Multiple nodal solutions having shared componentwise nodal numbers for coupled Schrödinger equations

Haoyu Li

Center for Applied Mathematics, Tianjin University
Tianjin 300072, P. R. China
E-mail: hyli1994@hotmail.com

Zhi-Qiang Wang

Department of Mathematics and Statistics, Utah State University
Logan, Utah 84322, USA
E-mail: zhi-qiang.wang@usu.edu

Abstract. We investigate the structure of nodal solutions for coupled nonlinear Schrödinger equations in the repulsive coupling regime. Among other results, for the following coupled system of N equations, we prove the existence of infinitely many nodal solutions which share the same componentwise-prescribed nodal numbers

$$(0.1) \qquad \begin{cases} -\Delta u_j + \lambda u_j = \mu u_j^3 + \sum_{i \neq j} \beta u_j u_i^2 & in\ \Omega, \\ u_j \in H_{0,r}^1(\Omega), \quad j = 1, \dots, N, \end{cases}$$

where Ω is a radial domain in $\mathbb{R}^n$ for $n = 2, 3$ and a bounded interval for $n = 1$, $\lambda > 0$, $\mu > 0$, and $\beta < 0$. More precisely, let p be a prime factor of N and write $N = pB$. Suppose $\beta \leq -\frac{\mu}{p-1}$. Then for any given non-negative integers $P_1, P_2, \dots, P_B$, (0.1) has infinitely many solutions $(u_1, \dots, u_N)$ such that each of these solutions satisfies the same property: for $b = 1, \dots, B$, u_{pb-p+i} changes sign precisely P_b times for $i = 1, \dots, p$. The result reveals the complex nature of the solution structure in the repulsive coupling regime due to componentwise segregation of solutions. Our method is to combine a heat flow approach as deformation with a minimax construction of the symmetric mountain pass theorem using a $\mathbb{Z}_p$ group action index. Our method is robust, also allowing to give the existence of one solution without assuming any symmetry of the coupling.

Keywords: Multiple nodal solution; Componentwise-prescribed number of nodes; Coupled Schrödinger equations.

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1. Introduction

1.1. **Main Result.** In this paper, we consider the following coupled nonlinear Schrödinger system of N equations:

$$(1.1) \qquad \begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{i=1, i \neq j}^{N} \beta_{ij} u_j u_i^2 & in\ \Omega, \\ u_j \in H_{0,r}^1(\Omega), \quad j = 1, \dots, N, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a radially symmetric domain, bounded or unbounded for $n = 2, 3$, and a bounded interval for $n = 1$, and the constants satisfy $\lambda_j > 0$, $\mu_j > 0$

for $j = 1, ..., N$, and $\beta_{ij} = \beta_{ji}$ for $i \neq j$. $H_{0,r}^1(\Omega)$ denotes the subspace of $H_0^1(\Omega)$ of radially symmetric functions.

To demonstrate the spirit of our results, we state the result in a special case first, where all λ_j are equal to $\lambda > 0$, all μ_j are equal to $\mu > 0$, and all β_{ij} are equal to β for $i \neq j$, i.e.,

$$(1.2) \qquad \begin{cases} -\Delta u_j + \lambda u_j = \mu u_j^3 + \sum_{i \neq j} \beta u_j u_i^2 & in \ \Omega, \\ u_j \in H_{0,r}^1(\Omega), \quad j = 1, \ldots, N. \end{cases}$$

Theorem 1.1. *Let p be a prime factor of N and write $N = pB$. Suppose $\beta \leq -\frac{\mu}{p-1}$, Then for any given non-negative integers $P_1, P_2, \ldots, P_B$, (1.2) has infinitely many solutions $(u_1, \ldots, u_N)$ such that for $b = 1, ..., B$, u_{pb-p+i} changes sign precisely P_b times for $i = 1, ..., p$.*

The result gives new insight into the structure of nodal solutions for coupled Schrödinger equations. For a componentwise-prescribed number of nodes, we find infinitely many solutions which share the same number of nodal domains, revealing more complexity of nodal solutions compared with the classical scalar field equation $-\Delta u + u = |u|^{p-2}u$ for which a long standing folklore has been the uniqueness of sign-changing solutions with a prescribed number of nodes. We say the solutions given above have componentwise-prescribed number of nodes.

Our method works in more general form than that of (1.2). Denote by $\mathcal{B} = (\beta_{ij})_{N \times N}$ the coefficient matrix involved on the right hand side of Problem (1.2), where we denote $\beta_{ii} = \mu_i$. We do not need to require the same values for λ_j, μ_j and β_{ij}. We denote the transformation of exchanging the i-th row and the j-th row of a matrix by R_{ij}, the i-th column and the j-th column by C_{ij}.

Theorem 1.2. *Let p be a prime factor of N and write $N = pB$. Assume the following four conditions hold.*

    (A) $\lambda_{pb-p+1} = \lambda_{pb-p+2} = \cdots = \lambda_{pb} > 0$ for $b = 1, \ldots, B$.

    (B) For $i, j = 1, \ldots, N$ and $i \neq j$, $\beta_{ij} = \beta_{ji} \leq 0$ and $\mu_j > 0$.

    (C) For $b = 1 \ldots, B$, $\mathcal{B} = (\beta_{ij})_{N \times N}$ is invariant under the action of

$$\prod_{i=1}^{p-1} C_{pb-p+i,pb-p+i+1} \circ R_{pb-p+i,pb-p+i+1}.$$

    (D) For $b = 1, \ldots, B$ and $pb - p + 1 \leq j \leq pb$, it holds

$$\mu_j + \sum_{pb-p+1 \leq i \leq pb \, ; \, i \neq j} \beta_{ij} \leq 0.$$

Then for any given non-negative integers $P_1, \ldots, P_B$, the Problem (1.1) possesses infinitely many solutions $(u_1, \ldots, u_N)$ such that for $b = 1, ..., B$, u_{pb-p+i} changes sign precisely P_b times for $i = 1, ..., p$.

Remark 1.3. *In the special case of* (1.2), *we see (A) and (C) are satisfied readily, while (B) and (D) are satisfied under* $\beta \leq -\frac{\mu}{p-1} < 0$. *Thus Theorem 1.1 follows.*

Our approach in this paper to study multiplicity of nodal solutions having the same componentwise-prescribed number of nodes is to combine an associated parabolic flow serving as a descending flow of the variational problem with a minimax construction in the spirit of the symmetric mountain pass theorem via an $\mathbb{Z}_p$ index theory in the presence of invariant sets of the flow. While for multiplicity of nodal solutions having the same nodal numbers, we need in an essential way the symmetry of the coupling coefficients, our method will be set up in a more general framework and also allows us to treat in the general case Problem (1.1) without such a symmetry. In this general setting we prove the existence of one solution with a componentwise-prescribed number of nodes, and this gives a different proof of a result from [32] where such a solution was given by gluing on Nehari manifold. In the present paper, we employ the corresponding parabolic flow as a tool for deformation of the variational problem, which is essential for establishing multiplicity results.

Theorem 1.4. *Assume* λ_j, $\mu_j > 0$ *for* $j = 1 \ldots, N$. *Then for any non-negative integers* $P_1, \ldots, P_N$, *there exists* $b > 0$ *such that if* $\beta_{ij} \leq b$ *for all* $i \neq j$, *Problem (1.1) has a solution* $(u_1 \ldots, u_N)$ *with the* j-*th component* u_j *changing sign precisely* P_j *times for* $j = 1 \ldots, N$.

We note that while for the multiplicity results we need the condition of negative coupling, for the existence of one solution we can allow a wider range of coupling here.

To make the symmetry condition in Theorem 1.2 clear, we give three examples for the coupling coefficient matrix $\mathcal{B} = (\beta_{ij})_{N \times N}$ of Problem (1.1). The matrices are cut into blocks for suitable symmetry.

Example 1.5. *For the case* $N = 4$ *and* $p = B = 2$, *the assumptions* $(B - D)$ *are satisfied in the following form*

$$\left(\begin{array}{cc|cc} \mu_1 & \beta_1 & \beta_3 & \beta_3 \\ \beta_1 & \mu_1 & \beta_3 & \beta_3 \\ \hline \beta_3 & \beta_3 & \mu_2 & \beta_2 \\ \beta_3 & \beta_3 & \beta_2 & \mu_2 \end{array} \right)$$

with $\beta_i \leq -\mu_i < 0$ *for* $i = 1, 2$ *and* $\beta_3 \leq 0$. *Assume that* $\lambda_1 = \lambda_2 > 0$ *and* $\lambda_3 = \lambda_4 > 0$. *Then given any two nonnegative integers* P_1, P_2, *there exist infinitely many solutions with first two components* u_1, u_2 *each having exactly* P_1 *simple zeros, and with the last two components* u_3, u_4 *each having exactly* P_2 *simple zeros.*

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If we set $N = 6$, $p = 2$ and $B = 3$, the assumptions $(B - D)$ are satisfied in

$$\left(\begin{array}{cc|cc|cc} \mu_1 & \beta_1 & \beta_4 & \beta_4 & \beta_5 & \beta_5 \\ \beta_1 & \mu_1 & \beta_4 & \beta_4 & \beta_5 & \beta_5 \\ \hline \beta_4 & \beta_4 & \mu_2 & \beta_2 & \beta_6 & \beta_6 \\ \beta_4 & \beta_4 & \beta_2 & \mu_2 & \beta_6 & \beta_6 \\ \hline \beta_5 & \beta_5 & \beta_6 & \beta_6 & \mu_3 & \beta_3 \\ \beta_5 & \beta_5 & \beta_6 & \beta_6 & \beta_3 & \mu_3 \end{array} \right)$$

with $\beta_i \le -\mu_i < 0$ for $i = 1, 2, 3$ and $\beta_4, \beta_5, \beta_6 \le 0$.

If we set $N = 6$, $p = 3$ and $B = 2$, the assumptions $(B - D)$ are satisfied in

$$\left(\begin{array}{ccc|ccc} \mu_1 & \beta_1 & \beta_1 & \beta_3 & \beta_3 & \beta_3 \\ \beta_1 & \mu_1 & \beta_1 & \beta_3 & \beta_3 & \beta_3 \\ \beta_1 & \beta_1 & \mu_1 & \beta_3 & \beta_3 & \beta_3 \\ \hline \beta_3 & \beta_3 & \beta_3 & \mu_2 & \beta_2 & \beta_2 \\ \beta_3 & \beta_3 & \beta_3 & \beta_2 & \mu_2 & \beta_2 \\ \beta_3 & \beta_3 & \beta_3 & \beta_2 & \beta_2 & \mu_2 \end{array} \right)$$

with $\beta_i \le -\frac{\mu_i}{2} < 0$ for $i = 1, 2$ and $\beta_3 \le 0$.

1.2. **Historical Remarks and the Idea of the Present Paper.** The nonlinear coupled elliptic system (1.1) has its theoretical root in Bose-Einstein condensates. The solutions of Problem (1.1) give rise to standing waves solutions of the time-dependent nonlinear coupled Schrödinger system

$$(1.3) \qquad \begin{cases} -i\partial_t \Phi_j - \Delta \Phi_j = \mu_j |\Phi_j|^2 \Phi_j + \sum_{i \ne j} \beta_{ij} \Phi_j |\Phi_i|^2 & in \ \Omega, \\ \Phi_j(t, x) \in \mathbb{C}, \quad j = 1, \ldots, N, \end{cases}$$

for $j = 1, \ldots, N$ and $t > 0$. In physics models, the parameters μ_j and β_{ij} are the intraspecies and interspecies scattering lengths respectively. When $\beta_{ij} > 0$, it is called the attractive case, when $\beta_{ij} < 0$, it is called the repulsive case. In this paper, we mainly consider the repulsive case while small attractive coupling is also considered. [2, 35] is referred for more physics background.

In recent years, a large number of mathematical results on Problem (1.1) have appeared, e.g., in [5, 10, 9, 17, 25, 26, 28, 30, 31, 36, 44, 49, 50, 52] for studying various aspects of the problem such as existence theory for the attractive case and for the repulsive case, the bifurcation analysis, the synchronization and segregation for different coupling parameter regimes, and convergence and regularity of large couplings in the repulsive case etc. We refer to these papers for more references therein. In the repulsive coupling case, solutions tend to be segregated component-wisely creating more complex patterns of solutions. The application of variational methods to the coupled Schrödinger systems mainly involves minimizing methods and minimax methods. The symmetric mountain-pass theorem

has been well adopted for a large number of elliptic problems that goes back to the celebrated [6] by Ambrosetti-Rabinowitz. For Problem (1.1), the first difficulty is that there exist infinitely many so-called semi-trivial solutions (solutions with some components being zeros) so the system is degenerated to a system of smaller number of equations. In [30, 31], Liu and Wang proved the existence of infinitely many non-trivial (all components are non-zero) solutions to Problem (1.1) via invariant sets of descending flow and Nehari manifold method respectively. In [17] and [50], the authors proved multiplicity results of positive solutions to the special case Problem (1.2) which possesses the componentwise permutation symmetry. This can be considered as a typical result for the repulsive case which shows distinct difference between a scalar field equation and a coupled nonlinear elliptic system since for the classical scalar field equation $-\Delta u + u = |u|^{p-2}u$ the uniqueness of positive solutions is well known ([21, 23]) and a folklore has been the uniqueness of nodal solutions with a prescribed number of nodes ([8, 48]). In [28], the authors obtained a multiplicity result of solutions to Problem (1.1) in general domains with prescribed number of positive components and prescribed number of sign-changing components. Recently, for radially symmetric domains, the existence of a nodal solution with componentwise prescribed number of nodes is obtained by Liu and Wang in [32] via gluing on Nehari's method, extending the work for scalar equations ([11, 46]). More precisely, it is proved in [32] that for any given nonnegative integers $P_1, ..., P_N$ there is a nodal solution $(u_1, ..., u_N)$ to Problem (1.1) such that u_i has exactly P_i simple zeros, $i = 1, ..., N$.

In the present paper, our main concern and interest is that for a componentwise-prescribed number of nodes whether there are *multiple such solutions* sharing the given nodal number, in particular whether there are *infinitely many such solutions?* This is the main goal of our studies. Our result gives a construction of infinitely many solutions sharing a given componentwise-prescribed number of nodes (Theorem 1.1 and 1.2).

To deal with the sign-changing property of multiple solutions we will employ the heat flow of the corresponding coupled heat equations to Problem (1.1). An important part of the present paper lies in the studies of the associated heat flow, including the existence and regularity results, the global existence and blow-up results, the non-increasing property of the sign-changing numbers along flow lines, the boundedness of trajectories and dynamical property of some invariant sets of the flow. We refer [4, 22, 33, 42] for general discussions on the parabolic problems. There have been a lot of works in the literature in which elliptic problems are solved with the help of heat flow methods. In [16], Conti, Merizzi and Terracini proved the existence of radial solutions with prescribed number of nodal domains to a scalar field equation. Utilizing the semilinear parabolic flow and the topological degree, they proved the result which was only treated by Nehari

method before ([11, 46]). In [15], Chang established a variational framework and applied it to minimal surface problems. Quittner proved the existence and multiplicity of solutions of several semilinear elliptic problems and other dynamical properties by using parabolic flow in [38, 39, 41]. In [1], Ackermann and Bartsch developed the idea of superstable manifold and refined the symmetric mountain-pass theorem for sign-changing solutions (c.f. [12, Section 2]), which produced multiplicity results, nodal properties and order comparison results. More works on using parabolic flow to treat elliptic problems can be found in the references in these papers. However, there are few results on the coupled Schrödinger systems using the heat flow. We mention [52] in which for two equations a comparison between components of positive solutions was obtained. We will further develop the ideas in these papers by using heat flow as a tool of descending flow of our variational problem for Problem (1.1). In fact, with the growth of the nonlinearity, a finer analysis on the global existence of the parabolic flow is also required. Combining the Cazenave-Lions interpolation ([13] and [15]) and some estimate in [15, 38], we can address that the growth of the nonlinearity is admissible to the global existence in dimensions $n \leq 3$. A finer analysis of the invariant sets requires the H^1-bounds of global solutions which we will prove in Section 2.4. We use a variant of the method in [40], and we refer [19, 40] for more references on this topic. Another important part of our work involves using some natural permutation symmetry in the coupling patterns. We will make use of the symmetry of the problem, that is, the problem is invariant under a $\mathbb{Z}_p$ group action of a cyclic permutation σ. With the heat flow serving as a deformation we will construct minimax critical values in the spirit of the symmetric mountain pass theorem via a $\mathbb{Z}_p$ index. We need to build up special symmetric subsets of large $\mathbb{Z}_p$ index contained in the invariant sets of the flow. Inspired by the approach for scalar equations in [16] our method is a sharper and symmetric variant of [16] for coupled systems. To accomplish this, a certain combination of the methods in [16, 31, 50] are needed. While the idea of Nehari manifold was used in [31], we will use the more natural ingredient, the boundary of the stable manifold of the origin, which has the advantage in keeping the non-increasing property of the sign-changing number along flow lines.

