A NOTE ON NON-ORDINARY PRIMES FOR SOME GENUS-ZERO ARITHMETIC GROUPS

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Abstract. Suppose that $\mathcal{O}_L$ is the ring of integers of a number field L, and suppose that

$$f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(\Gamma_0(N)^+) \cap \mathcal{O}_L[[q]]$$

is a normalized Hecke eigenform for $\Gamma_0(N)^+$. We say that f is non-ordinary at p if there is a prime ideal $\mathfrak{p} \subset \mathcal{O}_L$ above p for which $a_f(p) \equiv 0 \pmod{\mathfrak{p}}$. In the authors' previous paper with Ken Ono [10] it was proved that there are infinitely many Hecke eigenforms for $\mathrm{SL}_2(\mathbb{Z})$ such that are non-ordinary at any given finite set of primes. In this paper, we extend this result to some genus 0 subgroups of $\mathrm{SL}_2(\mathbb{R})$, namely, the normalizers $\Gamma_0(N)^+$ of the congruence subgroups $\Gamma_0(N)$. Our result also generalizes some of Choi and Kim's result in [2].

1. Introduction and Statement of the result

For any square-free positive integer N, we consider

$$\Gamma_0(N)^+ = \left\{ e^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) : ad - bc = e, a, b, c, d, e \in \mathbb{Z}. e \mid N, e \mid a, e \mid d, N \mid c \right\},$$

which is known as an arithmetic group related to the 'Monstrous moonshine conjectures'. Let $\overline{\Gamma_0(N)^+} = \Gamma_0(N)^+/\{\pm I\}$, where I denotes the identity matrix. In particular, $\mathrm{PSL}_2(\mathbb{Z}) = \overline{\Gamma_0(1)^+}$. It has been shown that there are 43 square-free integers $N > 1$ such that the quotient space $X_N := \overline{\Gamma_0(N)^+}\backslash\mathbb{H}$ has genus zero (see [4]). Each group has one cusp, which we can always choose to be at $i\infty$. The aim of this paper is to present results in the study of the non-ordinary prime theory associated to these 43 spaces.

Throughout, k is a positive even integer. As usual, we let $M_k(\Gamma_0(N)^+)$ (respectively, $S_k(\Gamma_0(N)^+)$) denote the $\mathbb{C}$-vector space of weight k modular forms (respectively, cusp forms) for $\Gamma_0(N)^+$. Furthermore, let $M_k^!(\Gamma_0(N)^+)$ denote the infinite dimensional space of weakly holomorphic modular forms (that is, meromorphic with poles only at the cusps) of weight k with respect to $\Gamma_0(N)^+$.

Throughout this paper, we call a modular form $f = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N)^+)$ *Hecke eigenform in* $M_k(\Gamma_0(N)^+)$ if it is an eigenform for every Hecke operator T_m. If $a_f(1) = 1$ for $f \in S_k(\Gamma_0(N)^+)$, we say that f is *normalized*.

Key words and phrases. Modular forms, Hecke eigenforms, Non-ordinary primes, Genus-zero arithmetic groups.

The first author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2020R1F1A1A01076646). The second author was funded by National Natural Science Funds of China(NSFC, Grant No. 11801401).

2020 Mathematics Subject Classification: 11F33, 11F11.

Suppose that $\mathcal{O}_L$ is the ring of integers of a number field L, and suppose that

$$f(z)=\sum_{n=1}^{\infty}a_f(n)q^n\in S_k(\Gamma_0(N)^+)\cap\mathcal{O}_L[[q]]$$

is a normalized Hecke eigenform for $\Gamma_0(N)^+$, where

$$N\in\{2,3,5,6,7,10,11,13,14,15,17,19,21,22,23,26,29,30,31,33,34,35,\\ 38,39,41,42,46,47,51,55,59,62,66,69,70,71,78,87,94,95,105,110,119\},$$

so that $\Gamma_0(N)^+$ has genus 0, and $q:=e^{2\pi iz}$. Unless otherwise stated, N shall denote an integer in the above set throughout.

