DEFECTS IN LIQUID CRYSTAL FLOWS

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Abstract. This paper concerns the dynamical properties of topological defects in 2D flows of liquid crystals modeled by the Ginzburg-Landau approximations. The fluid is transported by a nonlocal (an averaged) velocity and is coupled with effects of the elastic stress. The defects move along the trajectories of the flow associated with this averaged velocity, that is

$$\frac{d}{dt}a_j(t) = \mathbf{u}(a_j(t), t).$$

1. Introduction

We consider the dynamical properties of topological defects in two dimensional incompressible liquid crystal flows. Because of the elastic turbulence caused by defects and its motion in liquid crystal configurations, it is mathematically difficult to make sense of the point-wise value of the fluid velocity at the defects (the velocity function may be in the VMO space but not necessarily continuous, see [11]). Here we consider an average of the velocity field $\mathbf{u}$. This nonlocal velocity resembles the well studied lagrangian average velocity in the study of classical incompressible fluids, [2, 16, 17]. More precisely we study the following system:

$$\begin{cases} \partial_t v + \mathbf{u} \wedge \mathrm{curl} v + \nabla P = \mu \Delta v - \mathrm{div}(\nabla d \otimes \nabla d) \\ \partial_t d + \mathbf{u} \cdot \nabla d = \Delta d + \frac{d(1-|d|^2)}{\varepsilon^2} \\ \mathrm{div} v = 0 \\ -\alpha \Delta \mathbf{u} + \mathbf{u} = v \end{cases} \tag{1.1}$$

in $\Omega \times \mathbb{R}^+$ where $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain. The constants $\alpha > 0$ in the last equation may be small but fixed throughout our analysis, and the viscosity $\mu > 0$ for liquid crystal flows is often pretty large. To simplify notations and presentations we will be assumed both of these constants to be one since their magnitudes do not play a role in our analysis. The curl operator $\mathrm{curl} v$ is defined to be $\partial_{x_1} v_2 - \partial_{x_2} v_1$, and the outer product $\mathbf{u} \wedge \mathrm{curl} v = (\mathbf{u}_2, -\mathbf{u}_1)\mathrm{curl} v$ is also specified in two dimensional space. The system (1.1) is complemented with the initial and boundary data

$$(v, d)|_{t=0} = (v_0, d_0), \quad (v, d)|_{\partial\Omega} = (0, g), \quad \mathbf{u}|_{\partial\Omega} = 0. \tag{1.2}$$

The parameter $\varepsilon > 0$ in (1.1) is the main concern of the analysis in this work, and we focus on the asymptotic behaviour of the Ginzburg-Landau vortices, that is the behaviour

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of the topological defects of d, as $\varepsilon \to 0$. We set $|g| = 1$ on $\partial\Omega$ with $\deg(g, \partial\Omega) = N > 0$. The initial data (v_0, d_0) satisfies

$$\|v_0\|_{L^2(\Omega)} \le 2A \tag{1.3a}$$

$$E(d_0) = \int_\Omega e_\varepsilon(d_0)dx \le \pi N \log\frac{1}{\varepsilon} + B. \tag{1.3b}$$

with two positive constants A and B, and the local energy $e_\varepsilon(d)$ is defined as

$$e_\varepsilon(d) = \frac{1}{2}|\nabla d|^2 + W(d), \quad W(d) = \frac{1}{4\varepsilon^2}(1-|d|^2)^2.$$

The system (1.1) is a Lagrangian average of liquid crystal flows [12], and the averaged velocity $\mathbf{u}$ is viewed as the macro-scale averaging of the mesoscale velocity v. We refer the interested readers to [2, 16, 17] for other models of classical fluids with averaged velocity. Keeping in mind, the trilinear operator in (1.1)

$$(\mathbf{u} \wedge \mathrm{curl} v, \mathbf{u}) = 0$$

shares the similar structure as the classical Navier-Stokes equation, see [2, 17]. For every fixed parameter $\epsilon > 0$, the global smooth solution of the system (1.1)-(1.3) could be constructed in the spirit of [12], see also [10, 13, 14] for other related models. The classical solution of the system (1.1)-(1.3) satisfies the global energy law

$$\begin{aligned} &\frac{d}{dt}\left(\|\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + 2E(d(t))\right) \\ &\quad = -\left(\|\Delta\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + 2\|\partial_t d + \mathbf{u}\cdot\nabla d\|^2_{L^2(\Omega)}\right). \end{aligned} \tag{1.4}$$

As a consequence of the initial data and the energy law (1.4), the solutions (v, d) satisfies

$$2E(d(t)) + \|\mathbf{u}(t)\|^2_{L^2(\Omega)} + \|\nabla\mathbf{u}(t)\|^2_{L^2(\Omega)} \le 2\pi N \log\frac{1}{\epsilon} + 2A + 2B, \quad \forall t \in \mathbb{R}^+. \tag{1.5}$$

In this work, we are interested in the asymptotic behaviour of the smooth functions d_ε as $\varepsilon \to 0$. The asymptotic behaviour of solutions associated with the steady Ginzburg-Landau equation had been systematically studied in [3]. When the velocity v vanishes, the function d_ε satisfies the heat flow of Ginzburg-Landau equation, and the dynamic properties of the solution d_ε of the heat flow had been considered in [8, 6] for Dirichlet boundary data and in [9] for Neumann boundary data. They proved that the speed of the vortex, in the original time scale, is of order $|\ln\varepsilon|$, and the trajectories of vortices obey an ordinary differential equation associated with the renormalized energy. One crucial estimate in the dynamical motion of vortices of nonzero degree in [3, 6, 8, 9], see Lemma 2.1 in [8] for instance, is the lower energy bound

$$\int_\Omega e_\varepsilon(d)dx \ge \pi N|\ln\varepsilon| - K \quad \text{for all} \quad t \ge 0, \tag{1.6}$$

which guarantees that the unbounded parts of the upper and the lower bounds of energy agree. Nevertheless when the averaged velocity $\mathbf{u}$ presents, the vortices are transported both by the heat flow and the macroscopic velocity $\mathbf{u}$. Due to the higher regularity of the macroscopic velocity $\mathbf{u}$, the flow trajectory associated with $\mathbf{u}$ is well-defined. However in contrast to the heat flow [8, 6], from the dynamical viewpoint, the leading term as $\varepsilon \to 0$ is the convection $\mathbf{u}\cdot\nabla d$.

Our main result is stated as follows.

Theorem 1.1. *For the initial data* $(d_0)_\varepsilon$ *which satisfies* (2.5)-(2.8), *for any* $t \in \mathbb{R}^+$, *as* $\varepsilon \to 0$,

- *The convergence*

$$d_\varepsilon(x,t) \to d(x,t) = \prod_{j=1}^{N} \frac{x - a_j(t)}{|x - a_j(t)|} e^{ih(x,t)} \tag{1.7}$$

  holds true in $H^1_{loc}(\overline{\Omega} \backslash \{a_1(t), a_2(t), \cdots, a_N(t)\})$. *Here the convergence is understood in the sense that for any sequences of* ε*'s that go to zero, there is a subsequence for which* (1.7) *is true.*
- *Away from the set* $\{a_j(t) : j = 1, \cdots, N, \quad t > 0\} \subset \Omega \times \mathbb{R}^+$, *the limit function* $(v, \mathbf{u}, d)$ *of* $(v_\varepsilon, \mathbf{u}_\varepsilon, d_\varepsilon)$ *satisfies the system*

$$\begin{cases} \partial_t d + \mathbf{u} \cdot \nabla d = \Delta d + |\nabla d| d \\ \mathrm{div} v = 0 \\ -\alpha \Delta \mathbf{u} + \mathbf{u} = v \end{cases} \tag{1.8}$$

  in the sense of distributions. Moreover the limit function $h(x,t)$ *satisfies a linear parabolic equation*

$$\partial_t h(x,t) + \mathbf{u} \cdot \nabla(\Theta(x,t) + h(x,t)) + \mathcal{R}(x,t) = \Delta h(x,t)$$

  in $(\Omega \setminus \{a_i(t), i = 1, \cdots, N\}) \times \mathbb{R}^+$ *with* $\sup_{t \geq 0} \|\nabla h\|_{L^2(\Omega)} \leq K$, *where*

$$e^{i\Theta(x,t)} = \prod_{i=1}^{N} \frac{x - a_i(t)}{|x - a_i(t)|} = \prod_{i=1}^{N} \theta(x - a_i(t))$$

  and

$$\mathcal{R}(x,t) = \frac{\partial \Theta}{\partial t} = -\sum_{i=1}^{N} \left(\prod_{i \neq j}^{N} \theta(x - a_j(t)) \right) \mathbf{u}(a_i(t), t) \cdot \nabla \theta(x - a_i(t)).$$

- *The functions* $a_j(t) \in \Omega$, $j = 1, \cdots, N$, *are Hölder continuous with Hölder exponent* $\frac{3}{4}$ *and satisfy the following ODE:*

$$\begin{cases} \frac{d}{dt} a_j(t) = \mathbf{u}(a_j(t), t) \\ a_j(0) = b_j. \end{cases} \tag{1.9}$$

Theorem 1.1 states that assume that initially there are N isolated vortices of degree one, then, in the limit, these vortices persist and move with the velocity $\mathbf{u}$. The argument of Theorem 1.1 is based on the second moment (of the local energy density) estimate and the lower bound of the local energy (1.6). These estimates enable us to conclude that there are vortices $a_j(t)$, which are separated with each other and are Hölder continuous on t, such that the scaled local energy converges, up to a subsequence; that is,

$$|\ln \varepsilon|^{-1} e_\varepsilon(d_\varepsilon) \mathrm{dx} \overset{*}{\rightharpoonup} \pi \sum_{i=1}^{N} \delta_{\{a_j(t)\}}$$

in the sense of distribution. Away from the vortices d_ε converges uniformly to a function $d(x,t) \in S^{1}$[1], and $e_\varepsilon(d_\varepsilon)$ converges to $\frac{1}{2}|\nabla d|^2$. The ODE equations of vortices (1.9) are consequences of the associated evolutions of the first moments of the local energy density More precisely the uniform convergence of d_ε away from vortices and the energy identity (3.1), with an appropriately chosen test function, yield the ordinary differential equation (1.9) satisfied by the vortices in the original time scales. Moreover the verification of the equation (1.9) also relies on the convergence of the Hopf differential

$$\omega = \left|\frac{\partial d}{\partial x_1}\right|^2 - \left|\frac{\partial d}{\partial x_2}\right|^2 - 2i\frac{\partial d}{\partial x_1}\cdot\frac{\partial d}{\partial x_2},$$

which admits a cancellation in the scale of $|\ln\varepsilon|$, where $x = (x_1, x_2) \in \mathbb{R}^2$, i is the imaginary unit, and the dot product refers to the scalar product of vectors.

