Potential well and multiplicity of solutions for nonlinear Dirac equations

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Abstract

In this paper we consider the semi-classical solutions of a massive Dirac equations in presence of a critical growth nonlinearity

$$-i\hbar\sum_{k=1}^{3}\alpha_k\partial_k w + a\beta w + V(x)w = f(|w|)w.$$

Under a local condition imposed on the potential V, we relate the number of solutions with the topology of the set where the potential attains its minimum. In the proofs we apply variational methods, penalization techniques and Ljusternik-Schnirelmann theory.

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1 Introduction and main result

In these last years, attentions have been drawn to the study of standing wave solutions for the nonlinear Dirac equation of the form

$$-i\hbar\partial_t\varphi = ic\hbar\alpha\cdot\nabla\varphi - mc^2\beta\varphi - V(x)\varphi + g(x,\varphi), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^3 \tag{1.1}$$

where $\varphi(t,x)\in\mathbb{C}^4$ is a spinor function, $\hbar$ is a small positive constant which corresponds to the Plank's constant, $m, c > 0$ are constants representing the mass of a electron and the speed of light, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $\alpha\cdot\nabla = \sum_{k=1}^{3}\alpha_k\partial_k$ with α_1, α_2, α_3 and β being the 4×4 complex Pauli matrices:

$$\beta = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k\\ \sigma_k & 0\end{pmatrix}, \quad k=1,2,3,$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

Moreover, in Eq. (1.1), V is a potential function and g is the nonlinearity modeling some self-interaction in Quantum electrodynamics. In particle physics, (1.1) models many physical problems in the self-interacting scalar theories, where the nonlinear function g can be both a polynomial and non-polynomial. Various nonlinearities are considered as possible basis models for unified field theories, we just refer to [30, 31, 34] for more physical background.

A solution of the form $\varphi(t,x) = \exp(-i\omega t/\hbar)w(x)$ is called a standing wave. Assume that $g(x, \exp(-i\theta)\xi) = \exp(-i\theta)g(x,\xi)$ for $\theta \in \mathbb{R}$ and $\xi \in \mathbb{C}^4$, a change of notation (in particular ε instead of $\hbar$) leads to an equation of the form

$$(1.2) \qquad -i\varepsilon\alpha \cdot \nabla w + a\beta w + V(x)w = g(x,w) \quad w \in H^1(\mathbb{R}^3, \mathbb{C}^4).$$

This type of particle-like solution does not change its shape as it evolves in time, and thus has a soliton-like behavior.

It should be pointed out here that, in quantum mechanics, the existence and multiplicity of solutions to a dynamical equation in terms of an asymptotic representation as the Plank's constant tends to zero is of particular importance. To some extent, this corresponds to a deformation of quantum mechanics and quantum field theory to classical mechanics and classical field theory. Such deformation is parameterized by the Planck's constant and, in this deformation, solutions to dynamical equations are usually referred as semiclassical states. In the case of non-relativistic quantum field theories, standing wave solutions for the nonlinear Schrödinger equation

$$-i\hbar\partial_t\psi = \Delta\psi - V(x)\psi + f(\psi)$$

have been in the focus of nonlinear analysis since decades. Particularly, semiclassical states that concentrate near a critical point of the potential V have been widely investigated ever since the influential paper [33] by Floer and Weinstein who treated the cubic nonlinearity $|\psi|^2\psi$ in one-dimension. An incomprehensive list of references are [2, 3, 6, 10, 11, 19–22, 36–38], in which the authors used Lyapunov-Schmidt type methods, penalization and variational techniques to establish the concentration phenomenon of the semiclassical states for the Schrödinger equations.

Much less is known for the nonlinear Dirac equation (1.1) which arises in relativistic field theories. So far only a few results are available for the concentration phenomenon of semi-classical states around a minima x_0 of V; see [26, 27]. Related results, i.e., concentration of semiclassical states under the influence of nonlinear potentials, can be found for similar equations in [24–26]. Lyapunov-Schmidt type methods do not seem to be applicable to (1.2) because even for the homogeneous nonlinearity $g(x,w) = |w|^{p-2}w$ nothing is known about uniqueness or non-degeneracy of the least energy solution of

$$(1.3) \qquad -i\alpha \cdot \nabla w + a\beta w + V(x_0)w = |w|^{p-2}w, \quad w \in H^1(\mathbb{R}^3, \mathbb{C}^4)$$

which appears as limit equation for (1.2). As for variational methods, a major difference between nonlinear Schrödinger and Dirac equations is that the Dirac operator is strongly indefinite in the sense that both the negative and positive parts of the spectrum are unbounded and consist of essential spectrum. It follows that the quadratic part of the energy functional associated to (1.2) has no longer a positive sign, moreover, the Morse index and co-index at any critical point of the energy functional are infinite. It is not clear whether one can develop a penalization technique to find semiclassical states. And moreover, beyond the existence and concentration results in [26, 27], it is interesting to ask whether one can obtain a multiplicity of semiclassical solutions to Eq. (1.2). Very recently, in [40] Wang and Zhang obtained an interesting result in this direction. By using the symmetric structure of Eq. (1.2), they constructed an infinite sequence of bound state solutions for small values of ε, particularly, these solutions are of higher topological type.

In this paper, letting M be a set of local minima of the potential V, we are interested in the following aspects which have not been dealt with before and is new in the case of Dirac equations:

(1) to show the multiplicity of semiclassical solutions concentrating around M is influenced by the topology of the level sets of the potential V in a bounded domain;

(2) to apply variational methods, concentration-compactness and rescaling techniques to deal with nonlinearities more general than $|w|^{p-2}w$, in particular, $g(x,w)$ grows critically as $|w|\to\infty$.

We mention here that starting from the paper of Bahri and Coron [5], many papers are devoted to study the effect of the domain topology on the existence and multiplicity of solutions for semilinear elliptic problems. We refer to [7–9, 12, 16, 17] for related studies for Dirichlet and Neumann boundary value problems. We also refer to [4, 13–15] for the study of semiclassical states of Schrödinger equations.

Now, in order to state our results precisely, let us consider the following equation

$$-i\varepsilon\alpha\cdot\nabla w + a\beta w + V(x)w = f(|w|)w. \tag{1.4}$$

Throughout the paper, we assume that the potential V satisfies

(V_1) *V is locally Hölder continuous and $\|V\|_{L^\infty} < a$.*

(V_2) *There exists a bounded domain $\Lambda\subset\mathbb{R}^3$ such that*

$$\underline{\omega} := \min_{\Lambda} V < \min_{\partial\Lambda} V.$$

And we denote $M := \{x\in\Lambda : V(x) = \underline{\omega}\}$. For the nonlinear function f, we make the following assumptions:

(f_1) *$f\in C[0,\infty)\cap C^1(0,\infty)$, $f(0)=0$ and $f'(s)\geq 0$;*

(f_2) $\lim_{s\to\infty}\frac{f(s)}{s} = \kappa$;

(f_3) *there exist $p\in(2,3)$ such that $f(s)\geq s^{p-2}+\kappa s$ for all $s\geq 0$;*

(f_4) *there exists $\theta>2$ such that $0<\theta F(s)\leq f(s)s^2$ for $s>0$, where $F(s)=\int_0^s f(t)tdt$.*

Condition (f_1) implies that $s\mapsto f(s)s$ is superlinear and strictly increasing, an important role in our approach. If $\kappa>0$ in (f_2), then $F(s)\sim\kappa s^3$ as $s\to\infty$ is of critical growth. This terminology is befitting because the form domain of the quadratic form associated to the Dirac operator is $H^{\frac{1}{2}}(\mathbb{R}^3,\mathbb{C}^4)$. This space embeds into the corresponding L^q-spaces for $2\leq q\leq 3$. And if $\kappa=0$ then the problem is subcritical. (f_3) is a technical assumption, and (f_4) is the Ambrosetti-Rabinowitz condition.

Letting $cat_X(A)$ denote the Lusternick-Schnirelmann category of A in X for any topological pair (X,A), our main result can be stated as follows

Theorem 1.1. *Suppose $(V_1)-(V_2)$. There exists $\bar\kappa=\bar\kappa(\|V\|_{L^\infty})>0$ such that if f satisfies $(f_1)-(f_4)$ with $\kappa\in[0,\bar\kappa)$ then, letting*

$$M_\delta = \{x\in\mathbb{R}^3 : \operatorname{dist}(x,M)\leq\delta\},\quad \textit{for } \delta>0$$

Eq. (1.4) *has at least $cat_{M_\delta}(M)$ solutions w_ε^k, $k=1,\dots,cat_{M_\delta}(M)$, for sufficiently small $\varepsilon>0$. These solution have the following properties:*

(1) *for each k, $|w_\varepsilon^k|$ possesses a (global) maximum point x_ε^k in Λ such that*

$$\lim_{\varepsilon\to 0} V(x_\varepsilon^k) = \underline{\omega};$$

(2) *The rescaled function $v_\varepsilon^k(x) = w_\varepsilon^k(\varepsilon x + x_\varepsilon^k)$, converges in H^1 as $\varepsilon\to 0$ to a least energy solution $v:\mathbb{R}^3\to\mathbb{C}^4$ of*

$$-i\alpha\cdot\nabla v + a\beta v + \underline{\omega}v = f(|v|)v.$$

Remark 1.2. (1) In some circumstances, such as M is a finite set, M is smooth compact submanifold of $\mathbb{R}^3$ or M is the boundary of a bounded open set, $cat_{M_\delta}(M) = cat_M(M)$ for small $\delta > 0$. More specifically, if $M \sim S^2$ (the sphere in $\mathbb{R}^3$), then $cat_M(M) = 2$.