We say that f is non-ordinary at p if there is a prime ideal $\mathfrak{p}\subset\mathcal{O}_L$ above p for which $a_f(p)\equiv 0\pmod{\mathfrak{p}}$. Not so much is known about the distribution of non-ordinary primes. For the case of full modular group, there are several works including results done by Hatada [6], Hida [7]-[9], Choie-Kohnen-Ono [3], and Jin-Ma-Ono [10]. There has been progress for other groups, too. For example, El-Guindy [5] extended some results in the full modular group to some Hecke congruence groups, namely, $\Gamma_0(N)$ with $N\in\{2,3,5,7,13\}$ where the genus is 0. Later, Choi and Kim [2] dealt with some genus 0 plus groups, which are the groups generated by the Hecke congruence group $\Gamma_0(p)$ and the Fricke involution $W_p=\begin{pmatrix}0&-1\\p&0\end{pmatrix}$, where p is a prime. It is natural to consider the case for some other groups, too, and this is what we study in this paper.

For $\Gamma_0(N)^+$ with N in the above list, we present our main result.

Theorem 1.1. *Let p be a prime, and suppose that $f(z)=\sum_{n=1}^{\infty}a_f(n)q^n\in S_k(\Gamma_0(N)^+)\cap\mathcal{O}_L[[q]]$ is a normalized Hecke eigenform, where $k\in 2\mathbb{Z}$ and $\mathcal{O}_L$ is the ring of algebraic integers of a number field L. We denote by $\mathfrak{p}$ a prime ideal of $\mathcal{O}_L$ above p. m_N, k_N and $a_N(n)$ are as defined in Proposition 2.3.*

(1) If $p=2$ and $N=2$, then

$$a_f(p)\equiv 0\pmod{\mathfrak{p}},$$

i.e. every level 2 normalized Hecke eigenform is non-ordinary at 2.

(2) If $p=3$, suppose that $N\in\{3,7,13,19,21,31,39\}$ or $k\equiv 2$ mod 4, Then

$$\text{either } a_f(p)\equiv 0\pmod{\mathfrak{p}} \text{ or } \sum_{n=1}^{m_N}a_f(n)a_N(m_N-n)\equiv 0\pmod{\mathfrak{p}}.$$

In particular, when $m_N=1$, i.e., $N\in\{2,5,6\}$ and $k\equiv 2\pmod 4$, we must have $a_f(p)\equiv 0\pmod{\mathfrak{p}}$, i.e., f is non-ordinary at p.

(3) If $p\geq 5$, suppose that $(p-1)\mid(k-2-k_N)$ and

$$\left(\frac{l}{p}\right)\neq -1,\quad \forall \text{ prime } l\mid N,$$

then

$$\text{either } a_f(p)\equiv 0\pmod{\mathfrak{p}} \text{ or } \sum_{n=1}^{m_N}a_f(n)a_N(m_N-n)\equiv 0\pmod{\mathfrak{p}}.$$

In particular, when $m_N=1$, i.e., $N\in\{2,5,6\}$, we must have $a_f(p)\equiv 0\pmod{\mathfrak{p}}$, i.e., f is non-ordinary at p.

Remark. A normalized Hecke eigenform for $\Gamma_0(N)^+$ inside the space $S_k^{\text{new}}(\Gamma_0(N))$ is the same as a newform for $\Gamma_0(N)$, whose eigenvalue for Atkin-Lehner involutions are all 1, here by $S_k^{\text{new}}(\Gamma_0(N))$ we mean the space of newforms of weight k for $\Gamma_0(N)$. We show this in the following.

First, we see that

$$f\left(e^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right).$$

Hence for $\gamma = e^{-1/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^+$,

$$(f \mid_k \gamma)(z) = e^{k/2}(cz+d)^{-k} f(\gamma z) = (f \mid_k \gamma_0)(z),$$

where $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $|_k\ \gamma$ is the usual slash operator. That is to say, the modular property $(f \mid_k \gamma)(z) = f$, is equivalent to

$$(f \mid_k \gamma_0)(z) = f(z), \quad \forall \gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

where $ad - bc = e, a, b, c, d, e \in \mathbb{Z}, e \mid N, e \mid a, e \mid d, N \mid c$.

We apply this for $\gamma = \gamma_0$ running over $\Gamma_0(N)$ for $e = 1$, $\gamma_0 = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ for $e = N$, and $\gamma_0 = \begin{pmatrix} pa_0 & b \\ Nc_0 & pd_0 \end{pmatrix}$ for e being a prime divisor of N, respectively. Then it follows that if a normalized Hecke eigenform for $\Gamma_0(N)^+$ is inside the space $S_k^{\text{new}}(\Gamma_0(N))$, it is a newform for $\Gamma_0(N)$, whose eigenvalues for Atkin-Lehner involutions are all 1.

