# ALMOST SURE WELL-POSEDNESS AND SCATTERING OF THE 3D CUBIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We study the random data problem for 3D, defocusing, cubic nonlinear Schrödinger equation in $H_x^s(\mathbb{R}^3)$ with $s<\frac{1}{2}$. First, we prove that the almost sure local well-posedness holds when $\frac{1}{6}\leqslant s<\frac{1}{2}$ in the sense that the Duhamel term belongs to $H_x^{1/2}(\mathbb{R}^3)$.

Furthermore, we prove that the global well-posedness and scattering hold for randomized, radial, large data $f\in H_x^s(\mathbb{R}^3)$ when $\frac{3}{7}<s<\frac{1}{2}$. The key ingredient is to control the energy increment including the terms where the first order derivative acts on the linear flow, and our argument can lower down the order of derivative more than $\frac{1}{2}$. To our best knowledge, this is the first almost sure large data global result for this model.

## Contents



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## 1. Introduction

In this paper, we consider the nonlinear Schrödinger equations (NLS):

$$\begin{cases} i\partial_t u + \Delta u = \mu |u|^p u, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $p > 0$, $\mu = \pm 1$, and $u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is a complex-valued function. The positive sign "+" in nonlinear term of (1.1) denotes defocusing source, and the negative sign "−" denotes the focusing one.

The equation (1.1) has conserved mass

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, \mathrm{d}x = M(u_0), \tag{1.2}$$

and energy

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t,x)|^2 \, \mathrm{d}x + \mu \int_{\mathbb{R}^d} \frac{1}{p+2} |u(t,x)|^{p+2} \, \mathrm{d}x = E(u_0). \tag{1.3}$$

The class of solutions to equation (1.1) is invariant under the scaling

$$u(t,x) \to u_\lambda(t,x) = \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x) \quad \text{for} \quad \lambda > 0, \tag{1.4}$$

which maps the initial data as

$$u(0) \to u_\lambda(0) := \lambda^{\frac{2}{p}} u_0(\lambda x) \quad \text{for} \quad \lambda > 0. \tag{1.5}$$

Denote

$$s_c = \frac{d}{2} - \frac{2}{p},$$

then the scaling leaves $\dot{H}^{s_c}$ norm invariant, that is,

$$\|u(0)\|_{\dot{H}^{s_c}} = \|u_\lambda(0)\|_{\dot{H}^{s_c}}.$$

This gives the scaling critical exponent $s_c$. Let

$$2^* = \infty, \text{ when } d = 1 \text{ or } d = 2; \quad 2^* = \frac{4}{d-2}, \text{ when } d \geqslant 3.$$

Therefore, according to the conservation law, the equation is called mass or $L^2_x$ *critical* when $p = \frac{4}{d}$, and energy or $\dot{H}^1_x$ *critical* when $p = \frac{4}{d-2}$. Moreover, when $\frac{4}{d} < p < 2^*$, we say that the equation is *inter-critical.*

Let us now take a brief overview on the well-posedness and scattering theory of NLS (1.1). Kato [46] first proposed a method based on the contraction mapping and the Strichartz estimate, and obtained the local well-posedness when $p < \frac{4}{d-2}$ in $H^1_x$. See also [72] by Tsutsumi for the $L^2_x$-solution when $p < \frac{4}{d}$. Note that the above two results concerned the sub-critical cases when $s > s_c$. The local well-posedness in the critical sense was solved by Cazenave and Weissler, see [22]. Moreover, we refer the readers to Cazenave's textbook [21] for more complete local results of NLS.

The global well-posedness and scattering are basic topics for the long time behaviour of NLS. Lin and Strauss [59] obtained the large data scattering for the 3D, defocusing, cubic NLS with decaying data. Their argument relied on the Morawetz estimate, which was first discovered by Morawetz [61] for the Klein-Gordon equations. The global well-posedness and scattering in energy space were solved by Ginibre and Velo [42] in the defocusing inter-critical cases for $d \geqslant 3$. In this paper, we mainly focus on the results in $L^2_x$-based Sobolev spaces, thus we do not mention the vast scattering theory for NLS with decaying data.

The main breakthrough of the energy critical NLS was owed to Bourgain [10]. He introduced the powerful induction-on-energy method and the localised Morawetz estimate to study the defocusing equations with radial data for $d = 3, 4$. Bourgain's method was then further exploited extensively: Nakanishi [62] introduced a modified version of Morawetz estimate for low dimensions, and solved the energy scattering in the inter-critical cases for $d = 1, 2$; Bourgain's 3D result was extended to non-radial by Colliander, Keel, Staffilani, Takaoka, and Tao [27], based on a localised version of their interaction Morawetz estimate [26]. The results for defocusing energy critical NLS in higher dimensions were obtained by Ryckman and Visan [66, 74].

For the focusing equations, Kenig and Merle [48] introduced the concentration compactness method to give a complete dynamical characterization below the energy of ground state, for the energy critical NLS in $d = 3, 4, 5$ with radial data. Their study opened a way to study the scattering of focusing equations below the ground state. Then, Duyckaerts, Holmer, and Roudenko [40, 45] gave the result for 3D, focusing, cubic NLS, which is a typical model in the inter-critical cases. For the non-radial focusing energy-critical NLS, Killip and Visan [52] solved the $d \geqslant 5$ case, and Dodson [34] solved the 4D case. The scattering of 3D, focusing, energy-critical NLS in the non-radial case remains open.

The concentration compactness method also enlighten the development of mass critical NLS. Killip, Tao, Visan, and Zhang [53, 54, 71] studied the mass critical NLS in the radial case. Dodson then remove the radial assumption, and completely solved the global well-posedness and scattering of mass critical NLS in the defocusing case [30, 32, 33], and in the focusing case below the mass of ground state [31].

Next, we focus on the well-posedness results of 3D, defocusing, cubic NLS, for which the critical regularity exponent $s_c = 1/2$. We have learned that the defocusing equation is local well-posed in $\dot{H}^{1/2}_x$, while the global well-posedness and scattering hold in a smaller space $H^1_x$. A natural question is to ask the weakest space $X$ to guarantee the global well-posedness in $\dot{H}^{1/2}_x \cap X$. Bourgain [9] used the high-low decomposition method (introduced in [8]) to give $X = H^s_x$ with $s > \frac{11}{13}$. The lower bound was then improved by "I-method" gradually in [25, 26, 69], and so far, the best result is $s > \frac{5}{7}$. Under the radial assumption, Dodson [35] showed that the result holds for almost critical space $s > \frac{1}{2}$.

Note that $X$ spaces in the above mentioned results are all $\dot{H}^{1/2}_x$ super-critical. Recently, Dodson [36] gave a result in the critical space $X = \dot{W}^{11/7,7/6}_x$, based on the observation that linear solution becomes more regular with initial data in $L^p_x$ with $p < 2$. Using this observation, the authors [67] obtained that $X = \dot{W}^{s,1}$ for $s > \frac{12}{13}$, which is a sub-critical space with the order of derivative less than 1, by the method in [1] .

Currently, there is no result for the global well-posedness of 3D defocusing cubic NLS merely in $\dot{H}^{1/2}_x$ or $H^{1/2}_x$. Kenig and Merle [49] initiated an another approach towards this problem. They proposed the concept of "conditional scattering", namely the global well-posedness and scattering hold for the solution that is uniformly bounded in the critical space on the maximal existence interval. Generally for the inter-critical NLS, no global well-posed result is known in the critical space. See [2, 36] for some related results.

Now, we turn to the probability theory of NLS. Although there are ill-posedness results below the critical regularity for NLS due to the result of Christ, Colliander, and Tao [24], Bourgain [6, 7] first introduced a probabilistic method to study the well-posedness problem for periodic NLS for "almost" all the initial data in super-critical spaces. The probabilistic well-posedness result for super-critical wave equations on compact manifolds was also studied by Burq and Tzvetkov [16, 17]. There have been extensive studies about such subject since then, and we refer the readers to [5] for more complete overviews.

Next, we only review the study of random data theory for NLS on $\mathbb{R}^d$. There are several ways of randomization for the initial data. We recall the one relying on the unit-scale decomposition in frequency, which is named as the Wiener randomization, appeared first in [78]. Then, under the Wiener randomization, Bényi, Oh, and Pocovnicu [3] studied the cubic NLS when $d \geqslant 3$. They proved the almost sure local well-posedness, small data scattering, and a "conditional" global well-posedness under some a priori hypothesis. Afterwards, the almost sure local results were improved by Bényi-Oh-Pocovnicu [4] and Pocovnicu-Wang [65]. The random data well-posedness for quintic NLS was studied by Brereton [11]. Later, Oh, Okamoto, and Pocovnicu [64] studied the almost sure global well-posedness (with no a priori assumptions) for energy critical NLS on $d = 5, 6$.

The large data almost sure scattering was first obtained by Dodson, Lührmann, and Mendelson [37] in the context of 4D, defocusing, energy-critical, nonlinear wave equation with randomized radial data, using a double bootstrap argument combining the energy and Morawetz estimates. The result was extended by Bringmann to non-radial 4D case [13], and to radial 3D case [12]. The related results on non-radial energy-critical nonlinear Klein-Gordon equations were studied by Chen and Wang [23]. The first almost sure scattering result for NLS was given by Killip, Murphy, and Visan [51]. They proved the result for 4D, defocusing, energy-critical case with almost all the radial initial data in $H^s_x$ for $\frac{5}{6} < s < 1$. This result was then improved to $\frac{1}{2} < s < 1$ by Dodson, Lührmann, and Mendelson [38].

We remark that the Wiener randomization is closely related to the modulation space introduced by Feichtinger [41]. Such space has been applied to non-linear evolution equations before the development of Wiener randomization, dating back to the results of Wang, Zhao, Guo, and Hudzik [75, 76].

There are also other kinds of randomization for NLS on $\mathbb{R}^d$. Burq, Thomann, and Tzvetkov [15] constructed a Gibbs measure for NLS with harmonic potential, and proved almost sure $L^2$-scattering for 1D, defocusing NLS with $p \geqslant 5$, after changing the Schrödinger equations into the ones with harmonic oscillator potential by lens transform. Recently, Burq and Thomann [14] improved the result to all the short range exponents $p > 3$. See also [58] for higher dimensional extensions.

In addition, Murphy [60] introduced a new kind of randomization based on the physical space unit-scale decomposition, and studied the almost sure existence and uniqueness of wave operator for $L^2$ sub-critical NLS above the Strauss exponent. Then, Nakanishi and Yamamoto [63] extended the result below Strauss exponent, and applied the method on some quadratic Schrödinger models. We also mention that Bringmann's almost sure scattering results [12, 13] include other kinds of randomization for nonlinear wave equations on $\mathbb{R}^d$, involving the micro-local and the annuli decompositions of initial data.

To the best of our knowledge, the only study by far of global well-posedness and scattering for inter-critical NLS seems Burq, Thomann, and Tzvetkov's 1D $L^2$-scattering result [15], based on the Gibbs measure for the Schrödinger equations with harmonic potential. Very recently, we learnt that Duerinckx [39] also studied the global well-posedness of cubic NLS adding a tiny dissipation with spatial inhomogeneous random initial data. In this paper, we intend to study a typical model of inter-critical NLS, namely the 3D, defocusing, cubic NLS under the Wiener randomization, at super-critical regularity.

Before stating the main result, we give the definition of the randomization:

**Definition 1.1** (Wiener randomization). *Let $\widetilde{\psi} \in C_0^\infty(\mathbb{R}^3)$ be a real-valued function such that $\widetilde{\psi} \geqslant 0$, $\widetilde{\psi}(-\xi) = \widetilde{\psi}(\xi)$ for all $\xi \in \mathbb{R}^3$ and*

$$\widetilde{\psi}(\xi) = \begin{cases} 1, \text{ when } \xi \in [-\frac{1}{2}, \frac{1}{2}]^3, \\ 0, \text{ when } \xi \notin [-1,1]^3. \end{cases}$$

*Let*

$$\psi(\xi) := \frac{\widetilde{\psi}(\xi)}{\sum_{k\in\mathbb{Z}^3} \widetilde{\psi}(\xi - k)}.$$

*Then, $\psi \in C_0^\infty(\mathbb{R}^3)$ is a real-valued function, satisfying for all $\xi \in \mathbb{R}^3$, $0 \leqslant \psi \leqslant 1$, supp $\psi \subset [-1,1]^3$, $\psi(-\xi) = \psi(\xi)$, and $\sum_{k\in\mathbb{Z}^3} \psi(\xi - k) = 1$.*

*For any $k \in \mathbb{Z}^3$, define $\psi_k(\xi) = \psi(\xi - k)$. Denote the Fourier transform by $\mathcal{F}$. Then, we define*

$$\Box_k f = \mathcal{F}^{-1}\big(\psi_k \mathcal{F} f\big).$$

*Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $\{g_k\}_{k\in\mathbb{Z}^3}$ be a sequence of zero-mean, complex-valued Gaussian random variables on $\Omega$, where the real and imaginary parts of $g_k$ are independent. Then, for any function $f$, we define its randomization $f^\omega$ by*

$$f^\omega = \sum_{k\in\mathbb{Z}^3} g_k(\omega) \Box_k f. \tag{1.6}$$

In the following, we use the statement "almost every $\omega \in \Omega$, $PC(\omega)$ holds" to mean that

$$\mathbb{P}\Big(\big\{\omega \in \Omega : PC(\omega) \text{ holds}\big\}\Big) = 1.$$

Now, we study the 3D, defocusing, cubic NLS with randomized initial data:

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u, \\ u(0,x) = f^\omega(x). \end{cases} \tag{1.7}$$

For this model under the probabilistic setting, the local well-posedness, small data scattering, and conditional global well-posedness results have been established before.

We first recall the local results for (1.7). Bényi, Oh, and Pocovnicu [4] proved the local result with $f \in H_x^s$ when $\frac{2}{5}\sigma < s < \frac{1}{2}$ in the sense that Duhamel term belongs to $C(I; H_x^\sigma)$ for any fixed $\frac{1}{2} \leqslant \sigma \leqslant 1$. They also proved the improved local result when $\frac{1}{6} < s < \frac{1}{2}$ (except for the lower endpoint) by weakening the definition of local solution:

$$u - z_1 - z_3 - \cdots - z_{2k-1} \in C(I; H_x^{1/2}),$$

where the function $z_k \in C(I; H_x^{s_k})$ is defined by iteration with some $s_k < \frac{1}{2}$. Pocovnicu and Wang [65] also proved the local result in $L_x^2$ with Duhamel term in $C(I; L_x^4)$.

There are also global results of (1.7), either with small data restriction or with suitable a priori assumptions. Bényi, Oh, and Pocovnicu [3] proved the almost sure small data global well-posedness and scattering for $\frac{1}{4} < s < \frac{1}{2}$. Furthermore, they [3] also proved the random data global well-posedness when $\frac{1}{4} < s < \frac{1}{2}$ under two a priori assumptions:

- The Duhamel term is uniformly bounded in the critical space $H_x^{1/2}$ in the probabilistic setting.
- The 3D, defocusing, cubic NLS is globally well-posed with deterministic initial data in $H_x^{1/2}$.

Each of the above two a priori assumptions seems very difficult to verify.

In this paper, we improve the previous local results and give an optimal local result for (1.7). Moreover with the radial data, we prove the global well-posedness as well as scattering, without imposing any a priori assumption or size restriction, where the scattering result holds in the energy space.

1.1. **Almost sure local well-posedness.** The first main result in this paper concerns the almost sure local well-posedness. Previously for the random data local result, Bényi, Oh, and Pocovnicu [4] introduced the higher order expansion method; Pocovnicu and Wang's argument [65] is based on the dispersive inequality; Dodson, Lührmann, and Mendelson [38] used the high dimensional version of smoothing effect and maximal function estimates. In this paper, we give some simple new approaches combining the atom space method by Koch-Tataru [55] and the variants of bilinear Strichartz estimate.

**Theorem 1.2** (Local well-posedness). *Let $f \in H_x^s(\mathbb{R}^3)$. Then, for almost every $\omega \in \Omega$, it holds that:*

*(1) If $\frac{1}{6} \leqslant s < \frac{1}{2}$, then there exists $T > 0$ and a solution $u$ of (1.7) on $[0, T]$ such that*

$$u - e^{it\Delta} f^\omega \in C([0, T]; H_x^{\frac{1}{2}}(\mathbb{R}^3)).$$

*(2) If $\frac{1}{3} < s < \frac{1}{2}$, then there exists $T > 0$ and a solution $u$ of (1.7) on $[0, T]$ such that*

$$u - e^{it\Delta} f^\omega \in C([0, T]; H_x^1(\mathbb{R}^3)).$$

Our result improves the local results in [4], where Bényi, Oh, and Pocovnicu [4] proved the same results in Theorem 1.2 (1) for $\frac{1}{5} < s < \frac{1}{2}$ and Theorem 1.2 (2) for $\frac{2}{5} < s < \frac{1}{2}$.

The following are some remarks concerning the theorem.

*Remark* 1.3. (1) We believe that the first result in Theorem 1.2 is optimal in the following sense. In fact, we need to control the term

$$\big(\sqrt{-\Delta}\big)^{\frac{1}{2}} \big(|e^{it\Delta} f^\omega|^2 e^{it\Delta} f^\omega\big),$$

and there is at least $\frac{1}{6}$-order derivative acting on each $f^\omega$.

(2) It seems very difficult to extend the local solution obtained in Theorem 1.2 (1) to global directly. Therefore, we establish the local solution with higher

regularity in Theorem 1.2 (2). As remarked before, the lower bound $\frac{1}{3}$ seems also sharp in this case.

(3) Our second result can be compared to Dodson, Lührmann, and Mendelson's local result [38] for 4D, cubic NLS, which is energy critical, since both results put the Duhamel term in $C([0,T];H_x^1)$. They proved local well-posedness for $\frac{1}{3}<s<1$, also except the endpoint exponent $\frac{1}{3}$.

(4) Apparently, the local results in Theorem 1.2 also hold for focusing equations.

The proof of Theorem 1.2 (2) is more difficult than the first local result. We postponed here to illustrate the main idea. It reduces to consider the term

$$\nabla\big(|e^{it\Delta}f^\omega|^2 e^{it\Delta}f^\omega\big)$$

with $f$ merely in $H_x^{\frac{1}{3}}$. The task is how to allocate the first order derivative to each $e^{it\Delta}f^\omega$. However, the use of bilinear Strichartz estimate or local smoothing can only lower down semi-derivative. Then, we overcome the difficulty by following two tools:

- We employ the $U^p$-$V^p$ method introduced by Koch-Tataru [55] to exploit the duality structure.
- We also apply the bilinear Strichartz estimate in the form of

$$\big\|[e^{it\Delta}\phi][e^{\pm it\Delta}\psi]\big\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^3)} \tag{1.8}$$

  with $1\leqslant q,r\leqslant 2$, see Candy's result [18]. Particularly, the use of (1.8) with $q<2$ and $r=2$ can reduce the loss of derivative, at the cost of lower time integration exponents.

Let $v=e^{it\Delta}f^\omega$, and it suffices to control

$$\int_0^T\int_{\mathbb{R}^3} g_{\mathrm{hi}}\nabla v_{\mathrm{hi}}v_{\mathrm{low}}^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where $e^{-it\Delta}g\in V^2(\mathbb{R};L_x^2)$, and "hi", "low" represent the size of frequency. Heuristically, by Hölder's inequality,

$$\int_0^T\int_{\mathbb{R}^3} g_{\mathrm{hi}}\nabla v_{\mathrm{hi}}v_{\mathrm{low}}^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim \|g_{\mathrm{hi}}v_{\mathrm{low}}\|_{L_t^{\frac{3}{2}}L_x^2}\|\nabla v_{\mathrm{hi}}v_{\mathrm{low}}\|_{L_t^1L_x^2}^{\frac{1}{3}}\|\nabla v_{\mathrm{hi}}\|_{L_{t,x}^\infty}^{\frac{2}{3}}\|v_{\mathrm{low}}\|_{L_t^\infty L_x^2}^{\frac{2}{3}}.$$

This can cut down the order of high-frequency derivative to $\frac{1}{3}$ for $v_{\mathrm{hi}}$, with a total loss of low-frequency derivative of order $\frac{2}{3}$, which can be assigned to each $v_{\mathrm{low}}$.

Note that the above observation is sharp with respect to the regularity. Furthermore, we have a logarithmic loss of derivative when passing $g$ into $V_\Delta^2$ by interpolation. That is the main reason why we need $s>\frac{1}{3}$ to acquire additional regularity for summation.

Moreover, if we only requires

$$u-e^{it\Delta}f^\omega\in C([0,T];H_x^\sigma(\mathbb{R}^3))$$

for $\frac{1}{2}\leqslant\sigma<1$, the approach in above observation can provide enough additional regularity for summation. Thus, we expect that the argument works for the optimal lower endpoint, namely $\frac{1}{3}\sigma\leqslant s<\frac{1}{2}$, which clearly includes the result in Theorem 1.2 (1). However, in this paper, we only consider two endpoint cases when $\sigma=\frac{1}{2}$ or $\sigma=1$, and present two different methods, respectively. For Theorem 1.2 (1), we provide another proof without exploiting the duality structure. In fact, there is only $\frac{1}{2}$-order derivative acting on the nonlinear term, and we can transfer it simply using the bilinear Strichartz estimate.

1.2. **Almost sure scattering.** Now we turn to our second main result for the global well-posedness and scattering:

**Theorem 1.4** (Global well-posedness and scattering)**.** *Let $\frac{3}{7} < s \leqslant \frac{1}{2}$ and $f \in H_x^s(\mathbb{R}^3)$ be radial. Then, for almost every $\omega \in \Omega$, there exists a solution $u$ of* (1.7) *on $\mathbb{R}$ such that*

$$u - e^{it\Delta} f^\omega \in C(\mathbb{R}; H_x^1(\mathbb{R}^3)).$$

*Moreover, the solution $u$ scatters, in the sense that there exist $u_\pm \in H_x^1$ such that*

$$\lim_{t\to\pm\infty} \left\| u - e^{it\Delta} f^\omega - e^{it\Delta} u_\pm \right\|_{H_x^1} = 0.$$

The most significant point of this result is that we are able to control the energy increment containing the term $\nabla e^{it\Delta} f^\omega$, under the assumption that $f$ merely belongs to $H_x^s$ with some $s < \frac{1}{2}$.

Comparing to the energy-critical results in [38, 51], for 3D, defocusing, cubic NLS, it is easier to derive space-time estimates, since the Morawetz type estimates are energy sub-critical. On the other hand, however, this problem seems more difficult, in the sense that we need to reduce the order of derivative more than $\frac{1}{2}$ for $\nabla e^{it\Delta} f^\omega$, while the current results for energy-critical NLS lower down at most $\frac{1}{2}$-order derivative, in the view of local smoothing effect.

Our method is different from the recent results on the almost sure scattering of nonlinear Schrödinger and wave equations [12, 13, 37, 38, 51]. To establish the almost energy conservation of $u - e^{it\Delta} f^\omega$, we make a high-low frequency decomposition of the initial data, and keep track of the explicit increase of energy bound. Then, we implement a bootstrap argument for the energy, building upon a perturbed interaction Morawetz estimate, various nonlinear estimates and the bilinear Strichartz estimate.

Lastly, we remark that the lower bound of regularity $\frac{3}{7}$ is not sharp. Here, we do not achieve this optimality, and only give a well-presented result. However, it is of great interest to improve the regularity's lower bound down to $\frac{1}{3}$, or even $\frac{1}{6}$.

1.2.1. *Sketch the proof of Theorem 1.4.* The main ingredient of the proof is summarized as follows.

• **High-low frequency decomposition in the probabilistic setting.**

In probabilistic setting, we only have the boundedness in the almost every sense. Roughly speaking, in order to quantify the size of energy, we decompose the probability space $\Omega$ by setting

$$\widetilde{\Omega}_M = \left\{ \omega \in \Omega : \|f^\omega\|_{H^s} + N_0^{s-1} \|P_{\leqslant N_0} f^\omega\|_{H^1} + \|e^{it\Delta} f^\omega\|_{\widetilde{Y}^s(\mathbb{R})} \leqslant M \, \|f\|_{H^s} \right\},$$

where the $\widetilde{Y}^s$-norm is some required space-time norm defined by (3.13) and (5.1) below, and $N_0 \in 2^{\mathbb{N}}$ depends only on $M$ and $\|f\|_{H^s}$. See (5.4) for the precise definition of $\widetilde{\Omega}_M$. Then it follows from the Borel-Cantelli Lemma that

$$\mathbb{P}\big( \cup_{M\geqslant 1} \widetilde{\Omega}_M \big) = 1.$$

According to the decomposition above, we may consider $\omega \in \widetilde{\Omega}_M$ for each $M$ separately. Now, we give the high-low frequency decomposition $v = e^{it\Delta} P_{\geqslant N_0} f^\omega$ and $w = u - v$. Then, for any $\omega \in \widetilde{\Omega}_M$, there exists a constant $C(M) > 0$ such that

$$E(w(0)) \leqslant C(M, \|f\|_{H^s}) N_0^{2(1-s)}.$$

The application of Bourgain's high-low decomposition method [8] to random Cauchy problem was first made by Colliander and Oh [28] for 1D NLS on $\mathbb{T}$. However, in this paper, we do not intend to carry out Bourgain's iteration procedure. We only make the decomposition in order for two benefits:

(1) $\widehat{v}$ is supported on $\{|\xi| \gtrsim N_0\}$.
(2) We can explicitly keep track of the energy increment of $N_0$.

• **Strichartz estimates with $\frac{1}{2}-$derivative gain.**
Note that $v$ is not radial anymore under the Wiener randomization. However, due to the pioneer works [37, 38], we can prove that for the radial $f$,

$$\big\||\nabla|^{s+\frac{1}{2}}v\big\|_{L_t^2 L_x^\infty} < \infty, \tag{1.9}$$

which is followed by combining a "radialish" Sobolev inequality for the square function and the local smoothing estimate. Note that the estimate (1.9) gains $\frac{1}{2}$-order derivative.

