(\nu_i) \pmod 2,
\end{aligned}$$

which means that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is even in ν.

Case 2. ω satisfies the condition U in Definition 5.6. In this case, ν is obtained from ω by inserting $t\eta$ as a non-overlined part. For any $(k-1)$-band $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ of ν, we wish to show that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is even in ν. There are two cases.

Subcase 2.1. $\nu_i < t\eta$. By construction of ν, we see that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is also a $(k-1)$-band of ω and $\overline{V}_\omega(\nu_i) = \overline{V}_\nu(\nu_i)$. Using the same argument as in Subcase 1.1, it can be shown that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is even in ν.

Subcase 2.2. $\nu_i \geq t\eta$. Since $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of ν, we deduce that $\nu_{i+k-2} \geq \nu_i - \eta \geq (t-1)\eta$ and there is a part ν_{i+l_i} $(0 \leq l_i \leq k-2)$ marked with $k-1$ in $RG(\nu)$.

Using the condition (C), we find that $\overline{(t-1)\eta} < \tilde{r}_1(\nu) \leq t\eta$. It follows that $\tilde{r}_1(\nu) = \nu_{i+l_i}$. Hence $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is a $(k-1)$-band of ν including $\tilde{r}_1(\nu)$. As in the proof of the condition (C), we see that $\tilde{r}_1(\nu)$ is also in the $(k-1)$-band $\{\nu_{s+l}\}_{0\leq l\leq k-2}$ in (5.15). Therefore, it follows from Proposition 3.6 that $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ and $\{\nu_{s+l}\}_{0\leq l\leq k-2}$ have same parity. Hence we only need to show that $\{\nu_{s+l}\}_{0\leq l\leq k-2}$ in (5.15) is even in ν.

Assume that c is the smallest integer such that $\nu_c = t\eta$. As in the proof of the condition (C), we find that ν_c belongs to the $(k-1)$-band $\{\nu_{s+l}\}_{0\leq l\leq k-2}$ in (5.15) and $\overline{V}_\nu(\nu_s) = \overline{V}_\omega(\omega_s)$. Thus,

$$[\nu_s/\eta] + \cdots + [\nu_{s+k-2}/\eta] = t + [\omega_s/\eta] + \cdots + [\omega_{s+k-3}/\eta]. \tag{5.16}$$

Substituting (5.14) into (5.16) and using $\overline{V}_\nu(\nu_s) = \overline{V}_\omega(\omega_s)$, we are led to

$$[\nu_s/\eta] + \cdots + [\nu_{s+k-2}/\eta] \equiv r - 1 + \overline{V}_\nu(\nu_s) \pmod 2,$$

which means that the $(k-1)$-band $\{\nu_{s+l}\}_{0\leq l\leq k-2}$ in (5.15) is even, and so $\{\nu_{i+l}\}_{0\leq l\leq k-2}$ is even in ν.

In either case, we have shown that any $(k-1)$-band of ν is even, and thus the condition (D) is verified.

By now, we have shown that ν is an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. Clearly, $|\nu| = |\omega| + t\eta$ and $\ell(\nu) = \ell(\omega) + 1$. This completes the proof. ∎

We are now in a position to give a proof of Theorem 5.4 with the aid of Lemma 5.5 and Lemma 5.7.

Proof of Theorem 5.4. Let $\nu \in \overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. Invoking Lemma 5.5, we know that $D_t(\nu) \in \overline{\mathcal{B}}_0^{<}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. Setting $\omega = D_t(\nu)$, by Lemma 5.7, we see that $C_t(\omega) \in \overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$. It remains to show that $\nu = C_t(\omega)$. We consider the following two cases.

Case 1. $ol(\nu) < \overline{t\eta}$ and $\overline{(t-1)\eta} < \tilde{r}_1(\nu) \leq t\eta$. In this case, ω is obtained from ν by removing a non-overlined part $t\eta$. To prove that $\nu = C_t(\omega)$, it suffices to show that ω satisfies the condition U in Definition 5.6.

Assume that $\tilde{r}_1(\nu)$ is the r_1-th part of $\nu = (\nu_1,\nu_2,\ldots,\nu_\ell)$, that is, $\nu_{r_1} = \tilde{r}_1(\nu)$. Then the $(k-1)$-band induced by $\tilde{r}_1(\nu)$ consists of

$$\nu_{r_1-k+2} \geq \nu_{r_1-k+3} \geq \cdots \geq \nu_{r_1},$$

where $\nu_{r_1-k+2} \leq \nu_{r_1} + \eta$ with strict inequality if ν_{r_1} is overlined. Since ν is an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1,\ldots,\alpha_\lambda;\eta,k,r|t)$, we deduce that $\{\nu_{r_1-l}\}_{0\leq l\leq k-2}$ is even, namely,

$$[\nu_{r_1-k+2}/\eta] + [\nu_{r_1-k+3}/\eta] + \cdots + [\nu_{r_1}/\eta] \equiv r - 1 + \overline{V}_\nu(\nu_{r_1-k+2}) \pmod 2. \tag{5.17}$$

Under the assumption $\overline{(t-1)\eta} < \tilde{r}_1(\nu) \leq t\eta$, we see that $\nu_{r_1} = \tilde{r}_1(\nu) > \overline{(t-1)\eta}$ and $\nu_{r_1-k+2} \leq \nu_{r_1} + \eta \leq (t+1)\eta$, and thus

$$(t+1)\eta \geq \nu_{r_1-k+2} \geq \nu_{r_1-k+3} \geq \cdots \geq \nu_{r_1} > \overline{(t-1)\eta}.$$

It is clear from Proposition 5.3 that ν contains a non-overlined part $t\eta$. Assume that m is the smallest integer such that $\nu_m = t\eta$. The precondition $\nu \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ implies that $\nu_{r_1-k+1} > \overline{t\eta}$ and $\nu_{r_1+1} < t\eta$. Hence $r_1 - k + 2 \le m \le r_1$, $\omega_l = \nu_l$ for $r_1 - k + 2 \le l \le m - 1$, and $\omega_l = \nu_{l+1}$ for $m \le l \le r_1 - 1$. Consequently,

$$(t+1)\eta \ge \omega_{r_1-k+2} \ge \cdots \ge \omega_{r_1-1} > \overline{(t-1)\eta}$$

are the $k-2$ consecutive parts of ω such that $\omega_{r_1-k+2} \le \omega_{r_1-1} + \eta$ with strict inequality provided ω_{r_1-k+2} is overlined. By the construction of ω, we deduce that $\overline{V}_\omega(\omega_{r_1-k+2}) = \overline{V}_\nu(\nu_{r_1-k+2})$. Combining with (5.17), we get

$$\begin{aligned}
&[\omega_{r_1-k+2}/\eta] + [\omega_{r_1-k+1}/\eta] + \cdots + [\omega_{r_1-1}/\eta] \\
&\quad = [\nu_{r_1-k+2}/\eta] + [\nu_{r_1-k+1}/\eta] + \cdots + [\nu_{r_1}/\eta] - t \\
&\quad \equiv r - 1 + \overline{V}_\nu(\nu_{r_1-k+2}) - t \\
&\quad \equiv t + r - 1 + \overline{V}_\omega(\omega_{r_1-k+2}) \pmod 2.
\end{aligned}$$