On the other hand, if f is a newform for $\Gamma_0(N)$ such that the eigenvalue for Atkin-Lehner involutions are all 1, we show that it is a normalized Hecke eigenform for $\Gamma_0(N)^+$. As N has at most three different prime factors, we only need to show when $e = 1, p, p_1p_2, N$.

When $e = 1$ or $e = p \mid N$, it is straightforward from the fact that f is a modular form for $\Gamma_0(N)$ and f is eigenform of Atkin-Lehner involutions with eigenvalue 1.

When $e = N$, as

$$\gamma_0 = \begin{pmatrix} Na_0 & b \\ Nc_0 & Nd_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} c_0 & d_0 \\ -Na_0 & -b_0 \end{pmatrix} = \gamma_1\gamma_2,$$

knowing that f is invariant under Fricke involution, we get that

$$(f \mid_k \gamma_0)(z) = (f \mid_k \gamma_1\gamma_2)(z) = ((f \mid_k \gamma_1) \mid_k \gamma_2)(z) = (f \mid_k \gamma_2)(z) = f(z).$$

When $e = p_1p_2 \neq N = p_1p_2p_3$, as

$$\gamma_0 = \begin{pmatrix} p_1p_2a_0 & b_0 \\ Nc_0 & p_1p_2d_0 \end{pmatrix} = \begin{pmatrix} p_3a_1 & b_1 \\ Nc_1 & p_3d_1 \end{pmatrix}^{-1}\begin{pmatrix} Na_2 & b_2 \\ Nc_2 & Nd_2 \end{pmatrix} = \gamma_1^{-1}\gamma_2,$$

we have

$$f(z) = (f \mid_k \gamma_2)(z) = (f \mid_k \gamma_1\gamma_0)(z) = ((f \mid_k \gamma_1) \mid_k \gamma_0)(z) = (f \mid_k \gamma_0)(z).$$

Hence we know that a normalized Hecke eigenform for $\Gamma_0(N)^+$ inside the space $S_k^{\text{new}}(\Gamma_0(N))$ is the same as a newform for $\Gamma_0(N)$, whose eigenvalue for Atkin-Lehner involutions are all 1. In fact, from the discussion above, we see that f is a modular form for $\Gamma_0(N)^+$ shares the same meaning as f is a modular form for $\Gamma_0(N)$, such that f is eigenform for Atkin-Lehner involutions and the Fricke involution with eigenvalues 1.

Moreover, we can regard $\Gamma_0(N)^+$ as the group generated by $\Gamma_0(N)$, the Artkin-Lehner involution $W(Q_p) = \begin{pmatrix} pa & b \\ Nc & pd \end{pmatrix}$ and the Fricke involution $W(N) = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. In fact, this will apply to all $N \in \mathbb{Z}_+$ that is square free. In particular, when N is prime, this is just the plus group which was considered by Choi and Kim [2].

From Theorem 1.1, we can tell more about non-ordinary primes. For instance, we can establish the following. Let

$$\begin{aligned} S_2 &= \{2,3\} \cup \{\text{prime } p \geq 5 \mid p \equiv \pm 1 \pmod 8\}, \\ S_5 &= \{3,5\} \cup \{\text{prime } p \geq 7 \mid p \equiv \pm 1 \pmod 5\}, \\ S_6 &= \{3\} \cup \{\text{prime } p \geq 5 \mid p \equiv \pm 1 \pmod{24}\}. \end{aligned}$$

Corollary 1.2. *If S is a finite subset of S_N, where $N \in \{2,5,6\}$, then there are infinitely many normalized Hecke eigenforms for $\Gamma_0(N)^+$ which are non-ordinary for each $p \in S$.*

Remark. The proof of the corollary above is constructive. For example, suppose that $S = \{p_1, p_2, ..., p_m\}$ is a finite subset of S_2. Suppose that k is an even positive integer which is congruent to 2 mod 4. If we have $(p-1) \mid (k-10)$ for each $p \in S$, then every prime in S is non-ordinary for every normalized Hecke eigenform $f \in S_k(\Gamma_0(2)^+)$.

In Section 2 we show certain facts about modular forms and we prove Theorem 2.3. In Section 3 and 4 we obtain Theorem 1.1 and Corollary 1.2. We offer some numerical examples and give some discussion in Section 5.

2. Preliminaries

2.1. **Nuts and bolts.** We will make use of the Hauptmodul j_N associated to $\Gamma_0(N)^+$ (see [12] for more details). Knowing that $j_N \in M_0(\Gamma_0(N)^+)$ has a simple pole at infinity, for any $f \in M_0^!(\Gamma_0(N)^+)$, we have $f = P(j_N)$, where P is a polynomial whose degree is equal to the order of f at infinity.