• **Global space-time bound for the nonlinear solution.**
From the perturbed interaction Morawetz estimates, we can derive the bound of

$$\|w\|_{L^4_{t,x}},$$

which is $H^{\frac{1}{4}}$-critical, under the a priori hypothesis of $H^1$-bound. The high-low frequency decomposition also plays a crucial role for controlling the remainder. However, this is far from sufficient for the estimates of energy bound.

Then, an observation is that combining the above $L^4_{t,x}$-estimate and the integral equation, we can further control

$$\Big(\sum_{N\in 2^{\mathbb{N}}} N\,\|P_N w\|^2_{U^2_\Delta(L^2_x)}\Big)^{1/2},$$

which is $\dot H^{\frac{1}{2}}$-critical space-time estimates invoking the $U^p$-$V^p$ method, on suitable long-time interval. Keeping in mind that the equation is $\dot H^{\frac{1}{2}}$-critical, the space-time estimates under the same scaling play an important role throughout the whole argument.

Furthermore, applying the above $\dot H^{\frac{1}{2}}$-critical estimates, we can update the scaling up to $\dot H^1_x$:

$$\Big(\sum_{N\in 2^{\mathbb{N}}} N^{2l}\,\|P_N w\|^2_{U^2_\Delta(L^2_x)}\Big)^{1/2}, \text{ for any } l \in (\frac{1}{2}, 1].$$

For this purpose, we also need to use the maximal function techniques to deal with some critical cases.

In the above argument, the use of $U^2_\Delta$-space has two advantages: we can transfer the derivative by duality formula, and the $U^2_\Delta$-space allows estimates on any long-time interval.

• **Energy bound.**
The main goal is to prove

$$\sup_{t\in\mathbb{R}} E(w(t)) \lesssim_M N_0^{2(1-s)}.$$

It suffices to prove the bootstrap inequality

$$\sup_{t\in I} E(w(t)) \lesssim_M N_0^{-\alpha} N_0^{2(1-s)},$$

for some $\alpha > 0$ under the assumption $\sup_{t\in I} E(w(t)) \lesssim_M N_0^{2(1-s)}$. Under this bootstrap hypothesis, we can also give the precise increase of $N_0$ for the various global space-time estimates on $I$ obtained in the previous step, which are very useful for the control of energy increment. Now, the main term in the energy estimate is

$$\Big|\int_I \int_{\mathbb{R}^3} \nabla w_{\rm hi} \nabla v_{\rm hi} w_{\rm low}^2 \, \mathrm{d}x \, \mathrm{d}t\Big|, \tag{1.10}$$

where "hi" and "low" represent the size of frequency.

We remark that the Morawetz estimate of the form $\iint \frac{|w|^4}{|x|} \, \mathrm{d}x \, \mathrm{d}t$ plays an important role in the former energy-critical results [37, 38, 51], but it is not sufficient for the 3D cubic case. First, the Morawetz estimate cannot yield the global space-time bounds, as in the previous step. That is the reason why the use of interaction Morawetz estimate is necessary. Second, using their method, (1.10) can be controlled by

$$\|\nabla w_{\rm hi}\|_{L_t^\infty L_x^2} \big\|\sqrt{|x|}\nabla v_{\rm hi}\big\|_{L_t^2 L_x^\infty} \Big(\int_I \int_{\mathbb{R}^3} \frac{|w_{\rm low}|^4}{|x|} \, \mathrm{d}x \, \mathrm{d}t\Big)^{\frac{1}{2}}.$$

Then, the energy bound can be followed by the "radialish" Sobolev inequality, local smoothing effect, and Morawetz estimate when $s > \frac{1}{2}$. Unfortunitely, this argument does not work here, since we are lack of $\nabla v$-estimates in the view of (1.9), when $s < \frac{1}{2}$ in our situation.

To overcome the difficulty, we observe that there is still some gap in the estimate

$$(1.10) \lesssim \|\nabla w_{\rm hi}\|_{L_t^\infty L_x^2} \|\nabla v_{\rm hi}\|_{L_t^2 L_x^\infty} \|w_{\rm low}\|_{L_{t,x}^4}^2,$$

towards the desired bound $N_0^{2(1-s)}$. This gives us the room to use the global space-time estimates obtained above and the bilinear Strichartz estimate, which can further lower down the derivative for $\nabla v_{\rm hi}$.

1.3. **Organization of the paper.** In Section 2, we give some notation and useful results. In Section 3, we prove the almost sure space-time estimates for the linear solution. Then, we prove the local results in Theorem 1.2 in Section 4, and prove the global well-posedness and scattering results in Theorem 1.4 in Section 5.

## 2. Preliminary

2.1. **Notation.** For any $a \in \mathbb{R}$, $a\pm := a \pm \epsilon$ for arbitrary small $\epsilon > 0$. For any $z \in \mathbb{C}$, we define $\mathrm{Re}z$ and $\mathrm{Im}z$ as the real and imaginary part of $z$, respectively.

Let $C > 0$ denote some constant, and write $C(a) > 0$ for some constant depending on coefficient $a$. If $f \leqslant Cg$, we write $f \lesssim g$. If $f \leqslant Cg$ and $g \leqslant Cf$, we write $f \sim g$. Suppose further that $C = C(a)$ depends on $a$, then we write $f \lesssim_a g$ and $f \sim_a g$, respectively. If $f \leqslant 2^{-5}g$, we denote $f \ll g$ or $g \gg f$.

Moreover, we write "a.e. $\omega \in \Omega$" to mean "almost every $\omega \in \Omega$".

We use $\widehat{f}$ or $\mathcal{F}f$ to denote the Fourier transform of $f$:

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\mathrm{dx}.$$

We also define

$$\mathcal{F}^{-1}g(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}g(\xi)\mathrm{d}\xi.$$

Using the Fourier transform, we can define the fractional derivative $|\nabla| := \mathcal{F}^{-1}|\xi|\mathcal{F}$ and $|\nabla|^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$.

We next recall the unit-scale frequency decomposition in Definition 1.1. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ is a real-valued function, satisfying for all $\xi \in \mathbb{R}^d$, $0 \leqslant \psi \leqslant 1$, supp $\psi \subset [-1,1]^d$, $\psi(-\xi) = \psi(\xi)$, and $\sum_{k\in\mathbb{Z}^d}\psi(\xi-k) = 1$. For any $k \in \mathbb{Z}^d$, define $\psi_k(\xi) = \psi(\xi - k)$. Then, we define

$$f_k = \square_k f := \mathcal{F}^{-1}\big(\psi_k \mathcal{F}f\big).$$

We also define a fattening version

$$\widetilde{\square}_k f := \mathcal{F}^{-1}\big(\psi(2^{-1}(\xi-k))\mathcal{F}f\big),$$

with the property $\square_k = \square_k\widetilde{\square}_k$.

We also need the usual inhomogeneous Littlewood-Paley decomposition for the dyadic number. Take a cut-off function $\phi \in C_0^\infty(0,\infty)$ such that $\phi(r) = 1$ if $r \leqslant 1$ and $\phi(r) = 0$ if $r > 2$.

Then, we introduce the spatial cut-off function. Denote $\chi_0(r) = \phi(r)$, and $\chi_j(r) = \phi(2^{-j}r)$ for $j \in \mathbb{N}^+$. We also define a fattening version $\widetilde{\chi}_j := \phi(2^{-j-1}|\xi|)$ with the property $\chi_j = \chi_j\widetilde{\chi}_j$.

For dyadic $N \in 2^{\mathbb{N}}$, when $N \geqslant 1$, let $\phi_{\leqslant N}(r) = \phi(N^{-1}r)$ and $\phi_N(r) = \phi_{\leqslant N}(r) - \phi_{\leqslant N/2}(r)$. We define the Littlewood-Paley dyadic operator

$$f_{\leqslant N} = P_{\leqslant N}f := \mathcal{F}^{-1}(\phi_{\leqslant N}(|\xi|)\hat{f}(\xi)),$$

and

$$f_N = P_N f := \mathcal{F}^{-1}(\phi_N(|\xi|)\hat{f}(\xi)).$$

We also define that $f_{\geqslant N} = P_{\geqslant N}f := f - P_{\leqslant N}f$, $f_{\ll N} = P_{\ll N}f$, $f_{\gtrsim N} := P_{\gtrsim N}f$, $f_{\lesssim N} := P_{\lesssim N}f$, and $f_{\sim N} = P_{\sim N}f$.

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space, $\mathcal{S}'(\mathbb{R}^d)$ be the tempered distribution space, and $C_0^\infty(\mathbb{R}^d)$ be the space of all the smooth compact-supported functions.

Given $1 \leqslant p \leqslant \infty$, $L^p(\mathbb{R}^d)$ denotes the usual Lebesgue space. We define the Sobolev space

$$\dot{W}^{s,p}(\mathbb{R}^d) := \big\{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} := \||\nabla|^s f\|_{L^p(\mathbb{R}^d)} < +\infty\big\}.$$

We denote that $\dot{H}^s(\mathbb{R}^d) := \dot{W}^{s,2}(\mathbb{R}^d)$. The inhomogeneous spaces are defined by

$$W^{s,p}(\mathbb{R}^d) = \dot{W}^{s,p} \cap L^p(\mathbb{R}^d), \text{ and } H^s(\mathbb{R}^d) = \dot{H}^s \cap L^2(\mathbb{R}^d).$$

We often use the abbreviations $H^s = H^s(\mathbb{R}^d)$ and $L^p = L^p(\mathbb{R}^d)$. We also define $\langle\cdot,\cdot\rangle$ as real $L^2$ inner product:

$$\langle f, g\rangle = \mathrm{Re}\int f(x)\overline{g}(x)\ \mathrm{d}x.$$

For any $1 \leqslant p < \infty$, define $l^p_N = l^p_{N\in 2^{\mathbb{N}}}$ by its norm

$$\|c_N\|^p_{l^p_{N\in 2^{\mathbb{N}}}} := \sum_{N\in 2^{\mathbb{N}}} |c_N|^p.$$

The space $l_k^p = l_{k\in\mathbb{Z}^d}^p$ is defined in a similar way. In this paper, we use the following abbreviations

$$\sum_{N:N\leqslant N_1} := \sum_{N\in 2^{\mathbb{N}}:N\leqslant N_1}, \text{ and } \sum_{N\leqslant N_1} := \sum_{N,N_1\in 2^{\mathbb{N}}:N\leqslant N_1}.$$

We then define the mixed norms: for $1\leqslant q<\infty$, $1\leqslant r\leqslant\infty$, and the function $u(t,x)$, we define

$$\|u\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)}^q := \int_{\mathbb{R}} \|u(t,\cdot)\|_{L_x^r}^q \, \mathrm{d}t,$$

and for the function $u_N(x)$, we define

$$\|u_N\|_{l_N^qL_x^r(2^{\mathbb{N}}\times\mathbb{R}^d)}^q := \sum_N \|u_N(\cdot)\|_{L_x^r}^q .$$

The $q=\infty$ case can be defined similarly.

For any $0\leqslant\gamma\leqslant 1$, we call that the exponent pair $(q,r)\in\mathbb{R}^2$ is $\dot{H}^\gamma$*-admissible*, if $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}-\gamma$, $2\leqslant q\leqslant\infty$, $2\leqslant r\leqslant\infty$, and $(q,r,d)\neq(2,\infty,2)$. If $\gamma=0$, we say that $(q,r)$ is $L^2$*-admissible.*

2.2. **Atom space and bounded variation space.** We recall the definitions of $U^p$ and $V^p$, and some properties used in this paper. The $U^p$-$V^p$ method was first introduced by Koch-Tataru [55], and we also refer the readers to [20, 44, 56, 57] for their complete theories.

**Definition 2.1.** *Let $\mathcal{Z}$ be the set of finite partitions $-\infty<t_0<t_1<\ldots<t_K=\infty$.*

*(1) For $\{t_k\}_{k=0}^K\in\mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1}\subset L_x^2$ with $\sum_{k=0}^{K-1}\|\phi_k\|_{L_x^2}^p=1$, we call the function $a:\mathbb{R}\to L_x^2$ given by $a=\sum_{k=1}^K \mathbb{1}_{[t_{k-1},t_k)}\phi_{k-1}$ a $U^p$-atom. Furthermore, we define the atomic space*

$$U^p := \Big\{u=\sum_{j=1}^\infty \lambda_j a_j : a_j \ U^p\text{-atom}, \ \lambda_j\in\mathbb{C} \text{ with } \sum_{j=1}^\infty|\lambda_j|<\infty\Big\}, \tag{2.1}$$

*with norm*

$$\|u\|_{U^p(\mathbb{R};L^2)} := \inf\Big\{\sum_{j=1}^\infty|\lambda_j| : u=\sum_{j=1}^\infty\lambda_j a_j, a_j \ U^p\text{-atom}, \ \lambda_j\in\mathbb{C}\Big\}. \tag{2.2}$$

*(2) We define $V^p$ as the normed space of all functions $v:\mathbb{R}\to L^2$ such that*

$$\|v\|_{V^p(\mathbb{R};L_x^2)} := \sup_{\{t_k\}_{k=0}^K\in\mathcal{Z}} \Big(\sum_{k=1}^K \|v(t_k)-v(t_{k-1})\|_{L_x^2}^p\Big)^{1/p} \tag{2.3}$$

*is finite, where we use the convention $v(t_K)=v(\infty)=0$. $V_{rc}^p$ denotes the closed subspace of all right-continuous $V^p$ functions with $\lim_{t\to-\infty}v(t)=0$.*

*(3) We define $U_\Delta^2(\mathbb{R};L_x^2)$ as the adapted normed space:*

$$U_\Delta^2(\mathbb{R};L_x^2) := \Big\{u : \|u\|_{U_\Delta^2(\mathbb{R};L_x^2)} := \big\|e^{-it\Delta}u\big\|_{U^2(\mathbb{R};L_x^2)}<\infty\Big\}.$$

*Similarly, $V_\Delta^2(\mathbb{R};L_x^2)$ denotes the adapted normed space*

$$V_\Delta^2(\mathbb{R};L_x^2) := \Big\{u : \|u\|_{V_\Delta^2(\mathbb{R};L_x^2)} := \big\|e^{-it\Delta}u\big\|_{V^2(\mathbb{R};L_x^2)}<\infty, e^{-it\Delta}u\in V_{rc}^2\Big\}.$$

In this paper, we will use restriction spaces to some interval $I \subset \mathbb{R}$: $U^p(I;L_x^2)$, $V^p(I;L_x^2)$, $U_\Delta^p(I;L_x^2)$, and $V_\Delta^p(I;L_x^2)$. See Remark 2.23 in [44] for more details.

Note that for $1 \leqslant p < q < \infty$, the embeddings

$$U^p(\mathbb{R};L_x^2)\hookrightarrow L_t^\infty(\mathbb{R};L_x^2),\ V^2(\mathbb{R};L_x^2)\hookrightarrow L_t^\infty(\mathbb{R};L_x^2),$$

and $U^p\hookrightarrow V_{rc}^p\hookrightarrow U^q$ are continuous.

We need the following classical linear estimate and duality formula:

**Lemma 2.2** ([44])**.** *Let $I$ be an interval such that $0 = \inf I$. Then, for any $f \in L_x^2$,*

$$\left\|e^{it\Delta}f\right\|_{U_\Delta^2(I;L_x^2)} \lesssim \|f\|_{L_x^2},$$

*and for $F(t,x) \in L_t^1L_x^2(I\times\mathbb{R}^d)$,*

$$\Big\|\int_0^t e^{i(t-s)\Delta}F(s)\ \mathrm{d}s\Big\|_{U_\Delta^2(I;L_x^2)} = \sup_{\|g\|_{V_\Delta^2(I;L_x^2)}=1}\left|\int_I\int_{\mathbb{R}^3}F(t)\overline{g(t)}\ \mathrm{d}x\ \mathrm{d}t\right|.$$

We also need the following interpolation result to transfer from $U_\Delta^2$ into $V_\Delta^2$.

**Lemma 2.3** ([44])**.** *Let $q > 1$, $E$ be a Banach space and $T : U_\Delta^q\to E$ be a bounded, linear operator with $\|Tu\|_E \leqslant C_q\|u\|_{U_\Delta^q}$. In addition, assume that for some $1 \leqslant p < q$, there exists $C_p \in (0, C_q]$ such that the estimate $\|Tu\|_E \leqslant C_p\|u\|_{U_\Delta^p}$ holds true for all $u \in U_\Delta^p$. Then, $T$ satisfies the estimate for $u \in V_\Delta^p$,*

$$\|Tu\|_E \leqslant \frac{4}{(1-p/q)\ln 2}C_p\big(1 + 2(1-p/q)\ln 2 + \ln\frac{C_q}{C_p}\big)\|u\|_{V_\Delta^p}. \tag{2.4}$$

2.3. **Useful lemmas.** In this subsection, we gather some useful results.

**Lemma 2.4** (Schur's test)**.** *For any $a > 0$, let sequences $\{a_N\}$, $\{b_N\} \in l^2_{N\in 2^{\mathbb{N}}}$, then we have*

$$\sum_{N_1\leqslant N}\big(\frac{N_1}{N}\big)^a a_N b_{N_1} \lesssim \|a_N\|_{l_N^2}\|b_N\|_{l_N^2}. \tag{2.5}$$

**Lemma 2.5** (Hardy's inequality)**.** *For $1 < p < d$, we have that*

$$\left\||x|^{-1}u\right\|_{L_x^p(\mathbb{R}^d)} \lesssim \|\nabla u\|_{L_x^p(\mathbb{R}^d)}.$$

**Lemma 2.6** (Local smoothing, [29, 43, 50])**.** *We have that*

$$\sup_{R>0}R^{-\frac12}\left\|e^{it\Delta}f\right\|_{L_t^2(\mathbb{R};L^2_{|x|\leqslant R})} \lesssim \left\||\nabla|^{-\frac12}f\right\|_{L_x^2}.$$

**Lemma 2.7** (Strichartz estimate, [47, 57])**.** *Let $I \subset \mathbb{R}$. Suppose that $(q,r)$ and $(\widetilde{q},\widetilde{r})$ are $L_x^2$-admissible. Then,*

$$\left\|e^{it\Delta}\varphi\right\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\varphi\|_{L_x^2}, \tag{2.6}$$

*and*

$$\Big\|\int_0^t e^{i(t-s)\Delta}F(s)\ \mathrm{d}s\Big\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\widetilde{q}'}L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.7}$$

*Moreover, if we assume further $2 < q < \infty$, then*

$$\|u\|_{L_t^qL_x^r(I\times\mathbb{R}^3)} \lesssim \|u\|_{U_\Delta^q(I;L_x^2)} \lesssim \|u\|_{V_\Delta^2(I;L_x^2)}. \tag{2.8}$$

In this paper, we need the the following multi-scale bi-linear Strichartz estimate for Schrödinger equation, which is a particular case of Theorem 1.2 in [18]:

**Lemma 2.8.** *Let $1 \leqslant q, r \leqslant 2$, $\frac{1}{q} + \frac{2}{r} < 2$, and suppose that $M, N \in 2^{\mathbb{Z}}$ satisfy $M \ll N$. Then for any $\phi, \psi \in L_x^2(\mathbb{R}^3)$,*

$$\big\|[e^{it\Delta}P_N\phi][e^{\pm it\Delta}P_M\psi]\big\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^3)} \lesssim \frac{M^{4-\frac{4}{r}-\frac{2}{q}}}{N^{1-\frac{1}{r}}} \|P_N\phi\|_{L_x^2} \|P_M\psi\|_{L_x^2}. \tag{2.9}$$

The bilinear Strichartz estimate was first introduced by Bourgain [8], and further extended in [27, 74], when $q = r = 2$. The $q, r < 2$ case was referred to bilinear restriction estimates for paraboloid, first obtained by Tao [70], based on the method developed by Wolff [77].

Using the same argument by Visan [74], and combining the result by Candy [18], we can transfer the linear solutions in bi-linear estimate of Lemma 2.8 into general functions:

**Lemma 2.9.** *Let $I \subset \mathbb{R}$, $a \in I$, $1 \leqslant q, r \leqslant 2$, $\frac{1}{q} + \frac{2}{r} < 2$, and suppose that $M, N \in 2^{\mathbb{Z}}$ satisfy $M \ll N$. Moreover, for any $t \in I$, $\widehat{u}(t,\cdot)$ is supported on $\{\xi : |\xi| \sim N\}$, and $\widehat{v}(t,\cdot)$ is supported on $\{\xi : |\xi| \sim M\}$. Then,*

$$\|uv\|_{L_t^qL_x^r(I\times\mathbb{R}^3)} \lesssim \frac{M^{4-\frac{4}{r}-\frac{2}{q}}}{N^{1-\frac{1}{r}}} \|u\|_{S^*(I\times\mathbb{R}^3)} \|v\|_{S^*(I\times\mathbb{R}^3)}, \tag{2.10}$$

*where for any $L^2$-admissible $(\widetilde{q}, \widetilde{r})$,*

$$\|u\|_{S^*(I\times\mathbb{R}^3)} := \min\Big\{\|u(a)\|_{L_x^2} + \|(i\partial_t + \Delta)u\|_{L_t^{\widetilde{q}'}L_x^{\widetilde{r}'}(I\times\mathbb{R}^3)}, \|u\|_{U_\Delta^2(I;L_x^2(\mathbb{R}^3))}\Big\}. \tag{2.11}$$

2.4. **Maximal function estimates and Littlewood-Paley theory.** Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator:

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x-y)| \,\mathrm{d}y,$$

where $B(0,r) = \{x \in \mathbb{R}^d : |x| \leqslant r\}$. $\mathcal{M}$ is bounded on $L_x^p$ for $1 < p < \infty$. Furthermore, we have the vector-valued version of the boundedness:

**Lemma 2.10** ($L^pl^2$-boundedness for maximal function, see [68])**.** *Let $1 < p < \infty$ and $\{f_j\}_{j\in\mathbb{N}^+}$ be a sequence of functions such that $\|f_j\|_{l^2_{j\in\mathbb{N}^+}} \in L_x^p$. Then, we have*

$$\|\mathcal{M}(f_j)\|_{L_x^pl^2_{j\in\mathbb{N}^+}} \lesssim \|f_j\|_{L_x^pl^2_{j\in\mathbb{N}^+}}.$$

We also gather some useful classical results about the Littlewood-Paley projection operator.

**Lemma 2.11** (Maximal Littlewood-Paley estimates)**.** *Let $1 < p < \infty$ and $f \in L_x^p(\mathbb{R}^d)$. Then, we have*

$$\big\|\sup_{N\in2^{\mathbb{N}}} |P_Nf|\big\|_{L_x^p} + \big\|\sup_{N\in2^{\mathbb{N}}} |P_{\leqslant N}f|\big\|_{L_x^p} \lesssim \|f\|_{L_x^p}.$$

*Proof.* Note that $\mathcal{F}^{-1}(\phi_N)$ is a $L^1$-renormalised, radial Schwartz function, we have that for any $x \in \mathbb{R}^d$,

$$|P_Nf(x)| = |\mathcal{F}^{-1}(\phi_N) * f(x)| \lesssim \mathcal{M}(f)(x),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Then, by the $L^p$ boundedness of $\mathcal{M}$,

$$\big\|\sup_{N\in2^{\mathbb{N}}} |P_Nf|\big\|_{L_x^p} \lesssim \|\mathcal{M}(f)\|_{L_x^p} \lesssim \|f\|_{L_x^p}.$$

The proof for $P_{\leqslant N}f$ follows similarly. $\square$

**Lemma 2.12** (Littlewood-Paley estimates). *Let $1<p<\infty$ and $f\in L_x^p(\mathbb{R}^d)$. Then, we have*

$$\|f_N\|_{L_x^p l^2_{N\in 2^{\mathbb{N}}}} \sim_p \|f\|_{L_x^p}.$$

2.5. **Probabilistic theory.** We recall the large deviation estimate, which holds for the random variable sequence $\{\mathrm{Re}g_k, \mathrm{Im}g_k\}$ in the Definition 1.1.

**Lemma 2.13** (Large deviation estimate, [16]). *Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space. Let $\{g_n\}_{n\in\mathbb{N}^+}$ be a sequence of real-valued, independent, zero-mean random variables with associated distributions $\{\mu_n\}_{n\in\mathbb{N}^+}$ on $\Omega$. Suppose $\{\mu_n\}_{n\in\mathbb{N}^+}$ satisfies that there exists $c>0$ such that for all $\gamma\in\mathbb{R}$ and $n\in\mathbb{N}^+$*

$$\Big|\int_{\mathbb{R}} e^{\gamma x}\mathrm{d}\mu_n(x)\Big| \leqslant e^{c\gamma^2},$$

*then there exists $\alpha>0$ such that for any $\lambda>0$ and any complex-valued sequence $\{c_n\}_{n\in\mathbb{N}^+}\in l_n^2$, we have*

$$\mathbb{P}\Big(\Big\{\omega:\Big|\sum_{n=1}^{\infty}c_n g_n(\omega)\Big|>\lambda\Big\}\Big)\leqslant 2\exp\Big\{-\alpha\lambda\,\|c_n\|_{l_n^2}^{-2}\Big\}.$$

*Furthermore, there exists $C>0$ such that for any $2\leqslant p<\infty$ and complex-valued sequence $\{c_n\}_{n\in\mathbb{N}^+}\in l_n^2$, we have*

$$\Big\|\sum_{n=1}^{\infty}c_n g_n(\omega)\Big\|_{L^p_\omega(\Omega)}\leqslant C\sqrt{p}\,\|c_n\|_{l_n^2}. \tag{2.12}$$

The following lemma can be proved by the method in [73], see also [37, 38].

