A New Family of Mixed Method for the Biharmonic Eigenvalue Problem Based on the First Order Equations of Hellan-Herrmann-Johnson Type

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Abstract

In this paper, we consider the numerical approximation of a biharmonic eigenvalue problem by introducing a new family of the mixed method. This method is based on a formulation where the fourth-order eigenproblem is recast as a system of four first-order equations. The optimal convergence rates with $2k+2$ ($k \geq 0$ is the degree of the polynomials) of eigenvalue approximation are theoretically derived and numerically verified. The optimal or sub-optimal convergences of the other unknowns are theoretically proved. The new numerical schemes based on the deduced problems can be of lower complicacy, and the framework is fit for various fourth-order eigenvalue problems.

Keywords. biharmonic eigenvalue problem, mixed method, first-order system, finite element method.

AMS subject classifications. 35Q40, 35Q55, 65N30, 65N25, 65B99

1 Introduction

The biharmonic eigenvalue problem is one of the fundamental model problems in mathematics, physics, and elastic mechanics, and has wide applications in, e.g., modeling the vibration of thin plates [38], fluid-structure [9], inverse scattering theory [13] and electronic structure [37]. We consider the following the biharmonic eigenvalue problem:

$$\Delta^2 u = \lambda u, \ \text{in } \Omega, \tag{1.1a}$$

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$$u = \frac{\partial u}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is a polyhedral domain.

Many existing methods are based on the primal formulation (1.1), which only have the approximations to eigenvalue λ and the eigenfunction u as two unknowns. Among these methods include the conforming finite element (FE) [6, 16, 24, 26, 35], the C^0IPG [11], the classical non-conforming element [18, 36, 41], computation of guaranteed/asymptotic upper and lower bounds [15, 22, 30, 31, 40], spectral-Galerkin method [2], and adaptive method and its convergence analysis [18, 27]. In addition, [3] presents a high accuracy spectral method based on the min/max principle for biharmonic eigenvalue problems on a spherical domain. Recently, the discontinuous Galerkin (DG) method [39], two-grid method [28, 43], multi-level/multigrid method [42], and C^0 virtual element method [32] become the powerful alternative for numerically solving the biharmonic eigenvalue problems.

Since the design and the implementation of C^1 traditional FEMs for the biharmonic eigenvalue problem is computationally quite intensive due to keeping C^1-continuity across the inter-element boundaries, several approaches like mixed DG methods [39] and C^0-interior penalty methods [11] have been proposed but they are still computationally expensive. The lower order mixed finite element method is an effective method to avoid the higher regularity and is easier for programming and computing than the higher order element. A natural idea is to design more effective mixed element schemes for the eigenvalue problem based on the corresponding boundary value problem. As far as mixed methods are concerned, the fourth-order biharmonic equation can be recast in mixed form as the Hellan-Herrmann-Johnson (HHJ) type of equations of first-order (referred to as a problem with four unknown fields, cf [8]).

$$\begin{array}{lll} \boldsymbol{q} = \nabla u, & \underline{\boldsymbol{z}} = \nabla \boldsymbol{q}, & \text{in } \Omega, \\ \boldsymbol{w} = \nabla \cdot \underline{\boldsymbol{z}}, & \nabla \cdot \boldsymbol{w} = \lambda u, & \text{in } \Omega, \\ u = 0, & \boldsymbol{q} \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \end{array} \tag{1.2}$$

Following our convention, $(\nabla \boldsymbol{q})_{i\ell} = \partial_{x_\ell}(q_i)$ for $1 \le i, \ell \le d$, where q_i is the ith component of $\boldsymbol{q}$. Moreover, $(\nabla \cdot \underline{\boldsymbol{z}})_i = \sum_{\ell=1}^d \partial_{x_\ell} \underline{\boldsymbol{z}}_{i\ell}$, where the $\underline{\boldsymbol{z}}_{i\ell}$ is the $i\ell$-entry of $\underline{\boldsymbol{z}}$.

Several mixed element schemes for the biharmonic eigenvalue problems have been proposed in [14, 33, 34]. They are based on introducing the variable $z = \Delta u$ and obtaining a coupled system of Poisson problems. In [14, 23], the mixed finite element methods are first proposed to solve the biharmonic eigenvalue problem. Following the mixed method analysis of the source problem, error analysis of the mixed method for the biharmonic eigenvalue problem are developed by using piecewise continuous approximations for both variables in [14, 34]. [10] develops an isoparametric mixed method and present the estimate for taking into account the combined effect of boundary approximation and numerical integration on the approximation for general fourth-order elliptic eigenvalue problems. [33] presents the lowest order mixed finite element method of the biharmonic eigenvalue problem, but it is the piecewise linear and continuous finite element method, not piecewise constant finite element. A new postprocessing technique and the superconvergence of mixed finite element approximations of the eigenpairs and the biharmonic operator is proposed in [4]. [39]

gives the mixed DG method, propose a residual-based a posteriori error estimator and prove the convergence with the optimal order in L^2 and DG-norm.

In this paper, our method is based on a stationary variational principle (the Reissner principle) which was introduced by Hellan, Herrmann, and Visser [25]. Its alternative explanation is to transform the primal problems to order reduced formulations. Our method constructs a system on low-regularity spaces by introducing auxiliary variables, and then discretize the resulting system by the different finite element methods. In engineering applications, the first derivatives ∇u (the strain) and the second derivatives $\nabla\nabla u$ (the moments) of u are frequently more important than u itself. In fact, in the Reissner-Mindlin plate problem, we are interested in transverse displacement, rotation, bending moment, and shear stress. This encourages us to introduce the various-order derivatives of the primal variable as the auxiliary variables, and then expand the problem to the low-order spaces. So the four order eigenvalue problem (1.1) is modified by the first order system (1.2) by introducing three auxiliary variables. Our method will approximate the eigenfunction u, the second derivatives of u, namely $\underline{\boldsymbol{z}}$, with optimal order $k+1$ and the eigenvalue with optimal order $2(k+1)$ $(k \geq 0)$. As far as we know, this paper is the first study on the approximation of biharmonic eigenvalue problems by the piecewise constant and obtaining the convergence with optimal order. Furthermore, from the numerical examples, our methods can present lower or upper bounds of eigenvalues by using different finite element spaces.

The remaining paper is organized as follows. In the next section, we introduce the mixed method of the eigenproblem and essential notations used throughout the paper. Section 3 provides the convergence analysis of eigenvalues, eigenfunctions, and the other auxiliary functions based on the mixed first-order system with the optimal convergence order. In Section 4, we present numerical results to verify the theoretical results. Section 5 provides a discussion on the different choices of other finite element spaces. Finally, some concluding remarks are given in Section 6.

2 Mixed Element Method of the HHJ system

In order to discuss error analysis, we first recall the Dirichlet boundary value problem which is recast as first-order system of HHJ type and finds $(u^f, \boldsymbol{q}^f, \underline{\boldsymbol{z}}^f, \boldsymbol{w}^f) \in V \times \boldsymbol{Q} \times \underline{\boldsymbol{Z}} \times \boldsymbol{W}$, for any given "source" $f \in L^2(\Omega)$, such that

$$\nabla u^f = \boldsymbol{q}^f \quad \text{in} \quad \Omega \tag{2.1a}$$

$$\nabla \boldsymbol{q}^f = \underline{\boldsymbol{z}}^f \quad \text{in} \quad \Omega \tag{2.1b}$$

$$\nabla \cdot \underline{\boldsymbol{z}}^f = \boldsymbol{w}^f \quad \text{in} \quad \Omega \tag{2.1c}$$

$$\nabla \cdot \boldsymbol{w}^f = f \quad \text{in} \quad \Omega \tag{2.1d}$$

$$u^f = \boldsymbol{q}^f \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1e}$$

where $V = L^2(\Omega)$, $\boldsymbol{Q} = (L^2(\Omega))^d$, $\boldsymbol{W} = H(\text{div}, \Omega)$, $\underline{\boldsymbol{Z}} = \underline{\boldsymbol{H}}(\text{div}, \Omega)$. $\underline{\boldsymbol{H}}(\text{div}, \Omega)$ denotes all $d \times d$ matrix-valued functions such that each row belongs to the space $H(\text{div}, \Omega)$. Throughout, all functions are real-valued in this paper. We use the standard notations for Sobolev spaces $H^s(\Omega)$ and their associated norms $\|\cdot\|_s$ and seminorms $|\cdot|_s$. The $L^2(\Omega)$ inner-product is denoted by $(\cdot,\cdot)$, that is $(v,w) := \int_\Omega vw\mathrm{d}\Omega, \quad \forall v,w \in L^2(\Omega)$. Thus $\|\cdot\|_0 := \sqrt{(\cdot,\cdot)}$.

To facilitate our analysis, we introduce the following solution operators of the source problem with the source f:

$$
\begin{aligned}
&\mathbb{U}: L^2(\Omega) \to V, \text{ which is defined simply by } \mathbb{U}f = u^f,\\
&\mathbb{Q}: L^2(\Omega) \to \boldsymbol{Q}, \text{ which is defined simply by } \mathbb{Q}f = \boldsymbol{q}^f,\\
&\mathbb{Z}: L^2(\Omega) \to \underline{\boldsymbol{Z}}, \text{ which is defined simply by } \mathbb{Z}f = \underline{\boldsymbol{z}}^f,\\
&\mathbb{W}: L^2(\Omega) \to \boldsymbol{W}, \text{ which is defined simply by } \mathbb{W}f = \boldsymbol{w}^f.
\end{aligned}
$$

By the classical elliptic regularity results, if the domain Ω is smooth [1] or the largest interior angle of $\partial\Omega$ is less than $126.28°$ [12], and $f \in L^2(\Omega)$, then $u^f \in H^4(\Omega)$. For a convex polygonal domain, the weak solutions of the boundary value problem belong in general to $H^{3+s}(\Omega)$ for some $s \in (0,1]$. The value of s depends on depends on the largest interior angle of $\partial\Omega$. The regularity results in the source problem (2.1) will lead to the regularity of the eigenfunction u of (1.1).

2.1 The source problem

The mixed method based on HHJ type provides an approximation $(\mathbb{U}_h, \mathbb{Q}_h, \mathbb{W}_h, \mathbb{Z}_h)$ to $(\mathbb{U}, \mathbb{Q}, \mathbb{W}, \mathbb{Z})$. To understand this approximation, we first describe the mixed method of source problem based on HHJ type and introduce known results we shall use later.

