Comparison Theorem for Path Dependent SDEs Driven by G-Brownian Motion[*]

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Abstract

Sufficient and necessary conditions are presented for the comparison theorem of path dependent G-SDEs. Different from the corresponding study in path independent G-SDEs, a probability method is applied to prove these results. Moreover, the results extend the ones in the linear expectation case.

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1 Introduction

The order preservation of stochastic processes is an important property for one to compare a complicated process with simpler ones, and a result to ensure this property is called "comparison theorem" in the literature. There are two different type order preservations, one is in the distribution (weak) sense and another is in the pathwise (strong) sense, where the latter implies the former.

Under the linear expectation framework, the weak order preservation has been investigated in [2, 20, 21] and references within. There are also lots of results on the strong order

preservation, see, for instance, [1, 4, 9, 10, 14, 15, 17, 18, 22, 23] and references therein for comparison theorems on forward/backward SDEs (stochastic differential equations), with jumps and/or with memory. Recently, the first author and his co-authors extend the results in [9] to the path-distribution dependent case, one can refer to [8] for more details.

On the other hand, there are some results on the comparison theorem for G-SDEs, see [11, 12, 13]. Some sufficient condition is presented in [11, Theorem 7.1] for comparison theorem of one-dimensional G-SDEs. In [13], the authors obtain the sufficient and necessary conditions for comparison theorem by the viability property of SDEs, which is equivalent to the fact that the square of the distance to the constraint set is a viscosity supersolution to the associated Hamilton-Jacobi-Bellman equation, see [13, Theorem 2.5] and references therein for more details.

The aim of this paper is to present sufficient and necessary conditions of the order preservations for path dependent G-SDEs and we provide a probability method to prove them. The result extends the ones in [9] when the noise is a standard Brownian motion. We will adopt the method in [9] to complete the proof. However, some essential work needs to been done since the quadratic variation process $\langle B \rangle$ of the G-Brownian motion B is not determined under G-expectation. More precisely, we need to treat $\int_0^{\cdot} \langle h(s), \mathrm{d}\langle B\rangle(s)\rangle - 2\int_0^{\cdot} G(h(s))\mathrm{d}s$ which is well known as a non-increasing G-martingale. This is quite different from the linear expectation case. Moreover, in the proof of necessary condition of the comparison theorem, we will use the representation theorem (2.3) below of the G-expectation introduced in [3, 7, 19], by which the order preservation under G-expectation implies that in linear expectation case. Then the existed result in [9] can be applied to prove the necessary condition on diffusion coefficients.

Before moving on, we recall some basic facts on G-expectation and G-Brownian motion in the following section.

2 G-Expectation and G-Brownian motion

Let $\Omega = C_0([0,\infty); \mathbb{R}^m)$, the $\mathbb{R}^m$-valued and continuous functions on $[0,\infty)$ vanishing at zero, equipped with the metric

$$\rho(\omega^1, \omega^2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\max_{t \in [0,n]} |\omega_t^1 - \omega_t^2| \wedge 1 \right], \quad \omega^1, \omega^2 \in \Omega.$$

For any $T > 0$, set

$$L_{ip}(\Omega_T) = \{\omega \to \varphi(\omega_{t_1}, \cdots, \omega_{t_n}) : n \in \mathbb{N}^+, t_1, \cdots, t_n \in [0,T], \varphi \in C_{b,lip}((\mathbb{R}^m)^n)\},$$

and

$$L_{ip}(\Omega) = \bigcup_{T>0} L_{ip}(\Omega_T),$$

where $C_{b,lip}((\mathbb{R}^m)^n)$ denotes the set of bounded and Lipschitz continuous functions on $((\mathbb{R}^m)^n)$. We denote by $|A|^2 = \|A\|_{HS}^2$ for any matrix A. For two $m \times m$ matrices M and $\bar{M}$, define

$$\langle M, \bar{M} \rangle = \sum_{k,l=1}^{m} M_{kl} \bar{M}_{kl}.$$

Let $\mathbb{M}^m$ be the collection of all $m \times m$ matrices and $\mathbb{S}^m$ ($\mathbb{S}_+^m$) be the set of the symmetric (symmetric and positive definite) ones in $\mathbb{M}^m$. Fix two positive constants $\underline{\sigma} < \bar{\sigma}$ and define

$$(2.1) \qquad G(A) := \frac{1}{2} \sup_{\gamma \in \mathbb{S}_+^m \cap [\underline{\sigma}^2 \mathbf{I}_{m \times m}, \bar{\sigma}^2 \mathbf{I}_{m \times m}]} \langle \gamma, A \rangle, \ A \in \mathbb{S}^m.$$

It is not difficult to see that G has the following properties:

(a) (Positive homogeneity) $G(\lambda A) = \lambda G(A)$, $\lambda \geq 0, A \in \mathbb{S}^m$.

(b) (Sub-additivity) $G(A + \bar{A}) \leq G(A) + G(\bar{A})$, $G(A) - G(\bar{A}) \leq G(A - \bar{A})$, $A, \bar{A} \in \mathbb{S}^m$.

