AN EMBEDDED EXPONENTIAL-TYPE LOW-REGULARITY INTEGRATOR FOR MKDV EQUATION

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Abstract. In this paper, we propose an embedded low-regularity integrator (ELRI) under a new framework for solving the modified KdV (mKdV) equation under rough data. Different from the previous work [61], the present ELRI scheme is constructed based on an approximation of a scaled Schrödinger operator and a new strategy of iterative regularizing through the inverse Miura transform. Moreover, the ELRI scheme is explicitly defined in the physical space, and it is efficient under the Fourier pseudo-spectral discretization. By rigorous error analysis, we show that ELRI achieves first order accuracy by requiring the boundedness of one additional spatial derivative of the solution. Numerical results are presented to show the accuracy and efficiency of ELRI.

Keywords: mKdV equation, rough data, exponential-type integrator, Miura transform, iterative regularizing, error estimate

AMS Subject Classification: 65L05, 65L20, 65L70, 65M12, 65M15.

1. Introduction

The following type of Korteweg-de Vries equation is of fundamental importance in mathematical studies:

$$\partial_t u + \partial_x^3 u = \mu \partial_x (F(u)), \quad t > 0,\ x \in \mathbb{T}, \tag{1.1}$$

where $\mathbb{T} = (0, 2\pi)$ with periodic boundary condition, $u = u(t, x) : \mathbb{R}^+ \times \mathbb{T} \to \mathbb{R}$ is the unknown, $\mu \in \mathbb{R}$ is a given parameter, and $F(u)$ is a given polynomial of degree k. Typically, one would consider the power function $F(u) = u^k$ for some integer $k > 0$. In this case, $\mu > 0$ is referred to the defocusing and $\mu < 0$ is referred to the focusing. When $k = 2$, (1.1) is known as the classical KdV, and when $k = 3$, it is called the modified KdV (mKdV for short) equation. The cases $k \geq 4$ are classified as the generalized KdV equations.

The Cauchy problem of (1.1) on the torus has been extensively studied. In [5], Bourgain proved that the Cauchy problem of the KdV equation is globally well-posed in $H^s(\mathbb{T})$ for $s \geq 0$, whereas the mKdV equation is globally well-posed for $s \geq 1$ and locally well-posed for $s \geq \frac{1}{2}$. Kenig, Ponce and Vega [28] improved the local well-posedness of KdV to $s \geq -\frac{1}{2}$, and these local well-posedness results of KdV and mKdV were then extended as the global well-posedness by Colliander, Keel, Takaoka, Staffilani and Tao [10]. Later, Kappeler and Topalov [22, 23] showed that the KdV and the defocusing mKdV equations are globally well-posed for $s \geq -1$ and $s \geq 0$, respectively. The cases $k \geq 3$, the local and global well-posedess were studied by Colliander, Keel, Takaoka, Staffilani and Tao [11] and Bao, Wu [1].

Numerically, when the solution of (1.1) is smooth enough, extensive studies have also been carried out in the literature. Many different kinds of numerical methods have been proposed for solving (1.1), e.g., [2, 3, 4, 12, 15, 19, 20, 21, 25, 26, 27, 29, 34, 37, 38, 43, 49, 51, 52, 53, 54, 57, 62, 63]. We refer the readers to [61] for a more detailed review. Some recent attentions and efforts have been made to consider the numerical solution of some important dispersive models under rough data [30, 31, 33, 39, 40, 42, 44, 48, 59, 60, 61], where the roughness could be introduced in reality by randomness or measurements [14]. The goal is to raise the order of the temporal accuracy in H^γ-norm for solutions from $H^{\gamma+\alpha}$-space, and meanwhile to reduce γ and α. The index γ describes how rough the solution can be, and the index α denotes the order of spatial derivatives of the solution that have been lost essentially in the numerical approximation.

For the classical KdV equation, i.e., (1.1) with $F(u) = u^2$, [18, 60] proposed and analyzed a class of low-regularity integrators which can reach the first and the second order accuracy by requiring the boundedness of two and respectively four additional spatial derivatives of the solution. To further bring down such regularity requirements, in our recent work [61], a new class of *embedded low-regularity integrators* have been proposed for the classical KdV equation, which reduces the requirements for the first and the second order accuracy to only one and respectively three additional bounded derivatives. These low-regularity numerical methods from [18, 60, 61] are proposed through Fourier frequency analysis, and they strongly rely on the interaction and resonance structure from the quadratic nonlinearity in the classical KdV equation. However, for higher order interactions, such as the cubic nonlinearity in the mKdV equation, i.e., (1.1) with $F(u) = u^3$, the resonance structure would become much more complicated, which makes it far from straightforward to extend the existing low-regularity integrators. Let us mention the general framework introduced in [8, 46] for low-regularity integration on general models, but for a precise model, there could still be much room of improvement.

In this paper, we shall focus on the numerical solution of the following defocusing mKdV equation on the torus with periodic boundary condition under rough data:

$$\begin{cases} \partial_t u + \partial_x^3 u = 2\partial_x(u^3), \quad t > 0, \ x \in \mathbb{T}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}, \end{cases} \tag{1.2}$$

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial t^3} = 0.$$

$$\phi(t,x) = \frac{c}{2} \sec h^2 \left[\frac{1}{2}\sqrt{c}(x - ct) \right]$$

$$\phi(t,x) = e^{i\omega t} f(x)$$

where $u_0 \in H^s(\mathbb{T})$ for some $s \geq 0$ is a given initial data. We aim to propose the corresponding *embedded low-regularity integrator (ELRI)* for solving the mKdV equation (1.2), where we are able to maintain the usual convergence order and minimize the cost of derivatives as we could in the approximations. We shall work under the exponential-type integration framework [17]. While, the difficulty is that now to deal with the resonance structure from the cubic nonlinearity in Fourier space

$$\xi^3 - \xi_1^3 - \xi_2^3 - \xi_3^3,$$

the previous factorization technique for the classical KdV equation from [18, 60, 61] fails, which stops us from getting a low-regularity integration closed in the physical space. Note that a closed form of approximation in the physical space is very important for the efficiency of the scheme. What we propose as the first step in this work, is to make a 'coarse' approximation by losing three spatial derivatives for a stable integration scheme in the physical space. This is done by using a Schrödinger-type approximation to the model, which could be of independent future interests for the numerics. Then, to reduce the cost of the derivatives, we note that the mKdV equation can be transformed into the classical KdV equation through the Miura transform [36, 56]: $U = \partial_x u + u^2$, where U solves the KdV equation. By using the inverse of the Miura transform:

$$u = \partial_x^{-1} U - \partial_x^{-1} u^2 + \frac{1}{2\pi} \int_{\mathbb{T}} u_0(x) dx,$$

we shall propose an iterative regularizing framework for recovering the lost regularity from the coarse approximation. Our way to use the Miura transform is different from the existing works in the literature. Theoretically, the Miura transform is usually used to turn mKdV to KdV and apply the methods and results from KdV, e.g., [16, 36]. Here, we propose for the first time to consider the inverse of the Miura transform as an iterative process for smoothing the numerical solution. We end up with an ELRI scheme which is explicit in the physical space, and is efficient to program in practice incorporation with the Fourier pseudo-spectral method [50]. As we shall prove rigorously, our ELRI scheme has the **first** order accuracy by requiring only **one** additional spatial derivative, i.e.,

$$\|u(t_n, \cdot) - u^n\|_{H^\gamma} \lesssim \tau,$$

with τ the time step and u^n the numerical solution, for any initial data $u_0 \in H^{\gamma+1}$ with $\gamma > \frac{3}{2}$. Note $\gamma > \frac{3}{2}$ is a technical condition in our proof that can be further released by more delicate analytical tools. In this direction, let us mention the great efforts made recently in [39, 40] on the Schrödinger equation by using the discrete Strichartz estimates or discrete Bourgain spaces. In this work, we emphasize that our attention is focused on saving the lost derivatives in the numerical scheme. Numerical experiments will be done in the end to justify the error estimate and to illustrate the efficiency of ELRI for solving (1.2) under rough data.

The rest of the paper is organized as follows. In Section 2, we derive the ELRI scheme and present its convergence theorem. In Section 3, we present some important formulas and lemmas as preparations for the error analysis. Section 4 analyzes the coarse approximation for the mKdV equation. Section 5 exploits the Miura transform and analyzes the improved scheme after the first regularizing and the second regularizing iterations to reduce the regularity requirement. Numerical results will be given in Section 6, and some conclusions will be drawn in Section 7.

2. Numerical method: derivation and main result

In this section, we shall firstly present the connection between the mKdV and KdV equations through the Miura transform, and then based on which we shall derive our numerical approximation. The convergence theorem will be given in the end. We denote $\tau > 0$ as the time step and $t_n = n\tau$ as the time grids. The Fourier transform of a function $f(x)$ on $\mathbb{T}$ is denoted by

$$\widehat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{T}} \mathrm{e}^{-ix\xi} f(x)\,dx, \quad \xi \in \mathbb{Z}.$$

2.1. **Connection with KdV equation.**

Let $u = u(t,x)$ be the solution of the mKdV equation (1.2), then by the Miura transformation [36]

$$U := M[u] = \partial_x u + u^2, \tag{2.1}$$

the function $U = U(t,x)$ solves the following KdV equation:

$$\begin{cases} \partial_t U + \partial_x^3 U = 3\partial_x\left(U^2\right), & t > 0,\ x \in \mathbb{T}, \\ U(0,x) = \partial_x u_0 + u_0^2, & x \in \mathbb{T}, \end{cases} \tag{2.2}$$

with u_0 the initial data of the mKdV equation (1.2). By inverting the Miura transform (2.1), we find

$$u(t,x) = \partial_x^{-1} U(t,x) - \partial_x^{-1} u^2(t,x) + \frac{1}{2\pi}\int_{\mathbb{T}} u(t,x)dx, \quad t \geq 0,\ x \in \mathbb{T}.$$

Here and after, we define the operator ∂_x^{-1} for some function $f(x)$ on $\mathbb{T}$ as

$$\widehat{(\partial_x^{-1} f)}(\xi) = \begin{cases} (i\xi)^{-1}\widehat{f}(\xi), & \text{when } \xi \neq 0, \\ 0, & \text{when } \xi = 0. \end{cases}$$

Note that the mKdV equation (1.2) preserves the mass:

$$\int_{\mathbb{T}} u(t,x)dx \equiv \int_{\mathbb{T}} u_0(x)dx =: 2\pi m_0, \quad t \geq 0,$$

so we further get

$$u(t,x) = \partial_x^{-1} U(t,x) - \partial_x^{-1} u^2(t,x) + m_0, \quad t \geq 0,\ x \in \mathbb{T}. \tag{2.3}$$

Thanks to this connection, we are able to call the embedded low-regularity integrator from [61] for solving the KdV equation (2.2) to get $U(t,x)$, and meanwhile we can make some proper approximation for $u(t,x)$.

To the aim of saving regularity, it is more convenient to work on the twisted variables

$$V = V(t,x) := \mathrm{e}^{t\partial_x^3} U(t,x) \quad \text{and} \quad v = v(t,x) := \mathrm{e}^{t\partial_x^3} u(t,x), \quad t \geq 0,\ x \in \mathbb{T}, \tag{2.4}$$

respectively for the KdV equation (2.2) and the mKdV equation (1.2). Then, (2.3) becomes

$$v(t,x) = \partial_x^{-1} V(t,x) - \partial_x^{-1} \mathrm{e}^{t\partial_x^3}\left(\mathrm{e}^{-t\partial_x^3} v(t,x)\right)^2 + m_0, \quad t \geq 0,\ x \in \mathbb{T}, \tag{2.5}$$

which is the key relation for us to design numerical scheme.

2.2. **Numerical approximation.** For simplicity of notations, in the following we shall omit the spatial variable x of the involved time-space dependent functions, e.g., $u(t) = u(t,x)$. The framework of our approximation for v consists of three steps. We shall first present a coarse approximation for v where some regularity is lost, and then we perform two smoothing iterations based on (2.5) to get some regularity back.