We remark that the uniform estimate (2.3) implicitly implies the regularity of the flow map associated with the averaged velocity $\mathbf{u}$, and actually the flow map is almost Lipschitz with respect to the space variables, see Theorem 2 in [4]. From this viewpoint, we conclude that the vortices also satisfy

$$|a_j(t) - a_i(t)| \le K\gamma^{-1}|b_j - b_i|^\gamma \quad \text{as} \quad 1 \le i \ne j \le N$$

for all $\gamma \in (0,1)$ and $t \in \mathbb{R}^+$. Another remark concerns the limit as the parameter α approaches zero in (1.1). The limit system as $\alpha \to 0$ is the incompressible nematic liquid crystal system, and the dynamic property of defects of the limit system is out of reach of this work. Though by the theory of Di Perna - Lions [15] and recent extensions [1], one expects that the main result may valid for, at least, generic initial locations for defects.

The rest of this work is organized as follows. Section 2 focuses on the preliminary of the steady Ginzburg-Landau theories and properties of initial data. Section 3 is devoted to the proof of the main result. Throughout this work, the constant K denotes the universal positive constant which is independent of the parameter ε and may vary from line to line. The notation $K(\cdot,\cdot)$ stands for the dependence of K in terms of the argument. The convention in the summation over the repeated index is applied. The notation $G : H = \sum_{i,j=1}^2 G_{ij}H_{ij}$ is the inner product between the matrix G and the matrix H, and the notation $\alpha\cdot\beta = \sum_{i=1}^2 \alpha_i\beta_i$ stands for the inner product between two vectors α and β. Moreover the convention $(\nabla d\otimes\nabla d)_{ij}$ stands for the quantity $\sum_{k=1}^2 \partial_i d_k \partial_j d_k$.

2. Preliminary

We begin by introducing several functions. For $\theta \in \mathbb{R}$, and $\xi = (b,c) \in \mathbb{R}^2$, let

$$\xi^\perp = (-c, b), \quad \vec{n}(\theta) = (\cos\theta, \sin\theta), \quad \vec{t}(\theta) = \vec{n}(\theta)^\perp.$$

For a non-zero vector $x \in \mathbb{R}^2$, let $\theta(x)$ be the multi–valued function satisfying

$$\vec{n}(\theta(x)) = \frac{x}{|x|}, \quad \forall x \ne 0.$$

[1] S^1 stands for the unit sphere in $\mathbb{R}^2$.

Note that, locally on $\mathbb{R}^2 \setminus \{0\}$, there are smooth, single-valued representatives of $\theta(\cdot)$ and each representative satisfies

$$\nabla\theta(x) = \frac{\vec{t}(\theta(x))}{|x|} = \frac{x^\perp}{|x|^2}, \quad \forall x \neq 0. \tag{2.1}$$

Observe that a direct computation shows that $\theta(x)$ is a harmonic function as $x \neq 0$.

We recall a technical result for the existence of vortices and the local lower bound of the steady Ginzburg-Landau energy around vortices, see for instance Lemma 4.1 in [6] and also [3, 8].

Lemma 2.1. *For a fixed constant $\sigma > 0$, let $0 < \varepsilon < \min\{1, \sigma\}$ and $d : B_{2\sigma} \mapsto B_1$ be a continuously differentiable function satisfying*

$$|\nabla d| < \frac{K}{\varepsilon}, \quad deg(d, \partial B_\sigma) \neq 0, \quad |d(x)| \geq \frac{1}{2} \quad \forall |x| \in [\sigma, 2\sigma].$$

Then there is a constant K, independent of ε, such that

$$\int_{B_{2\sigma}} e_\varepsilon(d)dx \geq \pi \ln\left(\frac{\sigma}{\varepsilon}\right) - K.$$

Moreover there exists $x^* \in B_\sigma$ such that $d(x^*) = 0$ and for every $\lambda \in [\varepsilon, \sigma]$

$$\int_{B_\lambda(x^*)} e_\varepsilon(d)dx \geq \pi \ln\left(\frac{\lambda}{\varepsilon}\right) - K.$$

The gradient ∇d and the potential energy $W(d)$ satisfies the following uniform bounds, see Theorem III.2, Theorem VII.3, and Theorem IX.4 in [3] respectively.

Lemma 2.2. *Under the same condition as Lemma 2.1, up to a subsequence, one has*

$$\int_\Omega |\nabla |d_\varepsilon||^2 dx + \int_\Omega W(d_\varepsilon)dx \leq K$$

and in the weak $\star$ topology of $C(\Omega)$

$$\frac{1}{|\ln\varepsilon|}|\nabla d_\varepsilon|^2 \to 2\pi \sum_{j=1}^N \delta_{a_j}$$

$$\frac{1}{4\varepsilon^2}(|d_\varepsilon|^2 - 1)^2 \to \frac{\pi}{2} \sum_{j=1}^N \delta_{a_j}$$

as $\varepsilon \to 0$.

In terms of Lemma 2.1, the Ginzburg-Landau energy $E(d(t))$ has a lower bound

$$E(d(t)) \geq \pi N \log\frac{1}{\varepsilon} - K, \tag{2.2}$$

which, combined with the estimate (1.5), yields

$$\left(\|\mathbf{u}(t)\|^2_{L^2(\Omega)} + \|\nabla\mathbf{u}(t)\|^2_{L^2(\Omega)}\right) + \int_0^\infty \left(\|\Delta\mathbf{u}\|^2_{L^2(\Omega)} + \|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + 2\|\partial_t d + \mathbf{u}\cdot\nabla d\|^2_{L^2(\Omega)}\right) ds \\ \leq 2A + 2B + C, \tag{2.3}$$

where A, B, C are independent of ε. For the function d_ε with the estimate (1.3b), the general Ginzburg-Landau theory in [3] yields

$$\|\nabla d_\varepsilon\|_{L^p(\Omega)} \le K(p, \Omega, g, A, B, C) \quad \text{for all} \quad p \in [1, 2), \tag{2.4}$$

and there is a subsequence ε_m, which converges to 0, such that

$$d_{\varepsilon_m} \to \prod_{j=1}^{N} \frac{x - a_j(t)}{|x - a_j(t)|} e^{ih(x,t)}$$

in $L^2(\Omega) \cap H^1_{loc}(\overline{\Omega} \setminus \{a_1(t), \cdots, a_N(t)\})$ for each $t \in \mathbb{R}^+$. Moreover in the heat flow of Ginzburg-Landau theories the function $h(x, t)$ satisfies, see [3, 8]

$$\|h(t)\|_{H^1(\Omega)} \le K,$$

and the trajectory $a_j(t)$ of the vortex a_j, $j = 1, \cdots, N$, are continuous in $t \in \mathbb{R}^+$.

Remark 2.1. In view of the uniform estimate (2.4), the elastic stress term $\nabla d_\varepsilon \otimes \nabla d_\varepsilon$ is not known to be integrable uniformly in ε. From this viewpoint, in the large-time scale or in the limit as $\varepsilon \to 0$, the vortices would produce a highly singular external force in the momentum equation. Moreover in contrast with [5, 7], the lack of uniform estimates on $\nabla d \otimes \nabla d$ induces the loss of the uniform information on $\partial_t v$, which makes the convergence of the quadratic term $\mathbf{u} \wedge \mathrm{curl} v$ hard to be justified. This phenomenon would be the source of the possible turbulence of the fluid flow as $\varepsilon \to 0$.

To study the asymptotic properties of vortices, we consider the initial data $(d_0)_\varepsilon$ with the following properties:

- There is a positive constant K such that
$$|(d_0)_\varepsilon| \le 1, \quad |\nabla (d_0)_\varepsilon| \le \frac{K}{\varepsilon} \quad \text{for all} \quad x \in \mathbb{R}^2. \tag{2.5}$$
- $$(d_0)_\varepsilon \to \prod_{i=1}^{N} \frac{x - b_j}{|x - b_j|} e^{ih_0(x)} \tag{2.6}$$
weakly in $H^1_{loc}(\overline{\Omega} \setminus \{b_1, \cdots, b_N\})$ for some N distinct points $b_1, \cdots, b_N$ in Ω and $N = \deg(g, \partial\Omega) > 0$;
- $$\int_\Omega \rho(x)^2 \left(|\nabla (d_0)_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (|(d_0)_\varepsilon|^2 - 1)^2 \right) dx \le K \tag{2.7}$$
for a constant K which is independent of ε, where
$$\rho(x) = \min\{|x - b_j|, \quad j = 1, \cdots, N\};$$
- $$E((d_0)_\varepsilon) = \int_\Omega \left(|\nabla (d_0)_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (|(d_0)_\varepsilon|^2 - 1)^2 \right) dx \le \pi N |\log \varepsilon| + 2B. \tag{2.8}$$

The existence of the initial data $(d_0)_\varepsilon$ with the properties (2.5)-(2.8) could be guaranteed as in [3, 6]. We denote the distance between the initial vortices as

$$0 < 16\sigma = \min_{0 < \varepsilon \le 1} \min_{i \ne j} \{|b^i_\varepsilon - b^j_\varepsilon|, \quad \mathrm{dist}(b^i_\varepsilon, \partial\Omega)\}.$$

For convenience, we shall also assume the natural compatibility condition $(d_0)_\varepsilon = g(x)$ for all $x \in \partial\Omega$.