1.3. **The Structure of This Paper.** Section 2 mainly deals with the regularity and dynamical properties of the heat flow of the corresponding heat equations, constructing various invariant sets of the flow. We prove the existence result of Theorem 1.4 for the general system (1.1) in Section 3, and this also will set up the stage for the proof of the main result Theorem 1.2 in Section 4. In Section 4, we give the proof of the multiplicity result Theorem 1.2 from a minimax argument

by constructing symmetric sets of large $\mathbb{Z}_p$ index inside various flow invariant sets on the boundary of the domain of attraction of the origin.

2. Dynamical properties of the associated heat equations

The parabolic flow associated with the elliptic system will be used as a mean of descending flow for the variational problems. We start by collecting some relevant results on existence and regularity of the heat equations. Then we will develop some further estimates and construct some invariant sets of the flow which will be used in our proof later. Let us fix some notations first.

We always use capital letters to represent vector valued functions and the corresponding lower case letters with subscript for their components. For example, $U = (u_1, \ldots, , u_N)$ and $V = (v_1, \ldots, v_N)$. A solution $U = (u_1, \ldots, , u_N)$ to Problem (1.1) is called a non-trivial solution if and only if $u_j \neq 0$ for any $j = 1, \ldots, N$. It is semi-trivial if and only if $U \neq \theta$ and U has some zero components, where θ is the zero vector.

The norm of Lebesgue space $L^p(\Omega)$ is denoted by $|\cdot|_p$ and the norm of $H_0^1(\Omega)$ by $\|\cdot\|$. For the product of spaces, such as $(L^p(\Omega))^N$ $((H_0^1(\Omega))^N)$, we still use $|\cdot|_p (\|\cdot\|)$ to denote its norm. With no confusions, we sometimes omit the domain Ω, the boundary condition and the radial condition and only denote the corresponding spaces by L^p, H^1, H^2 and H^s for $s \in (1, 2)$.

2.1. Existence and Regularity Results of the Parabolic Flow.
Instead of the gradient flow, we will combine our variational structure with the following nonlinear coupled parabolic system:

$$(2.1) \quad \begin{cases} \frac{\partial}{\partial t} u_j - \Delta u_j + \lambda_j u_j = \mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_j u_i^2 & in \ \Omega, \\ u_j(t, x) \in H_{0,r}^s(\Omega), \quad j = 1, \ldots, N, \\ u_j(0, x) = u_{0,j}(x) \in H_{0,r}^s(\Omega), \quad j = 1, \ldots, N. \end{cases}$$

whose equilibria are solutions to Problem (1.1). Here, we require the coefficients λ_j's, μ_j's and β_{ij}'s satisfy the conditions in Theorem 1.4 and Theorem 1.2 when we prove two theorems respectively.

By the notation $\eta^t(U)$ we denote a solution to the parabolic system with $U = (u_{0,1}, \ldots, u_{0,N})$ as its initial data. Sometimes, for the sake of simplicity, we also write $U(t)$.

A special case of Problem (2.1) is of the form:

$$(2.2) \quad \begin{cases} \frac{\partial}{\partial t} u_j - \Delta u_j + \lambda u_j = \mu u_j^3 + \sum_{i \neq j} \beta u_j u_i^2 & in \ \Omega, \\ u_j(t, x) \in H_{0,r}^s(\Omega), \quad j = 1, \ldots, N, \\ u_j(0, x) = u_{0,j}(x) \in H_{0,r}^s(\Omega), \quad j = 1, \ldots, N. \end{cases}$$

It is obvious that an equilibrium point of Problem (2.2) is a solution to Problem (1.2). Both of the parameters s in Problems (2.1) and (2.2) will be taken to be in

[1, 2] depending upon the situation. Readers can find general theory of parabolic problems in [4, 18, 22, 33, 42]. We will state a slightly more general result on the existence and regularity for the parabolic system (2.1) than we need in this paper. Noticing that the spectrum of $-\Delta + \lambda$ is contained in $[\lambda, \infty)$, we conclude that the operator $-\Delta + \lambda$ is sectorial and, as a consequence, the existence and regularity results can be given. The results are stated and proved in terms of interpolation spaces X_α for $\alpha \in [0, 1]$ (e.g., [18]). We refer [4, 18, 22, 33, 42] once again for more information on sectorial operators and related properties on interpolation spaces. Note that with the range being L^2, the domain of the operator is $D(-\Delta + \lambda) = \{u \in H^2 | \gamma_2 u = 0\} := X_1$ (c.f. [18, (4.6), (4.7)] or [45]), where γ_2 is the trace operator in $L^2 = X_0$. Using the relation between these interpolation spaces and the Bessel-potential spaces $X_{s/2} = H_0^s(\Omega)$ for $s \in \left[1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, 2\right]$, we will state the following theorem with the H_0^s setting (c.f. [18, Theorem 4.20], [37] and [3]). The following result for Problem (2.1) is useful in the present paper.

Theorem 2.1. *Let $s \in [1, 2] \setminus \left\{\frac{3}{2}\right\}$ be fixed. If the initial value $U := (u_1, \ldots, u_N) \in (H^s)^N$, there is a unique solution $\eta^t(U) = (u_1(t), \ldots, u_N(t))$ to Problem (2.1) defined on its maximum interval $[0, T(U))$, satisfying*
  (I) it holds that

$$\eta^t(U) \in C^1((0, T(U)), (L^2)^N) \cap C([0, T(U)), (H^s)^N);$$

  (II) for any $U \in (H^s)^N$ and any $\delta \in [0, T(U))$, there are positive constants r, K such that for any $t \in [0, \delta]$

$$\|U - V\|_{(H^s)^N} < r \quad \Rightarrow \quad \|\eta^t(U) - \eta^t(V)\|_{(H^s)^N} \leq K\|U - V\|_{(H^s)^N};$$

  (III) the trivial solution $\theta \in (H^s)^N$ is asymptotically stable in $(H^s)^N$.

Remark 2.2. *Part (I) of this theorem is due to [18, Theorem 15.3, Theorem 16.2] and (II) is of [18, Proposition 16.8]. The assertion (III) is due to [22, Theorem 5.1.1].*

Remark 2.3. *In following, we mainly use the result for $s = 1$ and $s = 2$.*

Notice that the theorem also holds true if we restrict the spaces to the case of radial symmetric functions. A similar regularity theory can be found in [33].

2.2. **Global Existence of the Solutions Starting on the Boundary of the Stable Manifold.** The propositions are modified versions of some results in [16] and in [38]. In this section, we always assume $U(t) = (u_1(t), \ldots, u_N(t))$ is a solution to Problem (2.1). The energy of Problem (1.1) is the functional

$J(U) : (H_r^1)^N \to \mathbb{R}$ defined by

$$J(U) := J(u_1, \ldots, u_N)$$
$$= \frac{1}{2} \sum_{j=1}^N \int |\nabla u_j|^2 + \lambda_j |u_j|^2 - \frac{1}{4} \sum_{j=1}^N \left(\int \mu_j u_j^4 + \sum_{i \neq j} \int \beta_{ij} u_i^2 u_j^2 \right),$$

which is a C^2 functional and satisfies the (PS) condition.

Proposition 2.4. *For a solution $U(t) = (u_1(t), \ldots, u_N(t))$, we have*

$$\frac{\partial}{\partial t} J(U(t)) = -\sum_{j=1}^N \int |\partial_t u_j|^2.$$

Proof. Note that

$$\frac{\partial}{\partial t} J(U) = \sum_{j=1}^N \nabla_{u_j} J(U) \partial_t u_j.$$

By a direct computation we have

$$\nabla_{u_j} J(U) \partial_t u_j = \int \nabla u_j \cdot \nabla \partial_t u_j + \lambda_j u_j \partial_t u_j - \int \mu_j u_j^3 \partial_t u_j + \sum_{i \neq j} \beta_{ij} u_j \partial_t u_j u_i^2$$

$$= \int \partial_t u_j \left(-\Delta u_j + \lambda_j u_j - \mu_j u_j^3 - \sum_{i \neq j} \beta_{ij} u_j u_i^2 \right) = -\int |\partial_t u_j|^2.$$

Then the proposition follows.

$\square$

Corollary 2.5. *Let*

$$\mathcal{A} = \left\{ U \in (H^1)^N | T(U) = \infty \text{ and } \lim_{t \to T(U)} \eta^t(U) = \theta \text{ in } (H^1)^N \right\}.$$

Then $\mathcal{A}$ is invariant under the heat flow and is open in $(H^1)^N$.

This is a direct consequence of Theorem 2.1.

Lemma 2.6. *$\partial \mathcal{A}$ is invariant under the heat flow and $\inf_{U \in \partial \mathcal{A}} J(U) \geq 0$.*

Proof. The continuity of the energy $J(U)$ with respect to $(H^1)^N$ norm implies the second part of this lemma. Now we prove the first part.

Due to the definition of $\mathcal{A}$, if $U \in \partial \mathcal{A} \subset (H^1)^N \backslash \mathcal{A}$, an immediate consequence is that $\eta^t(U) \subset (H^1)^N \backslash \mathcal{A}$. And if there is a $t_0 \in (0, T(U))$ such that $\eta^{t_0}(U) \in (H^1)^N \backslash \overline{\mathcal{A}}$, due to Theorem 2.1 and the openness of $\mathcal{A}$ in $(H^1)^N$, we can find a $V \in \mathcal{A}$ such that $\eta^{t_0}(V) \in (H^1)^N \backslash \overline{\mathcal{A}}$. Therefore, we address a contradiction. The above deduction implies that $\eta^t(U) \subset \overline{\mathcal{A}} \backslash \mathcal{A} = \partial \mathcal{A}$.

$\square$

Now, we prove that the flow with its initial data on the boundary of the stable manifold $\partial\mathcal{A}$ has $[0,\infty)$ as its maximal existence interval. Before that, let us prove a lemma under a more general condition.

Lemma 2.7. *Suppose* $\{J(\eta^t(U))\}_{t\in[0,T(U))}$ *is bounded from below. Then* $U(t)$ *exists globally in* $(H^1)^N$.

Proof. The proof makes use of some arguments from [13], [15, Lemma 1] and [38, Section 3]. Since $\frac{\partial}{\partial t}J(\eta^t(U)) = -\sum_{j=1}^{N}\int |\partial_t u_j|^2$, the condition in the lemma implies that

$$(2.3) \qquad \sum_{j=1}^{N}\int_0^t\int|\partial_t u_j(s)|^2 dx ds = \left|J(\eta^t(U)) - J(U)\right| \leq C.$$

Now we give the L^2-estimate. First we have

$$\sum_{j=1}^{N}\int|u_j(t)|^2 dx = \sum_{j=1}^{N}\int_0^t\frac{d}{dt}\int|u_j(s)|^2 dx ds + \sum_{j=1}^{N}\int|u_j(0)|^2 dx$$

$$= 2\sum_{j=1}^{N}\int_0^t\int u_j\cdot\partial_t u_j dx ds + \sum_{j=1}^{N}\int|u_j(0)|^2 dx$$

$$\leq C\left(1 + \sum_{j=1}^{N}\int_0^t\int|u_j(s)|^2 dx ds\right).$$

Using the Gronwall's inequality, we have

$$(2.4) \qquad \sum_{j=1}^{N}\int|u_j(t)|^2 dx \leq Ce^{Ct}.$$

Notice that

$$\sum_{j=1}^{N}\int|u_j(t)|^2 dx - \sum_{j=1}^{N}\int|u_j(0)|^2 dx$$

$$= 2\sum_{j=1}^{N}\int_0^t\int u_j(s)\cdot\partial_t u_j(s) dx ds$$

$$= -8\int_0^t J(\eta^s(U)) ds + 2\sum_{j=1}^{N}\int_0^t\int\left(|\nabla u_j(s)|^2 + \lambda_j|u_j|^2\right) dx ds.$$

Therefore,

$$(2.5) \qquad \int_0^t\int|\nabla u_j(s)|^2 dx ds \leq Ce^{Ct}$$

follows immediately. Multiplying u_j on both sides of the j-th equation of Problem (2.1), integrating over Ω and summing up with j, we obtain

$$\sum_{j=1}^{N} \int u_j \cdot \partial_t u_j + \sum_{j=1}^{N} \|u_j\|^2 = \sum_{j=1}^{N} \int \mu_j u_j^4 + \sum_{i \neq j} \beta_j u_i^2 u_j^2.$$

Now we apply some methods in [13] and [15]. For any $T > 0$, we consider the norms on the time interval $[0, 2T]$. Due to the definition of the energy J and (2.4), we have

$$\sum_{j=1}^{N} \|u_j\|^2 \leq 4J(U) + \sum_{j=1}^{N} \int u_j \cdot \partial_t u_j$$

(2.6)
$$\leq C + \left(\sum_{j=1}^{N} |u_j|_2^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{N} |\partial_t u_j|_2^2 \right)^{\frac{1}{2}}$$

$$\leq C + C \left(\sum_{j=1}^{N} |\partial_t u_j|_2^2 \right)^{\frac{1}{2}}$$

for $t \in [0, 2T]$. This implies that

$$\int_0^{2T} \left(\int |u_j(t)|^{2^*} dx \right)^{\frac{4}{2^*}} dt \leq C \int_0^{2T} \left(\sum_{j=1}^{N} \|u_j(t)\|^2 \right)^2 dt$$

$$\leq C(T) + \sum_{j=1}^{N} \int_0^{2T} |\partial_t u_j(t)|_2^2 dt \leq C(T).$$

That is, $u_j \in L^4((0, 2T), L^{2^*}(\Omega))$, with $2^* = 6$ for dimension 3. At the same time, the embedding $H^1 \hookrightarrow L^p$ holds for any $p \geq 2$ for dimension 2. Therefore, we also have $u_j \in L^4((0, 2T), L^6(\Omega))$ for dimension 2. Notice that (2.3) implies that $\partial_t u_j \in L^2((0, 2T), L^2(\Omega))$.