**Lemma 2.14.** *Let $F$ be a real-valued measurable function on a probability space $(\Omega,\mathcal{A},\mathbb{P})$. Suppose that there exists $C_0>0$, $K>0$ and $p_0\geqslant 1$ such that for any $p\geqslant p_0$, we have*

$$\|F\|_{L^p_\omega(\Omega)}\leqslant\sqrt{p}C_0K.$$

*Then, there exist $c>0$ and $C_1>0$, depending on $C_0$ and $p_0$ but independent of $K$, such that for any $\lambda>0$,*

$$\mathbb{P}\big(\{\omega\in\Omega:|F(\omega)|>\lambda\}\big)\leqslant C_1e^{-c\lambda^2K^{-2}}.$$

*Particularly, we have*

$$\mathbb{P}\big(\{\omega\in\Omega:|F(\omega)|<\infty\}\big)=1.$$

## 3. Almost sure Strichartz estimates

### 3.1. Non-radial data.

**Lemma 3.1.** *Let $s\in\mathbb{R}$ and $f\in H_x^s$. Suppose that the randomization $f^\omega$ is defined in Definition 1.1. Then, we have the following estimates:*

*(1) For any* $2 \leqslant q, r < \infty$ *with* $\frac{2}{q} + \frac{3}{r} \leqslant \frac{3}{2}$, *and for any* $p \geqslant \max\{q, r\}$,

$$\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^q_t L^r_x (\Omega \times 2^{\mathbb{N}} \times \mathbb{R} \times \mathbb{R}^3)} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}. \tag{3.1}$$

*(2) For any* $p \geqslant 2$,

$$\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^\infty_t L^2_x (\Omega \times 2^{\mathbb{N}} \times \mathbb{R} \times \mathbb{R}^3)} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}. \tag{3.2}$$

*(3) For any* $2 \leqslant q < \infty$, *there exists* $p_0 \geqslant 2$ *such that for any* $p \geqslant p_0$,

$$\left\| \langle \nabla \rangle^{s-} e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^q_t L^\infty_x (\Omega \times 2^{\mathbb{N}} \times \mathbb{R} \times \mathbb{R}^3)} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}. \tag{3.3}$$

*(4) For any* $2 < r \leqslant \infty$, *there exists* $p_0 \geqslant 2$ *such that for any* $p \geqslant p_0$,

$$\left\| \langle \nabla \rangle^{s-} e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^\infty_t L^r_x (\Omega \times 2^{\mathbb{N}} \times \mathbb{R} \times \mathbb{R}^3)} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}. \tag{3.4}$$

*Remark* 3.2. We remark that for example, by Minkowski's inequality and Lemma 2.12, (3.1) also gives that

$$\left\| \langle \nabla \rangle^s e^{it\Delta} f^\omega \right\|_{L^p_\omega L^q_t L^r_x (\Omega \times \mathbb{R} \times \mathbb{R}^3)} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}.$$

*Proof.* In the proof of this lemma, we restrict the variables on $\omega \in \Omega$, $N \in 2^{\mathbb{N}}$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, and $k \in \mathbb{Z}^3$.

We first prove (3.1). By Minkowski's inequality and Lemma 2.13, we have

$$\begin{aligned} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^q_t L^r_x} &\lesssim \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{l^2_N L^q_t L^r_x L^p_\omega} \\ &\lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} \square_k P_N f \right\|_{l^2_N L^q_t L^r_x l^2_k} \\ &\lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N \square_k f \right\|_{l^2_N l^2_k L^q_t L^r_x}. \end{aligned} \tag{3.5}$$

Now, let $2 \leqslant r_0 \leqslant r$ such that $(q, r_0)$ is $L^2_x$-admissible. For any $k \in \mathbb{Z}^3$, by the support property of $\psi_k$ and Bernstein's inequality, we have

$$\left\| \langle \nabla \rangle^s e^{it\Delta} P_N \square_k f \right\|_{L^q_t L^r_x} \lesssim \left\| \langle \nabla \rangle^s e^{it\Delta} P_N \square_k f \right\|_{L^q_t L^{r_0}_x}. \tag{3.6}$$

Then, by (3.5), (3.6), Lemma 2.7, and orthogonality, we have

$$\begin{aligned} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^q_t L^r_x} &\lesssim \sqrt{p} \left\| \langle \nabla \rangle^s e^{it\Delta} P_N \square_k f \right\|_{l^2_N l^2_k L^q_t L^{r_0}_x} \\ &\lesssim \sqrt{p} \left\| \langle \nabla \rangle^s P_N \square_k f \right\|_{l^2_N l^2_k L^2_x} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}. \end{aligned}$$

This gives (3.1).

Next, we prove (3.2). By Plancherel's identity, we have

$$\left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^\infty_t L^2_x} \lesssim \left\| \langle \nabla \rangle^s P_N f^\omega \right\|_{L^p_\omega l^2_N L^2_x} \lesssim \left\| \langle \nabla \rangle^s f^\omega \right\|_{L^p_\omega L^2_x}. \tag{3.7}$$

Then, by Minkowski's inequality and Lemma 2.13,

$$\begin{aligned} \left\| \langle \nabla \rangle^s f^\omega \right\|_{L^p_\omega L^2_x} &\lesssim \left\| \langle \nabla \rangle^s f^\omega \right\|_{L^2_x L^p_\omega} \\ &\lesssim \sqrt{p} \left\| \langle \nabla \rangle^s \square_k f \right\|_{L^2_x l^2_k} \lesssim \sqrt{p} \left\| f \right\|_{H^s_x}. \end{aligned} \tag{3.8}$$

Then, (3.7) and (3.8) imply (3.2).

We then prove (3.3). Let $0 < \varepsilon \ll 1$. Using the Sobolev's embedding $W^{2\varepsilon, \frac{3}{\varepsilon}}_x \hookrightarrow L^\infty_x$ in $x$, we have

$$\left\| \langle \nabla \rangle^{s-2\varepsilon} e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^q_t L^\infty_x} \lesssim \left\| \langle \nabla \rangle^s e^{it\Delta} P_N f^\omega \right\|_{L^p_\omega l^2_N L^q_t L^{\frac{3}{\varepsilon}}_x}. \tag{3.9}$$

Let $p_0=\max\left\{q,\frac{3}{\varepsilon}\right\}$ and $2\leqslant r_0\leqslant\frac{3}{\varepsilon}$ such that $(q,r_0)$ is $L^2_x$-admissible. Then, similar as above, by Minkowski's, Bernstein's inequalities, Lemmas 2.7, and 2.13, for any $p\geqslant p_0$, we have

$$\begin{aligned}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^q_t L^{\frac{3}{\varepsilon}}_x}\lesssim&\left\|\langle\nabla\rangle^s e^{it\Delta}P_N f^\omega\right\|_{l^2_N L^q_t L^{\frac{3}{\varepsilon}}_x L^p_\omega}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N\Box_k f\right\|_{l^2_N L^q_t L^{\frac{3}{\varepsilon}}_x l^2_k}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N\Box_k f\right\|_{l^2_N l^2_k L^q_t L^{\frac{3}{\varepsilon}}_x}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N\Box_k f\right\|_{l^2_N l^2_k L^q_t L^{r_0}_x}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s P_N\Box_k f\right\|_{l^2_N l^2_k L^2_x}\lesssim\sqrt{p}\left\|f\right\|_{H^s_x}.\end{aligned}\tag{3.10}$$

By (3.9) and (3.10), we have that (3.3) holds.

Finally, we prove (3.4). We only consider the $r=\infty$ case. In fact, when $r<\infty$, we can prove it using interpolation between (3.2) and the $r=\infty$ case. Let $0<\varepsilon\ll 1$. Using Minkowski's inequality, the Sobolev's embedding $W^{2\varepsilon,\frac{1}{\varepsilon}}_t\hookrightarrow L^\infty_t$ in $t$ and $W^{2\varepsilon,\frac{3}{\varepsilon}}_x\hookrightarrow L^\infty_x$ in $x$, we have

$$\begin{aligned}\left\|\langle\nabla\rangle^{s-6\varepsilon} e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^\infty_t L^\infty_x}\lesssim&\left\|\langle\nabla\rangle^{s-4\varepsilon} e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^\infty_t L^{\frac{3}{\varepsilon}}_x}\\ \lesssim&\left\|\langle\nabla\rangle^{s-4\varepsilon}\langle\partial_t\rangle^{2\varepsilon} e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^{\frac{3}{\varepsilon}}_x L^{\frac{1}{\varepsilon}}_t}\\ \lesssim&\left\|\langle\nabla\rangle^{s} e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^{\frac{3}{\varepsilon}}_x L^{\frac{1}{\varepsilon}}_t}\\ \lesssim&\left\|\langle\nabla\rangle^{s} e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^{\frac{1}{\varepsilon}}_t L^{\frac{3}{\varepsilon}}_x}.\end{aligned}\tag{3.11}$$

Let $p_0=\frac{3}{\varepsilon}$ and $2<r_0\leqslant\frac{3}{\varepsilon}$ such that $(\frac{1}{\varepsilon},r_0)$ is $L^2_x$-admissible. Then, similar as above, for any $p\geqslant p_0$, we have

$$\begin{aligned}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N f^\omega\right\|_{L^p_\omega l^2_N L^{\frac{1}{\varepsilon}}_t L^{\frac{3}{\varepsilon}}_x}\lesssim&\left\|\langle\nabla\rangle^s e^{it\Delta}P_N f^\omega\right\|_{l^2_N L^{\frac{1}{\varepsilon}}_t L^{\frac{3}{\varepsilon}}_x L^p_\omega}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N\Box_k f\right\|_{l^2_N L^{\frac{1}{\varepsilon}}_t L^{\frac{3}{\varepsilon}}_x l^2_k}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N\Box_k f\right\|_{l^2_N l^2_k L^{\frac{1}{\varepsilon}}_t L^{\frac{3}{\varepsilon}}_x}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s e^{it\Delta}P_N\Box_k f\right\|_{l^2_N l^2_k L^{\frac{1}{\varepsilon}}_t L^{r_0}_x}\\ \lesssim&\sqrt{p}\left\|\langle\nabla\rangle^s P_N\Box_k f\right\|_{l^2_N l^2_k L^2_x}\lesssim\sqrt{p}\left\|f\right\|_{H^s_x}.\end{aligned}\tag{3.12}$$

By (3.11) and (3.12), we have that (3.4) holds. □

Next, we gather all the space-time norms that will be used below. Let $\frac{1}{6}\leqslant s<\frac{1}{2}$ and $\varepsilon>0$ be absolutely small. Define the $Y^s(I)$ space by its norm

$$\begin{aligned}\|v\|_{Y^s(I)}:=&\left\||\nabla|^s v_N\right\|_{l^2_N L^4_t L^{\frac{9}{2}}_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}+\left\||\nabla|^s v_N\right\|_{l^2_N L^6_{t,x}(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}\\ &+\left\||\nabla|^s v_N\right\|_{l^2_N L^4_t L^{18}_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}+\|v_N\|_{l^2_N L^{\frac{20}{7}}_t L^{10}_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}\\ &+\|v\|_{L^4_t L^{12}_x(I\times\mathbb{R}^3)}+\|v\|_{L^3_t L^\infty_x(I\times\mathbb{R}^3)}+\|v\|_{L^4_t L^6_x(I\times\mathbb{R}^3)}+\|v\|_{L^3_t L^6_x(I\times\mathbb{R}^3)}\\ &+\|v\|_{L^{\frac{3}{1-\varepsilon}}_t L^{\frac{6}{1-3\varepsilon}}_x(I\times\mathbb{R}^3)}+\|v\|_{L^{\frac{4(4-3\varepsilon)}{5+3\varepsilon}}_t L^{\frac{2(4-3\varepsilon)}{1-3\varepsilon}}_x(I\times\mathbb{R}^3)},\end{aligned}\tag{3.13}$$

and the $Z$-norm by

$$\|v\|_{Z^s(I)} := \big\|\langle\nabla\rangle^{s-} P_N v\big\|_{l^2_N L^\infty_t L^\infty_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)} + \|\langle\nabla\rangle^s P_N v\|_{l^2_N L^\infty_t L^2_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}\,. \tag{3.14}$$

Then, by Lemmas 3.1 and 2.14, we have

**Corollary 3.3.** *Let $\frac{1}{6}\leqslant s<\frac{1}{2}$ and $f\in H^s_x$. Then, there exist constants $C,c>0$ such that for any $\lambda$,*

$$\mathbb{P}\big(\big\{\omega\in\Omega : \big\|e^{it\Delta}f^\omega\big\|_{Y^s(\mathbb{R})}+\big\|e^{it\Delta}f^\omega\big\|_{Z^s(\mathbb{R})}>\lambda\big\}\big)\leqslant C\exp\big\{-c\lambda^2\,\|f\|^{-2}_{H^s_x(\mathbb{R}^3)}\big\}.$$

*Moreover,*

$$\big\|e^{it\Delta}f^\omega\big\|_{Y^s(\mathbb{R})}+\big\|e^{it\Delta}f^\omega\big\|_{Z^s(\mathbb{R})}<+\infty,\quad a.e.\ \omega\in\Omega.$$

3.2. **Radial data.** Here, we derive a super-critical estimate for the randomized radial data that can acquire $\frac{1}{2}$-derivative. Such class of estimates was first proved by Dodson, Lührmann, and Mendelson [38] in 4D case, based on their "radialish" Sobolev's inequality [37]. We adapt their method to the 3D case:

**Proposition 3.4.** *Let $4<r\leqslant\infty$, $s_0\in\mathbb{R}$. Suppose that $f\in H^{s_0}_x(\mathbb{R}^3)$ is radial. Then for any $s<s_0+\frac{1}{2}$, there exist constants $C,c>0$ such that for any $\lambda>0$,*

$$\mathbb{P}\big(\big\{\omega\in\Omega : \big\|\,|\nabla|^s\,e^{it\Delta}f^\omega\big\|_{L^2_tL^r_x(\mathbb{R}\times\mathbb{R}^3)}>\lambda\big\}\big)\leqslant C\exp\big\{-c\lambda^2\,\|f\|^{-2}_{H^{s_0}_x(\mathbb{R}^3)}\big\}. \tag{3.15}$$

*Moreover,*

$$\big\|\,|\nabla|^s\,e^{it\Delta}f^\omega\big\|_{L^2_tL^r_x(\mathbb{R}\times\mathbb{R}^3)}<\infty,\quad a.e.\ \omega\in\Omega. \tag{3.16}$$

To prove Proposition 3.4, we need the following 3D version of radial Sobolev estimate for the square function.

**Lemma 3.5.** *Suppose that the function $f$ is radial and that $2\leqslant r\leqslant\infty$. Then, for any $\epsilon>0$, there exists $C_\epsilon>0$ such that,*

$$\Big\||x|^{1-\frac{2}{r}}\,\|f_k\|_{l^2_k}\Big\|_{L^r_x(\mathbb{R}^3)}\leqslant C_\epsilon\,\|f\|_{H^\epsilon_x(\mathbb{R}^3)}\,. \tag{3.17}$$

*Proof.* It suffices to prove the $r=\infty$ case, since the general case can be obtained by interpolation with

$$\big\|\,\|f_k\|_{l^2_k}\big\|_{L^2_x(\mathbb{R}^3)}\sim\|f\|_{L^2_x(\mathbb{R}^3)}\,.$$

Since $f$ is radial, we can write $\widehat{f}(x)=\widehat{f}(|x|)$. Assume without loss of generality that $x=(0,0,|x|)$. Then, by integration-by-parts and the spherical coordinate

$$\xi(\rho,\theta,\alpha)=(\rho\sin\theta\cos\alpha,\rho\sin\theta\sin\alpha,\rho\cos\theta),$$

we have

$$\begin{aligned}
f_k =& \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi_k(\xi(\rho,\theta,\alpha))\widehat{f}(\rho)e^{i|x|\rho\cos\theta}\rho^2\sin\theta \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\alpha\\
=& -\frac{1}{i|x|}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \psi_k\big(\xi(\rho,\theta,\alpha)\big)\,\partial_\theta\left(e^{i|x|\rho\cos\theta}\right)\widehat{f}(\rho)\rho\, d\theta\mathrm{d}\alpha\,\mathrm{d}\rho\\
=&\frac{1}{i|x|}\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \partial_\theta\big(\psi_k(\xi(\rho,\theta,\alpha))\big)e^{i|x|\rho\cos\theta}\widehat{f}(\rho)\rho\,\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}\rho \qquad (3.18a)\\
&-\frac{1}{i|x|}\int_0^{\infty}\int_0^{2\pi}\psi_k(\xi(\rho,\pi,\alpha))e^{-i|x|\rho}\widehat{f}(\rho)\rho\,\mathrm{d}\alpha\mathrm{d}\rho \qquad (3.18b)\\
&+\frac{1}{i|x|}\int_0^{\infty}\int_0^{2\pi}\psi_k(\xi(\rho,0,\alpha))e^{i|x|\rho}\widehat{f}(\rho)\rho\,\mathrm{d}\alpha\mathrm{d}\rho. \qquad (3.18c)
\end{aligned}$$

Denote that

$$k=(k_1,k_2,k_3)=(|k|\sin\theta_k\cos\alpha_k, |k|\sin\theta_k\sin\alpha_k, |k|\cos\theta_k).$$

By the support of $\psi_k$, we have that $\rho$ is supported on the set $\{\rho : |\rho-|k|| \lesssim 1\}$, $\theta$ is supported on $\left\{\theta : |\theta-\theta_k| \lesssim \frac{1}{|k|}\right\}$, and $\alpha$ is supported on $\left\{\alpha : |\alpha-\alpha_k| \lesssim \frac{1}{\langle |k|\sin\theta_k\rangle}\right\}$. Furthermore, we also have

$$|\partial_\theta\left(\psi_k(\xi(\rho,\theta,\alpha))\right)| \lesssim \rho\left|\nabla_\xi\psi_k(\xi(\rho,\theta,\alpha))\right|.$$

Therefore, we have

$$\begin{aligned}
|(3.18a)| \lesssim&\frac{1}{|x|}\frac{1}{|k|}\frac{1}{\langle |k|\sin\theta_k\rangle}\big\|\widehat{f}(\rho)\rho^2\big\|_{L^1_{|\rho-|k||\lesssim 1}}\\
\lesssim&\frac{1}{|x|}\frac{1}{\langle |k|\sin\theta_k\rangle}\big\|\widehat{f}(\rho)\rho\big\|_{L^2_{|\rho-|k||\lesssim 1}}.
\end{aligned} \qquad (3.19)$$

Similarly, we also have

$$|(3.18b)|+|(3.18c)| \lesssim \frac{1}{|x|}\frac{1}{\langle |k|\sin\theta_k\rangle}\big\|\widehat{f}(\rho)\rho\big\|_{L^2_{|\rho-|k||\lesssim 1}}. \qquad (3.20)$$

Then, by (3.19) and (3.20),

$$\begin{aligned}
|x|^2\sum_{k\in\mathbb{Z}^3}|f_k|^2 \lesssim& \sum_{k\in\mathbb{Z}^3}\frac{1}{\langle |k|\sin\theta_k\rangle^2}\big\|\widehat{f}(\rho)\rho\big\|^2_{L^2_{|\rho-|k||\lesssim 1}}\\
\sim&\sum_{N\in\mathbb{N}}\sum_{\substack{k_1,k_2\in\mathbb{Z}^2\\ |k_1|\leqslant N,|k_2|\leqslant N}}\frac{1}{\langle k_1^2+k_2^2\rangle}\big\|\widehat{f}(\rho)\rho\big\|^2_{L^2_{|\rho-N|\lesssim 1}}\\
\lesssim&\sum_{N\in\mathbb{N}}\ln N\big\|\widehat{f}(\rho)\rho\big\|^2_{L^2_{|\rho-N|\lesssim 1}} \lesssim \|f\|^2_{H^\epsilon_x(\mathbb{R}^3)}.
\end{aligned}$$

This finishes the proof of (3.17). □

We also need the following mismatch estimates concerning the commutator of $\chi_j$ and $\square_k$. The same result was already proved in [38] for 4D, and their argument can be easily extended to general dimensions. Therefore, we omit the details of proof.

**Lemma 3.6** (Mismatch estimates)**.** *Let $2 \leqslant r \leqslant \infty$, $j, l \geqslant 0$ and $k, m \in \mathbb{Z}^3$. Suppose that $l > j+5$ and $|k-m| > 100$. Then for any integer $M > 0$, we have*

$$\|\chi_j \Box_k \chi_l\|_{L^2_x(\mathbb{R}^3)\to L^r_x(\mathbb{R}^3)} \leqslant C_M 2^{-Ml}, \tag{3.21}$$

*and*

$$\|\Box_k \chi_l \Box_m\|_{L^2_x(\mathbb{R}^3)\to L^r_x(\mathbb{R}^3)} \leqslant C_M 2^{-Ml}|k-m|^{-M}. \tag{3.22}$$

Now, we give the proof of Proposition 3.4 using Lemmas 3.5 and 3.6.

*Proof of Proposition 3.4.* Take some $\epsilon > 0$ that will be defined later. We first consider the $r < \infty$ case. Then, for any $p \geqslant r$, by Lemma 2.13,

$$\begin{aligned}\left\||\nabla|^s e^{it\Delta} f^\omega\right\|_{L^p_\omega L^2_t L^r_x} &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^s \left\|e^{it\Delta} P_N f^\omega\right\|_{L^2_t L^r_x L^p_\omega} \\ &\lesssim \sqrt{p} \sum_{N\in 2^{\mathbb{N}}} N^s \left\|e^{it\Delta}\Box_k P_N f\right\|_{L^2_t L^r_x l^2_k} \\ &\lesssim \sqrt{p} \sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 0} N^s \left\|\chi_j \Box_k \chi_{\leqslant j+5} e^{it\Delta} P_N f\right\|_{L^2_t L^r_x l^2_k} \qquad (3.23a) \\ &\quad + \sqrt{p} \sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 0} N^s \left\|\chi_j \Box_k \chi_{> j+5} e^{it\Delta} P_N f\right\|_{L^2_t L^r_x l^2_k}. \qquad (3.23b)\end{aligned}$$

We first bound the term (3.23a). When $j = 0$, by Lemmas 3.5 and 2.6,

$$\begin{aligned}\sum_{N\in 2^{\mathbb{N}}} N^s \left\|\chi_0 \Box_k \chi_{\leqslant 5} e^{it\Delta} P_N f\right\|_{L^2_t L^r_x l^2_k} &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^s \left\|\Box_k P_{\sim N} \chi_{\leqslant 5} e^{it\Delta} P_N f\right\|_{L^2_t L^r_x l^2_{k:|k|\sim N}} \\ &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^s \left\| |\nabla|^\epsilon P_{\sim N} \chi_{\leqslant 5} e^{it\Delta} P_N f\right\|_{L^2_{t,x}} \\ &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^{s+\epsilon} \left\|\chi_{\leqslant 5} e^{it\Delta} P_N f\right\|_{L^2_{t,x}} \\ &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^{s+\epsilon-\frac12} \left\|P_N f\right\|_{L^2_x} \lesssim \|f\|_{H^{s+2\epsilon-\frac12}_x}.\end{aligned} \tag{3.24}$$

For $j \geqslant 1$, also by Lemmas 3.5 and 2.6,

$$\begin{aligned}&\sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 1} N^s \left\|\chi_j \Box_k \chi_{\leqslant j+5} e^{it\Delta} P_N f\right\|_{L^2_t L^r_x l^2_k} \\ \lesssim &\sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 1} N^s 2^{-(1-\frac{2}{r})j} \left\|\chi_j |x|^{1-\frac{2}{r}} \Box_k P_{\sim N} \chi_{\leqslant j+5} e^{it\Delta} P_N f\right\|_{L^2_t L^r_x l^2_k} \\ \lesssim &\sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 1} N^s 2^{-(1-\frac{2}{r})j} \left\||\nabla|^\epsilon P_{\sim N} \chi_{\leqslant j+5} e^{it\Delta} P_N f\right\|_{L^2_{t,x}} \\ \lesssim &\sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 1} N^{s+\epsilon} 2^{-(1-\frac{2}{r})j} \left\|\chi_{\leqslant j+5} e^{it\Delta} P_N f\right\|_{L^2_{t,x}} \\ \lesssim &\sum_{N\in 2^{\mathbb{N}}} \sum_{j\geqslant 1} N^{s-\frac12+\epsilon} 2^{-(\frac12-\frac{2}{r})j} \left\|P_N f\right\|_{L^2_x} \lesssim \|f\|_{H^{s+2\epsilon-\frac12}_x}.\end{aligned} \tag{3.25}$$

Then, combining (3.24) and (3.25), we have

$$(3.23a) \lesssim \sqrt{p}\, \|f\|_{H^{s+2\epsilon-\frac12}_x}. \tag{3.26}$$

Next, we consider (3.23b). We decompose that

$$
\begin{aligned}
(3.23\text{b}) \lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0} N^s \left\|\chi_j \square_k \chi_{>j+5} e^{it\Delta} P_N f\right\|_{L_t^2 L_x^r l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \left\|\chi_j \square_k \chi_l \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_t^2 L_x^r l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|\leqslant 100} \chi_j \square_k \chi_l \square_m \widetilde{\chi}_l e^{it\Delta} P_N f\Big\|_{L_t^2 L_x^r l_k^2} \quad (3.27\text{a}) \\
& + \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|>100} \chi_j \square_k \chi_l \square_m \widetilde{\chi}_l e^{it\Delta} P_N f\Big\|_{L_t^2 L_x^r l_k^2}.
\end{aligned} \tag{3.27b}
$$

Now, we take some $M \gg 1$. For (3.27a), by Minkowski's inequality, Lemmas 3.6, and 2.6,

$$
\begin{aligned}
(3.27\text{a}) \lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|\leqslant 100} \chi_j \square_k \chi_l \square_m \widetilde{\chi}_l e^{it\Delta} P_N f\Big\|_{L_t^2 L_x^r l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|\leqslant 100} \left\|\chi_j \square_k \chi_l \square_m \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_t^2 L_x^r} \Big\|_{l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s 2^{-Ml} \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|\leqslant 100} \left\|\square_m \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_{t,x}^2} \Big\|_{l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s 2^{-Ml} \left\|\square_k \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{l_k^2 L_{t,x}^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s 2^{-Ml} \left\|\widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_{t,x}^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^{s-\frac{1}{2}} 2^{-(M-\frac{1}{2})l} \left\|P_N f\right\|_{L_x^2} \lesssim \sqrt{p}\left\|f\right\|_{H_x^{s-\frac{1}{2}+\epsilon}}.
\end{aligned} \tag{3.28}
$$

Similarly by Lemmas 3.6, 2.6, and Young's inequality in $k$,

$$
\begin{aligned}
(3.27\text{b}) = & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|>100} \chi_j \square_k \chi_l \square_m \widetilde{\chi}_l e^{it\Delta} P_N f\Big\|_{L_t^2 L_x^r l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|>100} \left\|\square_k \chi_l \square_m \widetilde{\square}_m \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_t^2 L_x^r} \Big\|_{l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s 2^{-Ml} \Big\| \sum_{m\in\mathbb{Z}^3:|m-k|>100} |k-m|^{-M} \left\|\widetilde{\square}_m \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_{t,x}^2} \Big\|_{l_k^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s 2^{-Ml} \Big\| \left\|\widetilde{\square}_m \widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_{t,x}^2} \Big\|_{l_m^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^s 2^{-Ml} \left\|\widetilde{\chi}_l e^{it\Delta} P_N f\right\|_{L_{t,x}^2} \\
\lesssim & \sqrt{p}\sum_{N\in 2^{\mathbb{N}}}\sum_{j\geqslant 0}\sum_{l>j+5} N^{s-\frac{1}{2}} 2^{-(M-\frac{1}{2})l} \left\|P_N f\right\|_{L_x^2} \lesssim \sqrt{p}\left\|f\right\|_{H_x^{s-\frac{1}{2}+\epsilon}}.
\end{aligned} \tag{3.29}
$$

Therefore, we have

$$(3.23\text{b}) \lesssim \sqrt{p}\, \|f\|_{H_x^{s-\frac{1}{2}+\epsilon}}. \tag{3.30}$$

By (3.26) and (3.30),

$$\begin{aligned}\left\||\nabla|^s e^{it\Delta} f^\omega\right\|_{L^p_\omega L^2_t L^r_x} &\lesssim \sum_{N\in 2^{\mathbb{N}}} N^s \left\|e^{it\Delta} P_N f^\omega\right\|_{L^2_t L^r_x L^p_\omega} \\ &\lesssim (3.23\text{a}) + (3.23\text{b}) \lesssim \sqrt{p}\, \|f\|_{H_x^{s-\frac{1}{2}+2\epsilon}}.\end{aligned}$$

Let $\epsilon \leqslant \frac{1}{2}(s_0 - s + \frac{1}{2})$, then we have (3.15) and (3.16) hold for $r < \infty$.