Moreover, we have that $\dim M_2(\Gamma_0(N)^+) = 0$ (see for example [12, Proposition 2]) and

$$\Theta j_N \in M_2^!(\Gamma_0(N)^+),$$

where Θ is the Ramanujan theta operator $\Theta := q\frac{d}{dq}$. Hence every $g \in M_2^!(\Gamma_0(N)^+)$ can be written as

$$P(j_N)\Theta j_N,$$

where P is a polynomial. Therefore, g is the derivative of a polynomial in j_N, and so its constant term in the Fourier expansion is zero. So we have the following proposition.

Proposition 2.1. *If $f(z) = \sum_{n \gg -\infty} a_f(n) q^n \in M_2^!(\Gamma_0(N)^+)$, then $a_f(0) = 0$.*

Next we define holomorphic Eisenstein series associated to $\Gamma_0(N)^+$. For $k \geq 2$,

$$E_{2k}^{(N)}(z) := \sum_{\gamma \in \Gamma_\infty(N) \backslash \Gamma_0(N)^+} (cz+d)^{-2k} \quad \text{with} \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix},$$

where $\Gamma_\infty(N)$ denotes the stabilizer group of the cusp at $i\infty$. It is known that

$$E_{2k}^{(N)}(z) = \frac{1}{\sigma_k(N)} \sum_{v|N} v^k E_{2k}(vz),$$

where σ_k denotes the generalized divisor function

$$\sigma_k(m) = \sum_{u|m} u^k,$$

and $E_{2k} \in M_{2k}(\Gamma_0(1))$ is the normalized Eisenstein series

$$E_{2k} := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where the rational numbers B_k are the usual Bernoulli numbers.

Now we breifly define the notion of a congruence between q-series. For two q-series $F(q) = \sum_{n\geq n_0} a(n)q^n$ and $G(q) = \sum_{n\geq n_0} b(n)q^n$ in $O_L[[q]]$ for a number field L, and for a prime ideal $\mathfrak{p} \subset \mathcal{O}_L$, we say that F *is congruent to* G *modulo* $\mathfrak{p}$ if $v_\mathfrak{p}(a(n) - b(n)) \geq 1$ for every n(where $v_\mathfrak{p}$ is the $\mathfrak{p}$-adic valuation), and we denote this by

$$F(q) \equiv G(q) (\text{mod } \mathfrak{p}).$$

As we have $E_{2k} \equiv 1 \pmod{24}$, for $p = 2, 3$, if $\gcd(p, \sigma_k(N)) = 1$, it is not hard to see that $E_{2k}^{(N)} \equiv 1 \pmod{p}$.

For prime $p \geq 5$, with the property that $E_{p-1}(z) \equiv 1 (\text{mod } p)$, we know that

$$E_{p-1}^{(N)}(z) = \frac{1}{\sigma_{(p-1)/2}(N)} \sum_{v|N} v^{(p-1)/2} E_{p-1}(vz) \equiv 1 (\text{mod } p)$$

always hold for $p \nmid \sigma_{(p-1)/2}(N)$. Moreover, as we know that

$$\sigma_{(p-1)/2}(N) = \sum_{u|N} u^{(p-1)/2} = \prod_{l|N} (1 + l^{(p-1)/2}),$$

where l is prime. So Euler's criterion implies that $p \nmid \sigma_{(p-1)/2}(N)$ is equivalent to

$$\left(\frac{l}{p}\right) \neq -1, \quad \forall \text{ prime } l \mid N,$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. Hence we have

Proposition 2.2. *Suppose that* $E_{2k}^{(N)}(z)$ *are weight* $2k$ *holomorphic Eisenstein series associated to* $\Gamma_0(N)^+$.

(1) If $p = 2, 3$, *then for any positive integer* $k \geq 2$ *such that* $\gcd(p, \sigma_k(N)) = 1$, $E_{2k}^{(N)}(z) \equiv 1 (\text{mod } p)$.

(2) If $p \geq 5$ *is prime and*

$$\left(\frac{l}{p}\right) \neq -1, \quad \forall \text{ prime } l \mid N,$$

then $E_{p-1}^{(N)}(z) \equiv 1 (\text{mod } p)$.

We also need to construct a modular form of some positive weight, say k_N, vanishing at the cusp $i\infty$ only. This is achieved using *the Kronecker limit functions (cf.* [11] *and* [12]*).* Let

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

denote the Dedekind eta function.