This implies that ω satisfies the condition U in Definition 5.6, and so $\nu = C_t(\omega)$. Hence we conclude that $C_t(D_t(\nu)) = \nu$ for $\nu \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$.

Case 2. $ol(\nu) = \overline{t\eta}$ and $\tilde{r}_1(\nu) \le \overline{t\eta}$. In this regard, ω is obtained from ν by removing $\overline{t\eta}$. To prove that $\nu = C_t(\omega)$, it is enough to show that ω satisfies the condition O in Definition 5.6. Suppose to the contrary that ω satisfies the condition U in Definition 5.6, that is, there exist $k-2$ consecutive parts of ω, say

$$(t+1)\eta \ge \omega_i \ge \omega_{i+1} \ge \cdots \ge \omega_{i+k-3} > \overline{(t-1)\eta},$$

such that $\omega_i \le \omega_{i+k-3} + \eta$ with strict inequality if ω_i is overlined, and

$$[\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] \equiv t + r - 1 + \overline{V}_\omega(\omega_i) \pmod 2. \tag{5.18}$$

Assume that $\overline{t\eta}$ is the m-th part ν_m of ν. Since $\omega \in \overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$, we see that $\tilde{r}_1(\omega) \le \overline{(t-1)\eta}$, and so there are no $(k-1)$-bands in $(\overline{(t-1)\eta}, (t+1)\eta]$. It follows that $\omega_{i-1} > \overline{t\eta}$ and $\omega_{i+k-2} < t\eta$, which implies that $i \le m \le i + k - 2$, $\omega_l = \nu_l > \overline{t\eta}$ for $i \le l < m$, and $\omega_l = \nu_{l+1} \le t\eta$ for $m \le l \le i + k - 3$. Thus,

$$(t+1)\eta \ge \nu_i \ge \cdots \ge \nu_{i+k-2} > \overline{(t-1)\eta},$$

where $\nu_i \le \nu_{i+k-2} + \eta$ with strict inequality if ν_i is overlined. In other words, $\{\nu_{i+l}\}_{0 \le l \le k-2}$ is a $(k-1)$-band of ν. Moreover, we get $\overline{V}_\nu(\nu_i) = \overline{V}_\omega(\omega_i) + 1$. The precondition that ν is an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ implies that $\{\nu_{i+l}\}_{0 \le l \le k-2}$ is even, and so

$$\begin{aligned}
&[\omega_i/\eta] + \cdots + [\omega_{i+k-3}/\eta] \\
&\quad = [\nu_i/\eta] + \cdots + [\nu_{i+k-2}/\eta] - t \\
&\quad \equiv r - 1 + \overline{V}_\nu(\nu_i) - t \\
&\quad \equiv t + r + \overline{V}_\omega(\omega_i) \pmod 2,
\end{aligned}$$

which contradicts (5.18). Hence ω satisfies the condition O in Definition 5.6, and so $\nu = C_t(\omega)$. This proves that $C_t(D_t(\nu)) = \nu$ for $\nu \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$.

Conversely, let $\omega \in \overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. By Lemma 5.7, we find that $C_t(\omega)$ belongs to $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. By the definitions of C_t and D_t, we deduce that $D_t(C_t(\omega)) = \omega$. This completes the proof. ∎

The following proposition provides a criterion to determine whether an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ is also an overpartition in $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t')$.

Proposition 5.8. *For $t \geq 1$, let ν be an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$. Then ν is an overpartition in $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t')$ if and only if $t < t'$.*

Proof. By definition, we see that ν is an overpartition in $\overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t)$ if and only if ν is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that

$$\max\{\lceil |ol(\nu)|/\eta \rceil, \lceil |\tilde{r}_1(\nu)|/\eta \rceil\} = t, \tag{5.19}$$

where $|\cdot|$ signified the value of a part regardless of overline, and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

On the other hand, ν is an overpartition in $\overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t')$ if and only if ν is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that

$$\max\{\lceil |ol(\nu)|/\eta \rceil, \lceil |\tilde{r}_1(\nu)|/\eta \rceil\} \leq t' - 1. \tag{5.20}$$

Combining (5.19) and (5.20) completes the proof. ∎

5.2 Proof of Theorem 5.1

In this subsection, we demonstrate that Theorem 5.1 can be justified by repeatedly using the $(k-1)$-reduction and the $(k-1)$-augmentation operations.

Proof of Theorem 5.1. Let ν be an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. We wish to construct a pair of overpartitions $\Theta(\nu) = (\zeta, \omega)$ in $\mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$ such that $|\nu| = |\zeta| + |\omega|$ and $\ell(\nu) = \ell(\zeta) + \ell(\omega)$. We consider the following two cases:

Case 1: There are no $(k-1)$-marked parts in $RG(\nu)$ and there are no overlined parts divisible by η in ν. Then set $\zeta = \emptyset$ and $\omega = \nu$. By definition, we see that ω is an overpartition in $\mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$. Moreover, $|\nu| = |\zeta| + |\omega|$ and $\ell(\nu) = \ell(\zeta) + \ell(\omega)$.

Case 2: There exists a $(k-1)$-marked part in $RG(\nu)$ or an overlined part divisible by η in ν. Set $b = 0$, $\nu^{(0)} = \nu$, $\zeta^{(0)} = \emptyset$, and execute the following procedure. Denote the intermediate pairs by $(\zeta^{(0)}, \nu^{(0)}), (\zeta^{(1)}, \nu^{(1)})$, and so on.