Due to the uniform estimate (2.3) and the assumption (2.5), the maximal principle of (1.1), see for instance Lemma 1 in [7], implies that

$$|d_\varepsilon|(x,t) \leq 1 \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}^+. \tag{2.9}$$

Moreover this uniform L^∞ estimate of d_ε further guarantees the uniform pointwise estimate on the gradient of d_ε.

Lemma 2.3. *For initial data $((v_0)_\varepsilon, (d_0)_\varepsilon)$ with the uniform estimates* (2.3), (2.9) *and the property* (2.5), *as ε is sufficiently small, there holds true*

$$|\nabla d_\varepsilon|(x,t) \leq \frac{K}{\varepsilon} \quad \textit{for all} \quad (x,t) \in \Omega \times \mathbb{R}^+. \tag{2.10}$$

Proof. In terms of the boundary condition for (v,d), we extend the functions (v,d) into $\mathbb{R}^2 \times \mathbb{R}^+$ by setting $(v,d)(x,t) = (0,e)$ if $x \in \mathbb{R}^2 \setminus \Omega$ and $t \geq 0$ with a constant vector $e \in S^1$. We can also extend the function $\mathbf{u}$ to be zero in $\mathbb{R}^2 \setminus \Omega$. Since for each fixed ε, the solution (v,d) is smooth for $t > 0$, we differentiate the equation of d in (1.1) with respect to the variable x to get

$$\partial_t \partial_i d_j - \Delta \partial_i d_j = -\partial_i(\mathbf{u} \cdot \nabla d_j) + \partial_i\Big(\frac{(1-|d|^2)d_j}{\varepsilon^2}\Big),$$

and hence in terms of Gaussan's kernel $G(x,t)$ associated with the heat equation, for any fixed point $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$ and for any $\sigma \in (0,t]$, one has

$$\begin{aligned} \partial_i d_j(t,x) &= [G(\cdot,\sigma) * \partial_i d_j(\cdot, t-\sigma)](x) \\ &\quad + \int_0^\sigma \left[G(\cdot,\tau) * \left[-\partial_i(\mathbf{u} \cdot \nabla d_j) + \partial_i\Big(\frac{(1-|d|^2)d_j}{\varepsilon^2}\Big)\right](\cdot, t-\tau)\right](x) d\tau \\ &= [\partial_i G(\cdot,\sigma) * d_j(\cdot, t-\sigma)](x) \\ &\quad + \int_0^\sigma \left[\partial_i G(\cdot,\tau) * \left[-\mathbf{u} \cdot \nabla d_j + \frac{(1-|d|^2)d_j}{\varepsilon^2}\right](\cdot, t-\tau)\right](x) d\tau. \end{aligned} \tag{2.11}$$

Notice that for any $\tau > 0$, there holds

$$\sqrt{\tau}\|\nabla G(\cdot,\tau)\|_{L^1} = \pi^{-1}\int_{\mathbb{R}^2} |y| e^{-|y|^2} dy = K.$$

Therefore Young's inequality for the convolution operator and the pointwise estimate (2.9) yield

$$\left|[\partial_i G(\cdot,\sigma) * d_j(\cdot, t-\sigma)]\right|(x) \leq \frac{K}{\sqrt{\sigma}}, \tag{2.12}$$

and

$$\begin{aligned} &\left| \int_0^\sigma \left[\partial_i G(\cdot,\tau) * \left[\frac{(1-|d|^2)d_j}{\varepsilon^2}\right](\cdot, t-\tau)\right](x) d\tau \right| \\ &\leq \frac{1}{\varepsilon^2}\int_0^\sigma \|\partial_i G\|_{L^1}(\tau) d\tau \leq \frac{K}{\varepsilon^2}\sigma^{\frac{1}{2}}. \end{aligned} \tag{2.13}$$

Moreover the uniform estimate (2.3) and the embedding $\|\mathbf{u}\|_{L^\infty(\mathbb{R}^2)} \le \|\mathbf{u}\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\Delta\mathbf{u}\|^{\frac12}_{L^2(\mathbb{R}^2)}$ give

$$\begin{aligned}
&\left|\int_0^\sigma \left[\partial_i G(\cdot,\tau) * [-\mathbf{u}\cdot\nabla d_j]\,(\cdot,t-\tau)\right](x)d\tau\right| \\
&\quad\le \int_0^\sigma \|\partial_i G\|_{L^1}(\tau)\|\mathbf{u}(t-\tau)\|_{L^\infty(\mathbb{R}^2)}|\nabla d_j(t-\tau)\|_{L^\infty(\mathbb{R}^2)}d\tau \\
&\quad\le K\left(\sup_{s\in\mathbb{R}^+}\|\nabla d_j(s)\|_{L^\infty(\mathbb{R}^2)}\right)\int_0^\sigma \frac{1}{\sqrt{\tau}}\|\mathbf{u}(t-\tau)\|^{\frac12}_{L^2}\|\Delta\mathbf{u}(t-\tau)\|^{\frac12}_{L^2}d\tau \\
&\quad\le K\left(\sup_{s\in\mathbb{R}^+}\|\nabla d_j(s)\|_{L^\infty(\mathbb{R}^2)}\right)\left(\sup_{s\in\mathbb{R}^+}\|\mathbf{u}(s)\|_{L^2}\right)^{\frac12} \\
&\qquad\times\left(\int_0^\sigma \frac{1}{\tau^{\frac23}}d\tau\right)^{\frac34}\left(\int_0^\sigma \|\Delta\mathbf{u}(t-\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\right)^{\frac14} \\
&\quad\le K\sigma^{\frac14}\left(\sup_{s\in\mathbb{R}^+}\|\nabla d_j(s)\|_{L^\infty(\mathbb{R}^2)}\right)\left(\sup_{s\in\mathbb{R}^+}\|\mathbf{u}(s)\|_{L^2}\right)^{\frac12}\left(\int_0^\infty \|\Delta\mathbf{u}(\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau\right)^{\frac14} \\
&\quad\le K(A,B,C)\sigma^{\frac14}\left(\sup_{s\in\mathbb{R}^+}\|\nabla d_j(s)\|_{L^\infty(\mathbb{R}^2)}\right).
\end{aligned} \tag{2.14}$$

Furthermore if $\sigma = t$, there holds

$$|[\partial_i G(\cdot,\sigma) * d_j(\cdot,t-\sigma)](x)| = |[G(\cdot,t) * \partial_i d_j(\cdot,0)](x)| \le \|\nabla d_0\|_{L^\infty}. \tag{2.15}$$

If $t \ge \varepsilon^2$, we set $\sigma = \varepsilon^2$ in (2.11)-(2.14) and obtain

$$|\nabla d|(t,x) \le \frac{K}{\varepsilon} + K(A,B,C)\varepsilon^{\frac12}\left(\sup_{s\in\mathbb{R}^+}\|\nabla d(s)\|_{L^\infty(\mathbb{R}^2)}\right).$$

If $t < \varepsilon^2$, we set $\sigma = t$ in (2.11) and (2.13)-(2.15) to get, using (2.5)

$$\begin{aligned}
|\nabla d|(t,x) &\le \|\nabla d_0\|_{L^\infty} + \frac{K}{\varepsilon^2}t^{\frac12} + K(A,B,C)t^{\frac14}\left(\sup_{s\in\mathbb{R}^+}\|\nabla d(s)\|_{L^\infty(\mathbb{R}^2)}\right) \\
&\le \frac{K}{\varepsilon} + K(A,B,C)\varepsilon^{\frac12}\left(\sup_{s\in\mathbb{R}^+}\|\nabla d(s)\|_{L^\infty(\mathbb{R}^2)}\right).
\end{aligned}$$

Therefore, in both cases, there holds

$$|\nabla d|(t,x) \le \frac{K}{\varepsilon} + K(A,B,C)\varepsilon^{\frac12}\left(\sup_{s\in\mathbb{R}^+}\|\nabla d(s)\|_{L^\infty(\mathbb{R}^2)}\right),$$

which yields the desired result (2.10) by letting ε be sufficiently small so that

$$K(A,B,C)\varepsilon^{\frac12} < \frac12.$$

□

From now on, we assume that ε is sufficiently small so that the pointwise estimate (2.10) holds true for all $t \in \mathbb{R}^+$.

3. Proof of Theorem 1.1

Let $\eta(x)$ be a smooth function with $\eta(x) = |\nabla\eta| = 0$ for $x \in \partial\Omega$. We multiply the second equation in (1.1) by $\eta(\partial_t d + \mathbf{u}\cdot\nabla d)$ and integrate by parts to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \eta e_\varepsilon(d)dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega \eta\left(|\nabla d|^2 + \frac{(1-|d|^2)^2}{2\varepsilon^2}\right)dx\\ &= -\int_\Omega \eta|\partial_t d + \mathbf{u}\cdot\nabla d|^2 dx + \int_\Omega \left(\nabla^2\eta : \nabla d\otimes\nabla d - \Delta\eta e_\varepsilon(d)\right)dx \\ &\quad+ \int_\Omega \mathbf{u}\cdot\nabla\eta e_\varepsilon(d)dx - \int_\Omega \eta\partial_j \mathbf{u}_k \partial_k d_i \partial_j d_i dx.\end{aligned} \tag{3.1}$$

Formally with $\eta = 1$ in (3.1), the global energy law (1.4) is established with the help of the momentum equation to cancel the last integral. In order to verify the separation and the evolution of vortices, we apply different test functions η in (3.1).

