

A Combinatorial Proof of a Schmidt Type Theorem of Andrews and Paule

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Abstract

This note is devoted to a combinatorial proof of a Schmidt type theorem due to Andrews and Paule. A four-variable refinement of Andrews and Paule's theorem is also obtained based on this combinatorial construction.

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The main objective of this note is to give a combinatorial proof of the following partition theorem due to Andrews and Paule [2]. Sylvester's bijection [3, 4, 5] for Euler's partition theorem and Wright's bijection [1, 6, 7] for Jacobi's triple product identity play important roles in the combinatorial construction.

Theorem 1 (Andrews-Paule). *Assume that $n \geq 1$. Let $s(n)$ denote the number of partitions $a_1 + a_2 + a_3 + \cdots$ satisfying $a_1 \geq a_2 \geq a_3 \geq \cdots$ and $n = a_1 + a_3 + a_5 + \cdots$. Let $t(n)$ denote the number of two-color partitions of n . Then*

$$s(n) = t(n)$$

For example, let $n = 3$. There are ten partitions counted by $s(3)$, which are

$$3, 3 + 3, 3 + 2, 3 + 1, 2 + 2 + 1, 2 + 2 + 1 + 1, 2 + 1 + 1, 2 + 1 + 1 + 1, \\ 1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1.$$

and there are also ten red and green partitions counted by $t(3)$, which are

$$3_r, 3_g, 2_r + 1_r, 2_g + 1_r, 2_r + 1_g, 2_g + 1_g, 1_r + 1_r + 1_r, 1_r + 1_r + 1_g, \\ 1_r + 1_g + 1_g, 1_g + 1_g + 1_g.$$

Proof of Theorem 1. Let $\mathcal{S}(n)$ denote the set of partitions counted by $s(n)$ and let $\mathcal{T}(n)$ denote the set of two-color partitions counted by $t(n)$. We aim to construct a bijection ϕ between $\mathcal{T}(n)$ and $\mathcal{S}(n)$.

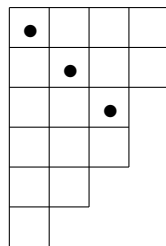
Let λ be a two-color partition in $\mathcal{T}(n)$ with r red parts and l green parts. Assume that $m = \max\{r, l\}$. We aim to define $\phi(\lambda) = \gamma = (\gamma_1, \gamma_2, \dots, \gamma_{2m-1}, \gamma_{2m})$ such that $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{2m-1} \geq \gamma_{2m} \geq 0$ and $\gamma_1 + \gamma_3 + \dots + \gamma_{2m-1} = n$.

Let α be a partition consisting of all red parts in λ and β be a partition consisting of all green parts in λ . First, add 0 at the end of α or β so that they are of the same length depending on which is of smaller length. Assume that $r \leq l$, so $m = l$. Then $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r, \underbrace{0, \dots, 0}_{l-r})$ and $\beta = (\beta_1, \beta_2, \dots, \beta_l)$.

We next define a pair $(\bar{\alpha}, \bar{\beta})$ of partitions with distinct parts corresponding to (α, β) , where $\bar{\alpha} = (\alpha_1 + l - 1, \alpha_2 + l - 2, \dots, \alpha_r + l - r, l - r - 1, \dots, 1, 0)$, and $\bar{\beta} = (\beta_1 + l - 1, \beta_2 + l - 2, \dots, \beta_l)$. Obviously, $|\bar{\alpha}| + |\bar{\beta}| = |\alpha| + |\beta| + l(l - 1)$.

We now apply Wright's bijection to represent $(\bar{\alpha}, \bar{\beta})$ as a Young diagram of an ordinary partition $Y(\bar{\alpha}, \bar{\beta})$: put l squares on the diagonal, and then for $j = 1, 2, \dots, l$, put $\bar{\alpha}_j$ squares in the j -th row to the right of the diagonal and $\bar{\beta}_j$ squares in the j -th column below the diagonal. For example, Figure 1. gives the Young diagram of $(\bar{\alpha}, \bar{\beta})$, where $\bar{\alpha} = (3, 2, 0)$ and $\bar{\beta} = (5, 3, 1)$.

The Young diagram



2-modular diagram

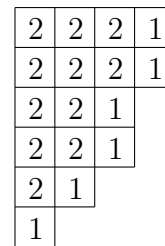


Figure 1: The Young diagram of $Y(\bar{\alpha}, \bar{\beta})$ and 2-modular diagram.