Proposition 2.3 ([11], Theorem 16)**.** *For any square-free N, assume that N has r prime factors. Let us define the constant l_N by*

$$l_N = 2^{1-r} \mathrm{lcm} \left(4, 2^{r-1} \frac{24}{(24, \sigma(N))}\right).$$

Then the Kronecker limit function *defined by*

$$\Delta_N(z) = \left(\prod_{v|N} \eta(vz) \right)^{l_N},$$

is a weight $k_N = 2^{r-1} l_N$ modular form on $\Gamma_0(N)^+$, vanishing at the cusp $i\infty$ only. Therefore, its inverse

$$\Delta_N(z)^{-1} = q^{-m_N} \sum_{n=0}^{\infty} a_N(n) q^n,$$

where $m_N = \frac{l_N \sigma(N)}{24}$ and $a_N(0) = 1$, is a weakly holomorphic modular form of weight $-k_N$ with respect to $\Gamma_0(N)^+$.

Note when $N \in \{2, 5, 6\}$, we have that $m_N = 1$ (with $k_2 = 8$ and $k_5 = k_6 = 4$), meaning that $\Delta_N(z)$ only have a simple zero at $i\infty$. In fact, $2, 5, 6$ are the only 3 out of the 43 numbers with $m_N = 1$.

2.2. **A technical result.** The result below shows how weakly holomorphic modualr forms for $\Gamma_0(N)^+$ act, which will lead to Theorem 1.1.

Theorem 2.4. *Let p be a prime, and suppose that $f(z) = \sum_{n \gg -\infty}^{\infty} a_f(n) q^n \in M_k^!(\Gamma_0(N)^+) \cap \mathcal{O}_L[[q]]$, where $k \in 2\mathbb{Z}$ and $\mathcal{O}_L$ is the ring of algebraic integers of a number field L. m_N, k_N and $a_N(n)$ are as defined in Proposition 2.3.*

(1) If $p = 2$, suppose that $a \geq 0$ is an integer for which

$$k - 2 \leq 8p^a.$$

Suppose that $N = 2$. If $\mathrm{ord}_\infty(f) > -p^a$, then for any integer $b \geq a$, we have

$$a_f(p^b) \equiv 0 \pmod{p}.$$

(2) If $p = 3$, suppose that $a \geq 0$ is an integer for which

$$k - 2 \leq k_N p^a.$$

Suppose that $N \in \{3, 7, 13, 19, 21, 31, 39\}$ or $k \equiv 2 \pmod 4$. If $\mathrm{ord}_\infty(f) > -p^a$, then for any integer $b \geq a$, we have

$$\sum_{n=0}^{m_N} a_f(np^b) a_N(m_N - n) \equiv 0 \pmod{p}.$$

(3) If $p \geq 5$, suppose that $a \geq 0$ is an integer for which

$$k - 2 \leq k_N p^a.$$

Suppose that

$$\left(\frac{l}{p}\right) \neq -1, \quad \forall \text{ prime } l \mid N,$$

If $\mathrm{ord}_\infty(f) > -p^a$ *and* $(p-1) \mid (k-2-k_N)$*, then for any integer* $b \geq a$*, we have*

$$\sum_{n=0}^{m_N} a_f(np^b) a_N(m_N - n) \equiv 0 \pmod{p}.$$

Proof. The proof begins with the construction of suitable weakly holomorphic modular forms of weight $2-k$. The product of such forms with f have weight 2, and so Proposition 2.1 implies that their constant terms vanish.

For $p = 2, 3$, when $k \geq 2$ and $p \nmid \sigma_k(N)$, we have that $E_{2k}^{(N)} \equiv 1 \pmod{24}$.

When $p = 2$, $p \nmid \sigma_k(N) \Leftrightarrow N = 2^u$, where u is a positive integer. As N is square-free, it has to be just 2. When $N = 2$, we can always find $h \in 2\mathbb{Z}$, so that

$$2 - k = h - k_N p^b.$$

It follows that

$$\Delta_N(z)^{-p^b} E_h^{(N)} \in M^!_{2-k}(\Gamma_0(N)^+).$$

Therefore, the constant term of $f\Delta_N(z)^{-p^b} E_h^{(N)}$ is zero. We have

$$\Delta_2(z)^{-1} = (\eta(z)\eta(2z))^{-8} = q^{-1} + 8 + O(q),$$

Hence

$$a_f(p^b) + 8a_f(0) \equiv a_f(p^b) \equiv 0 \pmod{2}.$$

When $p = 3$, if there is no $l \nmid N$ such that $l \equiv 2 \mod 3$ or if k is even, we have that the formula $\sigma_k(N) = \prod_{l \mid N}(1 + l^k)$ implies that $\gcd(p, \sigma_k(N)) = 1$. That is to say, when $N \in \{3, 7, 13, 19, 21, 31, 39\}$ or when $k \equiv 2 \mod 4$, we can always find $h \in 2\mathbb{Z}$, so that