3.1. **Second momentum of the local energy.** We begin with the second momentum of the local energy $e_\varepsilon(d)$; that is the identity (3.1) with the function η being a quadratic function near vortices. For this purpose, let $\eta = \eta_\sigma : \Omega \to \mathbb{R}^+$ be a smooth function such that

$$\eta_\sigma(x) = \begin{cases} \frac{1}{2}|x-b_j|^2 & \text{if} \quad x \in B_\sigma(b_j)\\ \geq \frac{1}{4}\sigma^2 & \text{if} \quad x = \Omega_\sigma \setminus \cup_{j=1}^N B_\sigma(b_j)\\ 0 & \text{if} \quad x \in \Omega\setminus\Omega_\sigma,\end{cases}$$

where

$$\Omega_\sigma = \{x\in\Omega : \mathrm{dist}(x,\partial\Omega) > \sigma\}.$$

Note that $\eta_\sigma(x) = |\nabla\eta_\sigma|(x) = 0$ as $x\in\partial\Omega$, and $\|\nabla\eta_\sigma\|_{L^\infty(\Omega)} + \|\nabla^2\eta_\sigma\|_{L^\infty(\Omega)} \leq K$. Moreover $\nabla^2\eta_\sigma = I$ in $\cup_{j=1}^N B_\sigma(b_j)$ and hence

$$\nabla^2\eta_\sigma : \nabla d\otimes\nabla d - \Delta\eta_\sigma e_\varepsilon = -2W(d) \quad \text{in} \quad \cup_{j=1}^N B_\sigma(b_j). \tag{3.2}$$

Furthermore, for $x\in\Omega\setminus\cup_{j=1}^N B_\sigma(b_j)$, there holds

$$|\nabla^2\eta_\sigma : \nabla d\otimes\nabla d - \Delta\eta_\sigma e_\varepsilon(d)| \leq K e_\varepsilon(d). \tag{3.3}$$

The estimate of the second momentum is stated as follows.

Lemma 3.1. *Suppose that for $0\leq t\leq T$ and for $j,l = 1,\cdots k$, $j\neq l$, we have*

$$\min\{|a_j(t)-a_l(t)|, \quad dist(a_j(t),\partial\Omega)\} \geq 4\sigma.$$

Then there holds true

$$A(t) \leq A(0) + K(\sigma,A,B,C)(t+t^{\frac{1}{2}})\left(\sup_{s\geq 0}\int_\Omega e_\varepsilon(d_\varepsilon)(x,t)dx\right)$$

for all $0\leq t\leq T$, where

$$A(t) = \int_\Omega \eta_\sigma(x)e_\varepsilon(d_\varepsilon(x,t))dx$$

Proof. Using integration by parts, one has

$$\begin{aligned}\frac{d}{dt}A(t) &= \int_\Omega \eta_\sigma(x)\left[\partial_i d_\varepsilon\cdot\partial_i\frac{\partial}{\partial t}d_\varepsilon + \frac{(|d_\varepsilon|^2-1)}{\varepsilon^2}d_\varepsilon\cdot\frac{\partial}{\partial t}d_\varepsilon\right]dx\\ &= -\int_\Omega \eta_\sigma(x)\left|\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\right|^2 dx - \int_\Omega \partial_i\eta_\sigma\partial_i d_\varepsilon\cdot\frac{\partial}{\partial t}d_\varepsilon dx\\ &\quad+\int_\Omega \eta_\sigma(x)\Big(\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\Big)\cdot[\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon]dx.\end{aligned}\tag{3.4}$$

Since

$$\begin{aligned}\partial_i\eta_\sigma\partial_i d_\varepsilon\cdot\frac{\partial}{\partial t}d_\varepsilon &= \partial_i\eta_\sigma\partial_i d_\varepsilon\cdot\left[-\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon + \Delta d_\varepsilon + \frac{d_\varepsilon}{\varepsilon^2}(1-|d_\varepsilon|^2)\right]\\ &= \partial_i\eta_\sigma\Big[-\partial_i d_\varepsilon(\mathbf{u}_\varepsilon)_j\partial_j d_\varepsilon + \partial_j(\partial_i d_\varepsilon\partial_j d_\varepsilon) - \frac{1}{2}\partial_i|\partial_j d_\varepsilon|^2 - \partial_i\frac{(1-|d_\varepsilon|^2)^2}{4\varepsilon^2}\Big],\end{aligned}$$

an integration of both sides of the identity above by parts gives, using $|\nabla\eta_\sigma| = 0$ on $\partial\Omega$

$$\begin{aligned}-\int_\Omega \partial_i\eta_\sigma\partial_i d_\varepsilon\cdot\frac{\partial}{\partial t}d_\varepsilon dx &= \int_\Omega \partial_i\eta_\sigma(\mathbf{u}_\varepsilon)_j\partial_i d_\varepsilon\cdot\partial_j d_\varepsilon dx - \int_\Omega \Delta\eta_\sigma e_\varepsilon(d_\varepsilon)dx\\ &\quad+\int_\Omega \partial_i\partial_j\eta_\sigma\partial_i d_\varepsilon\cdot\partial_j d_\varepsilon dx.\end{aligned}\tag{3.5}$$

Adding (3.4) and (3.5), we get

$$\begin{aligned}\frac{d}{dt}A(t) &= -\int_\Omega \eta_\sigma(x)\left|\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\right|^2 dx + \int_\Omega \partial_i\eta_\sigma(\mathbf{u}_\varepsilon)_j\partial_i d_\varepsilon\cdot\partial_j d_\varepsilon dx\\ &\quad-\int_\Omega \Delta\eta_\sigma e_\varepsilon(d_\varepsilon)dx + \int_\Omega \partial_i\partial_j\eta_\sigma\partial_i d_\varepsilon\cdot\partial_j d_\varepsilon dx\\ &\quad+\int_\Omega \eta_\sigma(x)\Big(\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\Big)\cdot[\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon]dx.\end{aligned}\tag{3.6}$$

The identity (3.2) implies that if $x\in\cup_{l=1}^N B_\sigma(b_l)$, one has

$$\partial_i\partial_j\eta_\sigma\partial_i d_\varepsilon\partial_j d_\varepsilon - \Delta\eta_\sigma e_\varepsilon(d_\varepsilon)\le 0.$$

The estimate (3.3) tells that if $x\in\Omega\setminus\cup_{j=1}^N B_\sigma(b_j)$, then we have

$$|\partial_i\partial_j\eta_\sigma\partial_i d_\varepsilon\partial_j d_\varepsilon - \Delta\eta_\sigma e_\varepsilon(d_\varepsilon)|\le K(\sigma)e_\varepsilon(d_\varepsilon).$$

Moreover the Cauchy-Swartz inequality implies

$$\begin{aligned}&\left|\int_\Omega \eta_\sigma(x)\Big(\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\Big)\cdot[\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon]dx\right|\\ &\quad\le \frac{1}{2}\int_\Omega \eta_\sigma(x)\left|\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\right|^2 dx + K\int_\Omega \eta_\sigma(x)|\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon|^2 dx\\ &\quad\le \frac{1}{2}\int_\Omega \eta_\sigma(x)\left|\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\right|^2 dx + K\|\mathbf{u}_\varepsilon\|_{L^\infty(\Omega)}^2\int_\Omega \eta_\sigma(x)e_\varepsilon(d_\varepsilon)dx.\end{aligned}$$

Therefore we deduce from (3.6) that

$$\frac{d}{dt}\int_\Omega \eta_\sigma e_\varepsilon(d_\varepsilon)dx \le -\frac{1}{2}\int_\Omega \eta_\sigma(x)\left|\frac{\partial}{\partial t}d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\right|^2 dx + K(\sigma)(1+\|\mathbf{u}_\varepsilon\|^2_{L^\infty(\Omega)})\int_\Omega e_\varepsilon(d_\varepsilon)dx$$
$$\le K(\sigma)(1+\|\mathbf{u}_\varepsilon\|^2_{L^\infty(\Omega)})\int_\Omega e_\varepsilon(d_\varepsilon)dx,$$

which, after integrating over t and using $\|\mathbf{u}\|_{L^\infty} \le \|\mathbf{u}\|^{\frac{1}{2}}_{L^2}\|\Delta\mathbf{u}\|^{\frac{1}{2}}_{L^2}$, further gives

$$A(t) \le A(0) + K(\sigma)\left(\int_0^t (1+\|\mathbf{u}_\varepsilon\|^2_{L^\infty(\Omega)})ds\right)\left(\sup_{s\ge 0}\int_\Omega e_\varepsilon(d_\varepsilon)(x,t)dx\right)$$
$$\le A(0) + K(\sigma)\left(t + t^{\frac{1}{2}}\Big(\sup_{s\in\mathbb{R}^+}\|\mathbf{u}_\varepsilon\|_{L^2}\Big)\Big(\int_0^t \|\Delta\mathbf{u}_\varepsilon\|^2_{L^2}dt\Big)^{\frac{1}{2}}\right)\left(\sup_{s\ge 0}\int_\Omega e_\varepsilon(d_\varepsilon)(x,t)dx\right)$$
$$\le A(0) + K(\sigma,A,B,C)(t+t^{\frac{1}{2}})\left(\sup_{s\ge 0}\int_\Omega e_\varepsilon(d_\varepsilon)(x,t)dx\right)$$

as desired. □

The following lemma concerns the L^2 integrability of the gradient of h in Ω.

Lemma 3.2. *Suppose that $|d| \ge \frac{1}{2}$ on $\tilde{\Omega}_{\sqrt{\varepsilon}} = \Omega_{\sqrt{\varepsilon}} \setminus \cup_{j=1}^N B_{\sqrt{\varepsilon}}(b_j)$ with $\varepsilon \le \sigma^2$, and that there is a constant K, independent of ε, satisfying*

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} e_\varepsilon(d)dx \le \frac{1}{2}\pi N|\ln\varepsilon| + K, \tag{3.7}$$

$$\int_{\partial B_{\sqrt{\varepsilon}}(b_j)} e_\varepsilon(d)d\mathcal{H}^1(x) \le \frac{K}{\sqrt{\varepsilon}}, \quad 1\le j\le N, \tag{3.8}$$

and

$$\int_{\partial\Omega_{\sqrt{\varepsilon}}} e_\varepsilon(d)d\mathcal{H}^1(x) \le K. \tag{3.9}$$

Then, there is a single-valued, smooth function $h(x)$ defined on $\tilde{\Omega}_\sigma$ such that

$$d(x) = |d(x)|\prod_{j=1}^N \frac{x-b_j}{|x-b_j|}e^{ih(x)} \tag{3.10}$$

and

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} |\nabla h|^2 dx \le K$$

with a constant K depending only on the boundary data g.