When $r = \infty$, using the similar argument above with $r = \frac{3}{\varepsilon}$,

$$\left\||\nabla|^s e^{it\Delta} f^\omega\right\|_{L^p_\omega L^2_t L^\infty_x} \lesssim \sum_{N\in 2^{\mathbb{N}}} N^{s+\epsilon} \left\|e^{it\Delta} P_N f^\omega\right\|_{L^2_t L_x^{\frac{3}{\epsilon}} L^p_\omega} \lesssim \sqrt{p}\, \|f\|_{H_x^{s-\frac{1}{2}+3\epsilon}}.$$

Let $\epsilon \leqslant \frac{1}{3}(s_0 - s + \frac{1}{2})$, then we finish the proof of the $r = \infty$ case. □

## 4. Local well-posedness

4.1. **Reduction to the deterministic problem.** Suppose that $u = v + w$ with $u_0 = v_0 + w_0$, $v = e^{it\Delta} v_0$, and $w$ satisfying

$$\begin{cases} i\partial_t w + \Delta w = |u|^2 u, \\ w(0,x) = w_0(x). \end{cases} \tag{4.1}$$

Define the working space

$$\|w\|_{X^{\frac{1}{2}}(I)} = \Big(\sum_{N\in 2^{\mathbb{N}}} N \|w_N\|^2_{U^2_\Delta(I;L^2_x)}\Big)^{\frac{1}{2}}. \tag{4.2}$$

We first have a local result for $H_x^{\frac{1}{6}}$-data:

**Proposition 4.1.** *Let $\frac{1}{6} \leqslant s < \frac{1}{2}$, $v \in Y^s \cap Z^s(\mathbb{R})$, and $w_0 \in H_x^{\frac{1}{2}}$. Then, there exists some $T > 0$ depending on $s$, $\|w_0\|_{H_x^{\frac{1}{2}}}$, and $\|v\|_{Y^s(\mathbb{R})\cap Z^s(\mathbb{R})}$ such that there uniquely exists a solution $w$ of (1.7) on $[0,T]$ with*

$$w \in C([0,T]; H_x^{\frac{1}{2}}) \cap X^{\frac{1}{2}}([0,T]).$$

Next, we turn to the improved local result for $H_x^{\frac{1}{3}+}$-data. Define

$$\|w\|_{X^1(I)} = \Big(\sum_{N\in 2^{\mathbb{N}}} N^2 \|w_N\|^2_{U^2_\Delta(I;L^2_x)}\Big)^{\frac{1}{2}}.$$

Then our local result for $H_x^{\frac{1}{3}+}$-data is

**Proposition 4.2.** *Let $\frac{1}{3} < s \leqslant \frac{1}{2}$, $v \in Y^s \cap Z^s(\mathbb{R})$, and $w_0 \in H^1_x$. Then, there exists some $T > 0$ depending on $s$, $\|w_0\|_{H^1_x}$, and $\|v\|_{Y^s(\mathbb{R})\cap Z^s(\mathbb{R})}$ such that there uniquely exists a solution $w$ of (1.7) on $[0,T]$ with*

$$w \in C([0,T]; H^1_x) \cap X^1([0,T]).$$

In fact, the $s=\frac{1}{2}$ case is trivial. However, we need it for the global result in Proposition 5.1.

Now, we give the proof of Theorem 1.2, assuming that Propositions 4.1 and 4.2 hold.

*Proof of Theorem 1.2.* Let

$$u(t)=e^{it\Delta}f^{\omega}+w(t),$$

then $w$ satisfies the equation (4.1) with

$$v_0=f^{\omega},\ w_0=0, \text{ and } v=e^{it\Delta}f^{\omega}.$$

We first prove Theorem 1.2($a$), using the result in Proposition 4.1. By Corollary 3.3, we have for almost every $\omega\in\Omega$,

$$\|v\|_{Y^s(\mathbb{R})}+\|v\|_{Z^s(\mathbb{R})}<\infty.$$

Since $w_0=0$, we can apply Proposition 4.1 to obtain the existence and uniqueness of $w\in C([0,T];H_x^{\frac{1}{2}})$ for almost every $\omega\in\Omega$.

The proof of Theorem 1.2 (2) by Proposition 4.2 is similar as above, so we omit the details. □

4.2. **Local well-posedness.** In this section, we prove Proposition 4.1. We make the choices of some parameters and define the auxiliary working space:

(1) Let $C_0>0$ be the constant such that

$$\left\|e^{it\Delta}w_0\right\|_{X^{\frac{1}{2}}(\mathbb{R})}\leqslant C_0\left\|w_0\right\|_{H_x^{\frac{1}{2}}}.$$

(2) Let

$$R:=\max\left\{C_0\left\|w_0\right\|_{H_x^{\frac{1}{2}}},1\right\}.$$

(3) Let $\delta$ and $\varepsilon$ be some constants such that $0<\delta,\varepsilon\ll 1$.

(4) Define the following space:

$$\begin{aligned}\widetilde{X}^{\frac{1}{2}}(I)=&\left\||\nabla|^{\frac{1}{3}}w_N\right\|_{l_N^2L_t^2L_x^9(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}+\left\||\nabla|^{\frac{1}{6}}w_N\right\|_{l_N^2L_t^4L_x^{\frac{9}{2}}(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}\\&+\left\||\nabla|^{\frac{1}{6}}w_N\right\|_{l_N^2L_t^3L_x^6(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}+\|w\|_{L_t^4L_x^6(I\times\mathbb{R}^3)}+\|w\|_{L_t^3L_x^6(I\times\mathbb{R}^3)}\\&+\left\|\langle\nabla\rangle^{\frac{1}{2}}w\right\|_{L_t^2L_x^6(I\times\mathbb{R}^3)}+\|w\|_{L_t^{\frac{3}{1-\varepsilon}}L_x^{\frac{6}{1-3\varepsilon}}(I\times\mathbb{R}^3)}\\&+\|w\|_{L_t^{\frac{4(4-3\varepsilon)}{5+3\varepsilon}}L_x^{\frac{2(4-3\varepsilon)}{1-3\varepsilon}}(I\times\mathbb{R}^3)}.\end{aligned}$$

(5) Let $T>0$ satisfy the smallness conditions

$$\left\|e^{it\Delta}w_0\right\|_{\widetilde{X}^{\frac{1}{2}}([0,T])}+\|v\|_{Y^s([0,T])}\leqslant\delta,$$

and

$$\delta T^{\varepsilon^2}\|v\|_{Z^s(\mathbb{R})}^2+T^{\frac{1}{100}}R^2\|v\|_{Z^s(\mathbb{R})}\lesssim\delta^3.$$

We remark that

$$X^{\frac{1}{2}}([0,T])\hookrightarrow\widetilde{X}^{\frac{1}{2}}([0,T]),$$

and $T$ depends on $s$, $\delta$, $\varepsilon$, $\|v\|_{Y^s(\mathbb{R})}$, $\|v\|_{Z^s(\mathbb{R})}$, and $\|w_0\|_{H_x^{\frac12}}$. Let the working space be defined by

$$B_{R,\delta,T} := \Big\{ w \in C([0,T]; H_x^{\frac12}) : \|w\|_{X^{\frac12}([0,T])} \leqslant 2R,\ \|w\|_{\widetilde{X}^{\frac12}([0,T])} \leqslant 2\delta \Big\}.$$

Define

$$\Phi_{w_0,v}(w) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta}(|u|^2 u)\, \mathrm{d}s.$$

Now, we are going to prove $\Phi_{w_0,v}$ is a contraction mapping on $B_{R,\delta,T}$, which is reduced to prove the following nonlinear estimate

$$\Big\| \int_0^t e^{i(t-s)\Delta}(|u|^2 u)\, \mathrm{d}s \Big\|_{X^{\frac12}([0,T])} \leqslant \delta.$$

In fact, we can use similar argument to prove

$$\big\|\Phi_{w_0,v}(w_1) - \Phi_{w_0,v}(w_2)\big\|_{X^{\frac12}([0,T])} \leqslant \frac12 \|w_1 - w_2\|_{X^{\frac12}([0,T])},$$

and then finish the proof of contraction mapping.

Therefore, we reduce the proof of Proposition 4.1 to the following lemma:

**Lemma 4.3.** *Let $\frac16 \leqslant s < \frac12$, and $\delta$, $\varepsilon$, $C_0$, $R$, $T$ be defined as above. Assume that the following estimates hold:*

$$\|v\|_{Y^s([0,T])} \leqslant \delta,\ \|w\|_{X^{\frac12}([0,T])} \lesssim R, \text{ and } \|w\|_{\widetilde{X}^{\frac12}([0,T])} \lesssim \delta.$$

*Then, we have the nonlinear estimate*

$$\Big\| \int_0^t e^{i(t-s)\Delta}(|u|^2 u)\, \mathrm{d}s \Big\|_{X^{\frac12}([0,T])} \leqslant \delta. \tag{4.3}$$

*Proof.* In the following, we shall slightly abuse notation and write $u$ for both itself and its complex conjugate, and all the space-time norms are taken over $[0,T]\times\mathbb{R}^3$ without writing its integral region. First, to prove (4.3), by Lemma 2.2, Hölder's inequality, and embedding $V_\Delta^2 \hookrightarrow L_t^\infty L_x^2$, we are reduced to consider

$$\begin{aligned}
\Big\| \int_0^t e^{i(t-s)\Delta}(|u|^2 u)\, \mathrm{d}s \Big\|_{X^{\frac12}} \lesssim& \Big( \sum_{N\in 2^{\mathbb{N}}} N \sup_{\|g\|_{V_\Delta^2}=1} \Big| \int_0^T \langle P_N(|u|^2u), g\rangle\, \mathrm{d}s \Big|^2 \Big)^{\frac12} \\
\lesssim& \Big( \sum_{N\in 2^{\mathbb{N}}} N \sup_{\|g\|_{V_\Delta^2}=1} \big\|P_N(|u|^2u)\big\|_{L_t^1L_x^2}^2 \|g\|_{L_t^\infty L_x^2}^2 \Big)^{\frac12} \\
\lesssim& \big\| N^{\frac12} P_N(|u|^2u)\big\|_{l_N^2 L_t^1 L_x^2}.
\end{aligned} \tag{4.4}$$

Noting that

$$\big|P_N(|u|^2u)\big| \lesssim \Big| \sum_{N_1: N_1 \gtrsim N} P_N\big(u_{N_1} u_{\leqslant N_1}^2\big)\Big|,$$

by Cauchy-Schwarz's inequality in $N_1$ and Lemma 2.12,

$$\begin{aligned}\big\|N^{\frac12}P_N(|u|^2u)\big\|_{l^2_N L^1_t L^2_x} \lesssim & \Big\|\Big\|\sum_{N_1:N_1\gtrsim N}\frac{N^{\frac12}}{N_1^{\frac12}}N_1^{\frac12}P_N(u_{N_1}u^2_{\leqslant N_1})\Big\|_{l^2_N}\Big\|_{L^1_tL^2_x}\\ \lesssim & \Big\|\Big\|N_1^{\frac12}P_N(u_{N_1}u^2_{\leqslant N_1})\Big\|_{l^2_N}\Big\|_{L^1_t l^2_{N_1}L^2_x}\\ \lesssim & \big\|N_1^{\frac12}u_{N_1}u^2_{\leqslant N_1}\big\|_{L^1_tL^2_x l^2_{N_1}} \lesssim I+II,\end{aligned} \tag{4.5}$$

where we denote

$$I := \big\|N_1^{\frac12}w_{N_1}u^2_{\leqslant N_1}\big\|_{L^1_tL^2_xl^2_{N_1}}, \text{ and } II := \big\|N_1^{\frac12}v_{N_1}u^2_{\leqslant N_1}\big\|_{L^1_tL^2_xl^2_{N_1}}.$$

First, we deal with the term $I$, where the $\frac12$-order derivative acts on $w$. This is the simpler case, since $w$ allows estimates with the derivative of order $\frac12$. By frequency support property,

$$I \lesssim \big\|N_1^{\frac12}w_{N_1}u_{\sim N_1}u_{\leqslant N_1}\big\|_{L^1_tL^2_xl^2_{N_1}} \tag{4.6a}$$

$$+ \big\|N_1^{\frac12}w_{N_1}u^2_{\ll N_1}\big\|_{L^1_tL^2_xl^2_{N_1}}. \tag{4.6b}$$

Now, we estimate (4.6a). By Hölder's inequality and Lemma 2.4, it holds that

$$\begin{aligned}(4.6a) \lesssim & \sum_{N_1} N_1^{\frac13}\,\|w_{N_1}\|_{L^2_tL^9_x}\,N_1^{\frac16}\,\|u_{\sim N_1}\|_{L^4_tL^{\frac92}_x}\,\|u_{\leqslant N_1}\|_{L^4_tL^6_x}\\ \lesssim & \big\|\langle\nabla\rangle^{\frac13}w_N\big\|_{l^2_NL^2_tL^9_x}\big\|\langle\nabla\rangle^{\frac16}u_{\sim N}\big\|_{l^2_NL^4_tL^{\frac92}_x}\,\|u\|_{L^4_tL^6_x}\\ \lesssim & \,\|w\|_{\widetilde{X}^{\frac12}}\,(\|w\|_{\widetilde{X}^{\frac12}}+\|v\|_{Y^s})^2.\end{aligned} \tag{4.7}$$

For (4.6b), by Hölder's inequality, Lemmas 2.11 and 2.12,

$$\begin{aligned}(4.6b) \lesssim & \big\|N_1^{\frac12}w_{N_1}\big\|_{L^2_tL^6_xl^2_{N_1}}\,\big\|\sup_{N_1}|u_{\ll N_1}|\big\|^2_{L^4_tL^6_x}\\ \lesssim & \big\|\langle\nabla\rangle^{\frac12}w\big\|_{L^2_tL^6_x}\,\|u\|^2_{L^4_tL^6_x} \lesssim \|w\|_{\widetilde{X}^{\frac12}}\,(\|w\|_{\widetilde{X}^{\frac12}}+\|v\|_{Y^s})^2.\end{aligned} \tag{4.8}$$

Then, by (4.7) and (4.8), we have

$$I \lesssim (4.6a)+(4.6b) \lesssim \|w\|_{\widetilde{X}^{\frac12}}\,(\|w\|_{\widetilde{X}^{\frac12}}+\|v\|_{Y^s})^2 \lesssim \delta^2. \tag{4.9}$$

Next, we consider the term $II$, where the $\frac12$-order derivative acts on $v$. However, the function $v$ can only have $\frac16$-order derivative. Therefore, we need to transfer the additional fractional order derivative to other functions. We make a frequency decomposition:

$$II \lesssim \big\|N_1^{\frac12}v_{N_1}u^2_{\sim N_1}\big\|_{L^1_tL^2_xl^2_{N_1}} \tag{4.10a}$$

$$+ \big\|N_1^{\frac12}v_{N_1}v_{\ll N_1}u_{\leqslant N_1}\big\|_{L^1_tL^2_xl^2_{N_1}} \tag{4.10b}$$

$$+ \big\|N_1^{\frac12}v_{N_1}w_{\ll N_1}u_{\leqslant N_1}\big\|_{L^1_tL^2_xl^2_{N_1}}. \tag{4.10c}$$

By Lemma 2.12, Hölder's inequality, and embedding $l^2_N \hookrightarrow l^3_N$,

$$\begin{aligned}(4.10a) \lesssim & \sum_{N_1 \in 2^{\mathbb{N}}} N_1^{\frac{1}{6}} \|v_{N_1}\|_{L^3_t L^6_x} \big(N_1^{\frac{1}{6}} \|u_{\sim N_1}\|_{L^3_t L^6_x}\big)^2 \\ \lesssim & \big\||\nabla|^{\frac{1}{6}} v_{N_1}\big\|_{l^2_{N_1} L^3_t L^6_x} \big\|\langle\nabla\rangle^{\frac{1}{6}} u_{\sim N_1}\big\|^2_{l^2_{N_1} L^3_t L^6_x} \\ \lesssim & \|v\|_{Y^s} (\|w\|_{\widetilde{X}^{\frac{1}{2}}} + \|v\|_{Y^s})^2 \lesssim \delta^3.\end{aligned} \tag{4.11}$$

Next, we consider (4.10b) and (4.10c). The proof is more difficult, where we use the bilinear Strichartz estimate to transfer derivative. However, this approach will create the term $\|v\|_{Z^s}$, which cannot get smallness by letting the interval small. Therefore, we also need some $T$ to control the $Z^s$-norm.

Now, we consider the term (4.10b). By Hölder's inequality,

$$\begin{aligned}(4.10b) \lesssim & \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \|v_{N_1} v_{N_2} u_{\leqslant N_1}\|_{L^1_t L^2_x} \\ \lesssim & \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \|v_{N_1} v_{N_2}\|^{\frac{2}{3}+\varepsilon}_{L_t^{\frac{1}{1-\varepsilon}} L^2_x} \|v_{N_1}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|v_{N_2}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|u_{\leqslant N_1}\|_{L_t^{\frac{3}{1-\varepsilon+3\varepsilon^2}} L_x^{\frac{6}{1-3\varepsilon}}}.\end{aligned} \tag{4.12}$$

By Lemma 2.9, for $N_2 \ll N_1$,

$$\|v_{N_1} v_{N_2}\|_{L_t^{\frac{1}{1-\varepsilon}} L^2_x} \lesssim N_2^{2\varepsilon} N_1^{-\frac{1}{2}} \|P_{N_1} v_0\|_{L^2_x} \|P_{N_2} v_0\|_{L^2_x} \lesssim N_1^{-\frac{2}{3}} \|v\|^2_{Z^s}. \tag{4.13}$$

Note that

$$\|u\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1-3\varepsilon}}} \lesssim \|w\|_{\widetilde{X}^{\frac{1}{2}}} + \|v\|_{Y^s} \lesssim \delta. \tag{4.14}$$

By (4.12), (4.13), (4.14), and Hölder's inequality,

$$\begin{aligned}(4.10b) \lesssim & \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \|v_{N_1} v_{N_2}\|^{\frac{2}{3}+\varepsilon}_{L_t^{\frac{1}{1-\varepsilon}} L^2_x} \|v_{N_1}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|v_{N_2}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|u_{\leqslant N_1}\|_{L_t^{\frac{3}{1-\varepsilon+3\varepsilon^2}} L_x^{\frac{6}{1-3\varepsilon}}} \\ \lesssim & T^{\varepsilon^2} \sum_{N_2 \ll N_1} N_1^{\frac{1}{18}-\frac{2}{3}\varepsilon} \|v\|^{\frac{4}{3}+2\varepsilon}_{Z^s} N_1^{-\frac{1}{18}+\frac{1}{2}\varepsilon-\varepsilon^2} \big\||\nabla|^{\frac{1}{6}-\varepsilon} v_{N_1}\big\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|v_{N_2}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|u\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1-3\varepsilon}}} \\ \lesssim & \delta T^{\varepsilon^2} \|v\|^{\frac{5}{3}+\varepsilon}_{Z^s} \sum_{N_2 \ll N_1} N_1^{-\frac{1}{6}\varepsilon-\varepsilon^2} \|v_{N_2}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \lesssim \delta T^{\varepsilon^2} \|v\|^2_{Z^s}.\end{aligned} \tag{4.15}$$

Finally, we consider the term (4.10c). By Hölder's inequality,

$$\begin{aligned}(4.10c) \lesssim & \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \|v_{N_1} w_{N_2} u_{\leqslant N_1}\|_{L^1_t L^2_x} \\ \lesssim & \sum_{N_2 \ll N_1} N_1^{\frac{1}{2}} \|v_{N_1} w_{N_2}\|^{\frac{2}{3}+\varepsilon}_{L_t^{\frac{4}{3}} L^2_x} \|v_{N_1}\|^{\frac{1}{3}-\varepsilon}_{L^\infty_{t,x}} \|w_{N_2}\|^{\frac{1}{3}-\varepsilon}_{L^{q_1}_t L^{r_1}_x} \|u_{\leqslant N_1}\|_{L^{q_1}_t L^{r_1}_x},\end{aligned} \tag{4.16}$$

where $(q_1, r_1)$ is defined by

$$\frac{2-3\varepsilon}{4} = (\frac{4}{3}-\varepsilon)\frac{1}{q_1}, \text{ and } \frac{1-3\varepsilon}{6} = (\frac{4}{3}-\varepsilon)\frac{1}{r_1}.$$

Noting that $q_1=\frac{4(4-3\varepsilon)}{3(2-3\varepsilon)}=\frac{8}{3}+$ and $r_1=\frac{2(4-3\varepsilon)}{1-3\varepsilon}=8+$, we have $\frac{3}{2}-\frac{2}{q_1}-\frac{3}{r_1}=\frac{3}{8}+$, and there exists $q_2=\frac{4(4-3\varepsilon)}{5+3\varepsilon}$ such that

$$\frac{2}{q_2}+\frac{3}{r_1}=1.$$

Then, we have

$$\|u_{\leqslant N_1}\|_{L_t^{q_2}L_x^{r_1}}\lesssim\|w\|_{L_t^{q_2}L_x^{r_1}}+\|v\|_{L_t^{q_2}L_x^{r_1}}\lesssim\delta. \tag{4.17}$$

By Lemma 2.9, for $N_2\ll N_1$,

$$\begin{aligned}\|v_{N_1}w_{N_2}\|_{L_t^{\frac{4}{3}}L_x^2}&\lesssim N_2^{\frac{1}{2}}N_1^{-\frac{1}{2}}\|P_{N_1}v_0\|_{L_x^2}\|w_{N_2}\|_{U_\Delta^2}\\&\lesssim N_1^{-\frac{2}{3}}\|v\|_{Z^s}\|w\|_{X^{\frac{1}{2}}}\lesssim N_1^{-\frac{2}{3}}R\|v\|_{Z^s}.\end{aligned} \tag{4.18}$$

We remark that for (4.18), if we do not invoke the $U^p$-$V^p$ method, by Lemma 2.9, it reduces to deal with the term

$$N_1^{\frac{1}{2}}\big(\|P_{N_1}w_0\|_{L_x^2}+\big\|P_{N_1}(|u|^2u)\big\|_{L_t^1L_x^2}\big).$$

Thus, the argument would be more complex, especially when $N_1^{1/2}$ acts on $|v|^2v$.