Next we claim $u_j \in L^\infty((0, T), L^{\frac{18}{5}}(\Omega))$ for $j = 1, \ldots, N$. We prove it in the following paragraph. Using the idea of Cazenave-Lions interpolation (c.f. [13], [15]), we set $\overline{u}_j(x, t) = u_j(x, t) \cdot \theta(t)$ and $v = |\overline{u}_j|^3$. Here, $\theta(t)$ is a smooth cut-off function such that $\theta(t) = 1$ for $t \in [0, T]$, $\theta(s) = 0$ for $t \in \left[\frac{3T}{2}, 2T \right]$ and $\theta(t) \in [0, 1]$ for $t \in [0, 2T]$. Using the Newton-Leibniz formula and Hölder's inequality, we can compute for any $t \in [0, T]$

$$|u_j(t)|_{\frac{18}{5}} = |v|_{\frac{6}{5}}^{\frac{1}{3}} \leq \left| \int_t^{2T} |v_t|_{\frac{6}{5}} \right|^{\frac{1}{3}} \leq C \left| \int_t^{2T} |\overline{u}_j^2 \cdot \partial_t \overline{u}_j|_{\frac{6}{5}} ds \right|^{\frac{1}{3}}$$

$$\leq C \left(\int_0^{2T} |\overline{u}_j^2|_3 |\partial_t \overline{u}_j|_2 ds \right)^{\frac{1}{3}}$$

with

$$\frac{5}{6} = \frac{1}{3} + \frac{1}{2}.$$

Hence, for $t \in [0, T]$ using Hölder's inequality with respect to the variable s

$$|u_j(t)|_{\frac{18}{5}} \le C\Big(\int_0^{2T} |\overline{u}_j|_6^2 |\partial_t \overline{u}_j|_2 ds\Big)^{\frac{1}{3}} \le C\Big(\int_0^{2T} |\overline{u}_j|_6^4 ds\Big)^{\frac{1}{6}} \Big(\int_0^{2T} |\partial_t \overline{u}_j|_2^2 ds\Big)^{\frac{1}{6}}.$$

Therefore, we have

$$\sup_{t \in (0,T)} |u_j(t)|_{\frac{18}{5}} \le C\Big(\int_0^{2T} |\overline{u}_j|_6^4 ds\Big)^{\frac{1}{6}} \Big(\int_0^{2T} |\partial_t \overline{u}_j|_2^2 ds\Big)^{\frac{1}{6}}.$$

Noticing that

$$\partial_t \overline{u}_j = \partial_t u_j \cdot \theta + u \cdot \partial_t \theta.$$

Using the fact that $\partial_t u_j \in L^2((0, 2T), L^2(\Omega))$ for any $j = 1, \ldots, N$ and (2.4),

$$\int_0^{2T} |\partial_t \overline{u}_j|_2^2 ds \le C \left(\int_0^{2T} |\partial_t u_j|_2^2 ds + \int_0^{2T} |u_j|_2^2 ds\right) \le C(T).$$

Therefore, we can conclude that for any $j = 1, \ldots, N$,

$$\sup_{t \in (0,T)} |u_j(t)|_{\frac{18}{5}} \le C(T)$$

for any $t \in [0, T]$, i.e. $u_j \in L^\infty((0, T), L^{\frac{18}{5}}(\Omega))$.

Multiplying u_j^3 on the both sides of the j-th equation and integrating over Ω,

$$\frac{d}{dt} |u_j^2|_2^2 + \|u_j^2\|^2 \le C\Big(\mu_j |u_j^2|_3^3 + \sum_{i \neq j} \int \beta_{ij} u_j^4 u_i^2\Big)$$

$$\le C \sum_{l=1}^N |u_l^2|_3^3.$$

Here, the second inequality is due to Young's inequality. Using the interpolation inequality, we have

$$\frac{1}{3} = \frac{\frac{4}{7}}{6} + \frac{\frac{3}{7}}{\frac{9}{5}}$$

and

$$\frac{d}{dt}|u_j^2|_2^2 + \|u_j^2\|^2 \leq C \sum_{l=1}^N |u_l^2|_{\frac{9}{5}}^{\frac{9}{7}} |u_l^2|_6^{\frac{12}{7}}$$

$$\leq C \sum_{l=1}^N |u_l^2|_{\frac{9}{5}}^{\frac{9}{7}} \|u_l^2\|^{\frac{12}{7}}$$

$$\leq \frac{1}{2N} \sum_{j=1}^N \|u_j^2\|^2 + C \sum_{l=1}^N |u_l|_{\frac{18}{5}}^{\frac{18}{5}}$$

$$\leq \frac{1}{2N} \sum_{j=1}^N \|u_j^2\|^2 + C.$$

It follows that

$$\frac{d}{dt} \sum_{j=1}^N |u_j^2|_2^2 + \sum_{j=1}^N \|u_j^2\|^2 \leq \frac{1}{2} \sum_{j=1}^N \|u_j^2\|^2 + C$$

and

$$\sum_{j=1}^N |u_j^2(t)|_2^2 \leq C \int_0^T dt + \sum_{j=1}^N |u_j^2(0)|_2^2 \leq C(T),$$

i.e. $u_j \in L^\infty((0,T), L^4(\Omega))$ for $j = 1 \ldots, N$. Using the definition of the energy J,

$$\sum_{j=1}^N \|u_j\|^2 \leq 2J(\eta^t(U)) + \frac{1}{2} \sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j^2 \leq C.$$

Hence $u_j \in L^\infty((0,T), H_0^1(\Omega))$. Therefore, $T(U) = \infty$ since $T > 0$ is arbitrary.

$\square$

Remark 2.8. *In fact, this lemma proves that, under the assumptions of Lemma 2.7, for any $T > 0$, there is a constant $C > 0$ depends only on $T > 0$ and the H^1-norm of the initial data such that $\|U(t)\| \leq C$ for any $t \in [0, T]$.*

Remark 2.9. *With the same conditions, we can address that $T(U) = \infty$, i.e. $\eta^t(U) \in C((0,\infty), (H^s)^N)$ for any $s \in (1,2] \backslash \{\frac{3}{2}\}$. To this end, we only need to use the formula of variation of constants and the fact that $\sum_{j=1}^N \left(\mu_j u_j^3 + \sum_{i \neq j} u_i^2 u_j \right) \in L^\infty((0,T), L^2(\Omega))$ (c.f. [33, Proposition 7.1.8]). Especially, with the same method, we can prove $\eta^t(U) \in C((0,\infty), (H^2)^N)$ when $U \in (H^2)^N$.*

The following corollary follows from the fact that $\inf_{\partial \mathcal{A}} J \geq 0$ and that the energy J is non-increasing along the flow line.

Corollary 2.10. *For any $U \in \partial \mathcal{A}$, $T(U) = \infty$.*

2.3. **The H^1 bounds of the Solutions Starting on $\partial\mathcal{A}$.** We will borrow some ideas used by Quittner in [40], which can be adapted for our situation (see also [19] for some related work).

Lemma 2.11. *Let $U(t)$ be a global solution to (2.1) such that $\lim_{t\to\infty} J(U(t)) = E_1$ is finite. Then there is $C > 0$ depending continuously upon the L^2-norm of the initial data, the initial energy $E_0 := J(U(0))$ and E_1, such that for any $t \geq 0$, $|U(t)|_2 \leq C$.*

Proof. Suppose $0 \leq t_0 \leq t < +\infty$. Denote $\Phi(t) = \int_{t_0}^t |U(s)|_2^2 ds$ and $E_0 = J(U(0))$. Using the computation in previous subsection, we have

$$\left| \sum_{j=1}^N |u_j(t)|_2^2 - \sum_{j=1}^N |u_j(t_0)|_2^2 \right| = 2 \left| \sum_{j=1}^N \int_{t_0}^t \int_\Omega u_j \partial_t u_j dx ds \right|$$

$$\leq 2 \Big(\sum_{j=1}^N \int_{t_0}^t |\partial_t u_j|_2^2 ds \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^N \int_{t_0}^t |u_j|_2^2 ds \Big)^{\frac{1}{2}}$$

$$\leq 2\sqrt{E_0 - E_1}\,\Phi(t)^{\frac{1}{2}},$$

which gives

$$(2.7) \qquad \Phi'(t) \leq |U(t_0)|_2^2 + 2\sqrt{E_0 - E_1}\,\Phi(t)^{\frac{1}{2}}.$$

Then we can compute that

$$2\Big(\sqrt{\Phi(t)} - |U(t_0)|_2\Big)'_+ = \frac{\Phi'(t)}{\sqrt{\Phi(t)}} \chi_{\{\sqrt{\Phi} > |U(t_0)|_2\}} \leq |U(t_0)|_2 + 2\sqrt{E_0 - E_1}.$$

This gives

$$(2.8) \qquad \sqrt{\Phi(t)} \leq |U(t_0)|_2 + \big(|U(t_0)|_2 + 2\sqrt{E_0 - E_1}\big)\frac{t - t_0}{2}.$$

Combining above deductions, we have

$$\left| \sum_{j=1}^N |u_j(t)|_2^2 - \sum_{j=1}^N |u_j(t_0)|_2^2 \right|$$

$$(2.9) \qquad \leq 2\sqrt{E_0 - E_1}\left(|U(t_0)|_2 + \big(|U(t_0)|_2 + 2\sqrt{E_0 - E_1}\big)\frac{t - t_0}{2} \right).$$

Set

$$(2.10) \qquad C_1 = \frac{1}{\min_j \lambda_j}(9 + 8|E_0| + 8|E_1|) + 2|U(0)|_2^2 + 81(E_0 - E_1) + 3$$

We claim $|U(t)|_2^2 \leq C_1$ for any $t \geq 0$. We prove the claim by contradiction. Suppose that there is a $\tau > 0$ such that $|U(\tau)|_2^2 > C_1$. First since $\sum_{j=1}^N \int_0^\infty |\partial_t u_j|_2^2 ds = E_0 - E_1 < \infty$, we can find a sequence $t_k \to \infty$ such that

$\nabla J(U(t_k)) \to 0$ and $J(U(t_k)) \to E_1$ as $k \to \infty$. Thus $(U(t_k)) \subset (H^1)^N$ is a (PS) sequence. It is easy to check (e.g., [53]) that $\|U(t_k)\|^2 \le 4(1+E_1)$. We also have
- $|U(0)|_2^2 \le |U(0)|_2^2 + 1 < \frac{C_1}{2}$;
- $|U(t_k)|_2^2 \le \frac{4(1+E_1)}{\min_j \lambda_j} < \frac{C_1}{2}$.

Let k be the integer such that $\tau \in [t_{k-1}, t_k]$ and, without loss of generality, let us assume $|U(\tau)|_2^2 = \max_{[t_{k-1}, t_k]} |U(s)|_2^2$. Then for any $t \in [\tau, \tau+1]$, applying (2.9) and the fact that $C_1 > 81(E_0 - E_1) + 1$,

$$|U(t)|_2^2 \ge |U(\tau)|_2^2 - 2\sqrt{E_0 - E_1}\Big(\frac{5}{2}|U(\tau)|_2 + 3\sqrt{E_0 - E_1}\Big)$$
$$\ge |U(\tau)|_2^2 - 5\sqrt{E_0 - E_1}|U(\tau)|_2 - 6(E_0 - E_1)$$
$$> \frac{|U(\tau)|_2^2}{2} > \frac{C_1}{2} > |U(t_k)|_2^2.$$

This implies that $t_k \notin [\tau, \tau+1]$. Therefore $\tau + 1 < t_k$ and $\tau + 1 \in [t_{k-1}, t_k]$. Consequently we have $|U(\tau+1)|_2 \le |U(\tau)|_2$. From above computation, we also have $|U(t)|_2^2 \ge \frac{C_1}{2}$ for $t \in [\tau, \tau+1]$. And now, since $C_1 > \frac{1 + 8(E_0 - E_1)}{\min_j \lambda_j}$, we have

$$0 \ge |U(\tau+1)|_2^2 - |U(\tau)|_2^2 = 2\sum_{j=1}^N \int_\tau^{\tau+1} \int_\Omega u_j \partial_t u_j \, dx \, ds$$
$$\ge -8 \int_\tau^{\tau+1} J(U(s)) \, ds + 2 \int_\tau^{\tau+1} \sum_{j=1}^N \lambda_j |u_j|_2^2(s) \, ds$$
$$\ge -8E_0 + 2\min_j \lambda_j \int_\tau^{\tau+1} |U(s)|_2^2 \, ds$$
$$\ge -8E_0 + \min_j \lambda_j C_1 > 1,$$

which is a contradiction. Hence we have $|U(t)|_2^2 \le C_1$ for any $t \ge 0$.

$\square$

For a further discussion on the boundedness of trajectories of the flow in $(H^1)^N$, we need a result on the maximal regularity of parabolic equation in [4, Theorem III.4.10.7]. Spaces involving time will be used here, such as $L^p(I, X)$, $W^{1,p}(I, X)$, where I is an interval and X is a Banach space with a norm $\|\cdot\|_X$, and the norms are defined respectively by

$$\|u\|_{L^p(I,X)} = \Big(\int_I \|u(t)\|_X^p \, dt\Big)^{\frac{1}{p}}$$

and

$$\|u\|_{W^{1,p}(I,X)} = \Big(\int_I \Big\|\frac{du}{dt}(t)\Big\|_X^p + \|u(t)\|_X^p \, dt\Big)^{\frac{1}{p}}.$$

Theorem 2.12. *Consider the linear parabolic problem*

(2.11)
$$
\begin{cases}
\frac{\partial}{\partial t}u - \Delta u + \lambda_0 u = f & in\ \Omega, \\
u(t,x) = 0 & on\ \partial\Omega, \\
u(0,x) = u_0(x) & in\ \Omega,
\end{cases}
$$

where $\lambda_0 > 0$. Given a compact interval $I = [0,T]$, $f \in L^q(I, L^p(\Omega))$ and $1 < p, q < \infty$, the solution u to the Problem (2.11) satisfies

(2.12) $\quad \|u\|_{W^{1,q}(I,L^p(\Omega))} + \|u\|_{L^q(I,W^{2,p}(\Omega))} \leq C_{MR}\big(\|u_0\|_{W^{s,p}(\Omega)} + \|f\|_{L^q(I,L^p(\Omega))}\big),$

where C_{MR} is a positive constant independent of f, u_0 and I and $s > 2\left(1 - \frac{1}{q}\right)$.

Remark 2.13. *In fact, this is a special case of [4, Theorem III.4.10.7]. We only give this version for our purpose here. In the original version stated in [4], the first term on the right hand side of (2.12) is in the form of $\|u_0\|_{X_{p,q}}$, where the interpolation space $X_{p,q} = (L^p(\Omega), W^{2,p}(\Omega))_{1-\frac{1}{q},q}$ satisfies $W^{s,p}(\Omega) \hookrightarrow X_{p,q}$ for $s > 2\left(1 - \frac{1}{q}\right)$. We refer [27, 33, 47] once again for details on interpolation spaces.*

Now we prove the H^1-boundedness of the global solutions.