Step 1. A coarse version. We begin by presenting a first coarse approximation for the solution of the mKdV equation (1.2), which can be regarded as a prediction step.

In the spirit of exponential integrators [17], we apply the Duhamel formula to (1.2) in terms of the twisted variable v:

$$v(t_{n+1}) = v(t_n) + 2\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left[\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\right]^3 ds, \quad n\geq 0. \tag{2.6}$$

It can be seen that here in Fourier space, the resonance structure of the phase function for the mKdV equation is

$$\xi^3-\xi_1^3-\xi_2^3-\xi_3^3,$$

which involves the interaction of three frequencies rather than two for the KdV equation. Due to this, the factoring technique used for the KdV equation [18, 60] can not be applied here. Consequently in (2.6), if one only takes the approximation

$$v(t_n+s)\approx v(t_n), \quad 0\leq s\leq \tau,$$

the rest of the integral cannot be evaluated explicitly in the physical space, and then the triple summation left in the Fourier space would be very costly for computations. Thus, some approximation has to be further made for the integral in (2.6), and unavoidably some more regularity will be lost. For example, one may consider to take

$$\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left[\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\right]^3 ds \approx \int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right]^3 ds, \tag{2.7}$$

and then evaluate the rest exactly. While, this apparently would cost **four** spatial derivatives. What is worse is that such simple approximation will also cause stability issue. Here, we make an effort to save as much regularity as we could, and meanwhile pursue a stable approximation. We replace the Airy operator in (2.7) with a Schrödinger operator, i.e., $\mathrm{e}^{s\partial_x^3}\approx \mathrm{e}^{is\partial_x^2}$. Then, we adopt the following approximation of (2.6) as our first approximation for the mKdV equation:

$$\begin{aligned} v(t_{n+1}) \approx& v(t_n) + 2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right]^3 ds \\ =& v(t_n) - 2\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3+i\tau\partial_x^2}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3\right] =: \Phi_I^n(v(t_n)), \quad n\geq 0. \end{aligned} \tag{2.8}$$

This approximation here shares the same spirit as the work in the literature [7, 9, 47, 55] to approximate the KdV equation by the Schrödinger equation.

In fact, by denoting the numerical solution $v_I^n\approx v(t_n)$ for $n\geq 0$, (2.8) defines the coarse version of our algorithm for solving the mKdV equation (1.2):

$$v_I^{n+1} = \Phi_I^n(v_I^n), \quad \text{with} \quad v_I^0 = u_0. \tag{2.9}$$

As we shall show rigorously in Section 4 that, up to any $T>0$, for $u_0\in H^{\gamma+3}$ with $\gamma>-1/2$, there exists a constant $C>0$ such that

$$\|v(t_n)-v_I^n\|_{H^\gamma}\leq C\tau, \quad 0\leq n\leq \frac{T}{\tau}.$$

That is to say, the first order accurate approximation (2.8) costs **three** spatial derivatives of the solution of the mKdV equation (1.2). Although this is a coarse version and the result obtained is much weaker than what we aim for in this paper, we claim that the technique used here has independent interests, which could be applied to other models with complicated resonance structure in the phase.

Step 2. Iterative regularizing: first layer. Now with the coarse approximation from the previous step, we introduce an iterative regularizing strategy based on the inverse Miura transform (2.5) to gain the regularity back. This step can be regarded as the first layer in our embedded scheme.

Firstly, the inverse Miura transform (2.5) involves the solution of the KdV equation (2.2). To get this part, we introduce the first order embedded low-regularity integrator from our work [61]. Denoting the numerical solution of the KdV equation (2.2) as $U^n \approx U(t_n)$ for $n \geq 0$ with $U^0 = U(0)$, then

$$U^n = \mathrm{e}^{-it_n\partial_x^3}V^n \quad \text{with} \quad V^n(x) = \widetilde{V}^n\left(x + \frac{3t_n}{\pi}\int_{\mathbb{T}} U^0 dx\right) + \frac{1}{2\pi}\int_{\mathbb{T}} U^0 dx, \quad n \geq 0, \tag{2.10}$$

where $\widetilde{V}^0 = \mathbb{P}U(0)$,

$$\widetilde{V}^{n+1} = \widetilde{V}^n + A_n\left(t_{n+1}, \widetilde{V}^n\right) - A_n\left(t_n, \widetilde{V}^n\right) - 2\tau\partial_x^{-1}\mathrm{e}^{t_n\partial_x^3}\left(\mathrm{e}^{-t_n\partial_x^3}\widetilde{V}^n\right)^3 + \frac{3\tau}{\pi}\partial_x^{-1}\widetilde{V}^n\int_{\mathbb{T}}\left(\widetilde{V}^n\right)^2 dx,$$

and

$$\begin{aligned} A_n(s, V) =& \mathrm{e}^{s\partial_x^3}\left(\mathrm{e}^{-s\partial_x^3}\partial_x^{-1}V\right)^2 - \frac{2}{3}\mathrm{e}^{s\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-s\partial_x^3}\partial_x^{-1}V\right)^3 \\ &- 2\mathbb{P}\left[\mathrm{e}^{t_{n+1}\partial_x^3}\left(\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}V \cdot \mathrm{e}^{-(s-t_n)\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{(s-t_n)\partial_x^3}\mathrm{e}^{-t_{n+1}\partial_x^3}\partial_x^{-1}V\right)^2\right)\right]. \end{aligned}$$

Here we denote the operator $\mathbb{P}f := f - \frac{1}{2\pi}\int_{\mathbb{T}} f(0)dx$, for some $f(x)$ on $\mathbb{T}$. We refer the readers to [61] for the detailed derivation of (2.10). The optimal convergence of this algorithm for solving the KdV equation has been rigorously established in [61]. Here for the further use in our analysis, let us directly quote its convergence result as the following proposition.

Proposition 2.1 ([61])**.** *Let U^n be the numerical solution of the KdV* (2.2) *obtained from* (2.10) *up to some fixed time $T > 0$. If $U(0) \in H^{\gamma+1}(\mathbb{T})$ for some $\gamma > \frac{1}{2}$, then there exist constants $\tau_0 > 0$ and $C > 0$ depending only on T and $\|U\|_{L^\infty((0,T);H^{\gamma+1})}$, such that for any $0 < \tau \leq \tau_0$,*

$$\|U(t_n) - U^n\|_{H^\gamma} \leq C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}.$$

Then, by plugging the numerical solution V^n from (2.10) and the prediction algorithm Φ_I^n defined in (2.8) into the right-hand-side of (2.5), we define an improved approximation for the mKdV equation (1.2):

$$v_{II}^n = \partial_x^{-1}V^n - \mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^2 + m_0, \quad n \geq 1, \quad v_{II}^0 = u_0. \tag{2.11}$$

By doing this, we are able to save one spatial derivative back compared with the coarse version obtained in Step 1. This will be proved rigorously in Section 5. With this version, we only need **two** additional spatial derivatives for the first order accuracy, while there is still a room for improvement.

Step 3. Iterative regularizing: second layer. With one more time of iteration where we now embed v_{II}^n from the previous step (2.11) into the right-hand-side of the inverse Miura transform (2.5), we are able to get one more derivative back. That is to say, we now define our final approximation as:

$$v^n = \partial_x^{-1}V^n - \partial_x^{-1}\mathrm{e}^{t_n\partial_x^3}\left(\mathrm{e}^{-t_n\partial_x^3}v_{II}^n\right)^2 + m_0, \quad n \geq 1, \quad v^0 = u_0. \tag{2.12}$$

It can be regarded as the second layer in our embedded scheme. We remark that more iterations will not save any more spatial derivatives, because after the second layer, the regularity requirement of the KdV part will become essential in (2.5). Thus, (2.12) completes our algorithm for solving the mKdV equation (1.2), and its convergence result will be stated as the main theorem below which tells that only **one** additional spatial derivative is needed for the first order accuracy.

2.3. Scheme and convergence result.

We now summarize the scheme obtained from the above framework and present its convergence result.

The detailed *embedded exponential-type low-regularity integrator (ELRI)* for solving the mKdV equation (1.2) reads as follows. Denoting $v^n \approx v(t_n)$ as the numerical solution for the twisted variable of the mKdV equation (1.2) for $n \geq 0$ and taking $v^0 = u_0$, then for $n \geq 1$,

$$v^n = \partial_x^{-1}V^n - \mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v_{II}^n\right)^2 + m_0, \quad v_{II}^n = \partial_x^{-1}V^n - \mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^2 + m_0, \tag{2.13}$$

where V^n is defined in (2.10) and

$$v_I^n = v_I^{n-1} - 2\mathrm{Re}\left[i\mathrm{e}^{t_{n-1}\partial_x^3 + i\tau\partial_x^2}\partial_x^{-1}\left(\mathrm{e}^{-t_{n-1}\partial_x^3}v_I^{n-1}\right)^3\right], \quad n \geq 1, \quad v_I^0 = u_0.$$

By inverting the twisted variable, it is equivalent to have the ELRI scheme for the original variable $u(t)$ of the mKdV equation (1.2): denoting $u^n \approx u(t_n)$ for $n \geq 0$ with $u^0 = u_0$, then for $n \geq 1$,

$$u^n = \partial_x^{-1}U^n - \partial_x^{-1}\left(u_{II}^n\right)^2 + m_0, \quad u_{II}^n = \partial_x^{-1}U^n - \partial_x^{-1}\left(u_I^n\right)^2 + m_0, \tag{2.14}$$

where U^n is defined in (2.10) and $u_I^n = \mathrm{e}^{-t_n\partial_x^3}v_I^n$.

The proposed scheme of ELRI (2.13) or (2.14) is fully explicit. In practice, it can be implemented efficiently under the Fourier pseudo-spectral method [50, 58] via the fast Fourier transform. The spatial shifting in (2.10) can be done efficiently by the non-uniform fast Fourier transform [13]. In total, the computational cost per time step is $O(N\log N)$ with $N > 0$ the number of grids points in space.

Now we state our **main result** for the convergence of the proposed ELRI scheme.

Theorem 2.2. *Let u^n be the numerical solution of the mKdV (1.2) obtained from the ELRI scheme (2.14) up to some fixed time $T > 0$. Under the assumption that $u_0 \in H^{\gamma+1}(\mathbb{T})$ with $\gamma > \frac{3}{2}$, there exist constants $\tau_0 > 0$ and $C > 0$ depending only on T and $\|u\|_{L^\infty((0,T);H^{\gamma+1})}$, such that for any $0 < \tau \leq \tau_0$,*

$$\|u(t_n,\cdot) - u^n\|_{H^\gamma} \leq C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}.$$

If one is only interested in the solution $u(t,x)$ of the mKdV equation at the final time $t = T > 0$, the two stages of regularizing at the intermediate time level can be skipped. That is to compute with u_I^n all the way to $n = T/\tau$, and then obtain u^n as (2.14). This could accelerate the practical computations, and the convergence result in Theorem 2.2 holds at the final time grid. The rest of the paper is devoted to rigorously proving Theorem 2.2.

Remark 1. The condition $\gamma > \frac{3}{2}$ is a technical condition for the simplicity of our rigorous error analysis. As indicated by our numerical result in Section 6, it could be improved by more delicate analytical technique, e.g., [39, 40, 41, 45]. This will be left for the future study.

Remark 2. (Second order accuracy) The ELRI method to the second order accuracy can be obtained in the same framework as above. For instance, one can get a simple second order coarse scheme by the approximation $v(t_n + s) \approx v(t_n) + s\partial_t v(t_n)$ in (2.6), and the regularity can then be recovered by applying the iterative regularizing strategy for several times. In this work, let us focus on the first order scheme to present and study the proposed framework.

3. Preliminaries

This section is devoted to some preparations for analyzing the numerical schemes. We shall first introduce some notations and then present some tool lemmas.