(c) $|G(A)| \leq \frac{1}{2} |A| \sup_{\gamma \in \mathbb{S}_+^m \cap [\underline{\sigma}^2 \mathbf{I}_{m \times m}, \bar{\sigma}^2 \mathbf{I}_{m \times m}]} |\gamma| = \frac{1}{2} |A| \sqrt{m} \bar{\sigma}^2$.

(d) $G(A) - G(\bar{A}) \geq \frac{\underline{\sigma}^2}{2} \mathrm{trace}[A - \bar{A}]$, $A \geq \bar{A}, A, \bar{A} \in \mathbb{S}^m$.

Remark 2.1. *(b) and (c) imply that G is continuous.*

Let $\bar{\mathbb{E}}^G$ be the nonlinear expectation on Ω such that coordinate process $(B(t))_{t \geq 0}$, i.e. $B(t)(\omega) = \omega_t, \omega \in \Omega$, is an m-dimensional G-Brownian motion on $(\Omega, L_G^1(\Omega), \bar{\mathbb{E}}^G)$, where $L_G^1(\Omega)$ is the completion of $L_{ip}(\Omega)$ under the norm $\bar{\mathbb{E}}^G|\cdot|$. One can refer to [19] for details on the construction of $\bar{\mathbb{E}}^G$. For any $p \geq 1$, let $L_G^p(\Omega)$ be the completion of $L_{ip}(\Omega)$ under the norm $(\bar{\mathbb{E}}^G|\cdot|^p)^{\frac{1}{p}}$. Similarly, we can define $L_G^p(\Omega_T)$ for any $T > 0$.

Let

$$M_G^{p,0}([0,T]) = \left\{ \eta_t := \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t); \ \xi_j \in L_G^p(\Omega_{t_j}), N \in \mathbb{N}^+, \right.$$

$$\left. 0 = t_0 < t_1 < \cdots < t_N = T \right\},$$

and $M_G^p([0,T])$ be the completion of $M_G^{p,0}([0,T])$ under the norm

$$\|\eta\|_{M_G^p([0,T])} := \left(\bar{\mathbb{E}}^G \int_0^T |\eta_t|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Let $\mathcal{M}$ be the collection of all probability measures on $(\Omega, \mathscr{B}(\Omega))$. According to [3, 7], there exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}$ such that

$$(2.2) \qquad \bar{\mathbb{E}}^G[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X], \ X \in L_G^1(\Omega),$$

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where $\mathbb{E}_P$ is the linear expectation under probability measure $P \in \mathcal{P}$. $\mathcal{P}$ is called a set that represents $\bar{\mathbb{E}}^G$. In fact, let W^0 be an m-dimensional Brownian motion on a complete filtered probability space $(\hat{\Omega}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and define

$$\mathbb{H} := \{\theta : \ \theta \ \text{is an} \ \mathbb{M}^m\text{-valued progressively measurable}$$
$$\text{stochastic process,} \ \ \theta_s\theta_s^* \in [\underline{\sigma}^2\mathbf{I}_{m\times m}, \bar{\sigma}^2\mathbf{I}_{m\times m}], \ \ s \geq 0\},$$

here $(\cdot)^*$ stands for the transpose of a matrix. For any $\theta \in \mathbb{H}$, define $\mathbb{P}_\theta$ as the law of $\int_0^\cdot \theta_s \mathrm{d}W_s^0$. Then by [3, 7], we can take $\mathcal{P} = \{\mathbb{P}_\theta, \theta \in \mathbb{H}\}$, i.e.

$$(2.3) \qquad \bar{\mathbb{E}}^G[X] = \sup_{\theta \in \mathbb{H}} \mathbb{E}_{\mathbb{P}_\theta}[X], \ X \in L_G^1(\Omega).$$

The associated Choquet capacity to $\bar{\mathbb{E}}^G$ is defined by

$$\mathcal{C}(A) = \sup_{P \in \mathcal{P}} P(A), \ A \in \mathscr{B}(\Omega).$$

A set $A \in \mathscr{B}(\Omega)$ is called polar if $\mathcal{C}(A) = 0$, and we say that a property holds $\mathcal{C}$-quasi-surely ($\mathcal{C}$-q.s.) if it holds outside a polar set, see [3] for more details on capacity.

Finally, letting $\langle B \rangle$ be the quadratic variation process of B, then by the property (d) and [16, Chapter III, Corollary 5.7], we have $\mathcal{C}$-q.s.

$$(2.4) \qquad \underline{\sigma}^2\mathbf{I}_{m\times m} \leq \frac{\mathrm{d}}{\mathrm{d}t}\langle B \rangle(t) \leq \bar{\sigma}^2\mathbf{I}_{m\times m}.$$

3 Main Results

Let $r_0 \geq 0$ be a constant and $d \geq 1$ be a natural number. $\mathscr{C} = C([-r_0, 0]; \mathbb{R}^d)$ is equipped with uniform norm $\|\cdot\|_\infty$. For any continuous map $f : [-r_0, \infty) \to \mathbb{R}^d$ and $t \geq 0$, let $f_t \in \mathscr{C}$ be such that $f_t(s) = f(s + t)$ for $s \in [-r_0, 0]$. We call $(f_t)_{t\geq 0}$ the segment of $(f(t))_{t\geq -r_0}$.