(A) Set

$$t_{b+1} = \max\{\lceil |ol(\nu^{(b)})|/\eta\rceil, \lceil |\tilde{r}_1(\nu^{(b)})|/\eta\rceil\}.$$

Since $\tilde{r}_1(\nu^{(b)}) \geq \overline{\alpha_1}$ or $ol(\nu^{(b)}) \geq \overline{\eta}$, we find that $t_{b+1} \geq 1$ and

$$\nu^{(b)} \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t_{b+1}).$$

Applying the $(k-1)$-reduction $D_{t_{b+1}}$ to $\nu^{(b)}$, we get

$$\nu^{(b+1)} = D_{t_{b+1}}(\nu^{(b)}).$$

In view of Lemma 5.5, we deduce that $\nu^{(b+1)} \in \overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r|t_{b+1})$,

$$|\nu^{(b+1)}| = |\nu^{(b)}| - \eta t_{b+1}, \tag{5.21}$$

and

$$\ell(\nu^{(b+1)}) = \ell(\nu^{(b)}) - 1. \tag{5.22}$$

Then, insert ηt_{b+1} into $\zeta^{(b)}$ as a part to get $\zeta^{(b+1)}$.

(B) Replace b by $b+1$. If there are no $(k-1)$-marked parts in $RG(\nu^{(b)})$ and there are no overlined parts divisible by η in $\nu^{(b)}$, then we are done. Otherwise, go back to (A).

Using Proposition 5.8, we obtain that

$$t_{b+1} > t_{b+2} \geq 1, \tag{5.23}$$

for $b \geq 0$, which means that the above procedure terminates. Assume that it terminates with $b = c$, that is, there are no $(k-1)$-marked parts in $RG(\nu^{(c)})$ and there are no overlined parts divisible by η in $\nu^{(c)}$. Set

$$\omega = \nu^{(c)} \quad \text{and} \quad \zeta = \zeta^{(c)} = (\eta t_1, \ldots, \eta t_c).$$

Since there are no $(k-1)$-marked parts in $RG(\nu^{(c)})$ and there are no overlined parts divisible by η in $\nu^{(c)}$, we conclude that $\omega = \nu^{(c)} \in \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$. In light of (5.23), we find that $\zeta \in \mathcal{D}_\eta$. Moreover, it is clear from (5.21) and (5.22) that $|\nu| = |\omega| + |\zeta|$ and $\ell(\nu) = \ell(\omega) + \ell(\zeta)$. Hence Θ is the desired map from $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ to $\mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$.

To prove that Θ is a bijection, we define a map Λ from $\mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$ to $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and intend to show that it is the inverse map of Θ. Given an overpartition ω in $\mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$ and a partition ζ in $\mathcal{D}_\eta$, we shall construct an overpartition $\nu \in \overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ such that $|\nu| = |\zeta| + |\omega|$ and $\ell(\nu) = \ell(\zeta) + \ell(\omega)$. There are two cases.

Case 1: $\zeta = \emptyset$. Then set $\nu = \omega$. Clearly, $\nu \in \overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ since there are no $(k-1)$-bands in ω. Moreover, $|\nu| = |\zeta| + |\omega|$ and $\ell(\nu) = \ell(\zeta) + \ell(\omega)$.

Case 2: $\zeta \neq \emptyset$. Assume that $\zeta = (\eta t_1, \eta t_2, \ldots, \eta t_c)$, where $t_1 > t_2 > \cdots > t_c \geq 1$. Starting with ω, apply the $(k-1)$-augmentation repeatedly to get ν. Denote the intermediate overpartitions by $\omega^{(0)}, \ldots, \omega^{(c)}$ with $\omega^{(0)} = \omega$ and $\omega^{(c)} = \nu$. Since there are no $(k-1)$-marked parts in $RG(\omega)$ and there are no overlined parts divisible by η in ω, we have $\tilde{r}_1(\omega) = -\infty$ and $ol(\omega) = \overline{0}$, which yields $\omega^{(0)} = \omega \in \overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | t_c)$.

Set $b = 0$, and execute the following procedure.

(A) Set

$$\omega^{(b+1)} = C_{t_{c-b}}(\omega^{(b)}).$$

Since

$$\omega^{(b)} \in \overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | t_{c-b}),$$

in light of Lemma 5.7, we see that $\omega^{(b+1)} \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | t_{c-b})$,

$$|\omega^{(b+1)}| = |\omega^{(b)}| + \eta t_{c-b}, \tag{5.24}$$

and

$$\ell(\omega^{(b+1)}) = \ell(\omega^{(b)}) + 1. \tag{5.25}$$

(B) Replace b by $b+1$. If $b = c$, then we are done. Otherwise, since $t_{c-b} > t_{c-b+1}$, it follows from Proposition 5.8 that

$$\omega^{(b)} \in \overline{\mathcal{B}}_0^{<}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | t_{c-b}).$$

Go back to (A).

The above procedure generates an overpartition $\nu = \omega^{(c)} \in \overline{\mathcal{B}}_0^{=}(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r | t_1)$, and so ν is an overpartition in $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. It is evident from (5.24) and (5.25) that

$$|\nu| = |\omega^{(c)}| = |\omega^{(0)}| + \eta t_c + \cdots + \eta t_1 = |\omega| + |\zeta|,$$

and

$$\ell(\nu) = \ell(\omega^{(c)}) = \ell(\omega^{(0)}) + c = \ell(\omega) + \ell(\zeta).$$

Therefore, Λ is a map from $\mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$ to $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$. By Theorem 5.4, we obtain that $\Lambda(\Theta(\nu)) = \nu$ for $\nu \in \overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and $\Theta(\Lambda(\zeta, \omega)) = (\zeta, \omega)$ for $(\zeta, \omega) \in \mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$. Hence Θ is a bijection between $\overline{\mathcal{B}}_0(\alpha_1, \ldots, \alpha_\lambda; \eta, k, r)$ and $\mathcal{D}_\eta \times \mathcal{B}_1(\alpha_1, \ldots, \alpha_\lambda; \eta, k-1, r-\delta_{r,k})$. This completes the proof. ∎

5.3 An example

We conclude this section with an example for the bijection Θ in Theorem 5.1. Let

$$\nu = (\overline{50}, \overline{30}, \overline{23}, 20, 20, \overline{10}, \overline{7}, \overline{3})$$

be an overpartition in $\overline{\mathcal{B}}_0(3,7;10,4,3)$. We have

$$RG(\nu) = (\overline{50}_1, \overline{30}_1, \overline{23}_2, 20_1, 20_3, \overline{10}_2, \overline{7}_1, \overline{3}_3).$$

The pair of overpartitions $\Theta(\nu) = (\zeta, \omega)$ is obtained by successively applying the $(k-1)$-reduction to ν. The detailed process is given below.