3.1. Notations.

We employ some useful notations from [10]. Firstly, we use $A \lesssim B$, $B \gtrsim A$ or $A = O(B)$ to denote the statement that $A \leq CB$ for some large absolute constant $C > 0$ which may vary from line to line but independent of τ or n. We denote $A \sim B$ for $A \lesssim B \lesssim A$. Moreover, we use $A \ll B$ or $B \gg A$ to denote the statement $A \leq C^{-1}B$.

We denote $(d\xi)$ to be the normalized counting measure on $\mathbb{Z}$ and then the inverse Fourier transform reads

$$f(x) = \int \mathrm{e}^{ix\xi}\widehat{f}(\xi)\,(d\xi) = \sum_{\xi\in\mathbb{Z}} \mathrm{e}^{ix\xi}\widehat{f}(\xi), \quad x \in \mathbb{T}.$$

The following usual properties of the Fourier transform hold:

$$\|f\|_{L^2(\mathbb{T})} = \sqrt{2\pi}\|\widehat{f}\|_{L^2((d\xi))} \quad \text{(Plancherel)};$$

$$\langle f, g\rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}\,dx = 2\pi\int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,(d\xi) \quad \text{(Parseval)};$$

$$\widehat{(fg)}(\xi) = \int \widehat{f}(\xi-\xi_1)\widehat{g}(\xi_1)\,(d\xi_1) \quad \text{(Convolution)}.$$

We define the operator

$$J^s := (1-\partial_{xx})^{\frac{s}{2}},$$

for some $s \in \mathbb{R}$, and the Sobolev space $H^s(\mathbb{T})$ has the equivalent norm

$$\|f\|_{H^s(\mathbb{T})} = \|J^s f\|_{L^2(\mathbb{T})} = \sqrt{2\pi}\left\|\left(1+\xi^2\right)^{\frac{s}{2}}\widehat{f}(\xi)\right\|_{L^2((d\xi))}.$$

Moreover, we denote $a\pm$ for $a \pm \epsilon$ with any small $\epsilon > 0$, and we denote the bracket $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

3.2. **Some tool lemmas.** To overcome the absence of the algebraic property of H^s when $s \notin \mathbb{Z}^+$, we will frequently use the following basic Kato-Ponce inequality, which was originally proved in [24], and recent important progress in the endpoint case was made in [6, 32].

Lemma 3.1. *(Kato-Ponce inequality) The following inequalities hold:*

(i) *For any $\gamma \geq 0, \gamma_1 > \frac{1}{2}$, $f, g \in H^\gamma \cap H^{\gamma_1}$,*

$$\|J^\gamma(fg)\|_{L^2} \lesssim \|f\|_{H^\gamma}\|g\|_{H^{\gamma_1}} + \|f\|_{H^{\gamma_1}}\|g\|_{H^\gamma}.$$

In particular, if $\gamma > \frac{1}{2}$, then,

$$\|J^\gamma(fg)\|_{L^2} \lesssim \|f\|_{H^\gamma}\|g\|_{H^\gamma}.$$

(ii) *For any $\gamma \geq 0, \gamma_1 > \frac{1}{2}$, $f \in H^{\gamma+\gamma_1}, g \in H^\gamma$,*

$$\|J^\gamma(fg)\|_{L^2} \lesssim \|f\|_{H^{\gamma+\gamma_1}}\|g\|_{H^\gamma}.$$

We also need the following specific Kato-Ponce inequality with negative derivatives.

Lemma 3.2. *For any $s_0 > \frac{1}{2}, s \in [-s_0, s_0]$, $f \in H^s, g \in H^{s_0}$, the following inequality holds:*

$$\|J^s(fg)\|_{L^2} \leq C\|J^s f\|_{L^2}\|J^{s_0}g\|_{L^2},$$

where the constant $C > 0$ depends on s, s_0.

Proof. When $s \geq 0$, it is followed from Lemma 3.1 and Sobolev's inequality. Hence, we may assume that $s < 0$. To do this, denote $h_2 = J^s f, h_3 = J^{s_0}g$, then by duality and Parseval's identity, it is sufficient to show that for any $h_j \in L^2(\mathbb{T})$, $j = 1, 2, 3$,

$$\left|\int_{\xi_1+\xi_2+\xi_3=0} M(\xi_1,\xi_2,\xi_3)\widehat{h_1}(\xi_1)\widehat{h_2}(\xi_2)\widehat{h_3}(\xi_3)\,d\sigma\right| \lesssim \prod_{j=1}^{3}\|h_j\|_{L^2}, \tag{3.1}$$

where $d\sigma = (d\xi_1)(d\xi_2)$ and the multiplier

$$M(\xi_1,\xi_2,\xi_3) = \langle\xi_1\rangle^s\langle\xi_2\rangle^{-s}\langle\xi_3\rangle^{-s_0}.$$

Moreover, we may assume that $\widehat{h_j}$, for $j = 1, 2, 3$ are positive, otherwise one may replace them by $|\widehat{h_j}|$. Now to prove (3.1), we split the left-hand-side of (3.1) into the following two parts:

$$I_1 = \int_{\Omega_1} M(\xi_1,\xi_2,\xi_3)\widehat{h_1}(\xi_1)\widehat{h_2}(\xi_2)\widehat{h_3}(\xi_3)\,d\sigma, \qquad I_2 = \int_{\Omega_2} M(\xi_1,\xi_2,\xi_3)\widehat{h_1}(\xi_1)\widehat{h_2}(\xi_2)\widehat{h_3}(\xi_3)\,d\sigma,$$

where

$$\Omega_1 = \{(\xi_1,\xi_2,\xi_3) : \xi_1+\xi_2+\xi_3 = 0, |\xi_2| \lesssim |\xi_1|\}, \quad \Omega_2 = \{(\xi_1,\xi_2,\xi_3) : \xi_1+\xi_2+\xi_3 = 0, |\xi_2| \gg |\xi_1|\}.$$

For I_1, we note that in Ω_1,

$$M(\xi_1,\xi_2,\xi_3) \lesssim \langle\xi_3\rangle^{-s_0}.$$

Thus by Parseval's identity again, we get

$$I_1 \lesssim \int_{\mathbb{T}} h_1(x)\, h_2(x)\, J^{-s_0} h_3(x)\, dx \lesssim \big\|h_1\big\|_{L^2}\big\|h_2\big\|_{L^2}\big\|J^{-s_0}h_3\big\|_{L^\infty} \lesssim \prod_{j=1}^{3}\|h_j\|_{L^2}.$$

For I_2, we have $|\xi_2| \sim |\xi_3|$ in Ω_2. We claim that in this case,

$$M(\xi_1,\xi_2,\xi_3) \lesssim \langle\xi_1\rangle^{-\frac{1}{2}-}. \tag{3.2}$$

Indeed, if $s \geq -\frac{1}{2}$, then for any $s_0 > \frac{1}{2}$,

$$M(\xi_1,\xi_2,\xi_3) \lesssim \langle\xi_1\rangle^{-\frac{1}{2}-}\langle\xi_3\rangle^{-s_0+\frac{1}{2}+} \lesssim \langle\xi_1\rangle^{-\frac{1}{2}-}.$$

If $s < -\frac{1}{2}$, then for any $s \geq -s_0$,

$$M(\xi_1,\xi_2,\xi_3) \lesssim \langle\xi_1\rangle^{s}\langle\xi_3\rangle^{-s_0-s} \lesssim \langle\xi_1\rangle^{-\frac{1}{2}-}.$$

Hence, we have (3.2). Similarly as above, we get that

$$I_2 \lesssim \int_{\mathbb{T}} J^{-\frac{1}{2}-}h_1(x)\, h_2(x)\, h_3(x)\, dx \lesssim \big\|J^{-\frac{1}{2}-}h_1\big\|_{L^\infty}\big\|h_2\big\|_{L^2}\big\|h_3\big\|_{L^2} \lesssim \prod_{j=1}^{3}\|h_j\|_{L^2}.$$

This finishes the proof of the lemma. □

3.3. **Some useful estimates with time integration.** In this subsection, we introduce some estimates with time integration that is crucial for the proof of stability. They are given as the following proposition. The main difficulty in establishing it is that the operator $\mathrm{e}^{is\partial_x^2}\partial_x$ in the following breaks the structure for integration-by-parts, and so the usual commutator estimate fails. Here we must take some delicate effect on the time integration into consideration, and the symmetric argument plays an important role in the proof of the proposition.

Proposition 3.3 (Estimates with time integration)**.** *The following estimates hold.*

(i) *Let* $\gamma \geq -1, \gamma_0 = \max\{\frac{3}{2}+, \gamma+1\}$. *Then, for any real-valued functions* $f_1 \in H^\gamma$ *and* $f_3, f_4 \in H^{\gamma_0}$,

$$\left|\int_0^\tau \left\langle \mathrm{e}^{is\partial_x^2}\partial_x J^\gamma\big(f_1f_3f_4\big), J^\gamma f_1\right\rangle ds\right| \lesssim \tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{\gamma_0}}\|f_4\|_{H^{\gamma_0}}.$$

(ii) *Let* $\gamma > \frac{3}{2}, \gamma_0 = \max\{\frac{3}{2}+, \gamma+1\}$. *Then, for any real-valued functions* $f_1 \in H^\gamma, f_4 \in H^{\gamma_0}$,

$$\left|\int_0^\tau \left\langle \mathrm{e}^{is\partial_x^2}\partial_x J^\gamma\big(f_1^2f_4\big), J^\gamma f_1\right\rangle ds\right| \lesssim \tau\|f_1\|_{H^\gamma}^3\|f_4\|_{H^{\gamma_0}}.$$

(iii) *Let* $\gamma > \frac{3}{2}$. *Then, for any real-valued functions* $f \in H^\gamma$,

$$\left|\int_0^\tau \left\langle \mathrm{e}^{is\partial_x^2}\partial_x J^\gamma\big(f^3\big), J^\gamma f\right\rangle ds\right| \lesssim \tau\|f\|_{H^\gamma}^4.$$

Proof. (i) We may assume that $\widehat{f_j}$ for $j = 1,3,4$ are positive, otherwise one may replace them by $|\widehat{f_j}|$. For short, we denote

$$\Gamma = \big\{(\xi_1,\xi_2,\xi_3,\xi_4) : \xi_1+\xi_2+\xi_3+\xi_4 = 0\big\} \quad \text{and} \quad d\sigma = (d\xi_1)(d\xi_2)(d\xi_3).$$

Then, by Parseval's identity, we get

$$\int_0^\tau \left\langle \mathrm{e}^{is\partial_x^2}\partial_x J^\gamma\big(f_1f_3f_4\big), J^\gamma f_1\right\rangle ds = 2\pi\int_0^\tau\int_\Gamma \mathrm{e}^{-i\xi_1^2 s} i\xi_1\langle\xi_1\rangle^{2\gamma}\widehat{f_1}(\xi_1)\widehat{f_1}(\xi_2)\widehat{f_3}(\xi_3)\widehat{f_4}(\xi_4)d\sigma ds.$$

We divide Γ into the following parts:

$$\Gamma_1 = \{(\xi_1,\xi_2,\xi_3,\xi_4) \in \Gamma : |\xi_1| \geq |\xi_2|, |\xi_3| \geq |\xi_4|, |\xi_2| \geq 10|\xi_3|\},$$

and

$$\Gamma_2 = \{(\xi_1,\xi_2,\xi_3,\xi_4) \in \Gamma : |\xi_1| \geq |\xi_2|, |\xi_3| \geq |\xi_4|, |\xi_2| < 10|\xi_3|\}.$$

In addition, we denote

$$M(s;\xi_1,\cdots,\xi_4) = \mathrm{e}^{-i\xi_1^2 s}i\xi_1\langle\xi_1\rangle^{2\gamma} + \mathrm{e}^{-i\xi_2^2 s}i\xi_2\langle\xi_2\rangle^{2\gamma}. \tag{3.3}$$