Consider the following path dependent SDEs:

$$(3.1) \qquad \begin{cases} \mathrm{d}X(t) = b(t, X_t)\,\mathrm{d}t + \langle h(t, X_t), \mathrm{d}\langle B \rangle(t)\rangle + \sigma(t, X_t)\,\mathrm{d}B(t), \\ \mathrm{d}\bar{X}(t) = \bar{b}(t, \bar{X}_t)\,\mathrm{d}t + \langle \bar{h}(t, \bar{X}_t), \mathrm{d}\langle B \rangle(t)\rangle + \bar{\sigma}(t, \bar{X}_t)\,\mathrm{d}B(t), \end{cases}$$

where

$$b, \bar{b} : [0, \infty) \times \mathscr{C} \to \mathbb{R}^d; \ \ h, \bar{h} : [0, \infty) \times \mathscr{C} \to (\mathbb{R}^m \otimes \mathbb{R}^m)^d;$$
$$\sigma, \bar{\sigma} : [0, \infty) \times \mathscr{C} \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable.

Without loss of generality, we assume that for any $i = 1, \cdots, d$, h^i and $\bar{h}^i$ are symmetric. Otherwise, we can replace h^i and $\bar{h}^i$ by $\frac{h^i + (h^i)^*}{2}$ and $\frac{\bar{h}^i + (\bar{h}^i)^*}{2}$ respectively to symmetrize them.

For any $s \geq 0$ and $\xi, \bar{\xi} \in \mathscr{C}$, a solution to (3.1) for $t \geq s$ with $(X_s, \bar{X}_s) = (\xi, \bar{\xi})$ is a continuous process $(X(t), \bar{X}(t))_{t \geq s}$ such that for all $t \geq s$,

$$X(t) = \xi(0) + \int_s^t b(r, X_r)\mathrm{d}r + \int_s^t \langle h(r, X_r), \mathrm{d}\langle B\rangle(r)\rangle + \int_s^t \sigma(r, X_r)\mathrm{d}B(r),$$

$$\bar{X}(t) = \bar{\xi}(0) + \int_s^t \bar{b}(r, \bar{X}_r)\mathrm{d}r + \int_s^t \langle \bar{h}(r, \bar{X}_r), \mathrm{d}\langle B\rangle(r)\rangle + \int_s^t \bar{\sigma}(r, \bar{X}_r)\mathrm{d}B(r),$$

where $(X_t, \bar{X}_t)_{t \geq s}$ is the segment process of $(X(t), \bar{X}(t))_{t \geq s - r_0}$ with $(X_s, \bar{X}_s) = (\xi, \bar{\xi})$. Throughout the paper, we make the following assumptions.

(H1) There exists an increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $t \geq 0, \xi, \eta \in \mathscr{C}$,

$$|b(t, \xi) - b(t, \eta)|^2 + |\bar{b}(t, \xi) - \bar{b}(t, \eta)|^2 + |h(t, \xi) - h(t, \eta)|^2 + |\bar{h}(t, \xi) - \bar{h}(t, \eta)|^2$$
$$+ \|\sigma(t, \xi) - \sigma(t, \eta)\|_{HS}^2 + \|\bar{\sigma}(t, \xi) - \bar{\sigma}(t, \eta)\|_{HS}^2 \leq \alpha(t)\|\xi - \eta\|_\infty^2.$$

(H2) There exists an increasing function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|b(t, 0)|^2 + |\bar{b}(t, 0)|^2 + |h(t, 0)|^2 + |\bar{h}(t, 0)|^2 + \|\sigma(t, 0)\|_{HS}^2 + \|\bar{\sigma}(t, 0)\|_{HS}^2 \leq K(t), \quad t \geq 0.$$

Remark 3.1. *According to [6, Lemma 2.1], under* **(H1)**-**(H2)**, *the equation (3.1) is strongly well-posed. For any $s \geq 0$ and $\xi, \bar{\xi} \in \mathscr{C}$, let $\{X(s, \xi; t), \bar{X}(s, \bar{\xi}; t)\}_{t \geq s - r_0}$ denote the unique solution to (3.1) with $X_s = \xi$ and $\bar{X}_s = \bar{\xi}$. Moreover, the segment process $\{X(s, \xi)_t, \bar{X}(s, \bar{\xi})_t\}_{t \geq s}$ satisfies*

$$(3.2) \qquad \bar{\mathbb{E}}^G \sup_{t \in [s, T]} \left(\|X(s, \xi)_t\|_\infty^2 + \|\bar{X}(s, \bar{\xi})_t\|_\infty^2 \right) < \infty, \quad T \in [s, \infty).$$