- Set $\nu^{(0)} = \nu$ and $\zeta^{(0)} = \emptyset$. Note that $ol(\nu^{(0)}) = \overline{50}$ and $\tilde{r}_1(\nu^{(0)}) = 20$. Let

  $$t_1 = \max\{\lceil |ol(\nu^{(0)})|/10\rceil, \lceil |\tilde{r}_1(\nu^{(0)})|/10\rceil\} = 5.$$

  Now, $\nu^{(0)} \in \overline{\mathcal{B}}_0^{=}(3,7;10,4,3|5)$. Apply the 3-reduction to $\nu^{(0)}$ to get $\nu^{(1)}$, namely, $\nu^{(1)}$ is obtained from $\nu^{(0)}$ by removing $\overline{50}$. We get

  $$RG(\nu^{(1)}) = (\overline{30}_1, \overline{23}_2, 20_1, 20_3, \overline{10}_2, \overline{7}_1, \overline{3}_3).$$

  Setting $\zeta^{(1)} = (50)$ and using Lemma 5.5, we obtain that $\nu^{(1)} \in \overline{\mathcal{B}}_0^{<}(3,7;10,4,3|5)$.

- Since $ol(\nu^{(1)}) = \overline{30}$ and $\tilde{r}_1(\nu^{(1)}) = 20$, we have

  $$t_2 = \max\{\lceil |ol(\nu^{(1)})|/10\rceil, \lceil |\tilde{r}_1(\nu^{(1)})|/10\rceil\} = 3,$$

  whence $\nu^{(1)} \in \overline{\mathcal{B}}_0^{=}(3,7;10,4,3|3)$. Removing $\overline{30}$ from $\nu^{(1)}$, we get $\nu^{(2)}$ and

  $$RG(\nu^{(2)}) = (\overline{23}_1, 20_2, 20_3, \overline{10}_1, \overline{7}_2, \overline{3}_3).$$

  Setting $\zeta^{(2)} = (50, 30)$ and using Lemma 5.5, we obtain that $\nu^{(2)} \in \overline{\mathcal{B}}_0^{<}(3,7;10,4,3|3)$.

- Since $ol(\nu^{(2)}) = \overline{10}$ and $\tilde{r}_1(\nu^{(2)}) = 20$, we have

  $$t_3 = \max\{\lceil |ol(\nu^{(2)})|/10\rceil, \lceil |\tilde{r}_1(\nu^{(2)})|/10\rceil\} = 2,$$

  whence $\nu^{(2)} \in \overline{\mathcal{B}}_0^{=}(3,7;10,4,3|2)$. Removing a non-overlined part 20 from $\nu^{(2)}$, we get $\nu^{(3)}$ and

  $$RG(\nu^{(3)}) = (\overline{23}_1, 20_2, \overline{10}_1, \overline{7}_2, \overline{3}_3).$$

  Setting $\zeta^{(3)} = (50, 30, 20)$ and using Lemma 5.5, we obtain that $\nu^{(3)} \in \overline{\mathcal{B}}_0^{<}(3,7;10,4,3|2)$.

- Since $ol(\nu^{(3)}) = \overline{10}$ and $\tilde{r}_1(\nu^{(3)}) = \overline{3}$, we have

  $$t_4 = \max\{\lceil |ol(\nu^{(3)})|/10\rceil, \lceil |\tilde{r}_1(\nu^{(3)})|/10\rceil\} = 1,$$

  whence $\nu^{(3)} \in \overline{\mathcal{B}}_0^{=}(3,7;10,4,3|1)$. Removing $\overline{10}$ from $\nu^{(3)}$, we get $\nu^{(4)}$ and

  $$RG(\nu^{(4)}) = (\overline{23}_1, 20_2, \overline{7}_1, \overline{3}_2).$$

  Setting $\zeta^{(4)} = (50, 30, 20, 10)$ and using Lemma 5.5, we obtain that $\nu^{(4)} \in \overline{\mathcal{B}}_0^{<}(3,7;10,4,3|1)$. Eventually, there are no 3-marked parts in $RG(\nu^{(4)})$ and there are no overlined parts divisible by 10 in $\nu^{(4)}$.

We now get a pair of partitions (ζ, ω) with

$$\zeta = \zeta^{(4)} = (50, 30, 20, 10) \quad \text{and} \quad \omega = \nu^{(4)} = (\overline{23}, 20, \overline{7}, \overline{3}) \tag{5.26}$$

such that $(\zeta, \omega) \in \mathcal{D}_{10} \times \overline{\mathcal{B}}_1(3, 7; 10, 3, 3)$, $|\nu| = |\zeta| + |\omega|$ and $\ell(\nu) = \ell(\omega) + \ell(\zeta)$.

Conversely, given $(\zeta, \omega) \in \mathcal{D}_{10} \times \overline{\mathcal{B}}_1(3, 7; 10, 3, 3)$ as in (5.26), we may recover the overpartition ν by successively applying the 3-augmentation operation. More precisely, the reverse process goes as follows.

- Insert 10 into $\omega^{(0)} = \omega$ to get $\omega^{(1)}$.

  Since there are no 3-marked parts in $RG(\omega^{(0)})$ and there are no overlined parts divisible by 10 in $\omega^{(0)}$, we have $\tilde{r}_1(\omega^{(0)}) = -\infty$ and $ol(\omega^{(0)}) = \overline{0}$, which implies that $\omega^{(0)} \in \overline{\mathcal{B}}_0^<(3, 7; 10, 4, 3|1)$. Notice that $\omega^{(0)}$ satisfies the condition O in Definition 5.6. Then insert $\overline{10}$ into $\omega^{(0)}$ as an overlined part to get

  $$\omega^{(1)} = C_1(\omega^{(0)}) = (\overline{23}, 20, \overline{10}, \overline{7}, \overline{3}).$$

  Using Lemma 5.7, we obtain that $\omega^{(1)} \in \overline{\mathcal{B}}_0^=(3, 7; 10, 4, 3|1)$.

- Insert 20 into $\omega^{(1)}$ to get $\omega^{(2)}$. By Proposition 5.8, we find that $\omega^{(1)} \in \overline{\mathcal{B}}_0^<(3, 7; 10, 4, 3|2)$. Since $\omega^{(1)}$ satisfies the condition U in Definition 5.6, inserting 20 into $\omega^{(1)}$ as a non-overlined part gives

  $$\omega^{(2)} = C_2(\omega^{(1)}) = (\overline{23}, 20, 20, \overline{10}, \overline{7}, \overline{3}).$$

  In light of Lemma 5.7, we deduce that $\omega^{(2)} \in \overline{\mathcal{B}}_0^=(3, 7; 10, 4, 3|2)$.