Now, let us demonstrate the mixed method based on the HHJ type. First we generate a shape-regular decomposition for the computational domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$) and the diameter of a cell $K \in \mathcal{T}_h$ is denoted by h_K. The mesh diameter h describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we construct the following finite element spaces denoted by $V_h \subset V$, $\boldsymbol{Q}_h \subset \boldsymbol{Q}$, $\underline{\boldsymbol{Z}}_h \subset \underline{\boldsymbol{Z}}$ and $\boldsymbol{W}_h \subset \boldsymbol{W}$. The family of finite-dimensional spaces $(V_h, \boldsymbol{Q}_h, \underline{\boldsymbol{Z}}_h, \boldsymbol{W}_h)$ is assumed to satisfy the following assumption:

$$
\begin{aligned}
&\lim_{h\to 0}\inf_{v_h\in V_h}\|v - v_h\|_0 = 0, \quad \forall v \in V, \qquad &&\lim_{h\to 0}\inf_{\boldsymbol{p}_h\in \boldsymbol{Q}_h}\|\boldsymbol{p} - \boldsymbol{p}_h\|_0 = 0, \quad \forall \boldsymbol{p} \in \boldsymbol{Q},\\
&\lim_{h\to 0}\inf_{\underline{\boldsymbol{s}}_h\in \underline{\boldsymbol{Z}}_h}\|\underline{\boldsymbol{s}} - \underline{\boldsymbol{s}}_h\|_0 = 0, \quad \forall \underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}}, \qquad &&\lim_{h\to 0}\inf_{\boldsymbol{m}_h\in \boldsymbol{W}_h}\|\boldsymbol{m} - \boldsymbol{m}_h\|_0 = 0, \quad \forall \boldsymbol{m} \in \boldsymbol{W}.
\end{aligned}
$$

The mixed method define an approximation $(u_h^f, \boldsymbol{q}_h^f, \underline{\boldsymbol{z}}_h^f, \boldsymbol{w}_h^f)$ to $(u^f, \boldsymbol{q}^f, \underline{\boldsymbol{z}}^f, \boldsymbol{w}^f)$ in the following spaces, respectively

$$V_h^k = \{v \in V : \ v|_K \in \mathcal{P}^k(K) \text{ for all } K \in \mathcal{T}_h\}, \tag{2.2a}$$

$$\boldsymbol{Q}_h^k = \{\boldsymbol{p} \in \boldsymbol{Q} : \ \boldsymbol{p}|_K \in (\mathcal{P}^k(K))^d \text{ for all } K \in \mathcal{T}_h\}, \tag{2.2b}$$

$$\underline{\boldsymbol{Z}}_h^k = \{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}} : \ \text{each row of } \underline{\boldsymbol{s}} \text{ belongs to } \boldsymbol{W}_h^k\}. \tag{2.2c}$$

$$\boldsymbol{W}_h^k = \{\boldsymbol{m} \in \boldsymbol{W} : \ \boldsymbol{m}|_K \in \mathrm{RT}^k(K) \text{ for all } K \in \mathcal{T}_h\}, \tag{2.2d}$$

The space of polynomials of degree less than or equal to k (≥ 0) is denoted by $\mathcal{P}^k(K)$. Furthermore, we let $\mathcal{P}^{-1}(K) := \{0\}$. The space $\mathrm{RT}^k(K) = (\mathcal{P}^k(K))^d + \mathcal{P}^k(K)\boldsymbol{x}$ is the Raviart-Thomas space of index k. The subscript h denotes the mesh size which is defined as the maximum of the diameters of all mesh elements. It should be noted that we omit the superscript k of (2.2) where there is no confusion.

The mixed method defines the approximation solution u_h^f, the approximation $\boldsymbol{w}_h^f$, the approximation $\underline{\boldsymbol{z}}_h^f$ and the approximation $\boldsymbol{w}_h^f$, as the functions in $(V_h, \boldsymbol{Q}_h, \underline{\boldsymbol{Z}}_h, \boldsymbol{W}_h)$, respectively, satisfying

$$(\boldsymbol{q}_h^f, \boldsymbol{p}_h) + (u_h^f, \nabla\cdot\boldsymbol{p}_h) = 0, \tag{2.3a}$$
$$(\underline{\boldsymbol{z}}_h^f, \underline{\boldsymbol{s}}_h) + (\boldsymbol{q}_h^f, \nabla\cdot\underline{\boldsymbol{s}}_h) = 0, \tag{2.3b}$$
$$-(\boldsymbol{w}_h^f, \boldsymbol{m}_h) + (\nabla\cdot\underline{\boldsymbol{z}}_h^f, \boldsymbol{m}_h) = 0, \tag{2.3c}$$
$$(\nabla\cdot\boldsymbol{w}_h^f, v_h) = (f, v_h), \tag{2.3d}$$

for all $(v_h, \boldsymbol{p}_h, \underline{\boldsymbol{s}}_h, \boldsymbol{m}_h) \in V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$. Note it is proved in [8] that the above discrete system (2.3) is uniquely solvable. So given any f in $L^2(\Omega)$, the unique solution $(u_h^f, \boldsymbol{q}_h^f, \underline{\boldsymbol{z}}_h^f, \boldsymbol{w}_h^f)$ of the above mixed discrete system (2.3) is used to define the discrete versions of the operators $\mathbb{U}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{W}$ in (2.1), namely

$$\begin{aligned}
&\mathbb{U}_h : L^2(\Omega) \to V_h, \text{ which is defined simply by } \mathbb{U}_h f = u_h^f,\\
&\mathbb{Q}_h : L^2(\Omega) \to \boldsymbol{Q}_h, \text{ which is defined simply by } \mathbb{Q}_h f = \boldsymbol{q}_h^f,\\
&\mathbb{Z}_h : L^2(\Omega) \to \underline{\boldsymbol{Z}}_h, \text{ which is defined simply by } \mathbb{Z}_h f = \underline{\boldsymbol{z}}_h^f,\\
&\mathbb{W}_h : L^2(\Omega) \to \boldsymbol{W}_h, \text{ which is defined simply by } \mathbb{W}_h f = \boldsymbol{w}_h^f.
\end{aligned}$$

which are the solution operators of the source problem with the source f. The following error estimate is presented in [8].

Theorem 2.1. *Assume that* $(u^f, \boldsymbol{q}^f, \underline{\boldsymbol{z}}^f, \boldsymbol{w}^f) \in V\times\boldsymbol{Q}\times\underline{\boldsymbol{Z}}\times\boldsymbol{W}$ *and* $(u_h^f, \boldsymbol{q}_h^f, \underline{\boldsymbol{z}}_h^f, \boldsymbol{w}_h^f) \in V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$ *are the (2.1) and (2.3), respectively. Then*

$$\begin{aligned}
\|u^f - u_h^f\|_{L^2(\Omega)} &\leq C(\|u^f - \Pi_h^V u^f\|_{L^2(\Omega)} + h\|\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}^f\|_{L^2(\Omega)}),\\
\|\boldsymbol{q}^f - \boldsymbol{q}_h^f\|_{L^2(\Omega)} &\leq C\|\boldsymbol{q}^f - \boldsymbol{\Pi}_h^Q\boldsymbol{q}^f\|_{L^2(\Omega)} + C\|\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}^f\|_{L^2(\Omega)},\\
\|\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f\|_{L^2(\Omega)} &\leq C\|\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}^f\|_{L^2(\Omega)} + \Big(\sum_{K\in\mathcal{T}_h} Ch^{2j_k}\|\boldsymbol{w}^f - \Pi^{RT}\boldsymbol{w}^f\|^2_{L^2(\Omega)}\Big)^{1/2},\\
\|\boldsymbol{w}^f - \boldsymbol{w}_h^f\|_{L^2(\Omega)} &\leq C(\|\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT}\boldsymbol{w}^f\|_{L^2(\Omega)} + C\|\nabla\cdot(\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}^f)\|_{L^2(\Omega)},
\end{aligned}$$

where all projections are defined in Section 3.

The convergence of the eigenvalue problem approximation method is based on the convergence of $\mathbb{U}_h$ to $\mathbb{U}$ in the operator norm. To apply this idea to the HHJ eigenvalue system, we need the following approximation result of the source problem, which shows that the spectrum of $\mathbb{U}_h$ approximates that of $\mathbb{U}$.

Theorem 2.2. *Suppose there is an $s \geq 1$ such that any solution $(\mathbb{U}f, \mathbb{Q}f, \mathbb{Z}f, \mathbb{W}f)$ of the problem (2.1) satisfies*

$$\|\mathbb{U}f\|_s + \|\mathbb{Q}f\|_s + \|\mathbb{Z}f\|_s + \|\mathbb{W}f\|_s \leq C\|f\|_0, \tag{2.4}$$

for all $f \in V$. Then

$$\|\mathbb{U} - \mathbb{U}_h\|_0 \leq Ch^{\min\{s,k+1\}}. \tag{2.5}$$

Proof. The convergence of the source problem in Theorem 3.7 in [8] implies

$$\begin{aligned} \|\mathbb{U}f - \mathbb{U}_h f\|_0 &\leq Ch^{\min\{s,k+1\}}(\|\mathbb{U}f\|_s + \|\mathbb{Q}f\|_s + \|\mathbb{Z}f\|_s + \|\mathbb{W}f\|_s) \\ &\leq Ch^{\min\{s,k+1\}}\|f\|_0. \end{aligned}$$

which completes the proof. □

Remark 2.1. *In fact, we assume H^{s+3} elliptic regularity for the source problem (2.1) which requires more than convexity; see [12] for results on polygons. Hence, we can assume (2.4) hold. Again, we would like to emphasize that the convexity of the domain and H^{s+3} elliptic regularity are just technical assumptions for the purpose of our error analysis.*

2.2 The eigenvalue problem

In order to present the mixed method, we introduce the weak form of HHJ eigenvalue system: find $(\lambda, u, \boldsymbol{q}, \underline{\boldsymbol{z}}, \boldsymbol{w}) \in \mathbb{R} \times V \times \boldsymbol{Q} \times \underline{\boldsymbol{Z}} \times \boldsymbol{W}$ that satisfy

$$(\boldsymbol{q}, \boldsymbol{p}) + (u, \nabla \cdot \boldsymbol{p}) = 0, \tag{2.6a}$$
$$(\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}}) + (\boldsymbol{q}, \nabla \cdot \underline{\boldsymbol{s}}) = 0, \tag{2.6b}$$
$$-(\boldsymbol{w}, \boldsymbol{m}) + (\nabla \cdot \underline{\boldsymbol{z}}, \boldsymbol{m}) = 0, \tag{2.6c}$$
$$(\nabla \cdot \boldsymbol{w}, v) = \lambda(u, v), \tag{2.6d}$$

for all $(v, \boldsymbol{p}, \underline{\boldsymbol{s}}, \boldsymbol{m}) \in V \times \boldsymbol{Q} \times \underline{\boldsymbol{Z}} \times \boldsymbol{W}$.

The mixed finite element method for the weak form is as follows: find $(\lambda_h, u_h, \boldsymbol{q}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{w}_h) \in \mathbb{R} \times V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$ that satisfies

$$(\boldsymbol{q}_h, \boldsymbol{p}_h) + (u_h, \nabla \cdot \boldsymbol{p}_h) = 0, \tag{2.7a}$$
$$(\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}_h) + (\boldsymbol{q}_h, \nabla \cdot \underline{\boldsymbol{s}}_h) = 0, \tag{2.7b}$$
$$-(\boldsymbol{w}_h, \boldsymbol{m}_h) + (\nabla \cdot \underline{\boldsymbol{z}}_h, \boldsymbol{m}_h) = 0, \tag{2.7c}$$
$$(\nabla \cdot \boldsymbol{w}_h, v_h) = (\lambda_h u_h, v_h), \tag{2.7d}$$

for all $(v_h, \boldsymbol{p}_h, \underline{\boldsymbol{s}}_h, \boldsymbol{m}_h) \in V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$.