To characterize the order preservation for solution of (3.1), we introduce the partial-order on $\mathscr{C}$. Firstly, for $x = (x^1, \cdots, x^d)$ and $y = (y^1, \cdots, y^d) \in \mathbb{R}^d$, we write $x \leq y$ if $x^i \leq y^i$ holds for all $1 \leq i \leq d$. Similarly, for $\xi = (\xi^1, \cdots, \xi^d)$ and $\eta = (\eta^1, \cdots, \eta^d) \in \mathscr{C}$, we write $\xi \leq \eta$ if $\xi^i(s) \leq \eta^i(s)$ holds for all $s \in [-r_0, 0]$ and $1 \leq i \leq d$. Moreover, for any $\xi_1, \xi_2 \in \mathscr{C}$, $\xi_1 \wedge \xi_2 \in \mathscr{C}$ is defined by

$$(\xi_1 \wedge \xi_2)^i = \min\{\xi_1^i, \xi_2^i\}, \quad 1 \leq i \leq d.$$

Definition 3.1. The stochastic differential system (3.1) is called order-preserving, if for any $s \geq 0$ and $\xi, \bar{\xi} \in \mathscr{C}$ with $\xi \leq \bar{\xi}$, it holds $\mathcal{C}$-q.s.

$$X(s, \xi; t) \leq \bar{X}(s, \bar{\xi}; t), \quad t \geq s.$$

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We first present the following sufficient conditions for the order preservation, which reduce back to the corresponding ones in [9] when the noise is an m-dimensional standard Brownian motion and in [13] where the system is path independent.

Theorem 3.2. *Assume* **(H1)**-**(H2)**. *The system* (3.1) *is order-preserving provided that the following two conditions are satisfied:*

(1) *For any* $1 \le i \le d$, $\xi, \eta \in \mathscr{C}$ *with* $\xi \le \eta$ *and* $\xi^i(0) = \eta^i(0)$,

$$b^i(t, \xi) - \bar{b}^i(t, \eta) + 2G(h^i(t, \xi) - \bar{h}^i(t, \eta)) \le 0, \quad a.e. \ t \ge 0.$$

(2) *For a.e.* $t \ge 0$ *it holds* $\sigma(t, \cdot) = \bar{\sigma}(t, \cdot)$ *and* $\sigma^{ij}(t, \xi) = \sigma^{ij}(t, \eta)$ *for any* $1 \le i \le d$, $1 \le j \le m$, $\xi, \eta \in \mathscr{C}$ *with* $\xi^i(0) = \eta^i(0)$.

Condition (2) means that for a.e. $t \ge 0$, $\sigma(t, \xi) = \bar{\sigma}(t, \xi)$ and $\sigma^{ij}(t, \xi)$ only depends on t and $\xi^i(0)$.

The next result shows that these conditions are also necessary if all coefficients are continuous on $[0, \infty) \times \mathscr{C}$.

Theorem 3.3. *Assume* **(H1)**-**(H2)** *and that* (3.1) *is order-preserving. If in addition,* b, h, σ *and* $\bar{b}, \bar{h}, \bar{\sigma}$ *are continuous on* $[0, \infty) \times \mathscr{C}$, *then conditions* (1) *and* (2) *in Theorem 3.2 hold.*

These two theorems will be proved in Section 4 and Section 5 respectively.

4 Proof of Theorem 3.2

Assume **(H1)**-**(H2)**, and let conditions (1) and (2) hold. For any $T > t_0 \ge 0$ and $\xi, \bar{\xi} \in \mathscr{C}$ with $\xi \le \bar{\xi}$, it suffices to prove

(4.1) $$\bar{\mathbb{E}}^G \sup_{t \in [t_0, T]} (X^i(t_0, \xi; t) - \bar{X}^i(t_0, \bar{\xi}; t))^+ = 0, \quad 1 \le i \le d,$$

where $s^+ := \max\{0, s\}$. In fact, by (4.1) and (2.2), for any $P \in \mathcal{P}$, it holds

(4.2) $$\mathbb{E}_P \sup_{t \in [t_0, T]} (X^i(t_0, \xi; t) - \bar{X}^i(t_0, \bar{\xi}; t))^+ = 0, \quad 1 \le i \le d.$$

This implies

$$P\{X^i(t_0, \xi; t) > \bar{X}^i(t_0, \bar{\xi}; t), \quad t \in [t_0, T]\} = 0,$$

from which we have

$$\mathcal{C}\{X^i(t_0, \xi; t) > \bar{X}^i(t_0, \bar{\xi}; t), \quad t \in [t_0, T]\} = 0.$$

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So the order preservation holds. For simplicity, in the following we denote $X(t) = X(t_0, \xi; t)$ and $\bar{X}(t) = \bar{X}(t_0, \bar{\xi}; t)$ for $t \geq t_0 - r_0$. Then it holds

(4.3)
$$X(t) = \xi(t - t_0), \quad \bar{X}(t) = \bar{\xi}(t - t_0), \quad t \in [t_0 - r_0, t_0].$$

To prove (4.1) using Itô's formula, we take the following C^2-approximation of s^+ as in the proof of [9, Theorem 1.1]. For any $n \geq 1$, let $\psi_n : \mathbb{R} \to [0, \infty)$ be constructed as follows: $\psi_n(s) = \psi_n'(s) = 0$ for $s \in (-\infty, 0]$, and

$$\psi_n''(s) = \begin{cases} 4n^2 s, & s \in [0, \frac{1}{2n}], \\ -4n^2(s - \frac{1}{n}), & s \in [\frac{1}{2n}, \frac{1}{n}], \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that

(4.4) $\quad 0 \leq \psi_n' \leq 1_{(0,\infty)}$, and as $n \uparrow \infty : \ 0 \leq \psi_n(s) \uparrow s^+, \quad s\psi_n''(s) \leq 1_{(0,\frac{1}{n})}(s) \downarrow 0.$