$$2 - k = h - k_N p^b \text{ and } E_h^{(N)} \equiv 1 \pmod{24}.$$

By a similar argument as before, we have

$$\sum_{n=0}^{m_N} a_f(np^b) a_N(m_N - n) \equiv 0 \pmod{3}.$$

For prime $p \geq 5$, since $(p-1) \mid (k-2-k_N)$ and $k - 2 \leq k_N p^b$, we can find a non-negative integer c such that

$$2 - k = c(p-1) - k_N p^b.$$

It follows that

$$\Delta_N(z)^{-p^b} (E_{p-1}^{(N)})^c \in M^!_{2-k}(\Gamma_0(N)^+).$$

That is to say, the constant term of $f\Delta_N(z)^{-p^b}(E_{p-1}^{(N)})^c$ is zero. If

$$\left(\frac{l}{p}\right) \neq -1, \quad \forall \text{ prime } l \mid N,$$

we have $E_{p-1}^{(N)}(z) \equiv 1 (\bmod\ p)$. Then we have that the constant term of $f\Delta_N(z)^{-p^b}$ is zero modulo p. Recall that

$$\Delta_N(z)^{-1} = q^{-m_N} \sum_{n=0}^{\infty} a_N(n) q^n,$$

where $a_N(0) = 1$. We will get

$$\sum_{n=0}^{m_N} a_f(np^b) a_N(m_N - n) \equiv 0 \pmod{p}.$$

□

3. Proof of Theorem 1.1

As $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(\Gamma_0(N)^+) \cap \mathcal{O}_L[[q]]$ is a normalized Hecke eigenform, we see that $a_f(0) = 0$ and $\mathrm{ord}_\infty(f) > -p^a$ for every $a \geq 0$. In addition, we have the property that

$$a_f(p^a n) \equiv a_f(p)^a a_f(n) \pmod{p}.$$

Recall that $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_L$ above p.

For the case $p = 2$, we have

$$a_f(p) \equiv a_f(p)^a \equiv a_f(p^a) \equiv 0 \pmod{\mathfrak{p}}.$$

For the case $p \geq 3$, knowing that

$$\sum_{n=0}^{m_N} a_f(np^b) a_N(m_N - n) \equiv 0 \pmod{p},$$

we get that

$$a_f(p)^b \sum_{n=0}^{m_N} a_f(n) a_N(m_N - n) \equiv 0 \pmod{\mathfrak{p}}.$$

Hence we have

$$\text{either } a_f(p) \equiv 0 \pmod{\mathfrak{p}} \text{ or } \sum_{n=1}^{m_N} a_f(n) a_N(m_N - n) \equiv 0 \pmod{\mathfrak{p}}.$$

In particular, when $N \in \{2, 5, 6\}$, it is not hard to see that

$$m_N = \frac{\sigma(N)}{24} \cdot 2^{1-r} \mathrm{lcm}\left(4, 2^{r-1} \frac{24}{(24, \sigma(N))}\right) = 1.$$

In fact, they are the only 3 numbers out of the given 43 with such a property. It follows that

$$\sum_{n=0}^{m_N} a_f(np^b) a_N(m_N - n) \equiv 0 \pmod{p}$$

turns to be

$$a_f(p^b) \equiv 0 \pmod{p}$$

as $a_f(0) = 0$ because f is a cusp form. Together with the fact that $a_f(p^b) \equiv a_f(p) \pmod{\mathfrak{p}}$, we get $a_f(p) \equiv 0 \pmod{\mathfrak{p}}$, that is to say, f is non-ordinary at p.

4. Proof of Corollary 1.2

When $N = 2$, for the given finite set of primes $S \subset S_2$, let

$$k_S = 2u \prod_{p \in S}(p-1) + 10,$$

where u is an arbitrary non-negative integer. For every normalized Hecke eigenform $f_S \in S_{k_S}(\Gamma_0(2)^+)$, we know that f_S is non-ordinary at 2 from Theorem 1.1(1). It is not hard to see that $k_S \equiv 2 \bmod 4$, hence f_S is non-ordinary at 3 by Theorem 1.1(2). We only need to consider the case that $p \geq 5$.