For matrix-valued functions, we use the notation

$$(\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}}) := \sum_{K \in \mathcal{T}_h} (\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}})_K, \text{ where } (\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}})_K := \int_K \underline{\boldsymbol{z}}(\boldsymbol{x}) : \underline{\boldsymbol{s}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},$$

which is the Frobenius inner product.

For vector-valued and scalar-valued functions we take a similar definition.

Lemma 2.1. *The eigenvalue λ_h of the discrete system is positive.*

Proof. Taking $v_h = u_h$, $\boldsymbol{p}_h = \boldsymbol{q}_h$, $\underline{\boldsymbol{s}}_h = \underline{\boldsymbol{z}}_h$, $\boldsymbol{m}_h = \boldsymbol{w}_h$ in (2.7), we have

$$\lambda_h(u_h, u_h) = (\nabla \cdot \boldsymbol{w}_h, u_h) = -(\boldsymbol{q}_h, \boldsymbol{w}_h) \tag{2.8}$$
$$= -(\nabla \cdot \underline{\boldsymbol{z}}_h, \boldsymbol{q}_h) = (\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{z}}_h) > 0. \tag{2.9}$$

So we obtain that $\lambda_h > 0$. □

Lemma 2.2. *The HHJ eigenvalue system (2.6) and discrete system (2.7) are Hermitian and positive definite.*

Proof. For any ψ and f in $L^2(\Omega)$, we have

$$\begin{aligned}
\lambda(\psi, \mathbb{U}f) &= (\nabla\cdot\mathbb{W}\psi, \mathbb{U}u) && (\text{by } v = \mathbb{U}f)\\
&= -(\mathbb{Q}f, \mathbb{W}\psi) && (\text{by } \boldsymbol{p} = \mathbb{W}\psi)\\
&= -(\mathbb{Q}f, \nabla\cdot\mathbb{Z}\psi) && (\text{by } \mathbb{W}\psi = \nabla\cdot\mathbb{Z}\psi)\\
&= (\mathbb{Z}f, \mathbb{Z}\psi) && (\text{by } \underline{\boldsymbol{s}} = \mathbb{Z}f)\\
&= \lambda(\mathbb{U}\psi, f).
\end{aligned}$$

$(\mathbb{Z}u, \mathbb{Z}u)$ shows that the HHJ eigenvalue system is Hermitian and positive definite. Similarly, for the local solution operator $\mathbb{U}_h$, we have

$$\begin{aligned}
\lambda_h(\psi, \mathbb{U}_h u) &= (\nabla\cdot\mathbb{W}_h\psi, \mathbb{U}_h u) && (\text{by } v_h = \mathbb{U}_h u \text{ in (2.7}d))\\
&= -(\mathbb{Q}_h u, \mathbb{W}_h\psi) && (\text{by } \boldsymbol{p}_h = \mathbb{W}_h\psi \text{ in (2.7}a))\\
&= -(\mathbb{Q}_h u, \nabla\cdot\mathbb{Z}_h\psi) && (\text{by } \underline{\boldsymbol{s}}_h = \mathbb{Z}_h\psi \text{ in (2.7}c)\\
&= (\mathbb{Z}_h u, \mathbb{Z}_h\psi) && (\text{by } \underline{\boldsymbol{s}}_h = \mathbb{Z}_h u \text{ in (2.7}b))\\
&= \lambda_h(\mathbb{U}_h\psi, f).
\end{aligned}$$

$(\mathbb{Z}_h u, \mathbb{Z}_h u)$ shows that the HHJ eigenvalue system is Hermitian and positive definite. □

3 Error estimates

This section provides a priori error results for the mixed method applied to the HHJ type (first-order equations) of biharmonic eigenvalue problems. We prove that under favorable regularity conditions, the eigenvalues of the first-order system converge at the rate $O(h^{2k+2})$, the eigenfunctions, the first derivatives ∇u and the second derivatives $\nabla\nabla u$ converge at the rate $O(h^{k+1})$ when we use polynomials of degree at most $k \geq 0$ for all variables. We are now ready to state our results.

In general, the error analysis starts to form the error equations which are written as follows:

$$\left\{\begin{aligned}
(\boldsymbol{q}-\boldsymbol{q}_h, \boldsymbol{p}_h) + (u-u_h, \nabla\cdot\boldsymbol{p}_h) &= 0,\\
(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}_h) + (\boldsymbol{q}-\boldsymbol{q}_h, \nabla\cdot\underline{\boldsymbol{s}}_h) &= 0,\\
-(\boldsymbol{w}-\boldsymbol{w}_h, \boldsymbol{m}_h) + (\nabla\cdot(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h), \boldsymbol{m}_h) &= 0,\\
(\nabla\cdot(\boldsymbol{w}-\boldsymbol{w}_h), v_h) - (\lambda u, v_h) + (\lambda_h u_h, v_h) &= 0,
\end{aligned}\right.$$

for all $(v_h, \boldsymbol{p}_h, \underline{\boldsymbol{s}}_h, \boldsymbol{m}_h) \in V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$.

The projections play an important role in error analysis. So we also need to define some projections. We let $\boldsymbol{\Pi}^{RT} : \boldsymbol{W}\cap L^p(\Omega) \to \boldsymbol{W}_h$ (for $p > 2$) be the Raviart-Thomas projection of index k defined on each $K \in \mathcal{T}_h$ by

$$(\boldsymbol{\Pi}^{RT}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\psi})_K = 0, \qquad \forall \boldsymbol{\psi} \in (\mathcal{P}^{k-1}(K))^d, \tag{3.1a}$$

$$\langle(\boldsymbol{\Pi}^{RT}\boldsymbol{\sigma} - \boldsymbol{\sigma})\cdot\mathbf{n}, \mu\rangle_F = 0, \qquad \forall \mu \in \mathcal{P}^k(K), \text{ for all faces } F \text{ of } K, \tag{3.1b}$$

for given any $\boldsymbol{\sigma} \in \boldsymbol{W}\cap L^p(\Omega)$. Here we used the notation $\langle\mu, \nu\rangle_F = \int_F \mu(s)\nu(s)\mathrm{d}s$.

Moreover, we let $\underline{\boldsymbol{\Pi}}^{RT}$ denote the matrix version of $\boldsymbol{\Pi}^{RT}$ as it acts on matrix-valued functions where $\underline{\boldsymbol{\Pi}}^{RT}$ acts on each row. Let $\boldsymbol{\Pi}_h^Q$ be the L^2-projection onto $\boldsymbol{Q}_h$. Finally, Π_h^V is the L^2-projection onto V_h. Throughout this paper, we will assume that $\boldsymbol{w}$ belongs to the domain of $\boldsymbol{\Pi}^{RT}$ and $\underline{\boldsymbol{z}}$ belongs to the domain of $\underline{\boldsymbol{\Pi}}^{RT}$.

We will need a few properties of $\boldsymbol{\Pi}^{RT}$. First, the commutative property is presented as follows:

$$\nabla \cdot (\boldsymbol{\Pi}^{RT}\boldsymbol{\sigma}) = \Pi_h^V \nabla \cdot \boldsymbol{\sigma}. \tag{3.2}$$

The following approximation properties hold: for given any $\boldsymbol{\sigma} \in \boldsymbol{W} \cap L^p(\Omega)$,

$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^{RT}\boldsymbol{\sigma}\|_{0,K} \leq h_K^{s+1}\|\boldsymbol{\sigma}\|_{s+1,K}. \tag{3.3}$$

for $0 \leq s \leq k$ and $K \in \mathcal{T}_h$.

By using the orthogonality of the projection $\boldsymbol{\Pi}^{RT}, \underline{\boldsymbol{\Pi}}^{RT}, \boldsymbol{\Pi}_h^Q$ and Π_h^V, the above error equations can be modified by

$$(\boldsymbol{\Pi}_h^Q\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{p}_h) + (\Pi_h^V u - u_h, \nabla \cdot \boldsymbol{p}_h) = (\boldsymbol{\Pi}_h^Q\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{p}_h), \tag{3.4a}$$

$$(\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}_h) + (\boldsymbol{\Pi}_h^Q\boldsymbol{q} - \boldsymbol{q}_h, \nabla \cdot \underline{\boldsymbol{s}}_h) = (\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}, \underline{\boldsymbol{s}}_h), \tag{3.4b}$$

$$-(\boldsymbol{\Pi}^{RT}\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{m}_h) + (\nabla \cdot (\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \boldsymbol{m}_h) = (\boldsymbol{\Pi}^{RT}\boldsymbol{w} - \boldsymbol{w}, \boldsymbol{m}_h), \tag{3.4c}$$

$$(\nabla \cdot (\boldsymbol{\Pi}^{RT}\boldsymbol{w} - \boldsymbol{w}_h), v_h) - (\lambda u, v_h) + (\lambda_h u_h, v_h) = 0, \tag{3.4d}$$

To evaluate the “distance” between eigenspaces, we recall some standard terminology. For any V_1 and V_2 of $V = L^2(\Omega)$, we define a suitable notion of “distance” or “gap” between two spaces as follows:

$$d(x, V_2) = \inf_{y \in V_2} \|x - y\|_{L^2(\Omega)}, \qquad d(V_1, V_2) = \sup_{x \in V_1} \frac{d(x, V_2)}{\|x\|_{L^2(\Omega)}}.$$

In order to go on the error analysis, we introduce the resolve operators $E(\mathbb{U})$ and $E_h(\mathbb{U}_h)$ as follows:

$$E(\mathbb{U}) = \frac{1}{2\pi i}\oint_\Gamma (z - \mathbb{U})^{-1} \mathrm{d}z,$$
$$E_h(\mathbb{U}_h) = \frac{1}{2\pi i}\oint_\Gamma (z - \mathbb{U}_h)^{-1} \mathrm{d}z.$$

For simplicity, we use E and E_h to denote $E(\mathbb{U})$ and $E_h(\mathbb{U}_h)$, respectively. We use $\mathcal{R}(E)$ and $\mathcal{R}(E_h)$ to denote the ranges or eigenspaces of the operators $E(\mathbb{U})$ and $E_h(\mathbb{U}_h)$, respectively. We define $J_h v = E_h \Pi_h^V v$, $\forall v \in \mathcal{R}(E)$.

The following Theorem collects and summaries a few convergence consequences including non-pollution of the spectrum, completeness of the spectrum, non-pollution, and completeness of the eigenspaces. These consequences, for the mixed method of biharmonic eigenvalue problem based on HHJ type, are simple extensions of the analogous results for the elliptic eigenvalue problem. Their proving arguments are standard and are already present in the literature (see [5]). Since they are applied to our method context with few modifications, we shall not present the proofs.

Theorem 3.1. *We have the following statements hold:*

1) (Non-pollution of the spectrum). Let Λ be an open set containing the spectrum of problem (1.1). Then for for sufficiently small h, Λ contains the spectrum of problem (2.7).

