LARGE TIME DYNAMICS OF 2D SEMI-DISSIPATIVE BOUSSINESQ EQUATIONS

BING LI, FANG WANG, AND KUN ZHAO

ABSTRACT. In this paper, we construct an alternative proof for the long-time behavior of large-data classical solutions to the 2D semi-dissipative Boussinesq equations without thermal diffusion on a bounded domain subject to the stress-free boundary conditions, which was previously studied in [16]. To demonstrate the effectiveness of the new approach, we study the long-time behavior of large-data classical solutions to the initial-boundary value problem of a related model with density variance and subject to the no-flow boundary condition, for which the analytic technique utilized in [16] is not directly accessible.

1. INTRODUCTION

This paper is oriented towards the 2D semi-dissipative Boussinesq equations for an incompressible, viscous and non thermal diffusive fluid flow driven by gravity:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} + \theta \mathbf{e}_2, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \end{cases} \tag{1.1}$$

$\mathbf{x} = (x, y) \in \mathbb{R}^2,\ t > 0$. The primary goal is to study the long-time behavior of large-data classical solutions to the initial-boundary value problems of (1.1) and one of its variants involving density variation under different boundary conditions, through developing a novel energy method.

1.1. Background.

The incompressible Boussinesq equations:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} + \theta \mathbf{e}_n, \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta, \end{cases} \tag{1.2}$$

$\mathbf{x} \in \mathbb{R}^n,\ t > 0$, were developed by French mathematician and physicist J.V. Boussinesq based on the thermal conductive Navier-Stokes equations by postulating that density difference is only substantial in terms associated with gravitational force, while the effect of such quantity on inertia is negligible (c.f. [36]). The postulation nowadays is known as the Boussinesq approximation, which makes the mathematics and physics simpler. In the model (1.2), the unknown functions $\mathbf{u}, P, \theta$ denote, respectively, the velocity, pressure and temperature fields; ν and κ are the kinematic viscosity and thermal conductivity, respectively; and $\mathbf{e}_n$ is the vertical unit vector. Since its initiation in the nineteenth century, the model has been frequently utilized in modeling physical phenomena across a broad spectrum of time and length scales, ranging from microfluidics and biophysics to geodynamics and astrophysics. The model is particularly effective in modeling oceanographic and atmospheric situations where rotation and stratification play a dominant role (c.f. [32, 33, 34]), as the Boussinesq approximation is highly accurate for many fluid flows in these areas, such as atmospheric fronts, oceanic circulation, and katabatic winds. In addition

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to natural sciences, the Boussinesq flows often appears in industrial applications and built environment, such as dense gas dispersion, ventilation, and central heating.

Besides its rich physical background, the incompressible Boussinesq equations (1.2) have been one of the most commonly studied models in mathematical fluid dynamics for many decades. Of particular interest to applied analysts is the two-dimensional version of the model. One of the characteristic features that make the two-dimensional model important is that a special case of the model can be identified with the 3D incompressible Euler equations for axisymmetric swirling flows. Indeed, when $\nu = \kappa = 0$, the vorticity formulation of (1.2) together with the temperature equation read:

$$\begin{cases} \partial_t \omega + v\,\partial_y \omega + u\,\partial_x \omega = \partial_x \theta, \\ \partial_t \theta + v\,\partial_y \theta + u\,\partial_x \theta = 0, \end{cases} \tag{1.3}$$

where $\omega = \partial_x v - \partial_y u$ is the 2D vorticity. On the other hand, the vorticity formulation of the 3D incompressible Euler equations for axisymmetric swirling flow in cylindrical coordinates, (r, ϕ, z), reads (c.f. [33]):

$$\begin{cases} \partial_t(\omega^\phi/r) + v^r\,\partial_r(\omega^\phi/r) + v^z\,\partial_z(\omega^\phi/r) = -\dfrac{1}{r^4}\partial_z[(r\,v^\phi)^2], \\ \partial_t[(r\,v^\phi)^2] + v^r\,\partial_r[(r\,v^\phi)^2] + v^z\,\partial_z[(r\,v^\phi)^2] = 0, \end{cases} \tag{1.4}$$

where $\mathbf{v} = (v^r, v^\phi, v^z)$ denotes the velocity field and ω^ϕ is the angular vorticity in cylindrical coordinates. Hence, the one-to-one correspondence:

$$\omega \longleftrightarrow \omega^\phi/r, \quad v \longleftrightarrow v^r, \quad u \longleftrightarrow v^z, \quad y \longleftrightarrow r, \quad x \longleftrightarrow z, \quad \theta \longleftrightarrow -(r\,v^\phi)^2$$

indicate that (1.3) is qualitatively identical to (1.4) away from the axis $r = 0$.

Another important qualitative property of the 2D Boussinesq equations is that the model shares a similar vortex stretching effect as in the three-dimensional flows, which is commonly recognized as one of the essential difficulties in establishing the global well-posedness of large-data classical solutions to the 3D incompressible Euler/Navier-Stokes equations. Again, for the case of ideal fluid flows ($\nu = \kappa = 0$), by.taking the L^2 inner product of the vorticity equation in (1.3) with ω and assuming appropriate boundary conditions under which integration by part is valid, one can show that

$$\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2 = 2\int \omega\,\partial_x\theta\,d\mathbf{x} \le 2\|\omega(\cdot,t)\|_{L^2}\|\partial_x\theta(\cdot,t)\|_{L^2},$$

which implies that

$$\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2} \le \|\partial_x\theta(\cdot,t)\|_{L^2}.$$

Taking the partial derivatives ∂_x and ∂_y of the second equation in (1.3), we have

$$\begin{cases} \partial_t(\partial_x\theta) + v\,\partial_y(\partial_x\theta) + u\,\partial_x(\partial_x\theta) = -(\partial_x v\,\partial_y\theta + \partial_x u\,\partial_x\theta), \\ \partial_t(\partial_y\theta) + v\,\partial_y(\partial_y\theta) + u\,\partial_x(\partial_y\theta) = -(\partial_y v\,\partial_y\theta + \partial_y u\,\partial_x\theta). \end{cases}$$

Upon taking the L^2 inner products, we can show that

$$\frac{d}{dt}\|\nabla\theta(\cdot,t)\|_{L^2} \le \|\nabla\mathbf{u}(\cdot,t)\|_{L^\infty}\|\nabla\theta(\cdot,t)\|_{L^2},$$

which implies that

$$\|\nabla\theta(\cdot,t)\|_{L^2} \le \|\nabla\theta_0(\cdot)\|_{L^2}\exp\left\{\int_0^t \|\nabla\mathbf{u}(\cdot,\tau)\|_{L^\infty}d\tau\right\}.$$

Hence, the growth of the vorticity is dictated by the following inequality:

$$\|\omega(\cdot,T)\|_{L^2} \le \|\omega_0(\cdot)\|_{L^2} + \|\nabla\theta_0(\cdot)\|_{L^2} \int_0^T \exp\left\{\int_0^t \|\nabla\mathbf{u}(\cdot,\tau)\|_{L^\infty} d\tau\right\} dt. \tag{1.5}$$

On the other hand, the vortex formulation of the 3D incompressible Euler equations reads:

$$\partial_t \mathbf{w} + \mathbf{U}\cdot\nabla\mathbf{w} = \mathbf{w}\cdot\nabla\mathbf{U},$$

where $\mathbf{w}$ and $\mathbf{U}$ denote the 3D vorticity and velocity, respectively. Taking the L^2 inner product of the above equation with $\mathbf{w}$, we can show that

$$\frac{d}{dt}\|\mathbf{w}(\cdot,t)\|_{L^2} \le \|\nabla\mathbf{U}(\cdot,t)\|_{L^\infty}\|\mathbf{w}(\cdot,t)\|_{L^2},$$

which implies that

$$\|\mathbf{w}(\cdot,T)\|_{L^2} \le \|\mathbf{w}_0(\cdot)\|_{L^2} \exp\left\{\int_0^T \|\nabla\mathbf{U}(\cdot,t)\|_{L^\infty} dt\right\}. \tag{1.6}$$

In view of (1.5) and (1.6) we see that for both the 2D and 3D models, the growth of vorticity depends on the temporal cumulation of the L^∞-norm of the gradient of the velocity field.

1.2. **Object.** The aforementioned characteristic features of the 2D Boussinesq equations make the model a rich area for mathematical investigation. From the rigorous mathematical perspective, it has been known that the fully dissipative 2D Boussinesq equations ($\nu > 0, \kappa > 0$) are globally well-posed in the regime of large-data solutions (c.f. [6, 30]). On the other hand, the global well-posedness/finite time blowup of the non-dissipative model ($\nu = \kappa = 0$) still remains as a significant challenge in mathematical fluid dynamics. In this paper, we consider an intermediate case in which the kinematic viscosity is strictly positive and the thermal diffusivity is zero, i.e., the 2D semi-dissipative Boussinesq system, (1.1), which arises naturally as a relevant model in geophysics in certain circumstances when the thermal diffusion is insignificant [19, 35, 38]. Because of its physical relevance and mathematical challenge, the rigorous study of (1.1) has been one of the focal points in applied analysis for more than one decade, starting from the independent works of Chae [8] and Hou-Li [20], concerning the global well-posedness of large-data classical solutions to the Cauchy problem of (1.1). Later on, the global well-posedness of different types of solutions to (1.2) with partial dissipation and its variants with anisotropic dissipation under various initial and/or boundary conditions has been investigated in a systematic fashion in the literature, and we refer the readers to [1, 2, 3, 7, 9, 12, 13, 14, 17, 18, 21, 22, 23, 26, 27, 28, 31, 39] for a non-exhaustive list of results in this direction.