In view of

$$\psi_n(X^i(t_0) - \bar{X}^i(t_0)) = \psi_n(\xi^i(0) - \bar{\xi}^i(0)) = 0,$$

and due to (2) $\sigma(t, \cdot) = \bar{\sigma}(t, \cdot)$ for a.e. $t \geq 0$, it follows from Itô's formula that

$$
\begin{aligned}
&\psi_n(X^i(t) - \bar{X}^i(t))^2 \\
&= 2 \sum_{j=1}^{m} \int_{t_0}^{t} (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar{X}_s))\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \mathrm{d}B^j(s) \\
&\quad + 2 \int_{t_0}^{t} \langle h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s), \mathrm{d}\langle B\rangle(s)\rangle \{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \mathrm{d}s \\
&\quad + 2 \int_{t_0}^{t} (b^i(s, X_s) - \bar{b}^i(s, \bar{X}_s))\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \mathrm{d}s \\
&\quad + \sum_{j=1,k=1}^{m} \int_{t_0}^{t} \{\psi_n \psi_n'' + \psi_n'^2\}(X^i(s) - \bar{X}^i(s)) \\
&\qquad\qquad\qquad \times (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar{X}_s))(\sigma^{ik}(s, X_s) - \sigma^{ik}(s, \bar{X}_s)) \mathrm{d}\langle B\rangle_{jk}(s) \\
&= M_i(t) + \bar{M}_i(t) + I_1 + I_2
\end{aligned}
$$

(4.5)

for any $n \geq 1$, $1 \leq i \leq d$ and $t \geq t_0$, where

$$M_i(t) := 2 \sum_{j=1}^{m} \int_{t_0}^{t} (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar{X}_s))\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \mathrm{d}B^j(s),$$

$$\bar{M}_i(t) := 2 \int_{t_0}^{t} \langle (h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s)), \mathrm{d}\langle B\rangle(s)\rangle \{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \mathrm{d}s$$

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$$- 4 \int_{t_0}^{t} G[\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s))(h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s))]\mathrm{d}s,$$

$$I_1 := 2 \int_{t_0}^{t} (b^i(s, X_s) - \bar{b}^i(s, \bar{X}_s))\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s))\mathrm{d}s$$

$$+ 4 \int_{t_0}^{t} G[\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s))(h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s))]\mathrm{d}s,$$

$$I_2 := \sum_{j=1,k=1}^{m} \int_{t_0}^{t} \{\psi_n \psi_n'' + \psi_n'^2\}(X^i(s) - \bar{X}^i(s))$$

$$\times (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar{X}_s))(\sigma^{ik}(s, X_s) - \sigma^{ik}(s, \bar{X}_s))\mathrm{d}\langle B\rangle_{jk}(s).$$

Noting that $0 \le \psi_n'(X^i(s) - \bar{X}^i(s)) \le 1_{\{X^i(s) > \bar{X}^i(s)\}}$ and when $X^i(s) > \bar{X}^i(s)$ one has $(X_s \wedge \bar{X}_s)^i(0) = (\bar{X}_s)^i(0)$, it follows from (1) that for a.e. $s \in [t_0, T]$, and $n \ge 1$

(4.6) $[b^i(s, X_s \wedge \bar{X}_s) - \bar{b}^i(s, \bar{X}_s) + 2G[h^i(s, X_s \wedge \bar{X}_s) - \bar{h}^i(s, \bar{X}_s)]]\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \le 0.$

In view of $\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s)) \ge 0$, it follows from the property (a) of G that

(4.7)
$$G[\{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s))(h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s))]$$
$$= \{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s))G(h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s)).$$

For simplicity, let $\Phi_s^n = \{\psi_n \psi_n'\}(X^i(s) - \bar{X}^i(s))$. Combining (4.6) with (4.7), **(H1)**, $0 \le \psi_n' \le 1$ and properties (b) and (c) of G, there exists a constant $C(T, \bar{\sigma}) > 0$ such that for any $n \ge 1, t \in [t_0, T]$,

$$I_1 = 2 \int_{t_0}^{t} [b^i(s, X_s) - \bar{b}^i(s, \bar{X}_s) + 2G(h^i(s, X_s) - \bar{h}^i(s, \bar{X}_s))]\Phi_s^n \mathrm{d}s$$

$$\le 2 \int_{t_0}^{t} [b^i(s, X_s) - b^i(s, X_s \wedge \bar{X}_s) + 2G(h^i(s, X_s) - h^i(s, X_s \wedge \bar{X}_s))]\Phi_s^n \mathrm{d}s$$

$$+ 2 \int_{t_0}^{t} [b^i(s, X_s \wedge \bar{X}_s) - \bar{b}^i(s, \bar{X}_s) + 2G(h^i(s, X_s \wedge \bar{X}_s) - \bar{h}^i(s, \bar{X}_s))]\Phi_s^n \mathrm{d}s$$