(2) There is an example showing that Eq. (1.4) has arbitrary large number of solutions. Under conditions $(V_1) - (V_2)$, if $M = \{x_n : n \geq 1\} \cup \{x_0\}$, where $x_n \to x_0$ as $n \to +\infty$, then for any $m \in \mathbb{N}$, there exists $\delta_m > 0$ such that $cat_{M_{\delta_m}}(M) \geq m$. Hence it follows from Theorem 1.1, for small $\varepsilon > 0$, Eq. (1.4) has at least m solutions.

(3) The constant $\bar{\kappa}(\|V\|_{L^\infty}) > 0$ will be explicitly defined; see (5.3). It satisfies $\bar{\kappa}(\|V\|_{L^\infty}) \to c(p) > 0$ as $\|V\|_{L^\infty} \to 0$. Thus we do allow critical growth $f(s)s \sim \kappa s^2$ but the factor κ cannot be too large. It is an interesting open problem whether this restriction on κ can be removed.

The proof will be done by variational techniques. Since we have no information on the potential V at infinity, we employ the truncation trick explored in [20]. It consists in making a suitable modification on the nonlinearity f , solving a modified problem and then check that, for ε small enough, the solutions of the modified problem are indeed solutions of the original one. We emphasize here that, in the usual concept, the truncation tricks are well adapted for the study of the subcritical variational problems, see for instance [10,11,19,21,22] for the studies of Schrödinger equations. However, due the strongly indefinite character of the Dirac operator, we note that it is not easy to obtain compactness in view of the critical growth of the nonlinearity even for the modified problem. To overcome this, we will need a delicate analysis for the limit problem (1.3) on the ground state energy level and use a version of the concentration-compactness principle originated from Lions [35] to control the factor $\kappa > 0$ in the critical growth. As a matter of fact, the truncation trick we adapt here is essentially depending on the factor κ as we will see in the Remark 4.3 in Section 4.

To obtain multiple solutions of the modified problem, the main ingredient is to make precisely comparisons between the category of some sublevel sets of the modified functional and the category of the set M. This kind of argument for the Schrödinger equations has been appeared in [8, 13–15], where subcritical problems were considered.

The remainder part of the paper is organized as follows. In Sect. 2 we first present the variational settings of the problem, both in the original and in the extended variables, and we truncate the original problem. For the sake of completeness, we collect some useful results which are needed in our proof. In Sect. 3, we investigate the associated autonomous problem. This study allow us to show the role which the critical factor κ plays in the ground state energy level. And the Palais-Smale condition, which does not hold in general case since we allow critical growth, will then be studied in Sect. 4. Next, in Sect. 5, we provide the main components of our proof. The first point is we introduce the min-max scheme that can be applied to the truncated problem. And as the second point, we construct two maps in terms of the truncated problem such that their composition is homotopically equivalent to the embedding $j : M \to M_\delta$. Finally, the main results are proved in Sect. 6..

2 Notations, known facts and main ingredients

Let $u(x) = w(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, it is clear to see that (1.4) is equivalent to

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x)u = f(|u|)u. \tag{2.1}$$

We shall in the sequel focus on this equivalent problem.

In what follows, by $|\cdot|_q$ we denote the usual L^q-norm, and $(\cdot,\cdot)_2$ the usual L^2-inner product. Let $\mathscr{L} = -i\alpha \cdot \nabla + a\beta$ denote the self-adjoint operator on $L^2 := L^2(\mathbb{R}^3, \mathbb{C}^4)$ with domain $\mathcal{D}(\mathscr{L}) = H^1 := H^1(\mathbb{R}^3, \mathbb{C}^4)$. It is well known that $\sigma(\mathscr{L}) = \sigma_c(\mathscr{L}) = \mathbb{R} \setminus (-a, a)$ where $\sigma(\cdot)$

and $\sigma_c(\cdot)$ denote the spectrum and the continuous spectrum. Thus the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^+ \oplus L^-, \quad u = u^+ + u^- \tag{2.2}$$

so that $\mathscr{L}$ is positive definite (resp. negative definite) in L^+ (resp. L^-). Let $E := \mathcal{D}(|\mathscr{L}|^{1/2}) = H^{1/2}$ be equipped with the inner product

$$\langle u, v\rangle = \Re(|\mathscr{L}|^{1/2}u, |\mathscr{L}|^{1/2}v)_2$$

and the induced norm $\|u\| = \langle u, u\rangle^{1/2}$, where $|\mathscr{L}|$ and $|\mathscr{L}|^{1/2}$ denote respectively the absolute value of $\mathscr{L}$ and the square root of $|\mathscr{L}|$. Since $\sigma(\mathscr{L}) = \mathbb{R} \setminus (-a, a)$, one has

$$a|u|_2^2 \le \|u\|^2 \quad \text{for all } u \in E. \tag{2.3}$$

Note that this norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^q for all $q \in [2, 3]$ and compactly into L^q_{loc} for all $q \in [1, 3)$.

It is clear that E possesses the following decomposition

$$E = E^+ \oplus E^- \text{ with } E^\pm = E \cap L^\pm, \tag{2.4}$$

orthogonal with respect to both $(\cdot,\cdot)_2$ and $\langle\cdot,\cdot\rangle$ inner products. And remarkably, this decomposition of E induces also a natural decomposition of L^q for every $q \in (1, +\infty)$:

Proposition 2.1 (see [27]). *Let $E^+ \oplus E^-$ be the decomposition of E according to the positive and negative part of $\sigma(\mathscr{L})$. Then, set $E_q^\pm := E^\pm \cap L^q$ for $q \in (1, \infty)$, there holds*

$$L^q = \mathrm{cl}_q\, E_q^+ \oplus \mathrm{cl}_q\, E_q^-$$

with cl_q denoting the closure in L^q. More precisely, there exists $d_q > 0$ for every $q \in (1, \infty)$ such that

$$d_q|u^\pm|_q \le |u|_q \quad \text{for all } u \in E \cap L^q.$$

Remark 2.2. It is of great importance for the projections from $H^{1/2} := E = E^+ \oplus E^-$ onto E^+ (or E^-) to be continuous in the L^q's and not only in $H^{1/2}$. This is not the case for every direct sum in $H^{1/2}$. In fact, the proof of Proposition 2.1 implies on the splitting of L^q's that: For every $q \in (1, \infty)$, L^q can be split into topologically direct sum of two (infinite dimensional) subspaces which, accordingly, are the positive and negative projected spaces of the Dirac operator $\mathscr{L}$.

In what follows, we define the energy functional

$$\begin{aligned}\widetilde{\Phi}_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \mathscr{L}u \cdot \overline{u} + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2 - \int_{\mathbb{R}^3} F(|u|) \\ &= \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2}\int V_\varepsilon(x)|u|^2 - \Psi(u)\end{aligned}$$

for $u = u^+ + u^- \in E$. Standard arguments show that, under our assumptions, $\widetilde{\Phi}_\varepsilon \in C^2(E, \mathbb{R})$ and critical point of $\widetilde{\Phi}_\varepsilon$ is a (weak) solution to (2.1).

To establish the multiplicity of solutions, we will adapt for our case an argument explored by the penalization method introduced by Del Pino and Felmer [20]. To this end, we need to fix some notations.

We first let $\delta_0 \in \big(0, \frac{a-|V|_\infty}{2}\big]$ and consider $\tilde{f} \in C^1(0, \infty)$ such that

$$\frac{d}{ds}\big(\tilde{f}(s)s\big) = \min\big\{f'(s)s + f(s),\, \delta_0\big\}. \tag{2.5}$$

Then we introduce

$$g(x,s) = \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x))\tilde{f}(s)\,, \tag{2.6}$$

and the corresponding energy functional

$$\Phi_\varepsilon(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^3} V_\varepsilon(x)|u|^2 - \Psi_\varepsilon(u)\,,$$

where χ_Λ denotes the characteristic function of Λ, $\Psi_\varepsilon(u) := \int_{\mathbb{R}^3} G(\varepsilon x, |u|)\,dx$ and $G(x,s) = \int_0^s g(x,t)t\,dt$. One should keep in mind here that Λ has to be rescaled when we consider the modified rescaled equation (2.1). It is well-known that such truncation trick will be helpful in both bringing compactness to the problem and locating the maximum points of the solutions, see [20, 21] and [27] for subcritical problems. Since we address here the critical growth, we remark that, in order to recover the compactness, δ_0 should be chosen even smaller and this will be seen in the proof of Proposition 4.1 where a implicit upper bound is established accordingly to the critical fact κ: if $\kappa = 0$, then $\delta_0 \le \frac{a-|V|_\infty}{2}$ is enough; if $\kappa > 0$, then δ_0 needs to be properly smaller.

It is elementary to check that (f_1) and (f_3) implies that g is a Carathéodory function and it satisfies

- (g_1) *$g_s(x,s)$ exists everywhere, $g(x,s)s = o(s)$ uniformly in x as $s \to 0$;*
- (g_2) *$0 \le g(x,s)s \le f(s)s$ for all x;*
- (g_3) *$0 < 2G(x,s) \le g(x,s)s^2 \le \delta_0 s^2$ for all $x \notin \Lambda$ and $s > 0$;*
- (g_4) *$g(x,s) \ge s^{p-2} + \kappa s$ for all $x \in \Lambda$ and $s > 0$;*
- (g_5) *$\frac{d}{ds}\big(g(x,s)s\big) \ge 0$ for all x and $s > 0$;*
- (g_6) *$\widehat{G}(x,s) \to \infty$ as $s \to \infty$ uniformly in x.*

Here we used the notation $\widehat{G}(x,s) = \frac{1}{2}g(x,s)s^2 - G(x,s)$.

In what follows, we shall collect some properties of Φ_ε when the assumptions on V and f hold. First, similar as that in [27], we give the following geometric behaviors of Φ_ε.