2) (Completeness of the spectrum) For any eigenvalue λ of problem (1.1), there is an eigenvalue λ_h of problem (2.7) such that

$$\lim_{h\to 0}\lambda_h = \lambda.$$

3) (Non-pollution and completeness of the eigenspaces) For eigenspaces $\mathcal{R}(E)$ and $\mathcal{R}(E_h)$ of problem (1.1) and problem (2.7) respectively, we have

$$\lim_{h\to 0} d(\mathcal{R}(E),\mathcal{R}(E_h)) = 0.$$

4) The operator E_h converges to E, i.e., $\lim_{h\to 0}\|E_h - E\| = 0$.

Remark 3.1. *Some results of Theorem 3.1 can be more refined. We can provide the convergence rate of this limit. As a matter of fact, $\|E - E_h\| \le Ch^{k+1}$ and $d(\mathcal{R}(E),\mathcal{R}(E_h)) \le Ch^{k+1}$ hold.*

We list the properties of the operators $\mathbb{U}_h$ and E_h (see Lemma 4.3 in [20])

- J_h is bijections,

$$C_1\|v\|_0 \le \|J_h v\|_0 \le C_2\|v\|_0, \quad \forall v \in \mathcal{R}(E), \tag{3.5}$$

- E_h is bijections,

$$\tilde{C}_1\|v\|_0 \le \|E_h v\|_0 \le \tilde{C}_2\|v\|_0, \quad \forall v \in \mathcal{R}(E). \tag{3.6}$$

The two inequalities can be proved by using the same techniques as Lemma 4.3 in [20]. So we omit it.

Define the similarity operators $\tilde{\mathbb{U}}$ and $\tilde{\mathbb{U}}_h$ by

$$\begin{aligned}\tilde{\mathbb{U}}_h &= J_h^{-1}\mathbb{U}J_h:\ \mathcal{R}(E)\to\mathcal{R}(E),\\ \tilde{\mathbb{U}} &= \mathbb{U}|_{\mathcal{R}(E)}:\ \mathcal{R}(E)\to\mathcal{R}(E).\end{aligned}$$

Next, we present the result which plays an important role in the proof of the main result (Theorem 3.2).

First, we need to consider the following dual problem:

$$\Delta^2\tilde{u} = \chi, \ \text{ in } \Omega, \tag{3.7a}$$

$$\tilde{u} = \frac{\partial\tilde{u}}{\partial\mathbf{n}} = 0, \quad \text{on } \partial\Omega, \tag{3.7b}$$

or

$$\nabla\tilde{u} = \tilde{\boldsymbol{q}} \tag{3.8a}$$

$$\nabla\tilde{\boldsymbol{q}} = \underline{\tilde{\boldsymbol{z}}} \tag{3.8b}$$

$$\nabla\cdot\underline{\tilde{\boldsymbol{z}}} = \tilde{\boldsymbol{w}} \tag{3.8c}$$

$$\nabla\cdot\tilde{\boldsymbol{w}} = \chi, \tag{3.8d}$$

Lemma 3.1. *Assume that* $(u^f, \boldsymbol{q}^f, \underline{\boldsymbol{z}}^f, \boldsymbol{w}^f)$ *and* $(u_h^f, \boldsymbol{q}_h^f, \underline{\boldsymbol{z}}_h^f, \boldsymbol{w}_h^f)$ *are the solutions of the source problem (2.1) and the corresponding discrete problem (2.3), respectively. Then we have*

$$
\begin{aligned}
(\Pi_h^V u^f - u_h^f, \chi) =& (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}^f, \tilde{\underline{\boldsymbol{z}}} - \underline{\boldsymbol{\Pi}}^{RT}\tilde{\underline{\boldsymbol{z}}}) \\
&+ (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT}\boldsymbol{w}^f, \tilde{\boldsymbol{q}} - \underline{\boldsymbol{\Pi}}_h^Q\tilde{\boldsymbol{q}}) \\
&- (\boldsymbol{q}^f - \boldsymbol{\Pi}_h^Q\boldsymbol{q}^f, \tilde{\boldsymbol{w}}^f - \boldsymbol{\Pi}^{RT}\tilde{\boldsymbol{w}}^f) \\
&+ (f - \Pi_h^V f, \tilde{u} - \Pi_h^V \tilde{u}).
\end{aligned} \tag{3.9}
$$

The proof of Lemma 3.1 is put in Appendix.

Theorem 3.2. *Suppose* $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ *is an eigenpair of discrete system with* $\|u_h\|_0 = 1$*, and* $(\lambda, u) \in \mathbb{R} \times L^2(\Omega)$ *is an eigenpair of HHJ eigenvalue system with* $\|u\|_0 = 1$*. Then we have the following a priori error estimates:*

$$
\begin{aligned}
\|u - u_h\|_0 &\leq Ch^{\min\{s,k+1\}}\|u\|_s, \\
|\lambda - \lambda_h| &\leq Ch^{2\min\{s,k+1\}}\|u\|_s.
\end{aligned}
$$

Proof. The arguments proving the convergence of the eigenfunctions are similar as [5] and [19], and since they apply to the mixed method context of the first-order system based on HHJ type with few modifications, we shall not repeat them. We only present the convergence of the eigenvalues.

It follows from the Bauer-Fike theorem [7] that

$$
|\lambda - \lambda_h| \leq C\|\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h\|_0.
$$

Here $\|\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h\|_0$ means the operator norm.

So we must bound $\|\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h\|_0$. In order to estimate it, we consider $(\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h)f$, where $f \in \mathcal{R}(E)$.

$$
\begin{aligned}
C\|(\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h)f\|_0 &\leq \|\mathbb{U}_h(\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h)f\|_0 \quad (\text{by } (3.5)) \\
&= \|E_h\Pi_h^V\mathbb{U}f - E_h\mathbb{U}_h\Pi_h^V f\|_0 \quad (\text{by } E_h\mathbb{U}_h = \mathbb{U}_hE_h) \\
&= \sup_{\nu_h \in \mathcal{R}(E_h)} \frac{(E_h\Pi_h^V\mathbb{U}f - E_h\mathbb{U}_h\Pi_h^V f, \nu_h)}{\|\nu_h\|_0} \\
&= \sup_{\nu \in \mathcal{R}(E)} \frac{(E_h(\Pi_h^V\mathbb{U}f - \mathbb{U}_h\Pi_h^V f), E_h\nu)}{\|E_h\nu_h\|_0} \quad (\text{by } E_h \text{ bijection}) \\
&= \frac{1}{\tilde{C}_1} \sup_{\nu \in \mathcal{R}(E)} \frac{(\Pi_h^V\mathbb{U}f - \mathbb{U}_h\Pi_h^V f, E_h\nu)}{\|\nu_h\|_0} \quad (\text{by } E_h \text{ bijection}).
\end{aligned}
$$

We express $(\Pi_h^V\mathbb{U}f - \mathbb{U}_h\Pi_h^V f, E_h\nu)$ by splitting it into four terms:

$$
\begin{aligned}
(\Pi_h^V\mathbb{U}f - \mathbb{U}_h\Pi_h^V f, E_h\nu) &= (\Pi_h^V u^f - \mathbb{U}_h\Pi_h^V f, E_h\nu) \\
&= (\Pi_h^V u^f - u_h^f, E_h\nu) + (u_h^f - \mathbb{U}_h\Pi_h^V f, E_h\nu) \\
&= (\Pi_h^V u^f - u_h^f, E_h\nu - \nu) + (\Pi_h^V u^f - u_h^f, \nu) \\
&\quad + (u_h^f - \mathbb{U}_h\Pi_h^V f, E_h\nu - \nu) + (u_h^f - \mathbb{U}_h\Pi_h^V f, \nu).
\end{aligned}
$$

We bound the four terms of the above equations. By using the approximation result (Lemma 3.1 in [8]), for the first term, we have

$$|(\Pi_h^V u^f - u_h^f, E_h\nu - \nu)| \leq Ch^{2\min\{s,r+1\}}(\|u\|_s + \|\underline{\boldsymbol{z}}\|_s)\|\nu\|_0. \qquad (3.10)$$

For the third term, by using $u_h^f = \mathbb{U}_h f$, we have

$$\begin{aligned} |(u_h^f - \mathbb{U}_h f \Pi_h^V u, E_h\nu - \nu)| &= |\mathbb{U}_h(f - \Pi_h^V f), E_h\nu - \nu)| \\ &\leq C\|f - \Pi_h^V f\|_0 \|E_h\nu - \nu\|_0 \\ &\leq Ch^{2\min\{s,r+1\}}\|u\|_s\|\nu\|_0. \end{aligned}$$

By using the adjoint of $\mathbb{U}_h$, for the fourth term, we have

$$\begin{aligned} |(u_h^f - \mathbb{U}_h \Pi_h^V f, \nu)| &= |\mathbb{U}_h(f - \Pi_h^V f), \nu)| = |(f - \Pi_h^V f, \mathbb{U}_h\nu)| \\ &= |(f - \Pi^V f, u^\nu - \Pi^V u^\nu)| \leq Ch^{2\min\{s,r+1\}}\|u\|_s\|\nu\|_0. \end{aligned}$$

The remainder is denoted to estimate the second term $(u_h^f - \mathbb{U}_h \Pi_h^V f, \nu)$. By using Lemma 3.1 with $\chi = \nu$, $(\Pi_h^V u^f - u_h^f, \nu)$ can be expressed by the four terms, i.e.,

$$\begin{aligned} (\Pi_h^V u^f - u_h^f, \nu) &= (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}^f, \tilde{\underline{\boldsymbol{z}}} - \underline{\boldsymbol{\Pi}}^{RT}\tilde{\underline{\boldsymbol{z}}}) + (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT}\boldsymbol{w}^f, \tilde{\boldsymbol{q}} - \underline{\boldsymbol{\Pi}}^{RT}\tilde{\boldsymbol{q}}) \\ &\quad -(\boldsymbol{q}^f - \boldsymbol{\Pi}_h^Q\boldsymbol{q}^f, \tilde{\boldsymbol{w}}^f - \boldsymbol{\Pi}^{RT}\tilde{\boldsymbol{w}}^f) + (f - \boldsymbol{\Pi}^{RT}f, \tilde{u} - \boldsymbol{\Pi}^{RT}\tilde{u}). \end{aligned}$$

So we bound these four terms, by the approximation properties, we have

$$\begin{aligned} |(\Pi_h^V u^f - u_h^f, \nu)| &\leq Ch^{2\min\{s,r+1\}}(\|\underline{\boldsymbol{z}}^f\|_s\|\tilde{\underline{\boldsymbol{z}}}^f\|_s + \|\boldsymbol{w}^f\|_s\|\tilde{\boldsymbol{q}}^f\|_s + \|\boldsymbol{q}^f\|_s\|\tilde{\boldsymbol{w}}\|_s + \|u^f\|_s\|\tilde{u}\|_s) \\ &\leq Ch^{2\min\{s,k+1\}}(\|\underline{\boldsymbol{z}}^f\|_s + \|\boldsymbol{w}^f\|_s + \|\boldsymbol{q}^f\|_s + \|u^f\|_s)\|\nu\|_0 \\ &\leq Ch^{2\min\{s,k+1\}}\|f\|_0\|\nu\|_0. \end{aligned}$$

Combining all the intermediate steps, we have

$$\|(\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h)f\|_0 \leq Ch^{2\min\{s,r+1\}}\|f\|_0,$$

i.e.

$$\|\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h\|_0 = \sup_{f\in\mathcal{R}(E)} \frac{\|(\tilde{\mathbb{U}} - \tilde{\mathbb{U}}_h)f\|_0}{\|f\|_0} \leq Ch^{2\min\{s,r+1\}}.$$

□

We end this section by stating the other main theorem of this section, i.e., error estimates of the auxiliary intermediary variables $\boldsymbol{q}$, $\underline{\boldsymbol{z}}$, and $\boldsymbol{w}$. The collection presents the a priori error estimate consequences of these variables' convergence. In the proof, we need the proposition which is listed as follows.