Comparing with the magnitude of research conducted on the local/global well-posedness, the long-time behavior of large-data solutions to (1.1) has been investigated relatively little. Recently, Doering *et al* [16] studied the long-time behavior of large-data solutions to (1.1) subject to the initial and stress-free boundary conditions:

$$\begin{cases} (\mathbf{u},\theta)(\mathbf{x},0) = (\mathbf{u}_0,\theta_0)(\mathbf{x}), & \mathbf{x}\in\overline{\Omega}; \\ \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, & t > 0, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^2$ is either a rectangle or a more general bounded and simply connected planar domain with $\partial\Omega$ belonging to $C^{2,\gamma}(\gamma \in (0,1))$ except at a finite number of points where $\partial\Omega$ is a corner of angle in $(0, \frac{\pi}{2}]$, and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. In particular, by combining the global well-posedness [26] and long-time behavior [16] results, one obtains the following:

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^2$ be either a rectangle or a more general Lipschitz domain as specified previously. Consider the initial-boundary value problem* (1.1) & (1.7)*. Assume the initial data $(\mathbf{u}_0, \theta_0) \in H^3$, $\nabla \cdot \mathbf{u}_0 = 0$, and are compatible with the boundary conditions. Then* (1.1) & (1.7) *possesses a unique and global-in-time solution $(\mathbf{u}, \theta)$ satisfying*

$$\mathbf{u} \in L^\infty((0,T); H^3) \cap L^2((0,T); H^4), \quad \theta \in L^\infty((0,T); H^3), \quad \forall T > 0.$$

Furthermore, the solution $(\mathbf{u}, \theta)$ obeys the following long-time behavior, as $t \to \infty$,

$$\|\mathbf{u}(\cdot,t)\|_{H^1} \to 0, \quad \|\partial_t \mathbf{u}(\cdot,t)\| \to 0, \quad \|(\nabla P - \theta \mathbf{e}_2)(\cdot,t)\|_{H^{-1}} \to 0. \tag{1.8}$$

We would also like to mention the closely related work [5], in which the existence of a global weak sigma-attractor containing infinitely many invariant manifolds associated with (1.1) on 2D periodic domains is established. To the authors' knowledge, the results reported in [5, 16] are so far the only ones regarding the long-time behavior of large-data solutions to the semi-dissipative system (1.1).

1.3. **Motivation and Goal.** The proof constructed in [16] employed the free energy functional, see (2.2) below, and the vorticity formulation of (1.1), and utilized the fact that a single variable, nonnegative and uniformly continuous function, $f \in L^1(0,\infty)$, must converge to zero as $t \to \infty$ (Lemma 3.1 of [16]). The primary goal of this paper is to construct an alternative proof of Proposition 1.1 by using a different technical device. Roughly speaking, the new gadget to be employed states that a single variable and nonnegative function, $f \in W^{1,1}(0,\infty)$, must converge to zero as $t \to \infty$. Indeed, such a fact is stronger than Lemma 3.1 of [16], as the former one implies the latter one. Motivation of the usage of such a heavier machinery comes from the discovery that the temporal derivatives of the L^2-norm of the first order (spatial and temporal) derivatives of the velocity field are uniformly integrable with respect to time, while such a structural characteristic of the solution was not observed previously in [16].

Another motivation of the current work originates from the boundary conditions supplemented to the system of equations in (1.1). It was mentioned in [16] that the stress-free boundary conditions in (1.7) are particularly effective for computational purpose, as they enhance the relaxation of the solution to the hydrostatic equilibrium. However, from the point of view of physical applications, the no-flow boundary condition: $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, is more relevant than the stress-free boundary conditions, since the former one is typically observed in laboratory experiments and real-world circumstances. In this case, the vorticity formulation based approach of [16] becomes not directly accessible for the initial-boundary value problem of (1.1) subject to the no-flow boundary condition, which in turn affects establishing the temporal uniform continuity of the L^2-norm of the first order derivatives of the velocity field, which is one of the essential components of the proof constructed in [16] leading to the long-time behavior results recorded in (1.8). Hence, a new method for studying the long-time behavior of the model under the no-flow boundary condition is in demand.

1.4. **Demonstration of the New Approach.** The analytic approach to be developed in the first part of this paper can be of independent interest. Not only can the approach be adopted to study the long-time behavior of large-data classical solutions to (1.1) under various boundary conditions, including the stress-free and no-flow boundary conditions, it is also capable of handling the same problem for related models with similar semi-dissipative structure.

To demonstrate the effectiveness of the new approach, in the next episode of this paper we study the long-time behavior of large-data classical solutions to the following model of hydrodynamics:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P = \mu \Delta \mathbf{u} + \theta \vec{f}, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1.9}$$

$\mathbf{x} = (x, y) \in \mathbb{R}^2,\ t > 0$, subject to the initial and no-flow boundary conditions:

$$\begin{cases} (\rho, \mathbf{u}, \theta)(\mathbf{x}, 0) = (\rho_0, \mathbf{u}_0, \theta_0)(\mathbf{x}), & \mathbf{x} \in \overline{\Omega}, \\ m \le \rho_0(\mathbf{x}) \le M, & \mathbf{x} \in \bar{\Omega}; \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, & t \ge 0, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and $0 < m < M < \infty$ are given constants. In the model (1.9), the additional unknown function ρ denotes the flow density; the constant $\mu > 0$ is the dynamic viscosity; and $\vec{f}$ stands for external forcing. The model describes the motion of an incompressible, viscous, non-homogeneous and non-heat-conductive fluid flow driven by an external force. We shall call the model (1.9) the generalized 2D density-dependent and semi-dissipative Navier-Stokes-Boussinesq equations, since when $\rho \equiv const.$ and $\vec{f}$ is the unit vertical vector $\mathbf{e}_2$, the model reduces to the standard 2D semi-dissipative Boussinesq equations, i.e., (1.1). On the other hand, when $\theta \equiv const.$ or $\theta = \rho$, the model (1.9) becomes the Navier-Stokes equations of an incompressible, viscous and non-homogeneous fluid flow [4, 11, 15, 24, 25].

We remark that one can establish the global well-posedness of large-data classical solutions to (1.9) & (1.10) by following similar arguments of [26] (e.g., regularization, Schauder fixed point theory, energy methods, *et al*). On the other hand, due to the absence of thermal diffusion and the coupling between the solution components, the long-time behavior of the solutions is elusive, which is one of the focal points of this paper. Heuristically, because of the no-flow boundary condition and dissipation induced by viscosity, it is expected that the velocity field will vanish as time goes to infinity. Hence, at the hydrostatic equilibrium, the flow is anticipated to be governed by the equation: $\nabla P_\infty(\mathbf{x}) = \theta_\infty(\mathbf{x})\vec{f}_\infty(\mathbf{x})$, assuming that the external force relaxes to certain equilibrium state as time goes to infinity. Nevertheless, since $\theta_\infty(\mathbf{x})$ can not be identified from the third equation in (1.9), due to the lack of thermal diffusion, the precise description of the hydrostatic equilibrium associated with the initial-boundary value problem (1.9) & (1.10) is puzzling, which in turn complicates the underlying analysis for the slowing down of the velocity field. As a starting point of research, we will show that the velocity field does converge to zero as time goes to infinity, provided that the external force is a gradient field and satisfies certain growth conditions. Since the model (1.9) contains (1.1) as a sub-model, the long-time behavior of large-data classical solutions to (1.1) under the no-flow boundary condition is also obtained as a special case of the results established for (1.9).

Next, we state the main results of the second part of this paper. Before doing so, we introduce some notations for convenience.

Notation 1.1. *Throughout this paper,* $\|\cdot\|_{L^p}, \|\cdot\|_{L^\infty}$ *and* $\|\cdot\|_{W^{s,p}}$ *denote respectively the norms of the Lebesgue measurable spaces* $L^p(\Omega)$*,* $L^\infty(\Omega)$ *and the Sobolev space* $W^{s,p}(\Omega)$*. When* $p = 2$*, we denote the norms* $\|\cdot\|$ *and* $\|\cdot\|_{W^{s,2}}$ *by* $\|\cdot\|$ *and* $\|\cdot\|_{H^s}$*, respectively. Unless otherwise specified,* C *will* *denote a generic constant which is independent of the unknown functions and time, but may depend on the model parameters, the spatial domain and initial data. The value of the constant may vary line by line according to the context.*

Our main results are recorded in the following theorem.