Lemma 2.3. *For $c \ge 0$, any $(P.S.)_c$-sequence for Φ_ε is bounded independent of $\varepsilon > 0$.*

Proof. We sketch the proof as follows. Let $\{u_n\}$ be such that

$$\Phi_\varepsilon(u_n) = c + o_n(1) \quad \text{and} \quad \Phi'_\varepsilon(u_n) = o_n(1) \quad \text{as } n \to \infty.$$

Then we have

$$\begin{aligned} c + o_n(1) = \Phi_\varepsilon(u_n) - \frac{1}{2}\Phi_\varepsilon(u_n)[u_n] &= \int_{\mathbb{R}^3} \widehat{G}(\varepsilon x, |u_n|) \\ &\ge \int_{\Lambda_\varepsilon} \frac{1}{2} f(|u_n|)|u_n|^2 - F(|u_n|) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} o_n(\|u_n\|) &= \Phi_\varepsilon(u_n)[u_n^+ - u_n^-] \\ &= \|u_n\|^2 + \Re\int_{\mathbb{R}^3} V_\varepsilon(x) u_n \cdot \overline{(u_n^+ - u_n^-)} - \Re\int_{\mathbb{R}^3} g(\varepsilon x, |u_n|) u_n \cdot \overline{(u_n^+ - u_n^-)}. \end{aligned} \tag{2.8}$$

By (g_3), we deduce from (2.8) that

$$\Big(1-\frac{|V|_\infty+\delta_0}{a}\Big)\|u_n\|^2 \le \int_{\Lambda_\varepsilon} f(|u_n|)u_n\cdot\overline{(u_n^+-u_n^-)}+o_n(\|u_n\|). \tag{2.9}$$

And by (f_1) and (f_2) we have $\big(f(s)s\big)^{\frac{3}{2}}\le Cf(s)s^2$ for some $C>0$, and hence it follows from (f_4) and (2.7)-(2.9) that

$$\begin{aligned}\Big(1-\frac{|V|_\infty+\delta_0}{a}\Big)\|u_n\|^2 &\le \Big(\int_{\Lambda_\varepsilon}\big|f(|u_n|)u_n\big|^{\frac{3}{2}}\Big)^{\frac{2}{3}}|u_n^+-u_n^-|_3+o_n(\|u_n\|)\\ &\le \bigg(\frac{2C\theta(c+o_n(1))}{\theta-2}\bigg)^{\frac{2}{3}}\|u_n\|+o_n(\|u_n\|)\end{aligned}$$

which implies the boundedness. Moreover, we can see from the above inequalities that $u_n\to 0$ in E if and only of $c=0$. □

In the sequel, let $\mathscr{K}_\varepsilon:=\big\{u\in E\setminus\{0\}:\ \Phi_\varepsilon'(u)=0\big\}$ be the critical set of Φ_ε, using the same iterative argument of [29, Proposition 3.2], we obtain the following

Lemma 2.4. *If $u\in\mathscr{K}_\varepsilon$ with $|\Phi_\varepsilon(u)|\le C$. Then, for any $q\ge 2$, $u\in W^{1,q}(\mathbb{R}^3,\mathbb{C}^4)$ with $\|u\|_{W^{1,q}}\le C_q$, where C_q depends only on C and q.*

In order to describe further the critical values, let us recall some known facts on a Lyapunov-Schmidt type reduction for Φ_ε. Such reduction technique depends on the convexity of the nonlinearities, specifically, it requires that the second order derivative of Φ_ε is negative definite on E^-. And by the anti-coercion and concavity properties of $\Phi_\varepsilon|_{E^-}$, we can define $\ell_\varepsilon: E^+\to E^-$ to be the bounded reduction map correspondingly such that, for any $u\in E^+$,

$$\Phi_\varepsilon\big(u+\ell_\varepsilon(u)\big)=\max_{v\in E^-}\Phi_\varepsilon(u+v).$$

And denote $I_\varepsilon(u)=\Phi_\varepsilon\big(u+\ell_\varepsilon(u)\big)$, we shall call $(\ell_\varepsilon,\, I_\varepsilon): E^+\times E^+\to E^-\times\mathbb{R}$ the reduction couple associated to Φ_ε on E^+ (for details we refer to [1, 27]). Then, it is all clear that $I_\varepsilon\in C^2(E^+,\mathbb{R})$ and critical points of I_ε and Φ_ε are in one-to-one correspondence via the injective map $u\mapsto u+\ell_\varepsilon(u)$ from E^+ to E.

Now, on E^+, let us consider the functional I_ε.

Lemma 2.5. *For small $\varepsilon>0$, I_ε has the mountain pass structure:*

(1) $I_\varepsilon(0)=0$ *and there are positive constants $r,\ \tau>0$ such that $I_\varepsilon|_{S_r^+}\ge\tau$.*

(2) *There exist $e_0\in E^+$ with $\|e_0\|>r$ independent of $\varepsilon>0$ such that $I_\varepsilon(e_0)<0$.*

Moreover, we have

Lemma 2.6. *For all $\varepsilon>0$, let*

$$\mathscr{N}_\varepsilon=\big\{u\in E^+\setminus\{0\}:\ I_\varepsilon'(u)[u]=0\big\}.$$

Then $\mathscr{N}_\varepsilon$ is a C^1 manifold, and there exist $\theta,\ \mu>0$ both independent of ε such that for any $u\in\mathscr{N}_\varepsilon$

$$\|u\|\ge\theta \quad \textit{and} \quad I_\varepsilon(u)\ge\mu;$$

Moreover, critical points of I_ε constrained on $\mathscr{N}_\varepsilon$ are free critical points of I_ε in E^+.

Remark 2.7. In general, letting $u \in E^+ \setminus \{0\}$, we find there exists at most one nontrivial critical point $t_\varepsilon = t_\varepsilon(u) > 0$ which realizes the maximum of the function $t \mapsto I_\varepsilon(tu)$. It can be also seen, that $\mathscr{N}_\varepsilon$ can be rewritten as

$$\mathscr{N}_\varepsilon = \big\{t_\varepsilon(u)u : \, u \in E^+ \setminus \{0\}, \, t_\varepsilon(u) < \infty\big\}.$$

It is worth pointing out that the set $\mathscr{N}_\varepsilon$ is slightly different from the usual concept of the Nehari manifold associated to the reduced functional I_ε. In fact, $\mathscr{N}_\varepsilon$ is no longer expected to be homeomorphic to the sphere $S^+ := \{u \in E^+ : \, \|u\| = 1\}$ due to the truncated nonlinear part is not superlinear at infinity for certain directions. The details of the above lemmas can be found in relevant material from [27], and we omit it. A general discussion of the properties of $\mathscr{N}_\varepsilon$ in an abstract setting can be found in [28, Section 4]

Lastly, let us remind the definition of the *Ljusternik-Schnirelman category* and a classical result of the related critical point theory.

Definition 2.8. Let X be a topological space and let $Y \neq \emptyset$ be a closed subset of X. The category of Y in X, $cat_X(Y)$, is the smallest integer n such that

$$Y \subset \bigcup_1^n A_k$$

where for each $k = 1, \dots, n$, A_k is a closed set contractible in X. If such a integer does not exist, then $cat_X(Y) = +\infty$. And set $cat_X(\emptyset) = 0$.

In this context, the category of X in itself, $cat_X(X)$, is simply denoted by $cat(X)$.

Theorem 2.9 (see [32])**.** *Let W be a complete C^1 manifold and let $\Phi \in C^1(W, \mathbb{R})$ be bounded from below on W and satisfying the Palais-Smale compactness condition. Denoted by, for $c \in \mathbb{R}$,*

$$\Phi^c = \big\{u \in W : \, \Phi(u) \le c\big\}$$

Then Φ has at least $cat(\Phi^c)$ distinct critical points in Φ^c.

3 Variational framework for superlinear problems

In this section we establish some preliminary results which are needed for the proof of our main theorems. Given $\omega \in (-a, a)$ and $\kappa \ge 0$, we consider the equation

$$\mathscr{L}u + \omega u = |u|^{p-2}u + \kappa|u|u \qquad u \in E = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4) \tag{3.1}$$

and the associated energy functional

$$\Phi(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\omega}{2}|u|_2^2 - \frac{1}{p}|u|_p^p - \frac{\kappa}{3}|u|_3^3 \quad \text{on } E = E^+ \oplus E^-.$$

And denoted by (ℓ, I) the reduction couple for Φ and set $\mathscr{N} = \{u \in E^+ \setminus \{0\} : \, I'(u)[u] = 0\}$. Then we have $\mathscr{N}$ is a smooth manifold of codimension 1 in E^+, and $\mathscr{N}$ is diffeomorphic to S^+ by a C^1 diffeomorphism. Particularly, the function $t \mapsto I(tu)$ attains its unique critical point $t = t(u) > 0$ for each $u \in E^+ \setminus \{0\}$, and $t : S^+ \to \mathbb{R}$ is a C^1 function. If denoted by

$$\gamma(\omega, \kappa) := \inf_{w \in E^+ \setminus \{0\}} \sup_{u \in \mathbb{R}w \oplus E^-} \Phi(u),$$

it can be also seen that $\gamma(\omega, \kappa) = \inf_{\mathscr{N}} I > 0$.

Proposition 3.1. *Set* $\omega_* = \min\{\omega, 0\}$*, then* $\gamma(\omega, \kappa)$ *is attained provided that*

$$\Big(\frac{a^2}{a^2-\omega_*^2}\Big)^{\frac{3}{2}} \cdot \kappa^2 \cdot \gamma(\omega,\kappa) < \frac{S^{\frac{3}{2}}}{6}, \tag{3.2}$$

where S *denotes the best Sobolev constant for the embedding* $H^1(\mathbb{R}^3, \mathbb{C}^4) \hookrightarrow L^6(\mathbb{R}^3, \mathbb{C}^4)$.