Lemma 2.14. *Let $U(t)$ be a global solution to Problem (2.1) with $U(0) \in (H^s)^N$ for $s \in (1,2]$ such that $\lim_{t\to\infty} J(U(t)) \geq 0$. Then there is a constant $C > 0$ depending only on the H^2-norm of $U(0)$ and the initial energy E_0 such that $\|U(t)\| \leq C$ for any $t \geq 0$.*

Proof. Denote the interval $I = [t_0, t_0 + T]$. Firstly, using the global L^2 bound of $U(t)$ and the same computation of (2.6), we have,

(2.13)
$$
\sum_{j=1}^{N} \|u_j\|^2 \leq C(C_1)\big(1 + |\partial_t U|_2\big).
$$

Here C_1 is the L^2 bound of the solution $U(t)$. Using Theorem 2.12 with respect to each equation in Problem (2.1) and putting $p = 2$ and $q = \frac{4}{3}$ (Therefore, $s \in \left(\frac{1}{2}, 2\right]$), we have

$$
\|u_j\|_{L^{\frac{4}{3}}(I,H^2(\Omega))} \leq C_{MR}\left(\sum_{i=1}^{N}\|U(t_0)_i\|_{H^s} + \Big\|\mu_j u_j^3 + \sum_{i\neq j}\beta_{ij}u_i^2 u_j\Big\|_{L^{\frac{4}{3}}(I,L^2(\Omega))}\right).
$$

The last term can be estimated as follows:

$$\left\|\mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j\right\|_{L^{\frac{4}{3}}(I, L^2(\Omega))} \leq C\left(\int_{t_0}^{T+t_0}\left(|u_j|_6^3 + \sum_{i=1}^N |u_i^2 u_j|_2\right)^{\frac{4}{3}} ds\right)^{\frac{3}{4}}$$

$$\leq C\left(\int_{t_0}^{T+t_0}\left(\sum_{i=1}^N |u_i|_6^3\right)^{\frac{4}{3}} ds\right)^{\frac{3}{4}}$$

$$\leq C\left(\int_{t_0}^{T+t_0}\left(\sum_{i=1}^N \|u_i\|^4\right) ds\right)^{\frac{3}{4}}$$

$$\leq C\left(\int_{t_0}^{T+t_0}\left(\sum_{i=1}^N \|u_i\|^2\right)^2 ds\right)^{\frac{3}{4}}.$$

Using (2.13) and Proposition 2.4, we have

$$\left\|\mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j\right\|_{L^{\frac{4}{3}}(I, L^2(\Omega))} \leq C(C_1)\left(\int_{t_0}^{t_0+T}\left(1 + |\partial_t U|_2^2\right) ds\right)^{\frac{3}{4}}$$

$$\leq C(C_1)(T + E_0)^{\frac{3}{4}} \leq C(C_1, E_0)(T+1)^{\frac{3}{4}}.$$

Now set

(2.14)
$$C_2 = 8(NC_{MR})^2\left((\|U(0)\|_{(H^s)^N} + 1)^2 + C(C_1, E_0)^2(2NC_{MR} + 1)^2\right) + C(C_1, E_0) + 1.$$

Let $T = \left(2NC_{MR} + 1\right)^{\frac{4}{3}}$ and $t_0 = 0$. Then we notice that $\|U(0)\|_{(H^s)^N} \leq C_2$. And we have

$$\left(\int_0^T \|U(\tau)\|_{(H^2)^N}^{\frac{4}{3}} d\tau\right)^{\frac{3}{4}} \leq \sum_{j=1}^N\left(\int_0^T \|u_j(\tau)\|_{(H^2)^N}^{\frac{4}{3}} d\tau\right)^{\frac{3}{4}}$$

$$\leq NC_{MR}\left(\|U(0)\|_{(H^s)^N} + C(C_1, E_0)(T+1)^{\frac{3}{4}}\right).$$

Therefore, there must be a positive number $t' \in (0, T)$ such that

$$\|U(t')\|_{(H^2)^N} \leq NC_{MR}\frac{\|U(0)\|_{(H^s)^N}}{T^{\frac{3}{4}}} + NC(C_1, E_0)C_{MR}\left(1 + \frac{1}{T}\right)^{\frac{3}{4}}$$

$$\leq \frac{NC_{MR}\left(\|U(0)\|_{(H^s)^N} + C(C_1, E_0)\right)}{T^{\frac{3}{4}}} + NC_{MR}C(C_1, E_0) \leq C_2.$$

We may assume t' is the largest such number in $(0, T]$. With above results, exchanging $t_0 = 0$ into $t_0 = t'$. Note that we can select $s = 2$ for the second and later steps. Via the same method, we can find a largest $t'' \in (t', t' + T]$ such that $\|U(t'')\|_{(H^2)^N} \leq C_2$. Inductively, we can find a sequence of $(t'_l)_l$ such that

- $0 < t'_l - t'_{l-1} \leq T$;
- $\lim_{l \to \infty} t'_l = \infty$;
- $\|U(t'_l)\|_{(H^2)^N} \leq C_2$.

The first and the last assertions are obvious. For $\lim_{l \to \infty} t'_l = \infty$, we first observe

$$\int_0^T \|U(\tau)\|_{(H^2)^N}^{\frac{4}{3}} d\tau \leq 4(NC_{MR})^2 \left(\|U(0)\|_{(H^s)^N}^2 + C(C_1, E_0)^2 \left((2NC_{MR} + 1)^{\frac{4}{3}} + 1 \right) \right)$$

$$\leq C_2.$$

This implies that

$$C_2 \geq \int_0^T \|U(\tau)\|_{(H^s)^N}^{\frac{4}{3}} d\tau = \int_{\|U\|_{(H^2)^N} < C_2} + \int_{\|U\|_{(H^2)^N} \geq C_2} \|U(\tau)\|_{(H^2)^N}^{\frac{4}{3}} d\tau$$

$$\geq 0 + (T - \delta) C_2^{\frac{4}{3}}.$$

where $\delta = |\{t \in [0, T] | \|U(t)\|_{(H^2)^N} < C_2\}|$. This gives $\delta \geq T - C_2^{-\frac{1}{3}} > 0$. Therefore for any $l = 0, 1, \ldots$, we have $t'_{l+1} - t'_l \geq \delta > 0$.

Using Remark 2.8 on every interval $[t'_l, t'_l + T]$, we can prove that $\|U(t)\|_{(H^1)^N} \leq C(C_2) = C(\|U(0)\|_{(H^s)^N})$ for any $t \in [t'_l, t'_l + T]$ and any $l = 1, 2, \ldots$. Therefore, $\|U(t)\|_{(H^1)^N}$ is bounded for $t \geq 0$ and the upper bound dependence on $\|U(0)\|_{(H^2)^N}$ continuously.

$\square$

We now give two corollaries which will be useful in following paragraph.

Corollary 2.15. *For any $U \in \partial \mathcal{A}$, $\|\eta^t(U)\|_{(H^1)^N} \leq C$ for any $t \geq 0$. Here, the constant $C > 0$ depends continuously on the initial data.*

Remark 2.16. *With the results in Remark 2.9, if we assume $U(0) \in (H^s)^N$, then we can conclude $\eta^t(U) \in L^\infty((0, \infty), (H^s)^N)$ for $s \in \left[1, \frac{3}{2}\right) \cap \left(\frac{3}{2}, 2\right]$ via the formula of variation of constants.*

2.4. **Finer Nodal Properties.** It is well known that for scalar equations along the heat flow the number of changing sign is non-increasing ([7, 16, 34]). For coupled systems it was proved in [52] for two equations. We can prove this is also the case for our system using the arguments in [16] and [17], and we omit details here. But we need a more specific version of this theorem from [16], which is based on the notation of bumps of a radial function.

Let us recall this from [16]. The number of changing sign of a continuous radial function $u = u(|x|)$, denoted by $n(u)$, is defined as the largest number k such that there exist a sequence of real number $0 < x_0 < x_1 < \cdots < x_k$ such that

$$u(x_j) \cdot u(x_{j+1}) < 0, \quad j = 0, \ldots, k-1.$$

We call $n(u)$ the nodal number of the function u. We always assume that the functions we discussed have finite nodal numbers. For a radial function u with $n(u) = k$ and $u(x_0) > 0$, we define its q-th bump for $q = 1, ..., k+1$, by

$$u_1(x) = \chi_{\{u>0\}} \cdot \chi_{\{|x|<x_1\}} \cdot u(x),$$
$$u_q(x) = \chi_{\{(-1)^{q-1}u>0\}} \cdot \chi_{\{x_{q-2}<|x|<x_q\}} \cdot u(x), \quad q = 2, \ldots, k,$$
$$u_{k+1}(x) = \chi_{\{u(x_k)\cdot u>0\}} \cdot \chi_{\{x_{q-1}<|x|\}} \cdot u(x).$$

For a radial function u with $n(u) = k$ and $u(x_0) < 0$, we define its q-th bump $q = 1, ..., k+1$ by

$$u_1(x) = \chi_{\{u<0\}} \cdot \chi_{\{|x|<x_1\}} \cdot u(x),$$
$$u_q(x) = \chi_{\{(-1)^{q-1}u<0\}} \cdot \chi_{\{x_{q-2}<|x|<x_q\}} \cdot u(x), \quad q = 2, \ldots, k,$$
$$u_{k+1}(x) = \chi_{\{u(x_k)\cdot u>0\}} \cdot \chi_{\{x_{q-1}<|x|\}} \cdot u(x).$$

To avoid confusion, for the j-th component u_j of $U = (u_1, \ldots, u_N)$, we denote its q-th bump by $u_{j,q}$.

For the solution $U(t)$ to Problem (2.1) with initial value $U \in (H_r^2)^N$ for $t \in [0, T(U))$, we denote its j-th component by $u(t)_j$. By $u(t)_{j,q}$ we denote the q-th bump of its j-th component.

In this subsection, we always assume that the initial data $U(0) \in (H^2)^N$. Theorem 2.1 ensure that $\eta^t(U) \in (H^2)^N$ for any $t \geq 0$.

We firstly consider the case for $\beta_{ij} \leq 0$ for all $i, j = 1, \ldots, N$ and $i \neq j$.

Proposition 2.17. *There is a positive number $\rho > 0$ such that if $0 < |u_{j,q}|_4 < \rho$ then $0 < |u_{j,q}(t)|_4 < \rho$ for $t \geq 0$.*

Proof. By Theorem 2.1 and the inclusion $H^2 \subset C(\Omega)$ for dimension $n = 2, 3$, then $U(t)$ is continuous in spatial variable. As a consequence, the nodal number of $u_j(t)$, $n(u_j(t))$, is well-defined. Hence, there exists a small $\varepsilon > 0$ such that if

$$(-1)^{q+1}u_j(x_q, 0) > 0,$$

then

$$(-1)^{q+1}u_j(x_q, t) > 0$$

for any $t \in [0, \varepsilon]$. Hence, due to the definition of bump $u_{j,q}$ (c.f. Section 2.1), the differential $\frac{\partial}{\partial t} \int |u_{j,q}|^4$ is well-defined. Notice that

$$\frac{\partial}{\partial t} \int |u_{j,q}|^4 = 4 \int u_{j,q}^3 \partial_t u_{j,q} = 4 \int u_{j,q}^3 \partial_t u_j$$

$$= 4 \int u_{j,q}^3 \left(\Delta u_j - \lambda_j u_j + \mu_j u_j^3 + \sum_{i \neq j} \beta_{ij} u_j u_i^2 \right)$$

$$= -3 \int |\nabla(u_{j,q}^2)|^2 - 4\lambda_j \int u_{j,q}^4 + 4\mu_j \int u_{j,q}^6 + 4 \sum_{i \neq j} \beta_{ij} \int u_{j,q}^4 u_i^2.$$

Denote $W = u_{j,q}^2$. By computing

$$\frac{1}{3} = \frac{\frac{1}{2}}{6} + \frac{1 - \frac{1}{2}}{2},$$

we have from Sobolev embedding of dimensions 2 and 3,

$$|W|_3^3 \leq C\|W\|^{\frac{3}{2}} |W|_2^{\frac{3}{2}}.$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t} \int |u_{j,p}|^4 &\leq -C\|W\|^2 + C\|W\|^{\frac{3}{2}} |W|_2^{\frac{3}{2}} \\
&\leq -C\|W\|^{\frac{3}{2}} |W|_2^{\frac{1}{2}} + C\|W\|^{\frac{3}{2}} |W|_2^{\frac{3}{2}} \\
&= -C\|W\|^{\frac{3}{2}} |W|_2^{\frac{1}{2}} \left(1 - C|W|_2\right) \\
&= -C\|W\|^{\frac{3}{2}} |W|_2^{\frac{1}{2}} \left(1 - C|u_{j,q}|_4^2\right) < 0
\end{aligned}$$

for $|u_{j,q}|_4$ small enough.

$\square$

If there is a couple (i_0, j_0) such that $i_0 \neq j_0$, $i_0, j_0 = 1, \ldots, N$ and $\beta_{i_0,j_0} > 0$, the property becomes more delicate.

Lemma 2.18. *For a solution $U(t)$ with its initial data $U(0) \in (H^2)^N$, there is a $b = b(\|U(0)\|_{(H^2)^N}) > 0$ such that if $\beta_{ij} < b$ for all $i, j = 1, \ldots, N$ and $i \neq j$, the conclusion of the last lemma holds true.*

Proof. With the same computation, we have

$$\frac{\partial}{\partial t} \int |u_{j,q}|^4 = -3 \int |\nabla(u_{j,q}^2)|^2 - 4\lambda_j \int u_{j,q}^4 + 4\mu_j \int u_{j,q}^6 + 4 \sum_{i \neq j} \beta_{ij} \int u_{j,q}^4 u_i^2$$

$$\leq -C_0 \|W\|^2 + 4\mu_j |W|_3^3 + 4 \max_{ij} \beta_{ij} \sum_{i=1}^N \int W^2 u_i^2,$$

where $C_0 = \min\{3, 4\lambda_j\}$ and $W = u_{j,q}^2$. Now we deal with the last term and have

$$\begin{aligned}
4 \max_{ij} \beta_{ij} \sum_{i=1}^N \int W^2 u_i^2 = 4 \max_{ij} \beta_{ij} \int W^2 \Big(\sum_{i=1}^N u_i^2\Big) &\leq 4N \max_{ij} \beta_{ij} |W|_3^2 |U|_6^2 \\
&\leq 4N \max_{ij} \beta_{ij} S_3^2 S_6^2 \|W\|^2 \|U\|^2 \\
&\leq 4Nb C_3(U(0)) S_3^2 S_6^2 \|W\|^2,
\end{aligned}$$

where S_p is the best constant for the inequality $|U|_p \leq S_p \|U\|$ and $C_3(U(0))$ is the upper bound of $\|U(t)\|^2$ we computed in the last subsection. If we assume

that $b = \frac{\min\{3,4\lambda_j\}}{8NS_3^2S_6^2C_3(U_0)} > 0$, we will have

$$\frac{\partial}{\partial t}\int |u_{j,q}|^4 \leq -\frac{C_0}{2}\|W\| + 4\mu_j|W|_3^3.$$

The rest part of the proof is the same with the last lemma.

$\square$

3. Proof of Theorem 1.4

3.1. **A Topological Lemma.** We give the linking structure without assuming any symmetry, which can be used in the proof of Theorem 1.4. The strategy of proving is to extend the original setting into a symmetric setting.

Lemma 3.1. *For $A = [0,+\infty)^n$ and a bounded open neighbourhood $\mathcal{O}$ of the origin 0 in $\mathbb{R}^n$, there is no continuous map $F : \partial\mathcal{O}\cap A \to A$ such that $F(\partial\mathcal{O}\cap A) \subset \partial A\backslash\{0\}$ and that the condition $x_j = 0$ implies $F_j(x_1\ldots,x_n) = 0$ for all $j = 1,\ldots,n$, where $F_j(x_1\ldots,x_n)$ is the j-th component of the vector $F(x_1\ldots,x_n)$.*

Proof. We argue it by contradiction. Suppose there is a continuous mapping $F : \partial\mathcal{O}\cap A \to A$ such that $F(\partial\mathcal{O}\cap A) \subset \partial A\backslash\{0\}$ and $F_j(x_1,\ldots,x_n) = 0$ if $x_j = 0$ for any $j = 1,\ldots,n$. Inspired by [24], we begin the proof by extending the setting to a symmetric version and obtain the contradiction via a genus argument.

Firstly, let us define another open neighbourhood $\mathcal{O}^*$ of the origin $0 \in \mathbb{R}^n$ by reflection with respect to each component of the coordinates, i.e.

$$\mathcal{O}^* = \{x = (x_1,\ldots,x_n)|(|x_1|,\ldots,|x_n|)\in\mathcal{O}\}.$$

It is easy to see that the open set $\mathcal{O}^*$ is antipodal symmetric. Then the following inclusion holds true:

(3.1) $$\partial\mathcal{O}^* \cap A \subset \partial\mathcal{O} \cap A.$$

Indeed, we observe that $\partial\mathcal{O}^*\cap int(A) = \partial\mathcal{O}\cap int(A)$. So we only need to show that $\partial\mathcal{O}^* \cap \partial A \subset \partial\mathcal{O} \cap \partial A$. For any $x \in \partial\mathcal{O}^*\cap\partial A$, for any $r > 0$, $B_{\mathbb{R}^n}(x,r)\cap\mathcal{O}^* \neq \emptyset$. Due to the construction of $\mathcal{O}^*$, the last intersection gives $B_{\mathbb{R}^n}(x,r)\cap\mathcal{O} \neq \emptyset$, which implies that $x \in \partial\mathcal{O}\cap\partial A$. Now we restrict the mapping F to the set $\partial\mathcal{O}^*\cap A$ and extend it to the whole $\partial\mathcal{O}^*$. Define the mapping $\tilde{F} : \partial\mathcal{O}^* \to X$ by

$$\tilde{F}(x_1,\ldots,x_n) = \big(sgn(x_1)F_1(|x_1|,\ldots,|x_n|),\ldots,sgn(x_n)F_n(|x_1|,\ldots,|x_n|)\big),$$

where $X = \{x = (x_1,\ldots,x_n)| \prod_{j=1,\ldots,n} x_j = 0\}$. Then we can claim
- $\tilde{F}$ is an odd and continuous extension of F;
- $\tilde{F}(\partial\mathcal{O}^*) \subset X\backslash\{0\}$.