Theorem 1.2. *Consider the initial-boundary value problem* (1.9) & (1.10). *Assume that the initial data* $(\rho_0, \mathbf{u}_0, \theta_0) \in H^3(\Omega)$ *are compatible with the boundary condition. Suppose that* $\vec{f} = \nabla\phi$, *where* $\phi(\mathbf{x}, t)$ *is a smooth function satisfying*

$$\|\phi\|_{L^\infty(0,T;C^2)} + \|\partial_t\phi\|_{L^1(0,T;L^1)} + \|\nabla\partial_t\phi\|_{L^2(0,T;L^2)} \le N, \quad \forall\, T > 0 \tag{1.11}$$

for some constant N *which is independent of* T. *Then there exists a unique and global-in-time solution* $(\rho, \mathbf{u}, \theta)$ *to* (1.9) & (1.10) , *such that* $(\rho, \theta) \in L^\infty([0, \mathcal{T}); H^3(\Omega))$, $m \le \rho \le M$, *and* $\mathbf{u} \in L^\infty([0, \mathcal{T}); H^3(\Omega)) \cap L^2([0, \mathcal{T}); H^4(\Omega))$ *for any* $\mathcal{T} > 0$. *Moreover, the function* $\mathbf{u}$ *obeys the long-time behavior as* $t \to \infty$*:*

$$\|\mathbf{u}(\cdot, t)\|_{H^1} \to 0, \quad \|\partial_t\mathbf{u}(\cdot, t)\| \to 0, \quad \|(\nabla P - \theta\nabla\phi)(\cdot, t)\|_{H^{-1}} \to 0. \tag{1.12}$$

Remark 1.1. *The assumption* (1.11) *can be verified for a family of functions, for example, perhaps the most important one,* $\vec{f} = \nabla y = \mathbf{e}_2 = (0, 1)^{\mathrm{T}}$, *representing the buoyancy due to density variations in the presence of gravitational force.*

Remark 1.2. *We also remark that despite the presence of density variance in the inertial terms in the momentum equation, the initial-boundary value problem* (1.9) & (1.10) *is consistent with* (1.1) & (1.7) *from the point of view of large-time asymptotic behavior. This somehow indicates that the original Boussinesq approximation (neglecting density variance in inertia) is an effective modeling assumption.*

Remark 1.3. *Last, but not least, we would like to mention that this paper is a starting point of research on the qualitative behavior of large-data classical solutions to the generalized density-dependent Boussinesq equations* (1.9). *We hope that our research will help offer future opportunities in this area.*

The rest of this paper is organized as follows. In Section 2 we present the alternative proof of Proposition 1.1. Section 3 contains the proof of Theorem 1.2. The paper finishes with concluding remarks in Section 4.

2. An Alternative Proof of Proposition 1.1

We first present a technical lemma, based on which the alternative proof of Proposition 1.1 is devised.

Lemma 2.1. *Let* $f \in W^{1,1}(0, \infty)$ *be a nonnegative function. Then* $f(t) \to 0$ *as* $t \to \infty$.

Proof. Since $f \in W^{1,1}(0, \infty)$, f can be identified with an absolutely continuous function a.e., so we may assume that f is absolutely continuous. The rest of the proof follows from the proof of Lemma 3.1 in [16]. □

Now we present the alternative proof of Proposition 1.1. First, note that since $\nabla \cdot \mathbf{u} = 0$, it holds that $\Delta\mathbf{u} = \nabla^\perp\omega$, where $\nabla^\perp = (-\partial_y, \partial_x)^{\mathrm{T}}$ and $\omega = \nabla^\perp \cdot \mathbf{u}$. Because of the boundary condition: $\omega|_{\partial\Omega} = 0$, it holds that

$$\int_\Omega \Delta\mathbf{u} \cdot \mathbf{u}\, d\mathbf{x} = \int_{\partial\Omega} \omega(v, -u) \cdot \mathbf{n}\, dS(\mathbf{x}) - \int_\Omega \omega(\nabla^\perp \cdot \mathbf{u})\, d\mathbf{x} = -\|\omega\|^2.$$

By rewriting the equations in (1.1) as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \left(P - \frac{1}{2} y^2 \right) = \nu \Delta \mathbf{u} + (\theta - y) \mathbf{e}_2, \\ \partial_t (\theta - y) + \mathbf{u} \cdot \nabla (\theta - y) = -\mathbf{u} \cdot \mathbf{e}_2, \end{cases}$$

then testing the first equation with $\mathbf{u}$ and the second with $\theta - y$, we can show that

$$\frac{d}{dt} \left(\|\mathbf{u}(\cdot, t)\|^2 + \|\theta(\cdot, t) - y\|^2 \right) + 2\nu \|\omega(\cdot, t)\|^2 = 0. \tag{2.1}$$

Upon integrating with respect to time, we have

$$\|\mathbf{u}(\cdot, t)\|^2 + \|\theta(\cdot, t) - y\|^2 + 2\nu \int_0^t \|\omega(\cdot, \tau)\|^2 d\tau = \|\mathbf{u}_0(\cdot)\|^2 + \|\theta_0(\cdot) - y\|^2, \quad \forall\, t > 0, \tag{2.2}$$

which is similar to the free energy functional derived in [16].

Second, by taking ∂_t to the momentum equation, we have

$$\partial_{tt} \mathbf{u} + \mathbf{u} \cdot \nabla \partial_t \mathbf{u} + \partial_t \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \partial_t P = \nu \Delta \partial_t \mathbf{u} + \partial_t \theta \mathbf{e}_2. \tag{2.3}$$

By taking the L^2 inner product of (2.3) with $\partial_t \mathbf{u}$ and using the temperature equation, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t \mathbf{u}(\cdot, t)\|^2 + \nu \|\partial_t \omega(\cdot, t)\|^2 &= \int_\Omega (\partial_t \theta)(\mathbf{e}_2 \cdot \partial_t \mathbf{u}) d\mathbf{x} - \int_\Omega (\partial_t \mathbf{u} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) d\mathbf{x} \\ &= \int_\Omega \theta \mathbf{u} \cdot \nabla (\mathbf{e}_2 \cdot \partial_t \mathbf{u}) d\mathbf{x} - \int_\Omega (\partial_t \mathbf{u} \cdot \nabla \mathbf{u}) \cdot (\partial_t \mathbf{u}) d\mathbf{x}. \end{aligned} \tag{2.4}$$

For the two terms on the RHS of (2.4), we can show that

$$\begin{aligned} \left| \int_\Omega \theta \mathbf{u} \cdot \nabla (\mathbf{e}_2 \cdot \partial_t \mathbf{u}) d\mathbf{x} \right| &\le \|\theta\|_{L^\infty} \|\mathbf{u}\| \|\nabla \partial_t \mathbf{u}\| \le C \|\theta_0\|_{L^\infty} \|\omega\| \|\partial_t \omega\| \\ &\le C \|\omega\|^2 + \frac{\nu}{4} \|\partial_t \omega\|^2, \end{aligned} \tag{2.5}$$

where the transport equation of θ and Calderón-Zygmund inequality (Lemma 2.1 of [16]) are applied, and

$$\begin{aligned} \left| \int_\Omega (\partial_t \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \partial_t \mathbf{u} d\mathbf{x} \right| &\le \|\nabla \mathbf{u}\| \|\partial_t \mathbf{u}\|_{L^4}^2 \le C \|\omega\| \|\partial_t \mathbf{u}\| \|\nabla \partial_t \mathbf{u}\| \\ &\le C \|\omega\|^2 \|\partial_t \mathbf{u}\|^2 + \frac{\nu}{4} \|\partial_t \omega\|^2, \end{aligned} \tag{2.6}$$

where the Ladyzhenskaya inequality is applied. Collecting the above estimates, we obtain

$$\frac{d}{dt} \|\partial_t \mathbf{u}(\cdot, t)\|^2 + \nu \|\partial_t \omega(\cdot, t)\|^2 \le C \|\omega(\cdot, t)\|^2 \|\partial_t \mathbf{u}(\cdot, t)\|^2 + C \|\omega(t)\|^2, \tag{2.7}$$

where the constant is independent of t. Applying Grönwall's inequality to (2.7) and using (2.2), we have

$$\|\partial_t \mathbf{u}(\cdot, t)\|^2 + \int_0^t \|\partial_\tau \omega(\cdot, \tau)\|^2 d\tau \le C, \ \ \forall\, t > 0, \tag{2.8}$$

where the constant is independent of time.

Third, since

$$\frac{d}{dt} \|\omega(\cdot, t)\|^2 = 2 \int_\Omega \omega\, \partial_t \omega\, d\mathbf{x},$$

it holds that

$$\int_0^t \left| \frac{d}{d\tau} \|\omega(\cdot, \tau)\|^2 \right| d\tau \le \int_0^t \left(\|\omega(\cdot, \tau)\|^2 + \|\partial_\tau \omega(\cdot, \tau)\|^2 \right) d\tau.$$

Returning to (2.4) and applying the estimates in (2.5), (2.6) and (2.8), we can show that

$$\int_0^t \left| \frac{d}{d\tau} \|\partial_\tau \mathbf{u}(\cdot,\tau)\|^2 \right| d\tau \le C \int_0^t \left(\|\omega(\cdot,\tau)\|^2 + \|\partial_\tau \omega(\cdot,\tau)\|^2 \right) d\tau$$

for some constant which is independent of t. According to (2.2) and (2.8), we then have

$$\frac{d}{dt} \left(\|\omega(\cdot,t)\|^2 + \|\partial_t \mathbf{u}(\cdot,t)\|^2 \right) \in L^1(0,\infty).$$

Moreover, from (2.2), (2.8) and Calderón-Zygmund inequality, we know

$$\left(\|\omega(\cdot,t)\|^2 + \|\partial_t \mathbf{u}(\cdot,t)\|^2 \right) \in L^1(0,\infty).$$

Therefore, as a result of Lemma 2.1 and Calderón-Zygmund inequality, we conclude that

$$\lim_{t\to\infty} \left(\|\mathbf{u}(\cdot,t)\|_{H^1}^2 + \|\partial_t \mathbf{u}(\cdot,t)\|^2 \right) = 0.$$

This completes the proof of Proposition 1.1. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. For the reader's convenience, we recall the system of equations:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \\ \rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P = \mu \Delta \mathbf{u} + \theta \nabla \phi, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{3.1}$$

$\mathbf{x} \in \Omega,\ t > 0$, which are supplemented with the initial and boundary conditions:

$$\begin{cases} (\rho_0, \mathbf{u}_0, \theta_0) \in H^3(\Omega), \\ 0 < m \le \rho_0(\mathbf{x}) \le M, \quad \mathbf{x} \in \bar{\Omega}; \\ \mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad t \ge 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and $\phi = \phi(\mathbf{x}, t)$ satisfies the conditions specified in (1.11). In what follows, we focus on the asymptotic analysis of the solution to (3.1) as time goes to infinity. We begin with the uniform-in-time energy estimates of the solution.