Before proving this proposition, we begin with some preliminary materials. Let us first consider the following functional

$$\mathcal{F}_\omega : E \setminus \{0\} \to \mathbb{R}, \quad z \mapsto \frac{\|z^+\|^2 - \|z^-\|^2 + \omega |z|_2^2}{|z|_3^2},$$

and the minimax scheme

$$T_\omega = \inf_{u \in E^+ \setminus \{0\}} \sup_{v \in E^-} \mathcal{F}_\omega(u+v).$$

We remark that $S|u|_6^2 \le |\nabla u|_2^2$, and if denote by $\mathscr{F} : L^2 \to L^2$ the Fourier transform, there holds

$$\frac{\int_{\mathbb{R}^3} |\xi|^2 |\mathscr{F}u(\xi)|^2 \, d\xi}{|u|_6^2} = \frac{|\mathscr{F}\nabla u|_2^2}{|u|_6^2} = \frac{|\nabla u|_2^2}{|u|_6^2} \ge S, \quad \forall u \in H^1(\mathbb{R}^3, \mathbb{C}^4).$$

Then, by virtue of Calderón-Lions interpolation theorem, we have

$$\frac{\int_{\mathbb{R}^3} |\xi| |\mathscr{F}u(\xi)|^2 \, d\xi}{|u|_3^2} \ge S^{\frac{1}{2}}, \quad \forall u \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4).$$

It follows that, in the Fourier domain, we have $\|u\|^2 = \int_{\mathbb{R}^3} (a^2 + |\xi|^2)^{\frac{1}{2}} |\mathscr{F}u(\xi)|^2 \, d\xi$. And hence, we have for any $u \in E^+ \setminus \{0\}$

$$\sup_{v \in E^-} \mathcal{F}_\omega(u+v) \ge \mathcal{F}_\omega(u) = \frac{\|u\|^2 + \omega |u|_2^2}{|u|_3^2} = \frac{\int_{\mathbb{R}^3} [(a^2+|\xi|^2)^{\frac{1}{2}} + \omega] |\mathscr{F}u(\xi)|^2 \, d\xi}{|u|_3^2}.$$

Taking into account that

$$\inf_{|\xi|>0} \frac{(a^2+|\xi|^2)^{\frac{1}{2}} + \omega}{|\xi|} = \begin{cases} 1 & \text{if } \omega \ge 0, \\ \Big(\dfrac{a^2-\omega^2}{a^2}\Big)^{\frac{1}{2}} & \text{if } \omega < 0, \end{cases}$$

we have

$$T_\omega \ge \inf_{u \in E^+ \setminus \{0\}} \mathcal{F}_\omega(u) \ge \Big(\frac{a^2 - \omega_*^2}{a^2}\Big)^{\frac{1}{2}} S^{\frac{1}{2}} \quad \text{with} \quad \omega_* = \min\{\omega, 0\}. \tag{3.3}$$

Next, let us consider the equation

$$\mathscr{L}u + \omega u = |u|u \quad \text{on } \mathbb{R}^3 \tag{3.4}$$

and the corresponding functional

$$\hat{\Phi}(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\omega}{2}|u|_2^2 - \frac{1}{3}|u|_3^3 \quad \text{on } E = E^+ \oplus E^-.$$

Denoted by $(\hat{\ell}, \hat{I})$ the reduction couple for $\hat{\Phi}$ and set $\hat{\mathscr{N}} = \{u \in E^+ \setminus \{0\} : \hat{I}'(u)[u] = 0\}$. It would be also standard to see that $\hat{\gamma}_\omega := \inf_{\hat{\mathscr{N}}} \hat{I} > 0$, and in particular, we have

Lemma 3.2. $T_\omega = (6\hat{\gamma}_\omega)^{\frac{1}{3}}$.

Proof. We sketch the proof as follows: Let $u \in E^+ \setminus \{0\}$ be fixed, and set $\pi_u(\cdot) = \mathcal{F}_\omega(u + \cdot)$, then elementary calculation shows that for $v \in E^-$ such that $\pi_u'(v)[w] = 0$ for all $w \in E^-$ there holds $\pi_u''(v)[w, w] < 0$. Hence, π_u has a unique critical point in E^- which realize its maximum (if there exists).

For any $u \in \hat{\mathscr{N}}$, we have $\|u\|^2 - \|\hat{\ell}(u)\|^2 + \omega|u + \hat{\ell}(u)|_2^2 - |u + \hat{\ell}(u)|_3^3 = 0$, and hence $\pi_u(\hat{\ell}(u)) = |u + \hat{\ell}(u)|_3$. Moreover, it is standard to check that $\pi_u'(\hat{\ell}(u))[w] = 0$ for all $w \in E^-$. Thus, we have $|u + \hat{\ell}(u)|_3 = \max_{v \in E^-} \pi_u(v)$.

Now, using the fact $\mathcal{F}_\omega(z) = \mathcal{F}_\omega(tz)$ for all $z \in E$ and $t > 0$, we can conclude

$$
\begin{aligned}
T_\omega &= \inf_{u \in S_1^+} \sup_{v \in E^-} \mathcal{F}_\omega(u + v) = \inf_{u \in \hat{\mathscr{N}}} \sup_{v \in E^-} \mathcal{F}_\omega(u + v) \\
&= \inf_{u \in \hat{\mathscr{N}}} |u + \hat{\ell}(u)|_3 = \inf_{u \in \hat{\mathscr{N}}} \big(6\hat{I}(u)\big)^{\frac{1}{3}} \\
&= (6\hat{\gamma}_\omega)^{\frac{1}{3}}
\end{aligned}
$$

as is desired. □

Now, we give the proof of the proposition.

Proof of Proposition 3.1. We only give the proof when $\kappa > 0$ since it is much easier for the case $\kappa = 0$.

Let $\{u_n\} \subset \mathscr{N}$ be a minimizing sequence for I. It is not difficult to check that $\{w_n = u_n + \ell(u_n)\}$ is bounded in E. Then by Lion's result (see [35]) it follows that $\{w_n\}$ is either vanishing or non-vanishing.

If $\{w_n\}$ is non-vanishing then we are done, so let us assume contrarily that $\{w_n\}$ is vanishing. Then $|w_n|_s \to 0$ for all $s \in (2, 3)$. And thus we have

$$
\kappa^2 \Phi(w_n) = \hat{\Phi}(\kappa w_n) + o_n(1) \le \hat{\Phi}(\hat{w}_n) + o_n(1) \le \kappa^2 \Phi(w_n) + o_n(1)
$$

where we used the notation $\hat{w}_n := \hat{t}_n u_n + \hat{\ell}(\hat{t}_n u_n)$ with $\hat{t}_n = \hat{t}(u_n)$ be such that $\hat{t}_n u_n \in \hat{\mathscr{N}}$.

By the above observation, and $\Phi(w_n) = I(u_n) = \gamma(\omega, \kappa) + o_n(1)$, we easily deduce from Lemma 3.2 and (3.3) that

$$
\kappa^2 \cdot \gamma(\omega, \kappa) + o_n(1) = \hat{\Phi}(\hat{w}_n) \ge \hat{\gamma}_\omega = \frac{T_\omega^3}{6} \ge \Big(\frac{a^2 - \omega_*^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6},
$$

which contradicts to (3.2). Therefore we have $\{w_n\}$ is non-vanishing, and this ends the proof. □

Remark 3.3. (1) Given $f \in C[0, \infty) \cap C^1(0, \infty)$ satisfying the hypotheses $(f_1) - (f_4)$, we introduce the following equation which corresponds to a limiting case to Eq. (1.4) as $\varepsilon \to 0$,

$$
\mathscr{L}u + \omega u = f(|u|)u, \tag{3.5}
$$

Its solutions are critical points of the functional

$$
\mathscr{T}_\omega(u) := \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\omega}{2}\int |u|^2 - \Psi(u)
$$

defined for $u = u^+ + u^- \in E = E^+ \oplus E^-$ where $\Psi(u) := \int_{\mathbb{R}^3} F(|u|)\, dx$. Let us denote by γ_ω the corresponding ground state critical level for $\mathscr{T}_\omega$, that is,

$$
\gamma_\omega = \inf\big\{\mathscr{T}_\omega(u) : \ u \neq 0 \text{ and } \mathscr{T}_\omega'(u) = 0\big\}. \tag{3.6}
$$

Then we have that $\gamma_\omega > 0$ is achieved provided the factor κ is small. Indeed, by (f_3), we have $\mathscr{T}_\omega(u) \le \Phi(u)$ for all $u \in E$. Moreover, thanks to the linking structure (see for example [39]), we have

$$\gamma_\omega = \inf_{e \in E^+ \setminus \{0\}} \sup_{u \in \mathbb{R}e \oplus E^-} \mathscr{T}_\omega(u) \le \inf_{e \in E^+ \setminus \{0\}} \sup_{u \in \mathbb{R}e \oplus E^-} \Phi(u) = \gamma(\omega, \kappa) \tag{3.7}$$

Clearly, by using the fact $\gamma(\omega,\kappa)$ decreases dependently with respect to κ, we can infer that $\gamma(\omega,\kappa) \le \gamma(\omega, 0)$ and the condition (3.2) is valid when κ is not large, say

$$\kappa^2 < \Big(\frac{a^2 - \omega_*^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6\gamma(\omega, 0)}. \tag{3.8}$$

Therefore, by using the invariance by translation of the problem and the concentration-compactness argument, we see that the conclusion follows. Moreover, it is evident to check that

$$\mathscr{R}_\omega := \{u \in E : \ \mathscr{T}_\omega(u) = \gamma_\omega, \ \mathscr{T}_\omega'(u) = 0, \ |u(0)| = |u|_\infty\}$$

is a compact set in E (similar results can be found in [26]).

(2) The upper bound for κ in (3.8) is explicitly defined. We may apply the argument in [27, Section 3] to deduce that the map $(-a, a) \to \mathbb{R}^+$, $\omega \mapsto \gamma(\omega, 0)$ is increasing. And as a consequence, the upper bound for κ increases as ω approaches 0 from the right side. For negative ω's, the picture becomes unclear. Our argument do allow critical growth $f(s)s \sim \kappa s^2$ at infinity but the factor κ cannot go too large.