Proof. Since $|d| \ge \frac{1}{2}$ on $\Omega_{\sqrt{\varepsilon}}$, the definition of $\Theta(x)$ implies that there is a single-valued, smooth function $h(x)$ defined on Ω such that the polar form (3.10) holds true.

By (3.7), (3.10) and the orthogonality between $ie^{if(x)}$ and $e^{if(x)}$, there holds

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} |d|^2[\frac{1}{2}|\nabla\Theta(x)|^2 + \frac{1}{2}|\nabla h(x)|^2 + \nabla h(x)\cdot\nabla\Theta(x)]dx \le \frac{1}{2}\pi N|\ln\varepsilon| + K. \tag{3.11}$$

Since $\Theta(x)$ is harmonic in $\tilde{\Omega}_{\sqrt{\varepsilon}}$, we use the integration by parts to get

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} \nabla h \cdot \nabla \Theta dx = \int_{\partial \Omega_{\sqrt{\varepsilon}}} h \nabla \Theta \cdot \vec{n} + \sum_{i=1}^{N} \int_{\partial B_{\sqrt{\varepsilon}}(b_i)} h \nabla \Theta \cdot \vec{n}^i,$$

where $\vec{n}$ and $\vec{n}^i$ are, respectively, the outward unit normal vectors of $\partial\Omega$ and $\partial B_{\sqrt{\varepsilon}}(b_i)$. The definition of Θ yields

$$\int_{\partial B_{\sqrt{\varepsilon}}(b_i)} \nabla \Theta \cdot \vec{n}^i d\mathcal{H}^1(x) = 0, \quad \forall\, 1 \le i \le N,$$

and hence, for any λ,

$$\begin{aligned} \left| \int_{\partial B_{\sqrt{\varepsilon}}(b_i)} h \nabla \Theta \cdot \vec{n}^i \right| &= \left| \int_{\partial B_{\sqrt{\varepsilon}}(b_i)} [h - \lambda] \nabla \Theta \cdot \vec{n}^i \right| \\ &\le K\varepsilon \sup_{\partial B_{\sqrt{\varepsilon}}(b_i)} |\nabla \Theta \cdot \vec{n}^i| \sup_{\partial B_{\sqrt{\varepsilon}}(b_i)} |h - \lambda|. \end{aligned}$$

For any fixed i, we choose

$$\lambda = \frac{1}{2\pi\sqrt{\varepsilon}} \int_{\partial B_{\sqrt{\varepsilon}}(b_i)} h d\mathcal{H}^1(x),$$

and then on $\partial B_{\sqrt{\varepsilon}}(b_i)$, there holds, using (3.8) and (3.10),

$$\begin{aligned} |h - \lambda| &\le K \int_{\partial B_{\sqrt{\varepsilon}}(b_i)} |\nabla h| d\mathcal{H}^1(x) \\ &\le K |\partial B_{\sqrt{\varepsilon}}(b_i)|^{\frac{1}{2}} \left(\int_{\partial B_{\sqrt{\varepsilon}}(b_i)} |\nabla h|^2 d\mathcal{H}^1(x) \right)^{\frac{1}{2}} \le K. \end{aligned}$$

We observe that $\vec{n}^i = -\frac{x - b_i}{|x - b_i|}$, and it follows from (2.1)

$$\nabla \Theta(x) \cdot \vec{n}^i(x) = -\sum_{j=1}^{N} \frac{\vec{t}(\theta(x - b_j)) \cdot \theta(x - b_i)}{|x - b_j|}.$$

Therefore, on $\partial B_{\sqrt{\varepsilon}}(b_i)$,

$$\left| \nabla \Theta(x) \cdot \vec{n}^i(x) \right| \le \sum_{i \ne j}^{N} \frac{1}{|x - b_j|} \le \frac{K}{\sigma},$$

and hence

$$\left| \sum_{i=1}^{N} \int_{\partial B_{\sqrt{\varepsilon}}(b_i)} h \nabla \Theta \cdot \vec{n}^i \right| \le K. \tag{3.12}$$

A similar argument as (3.12), with (3.8) replaced by (3.9), yields

$$\left| \int_{\partial \Omega_{\sqrt{\varepsilon}}} h \nabla \Theta \cdot \vec{n} \right| \le K.$$

Therefore

$$\left|\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} \nabla h \cdot \nabla \Theta(x) dx\right| \le K, \tag{3.13}$$

with a constant K depending only on g and σ.

Moreover the definition of Θ yields

$$\begin{aligned}\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} \frac{1}{2}|\nabla \Theta|^2 dx &\ge \sum_{i=1}^N \int_{\sqrt{\varepsilon}}^{\sigma} \int_{\partial B_\tau(b_i)} \frac{1}{2}|\nabla \theta(x-b_i)|^2 d\mathcal{H}^1(x) d\tau \\ &= \sum_{i=1}^N \int_{\sqrt{\varepsilon}}^{\sigma} \frac{\pi}{\tau} d\tau = \frac{1}{2}\pi N |\ln \varepsilon| - K,\end{aligned}$$

where $K = \pi N |\ln \sigma|$. This, combined with (3.11) and (3.13), gives,

$$\begin{aligned}\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} \frac{1}{8}|\nabla h|^2 dx &\le \int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} |d|^2 \Big[\frac{1}{2}|\nabla \Theta|^2 + \frac{1}{2}|\nabla h|^2 + \nabla h \cdot \nabla \Theta\Big] dx + K - \int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} \frac{1}{2}|d|^2 |\nabla \Theta|^2 dx \\ &\le K + \frac{1}{2}\pi N |\ln \varepsilon| - \frac{1}{2}\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} |d|^2 |\nabla \Theta|^2 dx \\ &\le K + \frac{1}{2}\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} (1-|d|^2)|\nabla \Theta|^2 dx.\end{aligned}$$

The definition of $\Theta(x)$ implies

$$|\nabla \Theta(x)| \le \frac{K}{\sqrt{\varepsilon}}, \quad x \in \tilde{\Omega}_{\sqrt{\varepsilon}},$$

we conclude by using (3.7) that

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} (1-|d|^2)|\nabla \Theta|^2 dx \le \frac{K}{\varepsilon}\left(\int_\Omega W(d) dx\right)^{\frac{1}{2}} \le K.$$

□

Since the estimate on ∇h in Lemma 3.2 is independent of ε, it follows that $\|\nabla h\|_{L^2(\Omega)} \le K$. Next we turn to the localization of the energy.

Lemma 3.3. *There are constants $t_0 > 0$, K, and functions*

$$a_\varepsilon^i : [0, t_0] \to B_{\sigma/2}(b_\varepsilon^i), \quad \forall i = 1, \cdots, N,$$

with $a_\varepsilon^i(0) = b_\varepsilon^i$ such that $d_\varepsilon(a_\varepsilon^i(t), t) = 0$. Moreover for any $\varepsilon \in (0, 1]$, $t \in [0, t_0]$, $\lambda \in [\varepsilon, \sigma]$,

$$\mu_\varepsilon(t)\Big(B_\lambda(a_\varepsilon^i(t))\Big) \ge \pi \ln\left(\frac{\lambda}{\varepsilon}\right) - K, \quad \forall i = 1, \cdots, N, \tag{3.14}$$

where

$$\mu_\varepsilon(t) = e_\varepsilon(d_\varepsilon)(x, t) dx.$$

Proof. For each fixed ε, the continuity of d_ε, the property (2.6), and the properties of the topological degree imply that

$$\deg(d_\varepsilon(\cdot, t); \partial B_\sigma(b_\varepsilon^i)) = 1 \quad \forall i = 1, \cdots, N,$$

and

$$|d_\varepsilon(x,t)| \geq \frac{1}{2} \quad \text{as} \quad |x - a_\varepsilon^i(t)| \in (\sigma/2, \sigma) \quad \text{and} \quad \forall i = 1, \cdots, N.$$

This, combined with the pointwise gradient estimate in Lemma 2.3 and Lemma 2.1, further imply that for every $\varepsilon \in (0,1]$ there exists

$$a_\varepsilon^i(t) \in B_\sigma(b_\varepsilon^i)$$

such that $d_\varepsilon(a_\varepsilon^i(t), t) = 0$ and (3.14) holds true.

The lower bound estimate (3.14) with $\lambda = \frac{\sigma}{2}$, combined with the global energy estimate (1.5), yields

$$\mu_\varepsilon(t)(\tilde{\Omega}_\varepsilon) \leq \mu_\varepsilon(t)\Big(\Big\{x : |x - a_\varepsilon^i(t)| \geq \frac{1}{2}\sigma, \quad \forall i = 1, \cdots, N\Big\}\Big) \leq K.$$

Moreover the second momentum estimate in Lemma 3.1 and the energy estimate (1.5) give

$$\begin{aligned}\int_\Omega \eta e_\varepsilon(d_\varepsilon)dx &\leq \int_\Omega \eta e_\varepsilon(d_0)dx + K(\sigma, A, B, C)(t + t^{\frac{1}{2}}) \sup_{s\geq 0} \int_\Omega e_\varepsilon(d_\varepsilon)dx \\ &\leq \int_\Omega \eta e_\varepsilon(d_0)dx + K_0|\ln\varepsilon|(t^{\frac{1}{2}} + t).\end{aligned}$$

The property (2.7) of the initial data d_0 gives

$$\lim_{\varepsilon\to 0} |\ln\varepsilon|^{-1} \int_\Omega \eta e_\varepsilon(d_0)dx = 0,$$

and hence

$$\int_\Omega \eta e_\varepsilon(d_\varepsilon)(x,t)dx \leq [K(\varepsilon) + K_0(t + t^{\frac{1}{2}})]|\ln\varepsilon| \quad \text{with} \quad \lim_{\varepsilon\to 0} K(\varepsilon) = 0. \tag{3.15}$$