3.1. **Uniform Estimates.** First, by using the third equation of (3.1), we can show that

$$\frac{d}{dt}\left(\int_\Omega \theta \phi d\mathbf{x} \right) = \int_\Omega (\partial_t \theta) \phi d\mathbf{x} + \int_\Omega \theta (\partial_t \phi) d\mathbf{x} = -\int_\Omega \nabla \cdot (\theta \mathbf{u}) \phi d\mathbf{x} + \int_\Omega \theta (\partial_t \phi) d\mathbf{x}.$$

Since $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, after integration by parts, we have

$$\frac{d}{dt}\left(\int_\Omega \theta \phi d\mathbf{x} \right) = \int_\Omega \theta \mathbf{u} \cdot \nabla \phi d\mathbf{x} + \int_\Omega \theta (\partial_t \phi) d\mathbf{x}. \tag{3.2}$$

By taking the L^2 inner product of $(3.1)_2$ with $\mathbf{u}$, after integration by parts, we have

$$\int_\Omega \rho \frac{\partial}{\partial t}(|\mathbf{u}|^2) d\mathbf{x} - \int_\Omega \nabla \cdot (\rho \mathbf{u})(|\mathbf{u}|^2) d\mathbf{x} + 2\mu \|\nabla \mathbf{u}\|^2 = 2\int_\Omega \theta \mathbf{u} \cdot \nabla \phi d\mathbf{x}. \tag{3.3}$$

By using (3.2) and the first equation of (3.1), we update (3.3) as

$$\frac{d}{dt}\left(\int_\Omega \rho |\mathbf{u}|^2 d\mathbf{x} - 2\int_\Omega \theta \phi d\mathbf{x} \right) + 2\mu \|\nabla \mathbf{u}\|^2 = -2\int_\Omega \theta (\partial_t \phi) d\mathbf{x}. \tag{3.4}$$

We remark that (3.4) somewhat provides the free energy formulation associated with the initial-boundary value problem (1.9) & (1.10). However, it is obvious that the quantity inside the temporal derivative on the LHS of (3.4) does not preserve a positive sign, which provides limited information about the long-time behavior of the solution. To recover the information hidden in (3.4), we integrate the equation with respect to t to deduce

$$\int_\Omega \rho|\mathbf{u}|^2 d\mathbf{x} - 2\int_\Omega \theta\phi d\mathbf{x} + 2\mu\int_0^t \|\nabla\mathbf{u}\|^2 d\tau = \int_\Omega \rho_0|\mathbf{u}_0|^2 d\mathbf{x} - 2\int_\Omega \theta_0\phi d\mathbf{x} - 2\int_0^t\int_\Omega \theta(\partial_t\phi) d\mathbf{x} d\tau,$$

which implies

$$\begin{aligned}&\int_\Omega \rho|\mathbf{u}|^2 d\mathbf{x} + 2\mu\int_0^t \|\nabla\mathbf{u}\|^2 d\tau\\ \le\ &\int_\Omega \rho_0|\mathbf{u}_0|^2 d\mathbf{x} + 2\big(\|\theta_0\|_{L^\infty} + \|\theta\|_{L^\infty}\big)\|\phi\|_{L^1} + 2\|\theta\|_{L^\infty}\|\partial_t\phi\|_{L^1(0,t;L^1)}.\end{aligned}$$

Note that, since θ is transported by a divergence-free vector field, $\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}$ for $\forall\, 1 \le p \le \infty$ and $\forall\, t > 0$. Therefore, the above estimate, together with the condition on ϕ, implies

$$\int_\Omega \rho|\mathbf{u}|^2 d\mathbf{x} + 2\mu\int_0^t \|\nabla\mathbf{u}\|^2 d\tau \le C,\ \ \forall\, t > 0, \tag{3.5}$$

where the constant on the RHS is independent of $t > 0$. Moreover, since ρ is transported by $\mathbf{u}$, we must have $m \le \rho(\mathbf{x}, t) \le M$ for $\forall\, (\mathbf{x}, t) \in \Omega \times (0, t)$ and $\forall\, t > 0$. Hence, the estimate (3.5) implies that

$$\|\mathbf{u}(\cdot, t)\|^2 + \int_0^t \|\nabla\mathbf{u}(\cdot, \tau)\|^2 d\tau \le C,\ \ \forall\, t > 0, \tag{3.6}$$

where the constant on the RHS is independent of $t > 0$.

Next, by taking the L^2 inner product of $(3.1)_2$ with $\partial_t\mathbf{u}$, we get

$$\frac{\mu}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|^2 + \int_\Omega \rho|\partial_t\mathbf{u}|^2 d\mathbf{x} = \int_\Omega \theta\nabla\phi\cdot\partial_t\mathbf{u} d\mathbf{x} - \int_\Omega \rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u} d\mathbf{x}. \tag{3.7}$$

For the first term on the RHS of (3.7), by using $(3.1)_3$, we have

$$\begin{aligned}\int_\Omega \theta\nabla\phi\cdot\partial_t\mathbf{u} d\mathbf{x} &= \frac{d}{dt}\left(\int_\Omega \theta\nabla\phi\cdot\mathbf{u} d\mathbf{x}\right) - \int_\Omega (\partial_t\theta)\nabla\phi\cdot\mathbf{u} d\mathbf{x} - \int_\Omega \theta\nabla\partial_t\phi\cdot\mathbf{u} d\mathbf{x}\\ &= \frac{d}{dt}\left(\int_\Omega \theta\nabla\phi\cdot\mathbf{u} d\mathbf{x}\right) + \int_\Omega \nabla\cdot(\theta\mathbf{u})(\nabla\phi\cdot\mathbf{u}) d\mathbf{x} - \int_\Omega \theta\nabla\partial_t\phi\cdot\mathbf{u} d\mathbf{x}\\ &= \frac{d}{dt}\left(\int_\Omega \theta\nabla\phi\cdot\mathbf{u} d\mathbf{x}\right) - \int_\Omega \theta\mathbf{u}\cdot\nabla(\nabla\phi\cdot\mathbf{u}) d\mathbf{x} - \int_\Omega \theta\nabla\partial_t\phi\cdot\mathbf{u} d\mathbf{x}.\end{aligned}$$

So we update (3.7) as

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla\mathbf{u}\|^2 - \int_\Omega \theta\nabla\phi\cdot\mathbf{u} d\mathbf{x}\right) + \int_\Omega \rho|\partial_t\mathbf{u}|^2 d\mathbf{x}\\ =\ &-\int_\Omega \theta\mathbf{u}\cdot\nabla(\nabla\phi\cdot\mathbf{u}) d\mathbf{x} - \int_\Omega \rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u} d\mathbf{x} - \int_\Omega \theta\nabla\partial_t\phi\cdot\mathbf{u} d\mathbf{x}.\end{aligned} \tag{3.8}$$

By using (3.6), the uniform estimate: $\|\theta(\cdot,t)\|_{L^\infty} \leq \|\theta_0(\cdot)\|_{L^\infty}$, the conditions on ϕ, and the Poincaré inequality, we deduce

$$\begin{aligned}\left|-\int_\Omega \theta\mathbf{u}\cdot\nabla(\nabla\phi\cdot\mathbf{u})d\mathbf{x}\right| &\leq \|\theta\|_{L^\infty}\|\mathbf{u}\|\,\|\nabla\phi\|_{L^\infty}\|\nabla\mathbf{u}\| + \|\theta\|_{L^\infty}\|D^2\phi\|_{L^\infty}\|\mathbf{u}\|^2 \\ &\leq C\|\nabla\mathbf{u}\|^2.\end{aligned} \tag{3.9}$$

Similarly, by using the Cauchy-Schwarz inequality, we can show that

$$\begin{aligned}\left|-\int_\Omega \rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}d\mathbf{x}\right| &\leq \frac{1}{4}\int_\Omega \rho|\partial_t\mathbf{u}|^2d\mathbf{x} + \int_\Omega \rho|\mathbf{u}\cdot\nabla\mathbf{u}|^2d\mathbf{x} \\ &\leq \frac{1}{4}\int_\Omega \rho|\partial_t\mathbf{u}|^2d\mathbf{x} + M\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2,\end{aligned} \tag{3.10}$$

where the upper bound of ρ is applied. By substituting (3.9) and (3.10) into (3.8), we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla\mathbf{u}\|^2 - \int_\Omega \theta\nabla\phi\cdot\mathbf{u}d\mathbf{x}\right) + \frac{3}{4}\int_\Omega \rho|\partial_t\mathbf{u}|^2d\mathbf{x} \\ &\leq C\|\nabla\mathbf{u}\|^2 + M\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 + \|\theta_0\|_{L^\infty}(\|\nabla\partial_t\phi\|^2 + \|\mathbf{u}\|^2).\end{aligned} \tag{3.11}$$

For the second term on the RHS of (3.11), by applying the Ladyzhenskaya and Poincaré inequalities and (3.6), we can show that

$$\begin{aligned}M\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 &\leq C\|\mathbf{u}\|\,\|\nabla\mathbf{u}\|\big(\|\nabla\mathbf{u}\|\,\|D^2\mathbf{u}\| + \|\nabla\mathbf{u}\|^2\big) \\ &\leq C\|\nabla\mathbf{u}\|^2\|D^2\mathbf{u}\| + C\|\nabla\mathbf{u}\|^4.\end{aligned} \tag{3.12}$$