(3) As before, we can introduce the reduction couple $(\mathscr{J}_\omega, J_\omega)$ for $\mathscr{T}_\omega$ as

$$\begin{aligned} &\mathscr{J}_\omega : E^+ \to E^-, \quad \mathscr{T}_\omega\big(u + \mathscr{J}_\omega(u)\big) = \max_{v \in E^-} \mathscr{T}_\omega(u + v)\,, \\ &J_\omega : E^+ \to \mathbb{R}, \quad J_\omega = \mathscr{T}_\omega\big(u + \mathscr{J}_\omega(u)\big)\,; \end{aligned}$$

and set $\mathscr{M}_\omega = \{u \in E^+ \setminus \{0\} : \ J_\omega'(u)[u] = 0\}$. Then we have

$$\gamma_\omega = \inf_{u \in E^+ \setminus \{0\}} \max_{t \ge 0} J_\omega(tu) = \inf_{u \in \mathscr{M}_\omega} J_\omega(u).$$

4 The Palais-Smale condition

Due to the non-compactness of the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4) \hookrightarrow L^3_{loc}(\mathbb{R}^3, \mathbb{C}^4)$, it is not difficult to see that Φ_ε does not satisfy the Palais-Smale condition on $E = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$. However, it will satisfy such compactness condition for certain energy levels. In this section, for notation convenience, let us denote $\Lambda_\varepsilon = \{x \in \mathbb{R}^3 : \ \varepsilon x \in \Lambda\}$, $g_\varepsilon(x, s) = g(\varepsilon x, s)$ and $G_\varepsilon(x, s) = G(\varepsilon x, s)$. Inspired by the priori bound for the factor κ in (3.8), our compactness result can be stated as follows.

Proposition 4.1. *For any $\varepsilon > 0$, if $c_0 \in \mathbb{R}$ satisfies $\kappa^2 \cdot c_0 < \Big(\frac{a^2 - |V|_\infty^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6}$, then there exists a $\delta_0 > 0$ in* (2.5) *such that the truncated functional Φ_ε satisfies the $(P.S.)_c$-condition on E for all $c \le c_0$.*

Proof. Let $\{w_n\} \subset E$ be a $(P.S.)_c$-sequence for Φ_ε, where $c \le c_0$, i.e.,

$$\kappa^2 \cdot \Phi_\varepsilon(w_n) \to \kappa^2 \cdot c \le \kappa^2 \cdot c_0 < \Big(\frac{a^2 - |V|_\infty^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6}, \quad \Phi_\varepsilon'(w_n) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.3, $\{w_n\}$ is bounded and there exists $w \in E$ such that, up to a subsequence, $w_n \rightharpoonup w$ in E. Moreover, $w_n \to w$ strongly in L^q_{loc} for $q \in [2, 3)$.

In the following, we will prove that $w_n \to w$ strongly in E. Let $z_n = w_n - w$, then $z_n \rightharpoonup 0$ in E and $\|w_n^{\pm}\|^2 = \|w^{\pm}\|^2 + \|z_n^{\pm}\|^2 + o_n(1)$. Note that

$$\lim_{s\to 0} \tilde{f}(s) = \lim_{s\to\infty} \frac{\tilde{f}(s)}{s} = 0$$

and

$$\lim_{s\to 0} f(s) = \lim_{s\to\infty} \frac{f(s)}{s} - \kappa = 0,$$

by the Brezis-Lieb type result (see for example [41]), we have

$$\int_{\mathbb{R}^3} G_\varepsilon(x, |w_n|) = \int_{\mathbb{R}^3} G_\varepsilon(x, |w|) + \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} G_\varepsilon(x, |z_n|) + \frac{\kappa}{3}\int_{\Lambda_\varepsilon} |z_n|^3 + o_n(1),$$

and

$$\int_{\mathbb{R}^3} g_\varepsilon(x, |w_n|)|w_n|^2 = \int_{\mathbb{R}^3} g_\varepsilon(x, |w|)|w|^2 + \int_{\mathbb{R}^3\setminus\Lambda_\varepsilon} g_\varepsilon(x, |z_n|)|z_n|^2 + \kappa\int_{\Lambda_\varepsilon} |z_n|^3 + o_n(1).$$

Thus,

$$\Phi_\varepsilon(w_n) = \Phi_\varepsilon(w) + \Phi_\varepsilon(z_n) + o_n(1),$$

and

$$\Phi'_\varepsilon(w_n)[w_n] = \Phi'_\varepsilon(w)[w] + \Phi'_\varepsilon(z_n)[z_n] + o_n(1).$$

Obviously, $\Phi'_\varepsilon(w) = 0$. Therefore, $\Phi'_\varepsilon(z_n)[z_n] = o_n(1)$.

Claim 4.1. *$\Phi'_\varepsilon(z_n) \to 0$ as $n \to \infty$.*

In fact, let $\varphi \in E$ with $\|\varphi\| \le 1$ be arbitrary and set $g^1(x, s) = g(x, s) - \kappa\chi(x)s$.We have

$$\begin{aligned}
\Phi'_\varepsilon(w_n)[\varphi] &= \langle w_n^+, \varphi^+\rangle - \langle w_n^-, \varphi^-\rangle + \Re\int_{\mathbb{R}^3} V_\varepsilon(x) w_n\cdot\bar{\varphi} - \Re\int_{\mathbb{R}^3} g_\varepsilon(x, |w_n|) w_n\cdot\bar{\varphi} \\
&= \langle z_n^+ + w^+, \varphi^+\rangle - \langle z_n^- + w^-, \varphi^-\rangle + \Re\int_{\mathbb{R}^3} V_\varepsilon(x)(z_n + w)\cdot\bar{\varphi} \\
&\quad - \Re\int_{\mathbb{R}^3} g_\varepsilon(x, |z_n + w|)(z_n + w)\cdot\bar{\varphi} \\
&= \langle z_n^+, \varphi^+\rangle - \langle z_n^-, \varphi^-\rangle + \langle w^+, \varphi^+\rangle - \langle w^-, \varphi^-\rangle \\
&\quad + \Re\int_{\mathbb{R}^3} V_\varepsilon(x) z_n\cdot\bar{\varphi} + \Re\int_{\mathbb{R}^3} V_\varepsilon(x) w\cdot\bar{\varphi} \\
&\quad - \Re\int_{\mathbb{R}^3} g^1_\varepsilon(x, |z_n|) z_n\cdot\bar{\varphi} - \Re\int_{\mathbb{R}^3} g^1_\varepsilon(x, |w|) w\cdot\bar{\varphi} \\
&\quad - \kappa\cdot\Re\int_{\Lambda_\varepsilon} |z_n + w|(z_n + w)\cdot\bar{\varphi} + o_n(\|\varphi\|).
\end{aligned} \tag{4.1}$$

Here the estimate for the subcritical part

$$\Re\int_{\mathbb{R}^3} g^1_\varepsilon(x, |w_n|) w_n\cdot\bar{\varphi} - \Re\int_{\mathbb{R}^3} g^1_\varepsilon(x, |z_n|) z_n\cdot\bar{\varphi} - \Re\int_{\mathbb{R}^3} g^1_\varepsilon(x, |w|) w\cdot\bar{\varphi} = o_n(\|\varphi\|)$$

follows from a standard argument in [23, Lemma 7.10]. To estimate the last integral in (4.1), we set $\psi_n := |z_n + w|(z_n + w) - |z_n|z_n - |w|w$. It is not difficult to see that $|\psi_n| \le 2|z_n|\cdot|w|$, and by the Egorov theorem, there exists $\Theta_\sigma \subset \Lambda_\varepsilon$ such that $\text{meas}(\Lambda_\varepsilon \setminus \Theta_\sigma) < \sigma$ and $z_n \to 0$ uniformly on Θ_σ as $n \to \infty$. Thus, by the Hölder inequality, we have

$$\begin{aligned}\int_{\Lambda_\varepsilon} |\psi_n| \cdot |\varphi| &= \int_{\Theta_\sigma} |\psi_n| \cdot |\varphi| + \int_{\Lambda_\varepsilon \setminus \Theta_\sigma} |\psi_n| \cdot |\varphi| \\ &\le \int_{\Theta_\sigma} |\psi_n| \cdot |\varphi| + 2\Big(\int_{\Lambda_\varepsilon \setminus \Theta_\sigma} |z_n|^3 \Big)^{\frac{1}{3}} \cdot \Big(\int_{\Lambda_\varepsilon \setminus \Theta_\sigma} |w|^3 \Big)^{\frac{1}{3}} \cdot \Big(\int_{\Lambda_\varepsilon \setminus \Theta_\sigma} |\varphi|^3 \Big)^{\frac{1}{3}}.\end{aligned}$$

The first integral in the above estimation converges to 0 as $n \to \infty$ and the remaining integrals go to 0 uniformly in n as $\sigma \to 0$. Therefore, we have

$$\int_{\Lambda_\varepsilon} |\psi_n| \cdot |\varphi| = o_n(\|\varphi\|) \quad \text{as } n \to \infty.$$

By obtaining this, we can get

$$\begin{aligned}\Phi'_\varepsilon(w_n)[\varphi] = &\langle z_n^+, \varphi^+ \rangle - \langle z_n^-, \varphi^- \rangle + \langle w^+, \varphi^+ \rangle - \langle w^-, \varphi^- \rangle \\ &+ \Re \int_{\mathbb{R}^3} V_\varepsilon(x) z_n \cdot \bar{\varphi} + \Re \int_{\mathbb{R}^3} V_\varepsilon(x) w \cdot \bar{\varphi} \\ &- \Re \int_{\mathbb{R}^3} g_\varepsilon^1(x, |z_n|) z_n \cdot \bar{\varphi} - \Re \int_{\mathbb{R}^3} g_\varepsilon^1(x, |w|) w \cdot \bar{\varphi} \\ &- \kappa \cdot \Re \int_{\Lambda_\varepsilon} |z_n| z_n \cdot \bar{\varphi} - \kappa \cdot \Re \int_{\Lambda_\varepsilon} |w| w \cdot \bar{\varphi} + o(\|\varphi\|) \\ = &\Phi'_\varepsilon(z_n)[\varphi] + \Phi'_\varepsilon(w)[\varphi] + o_n(\|\varphi\|) \\ = &\Phi'_\varepsilon(z_n)[\varphi] + o_n(\|\varphi\|)\end{aligned}$$

where we have used the fact $\Phi'_\varepsilon(w) = 0$. The above estimation shows $\Phi'_\varepsilon(z_n) \to 0$ as $n \to \infty$ as was claimed.