By (4.16), (4.17), (4.18), and Hölder's inequality in $t$,

$$\begin{aligned}\text{(4.10c)}\lesssim&\sum_{N_2\ll N_1}N_1^{\frac{1}{2}}\|v_{N_1}w_{N_2}\|_{L_t^{\frac{4}{3}}L_x^2}^{\frac{2}{3}+\varepsilon}\|v_{N_1}\|_{L_{t,x}^\infty}^{\frac{1}{3}-\varepsilon}\|w_{N_2}\|_{L_t^{q_1}L_x^{r_1}}^{\frac{1}{3}-\varepsilon}\|u_{\leqslant N_1}\|_{L_t^{q_1}L_x^{r_1}}\\ \lesssim&T^{(\frac{1}{3}-\varepsilon)(\frac{1}{q_1}-\frac{1}{q_2})}\sum_{N_2\ll N_1}N_1^{\frac{1}{18}-\frac{2}{3}\varepsilon}R^{\frac{2}{3}+\varepsilon}\|v\|_{Z^s}^{\frac{2}{3}+\varepsilon}\|v_{N_1}\|_{L_{t,x}^\infty}^{\frac{1}{3}-\varepsilon}\|w\|_{L_t^{q_2}L_x^{r_1}}^{\frac{1}{3}-\varepsilon}\|u\|_{L_t^{q_2}L_x^{r_1}}\\ \lesssim&T^{\frac{1}{100}}\delta^{\frac{4}{3}-\varepsilon}R^{\frac{2}{3}+\varepsilon}\|v\|_{Z^s}^{\frac{2}{3}+\varepsilon}\sum_{N_2\ll N_1}N_1^{-\frac{1}{6}\varepsilon-\varepsilon^2}\big\||\nabla|^{\frac{1}{6}-\varepsilon}v\big\|_{L_{t,x}^\infty}^{\frac{1}{3}-\varepsilon}\lesssim T^{\frac{1}{100}}R^2\|v\|_{Z^s}.\end{aligned} \tag{4.19}$$

Therefore, by (4.11), (4.15), and (4.19),

$$II\lesssim\text{(4.10a)}+\text{(4.10b)}+\text{(4.10c)}\lesssim\delta^3+\delta T^{\varepsilon^2}\|v\|_{Z^s}^2+T^{\frac{1}{100}}R^2\|v\|_{Z^s}. \tag{4.20}$$

Then, combining (4.5), (4.9), and (4.20), we have

$$\begin{aligned}\Big\|\int_0^t e^{i(t-s)\Delta}(|u|^2u)\,\mathrm{d}s\Big\|_{L_t^\infty H_x^{\frac{1}{2}}\cap X^{\frac{1}{2}}}\leqslant&I+II\\ \leqslant&C\left(\delta^3+\delta T^{\varepsilon^2}\|v\|_{Z^s}^2+T^{\frac{1}{100}}R^2\|v\|_{Z^s}\right)\\ \leqslant&C\delta^3\leqslant\delta.\end{aligned}$$

This completes the proof of the lemma. □

4.3. **Improved local theory.** In this section, we prove Proposition 4.2. Recall that

$$\|w\|_{X^1(I)}=\big(\sum_{N\in2^{\mathbb{N}}}N^2\|w_N\|_{U_\Delta^2(I;L_x^2)}^2\big)^{\frac{1}{2}}.$$

We write $U_\Delta^2=U_\Delta^2(I;L_x^2)$ and $V_\Delta^2=V_\Delta^2(I;L_x^2)$ for short. We make the choices of some parameters:

(1) Let $C_0 > 0$ be the constant such that

$$\left\| e^{it\Delta} w_0 \right\|_{X^1(\mathbb{R})} \leqslant C_0 \left\| w_0 \right\|_{H^1_x}.$$

(2) Let $0 < \varepsilon < \frac{1}{100}(s - \frac{1}{3})$ and

$$R := \max\left\{ C_0 \left\| w_0 \right\|_{H^1_x}, 1 \right\}.$$

(3) Let $T > 0$ satisfy the smallness conditions:

$$\begin{aligned} &\| v \|_{Y^s([0,T])} \leqslant R, \\ &CT^{\frac{1}{2}} R^2 \leqslant \frac{1}{10}, \; CT^{\frac{1}{36}} R^2 \leqslant \frac{1}{10}, \; CT^{\frac{\varepsilon}{100}} R^2 \leqslant \frac{1}{10}, \\ &CT^{\frac{1}{2}} \| v \|^2_{Z^s(\mathbb{R})} \leqslant \frac{1}{2}, \; CT^{\frac{1}{36}} \| v \|^3_{Z^s(\mathbb{R})} \leqslant \frac{1}{10} R, \text{ and } CT^{\frac{\varepsilon}{100}} \| v \|^3_{Z^s(\mathbb{R})} \leqslant \frac{1}{10} R. \end{aligned} \tag{4.21}$$

Note that $T$ depends on $s$, $\| w_0 \|_{H^1_x}$, $\| v \|_{Y^s(\mathbb{R})}$, and $\| v \|_{Z^s(\mathbb{R})}$. Let the working space be defined by

$$B_{R,T} := \left\{ w \in C([0,T]; H^1_x) : \| w \|_{X^1([0,T])} \leqslant 2R \right\}.$$

Define

$$\Phi_{w_0,v}(w) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, \mathrm{d}s.$$

Similar as in the former section, in order to get that $\Phi_{w_0,v}$ is a contraction mapping on $B_{R,T}$, it suffices to prove:

**Lemma 4.4.** *Let $\frac{1}{3} < s \leqslant \frac{1}{2}$, $0 < \varepsilon < \frac{1}{100}(s - \frac{1}{3})$, and $C_0, R$ be defined as above. Assume that $0 < T < 1$ satisfies the smallness condition* (4.21)*, and let*

$$\| w \|_{X^1([0,T])} \lesssim R.$$

*Then, we have the nonlinear estimate:*

$$\Big\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, \mathrm{d}s \Big\|_{X^1([0,T])} \leqslant R. \tag{4.22}$$

*Proof.* Again, we do not distinguish $u$ and $\overline{u}$, and all the space-time norms are restricted on $[0,T] \times \mathbb{R}^3$. By Lemma 2.2 and frequency decomposition, we have

$$\begin{aligned} &\Big\| \int_0^t e^{i(t-s)\Delta} (|u|^2 u) \, \mathrm{d}s \Big\|_{X^1} \\ \lesssim &\Big( \sum_{N \in 2^{\mathbb{N}}} N^2 \sup_{\| g \|_{V^2_\Delta} = 1} \Big| \int_0^T \langle P_N(|u|^2 u), g \rangle \, \mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}} \\ \lesssim &\Big( \sum_{N \in 2^{\mathbb{N}}} N^2 \sup_{\| g \|_{V^2_\Delta} = 1} \Big| \int_0^T \int P_N\big( \sum_{N_1} u_{N_1} u^2_{\leqslant N_1} \big) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}} \lesssim I + II \end{aligned}$$

where

$$I := \Big( \sum_{N \in 2^{\mathbb{N}}} N^2 \sup_{\| g \|_{V^2_\Delta} = 1} \Big| \int_0^T \int P_N\big( \sum_{N_1} w_{N_1} u^2_{\leqslant N_1} \big) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}},$$

and

$$II := \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N(\sum_{N_1} v_{N_1} u^2_{\leqslant N_1}) g \,\mathrm{d}x\,\mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}}.$$

We first consider the term $I$, where the first order derivative acts on $w$. By Hölder's inequality and embedding $V^2_\Delta \hookrightarrow L^\infty_t L^2_x$, we have

$$\begin{aligned} I \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \big\| P_N( \sum_{N_1:N\lesssim N_1} w_{N_1} u^2_{\leqslant N_1}) \big\|^2_{L^1_t L^2_x} \|g\|^2_{L^\infty_t L^2_x} \Big)^{\frac{1}{2}} \\ \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \big\| P_N( \sum_{N_1:N\lesssim N_1} w_{N_1} u^2_{\leqslant N_1}) \big\|^2_{L^1_t L^2_x} \|g\|^2_{V^2_\Delta} \Big)^{\frac{1}{2}} \\ \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \big\| P_N( \sum_{N_1:N\lesssim N_1} w_{N_1} u^2_{\leqslant N_1}) \big\|^2_{L^1_t L^2_x} \Big)^{\frac{1}{2}}. \end{aligned} \tag{4.23}$$

Then, by (4.23), Minkowski's inequality, Hölder's inequality in $N_1$, and Lemma 2.12,

$$\begin{aligned} I \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \big\| P_N( \sum_{N_1:N\lesssim N_1} w_{N_1} u^2_{\leqslant N_1}) \big\|^2_{L^1_t L^2_x} \Big)^{\frac{1}{2}} \\ \lesssim & \Big\| \sum_{N_1:N\lesssim N_1} \frac{N}{N_1} N_1 P_N(w_{N_1} u^2_{\leqslant N_1}) \Big\|_{L^1_t L^2_x l^2_N} \\ \lesssim & \big\| N_1 P_N(w_{N_1} u^2_{\leqslant N_1}) \big\|_{L^1_t L^2_x l^2_N l^2_{N_1}} \lesssim \big\| N_1 w_{N_1} u^2_{\leqslant N_1} \big\|_{L^1_t l^2_{N_1} L^2_x}. \end{aligned} \tag{4.24}$$

By (4.24), Hölder's inequality, and Lemma 2.11,

$$\begin{aligned} I \lesssim & \big\| N_1 w_{N_1} u^2_{\leqslant N_1} \big\|_{L^1_t l^2_{N_1} L^2_x} \\ \lesssim & \Big\| \, \|N_1 w_{N_1}\|_{L^6_x} \|u_{\leqslant N_1}\|^2_{L^6_x} \Big\|_{L^1_t l^2_{N_1}} \\ \lesssim & \, \|\langle\nabla\rangle w\|_{L^2_t L^6_x} \|u\|^2_{L^4_t L^6_x} \\ \lesssim & T^{\frac{1}{2}} \|\langle\nabla\rangle w\|_{L^2_t L^6_x} \|u\|^2_{L^\infty_t L^6_x} \\ \lesssim & T^{\frac{1}{2}} \|w\|_{X^1} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^2. \end{aligned} \tag{4.25}$$

Therefore, by (4.25) and the choice of $T$, we have

$$I \leqslant CT^{\frac{1}{2}} \|w\|_{X^1} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^2 \leqslant CT^{\frac{1}{2}} R (\|v\|^2_{Z^s} + R^2) \leqslant \frac{1}{2} R. \tag{4.26}$$

We next consider the term $II$, where the first order derivative acts fully on $v$. However, $v$ can only have estimates with the derivative of order $\frac{1}{3}$, thus there is a gap of $\frac{2}{3}$-order derivative. Note that the bilinear Strichartz estimate can only lower down $\frac{1}{2}$-order derivative. Therefore, this is the main case where we need to exploit the duality structure. To this end, we make a frequency decomposition:

$$
\begin{aligned}
II \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} v_{N_1} u^2_{\sim N_1}\big) g \,\mathrm{d}x\,\mathrm{d}s \Big|^2 \Big)^{\frac12} && \text{(4.27a)}\\
& + \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} v_{N_1} u_{\ll N_1} u_{\sim N_1}\big) g \,\mathrm{d}x\,\mathrm{d}s \Big|^2 \Big)^{\frac12} && \text{(4.27b)}\\
& + \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} v_{N_1} u^2_{\ll N_1}\big) g \,\mathrm{d}x\,\mathrm{d}s \Big|^2 \Big)^{\frac12}. && \text{(4.27c)}
\end{aligned}
$$

We first estimate the (4.27a). Using the same method as in (4.23) and (4.24),

$$
\begin{aligned}
\text{(4.27a)} \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} v_{N_1} u^2_{\sim N_1}\big) g \,\mathrm{d}x\,\mathrm{d}s \Big|^2 \Big)^{\frac12}\\
\lesssim & \sum_{N_1\in 2^{\mathbb{N}}} \big\| N_1 v_{N_1} u^2_{\sim N_1} \big\|_{L^1_t L^2_x}.
\end{aligned} \tag{4.28}
$$

Then, by (4.28), Lemma 2.12, Hölder's inequality in $N_1$, and $l^2_{N_1} \hookrightarrow l^3_{N_1}$, we have

$$
\begin{aligned}
\text{(4.27a)} \lesssim & \sum_{N_1\in 2^{\mathbb{N}}} \big\| N_1 v_{N_1} u^2_{\sim N_1} \big\|_{L^1_t L^2_x}\\
\lesssim & T^{\frac12} \sum_{N_1\in 2^{\mathbb{N}}} \big\| |\nabla|^{\frac13} v_{N_1} \big\|_{L^6_{t,x}} \big\| |\nabla|^{\frac13} u_{\sim N_1} \big\|_{L^6_{t,x}} \big\| |\nabla|^{\frac13} u_{\sim N_1} \big\|_{L^6_{t,x}}\\
\lesssim & T^{\frac12} \big\| |\nabla|^{\frac13} v_N \big\|_{l^2_N L^6_{t,x}} \big\| |\nabla|^{\frac13} u_{N_1} \big\|^2_{l^2_{N_1} L^6_{t,x}}\\
\lesssim & T^{\frac12} \|v\|_{Y^s} \big( \|v\|_{Y^s} + \|w\|_{X^1} \big)^2 \lesssim T^{\frac12} R^3.
\end{aligned} \tag{4.29}
$$

Next, we consider (4.27b). Noting that $(9, \frac{54}{23})$ is $L^2_x$-admissible, similar as above, by Hölder's inequality and Lemma 2.7,

$$
\begin{aligned}
\text{(4.27b)} \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} v_{N_1} u_{\ll N_1} u_{\sim N_1}\big) g \,\mathrm{d}x\,\mathrm{d}s \Big|^2 \Big)^{\frac12}\\
\lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^2 \big\| P_N\big( \sum_{N_1: N\lesssim N_1} v_{N_1} u_{\ll N_1} u_{\sim N_1}\big) \big\|^2_{L_t^{\frac98} L_x^{\frac{54}{31}}} \Big)^{\frac12}\\
\lesssim & \sum_{N_1} N_1 \big\| v_{N_1} u_{\ll N_1} u_{\sim N_1} \big\|_{L_t^{\frac98} L_x^{\frac{54}{31}}}.
\end{aligned} \tag{4.30}
$$

Note that by the bilinear Strichartz estimate in Lemma 2.9, for $N_2 \ll N_1$,

$$
\begin{aligned}
\|\nabla v_{N_1} u_{N_2}\|_{L_t^{\frac32} L^2_x} \lesssim & N_2^{\frac13} N_1^{\frac12} \|P_{N_1} v_0\|_{L^2_x} \big( \|P_{N_2} v_0\|_{L^2_x} + \|P_{N_2} w\|_{U^2_\Delta} \big)\\
\lesssim & N_1^{\frac16} \|v\|_{Z^s} \big( \big\| |\nabla|^{\frac13} P_{N_2} v_0 \big\|_{L^2_x} + N_2^{\frac13} \|P_{N_2} w\|_{U^2_\Delta} \big)\\
\lesssim & N_1^{\frac16} \|v\|_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big).
\end{aligned} \tag{4.31}
$$

By Hölder's inequality,

$$
\|u_{N_2}\|_{L^3_t L^{18}_x} \lesssim T^{\frac{1}{12}} \|u_{N_2}\|_{L^4_t L^{18}_x} \lesssim T^{\frac{1}{12}} N_2^{-\frac16} \big( \|v\|_{Y^s} + \|w\|_{X^1} \big). \tag{4.32}
$$

Combining (4.30), (4.31), (4.32), and Hölder's inequality,

$$
\begin{aligned}
(4.27\text{b}) \lesssim & \sum_{N_1} N_1 \big\| v_{N_1} u_{\ll N_1} u_{\sim N_1} \big\|_{L_t^{\frac{9}{8}} L_x^{\frac{54}{31}}} \\
\lesssim & \sum_{N_2 \ll N_1} \big\| \nabla v_{N_1} u_{N_2} \big\|^{\frac{2}{3}}_{L_t^{\frac{3}{2}} L_x^2} \big\| \nabla v_{N_1} \big\|^{\frac{1}{3}}_{L_t^2 L_x^\infty} \| u_{N_2} \|^{\frac{1}{3}}_{L_t^3 L_x^{18}} \| u_{\sim N_1} \|_{L_t^6 L_x^{\frac{9}{2}}} \\
\lesssim & \sum_{N_2 \ll N_1} N_1^{\frac{1}{9}} N_1^{\frac{2}{9}+} \big\| |\nabla|^{\frac{1}{3}-} v_{N_1} \big\|^{\frac{1}{3}}_{L_t^2 L_x^\infty} \| u_{N_2} \|^{\frac{1}{3}}_{L_t^3 L_x^{18}} \| u_{\sim N_1} \|_{L_t^6 L_x^{\frac{9}{2}}} \\
& \cdot \|v\|^{\frac{2}{3}}_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^{\frac{2}{3}} \qquad (4.33) \\
\lesssim & \sum_{N_2 \ll N_1} N_2^{-\frac{1}{18}} N_1^{-\frac{1}{6}+} \|v\|^{\frac{1}{3}}_{Y^s} T^{\frac{1}{36}} \big( \|v\|_{Y^s} + \|w\|_{X^1} \big)^{\frac{1}{3}} \big\| |\nabla|^{\frac{1}{2}} u_{N_1} \big\|_{L_t^6 L_x^{\frac{9}{2}}} \\
& \cdot \|v\|^{\frac{2}{3}}_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^{\frac{2}{3}} \\
\lesssim & T^{\frac{1}{36}} \|v\|^{\frac{1}{3}}_{Y^s} \|v\|^{\frac{2}{3}}_{Z^s} \big( \|v\|_{Y^s} + \|w\|_{X^1} \big)^{\frac{4}{3}} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^{\frac{2}{3}} \\
\lesssim & T^{\frac{1}{36}} R^{\frac{1}{3}} \|v\|^{\frac{2}{3}}_{Z^s} R^{\frac{4}{3}} \big( \|v\|_{Z^s} + R \big)^{\frac{2}{3}} \lesssim T^{\frac{1}{36}} \big( \|v\|^3_{Z^s} + R^3 \big).
\end{aligned}
$$

Finally, we consider the main term (4.27c). By frequency support property,

$$
\begin{aligned}
(4.27\text{c}) = & \Big( \sum_{N \in 2^{\mathbb{N}}} N^2 \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N \big( \sum_{N_1} v_{N_1} u^2_{\ll N_1} \big) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}} \\
\lesssim & \sum_{N \in 2^{\mathbb{N}}} N \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int \sum_{N_1} v_{N_1} u^2_{\ll N_1} g_N \, \mathrm{d}x \, \mathrm{d}s \Big| \\
\lesssim & \sum_{N \in 2^{\mathbb{N}}} N \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int v_N u^2_{\ll N} g_N \, \mathrm{d}x \, \mathrm{d}s \Big| \qquad (4.34) \\
\lesssim & \sum_{N_1 \leqslant N_2 \ll N} N \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int v_N g_N u_{N_1} u_{N_2} \, \mathrm{d}x \, \mathrm{d}s \Big|.
\end{aligned}
$$

To estimate this, we need to use the bilinear Strichartz estimate for both $g_N u_{N_1}$ and $\nabla v_N u_{N_2}$. Now, we give the estimate for $g_N u_{N_1}$, where we also need to pass $g_N$ into $V^2_\Delta$ by interpolation. By Lemma 2.9, for $N_1 \ll N$,

$$
\| g_N u_{N_1} \|_{L_t^{\frac{3}{2+\varepsilon}} L_x^2} \lesssim \frac{N_1^{\frac{2}{3}-\frac{2}{3}\varepsilon}}{N^{\frac{1}{2}}} \| g_N \|_{U^2_\Delta} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big), \qquad (4.35)
$$

and by Hölder's inequality and Lemma 2.7,

$$
\begin{aligned}
\| g_N u_{N_1} \|_{L_t^{\frac{3}{2+\varepsilon}} L_x^2} \lesssim & \| g_N \|_{L_t^{\frac{8}{3}} L_x^4} \| u_{N_1} \|_{L_t^{\frac{24}{7+8\varepsilon}} L_x^4} \\
\lesssim & N_1^{\frac{1}{6}-\frac{2}{3}\varepsilon} \| g_N \|_{U^{\frac{8}{3}}_\Delta} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big). \qquad (4.36)
\end{aligned}
$$

Then, noting that $\varepsilon < \frac{1}{100}(s-\frac{1}{3})$, by (4.35), (4.36), and Lemma 2.3, for $N_1 \ll N$,

$$\begin{aligned} \|g_N u_{N_1}\|_{L_t^{\frac{3}{2+\varepsilon}} L_x^2} \lesssim & \frac{N^\varepsilon}{N_1^\varepsilon} \frac{N_1^{\frac{2}{3}-\frac{2}{3}\varepsilon}}{N^{\frac{1}{2}}} \|g_N\|_{V_\Delta^2} \big( \|P_{N_1} v_0\|_{L_x^2} + \|w_{N_1}\|_{U_\Delta^2} \big) \\ \lesssim & \frac{N_1^{\frac{1}{3}-\frac{5}{3}\varepsilon}}{N^{\frac{1}{2}-\varepsilon}} N_1^{-100\varepsilon} \big( N_1^{\frac{1}{3}+100\varepsilon} \|P_{N_1} v_0\|_{L_x^2} + N_1^{\frac{1}{3}+100\varepsilon} \|w_{N_1}\|_{U_\Delta^2} \big) \\ \lesssim & \frac{N_1^{\frac{1}{3}-100\varepsilon}}{N^{\frac{1}{2}-\varepsilon}} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big). \end{aligned} \tag{4.37}$$

Next, we give the estimate for $\nabla v_N u_{N_2}$. Using Lemma 2.9 again, for $N_2 \ll N$, we also have

$$\begin{aligned} \|\nabla v_N u_{N_2}\|_{L_t^{\frac{1}{1-10\varepsilon}} L_x^2} \lesssim & N_2^{20\varepsilon} N^{\frac{1}{2}} \|P_N v_0\|_{L_x^2} \big( \|P_{N_2} v_0\|_{L_x^2} + \|w_{N_2}\|_{U_\Delta^2} \big) \\ \lesssim & N_2^{-\frac{1}{3}} N^{\frac{1}{6}-100\varepsilon} \|v\|_{Z^s} \big( N_2^{\frac{1}{3}+20\varepsilon} \|P_{N_2} v_0\|_{L_x^2} + N_2^{\frac{1}{3}+20\varepsilon} \|w_{N_2}\|_{U_\Delta^2} \big) \\ \lesssim & N_2^{-\frac{1}{3}} N^{\frac{1}{6}-100\varepsilon} \|v\|_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big). \end{aligned} \tag{4.38}$$

Then, we are ready to bound (4.27c). By (4.37), (4.38), and Hölder's inequality,

$$\begin{aligned} & N \int_0^T \int g_N v_N u_{N_1} u_{N_2} \,\mathrm{d}x \,\mathrm{d}s \\ \lesssim & \|g_N u_{N_1}\|_{L_t^{\frac{3}{2+\varepsilon}} L_x^2} \|\nabla v_N u_{N_2}\|^{\frac{1}{3}}_{L_t^{\frac{1}{1-10\varepsilon}} L_x^2} \|\nabla v_N\|^{\frac{2}{3}}_{L_t^{\frac{2}{9\varepsilon}} L_x^\infty} \|u_{N_2}\|^{\frac{2}{3}}_{L_t^\infty L_x^2} \\ \lesssim & \frac{N_1^{\frac{1}{3}-100\varepsilon}}{N^{\frac{1}{2}-\varepsilon}} \big( N_2^{-\frac{1}{3}} N^{\frac{1}{6}-100\varepsilon} \big)^{\frac{1}{3}} N^{\frac{2}{3}(\frac{2}{3}+\varepsilon)} \big\| |\nabla|^{\frac{1}{3}-\varepsilon} v_N \big\|^{\frac{2}{3}}_{L_t^{\frac{1}{\varepsilon}} L_x^\infty} \\ & \cdot N_2^{-\frac{2}{9}} \big\| |\nabla|^{\frac{1}{3}} u_{N_2} \big\|^{\frac{2}{3}}_{L_t^\infty L_x^2} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big) \|v\|^{\frac{1}{3}}_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^{\frac{1}{3}} \\ \lesssim & T^{\frac{\varepsilon}{100}} N_1^{\frac{1}{3}-100\varepsilon} N_2^{-\frac{1}{3}} N^{-20\varepsilon} \|v\|^{\frac{2}{3}}_{Y^s} \|v\|^{\frac{1}{3}}_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^2. \end{aligned} \tag{4.39}$$

Note that

$$\sum_{N_1 \leqslant N_2 \ll N} N_1^{\frac{1}{3}-100\varepsilon} N_2^{-\frac{1}{3}} N^{-20\varepsilon} \lesssim 1,$$

then by (4.34) and (4.39),

$$\begin{aligned} (4.27c) \lesssim & \sum_{N_1 \leqslant N_2 \ll N} N \sup_{\|g\|_{V_\Delta^2}=1} \Big| \int_0^T \int v_N g_N u_{N_1} u_{N_2} \,\mathrm{d}x \,\mathrm{d}s \Big| \\ \lesssim & T^{\frac{\varepsilon}{100}} \sum_{N_1 \leqslant N_2 \ll N} N_1^{\frac{1}{3}-100\varepsilon} N_2^{-\frac{1}{3}} N^{-20\varepsilon} \|v\|^{\frac{2}{3}}_{Y^s} \|v\|^{\frac{1}{3}}_{Z^s} \big( \|v\|_{Z^s} + \|w\|_{X^1} \big)^2 \\ \lesssim & T^{\frac{\varepsilon}{100}} R^{\frac{2}{3}} \|v\|^{\frac{1}{3}}_{Z^s} \big( \|v\|_{Z^s} + R \big)^2 \lesssim T^{\frac{\varepsilon}{100}} \big( \|v\|^3_{Z^s} + R^3 \big). \end{aligned} \tag{4.40}$$

Therefore, by (4.29), (4.33), and (4.40),

$$\begin{aligned} II \leqslant & (4.27a) + (4.27b) + (4.27c) \\ \leqslant & CT^{\frac{1}{2}} R^3 + CT^{\frac{1}{36}} \big( \|v\|^3_{Z^s} + R^3 \big) + CT^{\frac{\varepsilon}{100}} \big( \|v\|^3_{Z^s} + R^3 \big) \leqslant \frac{1}{2} R. \end{aligned} \tag{4.41}$$

Then, by (4.26) and (4.41), we have

$$\Big\| \int_0^t e^{i(t-s)\Delta}(|u|^2 u)\, \mathrm{d}s \Big\|_{X^1} \leqslant I + II \leqslant R.$$

This proves (4.22). □

## 5. Global well-posedness and scattering

5.1. **Reduction to the deterministic problem.** Let $\widetilde{Y}^s(I)$ be defined by its norm

$$\|v\|_{\widetilde{Y}^s(I)} := \|v\|_{Y^s(I)} + \big\| |\nabla|^{s+\frac12-} v_N \big\|_{l^2_N L^2_t L^\infty_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}. \tag{5.1}$$

Recall that

$$\|v\|_{Z^s(I)} = \big\| \langle\nabla\rangle^{s-} P_N v \big\|_{l^2_N L^\infty_t L^\infty_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)} + \|\langle\nabla\rangle^s P_N v\|_{l^2_N L^\infty_t L^2_x(2^{\mathbb{N}}\times I\times\mathbb{R}^3)}.$$

**Proposition 5.1.** *Let $\frac37 < s \leqslant \frac12$ and $A > 0$. Then, there exists $N_0 = N_0(A) \gg 1$ such that the following properties hold. Let $u_0 \in H^s_x$, $v_0$ satisfy that supp $\widehat{v}_0 \subset \big\{\xi \in \mathbb{R}^3 : |\xi| \geqslant \frac12 N_0\big\}$, and $w_0 = u_0 - v_0$. Moreover, let $v = e^{it\Delta}v_0$ and $w = u - v$. Suppose that $v \in \widetilde{Y}^s \cap Z^s(\mathbb{R})$, $w_0 \in H^1$ such that*

$$\|u_0\|_{H^s_x} + \|v\|_{\widetilde{Y}^s\cap Z^s(\mathbb{R})} \leqslant A, \text{ and } E(w_0) \leqslant A N_0^{2(1-s)}.$$

*Then, there exists a solution $u$ of (1.7) on $\mathbb{R}$ with $w \in C(\mathbb{R}; H^1_x)$. Furthermore, there exists $u_\pm \in H^1_x$ such that*

$$\lim_{t\to\pm\infty} \big\| u(t) - v(t) - e^{it\Delta}u_\pm \big\|_{H^1_x} = 0.$$

We will give the proof of Proposition 5.1 in Sections 5.2, 5.3, and 5.4. Now we prove Theorem 1.4 assuming that Proposition 5.1 holds.