We now define

$$t_0 = \sup\{t \geq 0 : a_\varepsilon^i(t) \in B_{\sigma/2}(b_\varepsilon^i) \quad i = 1, \cdots, N\}.$$

The continuity of d_ε implies that $t_0 > 0$ and at t_0 there exists some $i_0 \in \{1, \cdots, M\}$ such that

$$|a_\varepsilon^{i_0}(t_0) - b_\varepsilon^{i_0}| = \frac{\sigma}{2}.$$

For this i_0, we apply Lemma 3.1. Indeed we note that

$$\eta \geq K_1 := \frac{\sigma^2}{32} \quad \forall x \in B_{\sigma/4}(a_\varepsilon^{i_0}(t_0)),$$

and hence the lower bound estimate (3.14) with $\lambda = \sigma/4$ gives

$$\int_\Omega \eta e_\varepsilon(d_\varepsilon)(x,t_0)dx \geq K_1\mu_\varepsilon(t_0)\Big(B_{\sigma/4}(a_\varepsilon^{i_0}(t_0))\Big) \geq K_2|\ln\varepsilon| - K_3.$$

This, combined with (3.15), implies that for all $0 \leq t \leq t_0$, there holds

$$[K(\varepsilon) + K_0(t + t^{\frac{1}{2}})]|\ln\varepsilon| \geq K_2|\ln\varepsilon| - K_3,$$

which suggests the definition of t_0 as

$$t_0 = \min\left\{\frac{K_2}{2K_0}, \min\left\{1, \left(\frac{K_2}{2K_0}\right)^2\right\}\right\}.$$

□

Let t_0 be as in the previous lemma. By a diagonalization argument, up to a subsequence, we set

$$a^i(t) := \lim_{\varepsilon\to 0} a^i_\varepsilon(t) \quad \forall i = 1, \cdots, N.$$

Denote

$$\nu_\varepsilon(t) = |\ln\varepsilon|^{-1} e_\varepsilon(d_\varepsilon)(x,t)\mathrm{dx}.$$

The energy law (1.5) implies that $\nu_\varepsilon(t)$ is a bounded Radon measure for $t \in \mathbb{R}^+$ and Lemma 3.3 implies that

$$\nu_\varepsilon(t) \xrightarrow{*} \pi \sum_{i=1}^N \delta_{a^i(t)}, \quad t \in [0, t_0]$$

in the sense of Radon measures.

The next lemma concerns the regularity of the trajectories of vortices.

Lemma 3.4. *For every $i \in \{1, \cdots, N\}$, $a^i(\cdot)$ is a Hölder continuous function with the Hölder exponent $\frac{1}{4}$ on $[0, t_0]$. In particular $a^i_\varepsilon(t)$ converges to a^i uniformly on $[0, t_0]$.*

Proof. For any fixed i and for any $0 < s < t < t_0$ with $t - s$ being sufficiently small, there is a smooth function ϕ with $\mathrm{supp}\phi(x) \subset B_\sigma(b^i)$ such that

$$\phi(a^i(t)) = 2, \quad \phi(a^i(s)) = 1, \quad \|\nabla\phi\|_{L^\infty} = |a^i(t) - a^i(s)|^{-1}.$$

As a consequence, the function $\phi(x)$ is bounded by

$$\|\phi\|_{L^\infty} \le \sigma \|\nabla\phi\|_{L^\infty} \le \sigma |a^i(t) - a^i(s)|^{-1}.$$

Moreover for any $t \in [0, t_0]$,

$$\phi^2(a^i(t)) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_\Omega \phi^2(x) d\nu_\varepsilon(t). \tag{3.16}$$

The definition of $\nu_\varepsilon(t)$ gives

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \phi^2 d\nu_\varepsilon(t) &= -|\ln\varepsilon|^{-1}\int_\Omega \phi^2 |\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon|^2 dx - |\ln\varepsilon|^{-1}\int_\Omega \partial_j\phi^2 \partial_t d_\varepsilon \cdot \partial_j d_\varepsilon dx \\
&\quad + |\ln\varepsilon|^{-1}\int_\Omega \phi^2(\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot(\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)dx \\
&\le -\frac{1}{2}|\ln\varepsilon|^{-1}\int_\Omega \phi^2|\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon|^2 dx \\
&\quad + K|\ln\varepsilon|^{-1}\|\nabla\phi\|^2_{L^\infty}\int_\Omega |\partial_j d_\varepsilon|^2 dx \\
&\quad + K|\ln\varepsilon|^{-1}\|\phi\|^2_{L^\infty}\|\mathbf{u}_\varepsilon\|^2_{L^\infty(\Omega)}\int_\Omega |\partial_j d_\varepsilon|^2 dx,
\end{aligned}$$

which yields, as $0 \le s \le t$

$$\begin{aligned}
&\int_\Omega \phi^2 d\nu_\varepsilon(t) - \int_\Omega \phi^2 d\nu_\varepsilon(s) \\
&\quad \le K\|\nabla\phi\|_{L^\infty}^2 (t-s) \left(\sup_{\tau\ge 0} |\ln\varepsilon|^{-1} \int_\Omega |\nabla d_\varepsilon|^2 dx \right) \\
&\qquad + K\|\phi\|_{L^\infty}^2 \left(\int_s^t \|\mathbf{u}_\varepsilon\|_{L^\infty(\Omega)}^2 d\tau \right) \left(\sup_{\tau\ge 0} |\ln\varepsilon|^{-1} \int_\Omega |\nabla d_\varepsilon|^2 dx \right).
\end{aligned}$$

The global energy estimate (1.5) implies

$$\int_\Omega |\nabla d_\varepsilon|^2 dx \le K(|\ln\varepsilon| + 1), \forall t \ge 0$$

and the uniform estimate (2.3) yields

$$\sup_{\tau\ge 0} \|\mathbf{u}\|_{L^2(\Omega)}^2(\tau) + \int_0^\infty \|\Delta\mathbf{u}\|_{L^2(\Omega)}^2 d\tau \le K.$$

Using $\|\mathbf{u}\|_{L^\infty}^2 \le \|\mathbf{u}\|_{L^2}\|\Delta\mathbf{u}\|_{L^2}$, we have

$$\sup_{\tau\ge 0} |\ln\varepsilon|^{-1} \int_\Omega |\nabla d_\varepsilon|^2 dx + (t-s)^{-\frac{1}{2}} \int_s^t |\mathbf{u}_\varepsilon|_{L^\infty(\Omega)}^2 dt \le K.$$

Therefore,

$$\int_\Omega \phi^2 d\nu_\varepsilon(t) - \int_\Omega \phi^2 d\nu_\varepsilon(s) \le K\|\nabla\phi\|_{L^\infty}^2 (t-s) + K\|\phi\|_{L^\infty}^2 \sqrt{t-s},$$

which, combined with (3.16), yields

$$\phi^2(a^i(t)) - \phi^2(a^i(s)) \le K\|\nabla\phi\|_{L^\infty}^2 (t-s) + K\|\phi\|_{L^\infty}^2 \sqrt{t-s} \tag{3.17}$$

for all $0 \le s \le t \le t_0$. The Hölder continuity of $a^i(t)$ with Hölder exponent $\frac{1}{4}$ follows from the definition of ϕ and (3.17). The uniform convergence of a_ε^i follows from the Hölder continuity of $a^i(t)$ on $[0, t_0]$. □

3.2. **Hopf differential.** Recall that in the introduction, we introduce the Hopf differential

$$\omega(d) = \left|\frac{\partial d}{\partial x_1}\right|^2 - \left|\frac{\partial d}{\partial x_2}\right|^2 - 2i\frac{\partial d}{\partial x_1} \cdot \frac{\partial d}{\partial x_2},$$

and the potential function

$$W(d) = \frac{1}{4\varepsilon^2}(1 - |d|^2)^2.$$

A straightforward computation shows that any solution of

$$\partial_t d + \mathbf{u} \cdot \nabla d = \Delta d + \frac{1}{\varepsilon^2}(1 - |d|^2)d$$

satisfies

$$\frac{\partial\omega(d)}{\partial\overline{z}} = \frac{\partial}{\partial z}\Big(2W(d)\Big) + 2\frac{\partial d}{\partial z}(\partial_t d + \mathbf{u} \cdot \nabla d), \tag{3.18}$$

where, as usual,

$$\frac{\partial}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

In terms of Lemma 2.2, for any given time $t \in \mathbb{R}^+$, the limit of the scaled measure $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$ is a sum of Dirac measures with concentration on the locations of vortices. However the differential identity (3.18) implies that the measure $\lim_{\varepsilon\to 0}|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$ actually vanishes.

Lemma 3.5. *Under the same conditions as in Theorem 1.1, the scaled measure* $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$ *converges to* 0 *in the sense of Radon measures. In other words, for any* $\phi \in C(\Omega\times\mathbb{R}^+)$*,*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\times\mathbb{R}^+}|\ln\varepsilon|^{-1}\omega(d_\varepsilon)\phi(x,t)dxdt = 0.$$

Proof. Lemma 2.2 implies that the limit of $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$ is a sum of Dirac measure with concentration on the locations of vortices $\{a_1(t),\cdots,a_N(t)\}$ for each fixed time. Since these points are separated with each other, it is enough to consider one point, says $a_1(t)$.