Proposition 3.1 ([17]). *If $\boldsymbol{w}_h \in \boldsymbol{W}_h$ and $\nabla \cdot \boldsymbol{w}_h \in V_h^{k-1}$, then $\boldsymbol{w}_h \in \boldsymbol{W}_h \cap \boldsymbol{Q}_h$.*

Theorem 3.3. *Assume that $(\lambda_{j,h}, u_{j,h}, \boldsymbol{q}_h, \underline{\boldsymbol{z}}_{j,h}, \boldsymbol{w}_{j,h}) \in \mathbb{R} \times V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$ is a solution of (2.7) which converges to eigenvalue $(\lambda_j, u_j, \boldsymbol{q}_j, \underline{\boldsymbol{z}}_j, \boldsymbol{w}_j)$.*

Then we have the following estimates:

$$\|\boldsymbol{q}_j - \boldsymbol{q}_{j,h}\|_{L^2(\Omega)} \le C||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)} + C||\boldsymbol{q}_j - \boldsymbol{\Pi}_h^Q \boldsymbol{q}_j||_{L^2(\Omega)},$$

$$\|\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}\|_{L^2(\Omega)} \le C||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j||_{L^2(\Omega)} + C|\lambda_j - \lambda_{j,h}| + C||u_j - u_{j,h}||_{L^2(\Omega)} + C\Big(\sum_{K\in\mathcal{T}_h} h_k^{2l_k}||\boldsymbol{w}_j - \boldsymbol{\Pi}^{RT}\boldsymbol{w}_j||^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}$$

$$\|\boldsymbol{w}_j - \boldsymbol{w}_{j,h}\|_{L^2(\Omega)} \le C||\nabla\cdot(\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_{j,h} - \underline{\boldsymbol{z}}_{j,h})||_{L^2(\Omega)} + ||\boldsymbol{w}_j - \boldsymbol{\Pi}^{RT}\boldsymbol{w}_{j,h}||_{L^2(\Omega)} + C|\lambda_j - \lambda_{j,h}| + C||u_j - u_{j,h}||_{L^2(K)},$$

where $l_k = 0$ if $k = 0$ and $l_k = 1$ if $k \ge 1$.

Proof. There exists a matrix-valued function $\underline{\boldsymbol{J}} \in \underline{\boldsymbol{H}}(div,\Omega)$ such that

$$\nabla\cdot\underline{\boldsymbol{J}} = \boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_{j,h}$$

with

$$||\underline{\boldsymbol{J}}||_{H^1(\Omega)} \le C||\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_{j,h}||_{L^2(\Omega)}.$$

$$\begin{aligned}
||\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_{j,h}||^2_{L^2(\Omega)} &= (\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_{j,h}, \nabla\cdot\underline{\boldsymbol{J}}) = (\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_{j,h}, \nabla\cdot\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{J}}) && \text{(by (3.1))}\\
&= -(\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}, \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{J}}) \le ||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)}||\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{J}}||_{L^2(\Omega)} && \text{(by (3.4b))}\\
&\le ||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)}(||\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{J}} - \underline{\boldsymbol{J}}||_{L^2(\Omega)} + ||\underline{\boldsymbol{J}}||_{L^2(\Omega)})\\
&\le ||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)}(Ch||\underline{\boldsymbol{J}}||_{H^1(\Omega)} + ||\underline{\boldsymbol{J}}||_{L^2(\Omega)}) && \text{(by (3.3))}\\
&\le C||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)}||\underline{\boldsymbol{J}}||_{H^1(\Omega)}.
\end{aligned}$$

Combining the above two inequalities implies that

$$||\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_{j,h}||_{L^2(\Omega)} \le C||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)}.$$

This proves the first result.

Taking $\underline{\boldsymbol{s}}_h = \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}$ and $\boldsymbol{m}_h = \boldsymbol{\Pi}_h^Q\boldsymbol{p} - \boldsymbol{p}_h$ in (3), we have

$$\begin{aligned}
||\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||^2_{L^2(\Omega)} &= (\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_j, \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}) - (\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{\Pi}\boldsymbol{q}_{j,h}, \nabla\cdot(\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}))\\
&= (\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_j, \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}) - (\boldsymbol{\Pi}^{RT}\boldsymbol{w}_j - \boldsymbol{w}_j, \boldsymbol{\Pi}_h^Q\boldsymbol{p} - \boldsymbol{p}_h)\\
&\quad + (\boldsymbol{\Pi}^{RT}\boldsymbol{w}_j - \boldsymbol{w}_{j,h}, \boldsymbol{\Pi}_h^Q\boldsymbol{p} - \boldsymbol{p}_h)\\
&= (\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_j, \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}) - (\boldsymbol{\Pi}^{RT}\boldsymbol{w} - \boldsymbol{w}, \boldsymbol{\Pi}_h^Q\boldsymbol{p} - \boldsymbol{p}_h)\\
&\quad - (\Pi_h^V u_j - u_{j,h}, \nabla\cdot(\boldsymbol{\Pi}_h^{RT}\boldsymbol{w}_j - \boldsymbol{w}_{j,h})) + (\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_j, \boldsymbol{\Pi}_h^{RT}\boldsymbol{w}_j - \boldsymbol{w}_{j,h})\\
&= (\underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_j, \underline{\boldsymbol{\Pi}}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}) - (\boldsymbol{\Pi}^{RT}\boldsymbol{w}_j - \boldsymbol{w}_j, \boldsymbol{\Pi}_h^Q\boldsymbol{p} - \boldsymbol{p}_h)\\
&\quad + (\boldsymbol{\Pi}_h^Q\boldsymbol{q}_j - \boldsymbol{q}_j, \boldsymbol{\Pi}_h^{RT}\boldsymbol{w}_j - \boldsymbol{w}_{j,h})) - (\lambda u_j - \lambda_h u_{j,h}, \Pi_h^V u - u_h)
\end{aligned}$$

by (3.4a) with $\boldsymbol{p}_h = \boldsymbol{\Pi}^{RT}\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}$ and (3.4d) with $v_h = \boldsymbol{\Pi}_h^V u_j - u_{j,h}$.

Lastly, we prove the last result. In order to prove it, we introduce the following source problem with the source term $f = \lambda u$: find $(\widetilde{u}_h, \widetilde{\boldsymbol{q}}_h, \widetilde{\underline{\boldsymbol{z}}}_h, \widetilde{\boldsymbol{w}}_h) \in \mathbb{R}\times V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$ that satisfy

$$(\widetilde{\boldsymbol{q}}_h, \boldsymbol{p}_h) + (\widetilde{u}_h, \nabla\cdot\boldsymbol{p}_h) = 0, \tag{3.11a}$$

$$(\widetilde{\underline{\boldsymbol{z}}}_h, \underline{\boldsymbol{s}}_h) + (\widetilde{\boldsymbol{q}}_h, \nabla \cdot \underline{\boldsymbol{s}}_h) = 0, \tag{3.11b}$$
$$-(\widetilde{\boldsymbol{w}}_h, \boldsymbol{m}_h) + (\nabla \cdot \widetilde{\underline{\boldsymbol{z}}}_h, \boldsymbol{m}_h) = 0, \tag{3.11c}$$
$$(\nabla \cdot \widetilde{\boldsymbol{w}}_h, v_h) = (\lambda u, v_h), \tag{3.11d}$$

Subtracting (3.11d) from (2.6d), we have

$$(\nabla \cdot (\boldsymbol{w}_j - \widetilde{\boldsymbol{w}}_{j,h}), v_h)) = 0, \quad \forall v_h \in V_h.$$

By using the definition (3.1) of the projection $\boldsymbol{\Pi}^{RT}$, we have

$$(\nabla \cdot (\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \widetilde{\boldsymbol{w}}_{j,h}), v_h)) = 0, \quad \forall v_h \in V_h.$$

It follows from the above equation and $\nabla \cdot (\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \widetilde{\boldsymbol{w}}_{j,h}) \in V_h$ that

$$\nabla \cdot (\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \widetilde{\boldsymbol{w}}_{j,h}) = 0.$$

We obtain

$$\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \widetilde{\boldsymbol{w}}_{j,h} \in \boldsymbol{W}_h \cap \boldsymbol{Q}_h. \tag{3.12}$$

by Proposition 3.1. Furthermore, by (3.12) of Theorem 3.5 in [8], we can similarly obtain that

$$||\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \widetilde{\boldsymbol{w}}_{j,h}||_{L^2(K)} \leq C||\nabla \cdot (\underline{\boldsymbol{\Pi}}^{RT} \underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h})||_{L^2(K)} + C||\boldsymbol{w}_j - \boldsymbol{\Pi}^{RT} \boldsymbol{w}_j||_{L^2(K)}. \tag{3.13}$$

Subtracting (3.11d) from (2.7d), we have

$$(\nabla \cdot (\boldsymbol{w}_{j,h} - \widetilde{\boldsymbol{w}}_{j,h}), v_h)) = (\lambda_{j,h} u_{j,h} - \lambda_j u_j, v_h), \quad \forall v_h \in V_h.$$

Taking $v_h = \nabla \cdot (\boldsymbol{w}_{j,h} - \widetilde{\boldsymbol{w}}_{j,h})$ in the above equation, we have

$$||\nabla \cdot (\boldsymbol{w}_{j,h} - \widetilde{\boldsymbol{w}}_{j,h})||_{L^2(\Omega)} \leq C|\lambda_j - \lambda_{j,h})| + C||u_j - u_{j,h}||_{L^2(\Omega)}. \tag{3.14}$$

In order to obtain the last estimate we use (3.12) and have

$$\begin{aligned} ||\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \boldsymbol{w}_{j,h}||^2_{L^2(K)} &= (\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \boldsymbol{w}_{j,h}, \boldsymbol{\Pi}^{RT} \boldsymbol{w}_{j,h} - \widetilde{\boldsymbol{w}}_{j,h}) + (\boldsymbol{\Pi}^{RT} \boldsymbol{w} - \boldsymbol{w}_h, \widetilde{\boldsymbol{w}}_{j,h} - \boldsymbol{w}_{j,h}) \\ &= (\nabla \cdot (\underline{\boldsymbol{\Pi}}^{RT} \underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}), \boldsymbol{\Pi}^{RT} \boldsymbol{w}_{j,h} - \widetilde{\boldsymbol{w}}_{j,h}) + (\boldsymbol{\Pi}^{RT} \boldsymbol{w} - \boldsymbol{w}_h, \widetilde{\boldsymbol{w}}_{j,h} - \boldsymbol{w}_{j,h}) \end{aligned}$$

by choosing $\boldsymbol{m}_h = \boldsymbol{\Pi}^{RT} \boldsymbol{w} - \widetilde{\boldsymbol{w}}_h$. By Cauchy-Schwartz inequality, (3.13) and (3.14), we have

$$\begin{aligned} ||\boldsymbol{\Pi}^{RT} \boldsymbol{w}_j - \boldsymbol{w}_{j,h}||^2_{L^2(K)} &\leq C||\nabla \cdot (\underline{\boldsymbol{\Pi}}^{RT} \underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h})||_{L^2(K)} + C||\boldsymbol{w}_j - \boldsymbol{\Pi}^{RT} \boldsymbol{w}_j||_{L^2(K)} \\ &\quad + C|\lambda_j - \lambda_{j,h}| + C||u_j - u_{j,h}||_{L^2(K)}. \end{aligned}$$

This proves the last result. □

The following corollary easily follows from the above theorem.