*Proof of Theorem 1.4.* Let $N_0 \in 2^{\mathbb{N}}$ to be defined later, and make a high-low frequency decomposition for the initial data

$$u(t) = e^{it\Delta}P_{\geqslant N_0} f^\omega + w(t),$$

then $w$ satisfies the equation (4.1) with

$$u_0 = f^\omega,\ v_0 = P_{\geqslant N_0} f^\omega,\ w_0 = P_{\leqslant N_0} f^\omega, \text{ and } v = e^{it\Delta}P_{\geqslant N_0} f^\omega.$$

Since $f$ is radial, by Corollary 3.3, boundedness of the operator $P_{\geqslant N_0}$, Proposition 3.4, and Lemma 2.14, we have

$$\mathbb{P}\big(\big\{\omega \in \Omega : \|u_0\|_{H^s_x} + \|v\|_{\widetilde{Y}^s\cap Z^s(\mathbb{R})} > \lambda\big\}\big) \lesssim e^{-C\lambda^2 \|f\|_{H^s_x}^{-2}}. \tag{5.2}$$

For any $p \geqslant 2$, we have

$$\begin{aligned}
\|w_0\|_{L^p_\omega \dot H^1_x} \lesssim& \Big\| \sum_{k\in\mathbb{Z}^3} g_k(\omega)\nabla P_{\leqslant N_0} f_k \Big\|_{L^2_x L^p_\omega} \\
\lesssim& \sqrt{p}\, \|\nabla P_{\leqslant N_0} f_k\|_{L^2_x l^2_{k\in\mathbb{Z}^3}} \\
\lesssim& \sqrt{p}\, \|\nabla P_{\leqslant N_0} f\|_{L^2_x} \lesssim \sqrt{p} N_0^{1-s} \|f\|_{H^s_x}.
\end{aligned}$$

For any $p \geqslant 4$, we also have

$$\|w_0\|_{L^p_\omega L^4_x} \lesssim \sqrt{p}\,\|\Box_k P_{\leqslant N_0} f\|_{L^4_x l^2_k} \lesssim \sqrt{p}\,\|f\|_{L^2_x}.$$

Note that $N_0$ only depends on $M$ and $\|f\|_{H^s_x}$. Then, by Lemma 2.14,

$$\mathbb{P}\big(\{\omega \in \Omega : \frac{1}{N_0^{1-s}}\|w_0\|_{\dot H^1_x} + \|w_0\|_{L^4_x} \geqslant \lambda\}\big) \lesssim e^{-C\lambda^2\|f\|^{-2}_{H^s_x}}. \tag{5.3}$$

For any $M \geqslant 1$, let $\widetilde\Omega_M$ be defined by

$$\begin{aligned}\widetilde\Omega_M = \Big\{\omega \in \Omega : \|u_0\|_{H^s_x} + \|v\|_{\widetilde Y^s \cap Z^s(\mathbb{R})} < M\|f\|_{H^s_x},\\ \frac{1}{N_0^{1-s}}\|w_0\|_{\dot H^1_x} + \|w_0\|_{L^4_x} < M\|f\|_{H^s_x}\Big\}.\end{aligned} \tag{5.4}$$

Therefore, by (5.2) and (5.3), we have

$$\mathbb{P}(\widetilde\Omega^c_M) \lesssim e^{-CM^2}. \tag{5.5}$$

For any $\omega \in \widetilde\Omega_M$, we have $\|v\|_{\widetilde Y^s\cap Z^s(\mathbb{R})} < M\|f\|_{H^s_x}$, and

$$E(w_0) \leqslant CM^2 N_0^{2(1-s)} \cdot \max\Big\{M^2\|f\|^4_{H^s_x}, 1\Big\}.$$

Therefore, for any $M > 1$ and any $\omega \in \widetilde\Omega_M$, let

$$A = A(M, \|f\|_{H^s_x}) := \max\big\{M\|f\|_{H^s_x}, CM^2\cdot\max\{M^2\|f\|^4_{H^s_x}, 1\}\big\},$$

then we have $v = e^{it\Delta}P_{\geqslant N_0} f^\omega$,

$$\|u_0\|_{H^s_x} + \|v\|_{\widetilde Y^s\cap Z^s(\mathbb{R})} \leqslant A, \text{ and } E(w_0) \leqslant A N_0^{2(1-s)}.$$

Therefore, we can apply Proposition 5.1. Let $N_0$ depend on $A$ as in the statement of Proposition 5.1, and we obtain a global solution $w$ that scatters. Then, for any $\omega \in \widetilde\Omega = \cup_{M>1}\widetilde\Omega_M$, we can also derive that (4.1) admits a global solution $w$ that scatters. By (5.5), we have that $\mathbb{P}(\widetilde\Omega) = 1$. Then for almost every $\omega \in \Omega$, we obtain the global well-posedness and scattering for (4.1). This finishes the proof of Theorem 1.4.

□

## 5.2. **Global space-time estimates.**

**Lemma 5.2** (Interaction Morawetz). *Let $w \in C([0,T]; H^1_x)$ be the solution of perturbation equation (4.1). Then, we have*

$$\|w\|^4_{L^4_{t,x}} \lesssim \|w\|^2_{L^\infty_t L^2_x}\|w\|^2_{L^\infty_t \dot H^{\frac12}_x} + \|\nabla w\|^2_{L^\infty_t L^2_x}\|w\|^4_{L^\infty_t L^2_x}\|v\|^2_{L^2_t L^\infty_x} + \|v\|^4_{L^4_{t,x}}, \tag{5.6}$$

*where all the space-time norms are taken over $[0,T]\times\mathbb{R}^3$.*

*Proof.* Recall that $w$ satisfies

$$i\partial_t w + \Delta w = |w|^2 w + e,$$

where $e = |u|^2u - |w|^2w$. Denote that

$$m(t,x) = \frac12|w(t,x)|^2; \quad p(t,x) = \frac12\mathrm{Im}(\overline{w}(t,x)\nabla w(t,x)).$$

Then, we have

$$\partial_t m = -2\nabla\cdot p + \mathrm{Im}\left(e\bar w\right), \tag{5.7}$$

and

$$\partial_t p = -\operatorname{Re}\nabla\cdot(\nabla\overline{w}\nabla w) - \frac{1}{4}\nabla\big(|w|^4\big) + \frac{1}{2}\nabla\Delta m + \operatorname{Re}\left(\overline{e}\nabla w\right) - \frac{1}{2}\operatorname{Re}\nabla\left(\overline{w}e\right). \tag{5.8}$$

Moreover, we note that

$$\partial_j\Big(\frac{x_j}{|x|}\Big) = \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}; \quad \nabla\cdot\frac{x}{|x|} = \frac{2}{|x|}; \quad \Delta\nabla\cdot\frac{x}{|x|} = \delta(x).$$

Let

$$M(t) := \iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot p(t,x)\, m(t,y)\, \mathrm{d}x\, \mathrm{d}y,$$

then by (5.7) and (5.8), we have the interaction Morawetz identity

$$\begin{aligned}
\partial_t M(t) = & \iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot \partial_t p(t,x)\, m(t,y)\, \mathrm{d}x\, \mathrm{d}y \\
& + \iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot p(t,x)\, \partial_t m(t,y)\, \mathrm{d}x\, \mathrm{d}y \\
= & \iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot \Big(-\operatorname{Re}\nabla\cdot(\nabla\overline{w}\nabla w) - \frac{1}{4}\nabla\big(|w|^4\big)\Big)(t,x)\, m(t,y)\, \mathrm{d}x\, \mathrm{d}y \qquad (5.9a)\\
& - 2\iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot p(t,x)\, \nabla\cdot p(t,y)\, \mathrm{d}x\, \mathrm{d}y \qquad (5.9b)\\
& + \frac{1}{2}\iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot \nabla\Delta m(t,x)\, m(t,y)\, \mathrm{d}x\, \mathrm{d}y \qquad (5.9c)\\
& + \iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot p(t,x)\, \operatorname{Im}\left(e\bar{w}\right)(t,y)\, \mathrm{d}x\, \mathrm{d}y \qquad (5.9d)\\
& + \iint_{\mathbb{R}^{3+3}} \frac{x-y}{|x-y|}\cdot \operatorname{Re}\left(\overline{e}\nabla w\right)(t,x)\, m(t,y)\, \mathrm{d}x\, \mathrm{d}y \qquad (5.9e)\\
& + \iint_{\mathbb{R}^{3+3}} \frac{1}{|x-y|}\cdot \operatorname{Re}\left(\overline{e}w\right)(t,x)\, m(t,y)\, \mathrm{d}x\, \mathrm{d}y. \qquad (5.9f)
\end{aligned}$$

Note that by the classical argument in [26], we have

$$(5.9a) + (5.9b) \geqslant 0,$$

and

$$(5.9c) \gtrsim \|w(t)\|_{L^4_x}^4.$$

Moreover,

$$\sup_{t\in[0,T]} M(t) \lesssim \|w\|_{L^\infty_t L^2_x}^2 \|w\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x}^2.$$

Then, integrating over $[0,T]$, it holds that

$$C\|w\|_{L^4_{t,x}}^4 \leqslant M(T) - M(0) + \int_0^T |(5.9d)| + |(5.9e)| + |(5.9f)|\, \mathrm{d}t,$$

thus

$$\|w\|_{L^4_{t,x}}^4 \lesssim \|w\|_{L^\infty_t L^2_x}^2 \|w\|_{L^\infty_t \dot{H}^{\frac{1}{2}}_x}^2 + \int_0^T |(5.9d)| + |(5.9e)| + |(5.9f)|\, \mathrm{d}t.$$

By Hölder's inequality and Lemma 2.5, we have

$$\begin{aligned}&\int_0^T |(5.9\mathrm{d})| + |(5.9\mathrm{e})| + |(5.9\mathrm{f})| \,\mathrm{d}t\\ &\lesssim \|\nabla w\|_{L_t^\infty L_x^2} \|w\|_{L_t^\infty L_x^2}^2 \|e\|_{L_t^1 L_x^2}\\ &\lesssim \|\nabla w\|_{L_t^\infty L_x^2} \|w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty} \big( \|v\|_{L_{t,x}^4}^2 + \|w\|_{L_{t,x}^4}^2 \big).\end{aligned}$$

Note that

$$\|e\|_{L_t^1 L_x^2} \lesssim \|v\|_{L_t^2 L_x^\infty} \big( \|v\|_{L_{t,x}^4}^2 + \|w\|_{L_{t,x}^4}^2 \big),$$

we further have that

$$\begin{aligned}&\int_0^T |(5.9\mathrm{d})| + |(5.9\mathrm{e})| + |(5.9\mathrm{f})| \,\mathrm{d}t\\ &\lesssim \|\nabla w\|_{L_t^\infty L_x^2} \|w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty} \big( \|v\|_{L_{t,x}^4}^2 + \|w\|_{L_{t,x}^4}^2 \big).\end{aligned}$$

Therefore, we have

$$\|w\|_{L_{t,x}^4}^4 \lesssim \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty \dot H_x^{\frac12}}^2 + \|\nabla w\|_{L_t^\infty L_x^2} \|w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty} \big( \|v\|_{L_{t,x}^4}^2 + \|w\|_{L_{t,x}^4}^2 \big).$$

Then, by Young's inequality, we have

$$\|w\|_{L_{t,x}^4}^4 \lesssim \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty \dot H_x^{\frac12}}^2 + \|\nabla w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty L_x^2}^4 \|v\|_{L_t^2 L_x^\infty}^2 + \|v\|_{L_{t,x}^4}^4 .$$

This completes the proof of this lemma. □

However, the use of $L_{t,x}^4$-norm is not enough for our argument. We need larger class of space-time norms, and it is more convenient to invoke the $U^p$-$V^p$ method:

$$X^l(I) = \Big( \sum_{N\in 2^{\mathbb N}} N^{2l} \|w_N\|_{U_\Delta^2(I;L_x^2)}^2 \Big)^{\frac12}. \tag{5.10}$$

First, we can update our space-time estimates to $H^{\frac12}$-level.

**Lemma 5.3** ($H^{\frac12}$-regular promotion)**.** *Let $w \in C([0,T];H_x^1)$ be the solution of the perturbation equation* (4.1)*. Then, for any $0 \leqslant l \leqslant \frac12$,*

$$\|w\|_{X^l} \lesssim \|w_0\|_{H_x^l} + \big( \|w\|_{L_t^\infty H_x^{l+\frac12}} + \big\| |\nabla|^l v_N \big\|_{l_N^2 L_t^2 L_x^\infty} \big) \big( \|v\|_{Y^s}^2 + \|w\|_{L_{t,x}^4}^2 \big), \tag{5.11}$$

*where all the space-time norms are taken over $[0,T]\times\mathbb R^3$.*

*Proof.* By Minkowski's inequality, Lemmas 2.2, 2.7, and 2.12, we have

$$\begin{aligned}\|w\|_{X^l} &\lesssim \|w_0\|_{\dot H_x^l} + \big\| P_N \langle\nabla\rangle^l (|u|^2 u) \big\|_{l_N^2 L_t^2 L_x^{\frac65} \cup l_N^2 L_t^1 L_x^2}\\ &\lesssim \|w_0\|_{\dot H_x^l} + \big\| P_N \langle\nabla\rangle^l (|w|^2 w) \big\|_{L_t^2 L_x^{\frac65} l_N^2}\\ &\quad + \big\| P_N \langle\nabla\rangle^l (|u|^2 u - |w|^2 w) \big\|_{l_N^2 L_t^2 L_x^{\frac65} \cup l_N^2 L_t^1 L_x^2}\\ &\lesssim \|w_0\|_{\dot H_x^l} + \big\| \langle\nabla\rangle^l (|w|^2 w) \big\|_{L_t^2 L_x^{\frac65}}\\ &\quad + \big\| P_N \langle\nabla\rangle^l (|u|^2 u - |w|^2 w) \big\|_{l_N^2 L_t^2 L_x^{\frac65} \cup l_N^2 L_t^1 L_x^2}\\ &\lesssim \|w_0\|_{\dot H_x^l} + \big\| \langle\nabla\rangle^l w \big\|_{L_t^\infty L_x^3} \|w\|_{L_{t,x}^4}^2 + I + II,\end{aligned} \tag{5.12}$$

where

$$I := \big\| P_N \langle\nabla\rangle^l \big( \sum_{N_1 : N \lesssim N_1} w_{N_1} v_{\leqslant N_1} u_{\leqslant N_1} \big) \big\|_{L_t^2 L_x^{\frac{6}{5}} l_N^2},$$

and

$$II := \big\| P_N \langle\nabla\rangle^l \big( \sum_{N_1 : N \lesssim N_1} v_{N_1} u_{\leqslant N_1} u_{\leqslant N_1} \big) \big\|_{L_t^1 L_x^2 l_N^2}.$$

By Hölder's inequality, we have

$$\begin{aligned} I &\lesssim \sum_{N_1} \big\| \langle\nabla\rangle^l (w_{N_1} v_{\leqslant N_1} u_{\leqslant N_1}) \big\|_{L_t^2 L_x^{\frac{6}{5}}} \\ &\lesssim \sum_{N_1} \big\| N_1^l w_{N_1} v_{\leqslant N_1} u_{\leqslant N_1} \big\|_{L_t^2 L_x^{\frac{6}{5}}} \\ &\lesssim \sum_{N_1} \big\| N_1^l w_{N_1} \big\|_{L_t^\infty L_x^2} \| v_{\leqslant N_1} \|_{L_t^4 L_x^{12}} \| u_{\leqslant N_1} \|_{L_{t,x}^4} \\ &\lesssim \big\| \langle\nabla\rangle^{l+\frac{1}{2}} w \big\|_{L_t^\infty L_x^2} \| v \|_{Y^s} \big( \| v \|_{Y^s} + \| w \|_{L_{t,x}^4} \big), \end{aligned} \tag{5.13}$$

and by Lemma 2.11,

$$\begin{aligned} II &\lesssim \big\| N_1^l v_{N_1} u_{\leqslant N_1} u_{\leqslant N_1} \big\|_{L_t^1 L_x^2 l_{N_1}^2} \\ &\lesssim \big\| \| N_1^l v_{N_1} \|_{l_{N_1}^2} \sup_{N_1 \in 2^{\mathbb{N}}} | u_{\leqslant N_1} |^2 \big\|_{L_t^1 L_x^2} \\ &\lesssim \big\| N_1^l v_{N_1} \big\|_{L_t^2 L_x^\infty l_{N_1}^2} \big\| \sup_{N_1 \in 2^{\mathbb{N}}} | u_{\leqslant N_1} | \big\|_{L_{t,x}^4}^2 \\ &\lesssim \big\| N_1^l v_{N_1} \big\|_{l_{N_1}^2 L_t^2 L_x^\infty} \big( \| v \|_{Y^s} + \| w \|_{L_{t,x}^4} \big)^2. \end{aligned} \tag{5.14}$$

Therefore, by (5.12), (5.13), and (5.14), we have that (5.11) holds. ☐

Based on the space-time estimates in $H^{\frac{1}{2}}$-level, and keeping in mind that the equation is $H^{\frac{1}{2}}$-critical, we can further obtain the estimates in $H^l$-level with $l > \frac{1}{2}$.

**Lemma 5.4** ($H^1$-regular promotion)**.** *Let $0 < \varepsilon \ll 1$, and $w \in C([0,T]; H_x^1)$ be the solution of the perturbation equation* (4.1)*. Then, for any $\frac{1}{2} < l \leqslant 1$ and $\frac{1}{3} < s \leqslant \frac{1}{2}$, we have*

$$\begin{aligned} \| w \|_{X^l} \lesssim{}& \| w_0 \|_{H_x^l} + \big( \| w \|_{L_t^\infty H_x^l} + \| v \|_{\widetilde{Y}^s} \big) \big( \| v \|_{Y^s \cap Z^s} + \| w \|_{X^{\frac{1}{2}}} \big) \big( \| v \|_{Z^s} + \| w \|_{L_t^\infty H_x^{\frac{1}{2}}} \big) \\ &+ \| w \|_{L_t^\infty H_x^l} \big( \| v \|_{Y^s} + \| w \|_{X^{\frac{1}{2}-\varepsilon}} \big) \big( \| v \|_{Z^s} + \| w \|_{L_t^\infty H_x^{\frac{1}{2}+\varepsilon}} \big), \end{aligned}$$

*where all the space-time norms are taken over $[0,T] \times \mathbb{R}^3$.*

*Proof.* Similar to the proof of Lemma 5.3, we have

$$\begin{aligned} \| w \|_{X^l} &\lesssim \| w_0 \|_{H_x^l} + \Big( \sum_{N \in 2^{\mathbb{N}}} N^{2l} \sup_{\| g \|_{V_\Delta^2} = 1} \Big| \int_0^T \int P_N(|u|^2 u) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}} \\ &\lesssim \| w_0 \|_{H_x^l} + \Big( \sum_{N \in 2^{\mathbb{N}}} N^{2l} \sup_{\| g \|_{V_\Delta^2} = 1} \Big| \int_0^T \int P_N\big(\sum_{N_1} u_{N_1} u_{\leqslant N_1}^2\big) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac{1}{2}} \\ &\lesssim \| w_0 \|_{H_x^l} + I + II, \end{aligned}$$

where

$$I := \Big( \sum_{N\in 2^{\mathbb{N}}} N^{2l} \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} w_{N_1} u^2_{\leqslant N_1}\big) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac12},$$

and

$$II := \Big( \sum_{N\in 2^{\mathbb{N}}} N^{2l} \sup_{\|g\|_{V^2_\Delta}=1} \Big| \int_0^T \int P_N\big(\sum_{N_1} v_{N_1} u^2_{\leqslant N_1}\big) g \, \mathrm{d}x \, \mathrm{d}s \Big|^2 \Big)^{\frac12}.$$

We first consider the term $I$, where the first order derivative acts on $w$. By Hölder's inequality, and Lemmas 2.7 and 2.12,

$$\begin{aligned}
I \lesssim & \Big( \sum_{N\in 2^{\mathbb{N}}} N^{2l} \big\| P_N\big( \sum_{N_1: N\lesssim N_1} w_{N_1} u^2_{\leqslant N_1}\big) \big\|^2_{L^2_t L^{\frac65}_x} \Big)^{\frac12} \\
\lesssim & \big\| \langle \nabla \rangle^l \big( \sum_{N_1} w_{N_1} u^2_{\leqslant N_1}\big) \big\|_{L^2_t L^{\frac65}_x} \\
\lesssim & \big\| \langle \nabla \rangle^l \big( \sum_{N_1} w_{N_1} u_{\sim N_1} u_{\leqslant N_1}\big) \big\|_{L^2_t L^{\frac65}_x} && (5.15a)\\
& + \big\| \langle \nabla \rangle^l \big( \sum_{N_1} w_{N_1} u^2_{\ll N_1}\big) \big\|_{L^2_t L^{\frac65}_x}. && (5.15b)
\end{aligned}$$

Now, the main task is to update the summation of $w_{N_1}$ to $l^2_{N_1}$. To this end, for (5.15a), we can simply use Hölder's inequality in $N_1$ for $w_{N_1}$ and $u_{\sim N_1}$. Precisely, by (4.24), Hölder's inequality, and Lemma 2.12,

$$\begin{aligned}
(5.15a) \lesssim & \Big\| \sum_{N_1} \big\| N_1^l w_{N_1} u_{\sim N_1} u_{\leqslant N_1} \big\|_{L^{\frac65}_x} \Big\|_{L^2_t} \\
\lesssim & \Big\| \sum_{N_1} \big\| N_1^l w_{N_1} \big\|_{L^2_x} \| u_{\sim N_1} \|_{L^\infty_x} \| u_{\leqslant N_1} \|_{L^3_x} \Big\|_{L^2_t} \\
\lesssim & \Big\| \big\| N_1^l w_{N_1} \big\|_{l^2_{N_1} L^2_x} \| u_{\sim N_1} \|_{l^2_{N_1} L^\infty_x} \| u \|_{L^3_x} \Big\|_{L^2_t} && (5.16)\\
\lesssim & \big\| \langle \nabla \rangle^l w \big\|_{L^\infty_t L^2_x} \| u_{N_1} \|_{l^2_{N_1} L^2_t L^\infty_x} \| u \|_{L^\infty_t L^3_x} \\
\lesssim & \| w \|_{L^\infty_t H^l_x} \big( \| v \|_{Y^s} + \| w \|_{X^{\frac12}} \big) \big( \| v \|_{Z^s} + \| w \|_{L^\infty_t H^{\frac12}_x} \big).
\end{aligned}$$

For the second term (5.15b), we need to invoke the vector-valued Hardy-Littlewood maximal function to cover the critical summation problem. Using Lemma 2.10 and Hölder's inequality,

$$\begin{aligned}
(5.15b) \lesssim & \big\| N^l P_N \big( \sum_{N_1: N_1 \sim N} w_{N_1} u^2_{\ll N_1} \big) \big\|_{L^2_t L^{\frac65}_x l^2_N} \\
\lesssim & \big\| N^l \mathcal{M} \big( w_N u^2_{\ll N} \big) \big\|_{L^2_t L^{\frac65}_x l^2_N} \\
\lesssim & \big\| N^l w_N u^2_{\ll N} \big\|_{L^2_t L^{\frac65}_x l^2_N} && (5.17)\\
\lesssim & \Big\| \big\| N^l w_N \big\|_{l^2_N} \sup_N \big| u_{\ll N} \big|^2 \Big\|_{L^2_t L^{\frac65}_x} \\
\lesssim & \big\| N^l w_N \big\|_{L^\infty_t L^2_x l^2_N} \big\| \sup_N \big| u_{\ll N} \big|^2 \big\|_{L^2_t L^3_x}.
\end{aligned}$$

Let $0<\varepsilon\ll 1$. By Lemmas 2.4, 2.12, and Hölder's inequality,

$$\begin{aligned}\big\|\sup_N\big|u_{\ll N}\big|^2\big\|_{L_t^2L_x^3}\lesssim&\big\|\sup_N\sum_{N_1,N_2:N_1\leqslant N_2\ll N}\big|u_{N_1}u_{N_2}\big|\big\|_{L_t^2L_x^3}\\ \lesssim&\big\|\sum_{N_1\leqslant N_2}\big|u_{N_1}u_{N_2}\big|\big\|_{L_t^2L_x^3}\\ \lesssim&\big\|\sum_{N_1\leqslant N_2}\big(\frac{N_1}{N_2}\big)^{\varepsilon}N_1^{-\varepsilon}u_{N_1}N_2^{\varepsilon}u_{N_2}\big\|_{L_t^2L_x^3}\\ \lesssim&\big\|\,\big\|N_1^{-\varepsilon}u_{N_1}\big\|_{l^2_{N_1}}\,\big\|N_2^{\varepsilon}u_{N_2}\big\|_{l^2_{N_2}}\,\big\|_{L_t^2L_x^3}\\ \lesssim&\,\big\|N_1^{-\varepsilon}u_{N_1}\big\|_{L_t^2L_x^\infty l^2_{N_1}}\,\big\|N_2^{\varepsilon}u_{N_2}\big\|_{L_t^\infty L_x^3 l^2_{N_2}}\\ \lesssim&\big(\,\|v_{N_1}\|_{l^2_{N_1}L_t^2L_x^\infty}+\big\|N_1^{-\varepsilon}w_{N_1}\big\|_{l^2_{N_1}L_t^2L_x^\infty}\big)\big(\,\|v\|_{Z^s}+\|w\|_{L_t^\infty H_x^{\frac12+\varepsilon}}\big)\\ \lesssim&\big(\,\|v\|_{Y^s}+\|w_{N_1}\|_{X^{\frac12-\varepsilon}}\big)\big(\,\|v\|_{Z^s}+\|w\|_{L_t^\infty H_x^{\frac12+\varepsilon}}\big).\end{aligned}\tag{5.18}$$