By symmetries, there exists some absolute constant $C > 0$ such that

$$\begin{aligned}
&\mathcal{F}\left(\int_0^\tau \left\langle e^{is\partial_x^2}\partial_x J^\gamma\big(f_1f_3f_4\big), J^\gamma f_1\right\rangle ds\right)\\
=&\pi\int_0^\tau\int_\Gamma M(s;\xi_1,\cdots,\xi_4)\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma ds\\
=&C\int_0^\tau\int_{\Gamma_1\cup\Gamma_2} M(s;\xi_1,\cdots,\xi_4)\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma ds =: CI_1 + CI_2,
\end{aligned}$$

where

$$I_1 = \int_0^\tau\int_{\Gamma_1} M(s;\xi_1,\cdots,\xi_4)\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma ds,$$

and

$$I_2 = \int_0^\tau\int_{\Gamma_2} M(s;\xi_1,\cdots,\xi_4)\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma ds.$$

For I_1, we claim that in Γ_1,

$$\int_0^\tau \big|M(s;\xi_1,\cdots,\xi_4)\big|\,ds \lesssim \tau\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle. \tag{3.4}$$

Indeed, according to the definition of M, we have

$$\begin{aligned}
M(s;\xi_1,\cdots,\xi_4) =& e^{-i\xi_2^2 s}\left(i\xi_1\langle\xi_1\rangle^{2\gamma} + i\xi_2\langle\xi_2\rangle^{2\gamma}\right) + \left(e^{-i\xi_1^2 s} - e^{-i\xi_2^2 s}\right)i\xi_1\langle\xi_1\rangle^{2\gamma}\\
=:& M_1(s;\xi_1,\cdots,\xi_4) + M_2(s;\xi_1,\cdots,\xi_4).
\end{aligned}$$

Thanks to the restriction on Γ_1, we have $|\xi_1| \sim |\xi_2|$ and $|\xi_1+\xi_2| \le 2|\xi_3|$. Hence, for any $\gamma\in\mathbb{R}$,

$$\big|\xi_1\langle\xi_1\rangle^{2\gamma} + \xi_2\langle\xi_2\rangle^{2\gamma}\big| \lesssim \langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle,$$

and so we have

$$\int_0^\tau \big|M_1(s;\xi_1,\cdots,\xi_4)\big|\,ds \lesssim \tau\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle.$$

For M_2, we find that

$$\begin{aligned}
\int_0^\tau (e^{-i\xi_1^2 s} - e^{-i\xi_2^2 s})ds =& \frac{1}{i\xi_1^2}\big(1-e^{-i\xi_1^2\tau}\big) - \frac{1}{i\xi_2^2}\big(1-e^{-i\xi_2^2\tau}\big)\\
=& \frac{1}{i\xi_1^2}\big(e^{-i\xi_2^2\tau} - e^{-i\xi_1^2\tau}\big) + \left(\frac{1}{i\xi_2^2} - \frac{1}{i\xi_1^2}\right)\left(e^{-i\xi_2^2\tau} - 1\right),
\end{aligned}$$

and so

$$\left|\int_0^\tau (e^{-i\xi_1^2 s} - e^{-i\xi_2^2 s})\,ds\right| \le \frac{2\tau}{|\xi_1|^2}\big|\xi_1^2 - \xi_2^2\big| \lesssim \tau\frac{|\xi_3|}{|\xi_1|}.$$

This gives

$$\int_0^\tau \big|M_2(s;\xi_1,\cdots,\xi_4)\big|\,ds \lesssim \tau\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle,$$

and so in total we get (3.4). Therefore, we get for any $\gamma\in\mathbb{R}$,

$$\begin{aligned}
|I_1| =& \left|\int_{\Gamma_1}\int_0^\tau M(s;\xi_1,\cdots,\xi_4)ds\,\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma\right|\\
\le& \tau\int_{\Gamma_1}\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma \lesssim \tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{\frac32+}}\|f_4\|_{H^{\frac32+}}.
\end{aligned} \tag{3.5}$$

For I_2, according to the parameter γ, we can split it into several cases as well.

If $-1\le\gamma\le 0$, noting that $|\xi_2|\le|\xi_1|\lesssim|\xi_3|$ on Γ_2, we have

$$\big|M(s;\xi_1,\cdots,\xi_4)\big| \lesssim \langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle,$$

which implies

$$|I_2| \lesssim \tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{\frac32+}}\|f_4\|_{H^{\frac32+}}.$$

If $0<\gamma<\frac{1}{2}$, then

$$\big|M(s;\xi_1,\cdots,\xi_4)\big|\lesssim\langle\xi_1\rangle^{\gamma}\langle\xi_3\rangle^{1+\gamma}.$$

Moreover, since $\widehat{f}_j, j=1,3,4$ are positive, we find

$$|I_2|\lesssim\int_0^\tau\int_{\Gamma_2}\langle\xi_1\rangle^{\gamma}\langle\xi_3\rangle^{1+\gamma}\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_3(\xi_3)\widehat{f}_4(\xi_4)d\sigma ds\le\tau\int_{\mathbb{T}}J^{\gamma}f_1(x)\,f_1(x)\,J^{1+\gamma}f_3(x)\,f_4(x)\,dx.$$

Hence, by the Hölder and Sobolev inequalities, we get

$$|I_2|\lesssim\tau\big\|J^{\gamma}f_1\big\|_{L^2}\big\|f_1\big\|_{L^{\frac{2}{1-2\gamma}}}\big\|J^{1+\gamma}f_3\big\|_{L^{\frac{1}{\gamma}}}\big\|f_4\big\|_{L^\infty}\lesssim\tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{\frac32+}}\|f_4\|_{H^{\frac32+}}.$$

If $\gamma=\frac{1}{2}$, then

$$\big|M(s;\xi_1,\cdots,\xi_4)\big|\lesssim\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{0-}\langle\xi_3\rangle^{\frac32+}.$$

Similarly as above, we can find

$$|I_2|\lesssim\tau\big\|J^{\gamma}f_1\big\|_{L^2}\big\|J^{0-}f_1\big\|_{L^\infty}\big\|J^{\frac32+}f_3\big\|_{L^2}\big\|f_4\big\|_{L^\infty}\lesssim\tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{\frac32+}}\|f_4\|_{H^{\frac32+}}.$$

Therefore, we get that for any $-1\le\gamma\le\frac{1}{2}$,

$$|I_2|\lesssim\tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{\frac32+}}\|f_4\|_{H^{\frac32+}}. \tag{3.6}$$

If $\gamma>\frac{1}{2}$, then

$$\big|M(s;\xi_1,\cdots,\xi_4)\big|\lesssim\langle\xi_1\rangle^{\gamma}\langle\xi_3\rangle^{1+\gamma},$$

and similarly as above, we can get

$$|I_2|\lesssim\tau\big\|J^{\gamma}f_1\big\|_{L^2}\big\|f_1\big\|_{L^\infty}\big\|J^{1+\gamma}f_3\big\|_{L^2}\big\|f_4\big\|_{L^\infty}\lesssim\tau\|f_1\|_{H^\gamma}^2\|f_3\|_{H^{1+\gamma}}\|f_4\|_{H^{1+\gamma}}.$$

This last estimate together with (3.5) and (3.6) give the desired estimate (i).

(ii) The proof is similar as (i), and so we only give its sketch here for brevity. We denote

$$\widetilde{\Gamma}=\{(\xi_1,\xi_2,\xi_3,\xi_4)\ :\ \xi_1+\xi_2+\xi_3+\xi_4=0,\ |\xi_1|\ge|\xi_2|\ge|\xi_3|\}.$$

In addition, we denote

$$\widetilde{M}(s;\xi_1,\cdots,\xi_4)=\sum_{j=1}^{3}\mathrm{e}^{-i\xi_j^2 s}i\xi_j\langle\xi_j\rangle^{2\gamma}.$$

Then, by symmetry, there exists some absolute constant $C>0$ such that

$$\int_0^\tau\Big\langle\mathrm{e}^{is\partial_x^2}\partial_xJ^{\gamma}\big(f_1^2f_4\big),J^{\gamma}f_1\Big\rangle\,ds=C\int_0^\tau\int_{\widetilde{\Gamma}}\widetilde{M}(s;\xi_1,\cdots,\xi_4)\widehat{f}_1(\xi_1)\widehat{f}_1(\xi_2)\widehat{f}_1(\xi_3)\widehat{f}_4(\xi_4)d\sigma ds.$$

We claim that in $\widetilde{\Gamma}$,

$$\int_0^\tau\big|\widetilde{M}(s;\xi_1,\cdots,\xi_4)\big|\,ds\lesssim\tau\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{\gamma}\langle\xi_3\rangle+\tau\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{\gamma}\langle\xi_4\rangle+\tau\langle\xi_1\rangle^{\gamma}\langle\xi_4\rangle^{\gamma+1}. \tag{3.7}$$

Indeed, in $\widetilde{\Gamma}$ we can write

$$\widetilde{M}(s;\xi_1,\cdots,\xi_4)=M(s;\xi_1,\cdots,\xi_4)+O\big(\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{\gamma}\langle\xi_3\rangle\big),$$

where M is defined in (3.3). We consider the following two cases separately:

$$\text{Case 1},\ \ |\xi_2|\ge10|\xi_4|;\qquad\text{Case 2},\ \ |\xi_2|\le10|\xi_4|.$$

Case 1: $|\xi_2|\ge10|\xi_4|$. By the same treatment as for (3.4), we have

$$\int_0^\tau\big|M(s;\xi_1,\cdots,\xi_4)\big|\,ds\lesssim\tau\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{\gamma}\langle\xi_3+\xi_4\rangle\lesssim\tau\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{\gamma}\langle\xi_3\rangle+\tau\langle\xi_1\rangle^{\gamma}\langle\xi_2\rangle^{\gamma}\langle\xi_4\rangle.$$

Case 2: $|\xi_2|\le10|\xi_4|$. Note $|\xi_1|\sim|\xi_4|$, and so

$$\big|M(s;\xi_1,\cdots,\xi_4)\big|\lesssim\langle\xi_1\rangle^{2\gamma+1}\lesssim\langle\xi_1\rangle^{\gamma}\langle\xi_4\rangle^{\gamma+1}.$$

Hence, we get

$$\int_0^\tau \big|M(s;\xi_1,\cdots,\xi_4)\big|\,ds \lesssim \tau\langle\xi_1\rangle^\gamma\langle\xi_4\rangle^{\gamma+1}.$$

Combining the findings in Case 1 and Case 2, we get (3.7) which further shows that for any $\gamma>\frac{3}{2}$,

$$\begin{aligned}\left|\int_0^\tau \left\langle e^{is\partial_x^2}\partial_x J^\gamma\big(f_1^2 f_4\big), J^\gamma f_1\right\rangle ds\right| \lesssim&\tau\big\|J^\gamma f_1\big\|_{L^2}^2\big\|J^1 f_1\big\|_{L^\infty}\|f_4\|_{L^\infty}+\tau\big\|J^\gamma f_1\big\|_{L^2}^2\big\|f_1\big\|_{L^\infty}\big\|J^1 f_4\big\|_{L^\infty}\\ &+\tau\big\|J^\gamma f_1\big\|_{L^2}\big\|f_1\big\|_{L^\infty}^2\big\|J^{\gamma+1}f_4\big\|_{L^2}\lesssim\tau\|f_1\|_{H^\gamma}^3\|f_4\|_{H^{\gamma_0}}.\end{aligned}$$

(iii) Again, we denote

$$\breve{\Gamma}=\{(\xi_1,\xi_2,\xi_3,\xi_4)\ :\ \xi_1+\xi_2+\xi_3+\xi_4=0,\ |\xi_1|\ge|\xi_2|\ge|\xi_3|\ge|\xi_4|\},$$

and

$$\breve{M}(s;\xi_1,\cdots,\xi_4)=\sum_{j=1}^4 e^{-i\xi_j^2 s}i\xi_j\langle\xi_j\rangle^{2\gamma}.$$