The first assertion is easy. Now we check the second one. We only need to verify that for any $x \in \partial\mathcal{O}^*$, $\tilde{F}(x) \neq 0$. Otherwise, if for any $j = 1, \ldots, n$, $sgn(x_j)F_j(|x_1|, \ldots, |x_n|) = 0$. Since $x \in \partial\mathcal{O}^*$, there are some integers, say $1, \ldots, s$, with $x_1 = \ldots, x_s = 0$ and other integers, say $s+1, \ldots, n$, with $x_{s+1} \neq 0, \ldots, x_n \neq 0$. Therefore, $F_j(|x_1|, \ldots, |x_n|) \equiv 0$ for any $j = 1, \ldots, n$. This is a contradiction with $(|x_1|, \ldots, |x_n|) \in \partial\mathcal{O} \cap A$ and $F(\partial\mathcal{O} \cap A) \subset \partial A \backslash \{0\}$.

Now we will have a contradiction via the genus with respect to the symmetry of antipodal. We denote the genus generated by the antipodal symmetry by γ'. On one hand, we have $n = \gamma'(\partial\mathcal{O}^*) \leq \gamma'(\tilde{F}(\partial\mathcal{O}^*))$ due to Borsuk's theorem for the symmetry with respect to antipodal. On the other hand, notice that $X \cap X' = \{0\}$, where $X' = \{x = (x_1, \ldots, x_n) | x_1 = \cdots = x_n\}$. Then we can construct an odd homotopy G such that

$$X \backslash \{0\} \stackrel{G}{\simeq} \mathbb{S}^{n-2}.$$

This implies that $\gamma'(\tilde{F}(\partial\mathcal{O}^*)) \leq n - 1$. This is a contradiction.

$\square$

We remark that some of the computations in the above proof were used in [30, 31] and will be also used in the next section.

Remark 3.2. *Since the set $\mathcal{O}$ is an open neighbourhood of the origin in $\mathbb{R}^n$, the conclusion still holds if we replace $[0, +\infty)^n$ by $[a, +\infty)^n$, where $a > 0$ is small enough.*

3.2. **Proof of Theorem 1.4.** We prove this theorem via the concept of invariant sets of a descending flow and we will use the parabolic flow as a mean of descending flow. Recall we fix N non-negative integers $P_1, \ldots, P_N$ which are the prescribed componentwise nodal numbers. We first introduce some auxiliary functions.

- Firstly, for the radial domain Ω, we cut it into N radial sub-domains Ω_j for $j = 1, \ldots, N$ with $\overline{\Omega} = \cup_{j=1}^N \overline{\Omega_j}$;
- For any fixed $j = 1, \ldots, N$, we cut the domain Ω_j into $P_j + 1$ sub-domains $\Omega_{j,q}$ with $q = 1, \ldots, P_j + 1$ with $\overline{\Omega_j} = \cup_{q=1}^{P_j+1} \overline{\Omega_{j,q}}$;
- For any $j = 1, \ldots, N$ and $q = 1, \ldots, P_j + 1$, we define a smooth non-zero radial function with compact support $w_{j,q} : \Omega_{j,q} \to [0, +\infty)$.

Without loss of generality, we can assume that $|w_{j,q}|_4 \equiv 1$ for any $j = 1, \ldots, N$ and $q = 1, \ldots, P_j + 1$. We define the following set S by

$$S = \left\{ \left(\sum_{q=1}^{P_1+1} (-1)^{q+1} \alpha_{1,q} w_{1,q}(x), \ldots, \sum_{q=1}^{P_N+1} (-1)^{q+1} \alpha_{N,q} w_{N,q}(x) \right) \right| $$

$$\alpha_{j,q} \geq \frac{\varepsilon}{100} \text{ for } q = 1, \ldots, P_j + 1 \text{ and } j = 1, \ldots, N \right\},$$

which is a closed cone in the real vector space of dimension $\sum_{j=1}^{N} P_j + N$. And there is an isomorphism

$$i : S \to [0, +\infty)^{\sum_{j=1}^{N} P_j + N}$$

$$\left(\sum_{q=1}^{P_j+1} (-1)^{q+1} \alpha_{j,q} w_{j,q} \right)_j \mapsto \left(\alpha_{j,q} - \frac{\varepsilon}{100} \right)_{j,q}.$$

We denote by Y the $\sum_{j=1}^{N} P_j + N$ dimensional vector space spanned by S with respect to the linearity in $i(S)$. It is easy to see that $\mathcal{A} \cap Y$ is also an open neighbourhood of the origin in Y and $\mathcal{A} \cap Y$ is bounded. Notice that $S \cap \partial \mathcal{A}$ is a compact set and can be embedded into a finite dimensional vector space, where all the norms are equivalence. If there is at least one β_{ij} is positive, due to the restriction of Lemma 2.18, we need to find an upper bound $b > 0$ depending on $\sup_{U \in S \cap \partial \mathcal{A}} \sup_{t \geq 0} \|\eta^t(U)\|^2 \in (0, \infty)$. If all the β_{ij}'s are non-positive, the limit on the upper bound is no longer necessary (Lemma 2.17).

Now we will locate the portions of the boundary $\partial \mathcal{A}$ in which along the flow lines the number of nodal domains can be controlled. For this purpose, as done for scalar equations in [16], we introduce the following notations:

- $D_{j,k} = \{U = (u_1, \ldots, u_N) \in (H_r^2)^N | n(u_j) = k\}$ and $D = \cap_{j=1}^{N} D_{j,P_j}$, where $P_1, \ldots, P_N$ are given in the theorem;
- $E_{j,q}^{\varepsilon} = \{U = (u_1, \ldots, u_N) \in D | \|u_{j,q}|_4 < \varepsilon\}$ for $q = 1, \ldots, P_j + 1$ and $j = 1, \ldots, N$;
- denote

$$H = \left\{ U = (u_1, \ldots, u_N) \in (H_r^2)^N | n(u_j) \leq P_j \text{ for } j = 1, \ldots, N \right.$$

$$\left. \text{and } \sum_{j=1}^{N} n(u_j) < \sum_{j=1}^{N} P_j \right\}$$

- denote

$$F_\varepsilon = \cup_{j=1}^{N} \cup_{q=1}^{P_j+1} E_{j,q}^{\varepsilon} \cup H;$$

- the complete invariant set of the set $E_{j,q}^{\varepsilon}$ is defined as

$$C(E_{j,q}^{\varepsilon}) = \left\{ U \in (H^2)^N | \exists t_0 \geq 0 \ s.t. \ \eta^{t_0}(U) \in E_{j,q}^{\varepsilon} \right\}$$

for $q = 1, \ldots, P_j + 1$ and $j = 1, \ldots, N$. Therefore, we can denote

$$A_{j,q}^{\varepsilon} = C(E_{j,q}^{\varepsilon}) \cap \partial \mathcal{A} \cap D$$

for $q = 1, \ldots, P_j + 1$ and $j = 1, \ldots, N$.

Due to the invariance property proved in the last section we will also define an arriving time

- for any $U \in D \cap \partial \mathcal{A}$, denote $T^*(U) = \inf\{t \geq 0 | \eta^t(U) \in F_{\frac{\varepsilon}{2}}\}$. Note that $T^*(U)$ is only well-defined for $U \in D$.

Remark 3.3. *The concept of complete invariant set is from* [29].

We note that the set $D_{j,k}$ consists of the vector-valued functions whose j-th components have exactly k sign-changing number. For any function $(u_1, \ldots, u_N)$ in the set D, let u_j be an arbitrary component. The set $E_{j,q}^\varepsilon$ contains the functions whose q-th bump of j-th component has a small L^4 norm. This set is invariant due to Proposition 2.17 when $n(\eta^t(\cdot)_j)$ dose not change. As to the set F_ε, an element in F_ε either has a small bump or has a component with a sign-changing number less than what we prescribed. It should be noted that this set is what we want to remove. T^* is the time when the flow line arrives in the set $F_{\frac{\varepsilon}{2}}$. Using the computation in Proposition 2.17, any flow line which flows into F_ε in a finite time will flow into $F_{\frac{\varepsilon}{2}}$ eventually. We will fix small ε so that the invariance property holds. Following the idea in [16, Lemma 5], we now have the continuity of the arriving time $T^*(U)$.

Lemma 3.4. *The arriving time $T^*(U)$ is continuous.*

Proof. Let $U \in D$. For a sequence $U_n \to U$ in $(H^1)^N$, we only check the lower limit
$$T^*(U) \leq \varliminf_{n \to \infty} T^*(U_n).$$
The upper one $T^*(U) \geq \varlimsup_{n \to \infty} T^*(U_n)$ can be proved in the same way.

We argue by contradiction. Suppose, up to a subsequence, we have
$$s := \lim_{n \to \infty} T^*(U_n) < T^*(U) \leq T(U) = \infty.$$

Then we can find a $t \in (s, T^*(U))$. Set $V_n = \eta^t(U_n)$ and $V = \eta^t(U)$. Then, due to Theorem 2.1, we have $V_n \to V$ in $(H^1)^N$. Since $t < T^*(U)$, we have $V = \eta^t(U) \in D \cap \partial \mathcal{A} \backslash F_{\frac{\varepsilon}{2}}$. Then $|V_{j,q}|_4 \geq \frac{\varepsilon}{2}$ for all $j = 1, \ldots, N$, $q = 1, \ldots, P_j + 1$. By using $T^*(U_n) \to s < t$, we have $V_n = \eta^t(U_n) \in F_{\frac{\varepsilon}{2}} \cap \partial \mathcal{A}$, which implies $|(V_n)_{j_0,q_0}|_4 \leq \frac{\varepsilon}{2}$ for some $j_0 = 1, \ldots, N$, $q_0 = 1, \ldots, P_j + 1$. Combining these with the fact that $V_n \to V$ in $(H^1)^N$, we conclude that $|\eta^t(U)_{j_0,q_0}|_4 = |V_{j_0,q_0}|_4 = \frac{\varepsilon}{2}$. Hence, $t \geq T^*(U)$ due to the definition of T^*, which is a contradiction.

$\square$

Lemma 3.5. *For any $U = (u_1, \ldots, u_N) \in \partial \mathcal{A}$ satisfying*
- $\sum_{j=1}^N n(u_j) < \sum_{j=1}^N P_j$, *and*
- *there is a sequence of functions $U_n \in D \cap \partial \mathcal{A}$ such that $U_n \to U$ in $(H^1)^N$,*

we have $\lim_{n \to \infty} T^*(U_n) = 0$.

Proof. We prove the lemma by contradiction. For a sequence $U_n \to U$ in $(H^1)^N$ with $U_n \in D \cap \partial \mathcal{A}$ for any $n = 1, 2, \dots$, suppose there is $t_0 > 0$ such that $T^*(U_n) \geq 2t_0$ for large n. We select and fix a $t \in (0, t_0)$. On one hand, we have

$$\eta^t(U_n) \to \eta^t(U) \qquad in \ (H^1)^N,$$

due to Theorem 2.1. On the other hand, by the definition of the arriving time T^* and the non-increasing property of nodal number along the flow line, using $U_n \in D$ we have

- $n\big((U_n)_j\big) = P_j$ for $j = 1, \dots, N$;
- $\big|\big(\eta^t(U_n)\big)_{j,q}\big|_4 \geq \frac{\varepsilon}{2}$ for $j = 1, \dots, N$ and $q = 1, \dots, P_j + 1$;
- $n\big(\big(\eta^t(U)\big)_j\big) \leq P_j$ for $j = 1, \dots, N$ and $q = 1, \dots, P_j + 1$ and at least one of the $\leq$'s holds strictly.

Here, $(\eta^t(W))_j$ and $(\eta^t(W))_{j,q}$ are the j-th component and the q-th bump of the j-th component of $\eta^t(W)$. Now we show that these assertions lead us to a contradiction.

Since $\eta^t(U_n) \to \eta^t(U)$ in $(H^1)^N$, we can select a large $n_0 > 0$ such that $\big|\eta^t(U_{n_0}) - \eta^t(U)\big|_4 \leq \frac{\varepsilon}{4}$. In the following, we will argue it in terms of components. Let us consider $\eta^t(U)_1$ and $\eta^t(U_{n_0})_1$ for the sake of simplicity, where $\eta^t(U)_1$ and $\eta^t(U_{n_0})_1$ are the first components of $\eta^t(U)$ and $\eta^t(U_{n_0})$ respectively. Let us assume that $n(\eta^t(U)_1) < n(\eta^t(U_{n_0})_1) = P_1$ without loss of generality. Due to the definition of the sign-changing number, we can find a sequence of numbers $x_{q-1} \in \operatorname{supp} \eta^t(U_{n_0})_{1,q}$ for $q = 1, \dots, P_1 + 1$, where $\eta^t(U_{n_0})_{1,q}$ is the q-th bump, such that

$$\eta^t(U_{n_0})_1(x_q) \cdot \eta^t(U_{n_0})_1(x_{q+1}) < 0$$

for $q = 0, \dots, P_1$. Using the facts $\big|\eta^t(U_{n_0})_{1,q}\big|_4 \geq \frac{\varepsilon}{2}$ for $q = 1, \dots, P_1 + 1$, and $\big|\eta^t(U_{n_0}) - \eta^t(U)\big|_4 \leq \frac{\varepsilon}{4}$, we claim there must be $x'_{q-1} \in \operatorname{supp} \eta^t(U_{n_0})_{1,q}$ such that

(3.2) $$\eta^t(U)_1(x'_q) \cdot \eta^t(U)_1(x'_{q+1}) < 0$$

for $q = 0, \dots, P_1$. Otherwise, if there is a $q_0 = 1, \dots, P_1 + 1$ such that

- $\eta^t(U_{n_0})_{1,q_0} \geq 0$;
- $\eta^t(U) \leq 0$ on $\operatorname{supp} \eta^t(U_{n_0})_{1,q_0}$;
- $|\eta^t(U_{n_0})_{1,q_0}|_4 \geq \frac{\varepsilon}{2}$;

then we have

$$\frac{\varepsilon}{4} \geq |\eta^t(U_{n_0}) - \eta^t(U)|_4 \geq |\eta^t(U_{n_0})_{1,q_0} - \eta^t(U) \cdot \chi \operatorname{supp} \eta^t(U_{n_0})_{1,q_0}|_4$$

$$\geq |\eta^t(U_{n_0})_{1,q_0}|_4 \geq \frac{\varepsilon}{2},$$

which is a contradiction. Here, the function χ_A is the characteristic function of the set A. Therefore, (3.2) holds, i.e., $P_1 = n(\eta^t(U)_1)$, but this is a contradiction with $n(\eta^t(U)_1) < P_1$. The proof is complete.

□

Finally to prove Theorem 1.4, we only need to show that

$$A := \partial\mathcal{A} \cap D \backslash \left(\cup_{j=1}^{N} \cup_{q=1}^{P_j+1} A_{j,q}^{\varepsilon} \right) \neq \emptyset.$$

The rest part of the proof requires a lower bound of the energy functional J on the set A and the fact that the energy functional satisfies the (PS) condition. The second part is obvious and the first part is given by

$$A \subset \partial\mathcal{A}$$

and

$$0 \leq \inf_{\partial\mathcal{A}} J \leq \inf_{A} J.$$

Now we verify that $\partial\mathcal{A} \cap D \backslash \left(\cup_{j=1}^{N} \cup_{q=1}^{P_j+1} A_{j,q}^{\varepsilon} \right) \neq \emptyset$.