(4.8) $\quad \le 2 \int_{t_0}^{t} [b^i(s, X_s) - b^i(s, X_s \wedge \bar{X}_s) + 2G(h^i(s, X_s) - h^i(s, X_s \wedge \bar{X}_s))]\Phi_s^n \mathrm{d}s$

$$\le \int_{t_0}^{t} [b^i(s, X_s) - b^i(s, X_s \wedge \bar{X}_s) + 2G(h^i(s, X_s) - h^i(s, X_s \wedge \bar{X}_s))]^2 \mathrm{d}s$$

$$+ \int_{t_0}^{t} \psi_n(X^i(s) - \bar{X}^i(s))^2 \mathrm{d}s$$

$$\le \int_{t_0}^{t} C_0(T, \bar{\sigma})\|X_s - X_s \wedge \bar{X}_s\|_{\infty}^2 \mathrm{d}s + \int_{t_0}^{t} \psi_n(X^i(s) - \bar{X}^i(s))^2 \mathrm{d}s$$

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$$\leq \int_{t_0}^t C(T, \bar\sigma) \| X_s - X_s \wedge \bar X_s \|_\infty^2 \mathrm{d}s,$$

here in the last step we used the fact $\psi_n(s) \leq s^+$. Next, by condition (2) in Theorem 3.2, for a.e. $s \in [t_0, T]$, $\sigma^{ij}(s, X_s) = \bar\sigma^{ij}(s, X_s)$ and $\sigma^{ij}(s, X_s)$ depends only on s and $X^i(s)$. So, (4.4), **(H1)** and the positive definite property of $\langle B \rangle$ yield

$$
\begin{aligned}
(4.9) \quad I_2 &\leq \int_{t_0}^t \left(1_{\{X^i(s) - \bar X^i(s) \in (0, \frac{1}{n})\}} + 1_{\{X^i(s) - \bar X^i(s) \in (0, \infty)\}} \right) \\
&\qquad \times \sum_{j=1, k=1}^m (\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar X_s))(\sigma^{ik}(s, X_s) - \sigma^{ik}(s, \bar X_s)) \mathrm{d}\langle B \rangle_{jk}(s) \\
&\leq \int_{t_0}^t C(T, \bar\sigma) \{ (X^i(s) - \bar X^i(s))^+ \}^2 \mathrm{d}s, \quad n \geq 1, t \in [t_0, T].
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality in [5, Theorem 2.1] or [16, Lemma 8.1.12], we deduce

$$
\begin{aligned}
\bar{\mathbb{E}}^G &\sup_{s \in [t_0, t]} M_i(s) \\
&\leq C(T, \bar\sigma) \bar{\mathbb{E}}^G \left\{ \int_{t_0}^t \left| (\psi_n \psi_n')(X^i(s) - \bar X^i(s)) \right|^2 \sum_{j=1}^m |\sigma^{ij}(s, X_s) - \sigma^{ij}(s, \bar X_s)|^2 \mathrm{d}s \right\}^{\frac{1}{2}} \\
(4.10) \quad &\leq C(T, \bar\sigma) \bar{\mathbb{E}}^G \left(\int_{t_0}^t \{ (X^i(s) - \bar X^i(s))^+ \}^2 \psi_n(X^i(s) - \bar X^i(s))^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq C(T, \bar\sigma) \bar{\mathbb{E}}^G \int_{t_0}^t \| X_s - X_s \wedge \bar X_s \|_\infty^2 \mathrm{d}s \\
&\quad + \frac{1}{8} \bar{\mathbb{E}}^G \sup_{s \in [t_0, t]} \psi_n(X^i(s) - \bar X^i(s))^2, \quad n \geq 1, t \in [t_0, T].
\end{aligned}
$$

Finally, since $\bar M_i$ is a non-increasing G-martingale, we obtain from (4.7) that

$$(4.11) \qquad\qquad \bar{\mathbb{E}}^G \sup_{s \in [t_0, t]} \bar M_i(s) \leq 0.$$

Combining (4.5) with (4.8)-(4.11), and noting that $\psi_n(X^i(r) - \bar X^i(r)) = 0, r \in [t_0 - r_0, t_0]$ due to (4.4), (4.3) and $\xi \leq \bar\xi$, we arrive at

$$
\begin{aligned}
(4.12) \quad \bar{\mathbb{E}}^G \sup_{r \in [t_0 - r_0, t]} \psi_n(X^i(r) - \bar X^i(r))^2 &= \bar{\mathbb{E}}^G \sup_{r \in [t_0, t]} \psi_n(X^i(r) - \bar X^i(r))^2 \\
&\leq C \bar{\mathbb{E}}^G \int_{t_0}^t \| X_s - X_s \wedge \bar X_s \|_\infty^2 \mathrm{d}s + \frac{1}{8} \bar{\mathbb{E}}^G \sup_{s \in [t_0, t]} \psi_n(X^i(s) - \bar X^i(s))^2,
\end{aligned}
$$

for some constant $C > 0$ and all $n \geq 1, t \in [t_0, T], 1 \leq i \leq d$. Letting
$$\phi(s) = \sup_{r \in [t_0 - r_0, s]} |X(r) - X(r) \wedge \bar{X}(r)|^2, \quad s \in [t_0, T],$$