Combining (5.17) and (5.18), we have

$$\begin{aligned}\text{(5.15a)}\lesssim&\,\big\|N^lw_N\big\|_{L_t^\infty L_x^2l_N^2}\,\big\|\sup_N\big|u_{\ll N}\big|^2\big\|_{L_t^2L_x^3}\\ \lesssim&\,\|w\|_{L_t^\infty H_x^l}\big(\,\|v\|_{Y^s}+\|w\|_{X^{\frac12-\varepsilon}}\big)\big(\,\|v\|_{Z^s}+\|w\|_{L_t^\infty H_x^{\frac12+\varepsilon}}\big).\end{aligned}\tag{5.19}$$

Therefore, by (5.16) and (5.19),

$$\begin{aligned}I\lesssim&\text{(5.15a)}+\text{(5.15b)}\\ \lesssim&\,\|w\|_{L_t^\infty H_x^l}\big(\,\|v\|_{Y^s}+\|w\|_{X^{\frac12}}\big)\big(\,\|v\|_{Z^s}+\|w\|_{L_t^\infty H_x^{\frac12}}\big)\\ &+\|w\|_{L_t^\infty H_x^l}\big(\,\|v\|_{Y^s}+\|w\|_{X^{\frac12-\varepsilon}}\big)\big(\,\|v\|_{Z^s}+\|w\|_{L_t^\infty H_x^{\frac12+\varepsilon}}\big).\end{aligned}\tag{5.20}$$

Next, we estimate the term $II$. We first consider the case when $l<s+\frac12$, by the same argument as in the local results (see (4.5) for example) and Hölder's inequality,

$$\begin{aligned}II\lesssim&\big\|N_1^lv_{N_1}u^2_{\leqslant N_1}\big\|_{L_t^1L_x^2l^2_{N_1}}\\ \lesssim&\big\||\nabla|^lv_{N_1}\big\|_{L_t^2L_x^\infty l^2_{N_1}}\,\big\|\sup_{N_1}|u_{\leqslant N_1}|^2\big\|_{L^2_{t,x}}\\ \lesssim&\big\||\nabla|^lv_{N_1}\big\|_{l^2_{N_1}L_t^2L_x^\infty}\big(\,\|v\|^2_{L^4_{t,x}}+\|w\|_{L_t^2L_x^6}\,\|w\|_{L_t^\infty L_x^3}\big)\\ \lesssim&\,\|v\|_{\widetilde{Y}^s}\big(\,\|v\|^2_{Y^s}+\|w\|_{X^0}\,\|w\|_{L_t^\infty H_x^{\frac12}}\big)\\ \lesssim&\,\|v\|_{\widetilde{Y}^s}\big(\,\|v\|^2_{Y^s}+\|w\|_{X^{\frac12}}\,\|w\|_{L_t^\infty H_x^{\frac12}}\big).\end{aligned}\tag{5.21}$$

Then, we consider the case when $s+\frac12\leqslant l\leqslant 1$, where we need to use the bilinear Strichartz estimate and the duality structure. By frequency support property, we obtain

$$II\lesssim\big\|N_1^lv_{N_1}u_{\sim N_1}u_{\leqslant N_1}\big\|_{L_t^1L_x^2l^2_{N_1}}\tag{5.22a}$$

$$+\Big(\sum_{N\in 2^{\mathbb{N}}}N^{2l}\sup_{\|g\|_{V^2_\Delta}=1}\Big|\int_0^T\int P_N\big(\sum_{N_1}v_{N_1}u^2_{\ll N_1}\big)g\;\mathrm{d}x\;\mathrm{d}s\Big|^2\Big)^{\frac12}.\tag{5.22b}$$

Now, we estimate (5.22a). Since $s > \frac{1}{3}$, we have $2s + \frac{1}{2} > 1 \geqslant l$. Then, by $l^1_{N_1} \hookrightarrow l^2_{N_1}$ and Hölder's inequality,

$$\begin{aligned}
(5.22a) \lesssim & \sum_{N_1} \big\| |\nabla|^{l-s} v_{N_1} \big\|_{L^2_t L^\infty_x} \| \langle \nabla \rangle^s u_{\sim N_1} \|_{L^2_t L^6_x} \| u_{\leqslant N_1} \|_{L^\infty_t L^3_x} \\
\lesssim & \big\| |\nabla|^{l-s} v_{N_1} \big\|_{l^2_{N_1} L^2_t L^\infty_x} \| \langle \nabla \rangle^s u_{\sim N_1} \|_{l^2_{N_1} L^2_t L^6_x} \big( \|v\|_{L^\infty_t L^3_x} + \|w\|_{L^\infty_t H^{\frac{1}{2}}_x} \big) \\
\lesssim & \big\| |\nabla|^{s+\frac{1}{2}-} v_{N_1} \big\|_{l^2_{N_1} L^2_t L^\infty_x} \big( \| \langle \nabla \rangle^s v_{\sim N_1} \|_{l^2_{N_1} L^2_t L^6_x} + \|w\|_{X^s} \big) \big( \|v\|_{L^\infty_t L^3_x} + \|w\|_{L^\infty_t H^{\frac{1}{2}}_x} \big) \\
\lesssim & \|v\|_{\widetilde{Y}^s} \big( \|v\|_{Y^s} + \|w\|_{X^s} \big) \big( \|v\|_{Z^s} + \|w\|_{L^\infty_t H^{\frac{1}{2}}_x} \big).
\end{aligned} \tag{5.23}$$

Next, we consider (5.22b). To this end, we establish a bilinear Strichartz estimate before the proof for (5.22b). By Lemma 2.9, for $N_1 \ll N$,

$$\| u_{N_1} g_N \|_{L^2_{t,x}} \lesssim \frac{N_1}{N^{\frac{1}{2}}} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big) \| g \|_{U^2_\Delta}, \tag{5.24}$$

and by Bernstein's, Hölder's inequalities, Lemma 2.7, and embedding $U^4_\Delta \hookrightarrow L^\infty_t L^2_x$,

$$\begin{aligned}
\| u_{N_1} g_N \|_{L^2_{t,x}} \lesssim & \| u_{N_1} \|_{L^2_t L^\infty_x} \| g_N \|_{L^\infty_t L^2_x} \\
\lesssim & N_1^{\frac{1}{2}} \| u_{N_1} \|_{L^2_t L^6_x} \| g \|_{U^4_\Delta} \\
\lesssim & N_1^{\frac{1}{2}} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big) \| g \|_{U^4_\Delta}.
\end{aligned} \tag{5.25}$$

By (5.24), (5.25), and Lemma 2.3, we have the bilinear Strichartz estimate

$$\begin{aligned}
\| u_{N_1} g_N \|_{L^2_{t,x}} \lesssim & \frac{N^\varepsilon}{N_1^\varepsilon} \frac{N_1}{N^{\frac{1}{2}}} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big) \| g \|_{V^2_\Delta} \\
\lesssim & \frac{N_1^{l-s+}}{N^{l-s-\frac{1}{2}+}} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big) \| g \|_{V^2_\Delta}.
\end{aligned} \tag{5.26}$$

Noting that $l \leqslant 1$ and $\frac{1}{3} < s$, we have $l - s+ < 2s + 0-$. Then, by (5.26),

$$\begin{aligned}
(5.22b) \lesssim & \Big\| \sum_{N_1, N_2 : N_1 \leqslant N_2 \ll N} \sup_{\|g\|_{V^2_\Delta} = 1} \Big| \int_0^T \int N^l v_N u_{N_1} u_{N_2} g_N \, dx \, ds \Big| \Big\|_{l^2_N} \\
\lesssim & \Big\| \sum_{N_1, N_2 : N_1 \leqslant N_2 \ll N} \sup_{\|g\|_{V^2_\Delta} = 1} \big\| |\nabla|^l v_N \big\|_{L^2_t L^\infty_x} \| u_{N_2} \|_{L^\infty_t L^2_x} \| u_{N_1} g_N \|_{L^2_{t,x}} \Big\|_{l^2_N} \\
\lesssim & \sum_{N_1 \leqslant N_2} \big\| |\nabla|^{s+\frac{1}{2}-} v_N \big\|_{l^2_N L^2_t L^\infty_x} \| u_{N_2} \|_{L^\infty_t L^2_x} N_1^{l-s+} \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big) \\
\lesssim & \big\| |\nabla|^{s+\frac{1}{2}-} v_N \big\|_{l^2_N L^2_t L^\infty_x} \sum_{N_1 \leqslant N_2} N_1^{0-} N_2^s \| u_{N_2} \|_{L^\infty_t L^2_x} N_1^s \big( \| P_{N_1} v_0 \|_{L^2_x} + \| w_{N_1} \|_{U^2_\Delta} \big) \\
\lesssim & \|v\|_{\widetilde{Y}^s} \big( \|v\|_{Z^s} + \|w\|_{X^s} \big) \big( \|v\|_{Z^s} + \|w\|_{L^\infty_t H^s_x} \big).
\end{aligned} \tag{5.27}$$

By (5.23) and (5.27), we have for $s + \frac{1}{2} \leqslant l \leqslant 1$,

$$II \lesssim \|v\|_{\widetilde{Y}^s} \big( \|v\|_{Y^s \cap Z^s} + \|w\|_{X^{\frac{1}{2}}} \big) \big( \|v\|_{Z^s} + \|w\|_{L^\infty_t H^{\frac{1}{2}}_x} \big). \tag{5.28}$$

Then (5.20), (5.21), and (5.28) give the desired estimates. □

5.3. **Energy bound.**

**Proposition 5.5.** *Let* $\frac{3}{7} < s \leqslant \frac{1}{2}$, $A > 0$, $v = e^{it\Delta}v_0 \in \widetilde{Y}^s \cap Z^s(\mathbb{R})$ *and* $w$ *be the solution of* (4.1). *Take some* $T > 0$ *such that* $w \in C([0,T]; H_x^1)$. *Then, there exists* $N_0 = N_0(A) \gg 1$ *with the following properties. Assume that* $\widehat{v}_0$ *is supported on* $\{\xi \in \mathbb{R}^3 : |\xi| \geqslant \frac{1}{2}N_0\}$,

$$\|u_0\|_{H_x^s} + \|v\|_{\widetilde{Y}^s \cap Z^s(\mathbb{R})} \leqslant A, \text{ and } E(w_0) \leqslant AN_0^{2(1-s)}.$$

*Then, we have*

$$\sup_{t\in[0,T]} E(w(t)) \leqslant 2AN_0^{2(1-s)}. \tag{5.29}$$

*Proof.* Let $I = [0,T]$ and $N_0 = N_0(A)$ that will be defined later. From now on, all the space-time norms are taken over $I \times \mathbb{R}^3$. We implement a bootstrap procedure on $I$: assume an a priori bound

$$\sup_{t\in I} E(w(t)) \leqslant 2AN_0^{2(1-s)}, \tag{5.30}$$

then it suffices to prove that

$$\sup_{t\in I} E(w(t)) \leqslant \frac{3}{2}AN_0^{2(1-s)}. \tag{5.31}$$

To start with, we collect useful estimates on $I$. Now, we use the notation $C = C(A)$ for short, and the implicit constants in "$\lesssim$" depend on $A$. Moreover, we take all the space-time norms over $I \times \mathbb{R}^3$. By interpolation, we have

$$\|v\|_{L_t^\infty L_x^3} + \|\langle\nabla\rangle^s v\|_{L_t^\infty L_x^2} + \|v\|_{L_{t,x}^4} \lesssim \|v\|_{Y^s\cap Z} \lesssim 1. \tag{5.32}$$

By the frequency support of $v$, we have for any $0 \leqslant l < s + \frac{1}{2}$,

$$\big\||\nabla|^l v_N\big\|_{l_N^2 L_t^2 L_x^\infty} \lesssim N_0^{l-s-\frac{1}{2}+} \|v\|_{\widetilde{Y}^s} \lesssim N_0^{l-s-\frac{1}{2}+} \lesssim 1. \tag{5.33}$$

By the conservation of mass, we have $\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2}$. Then, combining (5.32), we have for all $t \in [0,T]$,

$$\|w(t)\|_{L_x^2} \lesssim \|u(t)\|_{L_x^2} + \|v(t)\|_{L_x^2} \lesssim \|u_0\|_{L_x^2} + 1 \lesssim 1. \tag{5.34}$$

By bootstrap hypothesis (5.30),

$$\|w\|_{L_t^\infty \dot{H}_x^1} \lesssim N_0^{1-s}. \tag{5.35}$$

Then, by interpolation, (5.34), and (5.35), we have for any $0 \leqslant l \leqslant 1$,

$$\|w\|_{L_t^\infty \dot{H}_x^l} \lesssim N_0^{l(1-s)}. \tag{5.36}$$

Next, we derive various space-time bounds combining Lemmas 5.2, 5.3, and 5.4, under the above setting.

**Lemma 5.6.** *If the assumptions in Proposition 5.5 and the estimate* (5.30) *hold, then there exists* $N_0 = N_0(A) \gg 1$ *satisfying the following estimates.*

*(1) First, we have the interaction Morawetz estimate*

$$\|w\|_{L_{t,x}^4} \lesssim N_0^{\frac{1-s}{4}}. \tag{5.37}$$

*(2) If* $0 \leqslant l \leqslant \frac{1}{2}$, *then*

$$\|w\|_{X^l} \lesssim N_0^{(l+1)(1-s)}. \tag{5.38}$$

*(3) If $\frac{1}{2} < l \leqslant 1$, then*

$$\|w\|_{X^l} \lesssim N_0^{(l+2)(1-s)}. \tag{5.39}$$

*Remark* 5.7. Roughly speaking, the interaction Morawetz estimate in Lemma 5.2 yields

$$\|w\|_{L^4_{t,x}}^4 \lesssim N_0^{1-s} + N_0^{2(1-s)} \|v\|_{L_t^2 L_x^\infty}^2 .$$

Since $v$ is high-frequency truncated, we are able to cover the additional increment for the remainder in the view of (5.33). This is the main reason why we implement the high-low frequency decomposition for the initial data.

*Proof.* Note that $s > 0$, by the perturbed Morawetz estimate in Lemma 5.2 and (5.36),

$$\begin{aligned}\|w\|_{L^4_{t,x}}^4 \lesssim& \|w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty \dot H_x^{\frac12}}^2 + \|\nabla w\|_{L_t^\infty L_x^2}^2 \|w\|_{L_t^\infty L_x^2}^4 \|v\|_{L_t^2 L_x^\infty}^2 + \|v\|_{L^4_{t,x}}^4 \\ \lesssim& N_0^{1-s} + N_0^{2(1-s)} N_0^{-1-2s+} + 1 \lesssim N_0^{1-s},\end{aligned}$$

where we need to take $N_0 = N_0(A)$ suitably large such that $N_0 \gtrsim_A 1$.

By Lemma 5.3 and (5.37), for any $0 \leqslant l \leqslant \frac{1}{2}$, we also have that

$$\begin{aligned}\|w\|_{X^l} \lesssim& \|w_0\|_{\dot H_x^l} + \big( \|w\|_{L_t^\infty H_x^{l+\frac12}} + \big\| |\nabla|^l v_N \big\|_{l_N^2 L_t^2 L_x^\infty} \big) \big( \|v\|_{Y^s}^2 + \|w\|_{L^4_{t,x}}^2 \big) \\ \lesssim& N_0^{l(1-s)} + \big(N_0^{(l+\frac12)(1-s)} + 1\big)\big(1 + N_0^{\frac12(1-s)}\big) \lesssim N_0^{(l+1)(1-s)}.\end{aligned}$$

Let $0 < \varepsilon \ll 1$. Then, by Lemma 5.4 and (5.38), for any $\frac{1}{2} < l \leqslant 1$, we have

$$\begin{aligned}\|w\|_{X^l} \lesssim& \|w_0\|_{H_x^l} + \big( \|w\|_{L_t^\infty H_x^l} + \|v\|_{\widetilde{Y}^s} \big) \big( \|v\|_{Y^s \cap Z^s} + \|w\|_{X^{\frac12}} \big) \big( \|v\|_{Z^s} + \|w\|_{L_t^\infty H_x^{\frac12}} \big) \\ &+ \|w\|_{L_t^\infty H_x^l} \big( \|v\|_{Y^s} + \|w\|_{X^{\frac12 - \varepsilon}} \big) \big( \|v\|_{Z^s} + \|w\|_{L_t^\infty H_x^{\frac12+\varepsilon}} \big) \\ \lesssim& N_0^{l(1-s)} + (N_0^{l(1-s)} + 1)(1 + N_0^{\frac32(1-s)})(1 + N_0^{\frac12(1-s)}) \\ &+ N_0^{l(1-s)}(1 + N_0^{(\frac32-\varepsilon)(1-s)})(1 + N_0^{(\frac12+\varepsilon)(1-s)}) \lesssim N_0^{(l+2)(1-s)}.\end{aligned}$$

This proves the lemma. □

Now, we are prepared to give the proof of Proposition 5.5. By (4.1) and integration-by-parts, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} E(w(t)) =& \mathrm{Im} \int \Delta \overline{w} \big(|u|^2 u - |w|^2 w\big) \,\mathrm{d}x + \mathrm{Im} \int |u|^2 u \big(|u|^2 u - |w|^2 w\big) \,\mathrm{d}x \\ =& - \mathrm{Im} \int \nabla \overline{w} \cdot \nabla \big(|u|^2 u - |w|^2 w\big) \,\mathrm{d}x + \mathrm{Im} \int |u|^2 u \big(|u|^2 u - |w|^2 w\big) \,\mathrm{d}x.\end{aligned}$$

Again, we do not distinguish between $u$ and $\overline{u}$. Then, we have

$$\sup_{t \in I} E(t) \lesssim E(w_0)$$

$$+ \Big| \int_I \int \nabla w \cdot \nabla v (v + w)^2 \,\mathrm{d}x \,\mathrm{d}t \Big| \tag{5.40}$$

$$+ \Big| \int_I \int \nabla w \cdot \nabla w v (v + w) \,\mathrm{d}x \,\mathrm{d}t \Big| \tag{5.41}$$

$$+ \Big| \int_I \int |u|^2 u \big(|u|^2 u - |w|^2 w\big) \,\mathrm{d}x \,\mathrm{d}t \Big|. \tag{5.42}$$

**Estimate on** (5.40)**.** This is the main case, where we need the restriction $s > \frac{3}{7}$. We first make a frequency decomposition:

$$(5.40) \lesssim \sum_{N\in 2^{\mathbb{N}}} \Big| \int_I \int \nabla w \cdot \nabla v_N v_{\gtrsim N}(v+w) \, \mathrm{d}x \, \mathrm{d}t \Big| \tag{5.43a}$$

$$+ \sum_{N\in 2^{\mathbb{N}}} \Big| \int_I \int \nabla w_N \cdot \nabla v_N v_{\ll N}(v_{\ll N}+w_{\ll N}) \, \mathrm{d}x \, \mathrm{d}t \Big| \tag{5.43b}$$

$$+ \sum_{N\in 2^{\mathbb{N}}} \Big| \int_I \int \nabla w_N \cdot \nabla v_N w_{\gtrsim N}(w+v) \, \mathrm{d}x \, \mathrm{d}t \Big| \tag{5.43c}$$

$$+ \sum_{N\in 2^{\mathbb{N}}} \Big| \int_I \int \nabla w_N \cdot \nabla v_N w_{\ll N}^2 \, \mathrm{d}x \, \mathrm{d}t \Big|. \tag{5.43d}$$

For (5.43a), we can directly transfer the derivative from $v_N$ to $v_{\gtrsim N}$. By Hölder's inequality, Lemma 2.12, and (5.33),

$$\begin{aligned}
(5.43a) &\lesssim \sum_{N\lesssim N_1} \Big| \int_I \int \nabla w \cdot \nabla v_N v_{N_1}(v+w) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
&\lesssim \sum_{N\lesssim N_1} \|w\|_{L_t^\infty \dot{H}_x^1} \|\nabla v_N\|_{L_t^2 L_x^\infty} \|v_{N_1}\|_{L_t^2 L_x^\infty} \left( \|v\|_{L_t^\infty L_x^2} + \|w\|_{L_t^\infty L_x^2} \right) \\
&\lesssim N_0^{1-s} \sum_{N\lesssim N_1} \frac{N^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \big\| |\nabla|^{\frac{1}{2}} v_N \big\|_{L_t^2 L_x^\infty} \big\| |\nabla|^{\frac{1}{2}} v_{N_1} \big\|_{L_t^2 L_x^\infty} \\
&\lesssim N_0^{1-s} \big\| |\nabla|^{\frac{1}{2}} v_N \big\|_{l_N^2 L_t^2 L_x^\infty} \big\| |\nabla|^{\frac{1}{2}} v_{N_1} \big\|_{l_{N_1}^2 L_t^2 L_x^\infty} \lesssim N_0^{1-s}.
\end{aligned} \tag{5.44}$$

Next, we bound (5.43b), where we use the bilinear Strichartz estimate for $\nabla w_N v_{\ll N}$ to lower down the derivative of $\nabla v_N$. From Lemma 2.9, (5.39), and (5.32), for $N_1 \ll N$, we have that

$$\begin{aligned}
\|\nabla w_N v_{N_1}\|_{L_{t,x}^2} &\lesssim \frac{N_1}{N^{\frac{1}{2}}} N \|w_N\|_{U_\Delta^2} \|P_{N_1} v_0\|_{L_x^2} \\
&\lesssim N_1^{1-s} N^{-\frac{1}{2}} \|w\|_{X^1} \|P_{N_1} v_0\|_{H_x^s} \\
&\lesssim N_1^{1-s} N^{-\frac{1}{2}} N_0^{4(1-s)}.
\end{aligned} \tag{5.45}$$

Note that $\frac{3}{7} < s$ gives

$$\|\nabla v_N\|_{L_t^2 L_x^\infty} \lesssim N^{\frac{1}{14}} \big\| |\nabla|^{s+\frac{1}{2}-} v_N \big\|_{L_t^2 L_x^\infty}, \text{ and } \|v_{N_1}\|_{L_t^2 L_x^\infty} \lesssim N_1^{-\frac{13}{14}} \big\| |\nabla|^{s+\frac{1}{2}-} v_{N_1} \big\|_{L_t^2 L_x^\infty}. \tag{5.46}$$

Then, combining Hölder's inequaltiy, (5.45), (5.46), (5.32), (5.33), and (5.36), it holds that

$$
\begin{aligned}
(5.43\mathrm{b}) \lesssim & \sum_{N_1 \ll N} \Big| \int_I \int \nabla w_N \cdot \nabla v_N v_{N_1} (v_{\ll N} + w_{\ll N}) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
\lesssim & \sum_{N_1 \ll N} \|\nabla w_N\|_{L_t^\infty L_x^2}^{\frac{3}{4}} \|\nabla v_N\|_{L_t^2 L_x^\infty} \|v_{N_1}\|_{L_t^2 L_x^\infty}^{\frac{3}{4}} \|\nabla w_N v_{N_1}\|_{L_{t,x}^2}^{\frac{1}{4}} \\
& \cdot \big( \|v_{\ll N}\|_{L_t^\infty L_x^2} + \|w_{\ll N}\|_{L_t^\infty L_x^2} \big) \\
\lesssim & N_0^{\frac{3}{4}(1-s)} \sum_{N_1 \ll N} \|\nabla v_N\|_{L_t^2 L_x^\infty} \|v_{N_1}\|_{L_t^2 L_x^\infty}^{\frac{3}{4}} \big( N_1^{1-s} N^{-\frac{1}{2}} N_0^{4(1-s)} \big)^{\frac{1}{4}} \\
\lesssim & N_0^{\frac{7}{4}(1-s)} \sum_{N_1 \ll N} N^{\frac{1}{14}} N_1^{-\frac{39}{56}} N_1^{\frac{1}{4}(1-s)} N^{-\frac{1}{8}} \\
\lesssim & N_0^{\frac{7}{4}(1-s)} \sum_{N_1 \ll N} N_1^{-\frac{1}{2}} N^{-\frac{3}{56}} \lesssim N_0^{-\frac{1}{4}(1-s)} N_0^{2(1-s)}.
\end{aligned} \tag{5.47}
$$

Next, we deal with the term (5.43c), where we can directly transfer the derivative from $v_N$ to $w_{\gtrsim N}$. Note that by $s > \frac{3}{7}$,

$$
\sum_{N, N_1 : N_0 \lesssim N \lesssim N_1} \frac{N^{\frac{1}{2}-s+}}{N_1^{\frac{1}{10}}} \lesssim N_0^{-\frac{1}{100}}. \tag{5.48}
$$

By interpolation, (5.37), and (5.38), we also have

$$
\big\| N_1^{\frac{1}{10}} w_{N_1} \big\|_{L_{t,x}^4} \lesssim \|w\|_{L_{t,x}^4}^{\frac{3}{5}} \big\| \langle \nabla \rangle^{\frac{1}{4}} w \big\|_{L_{t,x}^4}^{\frac{2}{5}} \lesssim N_0^{\frac{3}{4}(1-s)}. \tag{5.49}
$$

Therefore, by Hölder's inequality, (5.33), (5.48), (5.49), and (5.37),

$$
\begin{aligned}
(5.43\mathrm{c}) \lesssim & \sum_{N, N_1 \in 2^{\mathbb{N}} : N \lesssim N_1} \Big| \int_I \int \nabla w_N \cdot \nabla v_N w_{N_1} (w + v) \, \mathrm{d}x \, \mathrm{d}t \Big| \\
\lesssim & \sum_{N, N_1 \in 2^{\mathbb{N}} : N \lesssim N_1} \frac{N^{\frac{1}{2}-s+}}{N_1^{\frac{1}{10}}} \|\nabla w_N\|_{L_t^\infty L_x^2} \big\| |\nabla|^{s+\frac{1}{2}-} v_N \big\|_{L_t^2 L_x^\infty} \\
& \cdot \big\| N_1^{\frac{1}{10}} w_{N_1} \big\|_{L_{t,x}^4} \big( \|w\|_{L_{t,x}^4} + \|v\|_{L_{t,x}^4} \big) \\
\lesssim & N_0^{-\frac{1}{100}} N_0^{1-s} N_0^{\frac{3}{4}(1-s)} N_0^{\frac{1}{4}(1-s)} \lesssim N_0^{-\frac{1}{100}} N_0^{2(1-s)}.
\end{aligned} \tag{5.50}
$$

Now, we deal with the term (5.43d), which is the main part of the whole argument. We postponed here to illustrate the key idea. Roughly speaking, by Hölder's inequality, (5.36), and (5.37),

$$
\begin{aligned}
\Big| \int_I \int \nabla w_N \cdot \nabla v_N w_{\ll N}^2 \, \mathrm{d}x \, \mathrm{d}t \Big| \lesssim & \|\nabla w\|_{L_t^\infty L_x^2} \|\nabla v\|_{L_t^2 L_x^\infty} \|w\|_{L_{t,x}^4}^2 \\
\lesssim & N_0^{\frac{3}{2}(1-s)} \|\nabla v\|_{L_t^2 L_x^\infty}.
\end{aligned} \tag{5.51}
$$

Although we are lack of the $\|\nabla v\|_{L_t^2 L_x^\infty}$-estimate, the bilinear Strichartz estimate for $\nabla w_N w_{\ll N}$ can be introduced to lower down the derivative of $\nabla v_N$. In the view of (5.38) and (5.39), this procedure will cause the increase of $N_0$. This is allowed, since there is still $N_0^{\frac{1}{2}(1-s)}$-gap towards the energy increment $N_0^{2(1-s)}$ in (5.51).