- Insert 30 into $\omega^{(2)}$ to get $\omega^{(3)}$. By Proposition 5.8, we find that $\omega^{(2)} \in \overline{\mathcal{B}}_0^<(3, 7; 10, 4, 3|3)$. Notice that $\omega^{(2)}$ satisfies the condition O in Definition 5.6. Then insert $\overline{30}$ into $\omega^{(2)}$ as an overlined part to get

  $$\omega^{(3)} = C_3(\omega^{(2)}) = (\overline{30}, \overline{23}, 20, 20, \overline{10}, \overline{7}, \overline{3}).$$

  Using Lemma 5.7, we obtain that $\omega^{(3)} \in \overline{\mathcal{B}}_0^=(3, 7; 10, 4, 3|3)$.

- Finally, insert 50 into $\omega^{(3)}$ to get $\omega^{(4)}$. By Proposition 5.8, we find that $\omega^{(3)} \in \overline{\mathcal{B}}_0^<(3, 7; 10, 4, 3|5)$. Notice that $\omega^{(3)}$ satisfies the condition O in Definition 5.6. Then insert $\overline{50}$ into $\omega^{(3)}$ as an overlined part to get

  $$\omega^{(4)} = C_5(\omega^{(3)}) = (\overline{50}, \overline{30}, \overline{23}, 20, 20, \overline{10}, \overline{7}, \overline{3}).$$

  Using Lemma 5.7, we obtain that $\omega^{(4)} \in \overline{\mathcal{B}}_0^=(3, 7; 10, 4, 3|5)$.

Set $\nu = \omega^{(4)}$. Then ν is an overpartition in $\overline{\mathcal{B}}_0(3, 7; 10, 4, 3)$ such that $|\nu| = |\omega| + |\zeta|$ and $\ell(\nu) = \ell(\omega) + \ell(\zeta)$.

6 Proof of Theorem 1.21

In this section, we will give a proof of Theorem 1.21 by using Bailey pairs. It remains to be a question to find a combinatorial proof of this fact. For historical perspectives and recent advances on Bailey pairs, we refer to Agarwal, Andrews and Bressoud [1], Andrews [7,8], Bressoud, Ismail and Stanton [11], Lovejoy [34], Paule [40], Warnaar [43], to name of few. A pair of sequences $(\alpha_n(a,q),\beta_n(a,q))$ is said to be a Bailey pair relative to (a,q) if for $n\geq 0$,

$$\beta_n(a,q)=\sum_{r=0}^{n}\frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}.$$

The following formulation of Bailey's lemma was given by Andrews [6,7].

Theorem 6.1 (Bailey's lemma). *If $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair relative to (a,q), then $(\alpha'_n(a,q),\beta'_n(a,q))$ is also a Bailey pair relative to (a,q), where*

$$\alpha'_n(a,q)=\frac{(\rho_1;q)_n(\rho_2;q)_n}{(aq/\rho_1;q)_n(aq/\rho_2;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n\alpha_n(a,q),$$
$$\beta'_n(a,q)=\sum_{j=0}^{n}\frac{(\rho_1;q)_j(\rho_2;q)_j(aq/\rho_1\rho_2;q)_{n-j}}{(aq/\rho_1;q)_n(aq/\rho_2;q)_n(q;q)_{n-j}}\left(\frac{aq}{\rho_1\rho_2}\right)^j\beta_j(a,q).$$

When $\rho_1,\rho_2\to\infty$, Bailey's lemma reduces to the following form, which has been used by Andrews [6] to derive the Andrews-Gordon identity (1.2) when $r=1$ or $r=k$.

Lemma 6.2. *If $(\alpha_n(a,q),\beta_n(a,q))$ is a Bailey pair relative to (a,q), then $(\alpha'_n(a,q),\beta'_n(a,q))$ is also a Bailey pair relative to (a,q), where*

$$\alpha'_n(a,q)=a^nq^{n^2}\alpha_n(a,q),$$
$$\beta'_n(a,q)=\sum_{j=0}^{n}\frac{a^jq^{j^2}}{(q;q)_{n-j}}\beta_j(a,q).$$

Agarwal, Andrews and Bressoud [1] developed the technique of the Bailey lattice to establish the Andrews-Gordon identity (1.2) in general for $1\leq r\leq k$. Bressoud, Ismail and Stanton [11] found an alternative proof of the Andrews-Gordon identity (1.2) in the general case by successively using Bailey's lemma and the following proposition.

Proposition 6.3. [11, Proposition 4.1] *If $(\alpha_n(1,q),\beta_n(1,q))$ is a Bailey pair relative to $(1,q)$, where*

$$\alpha_n(1,q)=\begin{cases}1, & \text{if } n=0,\\ (-1)^nq^{An^2}(q^{(A-1)n}+q^{-(A-1)n}), & \text{if } n\geq 1,\end{cases}$$

then $(\alpha'_n(1,q), \beta'_n(1,q))$ *is also a Bailey pair relative to* $(1,q)$, *where*

$$\alpha'_n(1,q) = \begin{cases} 1, & \text{if } n=0, \\ (-1)^n q^{An^2}(q^{An}+q^{-An}), & \text{if } n \geq 1, \end{cases}$$

and for $n \geq 0$,

$$\beta'_n(1,q) = q^n \beta_n(1,q).$$

To prove Theorem 1.21, we also need the following proposition in [24] and a limiting case of an identity of Andrews [6].