Then, by symmetry, there exists some absolute constant $C>0$ such that

$$\int_0^\tau \left\langle e^{is\partial_x^2}\partial_x J^\gamma\big(f^3\big), J^\gamma f\right\rangle ds = C\int_0^\tau\int_{\breve{\Gamma}}\breve{M}(s;\xi_1,\cdots,\xi_4)\widehat{f}(\xi_1)\widehat{f}(\xi_2)\widehat{f}(\xi_3)\widehat{f}(\xi_4)d\sigma ds.$$

We claim that in $\breve{\Gamma}$,

$$\int_0^\tau\left|\breve{M}(s;\xi_1,\cdots,\xi_4)\right|\,ds\lesssim\tau\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle. \tag{3.8}$$

With (3.8) in hand, similarly as the above, we can get for any $\gamma>\frac{3}{2}$,

$$\begin{aligned}\left|\int_0^\tau \left\langle e^{is\partial_x^2}\partial_x J^\gamma\big(f^3\big), J^\gamma f\right\rangle ds\right| \lesssim&\tau\big\|J^\gamma f\big\|_{L^2}\big\|J^\gamma f\big\|_{L^2}\big\|J^1 f\big\|_{L^\infty}\big\|f\big\|_{L^\infty}\\ \lesssim&\tau\|f\|_{H^\gamma}^2\|f\|_{H^{\frac{3}{2}+}}\|f\|_{H^{\frac{1}{2}+}}\lesssim\tau\|f\|_{H^\gamma}^4.\end{aligned}$$

Now it is left to prove the claim (3.8). In the case of $|\xi_2|\gg|\xi_3|$, we write

$$\begin{aligned}\breve{M}(s;\xi_1,\cdots,\xi_4)=&\left[e^{-i\xi_2^2 s}\big(\xi_1\langle\xi_1\rangle^{2\gamma}+\xi_2\langle\xi_2\rangle^{2\gamma}\big)\right]+\left[\big(e^{-i\xi_1^2 s}-e^{-i\xi_2^2 s}\big)i\xi_1\langle\xi_1\rangle^{2\gamma}\right]\\ &+\left[e^{-i\xi_3^2 s}i\xi_3\langle\xi_3\rangle^{2\gamma}+e^{-i\xi_4^2 s}i\xi_4\langle\xi_4\rangle^{2\gamma}\right].\end{aligned}$$

For the first two terms, as proved before in (i), we have

$$\left|e^{-i\xi_2^2 s}\big(\xi_1\langle\xi_1\rangle^{2\gamma}+\xi_2\langle\xi_2\rangle^{2\gamma}\big)\right|\lesssim\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle,$$

and

$$\left|\int_0^\tau\left[\big(e^{-i\xi_1^2 s}-e^{-i\xi_2^2 s}\big)i\xi_1\langle\xi_1\rangle^{2\gamma}\right]\,ds\right|\lesssim\tau\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle.$$

For the third term, since $|\xi_4|\le|\xi_3|\le|\xi_2|\sim|\xi_1|$, we have

$$\left|e^{-i\xi_3^2 s}i\xi_3\langle\xi_3\rangle^{2\gamma}+e^{-i\xi_4^2 s}i\xi_4\langle\xi_4\rangle^{2\gamma}\right|\lesssim\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle.$$

Hence, we get (3.8) in this case. In the case of $|\xi_2|\lesssim|\xi_3|$, we have $|\xi_1|\sim|\xi_2|\sim|\xi_3|$, and so

$$\left|\breve{M}(s;\xi_1,\cdots,\xi_4)\right|\lesssim\langle\xi_1\rangle^\gamma\langle\xi_2\rangle^\gamma\langle\xi_3\rangle.$$

This again shows (3.8), and thus completes the proof of (iii). □

4. Exponential scheme: the coarse version

With the preparations from the previous section, we now build the convergence result for the coarse version scheme (2.8) introduced as the first step in Section 2.2. Recall

$$v_I^{n+1} = \Phi_I^n(v_I^n) = v_I^n - 2\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3 + i\tau\partial_x^2}\partial_x^{-1}\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right],\ v_I^0 = u_0, \quad n = 0,\ldots,\frac{T}{\tau}-1, \tag{4.1}$$

up to some fixed final time $T > 0$. The convergence result of the coarse scheme (4.1) is stated as follows.

Proposition 4.1 (Convergence of coarse scheme)**.** *Assume that $u_0 \in H^{\gamma_1+3}(\mathbb{T})$ with $\gamma_1 > -\frac{1}{2}$. There exist constants $\tau_0 > 0$ and $C > 0$ depending only on T and $\|v\|_{L^\infty((0,T);H^{\gamma_1+3})}$, such that for any $0 < \tau \le \tau_0$,*

$$\|v(t_n) - v_I^n\|_{H^{\gamma_1}} \le C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}.$$

This result serves as an intermediate step for proving our main theorem, and meanwhile it could be of independent interest as we explained in Section 2.2. To prove Proposition 4.1, we first consider the following local error estimate.

Lemma 4.2 (Local error estimate)**.** *Let $\eta \ge -1$, $\beta \in [0,1]$ and $\widetilde{\gamma} = \max\{1+\eta+2\beta, \beta+\frac{1}{2}+\}$. Assume that $u_0 \in H^{\widetilde{\gamma}}(\mathbb{T})$, then, there exist constants $\tau_0 > 0$ and $C > 0$ depending only on T and $\|v\|_{L^\infty((0,T);H^{\widetilde{\gamma}})}$, such that for any $0 < \tau \le \tau_0$,*

$$\|v(t_{n+1}) - \Phi_I^n(v(t_n))\|_{H^\eta} \le C\tau^{1+\beta}, \quad n = 0, 1, \ldots, \frac{T}{\tau} - 1.$$

Proof. Using (2.6) yields for $0 \le n < T/\tau$,

$$\begin{aligned}\zeta^n :=& v(t_{n+1}) - \Phi_I^n(v(t_n))\\ =& 2\int_0^\tau \mathrm{e}^{(t_n+s)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+s)\partial_x^3}v(t_n+s)\right)^3 ds - 2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 ds.\end{aligned}$$

Since $\mathrm{e}^{t\partial_x^3}f \in \mathbb{R}$, if $f \in \mathbb{R}$, and so by taking the Fourier transform, we have

$$\begin{aligned}\widehat{\zeta^n}(\xi) = 2i\xi\int_0^\tau\int_{\xi=\xi_1+\xi_2+\xi_3}\Big[&\mathrm{e}^{-i(t_n+s)\alpha}\widehat{v}(t_n+s,\xi_1)\widehat{v}(t_n+s,\xi_2)\widehat{v}(t_n+s,\xi_3)\\ &- \mathrm{e}^{-it_n\alpha}\mathrm{Re}(\mathrm{e}^{-is\xi^2})\widehat{v}(t_n,\xi_1)\widehat{v}(t_n,\xi_2)\widehat{v}(t_n,\xi_3)\Big]\,d\sigma ds.\end{aligned}$$

Here we denote $d\sigma = (d\xi_1)(d\xi_2)$ and $\alpha = \xi^3 - \xi_1^3 - \xi_2^3 - \xi_3^3$ for short. Now we split the expression above into two pieces:

$$\widehat{\zeta^n}(\xi) := \widehat{I}_1(\xi) + \widehat{I}_2(\xi),$$

where

$$\begin{aligned}\widehat{I}_1(\xi) =& 2i\xi\int_0^\tau\int_{\xi=\xi_1+\xi_2+\xi_3}\mathrm{e}^{-it_n\alpha}\left(\mathrm{e}^{-is\alpha} - \mathrm{Re}\left(\mathrm{e}^{-is\xi^2}\right)\right)\widehat{v}(t_n+s,\xi_1)\widehat{v}(t_n+s,\xi_2)\widehat{v}(t_n+s,\xi_3)d\xi_1 d\xi_2 ds,\\ \widehat{I}_2(\xi) =& 2i\xi\int_0^\tau\int_{\xi=\xi_1+\xi_2+\xi_3}\mathrm{e}^{-it_n\alpha}\mathrm{Re}\left(\mathrm{e}^{-is\xi^2}\right)\Big[\widehat{v}(t_n+s,\xi_1)\widehat{v}(t_n+s,\xi_2)\widehat{v}(t_n+s,\xi_3)\\ &- \widehat{v}(t_n,\xi_1)\widehat{v}(t_n,\xi_2)\widehat{v}(t_n,\xi_3)\Big]\,d\sigma ds.\end{aligned}$$

We then estimate I_1 and I_2 in a sequel. Without loss of generality, we assume $|\xi_1| \ge |\xi_2| \ge |\xi_3|$, and then we have

$$|\alpha| = |3(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)| \lesssim |\xi_1^2\xi_2|.$$

Thus, for any $\eta \ge -1$,

$$|\xi|\langle\xi\rangle^\eta\left|\mathrm{e}^{-is\alpha} - \mathrm{Re}(\mathrm{e}^{-is\xi^2})\right| \le \langle\xi\rangle^{1+\eta}(s|\alpha| + s|\xi|^2)^\beta \le s^\beta\langle\xi_1\rangle^{1+\eta+2\beta}\langle\xi_2\rangle^\beta, \quad \beta \in [0,1].$$

This implies that

$$\langle\xi\rangle^\eta|\widehat{I}_1(\xi)| \lesssim \int_0^\tau\int_{\xi=\xi_1+\xi_2+\xi_3} s^\beta\langle\xi_1\rangle^{1+\eta+2\beta}\langle\xi_2\rangle^\beta|\widehat{v}(t_n+s,\xi_1)\widehat{v}(t_n+s,\xi_2)\widehat{v}(t_n+s,\xi_3)|d\sigma ds,$$

and so we have

$$\|I_1\|_{H^\eta} \le C\tau^{1+\beta}\|v\|_{L^\infty((0,T);H^{1+\eta+2\beta})}\|v\|_{L^\infty((0,T);H^{\beta+\frac12+})}\|v\|_{L^\infty((0,T);H^{\frac12+})} \le C\tau^{1+\beta}\|v\|^3_{L^\infty((0,T);H^{\tilde\gamma})}.$$

For I_2, by symmetry we have

$$\begin{aligned}|\widehat{I_2}(\xi)| &\lesssim \left|\xi\int_0^\tau\int_{\xi=\xi_1+\xi_2+\xi_3}\Big[\widehat{v}(t_n+s,\xi_1)\widehat{v}(t_n+s,\xi_2)\widehat{v}(t_n+s,\xi_3)-\widehat{v}(t_n,\xi_1)\widehat{v}(t_n,\xi_2)\widehat{v}(t_n,\xi_3)\Big]\,d\sigma ds\right|\\ &\lesssim \int_0^\tau\int_{\xi=\xi_1+\xi_2+\xi_3}|\xi|\cdot|\widehat{v}(t_n+s,\xi_1)-\widehat{v}(t_n,\xi_1)|\cdot\Big(|\widehat{v}(t_n+s,\xi_2)\widehat{v}(t_n+s,\xi_3)|+|\widehat{v}(t_n,\xi_2)\widehat{v}(t_n,\xi_3)|\Big)\,d\sigma ds.\end{aligned}$$

Then, by Lemma 3.1 we have

$$\begin{aligned}\big\|I_2\big\|_{H^\eta} \lesssim&\tau\sup_{s\in[0,\tau]}\big\|v(t_n+s)-v(t_n)\big\|_{H^{1+\eta}}\big\|v\big\|^2_{L^\infty((0,T);H^{\frac12+})}\\ &+\tau\sup_{s\in[0,\tau]}\big\|v(t_n+s)-v(t_n)\big\|_{H^{\frac12+}}\big\|v\big\|_{L^\infty((0,T);H^{1+\eta})}\big\|v\big\|_{L^\infty((0,T);H^{\frac12+})}\\ \lesssim&\tau\sup_{s\in[0,\tau]}\big\|v(t_n+s)-v(t_n)\big\|_{H^{1+\eta}}\big\|v\big\|^2_{L^\infty((0,T);H^{\tilde\gamma})}+\tau\sup_{s\in[0,\tau]}\big\|v(t_n+s)-v(t_n)\big\|_{H^{\frac12+}}\big\|v\big\|^2_{L^\infty((0,T);H^{\tilde\gamma})}.\end{aligned}\tag{4.2}$$