Corollary 3.1. *Assume that* $(\boldsymbol{q}, \underline{\boldsymbol{z}}, \boldsymbol{w})$ *and* $(\boldsymbol{q}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{w}_h)$ *are the solutions of (2.6) and (2.7). Then if* $k \geq 1$, *we have*

$$\begin{aligned} ||\boldsymbol{q}_j - \boldsymbol{q}_{j,h}||_{L^2(\Omega)} &\leq Ch^{k+1} ||\underline{\boldsymbol{z}}_j||_{H^{k+1}(\Omega)}, \\ ||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)} &\leq Ch^{k+1} ||\underline{\boldsymbol{z}}_j||_{H^{k+1}(\Omega)}, \\ ||\boldsymbol{w}_j - \boldsymbol{w}_{j,h}||_{L^2(\Omega)} &\leq Ch^{k} ||\underline{\boldsymbol{z}}_j||_{H^{k+1}(\Omega)}, \end{aligned}$$

if $k = 0$, *we have*

$$\begin{aligned} ||\boldsymbol{q}_j - \boldsymbol{q}_{j,h}||_{L^2(\Omega)} &\leq Ch ||\underline{\boldsymbol{z}}_j||_{H^{1}(\Omega)}, \\ ||\underline{\boldsymbol{z}}_j - \underline{\boldsymbol{z}}_{j,h}||_{L^2(\Omega)} &\leq Ch ||\underline{\boldsymbol{z}}_j||_{H^{1}(\Omega)}. \end{aligned}$$

4 Numerical experiments

In this section, some numerical examples are presented to validate the result of our theoretical analysis in the previous sections. First, we consider two smooth model eigenproblems on a square and hexagon domain respectively. The spectral approximations using the mixed method discretization are computed. Then, we consider a corner singularity model eigenproblem on an L-shaped domain and we investigate the performance of our method with uniform meshes. It should be noted that the errors in the text refer to relative errors. All the computations have been performed by using the finite element package FreeFem++ [21].

4.1 Biharmonic eigenvalue problem on unit square

We first consider biharmonic eigenvalue problem based on the first-order system on unit square $\Omega = (0,1) \times (0,1)$. First, we obtain an initial mesh by subdividing the computing domain Ω into shape-regular triangles. Figure 1 shows this initial mesh ($h_1 = 1/8$). Other nested meshes are produced by regular refinements.

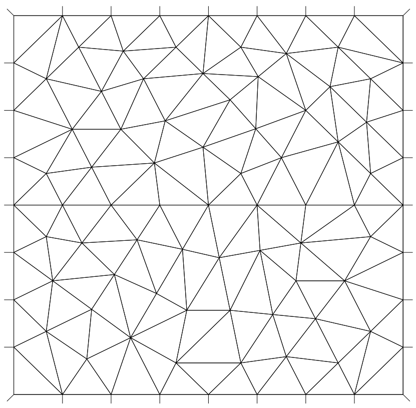

Figure 1: The initial mesh for Example 4.1

Since the exact eigenvalue is unknown, we use an accurate enough approximation

$$\lambda_1 = 1294.9339795917, \ \lambda_2 = 5386.6565607533$$

given by the extrapolation method (see, e.g. [29]) as the first two exact eigenvalues to investigate the errors. Since the square domain is convex, the eigenfunctions are enough smooth, i.e., $u \in H^3(\Omega)$. So the convergence rates should be limited only by the degrees ($k = 0, 1$) of the approximating polynomials. We solve first-order system (2.7) in each of these meshes by using $k = 0$ and $k = 1$ finite element system, respectively. The results obtained are collected below.

Figure 2 gives the corresponding numerical results for the first two eigenvalues. We see that the approximate eigenvalues λ_h converge to the exact values at the optimal rate of $O(h^{2k+2})$. This is a verification of the theoretical consequence of Theorem 3.2. Table 1 shows the approximate of the first two eigenvalues by solving the first-order system. Otherwise, from Table 1, we can find the exact eigenvalue approximated by the numerical eigenvalue below. This shows that what we get is effective lower bounds.

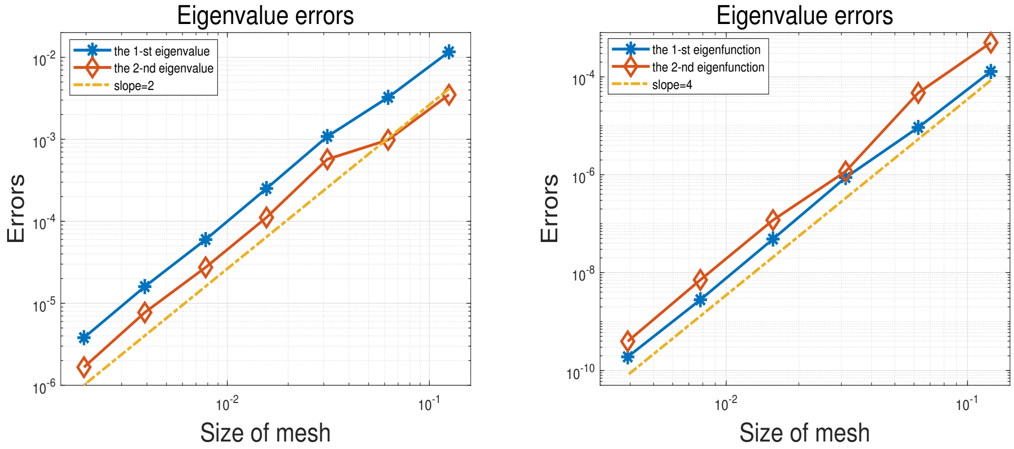

Figure 2: The errors of the first two eigenvalues with $k = 0$ (left) and $k = 1$ (right) on unit square for the initial mesh in Figure 1.

Table 1: Biharmonic eigenvalue problem on unit square

	$k=0$		$k=1$	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
1/8	1279.8315553	5367.8139340	1294.7671242	5383.9713287
1/16	1290.7265788	5381.3561084	1294.9220096	5386.5412478
1/32	1293.5326319	5383.5828748	1294.9328449	5386.6502777
1/64	1294.6089833	5386.0567480	1294.9339167	5386.6559598
1/128	1294.8564677	5386.5086483	1294.9339760	5386.6565237
1/256	1294.9132879	5386.6149540	1294.9339793	5386.6565588
1/512	1294.9290543	5386.6476059		
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

4.2 Biharmonic eigenvalue problem on hexagon

In the second example, we consider that the domain Ω is a regular hexagonal region with a side length of 1 and the center is the origin of coordinates. The initial mesh has been shown in Figure 3 ($h_1 = 1/4$). We also use regular refinement to obtain nested meshes to construct the corresponding finite element space.

Since the exact eigenvalue is unknown, we use an accurate enough approximation

$$\lambda_1 = 163.597568158247, \ \lambda_2 = 703.328903370623$$

given by the extrapolation method (see, e.g. [29]) as the first two exact eigenvalues to investigate the errors. Since the square domain is convex, the eigenfunctions are enough smooth, i.e., $u \in H^3(\Omega)$. So the convergence rates should be limited only by the degrees ($k = 0, 1$) of the approximating polynomials. We solve the first-order system (2.7) in each of these meshes by using $k = 0$ and $k = 1$ finite element system, respectively. The results obtained are collected below.

Figure 4 gives the corresponding numerical results for the first two eigenvalues. From Figure 4, we see that the approximate eigenvalues λ_h converge to the exact values at the optimal rate of $O(h^{2k+2})$. Table 2 shows the eigenvalue approximations

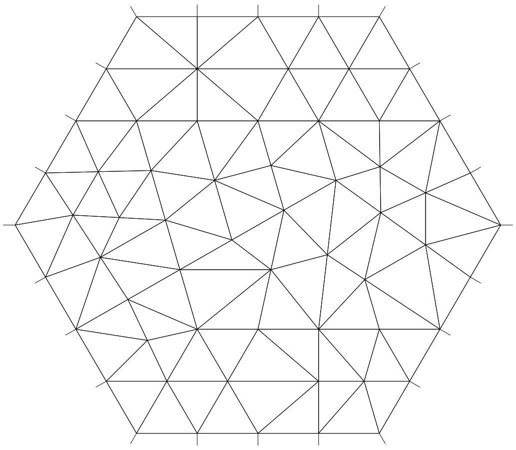

Figure 3: The initial mesh for Example 4.2

of the first 2 eigenvalues by solving the first-order system. From Table 2, we can also find the numerical approximations are lower bounds of the exact eigenvalues.

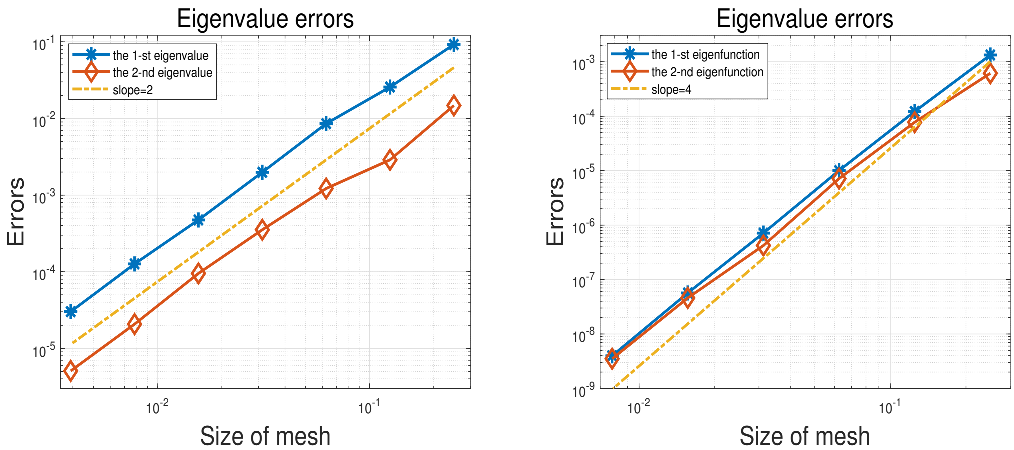

Figure 4: The errors of the first two eigenvalues with $k = 0$ (left) and $k = 1$ (right) on regular hexagon for the initial mesh in Figure 3.