(4.12) yields

$$(4.13) \qquad \sum_{i=1}^{d} \bar{\mathbb{E}}^G \sup_{r \in [t_0 - r_0, t]} \psi_n(X^i(r) - \bar{X}^i(r))^2 \leq C \int_{t_0}^{t} \bar{\mathbb{E}}^G \phi(s) \mathrm{d}s, \quad n \geq 1, t \in [t_0, T].$$

By (4.4), it is not difficult to see that

$$\lim_{n \to \infty} \sup_{r \in [t_0 - r_0, t]} \psi_n(X^i(r) - \bar{X}^i(r))^2 = \sup_{r \in [t_0 - r_0, t]} \{(X^i(r) - \bar{X}^i(r))^+\}^2.$$

So, letting $n \uparrow \infty$ in (4.13), by the monotone convergence theorem in [16, Theorem 6.1.14], the sub-additivity of G, we arrive at

$$\bar{\mathbb{E}}^G \phi(t) \leq \sum_{i=1}^{d} \bar{\mathbb{E}}^G \sup_{r \in [t_0 - r_0, t]} \{(X^i(r) - \bar{X}^i(r))^+\}^2 \leq C \int_{t_0}^{t} \bar{\mathbb{E}}^G \phi(s) \mathrm{d}s, \quad t \in [t_0, T].$$

By the definition of ϕ and (3.2), Gronwall's inequality implies

$$\bar{\mathbb{E}}^G \phi(T) = 0.$$

Thus, we prove (4.1).

5 Proof of Theorem 3.3

Proof of (1). Let $1 \leq i \leq d$ be fixed. For any $t_0 \geq 0$ and $\xi, \eta \in \mathscr{C}$ with $\xi \leq \eta$ and $\xi^i(0) = \eta^i(0)$, it holds $\mathcal{C}$-q.s.

$$(5.1) \qquad\qquad\qquad X(t_0, \xi; t) \leq \bar{X}(t_0, \eta; t), \quad t \geq t_0.$$

For simplicity, let $X(t) = X(t_0, \xi; t)$ and $\bar{X}(t) = \bar{X}(t_0, \eta; t)$ for $t \geq t_0 - r_0$. For any $\gamma \in \mathbb{S}_+^m \bigcap [\underline{\sigma}^2 \mathbf{I}_{m \times m}, \bar{\sigma}^2 \mathbf{I}_{m \times m}]$, take $\theta_s = \sqrt{\gamma}, s \geq 0$ and denote $\mathbb{E}_{\mathbb{P}_\theta} = \mathbb{E}_{\mathbb{P}_\gamma}$. Then $\mathbb{P}_\gamma$-a.s. $\langle B \rangle(r) = r\gamma$. By (3.1), (2.3) and (5.1), for any $s \geq 0$, we obtain $\mathbb{P}_\gamma$-a.s.

$$
\begin{aligned}
0 \geq X^i(t_0 + s) - \bar{X}^i(t_0 + s) &= \xi^i(0) - \eta^i(0) \\
&\quad + \int_{t_0}^{t_0+s} [b^i(r, X_r) - \bar{b}^i(r, \bar{X}_r)] \mathrm{d}r + \int_{t_0}^{t_0+s} \langle h^i(r, X_r) - \bar{h}^i(r, \bar{X}_r), \mathrm{d}\langle B\rangle(r) \rangle \\
&\quad + \sum_{j=1}^{m} \int_{t_0}^{t_0+s} [\sigma^{ij}(r, X_r) - \bar{\sigma}^{ij}(r, \bar{X}_r)] \mathrm{d}B^j(r) \\
&= \int_{t_0}^{t_0+s} [b^i(r, X_r) - \bar{b}^i(r, \bar{X}_r)] \mathrm{d}r + \int_{t_0}^{t_0+s} \langle h^i(r, X_r) - \bar{h}^i(r, \bar{X}_r), \gamma \rangle \mathrm{d}r \\
&\quad + \sum_{j=1}^{m} \int_{t_0}^{t_0+s} [\sigma^{ij}(r, X_r) - \bar{\sigma}^{ij}(r, \bar{X}_r)] \mathrm{d}B^j(r).
\end{aligned}
$$
(5.2)

10

By **(H1)**, **(H2)** and (3.2), taking expectation in (5.2) under $\mathbb{P}_\gamma$, we obtain

$$(5.3) \quad \frac{1}{s} \int_{t_0}^{t_0+s} \mathbb{E}_{\mathbb{P}_\gamma}\{[b^i(r, X_r) - \bar{b}^i(r, \bar{X}_r)] + \langle h^i(r, X_r) - \bar{h}^i(r, \bar{X}_r), \gamma \rangle\}\mathrm{d}r \leq 0, \quad s > 0.$$

Thus, taking $s \downarrow 0$ in (5.3), it follows from (3.2), (2.3), the continuity of $b, \bar{b}, h, \bar{h}$ and dominated convergence theorem that

$$[b^i(t_0, \xi) - \bar{b}^i(t_0, \eta)] + \langle h^i(t_0, \xi) - \bar{h}^i(t_0, \eta), \gamma \rangle \leq 0.$$