From (2.6), we have $\partial_t v = \mathrm{e}^{t\partial_x^3}\partial_x\left(\mathrm{e}^{-t\partial_x^3}v\right)^3$. This fact yields

$$v(t_n+s)-v(t_n)=\int_0^s\partial_t v(t_n+t)dt = 2\int_0^s \mathrm{e}^{(t_n+t)\partial_x^3}\partial_x\left(\mathrm{e}^{-(t_n+t)\partial_x^3}v(t_n+t)\right)^3 dt.$$

Hence, by Lemma 3.1 we obtain

$$\big\|v(t_n+s)-v(t_n)\big\|_{H^{1+\eta}} \le C\tau\big\|v\big\|_{L^\infty((0,T);H^{2+\eta})}\big\|v\big\|^2_{L^\infty((0,T);H^{\frac12+})}.\tag{4.3}$$

Moreover, we have the direct estimate

$$\big\|v(t_n+s)-v(t_n)\big\|_{H^{1+\eta}} \le 2\big\|v\big\|_{L^\infty((0,T);H^{1+\eta})}.\tag{4.4}$$

Interpolating (4.3) and (4.4) gives

$$\big\|v(t_n+s)-v(t_n)\big\|_{H^{1+\eta}} \le C\tau^\beta,$$

where C depends on $\|v\|_{L^\infty((0,T);H^{1+\eta+\beta}\cap H^{\frac12+})}$. Similarly, we have $\big\|v(t_n+s)-v(t_n)\big\|_{H^{\frac12+}} \le C\tau^\beta$, where C depends on $\|v\|_{L^\infty((0,T);H^{\frac12+\beta+})}$. Inserting the last two estimates above into (4.2), we find

$$\|I_2\|_{H^\eta} \le C\tau^{1+\beta},$$

where C depends on $\|v\|_{L^\infty((0,T);H^{\tilde\gamma})}$. Consequently, combining with the estimates on I_1 and I_2, we obtain

$$\|v(t_{n+1})-\Phi_I^n(v(t_n))\|_{H^\eta} \le C\tau^{1+\beta}, \qquad \beta\in[0,1].$$

This finishes the proof of the lemma. □

To build the stability result, we need the following lemma.

Lemma 4.3. *Let $u_0\in H^{\gamma_0+1}(\mathbb{T})$ with $\gamma_0>\frac32$. There exist constants $\tau_0>0$ and $C>0$ depending only on T and $\|v\|_{L^\infty((0,T);H^{\gamma_0+1})}$, such that for any $0<\tau\le\tau_0$,*

$$\|\Phi_I^n(v(t_n))-\Phi_I^n(v_I^n)\|_{H^{\gamma_0}} \le (1+C\tau)\|v(t_n)-v_I^n\|_{H^{\gamma_0}}+C\tau\|v(t_n)-v_I^n\|^3_{H^{\gamma_0}}, \quad n=0,\dots,\frac{T}{\tau}-1.$$

Proof. According to the definition of Φ_I^n, we have

$$\begin{aligned}&\Phi_I^n(v(t_n))-\Phi_I^n(v_I^n)\\ =&v(t_n)+2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}\partial_x\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 ds - v_I^n - 2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}\partial_x\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3 ds\\ =&v(t_n)-v_I^n+2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3-\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right] ds =: h^n+\Psi^n,\end{aligned}$$

where we denote

$$h^n = v(t_n) - v_I^n, \quad \Psi^n = 2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right]ds.$$

Using this equality, we have

$$\begin{aligned}\left\|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\right\|_{H^{\gamma_0}}^2 =&\langle J^{\gamma_0}\left(\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\right), J^{\gamma_0}\left(\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\right)\rangle\\ =&\left\|h^n\right\|_{H^{\gamma_0}}^2 + 2\langle J^{\gamma_0}h^n, J^{\gamma_0}\Psi^n\rangle + \langle J^{\gamma_0}\Psi^n, J^{\gamma_0}\Psi^n\rangle.\end{aligned} \tag{4.5}$$

Note that

$$\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3 = \left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^3 - 3\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^2\mathrm{e}^{-t_n\partial_x^3}v(t_n) + 3\mathrm{e}^{-t_n\partial_x^3}h^n\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^2,$$

so we can write

$$\begin{aligned}\Psi^n =&2\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^3\right]ds - 6\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^2\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right]ds\\ &+ 6\mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}h^n\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^2\right]ds.\end{aligned}$$

Hence, we have

$$\begin{aligned}\langle J^{\gamma_0}h^n, J^{\gamma_0}\Psi^n\rangle =&2\mathrm{Re}\int_0^\tau\left\langle J^{\gamma_0}h^n, \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}J^{\gamma_0}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^3\right]\right\rangle ds\\ &- 6\mathrm{Re}\int_0^\tau\left\langle J^{\gamma_0}h^n, \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}J^{\gamma_0}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^2\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right]\right\rangle ds\\ &+ 6\mathrm{Re}\int_0^\tau\left\langle J^{\gamma_0}h^n, \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}J^{\gamma_0}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}h^n\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^2\right]\right\rangle ds.\end{aligned}$$

Then, using Proposition 3.3, we obtain

$$\begin{aligned}\left|\langle J^{\gamma_0}h^n, J^{\gamma_0}\Psi^n\rangle\right| \lesssim&\tau\left(\|h^n\|_{H^{\gamma_0}}^4 + \|h^n\|_{H^{\gamma_0}}^3\|v(t_n)\|_{H^{\gamma_0+1}} + \|h^n\|_{H^{\gamma_0}}^2\|v(t_n)\|_{H^{\gamma_0+1}}^2\right)\\ \le& C\tau\left(\|h^n\|_{H^{\gamma_0}}^2 + \|h^n\|_{H^{\gamma_0}}^4\right),\end{aligned} \tag{4.6}$$

where C depends on $\|v\|_{L^\infty((0,T);H^{\gamma_0+1})}$.

Now we consider the term $\langle J^{\gamma_0}\Psi^n, J^{\gamma_0}\Psi^n\rangle$. By (4.1), it is equal to

$$\begin{aligned}&- 4\mathrm{Re}\int_0^\tau\Big\langle J^{\gamma_0}\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3 + i\tau\partial_x^2}\partial_x^{-1}\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right)\right],\\ &\qquad\mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}J^{\gamma_0}\partial_x\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^3\Big\rangle ds &(4.7a)\\ &+ 12\mathrm{Re}\int_0^\tau\Big\langle J^{\gamma_0}\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3 + i\tau\partial_x^2}\partial_x^{-1}\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right)\right],\\ &\qquad\mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^2\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right]\Big\rangle ds &(4.7b)\\ &- 12\mathrm{Re}\int_0^\tau\Big\langle J^{\gamma_0}\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3 + i\tau\partial_x^2}\partial_x^{-1}\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right)\right],\\ &\qquad\mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}h^n\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^2\right]\Big\rangle ds. &(4.7c)\end{aligned}$$

The three terms can be treated in the same way, and so we only consider (4.7a). By integration-by-parts and Lemma 3.1, it follows that

$$\begin{aligned}|(4.7\mathrm{a})| =&\left|4\int_0^\tau\Big\langle J^{\gamma_0}\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3 + i\tau\partial_x^2}\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right)\right], \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}J^{\gamma_0}\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^3\Big\rangle ds\right|\\ \lesssim&\tau\left\|J^{\gamma_0}\left[\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right]\right\|_{L^2}\left\|J^{\gamma_0}\left(\mathrm{e}^{-t_n\partial_x^3}h^n\right)^3\right\|_{L^2} \le C\tau\left(\|h^n\|_{H^{\gamma_0}}^4 + \|h^n\|_{H^{\gamma_0}}^6\right).\end{aligned}$$

Combining it with the analogous estimates for the terms (4.7b) and (4.7c), we get

$$\left|\langle J^{\gamma_0}\Psi^n, J^{\gamma_0}\Psi^n\rangle\right| \le C\tau\|h^n\|_{H^{\gamma_0}}^2 + C\tau\|h^n\|_{H^{\gamma_0}}^6. \tag{4.8}$$

Together with (4.5), (4.6), (4.8), Hölder's and Cauchy-Schwartz's inequalities, we get

$$\|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\gamma_0}}^2 \le (1 + C\tau)\|h^n\|_{H^{\gamma_0}}^2 + C\tau\|h^n\|_{H^{\gamma_0}}^6.$$

Since $\sqrt{1 + C\tau} \sim 1 + C\tau$ when τ is small enough, this implies that

$$\|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\gamma_0}} \le (1 + C\tau)\|v(t_n) - v_I^n\|_{H^{\gamma_0}} + C\tau\|v(t_n) - v_I^n\|_{H^{\gamma_0}}^3,$$

and the lemma is proved. □

With the prepared lemmas before, we can obtain the *a priori* estimate of the numerical solution v_I^n, which is done by establishing a weaker convergence rate of the scheme as in [35].

Lemma 4.4 (*A priori* estimate)**.** *Let v_I^n be defined in (4.1) and $u_0 \in H^{\gamma_1+3}(\mathbb{T})$ with $\gamma_1 > -\frac{1}{2}$. Then, there exist constants $\tau_0 > 0$ and $C > 0$ depending only on ϵ, T and $\|v\|_{L^\infty((0,T);H^{\gamma_1+3})}$, such that for any $0 < \tau \le \tau_0$ and any $\epsilon > 0$,*

$$\left\|v_I^n\right\|_{H^{\gamma_1+2-\epsilon}} \le C, \quad n = 0, 1, \ldots, \frac{T}{\tau}.$$

Proof. Denote $\eta_1 = \gamma_1 + 2 - \epsilon$ and we consider only $\eta_1 > \frac{3}{2}$ by choosing $\epsilon > 0$ small enough. We write

$$v(t_{n+1}) - v_I^{n+1} = v(t_{n+1}) - \Phi_I^n(v(t_n)) + \Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n).$$

Then, we have

$$\|v(t_{n+1}) - v_I^{n+1}\|_{H^{\eta_1}} \le \|v(t_{n+1}) - \Phi_I^n(v(t_n))\|_{H^{\eta_1}} + \|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\eta_1}}.$$

According to Lemma 4.2, we choose $\eta = \eta_1$ and $\beta = \frac{\epsilon}{2}$ to obtain

$$\|v(t_{n+1}) - \Phi_I^n(v(t_n))\|_{H^{\eta_1}} \le C\tau^{1+\frac{\epsilon}{2}}, \tag{4.9}$$

where C depends on $\|v\|_{L^\infty((0,T);H^{\gamma_1+3})}$.