By the above observations, we have $\Phi'_\varepsilon(z_n)[z_n^+ - z_n^-] = o_n(1)$, that is,

$$\begin{aligned}\|z_n\|^2 + \Re \int_{\mathbb{R}^3} V_\varepsilon(x) z_n \cdot \overline{(z_n^+ - z_n^-)} = &\int_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} g_\varepsilon(x, |z_n|) z_n \cdot \overline{(z_n^+ - z_n^-)} \\ &+ \kappa \cdot \Re \int_{\Lambda_\varepsilon} |z_n| z_n \cdot \overline{(z_n^+ - z_n^-)} + o_n(1).\end{aligned}$$

Recall the fact that $g_\varepsilon(x, s) \le \delta_0$ for $x \in \mathbb{R}^3 \setminus \Lambda_\varepsilon$, together with the estimate in (3.3), we shall have

$$\Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2} \Big)^{\frac{1}{2}} S^{\frac{1}{2}} \Big(\int_{\Lambda_\varepsilon} |z_n|^3 \Big)^{\frac{2}{3}} \le \kappa \cdot \int_{\Lambda_\varepsilon} |z_n|^3 + o_n(1). \tag{4.2}$$

Without loss of generality, we assume that $\lim_{n\to\infty} \int_{\Lambda_\varepsilon} |z_n|^3 = b \ge 0$. If $b > 0$, we get

$$\kappa^3 \cdot b + o_n(1) = \kappa^3 \cdot \int_{\Lambda_\varepsilon} |z_n|^3 \ge \Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2} \Big)^{\frac{3}{2}} S^{\frac{3}{2}}.$$

Plainly, if $\kappa = 0$, we will have a contradiction. Hence, let us suppose that $\kappa > 0$, and due to

$$\Phi_\varepsilon(w) = \int_{\mathbb{R}^3} \frac{1}{2} g_\varepsilon(x, |w|)|w|^2 - G_\varepsilon(x, |w|) \ge 0,$$

we then deduce from the facts $\Phi_\varepsilon(w_n) \ge \Phi_\varepsilon(z_n) + o_n(1)$ and $\Phi'_\varepsilon(z_n)[z_n] = o_n(1)$ that

$$\kappa^2 \cdot c + o_n(1) \ge \frac{\kappa^3}{6} \cdot b + o_n(1) \ge \Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2} \Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6} + o_n(1) \tag{4.3}$$

However, by noticing we have assumed $\kappa^2 \cdot c_0 < \left(\frac{a^2-|V|_\infty^2}{a^2}\right)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6}$, there is a chance to choose δ_0 small such that

$$\Big(\frac{a^2-|V|_\infty^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6} \geq \Big(\frac{a^2-(|V|_\infty+\delta_0)^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6} > \kappa^2 \cdot c_0,$$

then we will also get a contradiction with (4.3). Thus, $b=0$, i.e. $\|z_n\| = o_n(1)$ and $w_n \to w$ strongly in E. The proof is hereby completed. □

Recall the reduction couple $(\ell_\varepsilon,\, I_\varepsilon)$ defined for Φ_ε in Section 2, the following result can be viewed as an immediate corollary of Proposition 4.1.

Corollary 4.2. *Under the assumptions of Proposition* 4.1 *and fix δ_0 properly small, the reduced functional I_ε satisfies $(P.S.)_c$ condition for $c \leq c_0$.*

Proof. Let $\{u_n\} \subset E^+$ be the $(P.S.)_c$ sequence for I_ε, denote $w_n = u_n + \ell_\varepsilon(u_n)$. From the definition of ℓ_ε, we know that $\Phi_\varepsilon'\big(u+\ell_\varepsilon(u)\big)[v] = 0$ for all $u \in E^+$ and $v \in E^-$. Hence, we have $\{w_n\}$ is a $(P.S.)_c$ sequence for Φ_ε. Then the conclusion follows. □

Remark 4.3. Thanks to Lemma 2.5, we can define

$$c_\varepsilon = \inf_{\nu\in\Gamma_\varepsilon} \max_{t\in[0,1]} I_\varepsilon\big(\nu(t)\big),$$

where $\Gamma_\varepsilon = \{\nu(t) \in C\big([0,1], E^+\big) : \ \nu(0)=0,\ I_\varepsilon\big(\nu(1)\big) < 0\}$. Particularly, we have the following characterizations (see an argument in [27, Lemma 3.8]):

$$c_\varepsilon = \inf_{u\in E^+\setminus\{0\}} \max_{t\geq 0} I_\varepsilon(tu) = \inf_{u\in\mathscr{N}_\varepsilon} I_\varepsilon(u) = \inf_{w\in E^+\setminus\{0\}} \sup_{u\in\mathbb{R}w\oplus E^-} \Phi_\varepsilon(u). \tag{4.4}$$

Plainly, if we show that $\limsup_{\varepsilon\to 0} c_\varepsilon < c_0$ for a proper $c_0 > 0$, then according to Proposition 4.1 and Corollary 4.2 we can fix some $\delta_0 > 0$ in the truncation of f in (2.5) and set an upper bound for κ as

$$\kappa < \Big(\frac{a^2-|V|_\infty^2}{a^2}\Big)^{\frac{3}{4}} \Big(\frac{S^{\frac{3}{2}}}{6c_0}\Big)^{\frac{1}{2}}$$

such that Φ_ε (resp. I_ε) satisfies the Palais-Smale condition at the level c_ε as ε goes to zero.

5 Construction of a homotopy

In this section, we introduce two maps $\phi_\varepsilon : M \to \mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma}$ and $\zeta_\varepsilon : \mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma} \to M_\delta$, where

$$\mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma} := \big\{u \in \mathscr{N}_\varepsilon :\ I_\varepsilon(u) \leq \gamma_{\underline{\omega}} + \sigma\big\}$$

for some $\sigma > 0$ (here $\gamma_{\underline{\omega}}$ is defined in (3.6) with subscript $\underline{\omega}$), such that their composition is homotopically equivalent to the embedding $j : M \to M_\delta$ where $\delta > 0$ is small. In our case, the main difficulty is to deal with the infinite dimensional negative space of Φ_ε on $E = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ due to its strongly indefinite character. To overcome this, by using the reduction couple $(\ell_\varepsilon, I_\varepsilon)$, we will build our construction on E^+ instead of E.

5.1 The function ϕ_ε

Recalling the notations introduced in Remark 3.3 and the hypothesis

$$\underline{\omega} = \inf_{\Lambda} V < \inf_{\partial\Lambda} V$$

and $M = \{x \in \Lambda : V(x) = \underline{\omega}\}$, in what follows, let us fix a point $\xi \in M$ and $\varepsilon > 0$ and consider the function

$$w_{\xi,\varepsilon}(x) := w\Big(x - \frac{\xi}{\varepsilon}\Big) \quad \text{for } x \in \mathbb{R}^3$$

where $w \in \mathscr{R}_{\underline{\omega}}$ is a minimal energy solution that realizes $\gamma_{\underline{\omega}}$. Let us define

$$\phi_\varepsilon : M \to \mathscr{N}_\varepsilon \quad \xi \mapsto \phi_\varepsilon(\xi) := t_{\xi,\varepsilon} \cdot w^+_{\xi,\varepsilon}$$

where $t_{\xi,\varepsilon} = t_\varepsilon(w^+_{\xi,\varepsilon})$ is the unique $t > 0$ such that

$$t_{\xi,\varepsilon} \cdot w^+_{\xi,\varepsilon} \in \mathscr{N}_\varepsilon.$$

Proposition 5.1. *For each $\varepsilon > 0$, $\phi_\varepsilon : M \to \mathscr{N}_\varepsilon$ is continuous. Moreover, for any $\sigma > 0$, there exists $\varepsilon_\sigma > 0$ such that for $\varepsilon < \varepsilon_\sigma$ there holds*

$$\phi_\varepsilon(\xi) \in \mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma}, \quad \forall \xi \in M.$$

Proof. By the continuity of $u \mapsto t_\varepsilon(u)$ on $E^+ \setminus \{0\}$, the first conclusion follows evidently because $\phi_\varepsilon(\xi_k) \to \phi_\varepsilon(\xi)$ as $k \to \infty$ in E for any sequence $\{\xi_k\} \subset M$ convergent to ξ.