By the definition of G in (2.1), we derive

$$[b^i(t_0, \xi) - \bar{b}^i(t_0, \eta)] + 2G(h^i(t_0, \xi) - \bar{h}^i(t_0, \eta))$$

$$= [b^i(t_0, \xi) - \bar{b}^i(t_0, \eta)] + 2 \sup_{\gamma \in \mathbb{S}_+^m \bigcap [\underline{\sigma}^2 \mathbf{I}_{m \times m}, \bar{\sigma}^2 \mathbf{I}_{m \times m}]} \frac{1}{2}\langle h^i(t_0, \xi) - \bar{h}^i(t_0, \eta), \gamma \rangle \leq 0.$$

The proof is completed. $\qquad\square$

Proof of (2). For any $t_0 \geq 0$ and $\xi, \bar{\xi} \in \mathscr{C}$ with $\xi \leq \bar{\xi}$, it holds $\mathcal{C}$-q.s.

$$X(t_0, \xi; t) \leq \bar{X}(t_0, \bar{\xi}; t), \quad t \geq t_0.$$

Taking $\theta_s = \bar{\sigma}$, (2.3) implies $\mathbb{P}_\theta$-a.s. $X(t_0, \xi; t) \leq \bar{X}(t_0, \bar{\xi}; t), \quad t \geq t_0$. Noting that $\mathbb{P}_\theta$-a.s. $\langle B \rangle(r) = \bar{\sigma}^2 r$, (3.1) reduces to the SDE driven by Brownian motion under $\mathbb{P}_\theta$. According to the necessary condition of order preservation for functional SDEs in [9, Theorem 1.2 (II)], we immediately get the results desired. $\qquad\square$

References

[1] J. Bao, C. Yuan, *Comparison theorem for stochastic differential delay equations with jumps,* Acta Appl. Math. 116 (2011), 119–132.

[2] M.-F. Chen, F.-Y. Wang, *On order-preservation and positive correlations for multidimensional diffusion processes,* Prob. Theory. Relat. Fields 95 (1993), 421–428.

[3] L. Denis, M. Hu, S. Peng, *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes,* Potential Anal. 34 (2011), 139–161.

[4] L. Gal'cuk, M. Davis, *A note on a comparison theorem for equations with different diffusions,* Stochastics 6 (1982), 147–149.

[5] F. Gao, *Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian,* Stochastic Process. Appl. 119 (2009), 3356–3382.

[6] X. He, S. Han, *Averaging principle for SDEs of neutral type driven by G-Brownian motion,* Stoch. Dyn. 19 (2019), 1950004, 22 pp.

[7] M. Hu, S. Peng, *On representation theorem of G-expectations and paths of G-Brownian motion,* Acta Math. Appl. Sin. Engl. Ser. 25 (2009), 539–546.

[8] X. Huang, C. Liu, F.-Y. Wang, *Order preservation for path-distribution dependent SDEs,* Commun. Pure Appl. Anal. 17 (2018), 2125–2133.

[9] X. Huang, F.-Y. Wang, *Order-preservation for multidimensional stochastic functional differential equations with jumps,* J. Evol. Equat. 14 (2014),445–460.

[10] N. Ikeda, S. Watanabe, *A comparison theorem for solutions of stochastic differential equations and its applications,* Osaka J. Math. 14 (1977), 619–633.

[11] Q. Lin, *Some properties of stochastic differential equations driven by the G-Brownian motion,* Acta Math. Sin. (Engl. Ser.) 29 (2013), 923–942.

[12] P. Luo, G. Jia, *On monotonicity and order-preservation for multidimensional G-diffusion processes,* Stoch. Anal. Appl. 33 (2015), 67–90.

[13] P. Luo, F. Wang, *On the comparison theorem for multi-dimensional G-SDEs,* Statist. Probab. Lett. 96 (2015), 38–44.

[14] X. Mao, *A note on comparison theorems for stochastic differential equations with respect to semimartingales,* Stochastics 37 (1991), 49–59.

[15] G. L. O'Brien, *A new comparison theorem for solution of stochastic differential equations,* Stochastics 3 (1980), 245–249.

[16] S. Peng, *Nonlinear expectations and stochastic calculus under uncertainty with robust CLT and G-Brownian motion,* Probability Theory and Stochastic Modelling, Springer, 2019.

[17] S. Peng, Z. Yang, *Anticipated backward stochastic differential equations,* Ann. Probab. 37 (2009), 877–902.

[18] S. Peng, X. Zhu, *Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations,* Stochastic Process. Appl. 116 (2006), 370–380.

[19] Y. Song, *Gradient estimates for nonlinear diffusion semigroups by coupling methods,* Sci. China Math. 64 (2021), 1093–1108.

[20] F.-Y. Wang, *The stochastic order and critical phenomena for superprocesses,* Infin. Dimens. Anal. Quantum Probab. Relat. Top. 9 (2006), 107–128.

[21] J.-M. Wang, *Stochastic comparison for Lévy-type processes,* J. Theor. Probab. 26 (2013), 997–1019.

[22] Z. Yang, X. Mao, C. Yuan, *Comparison theorem of one-dimensional stochastic hybrid systems,* Systems Control Lett. 57 (2008), 56–63.

[23] X. Zhu, *On the comparison theorem for multi-dimensional stochastic differential equations with jumps (in Chinese),* Sci. Sin. Math. 42 (2012), 303–311.