Proposition 6.4. *If* $(\alpha_n(1,q), \beta_n(1,q))$ *is a Bailey pair relative to* $(1,q)$, *where*

$$\alpha_n(1,q) = \begin{cases} 1, & \text{if } n=0, \\ (-1)^n q^{An^2}(q^{(A-1)n}+q^{-(A-1)n}), & \text{if } n \geq 1, \end{cases}$$

then $(\alpha'_n(1,q), \beta'_n(1,q))$ *is also a Bailey pair relative to* $(1,q)$, *where*

$$\alpha'_n(1,q) = \begin{cases} 1, & \text{if } n=0, \\ (-1)^n q^{An^2}(q^{(A-1)n}+q^{-An})(1+q^n)/2, & \text{if } n \geq 1, \end{cases}$$

and for $n \geq 0$,

$$\beta'_n(1,q) = \beta_n(1,q)(1+q^n)/2.$$

Theorem 6.5 (Andrews)**.** *If* $(\alpha_n(1,q), \beta_n(1,q))$ *is a Bailey pair relative to* $(1,q)$, *then for* $N \geq 0$,

$$\begin{aligned}
&\sum_{n\geq 0} \frac{(b_1;q)_n (c_1;q)_n \cdots (b_k;q)_n (c_k;q)_n \left(q^{-N};q\right)_n}{(aq/b_1;q)_n (aq/c_1;q)_n \cdots (aq/b_k;q)_n (aq/c_k;q)_n (aq^{N+1};q)_n} \\
&\quad \times \left(\frac{a^k q^{k+N}}{b_1 c_1 \cdots b_k c_k}\right)^n q^{-\binom{n}{2}} (-1)^n \alpha_n(1,q) \\
&\qquad = \frac{(aq;q)_N (aq/b_k c_k;q)_N}{(aq/b_k;q)_N (aq/c_k;q)_N} \sum_{n_k \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{(b_k;q)_{n_k} (c_k;q)_{n_k} \cdots (b_1;q)_{n_1} (c_1;q)_{n_1}}{(q;q)_{n_k-n_{k-1}} (q;q)_{n_{k-1}-n_{k-2}} \cdots (q;q)_{n_2-n_1}} \\
&\qquad\quad \times \frac{\left(q^{-N};q\right)_{n_k} (aq/b_{k-1}c_{k-1};q)_{n_k-n_{k-1}} \cdots (aq/b_1c_1;q)_{n_2-n_1}}{(b_k c_k q^{-N}/a;q)_{n_k} (aq/b_{k-1};q)_{n_k} (aq/c_{k-1};q)_{n_k} \cdots (aq/b_1;q)_{n_2} (aq/c_1;q)_{n_2}} \\
&\qquad\quad \times q^{n_1+\cdots+n_k} a^{n_1+\cdots+n_{k-1}} (b_{k-1}c_{k-1})^{-n_{k-1}} \cdots (b_1c_1)^{-n_1} \beta_{n_1}(1,q). \qquad (6.1)
\end{aligned}$$

Below is a limiting case of Theorem 6.5.

Proposition 6.6. *If $(\alpha_n(1,q^\eta),\beta_n(1,q^\eta))$ is a Bailey pair relative to $(1,q^\eta)$, then for $r>\lambda\geq 0$,*

$$\sum_{n=0}^{\infty}\frac{2q^{(r-\frac{\lambda+1}{2})\eta n^2+\frac{\lambda+1}{2}\eta n-(\alpha_1+\cdots+\alpha_\lambda)n}(-q^{\alpha_1};q^\eta)_n\cdots(-q^{\alpha_\lambda};q^\eta)_n}{(1+q^{\eta n})(-q^{\eta-\alpha_1};q^\eta)_n\cdots(-q^{\eta-\alpha_\lambda};q^\eta)_n}\alpha_n(1,q^\eta)$$

$$=\frac{(q^\eta;q^\eta)_\infty}{(-q^{\eta-\alpha_1};q^\eta)_\infty}\sum_{N_1\geq N_2\geq\cdots\geq N_r\geq 0}\frac{q^{\eta(N_{\lambda+2}^2+\cdots+N_r^2)+\eta\left(\binom{N_1+1}{2}+\cdots+\binom{N_{\lambda+1}+1}{2}\right)-(\alpha_1N_1+\cdots+\alpha_\lambda N_\lambda)}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{r-1}-N_r}}$$

$$\times\frac{(-1;q^\eta)_{N_{\lambda+1}}(-q^{\alpha_1};q^\eta)_{N_1}\cdots(-q^{\alpha_\lambda};q^\eta)_{N_\lambda}}{(-q^\eta;q^\eta)_{N_\lambda}(-q^{\eta-\alpha_2};q^\eta)_{N_1}\cdots(-q^{\eta-\alpha_\lambda};q^\eta)_{N_{\lambda-1}}}\beta_{N_r}(1,q^\eta),\tag{6.2}$$

where we assume that $N_{r+1}=0$.

Proof. Replacing q by q^η and setting $k=r$, $a=1$, $c_{r-\lambda}=-1$ and $c_{r-s+1}=-q^{\alpha_s}$ for $1\leq s\leq\lambda$, as $b_i\to\infty$ for $1\leq i\leq r$, $c_m\to\infty$ for $1\leq m\leq r-\lambda-1$ and $N\to\infty$, (6.1) becomes

$$\sum_{n=0}^{\infty}\frac{2q^{(r-\frac{\lambda+1}{2})\eta n^2+\frac{\lambda+1}{2}\eta n-(\alpha_1+\cdots+\alpha_\lambda)n}(-q^{\alpha_1};q^\eta)_n\cdots(-q^{\alpha_\lambda};q^\eta)_n}{(1+q^{\eta n})(-q^{\eta-\alpha_1};q^\eta)_n\cdots(-q^{\eta-\alpha_\lambda};q^\eta)_n}\alpha_n(1,q^\eta)$$

$$=\frac{(q^\eta;q^\eta)_\infty}{(-q^{\eta-\alpha_1};q^\eta)_\infty}\sum_{n_r\geq n_{r-1}\geq\cdots\geq n_1\geq 0}\frac{q^{\eta(n_1^2+\cdots+n_{r-\lambda-1}^2)+\eta\left(\binom{n_{r-\lambda}+1}{2}+\cdots+\binom{n_r+1}{2}\right)-(\alpha_1n_r+\cdots+\alpha_\lambda n_{r-\lambda+1})}}{(q^\eta;q^\eta)_{n_r-n_{r-1}}\cdots(q^\eta;q^\eta)_{n_2-n_1}}$$

$$\times\frac{(-1;q^\eta)_{n_{r-\lambda}}(-q^{\alpha_1};q^\eta)_{n_r}\cdots(-q^{\alpha_\lambda};q^\eta)_{n_{r-\lambda+1}}}{(-q^\eta;q^\eta)_{n_{r-\lambda+1}}(-q^{\eta-\alpha_2};q^\eta)_{n_r}\cdots(-q^{\eta-\alpha_\lambda};q^\eta)_{n_{r-\lambda+2}}}\beta_{n_1}(1,q^\eta).$$

Writing $n_t=N_{r+1-t}$ for $1\leq t\leq r$, we are led to (6.2). This completes the proof. ∎

The following Bailey pair is also required in the proof of Theorem 1.21.