At $a_1(t)$, we choose a smooth spatial cut-off function $\phi(x)\in C_0^\infty(B_\sigma(a_1(t)))$ with $\phi(x) = 1$ as $x\in B_{\sigma/2}(a_1(t))$. For any smooth function $\psi(t)\in C_0(\mathbb{R}^+)$, we multiply the identity (3.18) by $x_1\phi(x)\psi(t)$ and integrate on $\mathbb{R}^2\times\mathbb{R}^+$ to obtain

$$\begin{aligned}\int_0^\infty\int_{B_\sigma(a_1(t))}&\frac{\partial\omega(d_\varepsilon)}{\partial\overline{z}}x_1\phi(x)\psi(t)dxdt\\ &=\int_0^\infty\int_{B_\sigma(a_1(t))}\frac{\partial}{\partial z}\Big(2W(d_\varepsilon)\Big)x_1\phi(x)\psi(t)dxdt\\ &\quad+2\int_0^\infty\int_{B_\sigma(a_1(t))}\frac{\partial d_\varepsilon}{\partial z}(\partial_t d_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)x_1\phi(x)\psi(t)dxdt.\end{aligned}\tag{3.19}$$

Integration by parts gives

$$\begin{aligned}&\int_0^\infty\int_{B_\sigma(a_1(t))}\frac{\partial\omega(d_\varepsilon)}{\partial\overline{z}}x_1\phi(x)\psi(t)dxdt\\ &=-\int_0^\infty\int_{B_\sigma(a_1(t))}\omega(d_\varepsilon)\frac{\partial(x_1\phi(x)\psi(t))}{\partial\overline{z}}dxdt\\ &=-\frac{1}{2}\int_0^\infty\int_{B_\sigma(a_1(t))}\omega(d_\varepsilon)\phi(x)\psi(t))dxdt-\int_0^\infty\int_{B_\sigma(a_1(t))}\omega(d_\varepsilon)\frac{\partial\phi(x)}{\partial\overline{z}}(x_1\psi(t))dxdt\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\int_0^\infty\int_{B_\sigma(a_1(t))}\frac{\partial}{\partial z}\Big(2W(d_\varepsilon)\Big)x_1\phi(x)\psi(t)dxdt\\ &=-\int_0^\infty\int_{B_\sigma(a_1(t))}\Big(2W(d_\varepsilon)\Big)\frac{\partial(x_1\phi(x)\psi(t))}{\partial z}dxdt\\ &=-\frac{1}{2}\int_0^\infty\int_{B_\sigma(a_1(t))}\Big(2W(d_\varepsilon)\Big)\phi(x)\psi(t)dxdt-\int_0^\infty\int_{B_\sigma(a_1(t))}\Big(2W(d_\varepsilon)\Big)x_1\psi(t)\frac{\partial\phi(x)}{\partial z}dxdt.\end{aligned}\tag{3.21}$$

The identity (3.21), combined with the uniform bound $\int_\Omega W(d_\varepsilon)dx \leq K$ in Lemma 2.2, implies that

$$|\ln\varepsilon|^{-1}\left|\int_0^\infty\int_{B_\sigma(a_1(t))}\frac{\partial}{\partial z}\Big(2W(d_\varepsilon)\Big)x_1\phi(x)\psi(t)dxdt\right|\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Since the limit of $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$ is a Dirac measure at locations of vortices and $\frac{\partial\phi(x)}{\partial\bar z}$ vanishes at $B_{\sigma/2}(a_1(t))$, we have

$$|\ln\varepsilon|^{-1}\int_0^\infty\int_{B_\sigma(a_1(t))}\omega(d_\varepsilon)\frac{\partial\phi(x)}{\partial\bar z}(x_1\psi(t))dxdt\to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Moreover the uniform estimate (2.3) implies

$$\begin{aligned}
&|\ln\varepsilon|^{-1}\left|\int_0^\infty\int_{B_\sigma(a_1(t))}\frac{\partial d_\varepsilon}{\partial z}(\partial_t d_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)x_1\phi(x)\psi(t)dxdt\right|\\
&\leq K|\ln\varepsilon|^{-\frac12}\left(\sup_{t\in\mathbb{R}^+}|\ln\varepsilon|^{-1}\int_\Omega|\nabla d_\varepsilon(t)|^2dx\right)^{\frac12}\left(\int_0^\infty\int_{\mathbb{R}^2}|\partial_t d_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon|^2dxdt\right)^{\frac12}\\
&\leq K|\ln\varepsilon|^{-\frac12}\to 0 \quad \text{as} \quad \varepsilon\to 0.
\end{aligned}$$

Therefore the identity (3.19), dividing by $|\ln\varepsilon|$ and using (3.20)-(3.21), implies

$$|\ln\varepsilon|^{-1}\int_0^\infty\int_{B_\sigma(a_1(t))}\omega(d_\varepsilon)\phi(x)\psi(t))dxdt\to 0 \quad \text{as} \quad \varepsilon\to 0$$

as claimed. □

In view of Lemma 2.2, the measures

$$\nu_{ij}(d_\varepsilon)=|\ln\varepsilon|^{-1}\partial_i d_\varepsilon\cdot\partial_j d_\varepsilon \mathrm{dx}$$

are well defined in the sense of Radon measures, and their limits are actually Dirac measures concentrated on the locations of vortices. Moreover Lemma 3.5 implies that the matrix with the $ij-$th entry

$$\nu_{ij}=\lim_{\varepsilon\to 0}\nu_{ij}(d_\varepsilon)$$

is actually diagonal and $\nu_{11}=\nu_{22}$. In other words, the matrix $(\nu_{ij})=\nu I$, where I is the identity matrix.

Remark 3.1. A similar identity as (3.18) for Hopf differential also hold true for the steady and parabolic Ginzburg-Landau equations. Therefore the statement in Lemma 3.5 about the matrix of Radon measures $\nu_{ij}(d_\varepsilon)$ is still true for the steady and parabolic Ginzburg-Landau equations.

3.3. **First momentum of energy and ODE** (1.9)**.** We calculate the first momentum of $e_\varepsilon(d_\varepsilon)$ near the vortices b_i. In fact we have

$$\begin{aligned}&\frac{d}{dt}\int_{B_r(b_i)} x e_\varepsilon(d_\varepsilon)\phi(x)dx\\ &= -\int_{B_r(b_i)} x|\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon|^2\phi(x)dx + \int_{B_r(b_i)} x(\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot(\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\phi(x)dx\\ &\quad -\int_{B_r(b_i)}\partial_t d_\varepsilon\cdot\partial_l d_\varepsilon\partial_l(x\phi(x))dx + \int_{\partial B_r(b_i)} x\frac{\partial d_\varepsilon}{\partial\vec{n}}\cdot\partial_t d_\varepsilon\phi(x)dx,\end{aligned} \tag{3.22}$$

where $\phi(x)$ is a positive smooth function in $\mathbb{R}^2$. Next we choose $\phi(x)\in C_0^\infty(B_r(b_i))$ with $\phi(x)=1$ as $x\in B_{r/2}(b_i)$ in (3.22) with $r=\sigma$ and integrate with respect to t on $[0,t]$ to get

$$\begin{aligned}&|\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)} x e_\varepsilon(d_\varepsilon)(x,t)\phi(x)dx - |\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)} x e_\varepsilon(d_\varepsilon)(x,0)\phi(x)dxds\\ &= -|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)} x|\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon|^2\phi(x)dxds\\ &\quad + |\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)} x(\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot(\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\phi(x)dxds\\ &\quad - |\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}\partial_t d_\varepsilon\cdot\partial_l d_\varepsilon\partial_l x\phi(x)dxds - |\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}\partial_t d_\varepsilon\cdot\partial_l d_\varepsilon x\partial_l\phi(x)dxds.\end{aligned} \tag{3.23}$$

The uniform estimate (2.3) implies that the first term in the right hand side of (3.23) tends to zero as $\varepsilon\to 0$. For the second term in the righthand side of (3.23), we have

$$\begin{aligned}&|\ln\varepsilon|^{-1}\left|\int_0^t\int_{B_\sigma(b_i)} x(\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot(\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\phi(x)dxds\right|\\ &\le K|\ln\varepsilon|^{-1}\int_0^t\|\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\|_{L^2(\Omega)}\|\mathbf{u}_\varepsilon\|_{L^\infty}\|\nabla d_\varepsilon\|_{L^2(\Omega)}ds\\ &\le K|\ln\varepsilon|^{-\frac12}\|\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\|_{L^2(\Omega\times\mathbb{R}^+)}\left(\sup_{t\ge 0}|\ln\varepsilon|^{-1}\|\nabla d_\varepsilon\|^2_{L^2(\Omega)}\right)^{\frac12}\left(\int_0^t\|\mathbf{u}_\varepsilon\|^2_{L^\infty}\right)^{\frac12}\\ &\le K|\ln\varepsilon|^{-\frac12}\left(\sup_{t\in\mathbb{R}^+}\|\mathbf{u}\|_{L^2(\Omega)}\right)^{\frac12} t^{\frac14}\|\Delta\mathbf{u}\|^{\frac12}_{L^2(\Omega\times\mathbb{R}^+)}\\ &\to 0\quad\text{as}\quad\varepsilon\to 0.\end{aligned}$$

The fourth term in the righthand side of (3.23) is equal to the sum of

$$-|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}(\partial_t d_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot\partial_l d_\varepsilon x\partial_l\phi(x)dxds \tag{3.24}$$

and

$$|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\cdot\partial_l d_\varepsilon x\partial_l\phi(x)dxds. \tag{3.25}$$

Due to the convergence of $|\ln\varepsilon|^{-1}|\nabla d_\varepsilon|^2$ to a sum of Dirac measures and $\nabla\phi$ vanishes near the vortices, the quantity (3.25) converges to zero as $\varepsilon\to 0$. For the quantity (3.24), there follows

$$\begin{aligned}|(3.24)|&\le K|\ln\varepsilon|^{-\frac12}\|\partial_t d_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon\|_{L^2(\Omega\times\mathbb{R}^+)}\left(\sup_{t\ge0}|\ln\varepsilon|^{-1}\|\nabla d_\varepsilon\|^2_{L^2(\Omega)}\right)^{\frac12}t^{\frac12}\\&\le K|\ln\varepsilon|^{-\frac12}\to0\quad\text{as}\quad\varepsilon\to0.\end{aligned}$$

For the third term in the righthand side of (3.23), it can be decomposed into the sum of two terms:

$$-|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}(\partial_t d_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot\partial_l d_\varepsilon\partial_l x\phi(x)dxds \tag{3.26}$$

and

$$|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}(\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot\partial_l d_\varepsilon\partial_l x\phi(x)dxds. \tag{3.27}$$

We can proceed as in (3.24) to verify that (3.26) converges to zero as $\varepsilon\to0$. In terms of Lemma 3.5 and the fact that $\mathbf{u}_\varepsilon(t)\in H_0^2(\Omega)\subset C(\Omega)$, for each time t, we know that

$$I_\varepsilon(t)=|\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)}(\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot\partial_l d_\varepsilon\partial_l x_k\phi(x)dx\to\int_\Omega\mathbf{u}_j(x,s)\nu_{jk}(s)dx=\mathbf{u}_k(a_i(s),s).$$

Since $I_\varepsilon(t)$ is bounded as a function of t, the dominated convergence theorem implies that

$$(3.27)\to\int_0^t\mathbf{u}_k(a_i(s),s)ds.$$

For the quantities in the left hand side of (3.23), as $\varepsilon\to0$, it converges to $a_i(t)-a_i(0)$. Therefore letting $\varepsilon\to0$ in (3.23), we arrive at

$$a_i(t)-a_i(0)=\int_0^t\mathbf{u}(a_i(s),s)ds. \tag{3.28}$$

The desired ordinary differential equation (1.9) follows from (3.28) by taking the derivative with respect to t in the sense of distributions since $a_i(t)$ is Hölder continuous.