This proof relies heavily on a technique used in [30, 31]. We will use the subset $S \subset D$ constructed at the beginning of this subsection and prove the theorem by proving $\partial\mathcal{A} \cap S \backslash \left(\cup_{j=1}^{N} \cup_{q=1}^{P_j+1} A_{j,q}^{\varepsilon} \right) \neq \emptyset$.

Now we argue by contradiction, i.e., we assume that $\partial\mathcal{A} \cap S \subset \cup_{j=1}^{N} \cup_{q=1}^{P_j+1} A_{j,q}^{\varepsilon}$. We use $\partial_Y S$ to denote the boundary of S with respect to the space of Y. Define a continuous cut-off function $\phi : [0, \infty) \to [0, 1]$:

(3.3)
$$\phi(x) = \begin{cases} 1 & s \geq \varepsilon, \\ 0 & s \leq \dfrac{\varepsilon}{2}, \\ \dfrac{2s}{\varepsilon} - 1 & s \in \left(\dfrac{\varepsilon}{2}, \varepsilon \right). \end{cases}$$

Let us define the mapping $h : \partial\mathcal{A} \cap S \to \partial_Y S$ by

$$h(U) = \left(\sum_{q=1}^{P_1+1} (-1)^{q+1} \left(\phi(|\eta^{T^*(U)}(U)_{1,q}|_4) + \frac{\varepsilon}{100} \right) w_{1,q}, \ldots, \right.$$

$$\left. \sum_{q=1}^{P_N+1} (-1)^{q+1} \left(\phi(|\eta^{T^*(U)}(U)_{N,q}|_4) + \frac{\varepsilon}{100} \right) w_{N,q} \right),$$

where $U = (U_1, \ldots, U_N) \in \partial\mathcal{A} \cap S$.

We are here in position to use Lemma 3.1. To do this, we only need to check that $h(\partial_Y(\mathcal{A} \cap Y) \cap S) \subset \partial_Y S \backslash \{\theta\}$. Firstly, we notice that $\partial_Y(\mathcal{A} \cap Y) \subset \partial\mathcal{A} \cap Y$. Then we claim that for any $U \in \partial\mathcal{A} \cap S$, there is a $j = 1, \ldots, N$ and a $q = 1, \ldots, P_j + 1$ such that $\phi(|\eta^{T^*(U)}(U)_{j,q}|_4) > 0$. If the claim is not true, for any $j = 1, \ldots, N$ and $q = 1, \ldots, P_j + 1$, we have $\int |U_{j,q}|^4 < \varepsilon^4$ as t goes large. Multiplying u_j on the both sides of the j-th equation of Problem (2.1) and

integrating and summing up with respect to j, we have

$$\frac{1}{2}\frac{\partial}{\partial t}|U(t)|_2^2 + \|U(t)\|^2 \leq \sum_{j=1}^{N}\left(\mu_j\int|u_j(t)|^4 + \sum_{i\neq j}\beta_{ij}\int u_i(t)^2 u_j(t)^2\right) < C\varepsilon$$

for t large. Using the openness of $\mathcal{A}$ in $(H^1)^N$ and the invariance of $\partial\mathcal{A}$, $\|U(t)\| \geq C > 0$ uniformly for $t \geq 0$. This implies that $\frac{\partial}{\partial t}|U(t)|_2^2 \leq -C$ for $t > 0$, a contradiction. Therefore, the mapping h satisfies the condition for F in Lemma 3.1, so we have a contradiction on the existence of the mapping h. The proof is complete.

$\square$

4. PROOF OF THEOREM 1.2

4.1. **The Idea of the Proof.** We are now in position to prove Theorem 1.2. We outline our approach briefly first. Using the flow invariance property we reduce the variational problem to one defined on a subset on the boundary of the stable set of the origin where the nodal number of the functions is controlled by the componentwise-prescribed number of nodes. In order to establish multiple nodal solutions having the same componentwise nodal number we will make use of the symmetry property imposed in Theorem 1.2.

More precisely, our problem possesses a $\mathbb{Z}_p$ symmetry under a cyclic permutation $\sigma : (u_1, \ldots, u_N) \mapsto (\sigma_1(u_1), \ldots, \sigma_N(u_N))$ in $(H_0^1(\Omega))^N$ defined by
- $\sigma_i(u_i) = u_{i+1}$ for $i \neq pb$ for $b = 1, \ldots, B$,
- $\sigma_{pb}(u_{pb}) = u_{p(b-1)+1}$ for $b = 1, \ldots, B$.

In other words, we define the permutation σ as

$$\sigma(u_1, u_2, \ldots, u_p; \ldots \ldots; u_{N-p+1}, u_{N-p+2}, \ldots, u_N)$$
$$= (u_2, \ldots, u_p, u_1; \ldots \ldots; u_{N-p+2}, \ldots, u_N, u_{N-p+1}).$$

It is easy to see that this can be regarded as a $\mathbb{Z}_p$ cyclic group action, our variational functional J is invariant under this action.

We will use a $\mathbb{Z}_p$ group action index (or genus), which is from [50] (see also related works in [51, 52]). We summarize some basic property of the index. Let E be a Banach space on which there is a $\mathbb{Z}_p$ action generated by σ. Let $F_\sigma = \{U \in E \mid \sigma U = U\}$ be the set of fixed points of the σ action. For a σ-symmetric compact set $A \subset E\backslash F_\sigma$, the index $\gamma(A)$ is defined as the smallest $m \in \mathbb{N}$ such that there exists a continuous mapping $h : A \to \mathbb{C}^m\backslash\{0\}$ with

$$h(\sigma U) = e^{\frac{2\pi i}{p}}h(U).$$

If there is no such mapping, we set $\gamma(A) = \infty$. We need the following properties of the Index γ.
- If $A \subset B$, then $\gamma(A) \leq \gamma(B)$;

- $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;
- if $g : A \to E \backslash F_\sigma$ is continuous and satisfies $g(\sigma(u)) = \sigma g(u)$ for all $u \in A$, then

$$\gamma(A) \leq \gamma(\overline{g(A)});$$

- if $\gamma(A) > 1$, then A is an infinite set;
- if A is compact and $\gamma(A) < \infty$, then there exist an open σ-invariant neighbourhood $\mathcal{N}$ of A such that $\gamma(A) = \gamma(\overline{\mathcal{N}})$;
- if S is the boundary of a bounded neighbourhood of the origin in a m-dimensional complex linear space such that $e^{\frac{2\pi i}{p}} U \in S$ for any $U \in S$, and $\Psi : S \to E \backslash F_\sigma$ is continuous and satisfies for any $U \in S$, $\Psi(e^{\frac{2\pi i}{p}} U) = \sigma(\Psi(U))$, then $\gamma(\Psi(S)) \geq m$;

One can find the proofs of all these properties in for example [50]. Nevertheless, to complete our proof, another lemma on the $\mathbb{Z}_p$ index is necessary. It can be considered as a generalization of the normalization of the $\mathbb{Z}_2$ genus, c.f. [43, Proposition 7.5].

Lemma 4.1. *Let A be a closed set such that $A \subset E \setminus F_\sigma$, and $\cap_{i=0}^{p-1} \sigma^i(A) = \emptyset$. Then $\gamma(\mathbb{Z}_p(A)) \leq p - 1$.*

Proof. The idea to proof is to decompose $\mathbb{Z}_p(A)$ into $p - 1$ sets with genus no larger than 1. Firstly, note that it is clear that if $\sigma^i(A) \cap \sigma^j(A) = \emptyset$ for $i \neq j$, $1 \leq i, j \leq p$, then $\gamma(\mathbb{Z}_p(A)) = 1$.

Let $L_0 = \{A, \sigma(A), ..., \sigma^{p-1}(A)\}$. Let $k < p$ be the largest integer such that there exist k elements in L_0 whose intersection is nonempty, and there is no $k + 1$ elements of L_0 whose intersection is nonempty. Without loss of generality, we assume $k = p - 1$ and we will do induction in k.

First there exist p sets $B_1, ..., B_p$ each of which is the intersection of $p - 1$ elements of L_0 so some B_i is nonempty. Due to σ symmetry, the union of these p sets form a $\mathbb{Z}_p$ invariant set. We claim that $B_i \cap B_j = \emptyset$ for all $i \neq j$. This follows from $k = p - 1$ being the largest. So each B_i is a closed set and $\cup_{i=1}^{p} B_i = \mathbb{Z}_p(B_j)$ for any j. Thus $\gamma(\cup_{i=1}^{p} B_i) = 1$. We may also find an open set U_1 such that $B_1 \subset U_1$ and $\sigma^i(\overline{U}_1) \cap \sigma^j(\bar{U}_1) = \emptyset$ for $i \neq j$. Thus we also have $\gamma(\cup_{i=1}^{p} \overline{U}_i) = 1$.

Define $A_1 = A \setminus \cup_{i=1}^{p} U_i$ and $L_1 = \{A_1, \sigma(A_1), ..., \sigma^{p-1}(A_1)\}$. Then A_1 is a closed set. If for A_1, the largest integer k_1 is such that some k_1 elements of L_1 have nonempty intersection, then k_1 is less or equal to $p - 2$. This is due to the construction. Without loss of generality, assume $k_1 = p - 2$. Now we repeat the arguments above, there exist at most $C_p^{p-2} = p(p-1)/2$ sets which are intersections of $p - 2$ elements of L_1: $C_1, C_2, ...$ We claim that C_i's are closed sets, and $C_i \cap C_j = \emptyset$ for all $i \neq j$. This follows from $p - 2$ being the largest in this case.

Due to σ symmetry, these sets C_i's form at most $(p-1)/2$ number of $\mathbb{Z}_p$ invariant sets, each such $\mathbb{Z}_p$ invariant set is given by $C_1, ..., C_p$ with the property $C_i = \sigma^{i-1}(C_1)$ for $i = 1, ..., p$, $C_i \cap C_j = \emptyset$ for $i \neq j$. Thus $\gamma(\cup_{i=1}^{p} C_i) = 1$. We have at most $(p-1)/2$ such disjoint closed $\mathbb{Z}_p$ invariant sets, whose union has genus 1. Then we choose an open neighborhood V_i of C_i so that the closures of these open sets are also disjoint and the union of these closed neighborhoods also has genus 1.

Define $A_2 = A_1 \setminus \cup_i V_i$ and $L_2 = \{A_2, \sigma(A_2), ..., \sigma^{p-1}(A_2)\}$. Doing this process until we get to $k = 1$ then we are done.

$\square$

Remark 4.2. *In fact we have proved that $\gamma(\mathbb{Z}_p(A)) \leq k$ if $\cap_{j \in J} \sigma^j(A) = \emptyset$ holds for any k-element subset $J \subset \{0, 1, \ldots, p-1\}$.*

Using this $\mathbb{Z}_p$-cyclical permutation symmetry and the genus type index generated as above, we construct multiple nodal solutions with a given componentwise prescribed number of nodes by a minimax type argument. With the aid of the flow invariance, the central part of the proof is to construct a certain set of vector-valued functions which has infinite $\mathbb{Z}_p$ genus and in which the flow line always possesses prescribed number of nodal domains. Then by a minimax construction in variational methods (c.f. [14, 43]), we will have a sequence of critical levels and therefore a sequence of solutions to Problem (1.1). For the construction of sets with large $\mathbb{Z}_p$-index, we will use a variant of the construction in [50, Proposition 4.2] where only positive solutions were considered, by making sets of sign-changing functions with σ symmetry property. For computations of the $\mathbb{Z}_p$ index, we will adapt some ideas from [16, 30, 31, 50] incorporating the invariance of nodal domains and the $\mathbb{Z}_p$ symmetry.

4.2. **Invariant Sets and Other Constructions.** We need a symmetric version of the settings in Section 3. We begin with constructing sets of vector-valued functions with componentwise-prescribed number of nodal domains and with arbitrarily large genus.

Recall that p is a prime factor of N and B is such that $N = pB$ and that $P_1, \ldots, P_B$ are B non-negative integers and fixed in the proof. For any given positive integer K, we will construct a subset having $\mathbb{Z}_p$ genus not less than K that consists of vector-valued functions $U = (u_1, \ldots, u_N)$ such that for $b = 1, \ldots, B$, $n(u_{pb-p+i}) = P_b$ for $i = 1, ..., p$, and satisfies other dynamic property.

Firstly, we divide the domain Ω into B radial parts and denote them by Ω_b for $b = 1, ..., B$ so $\overline{\cup_{b=1}^{B} \Omega_b} = \overline{\Omega}$. For a fixed integer $b = 1 \ldots, B$, we divide Ω_b into $P_b + 1$ radial sub-domains $\Omega_{b,q}$ for $q = 1, \ldots, P_b + 1$ so $\overline{\cup_{q=1}^{P_b+1} \Omega_{b,q}} = \overline{\Omega_b}$. For each sub-domain $\Omega_{b,q}$, divide it into K radial sub-domains $\Omega_{b,q,k}$ for $k = 1, \ldots, K$

so $\overline{\cup_{k=1}^{K} \Omega_{b,q,k}} = \overline{\Omega_{b,q}}$. Denote $\mathcal{O}_{b,q} = \mathbb{S}^1 \times \Omega_{b,q}$ and $\mathcal{O}_{b,q,k} = \mathbb{S}^1 \times \Omega_{b,q,k}$. For $b = 1\ldots,B$, $q = 1,\ldots,P_b+1$ and $k = 1,\ldots,K$, we define functions for $(t,x) \in \mathcal{O}_{b,q,k} = \mathbb{S}^1 \times \Omega_{b,q,k}$ as follows:

- $w_{b,q,k}(t,x) = w_{b,q,k}(t,|x|) = w_{b,q,k}(t,r) : \mathcal{O}_{b,q,k} \to \mathbb{R}$ of class C^4 and of compact support in $\mathcal{O}_{b,q,k}$;
- $w_{b,q,k} \geq 0$ and $w_{b,q,k}(t,\cdot) \not\equiv 0$ for any $t \in \mathbb{S}^1$;
- $\operatorname{supp} w_{b,q,k}(t,\cdot) \cap \operatorname{supp} w_{b,q,k}\left(\frac{2\pi}{p} + t, \cdot\right) = \emptyset$ for any $t \in \mathbb{S}^1$.

Now we have a few words about the notation for clarity. We point out that the subscript "b" denotes the number of the blocks of components with each block having p components so it is invariant under the $\mathbb{Z}_p$-permutation of components. The subscript "q" denotes the number of nodal domains. And the subscript "k" is for the factor K of the dimension of the simplex. To give a simplex in Sobolev space $(H^1)^N$ involving vector-valued functions, we start by its componentwise construction. In order to use the $\mathbb{Z}_p$ index γ, we need to consider the complex Euclidean space $\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)}$. For any vector $z = (z_{b,q,k})$ we decompose them in the polar-coordinates with respect to the components. This leads to $z_{b,q,k} = e^{i\theta_{b,q,k}}\alpha_{b,q,k}$ with $\alpha_{b,q,k}$'s are nonnegative real numbers and $\theta_{b,q,k} \in [0,2\pi)$ for any $b = 1\ldots,B$, $q = 1,\ldots,P_b+1$ and $k = 1,\ldots,K$. For $b = 1,\ldots,B$ fixed we define

$$V_b(t,z_b)(r) = \sum_{q=1}^{P_b+1}(-1)^{q+1}\sum_{k=1}^{K}\alpha_{b,q,k}w_{b,q,k}(t + \theta_{b,q,k}, r)$$

where the vector $z_b = \{(z_{b,q,k}) \,|\, q = 1,\ldots,P_b+1; k = 1,\ldots,K\}$. Then we can define a mapping

$$\psi : \mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)} \to (H_r^2)^N$$

by

$$\psi(z) = \left(V_1(0,z_1), V_1\left(\frac{2\pi}{p}, z_1\right), \ldots, V_1\left(\frac{2\pi(p-1)}{p}, z_1\right),\right.$$

$$\cdots \qquad \cdots$$

$$\left. V_B(0,z_B), V_B\left(\frac{2\pi}{p}, z_B\right), \ldots, V_B\left(\frac{2\pi(p-1)}{p}, z_B\right)\right).$$

We note that

$$V_b(t, e^{\frac{2\pi i}{p}}z_b)(r) = \sum_{q=1}^{P_b+1}(-1)^{q+1}\sum_{k=1}^{K}\alpha_{b,q,k}w_{b,q,k}\left(t + \theta_{b,q,k} + \frac{2\pi}{p}, r\right)$$

$$= V_b\left(\frac{2\pi}{p} + t, z_b\right)(r),$$

which implies that $\psi\left(e^{\frac{2\pi i}{p}}z\right) = \sigma\psi(z)$. Here recall σ is the $\mathbb{Z}_p$ cyclic permutation.