For $p \geq 5$, from Theorem 1.1(3), it is known that if $(p-1) \mid (k-10)$ (note that $k_2 = 8$) and $\left(\frac{2}{p}\right) \neq -1$, every normalized Hecke eigenform $f \in S_k(\Gamma_0(2)^+)$ will be non-ordinary at p. By quadratic reciprocity, we have that

$$\left(\frac{2}{p}\right) \neq -1 \Longleftrightarrow p \equiv \pm 1 \pmod 8.$$

It follows that every f_S will be non-ordinary at every $p \in S \subset S_2$. As u can be chosen freely, we know that there are infinitely many k_S, for which every normalized Hecke eigenform $f_S \in S_{k_S}(\Gamma_0(N)^+)$ will be non-ordinary at p. The only thing we need to show is that we can always find Hecke eigenforms inside each $S_{k_S}(\Gamma_0(N)^+)$.

Note first that $E^{(2)}_{k_S-8}\Delta_2 \in S_{k_S}(\Gamma_0(2)^+)$. Hence $\dim S_{k_S}(\Gamma_0(2)^+) \geq 1$, that is to say, $S_{k_S}(\Gamma_0(2)^+)$ won't be trivial. Furthermore, as Hecke operators are normal operators and commute with each other and the Atkin-Lehner operators, we see that one can find a basis for $S_{k_S}(\Gamma_0(2)^+)$, consisting of simultaneous Hecke eigenforms. Then we know that there is at least one normalized Hecke eigenform $f \in S_{k_S}(\Gamma_0(2)^+)$ for each k_S, and the conclusion follows.

When $N = 5, 6$, for the given finite set of primes $S \subset S_5, S_6$, respectively, let

$$k_S = 2u \prod_{p \in S}(p-1) + 6.$$

Applying the quadratic reciprocity law, for $p \geq 7$, we will get

$$\left(\frac{5}{p}\right) \neq -1 \Longleftrightarrow p \equiv \pm 1 \pmod 5,$$

$$\left(\frac{3}{p}\right) \neq -1 \Longleftrightarrow p \equiv \pm 1 \pmod{12}.$$

Together with the fact that $k_5 = k_6 = 4$, we will get the conclusion after a similar discussion as the case $N = 2$.

5. Examples and Remarks

Example (1). Let $S = \{3, 7, 17\}$. In the following table we list some of the weights k for which Hecke eigenforms for $\Gamma_0(2)^+$ are non-ordinary at each prime p.

p	$22 \leq k \leq 76$ such that all Hecke eigenforms for $S_k(\Gamma_0(2)^+)$ are non-ordinary at p																		
3	22	26		30	34	38		42	46	50		54	58	62		66	70	74	
7	22		28		34		40		46		52		58		64		70		76
17		26						42					58					74	

In particular, take the case $k = 58$. Suppose that q-expansion of a normalized weight 58 Hecke eigenform for $\Gamma_0(2)^+$

$$f_{58}(z) = \sum_{n=1}^{\infty} a_f(n) q^n.$$

With the L-functions and Modular Forms Database (LMFDB) (*http://www.lmfdb.org*), we get that

$$\begin{aligned} a_f(3) &= -24128544277404 - 23\beta, \\ a_f(7) &= -3615761968792960858431128 - 235936924286\beta, \\ a_f(17) &= -66690418465648267911621944661807181 + 914552894715917879281360\beta, \end{aligned}$$

where $\beta = 25920\sqrt{3104405074519849}$. It is not hard to see $a_f(p) \equiv 0 \pmod{p}$ for each $p \in S$. In particular, the trace forms are non-ordinary at each p, too. For the trace form of f, the sum of its distinct conjugates under $\mathrm{Aut}(\mathbb{C})$, say $\mathrm{Tr}(f) = \sum_{n=1}^{\infty} a_{\mathrm{Tr}(f)}(n) q^n$, we get that

$$\begin{aligned} a_{\mathrm{Tr}(f)}(3) &= -48257088554808 \equiv 0 \pmod{3}, \\ a_{\mathrm{Tr}(f)}(7) &= -7231523937585921716862566 \equiv 0 \pmod{7}, \\ a_{\mathrm{Tr}(f)}(17) &= -1333808369312965358232438893236143 6 \equiv 0 \pmod{17}. \end{aligned}$$