3.4. **Proof of Theorem 1.1.** We focus on the time interval $[0,t_0]$, where t_0 is given in Lemma 3.3. The conclusion for all $t\in\mathbb{R}^+$ follows from the classical iteration since at t_0 all requirements in (2.5)-(2.8) are satisfied and we can use the data at t_0 as the initial data to continue on $[t_0,2t_0]$.

For any sequence $\varepsilon_m\searrow0$, there is a subsequence (still denoted by ε_m) so that $d_{\varepsilon_m}(x,t)\to d(x,t)$ weakly in $H^1_{loc}(\Omega\setminus\cup_{i=1}^N\{b_i\}\times\mathbb{R}^+)$ and strongly in $L^2_{loc}(\Omega\times\mathbb{R}^+)$ due to the fact that $|d_\varepsilon|\le1$. Moreover $|d(x,t)|=1$ a.e. in $\Omega\times\mathbb{R}^+$.

In term of the uniform estimate (2.3), the velocity $(v_\varepsilon,\mathbf{u}_\varepsilon)$ converges weakly to $(v,\mathbf{u})$ in $L^2(\Omega\times\mathbb{R}^+)$ and $L^\infty(\mathbb{R}^+,H_0^1(\Omega))\cap L^2(\mathbb{R}^+,H^2(\Omega))$ respectively. Due to the identity

$$d_\varepsilon\wedge(\partial_t d_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla d_\varepsilon)=\sum_{j=1}^2\partial_j(d_\varepsilon\wedge\partial_j d_\varepsilon)\quad\text{in}\quad\Omega\times\mathbb{R}^+,$$

we take the limit as $\varepsilon_m \searrow 0$ to get, using $|d|(x,t) = 1$ a.e.,

$$\begin{cases} \partial_t d + \mathbf{u} \cdot \nabla d = \Delta d + |\nabla d|^2 d \quad \text{in} \quad \Omega \setminus \cup_{i=1}^N \{b_i\} \times \mathbb{R}^+, \\ d(x,0) = \prod_{i=1}^N \frac{x-b_j}{|x-b_j|} e^{ih_0(x)}, \\ d = g \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \end{cases}$$

We remark that in the limit of $d_\varepsilon \wedge (\partial_t d_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla d_\varepsilon)$, the strong convergence of d_ε away from the vortices and the divergence free condition $\mathrm{div}\mathbf{u} = 0$ are applied. More precisely, due to the incompressibility, one knows $\partial_t d_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla d_\varepsilon = \partial_t d_\varepsilon + \mathrm{div}(\mathbf{u}_\varepsilon d_\varepsilon)$ and hence its weak limit in $L^2(\mathbb{R}^2 \times \mathbb{R}^+)$ takes the form $\partial_t d + \mathrm{div}(\mathbf{u}d) = \partial_t d + \mathbf{u} \cdot \nabla d$ by applying the Aubin-Lions lemma. This, combined with the strong convergence of d_ε, yields the convergence of $d_\varepsilon \wedge (\partial_t d_\varepsilon + \mathbf{u}_\varepsilon \cdot \nabla d_\varepsilon)$ away from vortices as claimed.

The Hölder continuity of $\vec{a}(t)$ with the exponent $\frac{1}{4}$ follows from Lemma 3.4 and the evolutional equation (1.9) of $\vec{a}(t)$ is justified in (3.28). Moreover as a consequence of the identity (3.23), the trajectories of $a_i(t)$ is indeed $\frac{3}{4}-$ Hölder continuous since

$$\int_0^t \|\mathbf{u}\|_{L^\infty} dxds \le K \left(\sup_{t\in\mathbb{R}^+} \|\mathbf{u}\|_{L^2} \right)^{\frac{1}{2}} \left(\int_0^\infty \|\Delta \mathbf{u}\|_{L^2}^2 dxds \right)^{\frac{1}{4}} t^{\frac{3}{4}} \le K t^{\frac{3}{4}}.$$

Since the images $d(x,t)$ lie in the unit circle, the function $d(x,t)$ is smooth in $\Omega \setminus \{b_1, \cdots, b_N\} \times \mathbb{R}^+$. Moreover, by Lemma 3.1, we know that for any $t > 0$, the degrees $\deg(d(\cdot,t), \partial B_\sigma(b_j))$, $1 \le j \le N$ are well-defined and all equal to 1 by the property (2.6). Therefore we could write

$$d(x,t) = \prod_{j=1}^N \frac{x-b_j}{|x-b_j|} e^{ih(x,t)}.$$

The differential equation (1.9) implies that

$$\begin{aligned} \partial_t \theta(x - a_i(t)) &= -\partial_{x_j} \theta(x - a_i(t)) \partial_t (a_i(t))_j = -\mathbf{u}_j(a_i(t), t) \partial_{x_j} \theta(x - a_i(t)) \\ &= -\mathbf{u}(a_i(t), t) \cdot \nabla \theta(x - a_i(t)), \end{aligned}$$

and hence a direct computation shows that the function $h(x,t)$ satisfies

$$\begin{cases} \partial_t h(x,t) + \mathbf{u} \cdot \nabla(\Theta(x,t) + h(x,t)) + \mathcal{R}(x,t) = \Delta h(x,t) \quad \text{in} \quad \Omega \setminus \{b_1, \cdots, b_N\} \times \mathbb{R}^+, \\ \mathcal{R}(x,t) = -\sum_{i=1}^N \left(\prod_{i\neq j}^N \theta(x - a_j(t)) \right) \mathbf{u}(a_i(t), t) \cdot \nabla \theta(x - a_i(t)), \\ h(x,t) = h_0(x) \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ h(x,0) = h_0(x), \end{cases} \tag{3.29}$$

where $\Theta(x,t)$ is a multivalued-harmonic function on Ω so that

$$e^{i\Theta(x,t)} = \prod_{i=1}^N \frac{x - a_i(t)}{|x - a_i(t)|} = \prod_{i=1}^N \theta(x - a_i(t)).$$

We are left to verify that the function $h(x,t)$ satisfies the equation (3.29) in $\Omega \times \mathbb{R}^+$, and that the function $h(x,t)$ is determined by the limit of the whole family d_ε instead of a special subsequence d_{ε_m}.

Proposition 3.1. *The function $h(x,t)$ in (3.29) satisfies*

$$\sup_{t\geq 0} \|\nabla h(x,t)\|^2_{L^2(\Omega)} \leq K, \tag{3.30}$$

where K depends only on g, A, B, C, and Ω. Consequently, there follows

$$d_\varepsilon(x,t) \to \prod_{i=1}^{N} \frac{x-b_j}{|x-b_j|} e^{ih(x,t)}$$

in $L^2_{loc}(\Omega\times\mathbb{R}^+)$ and weakly in $H^1_{loc}(\Omega\setminus\{b_1,\cdots,b_N\})$ as $\varepsilon\to 0$.

Proof. It is sufficient to verify the estimate (3.30) by justifying the conditions in Lemma 3.2. The estimate (3.7) is a direct consequence of the Ginzburg-Landau theory, see for instance Theorem V.1 in [3]. The estimate (3.9) follows from the fact that the vortices $\vec{a}(t)$ are away from the boundary $\partial\Omega$.

For the estimate (3.8), we fix $i\in\{1,\cdots,N\}$ and we set

$$l(i) = \inf\left\{ r\int_{\partial B_r(a_i(t))} e_\varepsilon(d_\varepsilon)(x,t)d\mathcal{H}^1(x) : r\in[\varepsilon,\sqrt{\varepsilon}]\right\}.$$

Since

$$\mu_\varepsilon(t)(\{x\in\Omega : |x-a_i^\varepsilon(t)|\in[\sqrt{\varepsilon},\lambda\sqrt{\varepsilon}]\}) \leq \mu_\varepsilon(t)(\Omega_{\sqrt{\varepsilon}}) \leq K + 2\pi N|\ln\sqrt{\varepsilon}|,$$

we have

$$l(i)|\ln\sqrt{\varepsilon}| \leq \int_\varepsilon^{\sqrt{\varepsilon}} \frac{l(i)}{r}dr \leq \mu_\varepsilon(t)(\{x\in\Omega : |x-a_i^\varepsilon(t)|\in[\sqrt{\varepsilon},\lambda\sqrt{\varepsilon}]\}) \leq K + 2\pi N|\ln\sqrt{\varepsilon}|.$$

Therefore $l(i)\leq K$ as ε is sufficiently small and hence

$$\int_{\partial B_\delta(a_j^\varepsilon(t))} e_\varepsilon(d_\varepsilon)d\mathcal{H}^1(x) \leq \frac{K}{\delta}$$

for some $\delta\in(\varepsilon,\sqrt{\varepsilon})$, which we can take δ to be $\sqrt{\varepsilon}$ for simplicity, see Theorem 2.4 in [8]. □

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