Using Lemma 4.3 and (4.9) yields that there exist some positive constants $C_j, j = 1, 2, 3$ such that

$$\|v(t_{n+1}) - v_I^{n+1}\|_{H^{\eta_1}} \le C_1\tau^{1+\frac{\epsilon}{2}} + (1 + C_2\tau)\|v(t_n) - v_I^n\|_{H^{\eta_1}} + C_3\tau\|v(t_n) - v_I^n\|_{H^{\eta_1}}^3. \tag{4.10}$$

We claim that there exists some $\tau_0 > 0$ (to be determined) such that for any $\tau \in (0, \tau_0]$,

$$\|v(t_n) - v_I^n\|_{H^{\eta_1}} \le C_1\tau^{1+\frac{\epsilon}{2}} \sum_{j=0}^{n} (1 + 2C_2\tau)^j, \quad n = 0, 1, \ldots, \frac{T}{\tau}. \tag{4.11}$$

We prove it by the induction. Note that (4.11) clearly holds for $n = 0$. Now we assume that it holds till some $n_0 : 0 \le n_0 \le \frac{T}{\tau} - 1$, i.e.,

$$\|v(t_n) - v_I^n\|_{H^{\eta_1}} \le C_1\tau^{1+\frac{\epsilon}{2}} \sum_{j=0}^{n} (1 + 2C_2\tau)^j, \text{ for any } 0 \le n \le n_0. \tag{4.12}$$

From (4.12), we have that for any $0 \le n \le n_0$,

$$\|v(t_n) - v_I^n\|_{H^{\eta_1}} \le C_5\tau^{\frac{\epsilon}{2}}, \tag{4.13}$$

where $C_5 = C_1(2C_2)^{-1}e^{2C_2T}$. Then by (4.10), we find

$$\left\|v(t_{n_0+1}) - v_I^{n_0+1}\right\|_{H^{\eta_1}} \le C_1\tau^{1+\frac{\epsilon}{2}} + \left(1 + C_2\tau + C_3C_5^2\tau^{1+\epsilon}\right) \cdot C_1\tau^{1+\frac{\epsilon}{2}} \sum_{j=0}^{n_0} (1 + 2C_2\tau)^j.$$

Choose $\tau_0 > 0$ such that $C_3C_5^2\tau_0^\epsilon \le C_2$, then for any $\tau \in (0, \tau_0]$, we obtain that

$$\begin{aligned}\left\|v(t_{n_0+1}) - v_I^{n_0+1}\right\|_{H^{\eta_1}} \le& C_1\tau^{1+\frac{\epsilon}{2}} + (1 + 2C_2\tau) \cdot C_1\tau^{1+\frac{\epsilon}{2}} \sum_{j=0}^{n_0} (1 + 2C_2\tau)^j \\ =& C_1\tau^{1+\frac{\epsilon}{2}} + C_1\tau^{1+\frac{\epsilon}{2}} \sum_{j=0}^{n_0} (1 + 2C_2\tau)^{j+1} = C_1\tau^{1+\frac{\epsilon}{2}} \sum_{j=0}^{n_0+1} (1 + 2C_2\tau)^j.\end{aligned}$$

This finishes the induction and proves (4.11). Hence, we get (4.13) for any $n = 0, 1, \ldots, \frac{T}{\tau}$. Therefore,

$$\|v_I^n\|_{H^{\eta_1}} \le \|v(t_n) - v_I^n\|_{H^{\eta_1}} + \|v(t_n)\|_{H^{\eta_1}} \le C,$$

where $C>0$ depends only on ϵ, T and $\|v\|_{L^\infty((0,T);H^{\gamma_1+3})}$. This gives the desired estimate. $\square$

The stability result is given as the following lemma.

Lemma 4.5 (Stability)**.** *Let $u_0 \in H^{\gamma_1+3}(\mathbb{T})$ with $\gamma_1 > -\frac{1}{2}$. Then, there exist constants $\tau_0 > 0$ and $C > 0$ depending only on γ_1, T and $\|v\|_{L^\infty((0,T);H^{\gamma_1+3})}$, such that for any $0 < \tau \le \tau_0$,*

$$\|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\gamma_1}} \le (1+C\tau)\|v(t_n) - v_I^n\|_{H^{\gamma_1}}, \quad n = 0,1,\ldots,\frac{T}{\tau}-1.$$

Proof. As in the proof of Lemma 4.3, we write

$$\|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\gamma_1}}^2 = \langle J^{\gamma_1} h^n, J^{\gamma_1} h^n\rangle + 2\langle J^{\gamma_1} h^n, J^{\gamma_1}\Psi^n\rangle + \langle J^{\gamma_1}\Psi^n, J^{\gamma_1}\Psi^n\rangle. \tag{4.14}$$

Here we use the same notations as in Lemma 4.3. We will estimate each term in (4.14) individually. For the second term, we rewrite Ψ^n as

$$\Psi^n = \mathrm{Re}\int_0^\tau \mathrm{e}^{t_n\partial_x^3 + is\partial_x^2}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}h^n \cdot F_n\big(v(t_n), v_I^n\big)\right] ds,$$

where

$$F_n\big(v(t_n), v_I^n\big) = 2\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^2 + 2\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right) + 2\left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^2.$$

Hence, we get that

$$\langle J^{\gamma_1}h^n, J^{\gamma_1}\Psi^n\rangle = \mathrm{Re}\int_0^\tau \left\langle J^{\gamma_1}h^n, \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}J^{\gamma_1}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}h^n\cdot F_n\big(v(t_n),v_I^n\big)\right]\right\rangle ds.$$

Then, by Proposition 3.3 (i) (in which $\gamma=\gamma_1, \gamma_0 = \gamma_1 + 2 - \epsilon$) and Lemma 4.4 (in which $\epsilon \le \frac{\gamma_1}{2} + \frac{1}{4}$ is small enough), we get

$$\big|\langle J^{\gamma_1}h^n, J^{\gamma_1}\Psi^n\rangle\big| \le C\tau\|h^n\|_{H^{\gamma_1}}^2, \tag{4.15}$$

where C depends only on γ_1, T and $\|v\|_{L^\infty((0,T);H^{\gamma_1+3})}$.

Now we consider the term $\langle J^{\gamma_1}\Psi^n, J^{\gamma_1}\Psi^n\rangle$. Using the formula (4.1), we have

$$\Psi^n = -2\mathrm{Re}\left[i\mathrm{e}^{t_n\partial_x^3+i\tau\partial_x^2}\partial_x^{-1}\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right)\right].$$

Then by integration-by-parts, we have

$$\begin{aligned}\langle J^{\gamma_1}\Psi^n, J^{\gamma_1}\Psi^n\rangle = -2\mathrm{Re}\int_0^\tau \Big\langle & i\mathrm{e}^{t_n\partial_x^3+i\tau\partial_x^2}J^{\gamma_1}\partial_x^{-1}\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right),\\ & \mathrm{e}^{t_n\partial_x^3+is\partial_x^2}J^{\gamma_1}\partial_x\left[\mathrm{e}^{-t_n\partial_x^3}h^n\cdot F_n\big(v(t_n),v_I^n\big)\right]\Big\rangle ds\\ =2\mathrm{Re}\int_0^\tau \Big\langle i\mathrm{e}^{i\tau\partial_x^2}J^{\gamma_1}&\left(\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right), \mathrm{e}^{is\partial_x^2}J^{\gamma_1}\left[\mathrm{e}^{-t_n\partial_x^3}h^n\cdot F_n\big(v(t_n),v_I^n\big)\right]\Big\rangle ds.\end{aligned}$$

Therefore, by Hölder's inequality, we get

$$|\langle J^{\gamma_1}\Psi^n, J^{\gamma_1}\Psi^n\rangle| \lesssim \tau\left\|\left(\mathrm{e}^{-t_n\partial_x^3}v(t_n)\right)^3 - \left(\mathrm{e}^{-t_n\partial_x^3}v_I^n\right)^3\right\|_{H^{\gamma_1}}\left\|\mathrm{e}^{-t_n\partial_x^3}h^n\cdot F_n\big(v(t_n),v_I^n\big)\right\|_{H^{\gamma_1}}.$$

Now by Lemma 3.2 and Lemma 4.4, we obtain

$$\big|\langle J^{\gamma_1}\Psi^n, J^{\gamma_1}\Psi^n\rangle\big| \le C\tau\|h^n\|_{H^{\gamma_1}}^2. \tag{4.16}$$

Inserting (4.15) and (4.16) into (4.14), it follows that

$$\|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\gamma_1}}^2 \le (1+C\tau)\|h^n\|_{H^{\gamma_1}}^2.$$

This proves the lemma. $\square$

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. From Lemma 4.2 (choosing $\beta = 1$), we obtain

$$\|v(t_{n+1}) - \Phi_I^n(v(t_n))\|_{H^{\gamma_1}} \le C\tau^2.$$

Furthermore, from Lemma 4.5, we have

$$\begin{aligned}\left\|v(t_{n+1}) - v_I^{n+1}\right\|_{H^{\gamma_1}} &\le \|v(t_n) - \Phi_I^n(v(t_n))\|_{H^{\gamma_1}} + \|\Phi_I^n(v(t_n)) - \Phi_I^n(v_I^n)\|_{H^{\gamma_1}} \\ &\le C\tau^2 + (1 + C\tau)\|v(t_n) - v_I^n\|_{H^{\gamma_1}}.\end{aligned}$$

Then, the claimed result is followed from the iteration and Gronwall's inequality. □

5. Exponential scheme with iterative regularizing

In this section, we analyze the scheme after the first and the second regularizing iterations in Section 2.2. The technique based on the Miura transform turns out to reduce the loss of regularity. Recall our scheme in the second step of Section 2.2:

$$v_{II}^{n+1} = \partial_x^{-1} V^{n+1} - \mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_{n+1}\partial_x^3} v_I^{n+1}\right)^2 + m_0, \quad n \ge 0, \quad v_{II}^0 = u_0.$$

With the analysis of the coarse scheme, we have the following error estimate for this first regularizing.

Proposition 5.1. *Assume that $u_0 \in H^{\gamma+1}(\mathbb{T})$ with $\gamma > \frac{3}{2}$. Then, there exist positive constants τ_0, C, $\tilde{C}$ which depend only on γ, T and $\|v\|_{L^\infty((0,T);H^{\gamma+1})}$, such that for any $0 < \tau \le \tau_0$,*

$$|v_{II}^n\|_{H^\gamma} \le \widetilde{C} \quad \text{and} \quad \|v(t_n) - v_{II}^n\|_{H^{\gamma-1}} \le C\tau, \quad 0 \le n \le \frac{T}{\tau}. \tag{5.1}$$

Proof. By Lemma 4.4 (where $\gamma_1 = \gamma - 2$) and Proposition 2.1, we have that for any arbitrarily small $\epsilon > 0$,

$$\left\|v_I^n\right\|_{H^{\gamma-\epsilon}} \le C, \quad \left\|V^n\right\|_{H^{\gamma-1}} \le C. \tag{5.2}$$

Here and after in this proof, the constant C depends only on T and $\|v\|_{L^\infty((0,T);H^{\gamma+1})}$ that may vary line by line. Therefore, by Lemma 3.1 we get that

$$\left\|v_{II}^n\right\|_{H^\gamma} \lesssim \left\|V^n\right\|_{H^{\gamma-1}} + \left\|\left(\mathrm{e}^{-t_n\partial_x^3} v_I^n\right)^2\right\|_{H^{\gamma-1}} \lesssim \left\|V^n\right\|_{H^{\gamma-1}} + \left\|v_I^n\right\|_{H^{\gamma-1}}^2 \le \widetilde{C}.$$

This proves the first part of (5.1).