4.3 Biharmonic eigenvalue problem on L-shape domain

In the last example, we consider the HHJ eigenvalue system defined on the L-shape domain $\Omega = [-1/2, 1/2]^2/(0, 1/2) \times (-1/2, 0)$. The re-entrant corner on Ω causes the singularity of the eigenfunctions. Consequently, the convergence order for the first and second eigenvalue approximation is not optimal. However, as a numerical example, we also show the effectiveness of our method. Figure 5 shows the initial mesh.

Since the exact eigenvalue is unknown, we use the accurate enough approximation

$$\lambda_1 = 6700.09875796623, \ \lambda_2 = 11054.4911180150$$

Table 2: Biharmonic eigenvalue problem on regular hexagon

	$k=0$		$k=1$	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
1/4	160.9458396	692.9333602	163.3798197	703.1951012
1/8	162.6812267	701.2997887	163.5776036	703.2744127
1/16	163.3016611	702.4711335	163.5959258	703.3238815
1/32	163.5226051	703.0807014	163.5974516	703.3284793
1/64	163.5776298	703.2620556	163.5975589	703.3288689
1/128	163.5926992	703.3143307	163.5975675	703.3289008
1/256	163.5963661	703.3253328		
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

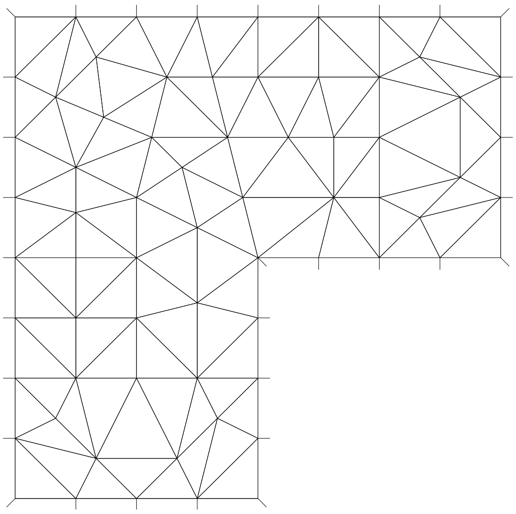

Figure 5: The initial mesh for Example 4.3

given by the extrapolation method (see, e.g. [29]) as the first two exact eigenvalues to investigate the errors. Here we also solve HHJ eigenvalue system (2.7) by using $k=0$ and $k=1$ finite element system, respectively.

Figure 6 gives the corresponding numerical results for the first two eigenvalues. From Figure 6, we can obtain the optimal error estimates that meets Theorem 3.2. Table 3 shows the eigenvalue approximations of the first 2 eigenvalues by solving the first-order system. From Table 3, we can find the numerical approximations are indeed lower bounds of the exact eigenvalues.

Remark 4.1. *From the above three numerical examples, we can indeed find that the eigenvalues obtained by solving the HHJ eigenvalue system are effective lower bounds. However, at present, we are unable to prove this conclusion.*

5 Other effective finite element spaces

Section 3 and Section 4 show that our method approximations eigenvalue λ and eigenfunction u and $\boldsymbol{q}$ with optimal order and $\boldsymbol{w}$ in a sub-optimal way. In this

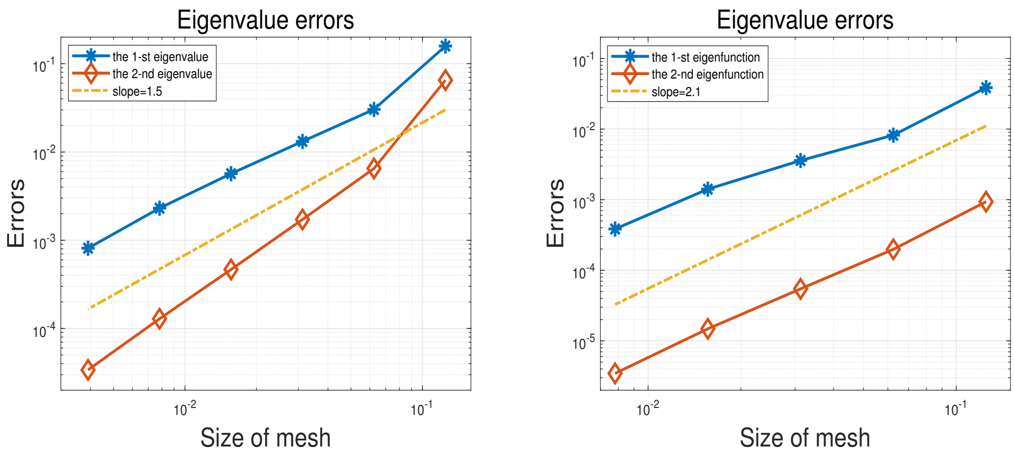

Figure 6: The errors of the first two eigenvalues with $k = 0$ (left) and $k = 1$ (right) on L-shape domain for the initial mesh in Figure 5.

Table 3: Biharmonic eigenvalue problem on L-shape domain

	$k=0$		$k=1$	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
1/8	5636.5006655	10334.1889032	6442.0857195	11044.1485401
1/16	6496.5069009	10982.6026156	6645.2214258	11052.3034939
1/32	6611.7862029	11035.4824786	6676.1086781	11053.8882698
1/64	6662.0614415	11049.3382099	6690.6691688	11054.3274377
1/128	6684.6224134	11053.0660625	6697.5268563	11054.4528623
1/256	6694.6542454	11054.1154644		
Trend	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

section, we present the other two groups of spaces in which computing eigenvalues is also very effective, and the upper bound of eigenvalues can be obtained.

5.1 Abstract finite element spaces

These spaces are interesting because the source problem (2.1) is uniquely solvable. The uniquely solvable consequence is easily proved, which is similar to Theorem 2.2 in [8]. So far we focused on the case when the same spaces are employed to approximate the components of the second derivative and the third derivative, the eigenfunction and its derivative are approximated by the same polynomials. Before describing the new spaces we introduce the following concept which the new spaces need to satisfy.

Definition 5.1. *The FE space pair $\boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h$ is a stable pair for the vector-valued Poisson problem if there exists a constant C such that for any $\boldsymbol{q} \in \boldsymbol{Q}_h$ there exists $\underline{\boldsymbol{z}} \in \underline{\boldsymbol{Z}}_h$ such that*

$$\nabla \cdot \underline{\boldsymbol{z}} = \boldsymbol{q},$$

with

$$\|\underline{\boldsymbol{z}}\|_{H(div,\Omega)} \leq C\|\boldsymbol{q}\|_{L^2(\Omega)}.$$

Similarly, $V_h \times \boldsymbol{W}_h$ is a stable pair for Poisson's problem if it is a row of a stable pair for the vector-valued Poisson problem.

From the above definition and discussion, we observe that Q_h nearly determines the scale of the global system. Then we fix Q_h as a set of k-degree polynomials. To hold a stable discrete space pair, we only vary the space of $\underline{\boldsymbol{Z}}_h$ and enlarge it. So $\underline{\boldsymbol{Z}}_h$ is changed to the polynomial space with degree $k+1$, i.e.,

Case 1:

$$\begin{aligned} V_h &= \{v \in V : \ v|_K \in \mathcal{P}^k(K) \text{ for all } K \in \mathcal{T}_h\}, && (5.1a)\\ \boldsymbol{Q}_h &= \{\boldsymbol{p} \in \boldsymbol{Q} : \ \boldsymbol{p}|_K \in (\mathcal{P}^k(K))^d \text{ for all } K \in \mathcal{T}_h\}, && (5.1b)\\ \underline{\boldsymbol{Z}}_h &= \{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}} : \ \underline{\boldsymbol{s}} \in (C^0(\Omega))^{d\times d}, \underline{\boldsymbol{s}}|_K \in (\mathcal{P}^{k+1}(K))^{d\times d} \text{ for all } K \in \mathcal{T}_h\}, && (5.1c)\\ \boldsymbol{W}_h &= \{\boldsymbol{m} \in \boldsymbol{W} : \ \boldsymbol{m}|_K \in \mathrm{RT}^k(K) \text{ for all } K \in \mathcal{T}_h\}. && (5.1d) \end{aligned}$$

It is easy to verify that $\boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h$ and $V_h \times \boldsymbol{W}_h$ are stable pairs. Indeed, $\boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h$ can be used for solving vector-valued Poisson's problem and is called the Brezzi-Marini-Douglas spaces. $V_h \times \boldsymbol{W}_h$ can be used for solving scalar Poisson's problem and is called the Raviart-Thomas spaces.

Based on case 1, we can appropriately reduce the size of the global system. So we can adjust the size of the two spaces $\boldsymbol{Q}_h$ and V_h, that is, space $\boldsymbol{Q}_h$ is adjusted to a k-degree polynomial space, then space V_h is naturally a $k-1$ degree polynomial space. We obtain the spaces of V_h, $\boldsymbol{W}_h$, and $\underline{\boldsymbol{Z}}_h$ as follows,

Case 2:

$$\begin{aligned} V_h &= \{v \in V : \ v|_K \in \mathcal{P}^k(K) \text{ for all } K \in \mathcal{T}_h\}, && (5.2a)\\ \boldsymbol{Q}_h &= \{\boldsymbol{p} \in \boldsymbol{Q} : \ \boldsymbol{p}|_K \in (\mathcal{P}^{k+1}(K))^d \text{ for all } K \in \mathcal{T}_h\}, && (5.2b)\\ \underline{\boldsymbol{Z}}_h &= \{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}} : \ \underline{\boldsymbol{s}} \in (C^0(\Omega))^{d\times d}, \underline{\boldsymbol{s}}|_K \in (\mathcal{P}^{k+2}(K))^{d\times d} \text{ for all } K \in \mathcal{T}_h\}, && (5.2c)\\ \boldsymbol{W}_h &= \{\boldsymbol{m} \in \boldsymbol{W} : \ \boldsymbol{m}|_K \in (\mathcal{P}^{k+1}(K))^d \text{ for all } K \in \mathcal{T}_h\}. && (5.2d) \end{aligned}$$

It is easy to verify that $\boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h$ and $V_h \times \boldsymbol{W}_h$ are stable pairs. Indeed, $\boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h$ can be used for solving vector-valued Poisson's problem and is called the Brezzi-Marini-Douglas spaces. $V_h \times \boldsymbol{W}_h$ can be used for solving scalar Poisson's problem and is called the Brezzi-Marini-Douglas spaces.

We can now follow similar techniques in [8] to obtain results analogous to the identity (3.9) in Lemma 3.1 in both these cases. On the other hand, rigorous proofs of the result are developed for these cases involving the matrix-valued BDM projection (d copies of BDM projection: one for each row) onto the space $\underline{\boldsymbol{Z}}_h$, instead of the projection, $\underline{\boldsymbol{\Pi}}^{RT}$. It follows then that we get the convergence rates for the eigenvalue. In fact, (3.9) will hold for these new spaces and therefore we will get the error estimates for eigenvalue as well. Finally, one can also prove optimal error estimates for the other variables. The numerical results for different cases are shown in the following figures and tables.