Since $\mathbf{u}|_{\partial\Omega} = \mathbf{0}$, by applying the classic estimates of the linear Stokes equation, (3.12) and the conditions on ϕ, we can show that

$$\begin{aligned}\|\mathbf{u}\|_{H^2} &\leq C\big(\|\rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \theta\nabla\phi\|\big) \\ &\leq C\big(\sqrt{M}\|\sqrt{\rho}\partial_t\mathbf{u}\| + M\|\mathbf{u}\|_{L^4}\|\nabla\mathbf{u}\|_{L^4} + \|\theta\|_{L^\infty}\|\nabla\phi\|\big) \\ &\leq C\|\sqrt{\rho}\partial_t\mathbf{u}\| + C\|\nabla\mathbf{u}\|\|D^2\mathbf{u}\|^{1/2} + C\|\nabla\mathbf{u}\|^2 + C \\ &\leq C\|\sqrt{\rho}\partial_t\mathbf{u}\| + \frac{1}{2}\|\mathbf{u}\|_{H^2} + C\|\nabla\mathbf{u}\|^2 + C.\end{aligned} \tag{3.13}$$

After rearranging terms, we get from (3.13) that

$$\|\mathbf{u}\|_{H^2} \leq C\|\sqrt{\rho}\partial_t\mathbf{u}\| + C\|\nabla\mathbf{u}\|^2 + C. \tag{3.14}$$

By substituting (3.14) into (3.12), we have

$$\begin{aligned}M\|\mathbf{u}\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^4}^2 &\leq C\|\nabla\mathbf{u}\|^2\big(\|\sqrt{\rho}\partial_t\mathbf{u}\| + \|\nabla\mathbf{u}\|^2 + C\big) + C\|\nabla\mathbf{u}\|^4 \\ &\leq C\|\nabla\mathbf{u}\|^2\|\sqrt{\rho}\partial_t\mathbf{u}\| + C\|\nabla\mathbf{u}\|^4 + C\|\nabla\mathbf{u}\|^2 \\ &\leq \frac{1}{4}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2 + C\|\nabla\mathbf{u}\|^4 + C\|\nabla\mathbf{u}\|^2.\end{aligned} \tag{3.15}$$

By combining (3.11) and (3.15), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\mu}{2}\|\nabla\mathbf{u}\|^2-\int_\Omega\theta\nabla\phi\cdot\mathbf{u}d\mathbf{x}\right)+\frac{1}{2}\int_\Omega\rho|\partial_t\mathbf{u}|^2d\mathbf{x}\\&\le C\|\nabla\mathbf{u}\|^4+C_1\|\nabla\mathbf{u}\|^2+\|\theta_0\|_{L^\infty}(\|\nabla\partial_t\phi\|^2+\|\mathbf{u}\|^2).\end{aligned}\tag{3.16}$$

Now, by multiplying (3.4) by C_1/μ we have

$$\frac{d}{dt}\left(\frac{C_1}{\mu}\int_\Omega\rho|\mathbf{u}|^2d\mathbf{x}-\frac{2C_1}{\mu}\int_\Omega\theta\phi d\mathbf{x}\right)+2C_1\|\nabla\mathbf{u}\|^2=-\frac{2C_1}{\mu}\int_\Omega\theta(\partial_t\phi)d\mathbf{x}.\tag{3.17}$$

By combining (3.16) and (3.17), we obtain

$$\begin{aligned}&\frac{d}{dt}\big[E_1(t)\big]+\frac{1}{2}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+C_1\|\nabla\mathbf{u}\|^2\\&\le C\|\nabla\mathbf{u}\|^4+\|\theta_0\|_{L^\infty}(\|\nabla\partial_t\phi\|^2+\|\mathbf{u}\|^2)+\frac{2C_1}{\mu}\|\theta_0\|_{L^\infty}\|\partial_t\phi\|_{L^1},\end{aligned}\tag{3.18}$$

where

$$E_1(t)=\frac{\mu}{2}\|\nabla\mathbf{u}\|^2+\frac{C_1}{\mu}\int_\Omega\rho|\mathbf{u}|^2d\mathbf{x}-\int_\Omega\theta\nabla\phi\cdot\mathbf{u}d\mathbf{x}-\frac{2C_1}{\mu}\int_\Omega\theta\phi d\mathbf{x}.\tag{3.19}$$

For the last two terms on the RHS of (3.19), we observe that

$$\begin{aligned}\left|-\int_\Omega\theta\nabla\phi\cdot\mathbf{u}d\mathbf{x}-\frac{2C_1}{\mu}\int_\Omega\theta\phi d\mathbf{x}\right|&\le\|\theta\|\;\|\nabla\phi\|_{L^\infty}\;\|\mathbf{u}\|+\frac{2C_1}{\mu}\|\theta\|\;\|\phi\|\\&\le\|\theta_0\|\;\|\nabla\phi\|_{L^\infty}\;m^{-\frac{1}{2}}\;\|\sqrt{\rho}\mathbf{u}\|+C\le C_2,\end{aligned}$$

where the constant C_2 is independent of t, due to (3.5). Therefore, we have

$$E_1(t)+C_2\ge\frac{\mu}{2}\|\nabla\mathbf{u}\|^2+\frac{C_1}{\mu}\int_\Omega\rho|\mathbf{u}|^2d\mathbf{x}.\tag{3.20}$$

By using (3.20), we update (3.18) as

$$\begin{aligned}&\frac{d}{dt}\big[E_1(t)+C_2\big]+\frac{1}{2}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+C_1\|\nabla\mathbf{u}\|^2\\&\le C\|\nabla\mathbf{u}\|^2\big[E_1(t)+C_2\big]+C\|\theta_0\|_{L^\infty}(\|\nabla\partial_t\phi\|^2+\|\nabla\mathbf{u}\|^2)+\frac{2C_1}{\mu}\|\theta_0\|_{L^\infty}\|\partial_t\phi\|_{L^1},\end{aligned}\tag{3.21}$$

where the Poincaré inequality is applied. By applying the Grönwall inequality to (3.21) and using (3.6) and the conditions on ϕ, we deduce

$$\big[E_1(t)+C_2\big]+\int_0^t(\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2)d\tau\le C.$$

In particular, by using (3.20) and the lower bound of ρ, we have

$$\|\nabla\mathbf{u}(\cdot,t)\|^2+\int_0^t\|\partial_t\mathbf{u}(\cdot,\tau)\|^2d\tau\le C,\quad\forall\,t>0.\tag{3.22}$$

Next, we prove the decay estimates recorded in Theorem 1.2.

3.2. **Decay of $\|\mathbf{u}(\cdot,t)\|_{H^1}$ and $\|\partial_t\mathbf{u}(\cdot,t)\|$.** By taking ∂_t to $(3.1)_2$, we have

$$\rho\partial_{tt}\mathbf{u}+\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}+\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}+\partial_t\rho(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})+\nabla\partial_tP=\mu\Delta\partial_t\mathbf{u}+\partial_t\theta\nabla\phi+\theta\nabla\partial_t\phi. \quad (3.23)$$

By taking the L^2 inner product of (3.23) with $\partial_t\mathbf{u}$ and using $(3.1)_1$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+\mu\|\nabla\partial_t\mathbf{u}\|^2=&\int_\Omega\partial_t\theta\nabla\phi\cdot\partial_t\mathbf{u}d\mathbf{x}-\int_\Omega\rho(\partial_t\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}d\mathbf{x}-\int_\Omega\partial_t\rho|\partial_t\mathbf{u}|^2d\mathbf{x}\\&-\int_\Omega\partial_t\rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}d\mathbf{x}+\int_\Omega\theta\nabla\partial_t\phi\cdot\partial_t\mathbf{u}d\mathbf{x}\ \equiv\ \sum_{i=1}^5R_i.\end{aligned}$$

Since $\mathbf{u}|_{\partial\Omega}=\mathbf{0}$ and $\partial_t\mathbf{u}|_{\partial\Omega}=\mathbf{0}$, by using $(3.1)_3$, the uniform estimate: $\|\theta(\cdot,t)\|_{L^\infty}\le\|\theta_0(\cdot)\|_{L^\infty}$, the conditions on ϕ, and the Poincaré and Cauchy inequalities, we estimate R_1 as

$$\begin{aligned}|R_1|=\left|\int_\Omega\nabla\cdot(\theta\mathbf{u})\nabla\phi\cdot\partial_t\mathbf{u}d\mathbf{x}\right|&=\left|\int_\Omega\theta\mathbf{u}\cdot\nabla(\nabla\phi\cdot\partial_t\mathbf{u})d\mathbf{x}\right|\\&\le\|\theta\|_{L^\infty}(\|\mathbf{u}\|\,\|\nabla\phi\|_{L^\infty}\|\nabla\partial_t\mathbf{u}\|+\|\mathbf{u}\|\,\|D^2\phi\|_{L^\infty}\|\partial_t\mathbf{u}\|)\\&\le C\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\|\\&\le C(\varepsilon)\|\nabla\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2,\end{aligned}$$

where $\varepsilon>0$ is a constant to be determined later. In a similar fashion, for R_2, by using (3.22) and the Ladyzhenskaya and Poincaré inequalities, we can show that

$$\begin{aligned}|R_2|=\left|\int_\Omega\rho(\partial_t\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}d\mathbf{x}\right|&\le M\|\nabla\mathbf{u}\|\,\|\partial_t\mathbf{u}\|_{L^4}^2\\&\le C\|\partial_t\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\|\\&\le C(\varepsilon)\|\partial_t\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.\end{aligned}$$