Now we consider the second assertion in (5.1). According to (2.5), we have that

$$v(t_n) - v_{II}^n = \partial_x^{-1}\left[V(t_n) - V^n\right] - \mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_n\partial_x^3} v(t_n)\right)^2 - \left(\mathrm{e}^{-t_n\partial_x^3} v_I^n\right)^2\right].$$

Hence, by the Kato-Ponce inequality in Lemma 3.2, we obtain

$$\begin{aligned}\|v(t_n) - v_{II}^n\|_{H^{\gamma-1}} &\lesssim \|V(t_n) - V^n\|_{H^{\gamma-2}} + \left\|\mathrm{e}^{-t_n\partial_x^3}(v(t_n) - v_I^n) \cdot \mathrm{e}^{-t_n\partial_x^3}(v(t_n) + v_I^n)\right\|_{H^{\gamma-2}} \\ &\lesssim \|V(t_n) - V^n\|_{H^{\gamma-2}} + \|v(t_n) - v_I^n\|_{H^{\gamma-2}}\|v(t_n) + v_I^n\|_{H^{\widetilde{\gamma}}},\end{aligned} \tag{5.3}$$

where $\widetilde{\gamma} = \max\{\frac{1}{2}+, \gamma - 2\}$. Since $u_0 \in H^{\gamma+1}(\mathbb{T})$ and $U(0) = \partial_x u_0 + u_0^2$, we get $U(0) \in H^\gamma(\mathbb{T})$. Therefore, from Proposition 2.1, we have

$$\|V(t_n) - V^n\|_{H^{\gamma-1}} \le C\tau. \tag{5.4}$$

Moreover, by Proposition 4.1 (in which $\gamma_1 = \gamma - 2$), we have

$$\|v(t_n) - v_I^n\|_{H^{\gamma-2}} \le C\tau. \tag{5.5}$$

Furthermore, by (5.2), we get

$$\|v(t_n) + v_I^n\|_{H^{\widetilde{\gamma}}} \le C. \tag{5.6}$$

Inserting (5.4), (5.5) and (5.6) into (5.3), the proof is finished. □

In the continuation of the iterative regularizing process, recall from Section 2.2 that after the second regularizing, the scheme reads

$$v^{n+1} = \partial_x^{-1} V^{n+1} - \mathrm{e}^{t_{n+1}\partial_x^3}\partial_x^{-1}\left(\mathrm{e}^{-t_{n+1}\partial_x^3} v_{II}^{n+1}\right)^2 + m_0, \quad n \geq 0, \quad v^0 = u_0.$$

Its convergence is stated as Theorem 2.2, and the proof is given below with the help of the previous estimates.

Proof of Theorem 2.2. Due to (2.5), we get

$$v(t_n) - v^n = \partial_x^{-1}\left[V(t_n) - V^n\right] - \mathrm{e}^{t_n\partial_x^3}\partial_x^{-1}\left[\left(\mathrm{e}^{-t_n\partial_x^3} v(t_n)\right)^2 - \left(\mathrm{e}^{-t_n\partial_x^3} v_{II}^n\right)^2\right].$$

Since $\gamma > \frac{3}{2}$, using the Kato-Ponce inequality in Lemma 3.1, we get

$$\begin{aligned}\|v(t_n) - v^n\|_{H^\gamma} \leq & \|V(t_n) - V^n\|_{H^{\gamma-1}} + \left\|\mathrm{e}^{t_n\partial_x^3}\left(v(t_n) - v_{II}^n\right)\cdot \mathrm{e}^{t_n\partial_x^3}\left(v(t_n) + v_{II}^n\right)\right\|_{H^{\gamma-1}} \\ \leq & \|V(t_n) - V^n\|_{H^{\gamma-1}} + \|v(t_n) - v_{II}^n\|_{H^{\gamma-1}}\|v(t_n) + v_{II}^n\|_{H^{\gamma-1}}.\end{aligned}$$

Since $u_0 \in H^{\gamma+1}$, so we have $U(0) \in H^\gamma$. Hence by Proposition 2.1, we know that

$$\|V(t_n) - V^n\|_{H^{\gamma-1}} \leq C\tau.$$

Moreover, by Proposition 5.1, we have $\|v_{II}^n\|_{H^\gamma} \leq C$ and $\|v(t_n) - v_{II}^n\|_{H^{\gamma-1}} \leq C\tau$, where C depends on $\|v\|_{L^\infty((0,T);H^{\gamma+1})}$. Consequently, we get

$$\|v(t_n) - v^n\|_{H^\gamma} \leq C\tau.$$

Since the twisting of variable (2.4) is isometric, so the proof is complete. □

6. Numerical result

In this section, we shall present the numerical results of the proposed ELRI scheme (2.14) for solving the mKdV equation (1.2) under rough initial data. For comparisons, the results of the coarse version scheme (2.9) and the classical Strang splitting scheme [20, 21] will be presented as well.

To construct the initial data $u_0(x)$ with the desired regularity, we adopt the following strategy as used in [42, 48, 60, 61]. Choose $N > 0$ as an even integer and discretize the spatial domain $\mathbb{T}$ with grid points $x_j = j\frac{2\pi}{N}$ for $j = 0, \dots, N$. Take a uniformly distributed random vector $\mathrm{rand}(N,1) \in [0,1]^N$ and define

$$u_0(x) := \frac{|\partial_{x,N}|^{-\theta}\mathcal{U}^N}{\||\partial_{x,N}|^{-\theta}\mathcal{U}^N\|_{L^\infty}}, \quad x \in \mathbb{T}, \quad \mathcal{U}^N = \mathrm{rand}(N,1), \tag{6.1}$$

where the pseudo-differential operator $|\partial_{x,N}|^{-\theta}$ for $\theta \geq 0$ reads: for Fourier modes $l = -N/2, \dots, N/2-1$,

$$\left(|\partial_{x,N}|^{-\theta}\right)_l = \begin{cases} |l|^{-\theta}, & \text{if } l \neq 0, \\ 0, & \text{if } l = 0. \end{cases}$$

Thus, we can get $u_0 \in H^\theta(\mathbb{T})$ for any $\theta \geq 0$. We implement the spatial discretizations of the aforementioned numerical methods by the Fourier pseudo-spectral method [50, 58] with a fixed large number of grid points $N = 2^{13}$ in $\mathbb{T}$ so that the spatial error is rather negligible in the test below.

We compute the error $u(t_n, x) - u^n(x)$ of the numerical methods at the final time $t_n = T = 0.5$ under the smooth initial data case

$$u_0(x) = \frac{\cos(x)}{2 + \sin(x)}, \quad x \in \mathbb{T},$$

and under the non-smooth initial data case (6.1) for $\theta = 2$, 3 or 5. The reference solutions are obtained numerically for the smooth case and the non-smooth case of $\theta = 5$ by the Strang splitting scheme [20, 21] with $\tau = 10^{-4}$. The Strang splitting scheme is implemented similarly as in [60]. For the non-smooth case of $\theta = 2$ or 3, the reference solution is computed by the ELRI (2.14) with $\tau = 10^{-4}$. The errors of the ELRI scheme (2.14), the coarse version scheme (2.9) and the Strang splitting scheme in the H^2-norm are plotted in Figure 1 for the smooth initial data and for the non-smooth data with $\theta = 3$ or 5. The error of ELRI in the H^1-norm is plotted in Figure 2 for the non-smooth initial data with $\theta = 2$.

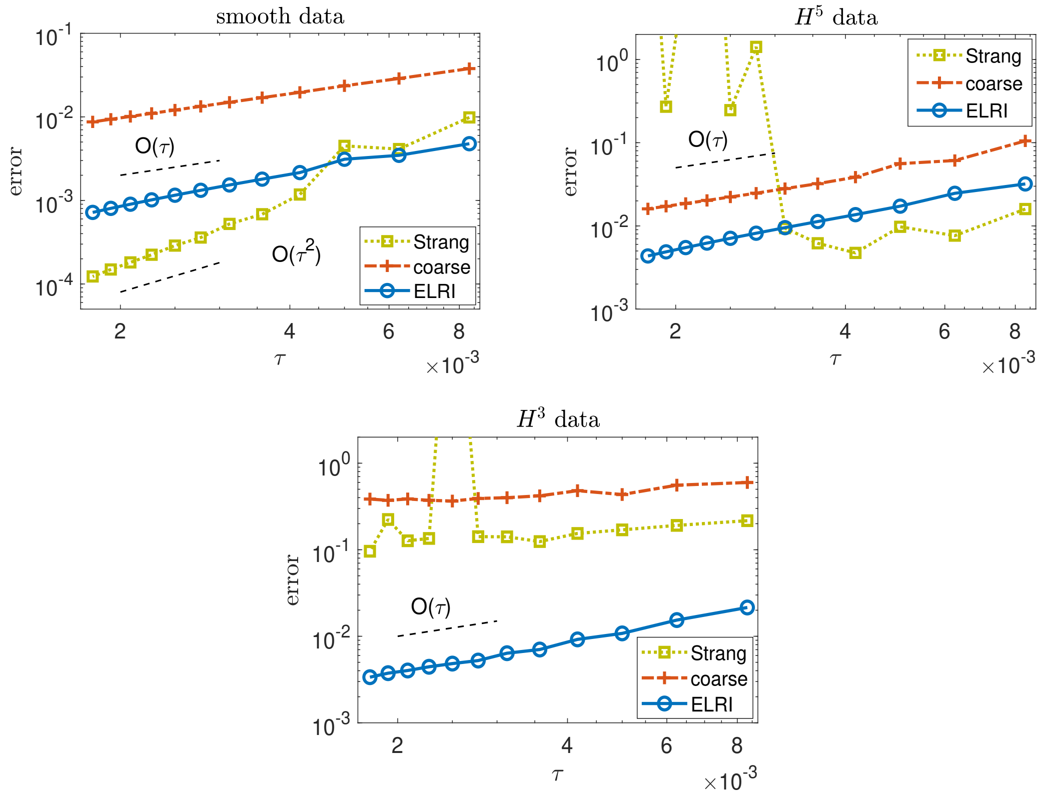

FIGURE 1. Convergence of the ELRI (2.14), the coarse scheme (2.9) and the Strang splitting: the relative error $\|u - u^n\|_{H^2}/\|u\|_{H^2}$ at $t_n = T = 0.5$ for smooth initial data (upper left), H^5-initial data (upper right) and H^3-initial data (2nd row).

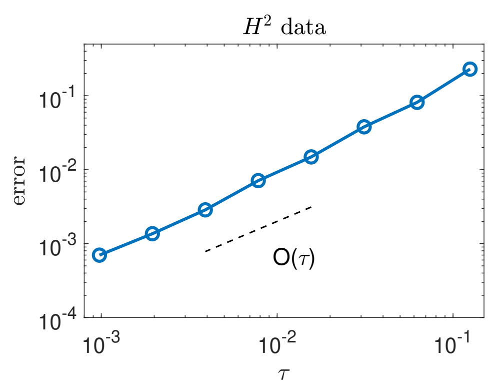

FIGURE 2. Convergence of the ELRI (2.14): the relative error $\|u - u^n\|_{H^1}/\|u\|_{H^1}$ at $t_n = T = 0.5$ for H^2-initial data.

From the numerical results, we can see that under the smooth data case, all the three methods can reach their own optimal convergence rates (Figure 1: upper left). While, for the non-smooth solution case, the classical Strang splitting scheme becomes much less accurate and the error barely converges (Figure 1: upper right and 2nd row). The ELRI scheme (2.14) always works well in the test, and it is always more accurate than the coarse version scheme (2.9). Under the non-smooth data case, the ELRI (2.14) only needs one additional bounded spatial derivative of the solution to reach its optimal first order accuracy (Figure 1: 2nd row). In contrast, the coarse version (2.9) needs three additional spatial derivatives to get the first order accuracy (Figure 1: upper right), otherwise it significantly loses convergence rate (Figure 1: 2nd row).

This verifies our theoretical results in Proposition 4.1 and Theorem 2.2. The technique condition $\gamma > \frac{3}{2}$ in Theorem 2.2 might be improved (based on the result in Figure 2) by more technical analysis in future. Thus, the proposed ELRI method is more efficient and accurate for solving the mKdV equation under rough data.

7. Conclusion

We considered the numerical solution of the periodic mKdV equation under rough data. Under the framework of exponential integration, previous low-regularity integration technique on the classical KdV equation failed on the mKdV equation, and some derivatives of the solution were unavoidably lost when integrating the cubic nonlinear interaction. By means of the Miura transform, we introduced a strategy of iterative regularizing for recovering the regularity of the numerical approximation. Under the new framework, an embedded exponential-type low-regularity integrator (ELRI) was proposed. The scheme is explicitly defined in the physical space and is efficient to implement under the Fourier pseudo-spectral method. We proved rigorously that the ELRI has first order accuracy in H^γ for the initial data from $H^{\gamma+1}$ under technical condition $\gamma > \frac{3}{2}$. Though we focused on the first order scheme, higher order methods can be derived under the same framework. Numerical experiments were done to confirm the theoretical result and show the accuracy of ELRI, where comparisons with Strang splitting scheme were made.

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