Proposition 6.7. *For $k\geq r\geq 1$ and $n\geq 0$,*

$$\alpha_n(1,q)=\begin{cases}1, & \text{if } n=0,\\ (-1)^nq^{(k-r)n^2}(q^{(k-r-1)n}+q^{-(k-r)n})(1+q^n)/2, & \text{if } n\geq 1,\end{cases}$$

$$\beta_n(1,q)=\sum_{n\geq N_{r+1}\geq\cdots\geq N_{k-1}\geq 0}\frac{(1+q^n)q^{(N_{r+1}^2+\cdots+N_{k-1}^2+N_{r+1}+\cdots+N_{k-1})}}{2(q;q)_{n-N_{r+1}}\cdots(q;q)_{N_{k-2}-N_{k-1}}(q^2;q^2)_{N_{k-1}}}.\tag{6.3}$$

is a Bailey pair relative to $(1,q)$.

Proof. We begin with the following Bailey pair [42, E(5)],

$$\alpha_n^{(0)}(1,q)=\begin{cases}1, & \text{if } n=0,\\ (-1)^n(q^{-n}+q^n), & \text{if } n\geq 1,\end{cases}$$

$$\beta_n^{(0)}(1,q)=\frac{(-1)^n}{q^n(q^2;q^2)_n}.\tag{6.4}$$

Applying Proposition 6.3 to (6.4), we obtain that

$$\alpha_n^{(1)}(1,q) = \begin{cases} 1, & \text{if } n=0, \\ 2(-1)^n, & \text{if } n \geq 1, \end{cases}$$

$$\beta_n^{(1)}(1,q) = \frac{(-1)^n}{(q^2;q^2)_n}.$$

Using Lemma 6.2, we get

$$\begin{aligned} \alpha_n^{(2)}(1,q) &= \begin{cases} 1, & \text{if } n=0, \\ 2(-1)^n q^{n^2}, & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(2)}(1,q) &= \sum_{j=0}^{n} \frac{(-1)^j q^{j^2}}{(q;q)_{n-j}(q^2;q^2)_j}. \end{aligned} \tag{6.5}$$

Employing the following q-Chu-Vandermonde formula with $c=-q$ and $a \to \infty$,

$$\sum_{j=0}^{n} \frac{(a;q)_j (q^{-n};q)_j}{(c;q)_j (q;q)_j} \left(\frac{cq^n}{a}\right)^j = \frac{(c/a;q)_n}{(c;q)_n},$$

we find that

$$\beta_n^{(2)}(1,q) = \frac{1}{(q^2;q^2)_n}.$$

Applying Proposition 6.3 and Lemma 6.2 $k-r-1$ times to (6.5) yields the following Bailey pair

$$\begin{aligned} \alpha_n^{(2k-2r)}(1,q) &= \begin{cases} 1, & \text{if } n=0, \\ (-1)^n q^{(k-r)n^2}(q^{(k-r-1)n} + q^{-(k-r-1)n}), & \text{if } n \geq 1, \end{cases} \\ \beta_n^{(2k-2r)}(1,q) &= \sum_{n \geq N_{r+1} \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{(N_{r+1}^2 + \cdots + N_{k-1}^2 + N_{r+1} + \cdots + N_{k-1})}}{(q;q)_{n-N_{r+1}} \cdots (q;q)_{N_{k-2}-N_{k-1}} (q^2;q^2)_{N_{k-1}}}. \end{aligned} \tag{6.6}$$

By Proposition 6.4 and (6.6), we obtain the following Bailey pair

$$\alpha_n(1,q) = \begin{cases} 1, & \text{if } n=0, \\ (-1)^n q^{(k-r)n^2}(q^{(k-r-1)n} + q^{-(k-r)n})(1+q^n)/2, & \text{if } n \geq 1, \end{cases}$$

$$\beta_n(1,q) = \sum_{n \geq N_{r+1} \geq \cdots \geq N_{k-1} \geq 0} \frac{(1+q^n) q^{(N_{r+1}^2 + \cdots + N_{k-1}^2 + N_{r+1} + \cdots + N_{k-1})}}{2(q;q)_{n-N_{r+1}} \cdots (q;q)_{N_{k-2}-N_{k-1}} (q^2;q^2)_{N_{k-1}}}.$$

This completes the proof. ∎

We conclude this section with the proof of Theorem 1.21 resorting to Proposition 6.6 and Proposition 6.7.

Proof of Theorem 1.21. For $k \geq r > \lambda$, plugging $\alpha_n(1,q)$ in (6.3) with q replaced by q^η into the left-hand side of (6.2), and using the assumption that $\alpha_i + \alpha_{\lambda+1-i} = \eta$ for $1 \leq i \leq \lambda$, the left-hand side of (6.2) simplifies to

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} & \frac{(-q^{\alpha_1};q^\eta)_n \cdots (-q^{\alpha_\lambda};q^\eta)_n}{(-q^{\eta-\alpha_1};q^\eta)_n \cdots (-q^{\eta-\alpha_\lambda};q^\eta)_n} \\
& \times (-1)^n q^{(k-\frac{\lambda+1}{2})\eta n^2 + \frac{\lambda}{2}\eta n - (\alpha_1+\cdots+\alpha_\lambda)n} \left(q^{(k-r-\frac{1}{2})\eta n} + q^{-(k-r-\frac{1}{2})\eta n}\right) \\
& = 1 + \sum_{n=1}^{\infty} (-1)^n q^{(k-\frac{\lambda+1}{2})\eta n^2} \left(q^{(k-r-\frac{1}{2})\eta n} + q^{-(k-r-\frac{1}{2})\eta n}\right) \\
& = \left(q^{(r-\frac{\lambda}{2})\eta}, q^{(2k-r-1-\frac{\lambda}{2})\eta}, q^{(2k-\lambda-1)\eta}; q^{(2k-\lambda-1)\eta}\right)_\infty,
\end{aligned} \tag{6.7}$$

where the last equality follows from Jacobi's triple product identity [5, Theorem 2.8].