To prove the second statement, it suffices to show that

$$I_\varepsilon\big(\phi_\varepsilon(\xi)\big) \le \gamma_{\underline{\omega}} + o_\varepsilon(1) \quad \text{for all } \xi \in M \text{ as } \varepsilon \to 0. \tag{5.1}$$

Indeed, by using the covariation of I_ε under translation,

$$\begin{aligned} I_\varepsilon\big(\phi_\varepsilon(\xi)\big) = \frac{1}{2}\big(\|t_{\xi,\varepsilon}w^+\|^2 - \|\ell_\varepsilon\big(t_{\xi,\varepsilon}w^+\big)\|^2\big) + \frac{1}{2}\int V(\varepsilon x + \xi)|t_{\xi,\varepsilon}w^+ + \ell_\varepsilon\big(t_{\xi,\varepsilon}w^+\big)|^2 \\ - \int G\big(\varepsilon x + \xi, |t_{\xi,\varepsilon}w^+ + \ell_\varepsilon\big(t_{\xi,\varepsilon}w^+\big)|\big). \end{aligned}$$

Since $\xi \in M \subset \mathrm{int}\Lambda$, $\chi_{\Lambda_\varepsilon}(\varepsilon x+\xi) \to 1$ a.e. in $\mathbb{R}^3$ as $\varepsilon \to 0$. Thus , by the fact that $V(\varepsilon x+\xi) \to \underline{\omega}$ as $\varepsilon \to 0$ uniformly on bounded set of $\mathbb{R}^3$ and the boundedness of $\{t_{\xi,\varepsilon}\}$, we have

$$\begin{aligned} I_\varepsilon\big(\phi_\varepsilon(\xi)\big) &= \mathscr{T}_{\underline{\omega}}\big(t_{\xi,\varepsilon}w^+ + \ell_\varepsilon\big(t_{\xi,\varepsilon}w^+\big)\big) + o_\varepsilon(1) \le \mathscr{T}_{\underline{\omega}}\big(t_{\xi,\varepsilon}w^+ + \mathscr{J}_{\underline{\omega}}\big(t_{\xi,\varepsilon}w^+\big)\big) + o_\varepsilon(1) \\ &= J_{\underline{\omega}}(t_{\xi,\varepsilon}w^+) + o_\varepsilon(1) \\ &\le \frac{1}{2}\big(\|w^+\|^2 - \|w^-\|^2\big) + \frac{\omega}{2}\int |w|^2 - \int F(|w|) + o_\varepsilon(1) \\ &= \gamma_{\underline{\omega}} + o_\varepsilon(1) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

where in the last inequality we have used the fact $\gamma_{\underline{\omega}} = J_{\underline{\omega}}(w^+) = \max_{t>0} J_{\underline{\omega}}(tw^+)$. This completes the proof of (5.1). □

Remark 5.2. By virtue of the inequality (3.7) and the monotonicity of the map $\omega \mapsto \gamma(\omega, 0)$, it is all clear that $\gamma_{\underline{\omega}} < \gamma(|V|_\infty, 0)$. Thus, it can be derived from Proposition 5.1 that the minimax value c_ε characterized in (4.4) satisfies

$$\limsup_{\varepsilon \to 0} c_\varepsilon < \gamma(|V|_\infty, 0). \tag{5.2}$$

Therefore from now on, according to Remark 4.3, we can precisely fix δ_0 and set κ in a range as

$$0 \le \kappa < \Big(\frac{a^2 - |V|_\infty^2}{a^2}\Big)^{\frac{3}{4}} \Big(\frac{S^{\frac{3}{2}}}{6\gamma(|V|_\infty, 0)}\Big)^{\frac{1}{2}} \tag{5.3}$$

so that I_ε satisfies the Palais-Smale condition at the level c_ε as ε goes to zero.

5.2 The function ζ_ε

Now we introduce the function $\zeta_\varepsilon : E^+ \setminus \{0\} \to \mathbb{R}^3$ to construct the homotopy, which is used to relate multiplicity of solutions to the topology of M. We define

$$\zeta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3} \eta_\varepsilon(x)|u|^2 dx}{|u|_2^2},$$

where $\eta_\varepsilon(x) = \eta(\varepsilon x)$ is the cut-off function:

$$\eta(x) = \begin{cases} x & \text{if } |x| \le \rho \\ \dfrac{\rho x}{|x|} & \text{if } |x| > \rho \end{cases}$$

where $\rho > 0$ is large enough such that $M \subset B_\rho$.

Proposition 5.3. *Let $\sigma(\varepsilon)$ be any positive function tending to 0 as $\varepsilon \to 0$,*

$$\lim_{\varepsilon\to 0} \sup_{u \in \mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma(\varepsilon)}} \text{dist}\big(\zeta_\varepsilon(u), M\big) = 0.$$

Proof. For arbitrary $\varepsilon_n \to 0$, we choose $w_n \in \mathfrak{N}_{\varepsilon_n}^{\gamma_{\underline{\omega}}+\sigma(\varepsilon_n)}$ to be such that

$$\text{dist}\big(\zeta_{\varepsilon_n}(w_n), M\big) \ge \sup_{u \in \mathfrak{N}_{\varepsilon_n}^{\gamma_{\underline{\omega}}+\sigma(\varepsilon_n)}} \text{dist}\big(\zeta_{\varepsilon_n}(u), M\big) - \frac{1}{n}. \tag{5.4}$$

Then our arguments start with the observation that the sequence $\{u_n\}$ is bounded in E, where $u_n := w_n + \ell_{\varepsilon_n}(w_n)$.

In what follows, we will divide our proof into four steps:

Step 1. The sequence $\{u_n\}$ is non-vanishing.

Suppose contrarily that

$$\sup_{x\in\mathbb{R}^3} \int_{B_R(x)} |u_n|^2 dx \to 0 \quad \text{as } \varepsilon \to 0$$

for all $R > 0$. Then, by Lion's concentration principle [35], we have $|u_n|_q \to 0$ for $q \in (2, 3)$. Since $u_n^+ = w_n \in \mathfrak{N}_{\varepsilon_n}^{\gamma_{\underline{\omega}}+\sigma(\varepsilon_n)}$, it follows that $\Phi_{\varepsilon_n}(u_n) \le \gamma_{\underline{\omega}} + \sigma(\varepsilon_n)$ and $\Phi'_{\varepsilon_n}(u_n)[u_n] = \Phi'_{\varepsilon_n}(u_n)[u_n^+ - u_n^-] = 0$. Therefore, similar to (4.2), we deduce

$$\Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\Big)^{\frac{1}{2}} S^{\frac{1}{2}} \Big(\int_{\Lambda_{\varepsilon_n}} |u_n|^3 dx\Big)^{\frac{2}{3}} \le \kappa \cdot \int_{\Lambda_{\varepsilon_n}} |u_n|^3 dx + o_n(1).$$

Without loss of generality, we assume $\kappa > 0$ and $\lim\limits_{n\to\infty} \int_{\Lambda_{\varepsilon_n}} |u_n|^3 = b > 0$. Then we get

$$\begin{aligned} \kappa^2(\gamma_{\underline{\omega}} + \sigma(\varepsilon_n)) &\ge \kappa^2 \Phi_{\varepsilon_n}(u_n) - \frac{\kappa^2}{2}\Phi'_{\varepsilon_n}(u_n)[u_n] = \frac{\kappa^3}{6} \cdot b + o_n(1) \\ &\ge \Big(\frac{a^2 - (|V|_\infty + \delta_0)^2}{a^2}\Big)^{\frac{3}{2}} \frac{S^{\frac{3}{2}}}{6} + o_n(1). \end{aligned}$$

This contradicts to the fact $\gamma_{\underline{\omega}} + \sigma(\varepsilon_n) < \gamma(|V|_\infty, 0)$ for large n and our choice of κ in (5.3).

Step 2. $\{\chi_{\Lambda_{\varepsilon_n}} \cdot u_n\}$ is non-vanishing.

Indeed, if $\{\chi_{\Lambda_{\varepsilon_n}} \cdot u_n\}$ is vanishing, by Step 1 we have that $\{(1 - \chi_{\Lambda_{\varepsilon_n}}) \cdot u_n)\}$ is non-vanishing. This implies there exit $x_n \in \mathbb{R}^3$ and $R, \mu > 0$ such that $B_R(x_n) \subset \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n}$ and

$$\int_{B_R(x_n)} |u_n|^2 dx \ge \mu \quad \text{as } n \to \infty.$$

Set $v_n(x) = u_n(x + x_n)$, then v_n satisfies

$$\mathscr{L} v_n + \hat{V}_\varepsilon(x) v_n = g(\varepsilon_n(x + x_n), |v_n|) v_n \tag{5.5}$$

where $\hat{V}_\varepsilon(x) = V(\varepsilon_n(x + x_n))$. Moreover, we have $v_n \rightharpoonup v$ in E and $v_n \to v$ in L^q_{loc} for $q \in [1, 3)$ for some $v \neq 0$. Now assuming without loss of generality that $V(\varepsilon_n x_n) \to V_\infty$, and using $\psi \in C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ as a test function in (5.5), one gets

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \Re \int \big(\mathscr{L} v_n + \hat{V}_\varepsilon(x) v_n - g(\varepsilon_n(x + x_n), |v_n|) v_n \big) \cdot \bar{\psi} dx \\ &= \Re \int \big(\mathscr{L} v + \hat{V}_\infty v - \tilde{f}(|v|) v \big) \cdot \bar{\psi} dx \end{aligned}$$

And hence $v \in E$ is a non-trivial solution to

$$\mathscr{L} v + \hat{V}_\infty v - \tilde{f}(|v|) v. \tag{5.6}$$

However, using the test function $v^+ - v^-$ in (5.6), we have

$$\begin{aligned} 0 &= \|v\|^2 + V_\infty \int v \cdot \overline{(v^+ - v^-)} - \int \tilde{f}(|v|) v \cdot \overline{(v^+ - v^-)} \\ &\geq \|v\|^2 - \frac{|V|_\infty}{a} \|v\|^2 - \frac{a - |V|_\infty}{2a} \|v\|^2 \\ &= \frac{a - |V|_\infty}{2a} \|v\|^2. \end{aligned}$$

Therefore, we have $v = 0$, which is a contradiction.

Step 3. Let $x_n \in \mathbb{R}^3$ and $R, \mu > 0$ be such that

$$\int_{B_R(x_n)} |\chi_{\Lambda_{\varepsilon_n}} \cdot u_n|^2 dx \geq \mu.$$

Then $\varepsilon_n x_n \to x_0 \in M$ as $n \to \infty$.