The estimate of R_3 also follows in a similar way by using $(3.1)_1$:

$$\begin{aligned}|R_3|=\left|\int_\Omega\rho_t|\partial_t\mathbf{u}|^2d\mathbf{x}\right|&=\left|\int_\Omega\rho\mathbf{u}\cdot\nabla(|\partial_t\mathbf{u}|^2)d\mathbf{x}\right|\\&\le C\|\nabla\partial_t\mathbf{u}\|\,\|\mathbf{u}\|_{L^4}\|\partial_t\mathbf{u}\|_{L^4}\\&\le C\|\nabla\partial_t\mathbf{u}\|\,\|\mathbf{u}\|^{\frac12}\|\nabla\mathbf{u}\|^{\frac12}\|\partial_t\mathbf{u}\|^{\frac12}\|\nabla\partial_t\mathbf{u}\|^{\frac12}\\&\le C\|\partial_t\mathbf{u}\|^{\frac12}\|\nabla\partial_t\mathbf{u}\|^{\frac32}\\&\le C(\varepsilon)\|\partial_t\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.\end{aligned}$$

Next, we estimate R_4 as follows:

$$\begin{aligned}|R_4|&=\left|\int_\Omega\rho_t(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}d\mathbf{x}\right|\\&=\left|\int_\Omega\rho\mathbf{u}\cdot\nabla\big[(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_t\mathbf{u}\big]d\mathbf{x}\right|\\&\le M\bigg(\int_\Omega|\mathbf{u}||\nabla\mathbf{u}|^2|\partial_t\mathbf{u}|d\mathbf{x}+\int_\Omega|\mathbf{u}|^2|D^2\mathbf{u}||\partial_t\mathbf{u}|d\mathbf{x}+\int_\Omega|\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\partial_t\mathbf{u}|d\mathbf{x}\bigg).\end{aligned} \quad (3.24)$$

By using (3.22) and (3.14), we estimate the first term on the RHS of (3.24) as:

$$
\begin{aligned}
M\int_\Omega |\mathbf{u}||\nabla\mathbf{u}|^2|\partial_t\mathbf{u}|d\mathbf{x} &\le M\|\nabla\mathbf{u}\|_{L^4}^2\|\mathbf{u}\|_{L^4}\|\partial_t\mathbf{u}\|_{L^4} \\
&\le C\big(\|\nabla\mathbf{u}\|\,\|D^2\mathbf{u}\|+\|\nabla\mathbf{u}\|^2\big)\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\| \\
&\le C\big(\|D^2\mathbf{u}\|+1\big)\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\| \\
&\le C\big(\|\sqrt{\rho}\partial_t\mathbf{u}\|+\|\nabla\mathbf{u}\|^2+1\big)\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\| \\
&\le C\big(\|\sqrt{\rho}\partial_t\mathbf{u}\|+1\big)\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\| \\
&\le C\|\sqrt{\rho}\partial_t\mathbf{u}\|\,\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\|+C\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\| \\
&\le C(\varepsilon)\|\sqrt{\rho}\partial_t\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2+C(\varepsilon)\|\nabla\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.
\end{aligned}
\tag{3.25}
$$

For the second term on the RHS of (3.24), by using the Gagliardo-Nirenberg and Poincaré inequalities, we can show that

$$
\begin{aligned}
M\int_\Omega |\mathbf{u}|^2|D^2\mathbf{u}||\partial_t\mathbf{u}|d\mathbf{x} &\le M\|D^2\mathbf{u}\|\,\|\mathbf{u}\|_{L^8}^2\|\partial_t\mathbf{u}\|_{L^4} \\
&\le C\big(\|\sqrt{\rho}\partial_t\mathbf{u}\|+\|\nabla\mathbf{u}\|^2+1\big)\|\nabla\mathbf{u}\|^{\frac{3}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\partial_t\mathbf{u}\| \\
&\le C\big(\|\sqrt{\rho}\partial_t\mathbf{u}\|+\|\nabla\mathbf{u}\|^2+1\big)\|\nabla\mathbf{u}\|^2\|\nabla\partial_t\mathbf{u}\| \\
&\le C\big(\|\sqrt{\rho}\partial_t\mathbf{u}\|+1\big)\|\nabla\mathbf{u}\|\|\nabla\partial_t\mathbf{u}\| \\
&\le C\|\sqrt{\rho}\partial_t\mathbf{u}\|\,\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\|+C\|\nabla\mathbf{u}\|\,\|\nabla\partial_t\mathbf{u}\| \\
&\le C(\varepsilon)\|\sqrt{\rho}\partial_t\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2+C(\varepsilon)\|\nabla\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.
\end{aligned}
\tag{3.26}
$$

For the third term on the RHS of (3.24), we can show that

$$
\begin{aligned}
M\int_\Omega |\mathbf{u}|^2|\nabla\mathbf{u}||\nabla\partial_t\mathbf{u}|d\mathbf{x} &\le C(\varepsilon)\|\nabla\mathbf{u}\|_{L^4}^2\|\mathbf{u}\|_{L^8}^4+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2 \\
&\le C(\varepsilon)\big(\|\nabla\mathbf{u}\|\,\|D^2\mathbf{u}\|+\|\nabla\mathbf{u}\|^2\big)\|\nabla\mathbf{u}\|^4+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2 \\
&\le C(\varepsilon)\big(\|D^2\mathbf{u}\|+1\big)\|\nabla\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2 \\
&\le C(\varepsilon)\big(\|\sqrt{\rho}\partial_t\mathbf{u}\|+1\big)\|\nabla\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2 \\
&\le C(\varepsilon)\|\sqrt{\rho}\partial_t\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2+C(\varepsilon)\|\nabla\mathbf{u}\|^2+\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.
\end{aligned}
\tag{3.27}
$$

By substituting (3.25)–(3.27) into (3.24), we obtain

$$|R_4|\le C(\varepsilon)\|\nabla\mathbf{u}\|^2\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+C(\varepsilon)\|\nabla\mathbf{u}\|^2+3\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.$$

Lastly, R_5 is simply estimated as:

$$|R_5|\le\|\theta_0\|_{L^\infty}(\|\nabla\partial_t\phi\|^2+\|\partial_t\mathbf{u}\|^2).$$

Collecting the estimates of R_i and substituting them into (3.23), we deduce

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+\mu\|\nabla\partial_t\mathbf{u}\|^2 &\le C(\varepsilon)\|\nabla\mathbf{u}\|^2\|\sqrt{\rho}\partial_t\mathbf{u}\|^2 \\
&\quad+C(\varepsilon)\big(\|\nabla\mathbf{u}\|^2+\|\partial_t\mathbf{u}\|^2+\|\nabla\partial_t\phi\|^2\big)+6\varepsilon\|\nabla\partial_t\mathbf{u}\|^2.
\end{aligned}
$$

By choosing $\varepsilon=\mu/12$, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+\frac{\mu}{2}\|\nabla\partial_t\mathbf{u}\|^2\le C\|\nabla\mathbf{u}\|^2\|\sqrt{\rho}\partial_t\mathbf{u}\|^2+C\big(\|\nabla\mathbf{u}\|^2+\|\partial_t\mathbf{u}\|^2+\|\nabla\partial_t\phi\|^2\big). \tag{3.28}$$

Applying the Grönwall inequality to (3.28) and using (3.6) and (3.22), we arrive at

$$\|\partial_t \mathbf{u}(\cdot,t)\|^2 + \int_0^t \|\nabla\partial_t \mathbf{u}(\cdot,\tau)\|^2 d\tau \le C, \quad \forall\, t>0. \tag{3.29}$$

Since

$$\begin{aligned}\int_0^t \Big|\frac{d}{d\tau}\|\nabla\mathbf{u}(\cdot,\tau)\|^2\Big| d\tau &\le 2\int_0^t \|\nabla\mathbf{u}(\cdot,\tau)\|\,\|\nabla\partial_t\mathbf{u}(\cdot,\tau)\| d\tau \\ &\le \int_0^t (\|\nabla\mathbf{u}(\cdot,\tau)\|^2 + \|\nabla\partial_t\mathbf{u}(\cdot,\tau)\|^2) d\tau,\end{aligned}$$

by using (3.5) and (3.29), we have

$$\int_0^t \Big|\frac{d}{d\tau}\|\nabla\mathbf{u}(\cdot,\tau)\|^2\Big| d\tau \le C, \quad \forall\, t>0,$$

which, together with (3.5), implies

$$\|\nabla\mathbf{u}(\cdot,t)\|^2 \in W^{1,1}(0,\infty).$$

Hence,

$$\lim_{t\to\infty} \|\nabla\mathbf{u}(\cdot,t)\|^2 = 0.$$

As a result of the Poincaré inequality, we have

$$\lim_{t\to\infty} \|\mathbf{u}(\cdot,t)\|_{H^1}^2 = 0.$$

Moreover, by using the Poincaré inequality and (3.29), we have

$$\int_0^t \|\partial_t\mathbf{u}(\cdot,\tau)\|^2 d\tau \le C, \quad \forall\, t>0,$$

which implies

$$\int_0^t \|\sqrt{\rho}\partial_t\mathbf{u}(\cdot,\tau)\|^2 d\tau \le C, \quad \forall\, t>0.$$

By repeating the arguments in deriving (3.28) and using the previous estimates and the conditions on ϕ, we can show that

$$\int_0^t \Big|\frac{d}{d\tau}\|\sqrt{\rho}\partial_t\mathbf{u}(\cdot,\tau)\|^2\Big| d\tau \le C.$$