As the settings and notations used in Section 3, we introduce the following notations:

- $D_{j,k} = \{U = (u_1, \ldots, u_N) \in (H_r^2)^N | n(u_j) = k\}$ and

$$D = \cap_{b=1}^B \cap_{i=1}^p D_{pb-p+i, P_b};$$

- For $\varepsilon > 0$, $E_{j,q}^\varepsilon = \{U = (u_1, \ldots, u_N) \in D | \|u_{j,q}\|_4 < \varepsilon\}$;
- denote

$$H = \left\{ U = (u_1, \ldots, u_N) \in (H_r^2)^N | n(u_{bp-p+i}) \le P_b, \text{ for } i = 1, \ldots, p, \ b = 1, \ldots, B, \right.$$

$$\left. \text{and } \sum_{j=1}^N n(u_j) < p \sum_{b=1}^B P_b \right\}$$

and

$$F_\varepsilon = \cup_{b=1}^B \left(\cup_{q=1}^{P_b+1} \cup_{i=1}^p E_{pb-p+i,q}^\varepsilon \right) \cup H;$$

- the complete invariant set of the set $E_{j,q}^\varepsilon$ is defined as

$$C(E_{j,q}^\varepsilon) = \left\{ U \in (H^2)^N | \exists t_0 \ge 0 \ s.t. \ \eta^{t_0}(U) \in E_{j,q}^\varepsilon \right\}.$$

Therefore, we can denote

$$A_{j,q}^\varepsilon = C(E_{j,q}^\varepsilon) \cap \partial \mathcal{A} \cap D$$

and

$$A_\varepsilon = \cup_{b=1}^B \cup_{q=1}^{P_b+1} \cup_{i=1}^p A_{pb-p+i,q}^\varepsilon.$$

- $T^*(U) = \inf\{t \in [0, T(U)) | \eta^t(U) \in F_{\frac{\varepsilon}{2}} \cap \partial \mathcal{A}\}$ for $U \in \partial \mathcal{A} \cap D$.

The notations are symmetric versions of the ones in Section 3, and the difference is that we restrict the sign-changing condition for the sake of componentwise permutation.

As we proved in Section 3, the continuity of the arriving time $T^*(U)$ holds. Besides, the invariance of $T^*(U)$ is easy to check.

Lemma 4.3. $T^*(U)$ *is continuous and invariant under the permutation* σ.

So holds the following lemma.

Lemma 4.4. *For any* $U = (u_1, \ldots, u_N) \in \partial \mathcal{A}$ *satisfies*
- $\sum_{j=1}^N n(u_j) < B \sum_{q=p}^N P_q$,
- *there is a sequence of vector-valued functions* $U_n \in D \cap \partial \mathcal{A}$ *such that* $U_n \to U$ *in* $(H^1)^N$,

then $\lim_{n \to \infty} T^*(U_n) = 0$.

To compute the $\mathbb{Z}_p$-index, we will use an idea from [30]. Nevertheless, it should be noticed that the simplex in [30] is different from ours. In order to make it work, we need to enlarge the previous set $\psi\left(\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b+B\right)}\right)$. Let us select a σ-invariant set G which contains $\psi\left(\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b+B\right)}\right)$ by the following

$$
G = \left\{ \left(\sum_{k=1}^{K}\sum_{q=1}^{P_1+1}(-1)^{q+1}\alpha_{q,k}^{1,1}w_{1,q,k}(s_{q,k}^1,r), \ldots, \sum_{k=1}^{K}\sum_{q=1}^{P_1+1}(-1)^{q+1}\alpha_{q,k}^{1,p}w_{1,q,k}\left(\frac{2\pi(p-1)}{p}+s_{q,k}^1,r\right), \right.\right.
$$
$$
\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots ,
$$
$$
\sum_{k=1}^{K}\sum_{q=1}^{P_B+1}(-1)^{q+1}\alpha_{q,k}^{B,1}w_{B,q,k}(s_{q,k}^B,r), \ldots, \left.\sum_{k=1}^{K}\sum_{q=1}^{P_B+1}(-1)^{q+1}\alpha_{q,k}^{B,p}w_{B,q,k}\left(\frac{2\pi(p-1)}{p}+s_{q,k}^B,r\right) \right) \Bigg|
$$
$$
\alpha_{q,k}^{b,j} \geq 0, \ s_{q,k}^b \in [0,2\pi), \ \text{for any } b=1,\ldots,B, \ j=1\ldots,p, \ q=1,\ldots,P_b+1,
$$
$$
\left. k=1,\ldots,K \right\}.
$$

Note that for any $t \geq 0$ and $U \in G$, we have $tU \in G$. The difference between the set G and the set $\psi\left(\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b+B\right)}\right)$ is that in G the coefficients $\alpha_{q,k}^{b,j}$'s are independent. Notice that the set G contains no nontrivial fixed points of σ due to the definitions of the functions $w_{b,q,k}$'s. We observe that due to the definition of $\mathcal{A}$ and the property of the heat flow η^t, every half-line in G starting at the origin intersects $\partial\mathcal{A}$. Moreover, the set $G \cap \partial\mathcal{A}$ is compact and σ-invariant. In particular, we denote

$$
G_0 = \left\{ U \in G \big| n(U_{bp-p+i}) = P_b \text{ for any } i=1,\ldots,p \text{ and } b=1,\ldots,B \right\}.
$$

This is to say that we define G_0 as the portion of G whose elements do not degenerate in the sense of no drop-off of the sign-changing number, i.e., $n(U_{bp-p+i}) = P_b$ for any $i=1,\ldots,p$ and $b=1,\ldots,B$. It is easy to see $G = \overline{G_0}$.

4.3. **Avoiding the Fixed Points.** In Section 3, we already proved that $\partial\mathcal{A} \cap D \cap (H^2)^N \backslash A_\varepsilon \neq \emptyset$. In this subsection we show that $\partial\mathcal{A} \cap D \backslash A_\varepsilon$ contains no fixed points of the permutation σ action. The following lemma ensures that the flow line does not go through the fixed points of the permutation σ.

Lemma 4.5. *For any $U \in \partial\mathcal{A} \cap D \backslash A_\varepsilon$, the flow line $\{\eta^t(U)\}_{t\geq 0}$ contains no fixed point of the permutation σ.*

Proof. We argue by contradiction. Suppose that there is a $t_0 > 0$ such that $\eta^{t_0}(U)$ is a fixed point of the permutation σ. Then we have $\eta^{t_0}(U)_1 = \cdots = \eta^{t_0}(U)_p$. Due to the uniqueness of the solution, $\eta^t(U)_1 = \cdots = \eta^t(U)_p$ for any

$t \geq t_0$. Multiplying u_1^3 on both sides of the first equation of Problem (2.1) and integrating over Ω, we get

$$\frac{d}{dt} \int u_1^4 + 4\min\left\{\frac{3}{4}, \lambda_1, \ldots, \lambda_N\right\} \|u_1^2\|^2 \leq 4\left(\mu_1 \int u_1^6 + \sum_{i=2}^{N} \beta_{i1} \int u_i^2 u_1^4\right)$$

$$= 4\left(\left(\mu_1 + \sum_{i=2}^{p} \beta_{i1}\right) \int u_1^6 + \beta_{i1} \sum_{i=p+1}^{N} \int u_i^2 u_1^4\right).$$

Combining with $\mu_1 + \sum_{i=2}^{p} \beta_{i1} \leq 0$ (assumption (D) of Theorem 1.2) and the Sobolev's embedding, we have

$$\frac{d}{dt} \int u_1^4 \leq -C\|u_1^2\|^2 \leq -C|u_1|_4^4.$$

Hence, $\int u_1^4 \leq Ce^{-Ct}$ follows. Therefore, for some $T_0 > 0$ and $q = 1, \ldots, P_1$, $\eta^t(U) \in E_{1,q}^{\frac{\varepsilon}{2}}$ for $t > T_0$. This is a contradiction with the fact that $U \in \partial\mathcal{A} \cap D \backslash A_\varepsilon$. $\qquad\square$

Remark 4.6. *In fact, we can do the same computation to the other components, therefore we will have $\eta^t(U) \to \theta$ in $(L^4)^N$.*

Corollary 4.7. *The set $\partial\mathcal{A} \cap D \backslash A_\varepsilon$ contains no fixed point.*

4.4. **Construction of σ-Symmetric Sets of Functions of Prescribed N-odal Numbers with Arbitrarily Large Genus.** The aim of this subsection is to prove that for any integer $k > 0$, there is a compact subset $B_k \subset \partial\mathcal{A} \cap D \backslash A_\varepsilon$ satisfies $\sigma(B_k) = B_k$ and $\gamma(B_k) \geq k$. To do this, we only need to check that for the set G constructed in the last subsection 4.2, it holds $\gamma(\partial\mathcal{A} \cap G \backslash A_\varepsilon) \geq K$ since K can be chosen to be arbitrarily large.

Lemma 4.8. $\gamma(G \cap \partial\mathcal{A}) = K\left(\sum_{b=1}^{B} P_b + B\right)$.

Proof. It is obvious that $\psi\left(\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)}\right) \subset G$. Hence, we have

$$\gamma(G \cap \partial\mathcal{A}) \geq \gamma\left(\psi\left(\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)}\right) \cap \partial\mathcal{A}\right) = K\left(\sum_{b=1}^{B} P_b + B\right),$$

where the equality holds due to Borsuk's theorem. To obtain the reversed inequality, we note that the set G is homeomorphic to the following subset X of $\mathbb{C}^{pK\left(\sum_{b=1}^{B} P_b + B\right)}$:

$$X = \left\{(z_{q,k}^{b,j}) \in \mathbb{C}^{pK\left(\sum_{b=1}^{B} P_b + B\right)} \middle| arc(z_{q,k}^{b,1}) = arc(z_{q,k}^{b,2}) = \cdots = arc(z_{q,k}^{b,p})\right.$$

$$\left. \text{for any } b = 1, \ldots, B, \ q = 1, \ldots, P_b + 1, \ k = 1, \ldots, K\right\}.$$

In fact we may define $\xi : G \to X$ by

$$
\xi : \left(\sum_{k=1}^{K} \sum_{q=1}^{P_b+1} (-1)^{q+1} \alpha_{q,k}^{b,j} w_{b,q,k} \left(\frac{2\pi(j-1)}{p} + s_{q,k}^b, \cdot \right) \right) \to \left(e^{is_{q,k}^b} \alpha_{q,k}^{b,j} \right).
$$

To distinguish between the spaces $\mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)}$ and $\mathbb{C}^{pK\left(\sum_{b=1}^{B} P_b + B\right)}$, we denote their vectors by $(z_{b,q,k})$ and $(z_{q,k}^{b,j})$ respectively. On the other hand, we have a continuous map $f : X \to \mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)}$ written as

$$
f(z_{q,k}^{b,j}) = \left(\sum_{j=1}^{p} z_{q,k}^{b,j} \right).
$$

Notice that $f^{-1}(0) = 0$ and $f(e^{\frac{2\pi i}{p}} z_{q,k}^{b,j}) = e^{\frac{2\pi i}{p}} f(z_{q,k}^{b,j})$. The reversed inequality follows from the identity $f \circ \xi(\sigma U) = e^{\frac{2\pi i}{p}} f \circ \xi(U)$.

$\square$

By Proposition 2.17 for $\varepsilon > 0$ small enough, $E_{bp-p+i,q}^{\varepsilon}$ is invariant under the heat flow $\eta^t(\cdot)$ for any $b = 1, \ldots, B$, $i = 1, \ldots, p$ and $q = 1, \ldots, P_b + 1$. For the sake of convenience, we will denote the set $E_{bp-p+i,q}^{\varepsilon}$ by $E_{j,q}^{\varepsilon}$ with $b = 1, \ldots, B$, $i = 1, \ldots, p$, $q = 1, \ldots, P_b + 1$ and $j = pb - p + i$. As we did in Section 3.2, define a continuous cut-off function $\phi : [0, \infty) \to [0, 1]$ by $\phi(s) = 1$ for $s \geq \varepsilon$, $\phi(s) = 0$ for $s \leq \frac{\varepsilon}{2}$, and $\phi(s) = \frac{2s}{\varepsilon} - 1$ for $s \in (\frac{\varepsilon}{2}, \varepsilon)$.

We will prove $\gamma(\partial\mathcal{A} \cap G \backslash A_\varepsilon) \geq K - (p-1)(\sum_{b=1}^{B} P_b + B)$, which completes the proof of the present subsection. To obtain this, we define a mapping $h : G \cap A_\varepsilon \to G$ as

$$
h(U) = \left(\sum_{q=1}^{P_1+1} (-1)^{q+1} \phi\big(|\eta^{T^*(U)}(U)_{1,q}|_4\big) U_{1,q}, \ldots, \sum_{q=1}^{P_B+1} (-1)^{q+1} \phi\big(|\eta^{T^*(U)}(U)_{N,q}|_4\big) U_{N,q} \right)
$$

for $U \in G_0$ and for $U \in G \backslash G_0$, let

$$
h(U) = \left(\sum_{q=1}^{P_1+1} (-1)^{q+1} \phi\big(|U_1 \cdot \chi_{\Omega_{1,q}}|_4\big) U_1 \cdot \chi_{\Omega_{1,q}}, \ldots, \sum_{q=1}^{P_B+1} (-1)^{q+1} \phi\big(|U_N \cdot \chi_{\Omega_{N,q}}|_4\big) U_N \cdot \chi_{\Omega_{N,q}} \right).
$$

Here, χ_A is the characteristic function of the set $A \subset \Omega$. In fact, for $U \in G$, $U_{bp-p+i,q} = U_{bp-p+i} \cdot \chi_{\Omega_{b,q}}$ for $q = 1 \ldots, P_b + 1$, $i = 1, \ldots, p$ and $b = 1 \ldots, B$. And $n(U_{bp-p+i}) \leq P_b$ for $i = 1, \ldots, p$ and $b = 1, \ldots, B$. The "$\leq$"'s hold strictly for at least one of admissible (b, j)'s. The mapping h is continuous due to Lemma 4.3 and Lemma 4.4. And it is easy to see that this mapping is also σ-equivariant. Then, we have the following claim.

Lemma 4.9. *For any $U \in \partial\mathcal{A} \cap G_0 \cap A_\varepsilon$, there are admissible couples (j_1, q_1) and (j_2, q_2) such that $\phi(|\eta^{T^*(U)}(U)_{j_1,q_1}|_4) = 0$ and $\phi(|\eta^{T^*(U)}(U)_{j_2,q_2}|_4) > 0$.*

Proof. We notice that $\eta^t(U)$ always stays on $\partial\mathcal{A}$, which implies that $\|\eta^t(U)\| \geq C > 0$ for any $t > 0$. Let us assume that $\phi(|\eta^{T^*(U)}(U)_{j,q}|_4) = 0$ for any (j,q) admissible. This gives $\sum_{j=1}^N \int |u_j|^4 < C\varepsilon^4$ as t goes large. Multiplying u_j on the both sides of the j-th equation of Problem (2.1) and summing up with respect to j, we have

$$\frac{1}{2}\frac{\partial}{\partial t}|U(t)|_2^2 + \|U(t)\|^2 \leq \sum_{j=1}^N \left(\mu_j \int |u_j(t)|^4 + \sum_{i \neq j} \beta_{ij} \int u_i(t)^2 u_j(t)^2 \right) < C\varepsilon$$

when t is large. Using the openness of $\mathcal{A}$ in $(H^1)^N$ and the invariance of $\partial\mathcal{A}$, we have $\frac{\partial}{\partial t}|U(t)|_2^2 \leq -C$ for $t > 0$. This is a contradiction, and the proof is complete.