In order to see this, first of all, we choose $x_n \in \Lambda_{\varepsilon_n}$, i.e., $\varepsilon_n x_n \in \Lambda$. And suppose that, up to a subsequence, $\varepsilon_n x_n \to x_0 \in \bar{\Lambda}$ as $n \to \infty$. Then, as argued in Step 2, it is possible to show that the sequence $v_n(\cdot) := u_n(\cdot + x_n)$ weakly converges to some v in E which satisfies

$$\mathscr{L} v + V(x_0) v = g_\infty(x, |v|) v \, , \tag{5.7}$$

with g_∞ in the form of $g_\infty(x, s) = \chi_\infty \cdot f(s) + (1 - \chi_\infty) \cdot \tilde{f}(s)$. Here χ_∞ is either a characteristic function of a half-space in $\mathbb{R}^3$ provided

$$\limsup_{n \to \infty} \operatorname{dist}(x_n, \partial \Lambda_{\varepsilon_n}) < +\infty$$

or $\chi_\infty \equiv 1$ (this is due to the fact χ_∞ is the pointwise limit of the function $\chi_\Lambda(\varepsilon_n(\cdot + x_n))$ as $n \to \infty$).

Denote S_∞ to be the associate energy functional to (5.7):

$$S_\infty(u) := \frac{1}{2} \big(\|u^+\|^2 - \|u^-\|^2 \big) + \frac{V(x_0)}{2} |u|_2^2 - \Psi_\infty(u) \, ,$$

where

$$\Psi_\infty(u) := \int G_\infty(x, |u|) \quad \text{and} \quad G_\infty(x, s) = \int_0^s g_\infty(x, \tau) \tau \, d\tau \, .$$

By noting that $\Psi_\infty(u) \leq \Psi(u) = \int_{\mathbb{R}^3} F(|u|) dx$, we have

$$S_\infty(u) \geq \mathscr{T}_{V(x_0)}(u) = \mathscr{T}_{\underline{\omega}}(u) + \frac{V(x_0) - \underline{\omega}}{2} |u|_2^2 \quad \text{for all } u \in E \, .$$

Now let us define (h_∞, I_∞) as the reduction couple of S_∞, then, by the definition, we have

$$S_\infty\big(u + h_\infty(u)\big) = \max_{v\in E^-} S_\infty(u+v) \quad \text{and} \quad I_\infty(u) = S_\infty\big(u + h_\infty(u)\big)\,.$$

It is standard to see that: if $u \in E^+ \setminus \{0\}$ satisfies $I_\infty'(u)[u] = 0$, then $I_\infty''(u)[u,u] < 0$ (see for instance [27]). Since we already have $v \neq 0$ is a critical point of S_∞, we then infer v^+ is a critical point of I_∞ and $I_\infty(v^+) = \max_{t\ge 0} I_\infty(tv^+)$. Let $\tau > 0$ such that $J_{\underline{\omega}}(\tau v^+) = \max_{t\ge0} J_{\underline{\omega}}(tv^+)$, we infer

$$\begin{aligned} S_\infty(v) = I_\infty(v^+) &= \max_{t\ge0} I_\infty(tv^+) \\ &\ge I_\infty(\tau v^+) = S_\infty\big(\tau v^+ + h_\infty(\tau v^+)\big) \\ &\ge S_\infty\big(\tau v^+ + \mathscr{J}_{\underline{\omega}}(\tau v^+)\big) \\ &\ge \mathscr{T}_{\underline{\omega}}\big(\tau v^+ + \mathscr{J}_{\underline{\omega}}(\tau v^+)\big) + \frac{V(x_0)-\underline{\omega}}{2}\big|\tau v^+ + \mathscr{J}_{\underline{\omega}}(\tau v^+)\big|_2^2 \\ &\ge \gamma_{\underline{\omega}} + \frac{V(x_0)-\underline{\omega}}{2}\big|\tau v^+ + \mathscr{J}_{\underline{\omega}}(\tau v^+)\big|_2^2\,. \end{aligned} \tag{5.8}$$

On the other hand, by Fatou's lemma, we deduce

$$\begin{aligned} \gamma_{\underline{\omega}} + \sigma(\varepsilon_n) \ge \Phi_{\varepsilon_n}(u_n) - \frac12 \Phi_{\varepsilon_n}'(u_n)u_n &= \int \frac12 g_{\varepsilon_n}(x,|u_n|)|u_n|^2 - G_{\varepsilon_n}(x,|u_n|) \\ &= \int \frac12 g(\varepsilon_n(x+x_n),|v_n|)|v_n|^2 - G(\varepsilon_n(x+x_n),|v_n|) \\ &\ge \int \frac12 g_\infty(x,|v|)|v|^2 - G_\infty(x,|v|) \\ &= S_\infty(v) - \frac12 S_\infty'(v)[v] = S_\infty(v). \end{aligned}$$

Therefore, together with (5.8), we get $V(x_0) - \underline{\omega} \le O(\sigma(\varepsilon_n))$ as $n\to\infty$. Notice that $x_0 \in \Lambda$ and $\underline{\omega} = \min_{x\in\Lambda} V(x)$, we soon conclude $V(x_0) = \underline{\omega}$ and thus $x_0 \in M$.

Step 4. The final conclusion.

Remark that $u_n^+ = w_n \in \mathfrak{N}_{\varepsilon_n}^{\gamma_{\underline{\omega}}+\sigma(\varepsilon_n)}$, an argument of concentration-compactness type gives that if $v_n \equiv u_n(\cdot + x_n)$ is dichotomy, i.e. $v_n \rightharpoonup v$ and $\{v_n - v\}$ is non-vanishing, then we must have $\Phi_\varepsilon(u_n) \ge 2\gamma_{\underline{\omega}}$. And hence the translated sequence $\{v_n\}$ is in fact compact in the norm topology of E, and thus we have $v_n \to v$ as $n\to\infty$. Consequently, we obtain

$$\begin{aligned} \zeta_{\varepsilon_n}(w_n) = \frac{\int \eta(\varepsilon_n x)|w_n|^2 dx}{\int |w_n|^2 dx} &= \frac{\int \eta(\varepsilon_n x + \varepsilon_n x_n)|v_n^+|^2 dx}{\int |v_n^+|^2 dx} \\ &= \varepsilon_n x_n + \frac{\int \big(\eta(\varepsilon_n x + \varepsilon_n x_n) - \varepsilon_n x_n\big)|v_n^+|^2 dx}{\int |v_n^+|^2 dx}. \end{aligned}$$

And, for n large, we have $\operatorname{dist}\big(\zeta_{\varepsilon_n}(w_n), M\big) = o_n(1)$. Therefore, by (5.4), we obtain the assertion. □

6 Proof of the main result

Proof of Theorem 1.1. It follows from Remarks 4.3 and 5.2, for arbitrary positive function $\sigma(\cdot)$ satisfying $\sigma(\varepsilon)\to 0$ as $\varepsilon\to 0$, we have $\gamma_{\underline{\omega}} + \sigma(\varepsilon) < \gamma(|V|_\infty, 0)$ for small ε and hence Φ_ε (resp. I_ε) satisfies the Palais-Smale condition below the level $\gamma_{\underline{\omega}} + \sigma(\varepsilon)$.

Now let us analysis the relation between the topology of M and the number of critical points for I_ε. It follows from Proposition 5.1 and Proposition 5.3 that there exists $\hat\varepsilon > 0$ such that for $\varepsilon \in (0,\hat\varepsilon)$ the following diagram of continuous maps is well defined,

$$M \xrightarrow{\phi_\varepsilon} \mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma(\varepsilon)} \xrightarrow{\zeta_\varepsilon} M_\delta.$$

Thus we see that $\zeta_\varepsilon \circ \phi_\varepsilon(\cdot)$ is homotopic to the inclusion $j: M \to M_\delta$ for small $\varepsilon > 0$. Hence, by applying Theorem 2.9, we obtain that

$$\sharp \big\{ u \in \mathscr{N}_\varepsilon :\ I_\varepsilon(u) \le \gamma_{\underline{\omega}} + \sigma(\varepsilon),\ I'_\varepsilon(u) = 0 \big\} \ge cat\big(\mathfrak{N}_\varepsilon^{\gamma_{\underline{\omega}}+\sigma(\varepsilon)}\big) \ge cat_{M_\delta}(M).$$

This means that for $\varepsilon \in (0, \hat{\varepsilon})$, I_ε has at least $cat_{M_\delta}(M)$ critical points and so does Φ_ε.

To complete the proof we still have to show that these critical points are also solutions to the original problem (2.1). Let $\{u_\varepsilon\}$ be any sequence of the critical points such that $\Phi_\varepsilon(u_\varepsilon) \le \gamma_{\underline{\omega}} + \sigma(\varepsilon)$, we point out that, thanks to Lemma 2.4, the non-vanishing points $\{x_\varepsilon\}$ can be chosen to be the maximum points of $|u_\varepsilon|$. Due to fact $u_\varepsilon(\cdot + x_\varepsilon) \to u_0 \in \mathscr{R}_{\underline{\omega}}$, a similar decay estimate of [27] implies there exist $C, c > 0$ such that

$$|u_\varepsilon(x)| \le Ce^{-c|x-x_\varepsilon|} \text{ for all } u_\varepsilon \in \big\{ u \in E :\ \Phi'_\varepsilon(u) = 0,\ \Phi_\varepsilon(u) \le \gamma_{\underline{\omega}} + \sigma(\varepsilon) \big\}.$$

Moreover, we have $V(\varepsilon x_\varepsilon) \to V(x_0) = \underline{\omega}$ as $\varepsilon \to 0$. From the assumption

$$\min_{\Lambda} V < \min_{\partial\Lambda} V,$$

we note that $d := \mathrm{dist}(M, \partial\Lambda) > 0$. And thus for $x \notin \Lambda_\varepsilon$ we have $|u_\varepsilon(x)| \le C\exp(-\frac{cd}{\varepsilon})$. And then we can choose $\varepsilon > 0$ small enough such that $g(x, |u_\varepsilon|) = f(|u_\varepsilon|)$, and the proof is hereby completed. □

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