Therefore,

$$\|\sqrt{\rho}\partial_t\mathbf{u}(\cdot,t)\|^2 \in W^{1,1}(0,\infty),$$

which implies

$$\lim_{t\to\infty} \|\sqrt{\rho}\partial_t\mathbf{u}(\cdot,t)\|^2 = 0.$$

Since $\rho \ge m$, it thus holds that

$$\lim_{t\to\infty} \|\partial_t\mathbf{u}(\cdot,t)\|^2 = 0.$$

Note that since $\|\partial_t\mathbf{u}\|$ and $\|\nabla\mathbf{u}\|$ are uniformly bounded according to (3.29) and (3.22) and $m \le \rho \le M$, one reads from (3.14) that

$$\|\mathbf{u}(\cdot,t)\|_{H^2} \le C\left(\|\partial_t\mathbf{u}(\cdot,t)\| + \|\nabla\mathbf{u}(\cdot,t)\|^2 + 1\right) \le C, \quad \forall\, t>0 \tag{3.30}$$

which implies, by Sobolev embedding,

$$\|\mathbf{u}(\cdot,t)\|_{L^\infty} \le C, \quad \forall\, t>0. \tag{3.31}$$

As a consequence of (3.31), we can show that

$$\begin{aligned}\|\nabla P-\theta\nabla\phi\|_{H^{-1}} &\le C\left(\|\rho\partial_t\mathbf{u}\|_{H^{-1}}+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^{-1}}+\|\Delta\mathbf{u}\|_{H^{-1}}\right)\\ &\le C\left(\|\rho\partial_t\mathbf{u}\|_{L^2}+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\right)\\ &\le C\left(M\|\partial_t\mathbf{u}\|_{L^2}+M\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\right)\\ &\le C\left(\|\partial_t\mathbf{u}\|_{L^2}+\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\right)\\ &\le C\left(\|\partial_t\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\right).\end{aligned}$$

Hence, the temporal decay of the velocity field implies that

$$\lim_{t\to\infty}\|(\nabla P-\theta\nabla\phi)(\cdot,t)\|_{H^{-1}}=0.$$

Next, we build up the higher order spatial regularity of the solution, in order to complete the proof of Theorem 1.2, which is achieved by applying various interpolation inequalities.

3.3. **Higher Order Regularity.** First of all, as an immediate consequence of (3.30) and (3.31), we see that, for any $2<p<\infty$,

$$\begin{aligned}\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p}^2 &\le M^2\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p}^2\\ &\le C\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{H^1}^2\\ &\le C\big(\|\mathbf{u}\|_{L^\infty}^2+\|\mathbf{u}\|_{H^2}^2\big)\|\mathbf{u}\|_{H^2}^2\ \le\ C.\end{aligned}\tag{3.32}$$

It follows from the Stokes estimates, (3.29) and (3.32) that

$$\begin{aligned}\int_0^t\|\mathbf{u}\|_{W^{2,p}}^2\,d\tau &\le C\int_0^t\left(\|\rho\partial_t\mathbf{u}\|_{L^p}^2+\|\rho\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^p}^2+\|\theta\nabla\phi\|_{L^p}^2\right)d\tau\\ &\le C\int_0^t\left(\|\partial_t\mathbf{u}\|_{H^1}^2+1+\|\theta_0\|_{L^p}^2\|\nabla\phi\|_{L^\infty}^2\right)d\tau\\ &\le C\int_0^t\left(\|\nabla\partial_t\mathbf{u}\|^2+1\right)d\tau\\ &\le C(1+t),\qquad \forall\,0<t<\infty,\ \ \forall\,2<p<\infty.\end{aligned}$$

It follows from the above estimate and Sobolev embedding that

$$\int_0^t\|\nabla\mathbf{u}(\tau)\|_{L^\infty}^2d\tau\le C(1+t),\quad\forall\,0<t<\infty,\tag{3.33}$$

where the constant C is independent of t.

Second, by using (3.33) and the property of the transport equation, we can easily derive

$$\begin{aligned}\|\nabla\rho(t)\|_{L^p}&\le\|\nabla\rho_0\|_{L^p}\exp\left\{\int_0^t\|\nabla\mathbf{u}\|_{L^\infty}d\tau\right\}\le C\,e^{Ct},\\ \|\nabla\theta(t)\|_{L^p}&\le\|\nabla\theta_0\|_{L^p}\exp\left\{\int_0^t\|\nabla\mathbf{u}\|_{L^\infty}d\tau\right\}\le C\,e^{Ct},\end{aligned}\tag{3.34}$$

for $\forall\,2\le p\le\infty$ and $\forall\,0<t<\infty$, where the constants on the RHS are independent of t.

Third, by taking the L^2 inner product of (3.23) with $\partial_{tt}\mathbf{u}$, we can show that

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\nabla\partial_t\mathbf{u}\|^2+\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2=&-\int_\Omega\rho(\mathbf{u}\cdot\nabla\partial_t\mathbf{u})\cdot\partial_{tt}\mathbf{u}d\mathbf{x}-\int_\Omega\rho(\partial_t\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\\&-\int_\Omega\partial_t\rho\partial_t\mathbf{u}\cdot\partial_{tt}\mathbf{u}d\mathbf{x}-\int_\Omega\partial_t\rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\\&+\int_\Omega\partial_t\theta\nabla\phi\cdot\partial_{tt}\mathbf{u}d\mathbf{x}+\int_\Omega\theta\nabla\partial_t\phi\cdot\partial_{tt}\mathbf{u}d\mathbf{x}.\end{aligned}\tag{3.35}$$

The first two terms on the RHS of (3.35) are estimated as

$$\begin{aligned}\left|-\int_\Omega\rho(\mathbf{u}\cdot\nabla\partial_t\mathbf{u})\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\right|&\le\sqrt{M}\|\mathbf{u}\|_{L^\infty}\|\nabla\partial_t\mathbf{u}\|\,\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\|\nabla\partial_t\mathbf{u}\|^2+\frac{1}{12}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|,\end{aligned}\tag{3.36}$$

where (3.31) is applied, and

$$\begin{aligned}\left|-\int_\Omega\rho(\partial_t\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\right|&\le\sqrt{M}\|\nabla\mathbf{u}\|_{L^\infty}\|\partial_t\mathbf{u}\|\,\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\|\nabla\mathbf{u}\|_{L^\infty}^2+\frac{1}{12}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|,\end{aligned}\tag{3.37}$$

where (3.29) is applied. For the middle two terms on the RHS of (3.35), by using the equations for ρ and θ, we can show that

$$\begin{aligned}\left|-\int_\Omega\partial_t\rho\partial_t\mathbf{u}\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\right|&\le\|\partial_t\rho\|_{L^4}\|\partial_t\mathbf{u}\|_{L^4}\|\partial_{tt}\mathbf{u}\|\\&\le C\|\mathbf{u}\cdot\nabla\rho\|_{L^4}\|\nabla\partial_t\mathbf{u}\|\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\|\mathbf{u}\|_{L^\infty}\|\nabla\rho\|_{L^4}\|\nabla\partial_t\mathbf{u}\|\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\,e^{Ct}\|\nabla\partial_t\mathbf{u}\|\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\,e^{Ct}\|\nabla\partial_t\mathbf{u}\|^2+\frac{1}{12}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2,\end{aligned}\tag{3.38}$$

where the Sobolev embedding $H^1\hookrightarrow L^4$, Poincaré inequality, (3.31) and (3.34) are applied, and

$$\begin{aligned}\left|-\int_\Omega\partial_t\rho(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\right|&\le\|\partial_t\rho\|\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^\infty}\|\partial_{tt}\mathbf{u}\|\\&\le C\,e^{Ct}\|\nabla\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\,e^{Ct}\|\nabla\mathbf{u}\|_{L^\infty}^2+\frac{1}{12}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2.\end{aligned}\tag{3.39}$$

For the last two terms on the RHS of (3.35), by using the temperature equation and the conditions on ϕ, we can show that

$$\begin{aligned}\left|\int_\Omega\partial_t\theta\nabla\phi\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\right|&\le\|\mathbf{u}\|_{L^\infty}\|\nabla\theta\|\|\nabla\phi\|_{L^\infty}\|\partial_{tt}\mathbf{u}\|\\&\le C\,e^{Ct}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|\\&\le C\,e^{Ct}+\frac{1}{12}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2,\end{aligned}\tag{3.40}$$

and

$$\begin{aligned}\left|\int_\Omega \theta\nabla\partial_t\phi\cdot\partial_{tt}\mathbf{u}d\mathbf{x}\right| &\le \|\theta\|_{L^\infty}\|\nabla\partial_t\phi\|\|\partial_{tt}\mathbf{u}\| \\ &\le C\|\nabla\partial_t\phi\|\|\sqrt{\rho}\partial_{tt}\mathbf{u}\| \\ &\le C\|\nabla\partial_t\phi\|^2+\frac{1}{12}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2.\end{aligned} \tag{3.41}$$

Substituting (3.36)–(3.41) into (3.35), we obtain

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\nabla\partial_t\mathbf{u}\|^2+\frac{1}{2}\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2 \le C\left(e^{Ct}+1\right)\left(\|\nabla\partial_t\mathbf{u}\|^2+\|\nabla\mathbf{u}\|_{L^\infty}^2\right)\\ +C\,e^{Ct}+C\|\nabla\partial_t\phi\|^2.\end{aligned} \tag{3.42}$$

Integrating (3.42) with respect to time, and using (3.29), (3.33) and the condition on ϕ, we can show that

$$\|\nabla\partial_t\mathbf{u}(t)\|^2+\int_0^t\|\sqrt{\rho}\partial_{tt}\mathbf{u}\|^2d\tau\le C\left(e^{Ct}+1\right)(t+1),\quad\forall\, t>0, \tag{3.43}$$

for some constant which is independent of t.