Now, we give the concrete argument for the estimate of (5.43d). By Lemma 2.9, (5.38), and (5.39),

$$\begin{aligned}\|\nabla w_N w_{N_1}\|_{L^2_{t,x}} \lesssim & \frac{N_1}{N^{\frac{1}{2}}} N \|w_N\|_{U^2_\Delta} \|w_{N_1}\|_{U^2_\Delta} \\ \lesssim & N_1 N_1^{-\varepsilon} N^{-\frac{1}{2}} \|w_N\|_{X^1} \|w_{N_1}\|_{X^\varepsilon} \\ \lesssim & N_1 N_1^{-\varepsilon} N^{-\frac{1}{2}} N_0^{(4+\varepsilon)(1-s)}.\end{aligned} \tag{5.52}$$

Therefore, by Hölder's inequality, (5.52) and (5.33),

$$\begin{aligned}(5.43\text{d}) \lesssim & \sum_{N_1\leqslant N_2 \ll N} \Big| \int_I \int \nabla w_N \cdot \nabla v_N w_{N_1} w_{N_2} \,\mathrm{d}x\,\mathrm{d}t \Big| \\ \lesssim & \sum_{N_1\leqslant N_2 \ll N} \|\nabla w_N\|_{L^\infty_t L^2_x}^{\frac{6}{7}+10\varepsilon} \|\nabla v_N\|_{L^2_t L^\infty_x} \|\nabla w_N w_{N_1}\|_{L^2_{t,x}}^{\frac{1}{7}-10\varepsilon} \\ & \cdot \|w_{N_1}\|_{L^4_{t,x}}^{\frac{5}{7}+20\varepsilon} \|w_{N_2}\|_{L^4_{t,x}} \|w_{N_1}\|_{L^\infty_t L^2_x}^{\frac{1}{7}-10\varepsilon} \\ \lesssim & \sum_{N_1\leqslant N_2 \ll N} N_0^{(\frac{6}{7}+10\varepsilon)(1-s)} N^{\frac{1}{2}-s+\varepsilon} \|v\|_{\widetilde{Y}^s} \left[N_1 N_1^{-\varepsilon} N^{-\frac{1}{2}} N_0^{(4+\varepsilon)(1-s)}\right]^{\frac{1}{7}-10\varepsilon} \\ & \cdot N_0^{\frac{1}{4}(\frac{5}{7}+20\varepsilon)(1-s)} N_0^{\frac{1}{4}(1-s)} N_1^{-(\frac{1}{7}-10\varepsilon)} N_0^{(\frac{1}{7}-10\varepsilon)(1-s)} \\ \lesssim & N_0^{(2-20\varepsilon)(1-s)} \sum_{N_1\leqslant N_2 \ll N} N_1^{-\frac{1}{10}\varepsilon} N^{\frac{3}{7}-s+6\varepsilon}.\end{aligned}$$

Since $s > \frac{3}{7}$, we take a suitably small $\varepsilon > 0$ such that $\frac{3}{7} - s + 6\varepsilon < 0$, then we obtain that

$$(5.43\text{d}) \lesssim N_0^{-20\varepsilon(1-s)} N_0^{2(1-s)}. \tag{5.53}$$

Combining (5.44), (5.47), (5.50), and (5.53), we have

$$\begin{aligned}(5.40) \lesssim & (5.43\text{a}) + (5.43\text{b}) + (5.43\text{c}) + (5.43\text{d}) \\ \lesssim & \big(N_0^{-(1-s)} + N_0^{-\frac{1}{4}(1-s)} + N_0^{-\frac{1}{100}} + N_0^{-20\varepsilon(1-s)}\big) N_0^{2(1-s)} \\ \lesssim & N_0^{-20\varepsilon(1-s)} N_0^{2(1-s)}.\end{aligned} \tag{5.54}$$

**Estimate on** (5.41)**.** The proof for (5.41) is easier, since there is no derivative acting on $v$. However, the integration contains two $\nabla w$ terms, which already leads to the increment of $N_0^{2(1-s)}$. Therefore, we need to cover the additional $N_0$.

Heuristically, by Hölder's inequality,

$$\Big| \int_I \int \nabla w \cdot \nabla w v w \,\mathrm{d}x\,\mathrm{d}t \Big| \lesssim \|\nabla w\|_{L^\infty_t L^2_x}^2 \|v\|_{L^2_t L^\infty_x} \|w\|_{L^2_t L^\infty_x}. \tag{5.55}$$

Note that $w$ has the bounds $\|w_N\|_{l^2_N L^2_t L^\infty_x} \lesssim \|w\|_{X^{\frac{1}{2}}} \lesssim N_0^{\frac{3}{2}(1-s)}$ and $\|w\|_{L^2_t L^\infty_x} \lesssim \|w\|_{X^{\frac{1}{2}+}} \lesssim N_0^{\frac{5}{2}(1-s)+}$, from which we observe that the latter one increases the energy bound too much. Therefore, the norm $l^2_N L^2_t L^\infty_x$ is a better choice, and we need to cover a logarithmic divergence problem for (5.55).

By frequency decomposition, we have

$$
\begin{aligned}
(5.41) \lesssim & \Big| \int_I \int \nabla w \cdot \nabla w v v \, \mathrm{d}x \, \mathrm{d}t \Big| && (5.56a) \\
& + \Big| \int_I \sum_{N \in 2^{\mathbb{N}}} \int \nabla w \cdot \nabla w_{\lesssim N} v w_N \, \mathrm{d}x \, \mathrm{d}t \Big| && (5.56b) \\
& + \Big| \int_I \sum_{N \in 2^{\mathbb{N}}} \int \nabla w \cdot \nabla w v_{\gtrsim N} w_N \, \mathrm{d}x \, \mathrm{d}t \Big| && (5.56c) \\
& + \sum_{N \in 2^{\mathbb{N}}} \Big| \int_I \int \nabla w_{\gg N} \cdot \nabla w_{\gg N} v_{\ll N} w_N \, \mathrm{d}x \, \mathrm{d}t \Big|. && (5.56d)
\end{aligned}
$$

First, by Hölder's inequality and (5.33), it holds that

$$
(5.56a) \lesssim \|\nabla w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty}^2 \lesssim N_0^{-2s-1+} N_0^{2(1-s)}. \tag{5.57}
$$

By the above observation, the main task for (5.56b), (5.56c), and (5.56d) is to update the summation of $w_N$ to $l_N^2$. For (5.56b), we move the first order derivative from $w_{\lesssim N}$ to $w_N$, and use the Schur's test to update the summation. Since $s > \frac{3}{7}$ implies $1 - \frac{5}{2}s+ < -\frac{1}{100}$, by Hölder's inequality, Lemma 2.12, Lemma 2.4, (5.33), and (5.38),

$$
\begin{aligned}
(5.56b) \lesssim & \Big| \int_I \sum_{N_1 \lesssim N} \int \nabla w \cdot \nabla w_{N_1} v w_N \, \mathrm{d}x \, \mathrm{d}t \Big| \\
\lesssim & \int_I \sum_{N_1 \lesssim N} \|\nabla w\|_{L_x^2} \|\nabla w_{N_1}\|_{L_x^2} \|w_N\|_{L_x^\infty} \|v\|_{L_x^\infty} \, \mathrm{d}t \\
\lesssim & \int_I \|\nabla w\|_{L_x^2} \|v\|_{L_x^\infty} \sum_{N_1 \lesssim N} \frac{N_1}{N} \|w_{N_1}\|_{L_x^\infty} \|\nabla w_N\|_{L_x^2} \, \mathrm{d}t \\
\lesssim & \int_I \|\nabla w\|_{L_x^2}^2 \|v\|_{L_x^\infty} \|w_{N_1}\|_{l_{N_1}^2 L_x^\infty} \, \mathrm{d}t \\
\lesssim & \|\nabla w\|_{L_t^\infty L_x^2}^2 \|v\|_{L_t^2 L_x^\infty} \|w_{N_1}\|_{l_{N_1}^2 L_t^2 L_x^\infty} \\
\lesssim & N_0^{2(1-s)} N_0^{-s-\frac{1}{2}+} N_0^{\frac{3}{2}(1-s)} \lesssim N_0^{-\frac{1}{100}} N_0^{2(1-s)}.
\end{aligned} \tag{5.58}
$$

Next, we estimate (5.56c). In this case, we can transfer the additional $\varepsilon$-regularity from $w_N$ to $v_{\gtrsim N}$. Let $\varepsilon > 0$ be a absolutely small constant. Similarly by Hölder's inequality, Lemma 2.12, Lemma 2.4, (5.33), and (5.38), we obtain

$$\begin{aligned}(5.56c) \lesssim & \sum_{N\lesssim N_1} \Big|\int_I\int \nabla w\cdot\nabla w v_{N_1} w_N \,\mathrm{d}x\,\mathrm{d}t\Big| \\ \lesssim & \sum_{N\lesssim N_1} \|\nabla w\|_{L_t^\infty L_x^2}^2 \|v_{N_1}\|_{L_t^2L_x^\infty} \|w_N\|_{L_t^2L_x^\infty} \\ \lesssim & \sum_{N\lesssim N_1} \|\nabla w\|_{L_t^\infty L_x^2}^2 \frac{N^\varepsilon}{N_1^\varepsilon} \|N_1^\varepsilon v_{N_1}\|_{L_t^2L_x^\infty} \|w_N\|_{L_t^2L_x^\infty} \\ \lesssim & \|\nabla w\|_{L_t^\infty L_x^2}^2 \||\nabla|^\varepsilon v_{N_1}\|_{l^2_{N_1}L_t^2L_x^\infty} \|w_N\|_{l^2_N L_t^2L_x^\infty} \\ \lesssim & N_0^{2(1-s)} N_0^{-s-\frac12+2\varepsilon} N_0^{\frac32(1-s)} \lesssim N_0^{-\frac{1}{100}} N_0^{2(1-s)}.\end{aligned} \tag{5.59}$$

Finally, we deal with the term (5.56d), where we need to get additional regularity for summation by the bilinear Strichartz estimate. By Lemma 2.9, (5.38), and (5.39),

$$\begin{aligned}\|\nabla w_{N_1} v_{N_2}\|_{L^2_{t,x}} \lesssim & \frac{N_2}{N_1^{\frac12}} N_1 \|w_{N_1}\|_{U^2_\Delta} \|P_{N_2} v_0\|_{L_x^2} \\ \lesssim & N_2 N_1^{-\frac12} \|w\|_{X^1} \|v\|_{Z^s} \\ \lesssim & N_2 N_1^{-\frac12} N_0^{3(1-s)}.\end{aligned} \tag{5.60}$$

Note also that

$$N_2^\varepsilon \|v_{N_2}\|_{L_t^2L_x^\infty}^{1-\varepsilon} \lesssim N_2^{-\frac12+3\varepsilon-(1-\varepsilon)s} \||\nabla|^{s+\frac12-\varepsilon} v_{N_2}\|_{L_t^2L_x^\infty}^{1-\varepsilon} \lesssim N_2^{-\frac12+3\varepsilon-(1-\varepsilon)s}. \tag{5.61}$$

Since $0<\varepsilon\ll 1$, then we have

$$1+5\varepsilon-(\frac52+\varepsilon)s < -\frac{1}{100}.$$

Therefore, by Hölder's inequality, (5.60), (5.61), (5.33), and (5.38),

$$\begin{aligned}(5.56d) \lesssim & \sum_{N_2\ll N\ll N_1} \Big|\int_I\int \nabla w_{N_1}\cdot\nabla w_{N_1} w_N v_{N_2} \,\mathrm{d}x\,\mathrm{d}t\Big| \\ \lesssim & \sum_{N_2\ll N\ll N_1} \|\nabla w_{N_1}\|_{L_t^\infty L_x^2}^{2-\varepsilon} \|\nabla w_{N_1} v_{N_2}\|_{L^2_{t,x}}^\varepsilon \|w_N\|_{L_t^2L_x^\infty} \|v_{N_2}\|_{L_t^2L_x^\infty}^{1-\varepsilon} \\ \lesssim & \sum_{N_2\ll N\ll N_1} \|\nabla w_{N_1}\|_{L_t^\infty L_x^2}^{2-\varepsilon} N_2^\varepsilon N_1^{-\frac12\varepsilon} N_0^{3\varepsilon(1-s)} \|w_N\|_{L_t^2L_x^\infty} \|v_{N_2}\|_{L_t^2L_x^\infty}^{1-\varepsilon} \\ \lesssim & \sum_{N_2\ll N\ll N_1} N_1^{-\frac12\varepsilon} N_0^{(2-\varepsilon)(1-s)} N_0^{3\varepsilon(1-s)} N_0^{\frac32(1-s)} N_0^{-\frac12+3\varepsilon-(1-\varepsilon)s} \\ \lesssim & N_0^{2(1-s)} N_0^{1+5\varepsilon-(\frac52+\varepsilon)s} \lesssim N_0^{-\frac{1}{100}} N_0^{2(1-s)}.\end{aligned} \tag{5.62}$$

Therefore, combining (5.57), (5.58), (5.59), and (5.62), we have

$$\begin{aligned}(5.41) \lesssim & (5.56a)+(5.56b)+(5.56c)+(5.56d) \\ \lesssim & \big(N_0^{-2s-1+} + N_0^{-\frac{1}{100}}\big) N_0^{2(1-s)} \lesssim N_0^{-\frac{1}{100}} N_0^{2(1-s)}.\end{aligned} \tag{5.63}$$

**Estimate on** (5.42)**.** This is a simple case, where no derivative appears. By Hölder's inequality, (5.37), and (5.36),

$$\begin{aligned}(5.42) \lesssim& \Big|\int_I\int |u|^2u\big(|u|^2u-|w|^2w\big)\,\mathrm{d}x\,\mathrm{d}t\Big|\\ \lesssim& \|v\|_{L_t^2L_x^\infty}\big(\|v\|_{L^4_{t,x}}^2+\|w\|_{L^4_{t,x}}^2\big)\big(\|v\|^3_{L_t^\infty L_x^6}+\|w\|^3_{L_t^\infty L_x^6}\big)\\ \lesssim& N_0^{-s-\frac12+}N_0^{\frac12(1-s)}N_0^{3(1-s)}\lesssim N_0^{1-\frac52 s+}N_0^{2(1-s)}\lesssim N_0^{-\frac{1}{100}}N_0^{2(1-s)}.\end{aligned}$$

Then, by choosing $N_0=N_0(A)$ suitably large, and combining (5.54), (5.63), and (5.64), we have

$$\begin{aligned}\sup_{t\in I}E(t)\leqslant& E(w_0)+(5.40)+(5.41)+(5.42)\\ \leqslant& AN_0^{2(1-s)}+C(A)\cdot\big(N_0^{-20\varepsilon(1-s)}+N_0^{-\frac{1}{100}}\big)N_0^{2(1-s)}\\ \leqslant& \frac32 AN_0^{2(1-s)}.\end{aligned}\tag{5.64}$$

Then, by the standard bootstrap argument, we finish the proof of (5.29). □

5.4. **Proof of Proposition 5.1.** We first prove the global well-posedness. Since $v\in Y^s\cap Z^s(\mathbb{R})$ and $w_0\in H_x^1$, by Proposition 4.2, there exists $T_1$ depending on $\|w_0\|_{H_x^1}$ and $\|v\|_{Y^s(\mathbb{R})\cap Z^s(\mathbb{R})}$, such that $w\in C([0,T_1];H_x^1)$ solves (4.1). By Proposition 5.5, we have

$$E(w(T_1))\leqslant \sup_{t\in[0,T_1]}E(w(t))\leqslant 2AN_0^{2(1-s)}.$$

Then, we have $\|w(T_1)\|^2_{H_x^1}\leqslant 2AN_0^{2(1-s)}$, and can apply Proposition 4.2 again starting from $T_1$. Since the energy bound in (5.29) does not reply on $T$, we can extend the solution on $\mathbb{R}$ by induction, and get

$$\sup_{t\in\mathbb{R}}E(w(t))\leqslant 2AN_0^{2(1-s)}.\tag{5.65}$$

Next, we prove the scattering statement. We only consider the $t\to+\infty$ case, and it suffices to prove that

$$\Big\|\nabla\int_0^\infty e^{-is\Delta}(|u|^2u)\,\mathrm{d}x\Big\|_{L_x^2}\leqslant C(A,N_0).\tag{5.66}$$

In fact, since the global well-posedness already holds, we do not care the explicit expression of $A$ and $N_0$. Now, all the space-time norms are taken over $\mathbb{R}\times\mathbb{R}^3$. Using Lemmas 5.2 and 5.3 on $[0,\infty)$, we have for any $0\leqslant l\leqslant 1$,

$$\|w\|_{X^l}\leqslant C(A,N_0).$$

Furthermore, we clearly have

$$\|v\|_{L_t^\infty H_x^{\frac13}}+\big\||\nabla|^{\frac12}v\big\|_{L_t^2L_x^\infty}+\big\||\nabla|^{\frac56+}v\big\|_{L_t^2L_x^\infty}+\|v\|_{L_t^4L_x^6}\leqslant C(A).$$

Next, we start to prove (5.66) using the above estimates without mentioning. We split

$$\text{L.H.S. of (5.66)} \lesssim \Big\| \int_0^\infty e^{-is\Delta}(\nabla w u^2)\, \mathrm{d}s \Big\|_{L_x^2} \tag{5.67a}$$

$$+ \Big\| \int_0^\infty e^{-is\Delta}(\nabla v u^2)\, \mathrm{d}s \Big\|_{L_x^2}. \tag{5.67b}$$

The proof for the first term (5.67a) is easy. By Lemma 2.7 and Hölder's inequality,

$$\begin{aligned}(5.67\text{a}) \leqslant & C \left\| \nabla w u^2 \right\|_{L_t^2 L_x^{\frac{6}{5}}} \\ \leqslant & C \left\| \nabla w \right\|_{L_t^\infty L_x^2} \left( \|w\|_{L_t^4 L_x^6}^2 + \|v\|_{L_t^4 L_x^6}^2 \right) \leqslant C(A, N_0).\end{aligned} \tag{5.68}$$

Next, we deal with the term (5.67b). By frequency decomposition,

$$(5.67\text{b}) \lesssim \sum_{N \in 2^{\mathbb{N}}} \Big\| \int_0^\infty e^{-is\Delta}(\nabla v_N u_{\gtrsim N} u)\, \mathrm{d}s \Big\|_{L_x^2} \tag{5.69a}$$

$$+ \sum_{N \in 2^{\mathbb{N}}} \Big\| \int_0^\infty e^{-is\Delta}(\nabla v_N u_{\ll N}^2)\, \mathrm{d}s \Big\|_{L_x^2}. \tag{5.69b}$$

By Hölder's inequality and Lemma 2.4, we have

$$\begin{aligned}(5.69\text{a}) \leqslant & C \sum_{N \lesssim N_1} \| \nabla v_N u_{N_1} u \|_{L_t^1 L_x^2} \\ \leqslant & C \sum_{N \lesssim N_1} \frac{N^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \big\| |\nabla|^{\frac{1}{2}} v_N \big\|_{L_t^2 L_x^\infty} \big\| |\nabla|^{\frac{1}{2}} v_{N_1} \big\|_{L_t^2 L_x^\infty} \|u\|_{L_t^\infty L_x^2} \\ & + C \sum_{N \lesssim N_1} \frac{N^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \big\| |\nabla|^{\frac{1}{2}} v_N \big\|_{L_t^2 L_x^\infty} \big\| |\nabla|^{\frac{1}{2}} w_{N_1} \big\|_{L_{t,x}^4} \|u\|_{L_{t,x}^4} \\ \leqslant & \big\| |\nabla|^{\frac{1}{2}} v_N \big\|_{l_N^2 L_t^2 L_x^\infty} \big\| |\nabla|^{\frac{1}{2}} v_{N_1} \big\|_{l_{N_1}^2 L_t^2 L_x^\infty} \|u\|_{L_t^\infty L_x^2} \\ & + C \big\| |\nabla|^{\frac{1}{2}} v_N \big\|_{l_N^2 L_t^2 L_x^\infty} \big\| |\nabla|^{\frac{1}{2}} w_{N_1} \big\|_{l_{N_1}^2 L_{t,x}^4} \|u\|_{L_{t,x}^4} \leqslant C(A, N_0).\end{aligned} \tag{5.70}$$

Finally, we estimate (5.69b), where we need to exploit the duality structure as in the proof of Proposition 4.2. In this case, it is unnecessary to invoke the $U^p$-$V^p$ method as before for two reasons: first, we are considering the dual operator of $e^{is\Delta}$; second, we have estimate for $\big\| |\nabla|^{\frac{5}{6}+} v \big\|_{L_t^2 L_x^\infty}$ under the radial assumption. We can simply use the duality representation of the $L_x^2$-norm:

$$\begin{aligned}(5.69\text{b}) \leqslant & C \sum_{N_1 \leqslant N_2 \ll N} \Big\| \int_0^\infty e^{-is\Delta}(\nabla v_N u_{N_1} u_{N_2})\, \mathrm{d}s \Big\|_{L_x^2} \\ \leqslant & C \sum_{N_1 \leqslant N_2 \ll N} \sup_{\|g\|_{L_x^2} = 1} \int_0^\infty \langle g, e^{-is\Delta}(\nabla v_N u_{N_1} u_{N_2}) \rangle \, \mathrm{d}s \\ \leqslant & C \sum_{N_1 \leqslant N_2 \ll N} \sup_{\|g\|_{L_x^2} = 1} \int_0^\infty \int e^{is\Delta} g_{\sim N} \nabla v_N u_{N_1} u_{N_2} \, \mathrm{d}x \, \mathrm{d}s.\end{aligned} \tag{5.71}$$

By Lemma 2.9, for $N_1 \ll N$,

$$\begin{aligned}\left\|e^{is\Delta}g_{\sim N}u_{N_1}\right\|_{L^2_{t,x}} \lesssim & \frac{N_1}{N^{\frac12}}\left\|g_{\sim N}\right\|_{L^2_x}\big(\left\|P_{N_1}v_0\right\|_{L^2_x}+\left\|w_{N_1}\right\|_{U^2_\Delta}\big)\\ \lesssim & \frac{N_1^{\frac12}}{N^{\frac13}}\big(N_1^{\frac13}\left\|P_{N_1}v_0\right\|_{L^2_x}+N_1^{\frac13}\left\|w_{N_1}\right\|_{U^2_\Delta}\big)\lesssim \frac{N_1^{\frac13}}{N^{\frac16}}C(A,N_0).\end{aligned} \tag{5.72}$$

By (5.72) and Hölder's inequality, for $N_1 \leqslant N_2 \ll N$,

$$\begin{aligned}&\int_0^\infty\int e^{is\Delta}g_{\sim N}\nabla v_N u_{N_1}u_{N_2}\,\mathrm{d}x\,\mathrm{d}s\\ \leqslant & C\left\|e^{is\Delta}g_{\sim N}u_{N_1}\right\|_{L^2_{t,x}}\left\|\nabla v_N\right\|_{L^2_tL^\infty_x}\left\|u_{N_2}\right\|_{L^\infty_tL^2_x}\\ \leqslant & C(A,N_0)\frac{N_1^{\frac13}}{N^{\frac16}}\left\|\nabla v_N\right\|_{L^2_tL^\infty_x}\left\|u_{N_2}\right\|_{L^\infty_tL^2_x}\\ \leqslant & C(A,N_0)\frac{N_1^{\frac13}}{N_2^{\frac13}}N^{0-}\left\||\nabla|^{\frac56+}v_N\right\|_{L^2_tL^\infty_x}\left\||\nabla|^{\frac13}u_{N_2}\right\|_{L^\infty_tL^2_x}\leqslant C(A,N_0)\frac{N_1^{\frac13}}{N_2^{\frac13}}N^{0-}.\end{aligned} \tag{5.73}$$

Then, by (5.71) and (5.73),

$$\begin{aligned}(5.69\text{b})\leqslant & C\sum_{N_1\leqslant N_2\ll N}\sup_{\|g\|_{L^2_x}=1}\int_0^\infty\int e^{is\Delta}g_{\sim N}\nabla v_N u_{N_1}u_{N_2}\,\mathrm{d}x\,\mathrm{d}s\\ \leqslant & C(A,N_0)\sum_{N_1\leqslant N_2\ll N}\frac{N_1^{\frac13}}{N_2^{\frac13}}N^{0-}\lesssim C(A,N_0).\end{aligned} \tag{5.74}$$

By (5.70) and (5.74), we have

$$(5.67\text{b})\leqslant C\cdot\big((5.69\text{a})+(5.69\text{b})\big)\leqslant C(A,N_0). \tag{5.75}$$

(5.68) and (5.75) imply (5.66). This finishes the proof of Proposition 5.1.

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