$\square$

Remark 4.10. *The lemma implies that for any $U \in \partial\mathcal{A} \cap G_0 \cap A_\varepsilon$, there are two admissible couples (j_1, q_1) and (j_2, q_2) such that $h(U)_{j_1,q_1} = 0$ and $h(U)_{j_2,q_2} \neq 0$.*

Lemma 4.11. *It holds $\gamma(\partial\mathcal{A} \cap D\backslash A_\varepsilon) = \infty$.*

Proof. We use the notations in the proof of Lemma 4.8. To proceed our computation, we need an upper bound of $\gamma(h(\partial\mathcal{A} \cap G \cap A_\varepsilon))$. Due to above deduction, for any $U \in h(\partial\mathcal{A} \cap G \cap A_\varepsilon)$, we can find an admissible couple (j,q) such that $|U_{j,q}|_4 = 0$. Translating into the version in G, we have that there are some $b_0 = 1, \ldots, B$, $j_0 = 1, \ldots, p$ and $q_0 = 1, \ldots, P_{b_0} + 1$ such that $|z_{q_0,k}^{b_0,j_0}| = 0$ for any $k = 1, \ldots, K$. Instead of estimating $\gamma(h(\partial\mathcal{A} \cap G \cap A_\varepsilon))$, we will give an upper bound of $\gamma(\xi \circ h(\partial\mathcal{A} \cap G \cap A_\varepsilon))$ since $\gamma(h(\partial\mathcal{A} \cap G \cap A_\varepsilon)) \leq \gamma(\xi \circ h(\partial\mathcal{A} \cap G \cap A_\varepsilon))$.

We divide $h(\partial\mathcal{A} \cap G \cap A_\varepsilon)$ into two parts. The first part is defined by

$$L = \left\{ (z_{q,k}^{b,j}) \in \xi \circ h(\partial\mathcal{A} \cap G \cap A_\varepsilon) \middle| \exists b_0 = 1, \ldots, B, \ \exists q_0 = 1, \ldots, P_{b_0} + 1 \right.$$

$$\left. s.t. \ |z_{q_0,k}^{b_0,j}| = 0 \ \forall j = 1, \ldots, p, \ \forall k = 1, \ldots, K \right\}.$$

It is easy to see that for any $(z_{b,q,k}) \in f(L) \subset \mathbb{C}^{K(\sum_{b=1}^B P_b + B)}$, we can find some $b_0 = 1, \ldots, B$ and $q_0 = 1, \ldots, P_{b_0} + 1$ such that for any $k = 1, \ldots, K$, $z_{b_0,q_0,k} = 0$. We define a subspace $W \subset \mathbb{C}^{K\left(\sum_{b=1}^B P_b + B\right)}$ by

$$W = \left\{ (z_{b,q,k}) \in \mathbb{C}^{K\left(\sum_{b=1}^B P_b + B\right)} | z_{1,1,k} = \cdots = z_{1,P_1+1,k} = \cdots = z_{B,1,k} = \cdots \right.$$

$$\left. \cdots = z_{B,P_B+1,k} \ \forall k = 1, \ldots, K \right\}.$$

To continue the proof, we need the following lemma:

Lemma 4.12. *There is an $\varepsilon_0 > 0$ such that for any $a = (a_1, \ldots, a_{K(\sum_{b=1}^{B} P_b + B)}) \in$*
$f(L)$, $\sum_{l=1}^{K(\sum_{b=1}^{B} P_b + B)} |a_l| \geq \varepsilon_0$.

Proof. We argue it by contradiction. Then there is a sequence $a^{(n)} \in f(L)$ such that $|a^{(n)}| \to 0$, which implies that there is a sequence $U^{(n)} \in \partial\mathcal{A} \cap G \cap A_\varepsilon$ such that $\sum_{j=1}^{N} |u_j^{(n)}|_4 \to 0$. Using the fact that $T(U) = \infty$ for $U \in \partial\mathcal{A}$ and similar computations in Proposition 2.17 and Lemma 4.9, we will have a contradiction.
$\square$

We now return to the proof of Lemma 4.11. On one hand, we have $dim(W) = K$. On the other hand, due to the definition of the space W and Lemma 4.12, we have

$$f(L) \subset \mathbb{C}^{K\left(\sum_{b=1}^{B} P_b + B\right)} \backslash W_\delta \overset{\sigma}{\simeq} \mathbb{S}^{K\left(\sum_{b=1}^{B} P_b + B - 1\right) - 1}.$$

Here, $\delta > 0$ is small, W_δ represents the δ-neighbourhood of W, and the symbol $\overset{\sigma}{\simeq}$ means that two topological spaces are homotopy equivalent via a homotopy F satisfies $F(t, e^{\frac{2\pi i}{p}} z) = e^{\frac{2\pi i}{p}} F(t, z)$ and $\mathbb{S}^{m-1}$ denote the unit sphere in $\mathbb{C}^m$. Hence, we have

$$\gamma(L) \leq \gamma(f(L)) \leq K\left(\sum_{b=1}^{B} P_b + B - 1\right).$$

Notice that L is a compact set, then there is a $\delta > 0$ such that $\gamma(L) = \gamma(L_\delta)$.

Now we give an estimate on another part of $\xi \circ h(\partial\mathcal{A} \cap G \cap A_\varepsilon)$. Define the set

$$M := \xi \circ h(\partial\mathcal{A} \cap G \cap A_\varepsilon) \backslash L_\delta,$$

where $\delta > 0$ is small. According to the previous deduction, the elements $(z_{q,k}^{b,j})$ in M satisfy that there are $b_0 = 1, \ldots, B$ and $q_0 = 1, \ldots, P_{b_0} + 1$ such that there are two $j_1, j_2 = 1, \ldots, p$ with $j_1 \neq j_2$ and
- $|z_{q_0,k}^{b_0,j_1}| = 0$ for any $k = 1, \ldots, K$;
- $\sum_{k=1}^{K} |z_{q_0,k}^{b_0,j_2}| \geq \delta_1$,

for some $\delta_1 > 0$. Using these for any $b = 1 \ldots, B$ and $q = 1, \ldots, P_{b_1} + 1$ we define

$$N_q^b = \left\{ (z_{q,k}^{b,j})_{j,k} \in M \,\middle|\, \text{there exist } j_1 \neq j_2 \text{ such that } \sum_{k=1}^{K} \left| z_{q,k}^{b,j_1} \right| = 0, \sum_{k=1}^{K} |z_{q,k}^{b,j_2}| \geq \delta_1 \right\}.$$

Then, N_q^b can be divided into p sets

$$N_q^b(j+1) = \sigma^j \left(N_q^b(1) \right),$$

where $j = 0, \ldots, p-1$ and

$$N_q^b(1) = \left\{ (z_{q,k}^{b,j}) \in N_q^b \,\middle|\, \sum_{k=1}^{K} \left| z_{q,k}^{b,1} \right| = 0 \right\}.$$

It is easy to see that, due to the definition, we have

$$N_q^b(j+1) = \left\{ (z_{q,k}^{b,j}) \in N_q^b \,\middle|\, \sum_{k=1}^{K} \left| z_{q,k}^{b,j+1} \right| = 0 \right\}.$$

for $j = 0, \ldots, p-1$,

$$\cap_{j=0}^{p-1} \sigma^j \left(N_q^b(1) \right) = \emptyset$$

and

$$N_q^b = \cup_{j=0}^{p-1} \sigma^j \left(N_q^b(1) \right).$$

Therefore, using the last property of the index γ in Section 4.1 we have

$$\gamma \left(N_q^b \right) \leq p - 1$$

for any fixed $b = 1, \ldots, B$ and $q = 1, \ldots, P_b + 1$ and

$$\gamma(M) = \gamma\big(\xi^{-1}(M)\big) \leq \sum_{b=1}^{B} \sum_{q=1}^{P_b+1} \gamma\left(\big(\xi^{-1}(N_q^b)\big)\right) \leq (p-1)\left(\sum_{b=1}^{B} P_b + B\right).$$

Now we can compute

$$\gamma\left(\overline{\partial\mathcal{A} \cap G \cap A_\varepsilon}\right) \leq \gamma\left(\overline{h(\partial\mathcal{A} \cap G \cap A_\varepsilon)}\right) \leq \gamma\left(\overline{\xi \circ h(\partial\mathcal{A} \cap G \cap A_\varepsilon)}\right)$$
$$\leq \gamma(L_\delta) + \gamma(M_\delta) = \gamma(L) + \gamma(M)$$
$$\leq K\left(\sum_{b=1}^{B} P_b + B - 1\right) + (p-1)\left(\sum_{b=1}^{B} P_b + B\right).$$

The above result implies that

$$\gamma\big((\partial\mathcal{A}\backslash A_\varepsilon) \cap G\big)$$
$$\geq \gamma(G \cap \partial\mathcal{A}) - \gamma\left(\overline{A_\varepsilon \cap \partial\mathcal{A} \cap G}\right)$$
$$\geq K\left(\sum_{b=1}^{B} P_b + B\right) - K\left(\sum_{b=1}^{B} P_b + B - 1\right) - (p-1)\left(\sum_{b=1}^{B} P_b + B\right)$$
$$= K - (p-1)\left(\sum_{b=1}^{B} P_b + B\right).$$

The proof of Lemma 4.11 is complete since K is arbitrarily large.

$\square$

Remark 4.13. *We note that $\partial\mathcal{A} \cap D\backslash A_\varepsilon$ is an invariant set of the heat flow, from which, a sequence of compact sets with unbounded genus can be selected.*

4.5. **The Existence of Multiple Equilibria Having the Same Componentwise Prescribed Number of Nodes.** In this subsection, we complete the proof of the main result Theorem 1.2.

Lemma 4.14. *Let $c \in J(\partial\mathcal{A} \cap D\backslash A_\varepsilon)$. If there are positive numbers α and ε' such that for any $U \in J^{-1}[c-\varepsilon', c+\varepsilon'] \cap \partial\mathcal{A} \cap D\backslash A_\varepsilon$, $|\Delta u_j - \lambda u_j + \mu u_j^3 + \sum_{i\neq j} \beta u_i^2 u_j|_2 \geq \alpha$ for some $j = 1, \ldots, N$, then there is $T > 0$ independent of U such that $\eta^T(U) \in J^{c-\varepsilon'} \cap \partial\mathcal{A} \cap D\backslash A_\varepsilon$.*

Proof. Let $T = \frac{4\varepsilon'}{\alpha^2} > 0$. Notice that $\partial\mathcal{A} \cap D\backslash A_\varepsilon$ is invariant under the flow η. If $\eta^T(U) \in J^{c-\varepsilon'} \cap \partial\mathcal{A} \cap D\backslash A_\varepsilon$, the proof is complete. Otherwise, assume $J(\eta^T(U)) > c - \varepsilon'$. Then $\eta^t(U) \in J^{-1}[c-\varepsilon', c+\varepsilon'] \cap \partial\mathcal{A} \cap D\backslash A_\varepsilon$ for any $t \in [0, T]$. Compute that

$$\frac{d}{dt} J(\eta^t(U)) = -\sum_{j=1}^{N} |\partial_t u_j(t, \cdot)|_2^2$$

$$= -\sum_{j=1}^{N} \left| \Delta u_j - \lambda_j u_j + \mu_j u_j^3 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j \right|_2^2(t)$$

$$\leq -\alpha^2.$$

Therefore,

$$c - \varepsilon' \leq c + \int_0^T \frac{d}{dt} J(\eta^t(U)) dt \leq c - \alpha^2 T = c - 4\varepsilon'.$$

This is a contradiction.

$\square$

Define

$$\Gamma_k = \{A \subset \partial\mathcal{A} \cap D\backslash A_\varepsilon | A \text{ is } \sigma \text{ invariant compact set}, \gamma(A) \geq k\}.$$

By Lemma 4.11, $\Gamma_k \neq \emptyset$ for large k, and the values

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} J(U)$$

are well-defined. Using Lemma 4.14 and some classical arguments as [43, Propersition 8.5] it is easy to verify the following.

Lemma 4.15. *(i). $K_{c_k} \cap \partial\mathcal{A} \cap D\backslash A_\varepsilon \neq \emptyset$ for large k.*
   (ii). If $c := c_j = \cdots = c_{j+l}$, then $\gamma(K_c \cap \partial\mathcal{A} \cap D\backslash A_\varepsilon) \geq l + 1$.

Proof. A standard argument ensures that there is a sequence $U_n \in \partial\mathcal{A} \cap D\backslash A_\varepsilon$, such that
   - $J(U_n) \to c_k$;
   - for any $j = 1, \ldots, N$, $\Delta u_{n,j} - \lambda_j u_{n,j} + \mu_j u_{n,j}^3 + \sum_{i\neq j} \beta_{ij} u_{n,i}^2 u_{n,j} \to 0$ in L^2,

where $u_{n,j}$ is the j-th component of U_n. The second assertion implies that $\nabla_{u_j} J(U_n) \to \theta$ in H^{-1}. Then $U_n \to U$ in $(H^1)^N$ for some U since the energy functional J satisfies the (PS) condition. It is easy to see that U is a critical point of the energy J. Therefore, U is of class $(H^2)^N$ due to the elliptic regularity theory. Now we prove that $U \in \partial\mathcal{A} \cap D \backslash A_\varepsilon$.

To show $U \in D$, denote $U = (u_1, ..., u_N)$. U_n's are continuous since they are of class $(H^2)^N$. We have for $b = 1, ..., B$, $n\big((U_n)_{pb-p+i}\big) = P_b$ for $i = 1, ..., p$, and $\big|(U_n)_{pb-p+i,q}\big|_4 \geq \frac{\varepsilon}{2}$ for $i = 1, ..., p$ and $q = 0, ..., P_b$. Here $(U_n)_{pb-p+i,q}$ denotes the q-th bump of the $(pb - p + i)$-th component of U_n.

Since $U_n \to U$ in $(H^1)^N$, for n large, we have $\big|(U_n)_{pb-p+i} - u_{pb-p+i}\big|_4 < \frac{\varepsilon}{100}$. Therefore, for fixed $b = 1, ..., B$, $i = 1, ..., p$, we can find a sequence $x_0, ..., x_{P_b} \in \Omega$ such that

- $0 < |x_0| < \cdots < |x_{P_b}| < \infty$;
- $u_{pb-p+i}(x_k) \cdot u_{pb-p+i}(x_{k+1}) < 0$ for $k = 0, ..., P_b - 1$.

Therefore, $U \in D$.

Now we prove $U \in \partial\mathcal{A} \backslash A_\varepsilon$ by contradiction. If $U \in A_\varepsilon$, due to the definition of A_ε, there is an admissible pair (j_0, q_0) such that $U \in A^\varepsilon_{j_0,q_0}$. If $|U_{j_0,q_0}|_4 \geq \varepsilon$, there is a $T > 0$ such that $\big|\eta^T(U)_{j_0,q_0}\big|_4 < \varepsilon$. This implies that U is not equilibrium. Therefore we have a contradiction. On the other hand, if $|U_{j_0,q_0}|_4 < \varepsilon$, due to Proposition 2.17, once again, U is not equilibrium. Hence, $U \notin A_\varepsilon$. And $U \in \partial\mathcal{A}$ is due to the closeness of $\partial\mathcal{A}$ with respect to the norm $\|\cdot\|$. Assertion (i) is proved.

Assertion (ii) can be proved by some arguments for the classical genus [43, Propersition 8.5], and we omit it here.

$\square$

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