Next, according to the classical estimates for the Stokes equation, we have

$$\|\mathbf{u}\|_{H^3}^2\le C\left(\|\rho\partial_t\mathbf{u}\|_{H^1}^2+\|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^1}^2+\|\theta\nabla\phi\|_{H^1}^2\right). \tag{3.44}$$

For the first term on the RHS of (3.44), we can show that

$$\begin{aligned}\|\rho\partial_t\mathbf{u}\|_{H^1}^2 &\le C\left(\|\partial_t\mathbf{u}\|^2+\|\nabla\partial_t\mathbf{u}\|^2+\|\nabla\rho\|_{L^4}^2\|\partial_t\mathbf{u}\|_{L^4}^2\right)\\ &\le C\left(1+e^{Ct}\right)\|\nabla\partial_t\mathbf{u}\|^2\\ &\le C\left(1+e^{Ct}\right)(t+1),\end{aligned} \tag{3.45}$$

where the Ladyzhenskaya and Poincaré inequalities and (3.34) and (3.43) are applied. The second term on the RHS of (3.44) is estimated as

$$\begin{aligned}\|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^1}^2 &\le C\left(\|\mathbf{u}\cdot\nabla\mathbf{u}\|^2+\|\nabla(\mathbf{u}\cdot\nabla\mathbf{u})\|^2+\|\nabla\rho\|_{L^4}^2\|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^4}^2\right)\\ &\le C\left(1+e^{Ct}\right),\end{aligned} \tag{3.46}$$

where (3.32) is applied. For the third term on the RHS of (3.44), by using the conditions on ϕ, we can show that

$$\begin{aligned}\|\theta\nabla\phi\|_{H^1}^2 &\le C\left(\|\theta\|_{L^\infty}^2\|\nabla\phi\|^2+\|\theta\|_{L^\infty}^2\|D^2\phi\|^2+\|\nabla\theta\|_{L^4}^2\|\nabla\phi\|_{L^4}^2\right)\\ &\le C\left(1+e^{Ct}\right).\end{aligned} \tag{3.47}$$

Substituting (3.45)–(3.47) into (3.44), we obtain

$$\|\mathbf{u}(t)\|_{H^3}^2\le C\left(1+e^{Ct}\right)(t+1), \tag{3.48}$$

where the constant C is independent of t.

Using the property of the transport equation, we can show that

$$\begin{aligned}\frac{d}{dt}\|\nabla\theta\|_{H^2}^2\le C\big(&\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\theta\|_{H^2}^2+\|D^2\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4}\|D^2\theta\|\\ &+\|D^2\mathbf{u}\|_{L^4}\|D^2\theta\|_{L^4}\|D^3\theta\|+\|D^3\mathbf{u}\|\|\nabla\theta\|_{L^\infty}\|D^3\theta\|\big),\end{aligned}$$

which can be simplified, by using the Ladyzhenskaya and Sobolev inequalities, as

$$\begin{aligned}\frac{d}{dt}\|\nabla\theta\|_{H^2}^2 &\le C\|\mathbf{u}\|_{H^3}\|\nabla\theta\|_{H^2}^2\\ &\le C\left(1+e^{Ct}\right)\sqrt{t+1}\,\|\nabla\theta\|_{H^2}^2.\end{aligned}$$

Solving the differential inequality, we obtain

$$\|\nabla\theta(t)\|_{H^2}^2 \le \|\nabla\theta_0\|_{H^2}^2 \exp\left\{C\left(1+e^{Ct}\right)t\sqrt{t+1}\right\}. \tag{3.49}$$

Since ρ satisfies the same transport equation, we have

$$\|\nabla\rho(t)\|_{H^2}^2 \le \|\nabla\rho_0\|_{H^2}^2 \exp\left\{C\left(1+e^{Ct}\right)t\sqrt{t+1}\right\}. \tag{3.50}$$

Furthermore, according to (3.23) and the classical estimates of the Stokes equation, we know that

$$\begin{aligned}\|\partial_t\mathbf{u}\|_{H^2}^2 \le C\big(&\|\rho\partial_{tt}\mathbf{u}\|^2 + \|\rho\mathbf{u}\cdot\nabla\partial_t\mathbf{u}\|^2 + \|\rho\partial_t\mathbf{u}\cdot\nabla\mathbf{u}\|^2 + \|\partial_t\rho\partial_t\mathbf{u}\|^2 \\ &+ \|\partial_t\rho(\mathbf{u}\cdot\nabla\mathbf{u})\|^2 + \|\partial_t\theta\nabla\phi\|^2 + \|\theta\nabla\partial_t\phi\|^2\big).\end{aligned}$$

Using the previously established energy estimates, we can show that

$$\int_0^t \|\partial_t\mathbf{u}\|_{H^2}^2 d\tau \le D_1(t), \quad \forall\, t>0, \tag{3.51}$$

where $D_1(t)>0$ is some increasing function of t and remains finite as long as t is finite. Since the proof of (3.51) follows from routine calculations, we omit the details to simplify the presentation. Finally, since

$$\|\mathbf{u}\|_{H^4}^2 \le C\left(\|\rho\partial_t\mathbf{u}\|_{H^2}^2 + \|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{H^2}^2 + \|\theta\nabla\phi\|_{H^2}^2\right),$$

by using (3.51), we can show that

$$\int_0^t \|\mathbf{u}\|_{H^4}^2 d\tau \le D_2(t), \quad \forall\, t>0,$$

where $D_2(t)>0$ is some increasing function of t and remains finite as long as t is finite. Again, the technical details leading to the above estimate are omitted to simplify the presentation, since they follow from routine calculations. This completes the proof of Theorem 1.2. □

4. Conclusion and Future Topics

We constructed an alternative proof for the long-time behavior of large-data classical solutions to the initial-boundary value problem of the 2D semi-dissipative Boussinesq equations, (1.1), on a bounded domain subject to the stress-free boundary conditions, which was previously studied in [16]. The new proof is shorter than the one devised in [16], and reveals more delicate solution structures associated with the problem, which was not observed in [16]. To demonstrate the effectiveness of the new approach, we studied the long-time behavior of large-data classical solutions to the initial-boundary value problem of the 2D generalized semi-dissipative Navier-Stokes-Boussinesq equations with density variance, (1.9). The new approach is proven to be versatile, as it is capable of handling not only different mathematical models, but also various types of boundary conditions.

Moreover, we expect that the new approach can be integrated with other energy methods to study the qualitative behavior of classical solutions to more complicated mathematical models in related areas. Here we list three of them for future investigations.

- In the area of mathematical physics, the following coupled system:

$$\begin{cases}\partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = \Delta(-\Delta\varphi + \varphi^3 - \varphi),\\ \rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \mu\Delta\mathbf{u} + \rho\vec{F} + \nabla\cdot(\nabla\varphi\otimes\nabla\varphi),\\ \partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0,\\ \nabla\cdot\mathbf{u} = 0,\end{cases}$$

can be derived from the quasi-incompressible Navier-Stokes-Cahn-Hilliard equations [31] under suitable modeling assumptions, which describes the motion of inhomogeneous hydrodynamic two-phase flows. Clearly, the model exhibits a semi-dissipative structure, due to the absence of density diffusion.

- Another related model with semi-dissipative structure:

$$\begin{cases} \partial_t c + \mathbf{u}\cdot\nabla c = \Delta c - \nabla\cdot(c\chi(n)\nabla n), \\ \partial_t n + \mathbf{u}\cdot\nabla n = \Delta n - cf(n), \\ \rho\partial_t \mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = \mu\Delta\mathbf{u} - c\nabla\psi + \rho\vec{F}, \\ \partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

can be viewed as the pre-simplified version of the model arising in the study of the mutual influences of chemotaxis and hydrodynamics [37] by taking into account of density variance.

- The third model that we would like to mention:

$$\begin{cases} \partial_t n = \Delta n + f(n,\rho), \\ \rho\partial_t \mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi = -\alpha\rho\mathbf{u} + \rho\nabla n, \\ \partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0, \\ \nabla\cdot\mathbf{u} = 0, \end{cases}$$

can be obtained from its compressible version (c.f. [10]), derived from a mean-field Fokker-Planck type equation describing the sticking and clamping effects in chemotaxis, by assuming the incompressibility condition.

We expect that the semi-dissipative structures of the above systems can be circumvented by applying the approach developed in this paper when studying the long-time behavior of classical solutions. The idea, to be mentioned again, is to derive energy estimates, such that certain time-dependent quantities (spatial Sobolev norms) associated with the solutions belong to the space $W^{1,1}(0,\infty)$.

On the other hand, this paper is a starting point of the research of qualitative behavior of large-data classical solutions to (1.9), while many important questions are still widely open. For example,

- Since the lower bound of the density function and the Poincaré inequality are frequently utilized in the proof, it is not clear whether Theorem 1.2 still holds true if the initial density is not strictly bounded away from zero, or the problem is set on the whole space, i.e., for the Cauchy problem of (1.9).
- Although the velocity field is shown to vanish as time evolves, we still do not know how fast the function converges to zero, i.e., the explicit decay rates are not identified.
- Because there are infinitely many conserved quantities associated with ρ and θ, the classification of the final states of the density and temperature fields is elusive.

We leave the investigations of these open problems in future works.

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(B. Li) Center for Applied Mathematics, Tianjin University, Tianjin 300072, P. R. China
Email address: binglimath@gmail.com

(F. Wang) School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, Hunan Province, P. R. China
Email address: wangfang1209@csust.edu.cn

(K. Zhao) Department of Mathematics, Tulane University, New Orleans, LA 70118, USA
Email address: kzhao@tulane.edu