Example (2). Let $S = \{3, 23\}$. In the following table we list some of the weights k for which Hecke eigenforms for $\Gamma_0(6)^+$ are non-ordinary at each prime $p \in S$.

p	$26 \leq k \leq 94$ such that all Hecke eigenforms for $S_k(\Gamma_0(6)^+)$ are non-ordinary at p																			
3	26		30	34	38	42	46	50	54	58	62	66	70		74	78	82	86	90	94
23		28						50						72						94

In particular, take the case $k = 50$. Suppose that q-expansion of the normalized weight 50 Hecke eigenform for $\Gamma_0(6)^+$

$$f_{50}(z) = \sum_{n=1}^{\infty} a_f(n) q^n.$$

With the database LMFDB again, we get that

$$\begin{aligned} a_f(3) &= -282429536481, \\ a_f(23) &= -20199880455489707315851048239641044 + 393824493859936505 8\beta, \end{aligned}$$

where $\beta = 12700800\sqrt{2444780087512801}$. It is not hard to see that they meet the result. For the trace form of f, say $\mathrm{Tr}(f) = \sum_{n=1}^{\infty} a_{\mathrm{Tr}(f)}(n) q^n$, we get that

$$\begin{aligned} a_{\mathrm{Tr}(f)}(3) &= -564859072962 \equiv 0 \pmod{3}, \\ a_{\mathrm{Tr}(f)}(23) &= -40399760910979414631702096479282088 \equiv 0 \pmod{23}. \end{aligned}$$

Remark (1). For the case $p \geq 5$ in Theorem 1.1, if we could find $u \in \mathbb{Z}_+$ so that $k+c(p-1)-uk_Np^b = 2$, then we would have that the constant term of $f\Delta_N(z)^{-up^b}$ is just zero modulo p. Let

$$\Delta_N(z)^{-u} = q^{-um_N} \sum_{n=0}^{\infty} a_{uN}(n)q^n.$$

Then we see that the following holds.

$$\sum_{n=0}^{um_N} a_f(np^b)a_{uN}(um_N - n) \equiv 0 \pmod{p}.$$

Suppose that f is a normalized Hecke eigenform, it turns out that

$$\text{either } a_f(p) \equiv 0 \pmod{\mathfrak{p}} \text{ or } \sum_{n=1}^{um_N} a_f(n)a_{uN}(um_N - n) \equiv 0 \pmod{\mathfrak{p}}.$$

The result is a bit ugly, but we may cover some more cases.

For example, for the case $u = 2$, we have

$$\text{either } a_f(p) \equiv 0 \pmod{\mathfrak{p}} \text{ or } \sum_{n=1}^{2m_N} a_f(n)a_{2,N}(2m_N - n) \equiv 0 \pmod{\mathfrak{p}},$$

where $a_{2,N}(n)$ is denoted by

$$\Delta_N(z)^{-2} = q^{-2m_N} \sum_{n=0}^{\infty} a_{2,N}(n)q^n.$$

The latter congruence can be figured out with $a_f(n), n \leq 2m_N$. If this congruence won't hold, we can say f is non-ordinary at p.

Remark (2). Moreover, there is no result for the cases such that the genus isn't 0, up to the completion of this paper. One main barrier is that the minimal order of pole for modular function at $i\infty$ may not be just 1. It is not possible in general to find a modular function like j_N so that every modular function will be a polynomial of it.

Take $\Gamma_0(l)$ for example, where l is a prime. Atkin [1] proved in 1973 that if l is prime, ∞ is never a Weierstrass point of $X_0(l)$. So the minimal order of pole of a non-constant modular function for $\Gamma_0(l)$, holomorphic away from ∞, is exactly $g+1$, where g is the genus.

When $l = 24u+1$, we know that the genus is just $2u-1$ by using the well-known genus formula. It is easy to see that $\varphi(z) = (\eta(z)/\eta(lz))^2$ is just a modular function for $\Gamma_0(l)$, holomorphic away from a pole of order $2u$ at ∞. Moreover, the order of vanishing for $\varphi(z)$ at 0 is again just $2u$. (See, for example [13, Theorem 1.65].) It follows that if $f(z)$ is a modular function for $\Gamma_0(l)$, holomorphic away from ∞, then the only thing we may find will be something of the form

$$f(z) = c + \sum_{r=1}^{\infty} P_r(j(z))\varphi(z)^r,$$

where c is a constant, j is the usual singular moduli and P_r are polynomials such that $\deg P_r \leq 2u-1$.

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