For each row in the Young diagram of $(\bar{\alpha}, \bar{\beta})$, write 2 in each box and a 1 at the end of the row to obtain the 2-modular diagram. Decompose the 2-modular diagram into hooks H_1, H_2, \dots with the diagonal boxes as corners. Let μ_1 be the number of squares in H_1 , let μ_2 be the number of 2's in H_1 , let μ_3 be the number of squares in H_2 , let μ_4 be the number of 2's in H_2 , and so on. Set $\mu = (\mu_1, \mu_2, \dots, \mu_{2l-1}, \mu_{2l})$, see Figure 2. Then μ is clearly a partition with distinct parts. Furthermore, $\mu_1 + \mu_3 + \dots + \mu_{2l-1} = |\alpha| + |\beta| + l^2$. Hence we may define $\gamma = (\mu_1 - (2l - 1), \mu_2 - 2l - 2, \dots, \mu_{2l-1} - 1, \mu_{2l})$. Clearly, $\gamma_1 + \gamma_3 + \dots + \gamma_{2l-1} = |\alpha| + |\beta|$, and so $\gamma \in \mathcal{S}(n)$. Furthermore, this process is reversible since Sylvester's bijection and Wright's bijection are reversible. Thus, we complete the proof of Theorem 1.

□

2-Modular Diagram

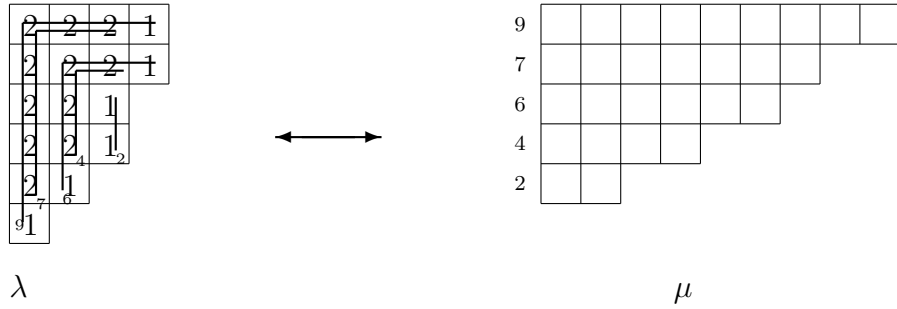


Figure 2: Sylvester's bijection.

Applying the above bijection, we get the following correspondence between the set $\mathcal{T}(3)$ and the set $\mathcal{S}(3)$.

$$\begin{aligned}
 3_r &\leftrightarrow 3 + 3 & 3_g &\leftrightarrow 3 & 2_r + 1_r &\leftrightarrow 2 + 2 + 1 + 1 \\
 2_g + 1_r &\leftrightarrow 3 + 1 & 2_r + 1_g &\leftrightarrow 3 + 2 & 2_g + 1_g &\leftrightarrow 2 + 1 + 1 \\
 1_r + 1_r + 1_r &\leftrightarrow 1 + 1 + 1 + 1 + 1 + 1 & 1_r + 1_r + 1_g &\leftrightarrow 2 + 1 + 1 + 1 \\
 1_r + 1_g + 1_g &\leftrightarrow 2 + 2 + 1 & 1_g + 1_g + 1_g &\leftrightarrow 1 + 1 + 1 + 1 + 1
 \end{aligned}$$

The following result immediately follows from the combinatorial construction of Theorem 1.

Theorem 2. Assume that $n \geq 1$, $r, l, p, q \geq 1$. Let $s_{r,l,p,q}(n)$ denote the number of partitions $a_1 + a_2 + a_3 + \cdots + a_{2\max\{r,l\}}$ satisfying $p + q \geq a_1 \geq a_2 \geq \cdots \geq a_{2\max\{r,l\}} \geq 0$ and $n = a_1 + a_3 + a_5 + \cdots + a_{2\max\{r,l\}-1}$. Let $t_{r,l,p,q}(n)$ denote the number of two-color partitions of n such that there are r red parts and l blue parts with the largest red part being not larger than p and the largest blue part being not larger than q . Then

$$s_{r,l,p,q}(n) = t_{r,l,p,q}(n)$$

Acknowledgements

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