On the other hand, substituting the expression for $\beta_n(1,q)$ in (6.3) with q replaced by q^η into the right-hand side of (6.2), we get

$$\begin{aligned}
\frac{(q^\eta;q^\eta)_\infty}{(-q^{\eta-\alpha_1};q^\eta)_\infty} & \sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} \frac{(1+q^{-\eta N_r})(-q^\eta;q^\eta)_{N_{\lambda+1}-1} q^{\eta(N_{\lambda+2}^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}}{(q^\eta;q^\eta)_{N_1-N_2} \cdots (q^\eta;q^\eta)_{N_{k-2}-N_{k-1}} (q^{2\eta};q^{2\eta})_{N_{k-1}}} \\
\times & \frac{q^{\eta\left(\binom{N_1+1}{2}+\cdots+\binom{N_{\lambda+1}+1}{2}\right)-(\alpha_1 N_1+\cdots+\alpha_\lambda N_\lambda)} (-q^{\alpha_1};q^\eta)_{N_1} \cdots (-q^{\alpha_\lambda};q^\eta)_{N_\lambda}}{(-q^\eta;q^\eta)_{N_\lambda} (-q^{\eta-\alpha_2};q^\eta)_{N_1} \cdots (-q^{\eta-\alpha_\lambda};q^\eta)_{N_{\lambda-1}}}.
\end{aligned} \tag{6.8}$$

Observing that

$$(-q^r;q^\eta)_n = q^{rn+\eta\binom{n}{2}} (-q^{\eta-r-n\eta};q^\eta)_n,$$

and

$$\frac{1}{(-q^{\eta-r};q^\eta)_n} = \frac{(-q^{\eta-r+n\eta};q^\eta)_\infty}{(-q^{\eta-r};q^\eta)_\infty},$$

the summation in (6.8) equals

$$\begin{aligned}
\sum_{N_1 \geq \cdots \geq N_{k-1} \geq 0} & \frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})} (1+q^{-\eta N_r})(-q^{\eta-\eta N_{\lambda+1}};q^\eta)_{N_{\lambda+1}-1}}{(q^\eta;q^\eta)_{N_1-N_2} \cdots (q^\eta;q^\eta)_{N_{k-2}-N_{k-1}} (q^{2\eta};q^{2\eta})_{N_{k-1}}} \\
\times & \frac{(-q^{\eta+\eta N_\lambda};q^\eta)_\infty \prod_{s=1}^{\lambda} (-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s} \prod_{s=2}^{\lambda} (-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty}{(-q^\eta;q^\eta)_\infty \prod_{s=2}^{\lambda} (-q^{\eta-\alpha_s};q^\eta)_\infty}.
\end{aligned} \tag{6.9}$$

Combining (6.7) and (6.9), we deduce that

$$\frac{(q^\eta;q^\eta)_\infty}{(-q^{\eta-\alpha_1};q^\eta)_\infty}\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}(1+q^{-\eta N_r})(-q^{\eta-\eta N_{\lambda+1}};q^\eta)_{N_{\lambda+1}-1}}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{2\eta};q^{2\eta})_{N_{k-1}}}$$

$$\times\frac{(-q^{\eta+\eta N_\lambda};q^\eta)_\infty\prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s}\prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty}{(-q^\eta;q^\eta)_\infty\prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s};q^\eta)_\infty}$$

$$=(q^{(r-\frac{\lambda}{2})\eta},q^{(2k-r-1-\frac{\lambda}{2})\eta},q^{(2k-\lambda-1)\eta};q^{(2k-\lambda-1)\eta})_\infty.$$

Multiplying both sides by

$$\frac{(-q^{\eta-\alpha_1},\ldots,-q^{\eta-\alpha_\lambda},-q^\eta;q^\eta)_\infty}{(q^\eta;q^\eta)_\infty},$$

we obtain

$$\sum_{N_1\geq\cdots\geq N_{k-1}\geq 0}\frac{q^{\eta(N_1^2+\cdots+N_{k-1}^2+N_r+\cdots+N_{k-1})}(1+q^{-\eta N_r})(-q^{\eta-\eta N_{\lambda+1}};q^\eta)_{N_{\lambda+1}-1}(-q^{\eta+\eta N_\lambda};q^\eta)_\infty}{(q^\eta;q^\eta)_{N_1-N_2}\cdots(q^\eta;q^\eta)_{N_{k-2}-N_{k-1}}(q^{2\eta};q^{2\eta})_{N_{k-1}}}$$

$$\times\prod_{s=1}^{\lambda}(-q^{\eta-\alpha_s-\eta N_s};q^\eta)_{N_s}\prod_{s=2}^{\lambda}(-q^{\eta-\alpha_s+\eta N_{s-1}};q^\eta)_\infty$$

$$=\frac{(-q^{\eta-\alpha_1},\ldots,-q^{\eta-\alpha_\lambda},-q^\eta;q^\eta)_\infty(q^{(r-\frac{\lambda}{2})\eta},q^{(2k-r-1-\frac{\lambda}{2})\eta},q^{(2k-\lambda-1)\eta};q^{(2k-\lambda-1)\eta})_\infty}{(q^\eta;q^\eta)_\infty}.$$

But $\alpha_i+\alpha_{\lambda+1-i}=\eta$ for $1\leq i\leq\lambda$, so we reach (1.10) in Theorem 1.21. This completes the proof. ∎

7 Concluding remarks

To conclude, we make a few remarks on the connection between the main results of this paper and the original conjecture of Bressoud, along with our subsequent work in this direction. Then we mention some potential problems for future study.

It should be stressed that the overpartition analogues considered in this paper are not merely a matter of extension and specialization. In fact, they play an essential role and serve as an indispensable structure in tackling the conjecture of Bressoud formulated in terms of ordinary partitions.

Based on the relationship between the overpartition analogue $\overline{B}_1$ and Bressoud's function B_0 (Theorem 1.16), we realize that the case $j=0$ of Bressoud's conjecture (that is, $A_0=B_0$) is a consequence of the relation $\overline{A}_1=\overline{B}_1$ on overpartitions. Nevertheless, the case $j=1$ of Bressoud's conjecture has been resolved by Kim [27] without resorting to overpartitions. One is immediately led to show that $\overline{A}_1 = \overline{B}_1$. This is the objective of our subsequent paper [25]. It is worth mentioning that the relation $\overline{A}_1 = \overline{B}_1$ can be regarded as an overpartition analogue of Bressoud's conjecture for the case $j = 1$. In other words, we may say that Bressoud's conjecture consists of two parts, one of which is the case $j = 1$ settled by Kim, and the other (the case $j = 0$) is an overpartition analogue. Naturally, it would be interesting to give a direct combinatorial proof of the case $j = 0$ of Bressoud's conjecture without relying on the overpartition setting. Also, it would be desirable to give direct combinatorial proofs of the generating functions of $\overline{B}_0$ and $\overline{B}_1$.

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