5.2 Two specific FE spaces for the biharmonic eigenvalue problem

We also consider biharmonic eigenvalue problem based on the first-order system on unit square $\Omega = (0, 1) \times (0, 1)$. First, we decompose the computing domain Ω into shape-regular triangles. Figure 7 shows this initial meshes ($h_1 = 1/4$). Other nested meshes are produced by regular refinements.

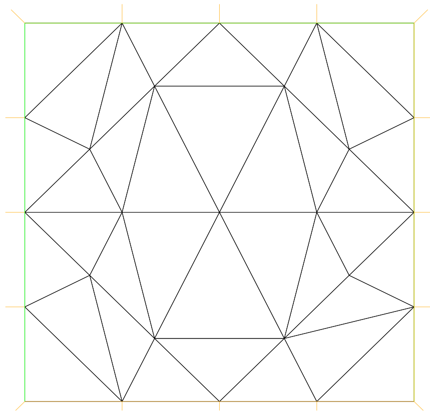

Figure 7: The initial mesh for Example 5.2

We solve the first-order system (2.7) by using first FE group (5.1) and second FE group (5.2) with $k = 1$, respectively. The corresponding numerical results are shown in Figure 8 which also exhibits the optimal convergence rate. Table 4 shows the eigenvalue approximations of the first 2 eigenvalues by solving the first-order system. From Table 4, we can find the numerical eigenvalues approximate the exact eigenvalues below. This shows that what we get is effective upper bounds.

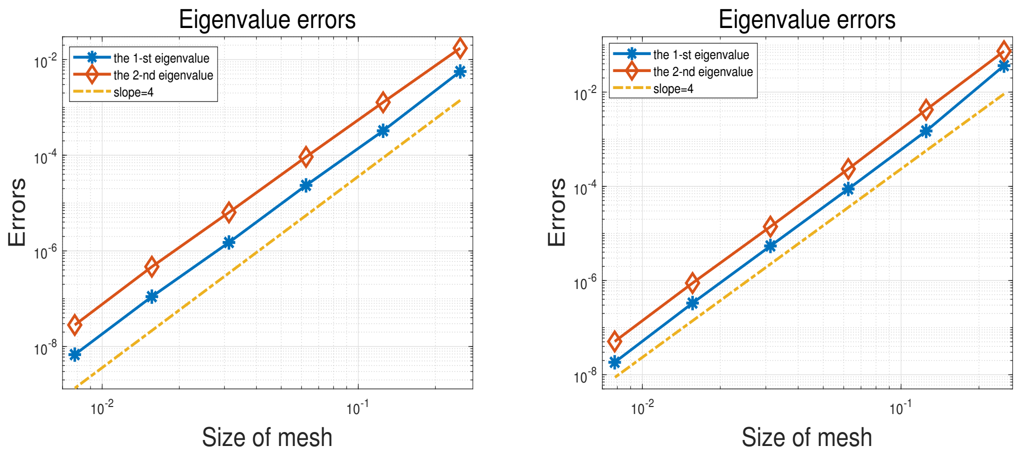

Figure 8: The errors of the first two eigenvalues by 1st FE group (5.1) (left) and 2nd FE group (5.2) (right) on unit square with $k = 1$.

Remark 5.1. *From the above numerical example, we can find that the eigenvalues obtained by solving the HHJ eigenvalue system are effective upper bounds. However, at present, we are unable to prove this conclusion.*

Table 4: Biharmonic eigenvalue problem on unit square

	1st FE group (5.1) with $k = 1$		2nd FE group (5.2) with $k = 1$	
h	$\lambda_{1,h}$	$\lambda_{2,h}$	$\lambda_{1,h}$	$\lambda_{2,h}$
1/4	1302.2386966	5480.0091932	1342.3203813	5785.8620433
1/8	1295.3541917	5393.5495941	1296.8731606	5409.4537648
1/16	1294.9642912	5387.1536319	1295.0478874	5387.9185701
1/32	1294.9359196	5386.6908683	1294.9409933	5386.7316576
1/64	1294.9341227	5386.6590695	1294.9344093	5386.6613214
1/128	1294.9339884	5386.6567126	1294.9340033	5386.6568345
Trend	↘	↘	↘	↘

6 Conclusion

In this paper, a new type of mixed method is designed to solve the fourth-order biharmonic eigenvalue problems based on the mixed first-order system. The higher-order eigenvalue problem is transformed into a mixed first-order system containing four first-order equations. We have proved the optimal error estimates. Three numerical experiments validate the optimality and show that this the method is efficient for many different domains. For good measure, we also find that this method can effectively obtain the lower bounds of eigenvalues.

Acknowledgements

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Appendix. The proof of Lemma 3.1

In order to get the best possible estimates, we assume the following elliptic regularity result:

$$\|u^f\|_4 \leq C\|f\|_0 \tag{A.1}$$

Then we have error equations as follows

$$(\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{p}_h) + (u^f - u_h^f, \nabla \cdot \boldsymbol{p}_h) = 0,, \tag{A.2a}$$

$$(\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f, \underline{\boldsymbol{s}}_h) + (\boldsymbol{q}^f - \boldsymbol{q}_h^f, \nabla \cdot \underline{\boldsymbol{s}}_h) = 0,, \tag{A.2b}$$

$$-(\boldsymbol{w}^f - \boldsymbol{w}_h^f, \boldsymbol{m}_h) + (\nabla \cdot (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f), \boldsymbol{m}_h) = 0, \tag{A.2c}$$

$$(\nabla \cdot (\boldsymbol{w}^f - \boldsymbol{w}_h^f), v_h) = 0, \tag{A.2d}$$

for all $(v_h, \boldsymbol{p}_h, \underline{\boldsymbol{s}}_h, \boldsymbol{m}_h) \in V_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{W}_h$.

Now we begin to prove Lemma 3.1.

Proof. Using the dual equations (A.2) of the source problem (2.1), we have

$$\begin{aligned}(\Pi_h^V u - u_h, \chi) &= (\Pi_h^V u - u_h, \nabla \cdot \tilde{\boldsymbol{w}}) \qquad \text{(by dual equation (3.8d))} \\ &= (\Pi_h^V u - u_h, \nabla \cdot \boldsymbol{\Pi}^{RT} \tilde{\boldsymbol{w}}) \qquad \text{(by (3.1a))} \\ &= -(\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{\Pi}^{RT} \tilde{\boldsymbol{w}}) \qquad \text{(by error equation (A.2a))} \\ &= -(\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{\Pi}^{RT} \tilde{\boldsymbol{w}} - \tilde{\boldsymbol{w}}) - (\boldsymbol{q}^f - \boldsymbol{q}_h^f, \tilde{\boldsymbol{w}} - \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{w}}) - (\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{w}}).\end{aligned}$$

We express the last term $(\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{w}})$. By the dual equation (3.8c), we have

$$\begin{aligned}(\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{w}}) &= (\boldsymbol{q}^f - \boldsymbol{q}_h^f, \boldsymbol{\Pi}_h^Q \nabla \cdot \tilde{\underline{\boldsymbol{z}}}) \qquad \text{(by dual equation (3.8c))} \\ &= -(\boldsymbol{q}^f - \boldsymbol{q}_h^f, \nabla \cdot \underline{\boldsymbol{\Pi}}^{RT} \tilde{\underline{\boldsymbol{z}}}) \qquad \text{(by the commutative property } \boldsymbol{\Pi}_h^Q) \\ &= (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f, \underline{\boldsymbol{\Pi}}^{RT} \tilde{\underline{\boldsymbol{z}}} - \tilde{\underline{\boldsymbol{z}}}) + (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f, \tilde{\underline{\boldsymbol{z}}}). \qquad \text{(by error equation (A.2b))}\end{aligned}$$

Furthermore, using the integration by parts, we have

$$\begin{aligned}(\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f, \tilde{\underline{\boldsymbol{z}}}) &= (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f, \nabla \tilde{\boldsymbol{q}}) = (\nabla \cdot (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f), \tilde{\boldsymbol{q}}) \qquad \text{(by } \tilde{\boldsymbol{q}}|_{\partial\Omega} = 0) \\ &= (\nabla \cdot (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f), \tilde{\boldsymbol{q}} - \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) + (\nabla \cdot (\underline{\boldsymbol{z}}^f - \underline{\boldsymbol{z}}_h^f), \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) \\ &= (\boldsymbol{w}^f, \tilde{\boldsymbol{q}} - \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) + (\boldsymbol{w}^f - \boldsymbol{w}_h^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) \qquad \text{(by error equation (A.2b) and (2.1c))} \\ &= (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \tilde{\boldsymbol{q}} - \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) + (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) + (\boldsymbol{\Pi}^{RT} \boldsymbol{w}^f - \boldsymbol{w}_h^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}).\end{aligned}$$

Next, we express the second term and the third term

$$\begin{aligned}(\boldsymbol{w}^f - \boldsymbol{\Pi}_h^Q \boldsymbol{w}^f, \boldsymbol{\Pi}^{RT} \tilde{\boldsymbol{q}}) &= (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}^f - \tilde{\boldsymbol{q}}) + (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \nabla \tilde{u}) \\ &= (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}) + (\nabla \cdot \boldsymbol{w}^f - \nabla \cdot \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \tilde{u}) \\ &= (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}) + (f - \Pi_h^V \nabla \cdot \boldsymbol{w}^f, \tilde{u}) \\ &= (\boldsymbol{w}^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}} - \tilde{\boldsymbol{q}}) + (f - \Pi_h^V f, \tilde{u} - \Pi_h^V \tilde{u}).\end{aligned}$$

The last term vanishes. In fact, it follows from $\boldsymbol{w}_h^f - \boldsymbol{\Pi}^{RT} \boldsymbol{w}^f \in \boldsymbol{Q}_h \cap \boldsymbol{W}_h$ and $\nabla \cdot (\underline{\boldsymbol{\Pi}}^{RT} \boldsymbol{w}^f - \boldsymbol{w}_h^f) = 0$ that

$$\begin{aligned}(\boldsymbol{\Pi}^{RT} \boldsymbol{w}^f - \boldsymbol{w}_h^f, \boldsymbol{\Pi}_h^Q \tilde{\boldsymbol{q}}) &= (\boldsymbol{\Pi}^{RT} \boldsymbol{w}^f - \boldsymbol{w}_h^f, \tilde{\boldsymbol{q}}^f) = (\boldsymbol{\Pi}^{RT} \boldsymbol{w}^f - \boldsymbol{w}_h^f, \nabla \tilde{u}) \\ &= (\nabla \cdot (\boldsymbol{\Pi}^{RT} \boldsymbol{w}^f - \boldsymbol{w}_h^f), \tilde{u}) = 0\end{aligned}$$

by the integration by parts, $\tilde{u}|_{\partial\Omega} = 0$ and (A.2d). Combining the all above steps implies the desired result. □

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