Distribution Dependent SDEs with Hölder Continuous Drift and α-Stable Noise*

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February 14, 2020

Abstract

In this paper, the existence and uniqueness of the distribution dependent SDEs with Hölder continuous drift driven by α-stable process is investigated. Moreover, by using Zvonkin type transformation, the convergence rate of Euler-Maruyama method and propagation of chaos are also obtained. The results cover the ones in the case of distribution independent SDEs.

AMS subject Classification: 60H35, 41A25, 60H10, 60C30.
Keywords: Distribution dependent SDEs, Hölder continuous, Zvonkin type transformation, Euler-Maruyama method, α-stable process.

1 Introduction

Distribution dependent stochastic differential equations (SDEs for abbreviation), also called McKean-Vlasov SDE, can be used to characterize nonlinear Fokker-Planck equations (see [23, 6]). Recently, there are many results on Distribution dependent SDEs (see [25, 33] and references within). Existence and uniqueness of McKean-Vlasov SDEs with regular coefficients have been investigated extensively (see e.g. [7, 25, 33, 34]). Meanwhile, the strong wellposedness of McKean-Vlasov SDEs with irregular coefficients has also received much attention (see, for example, [4, 8, 15, 30], where, in [8], the dependence of laws is of integral type and the diffusion is non-degenerate, and [15] is concerned with the integrability condition but excluding linear growth of the drift). For weak wellposedness of McKean-Vlasov SDEs, we refer to e.g. [15, 21, 24, 25]. [29, 35] studied the Lion's derivative and ergodicity for SDEs driven by Brownian motion. [32] investigated the derivative formula and gradient estimate for McKean-Vlasov SDEs driven by jump process (See [1, 5, 9, 10, 11] for more results on McKean-Vlasov SDEs).

*Supported in part by NNSFC (11801406).

Recently, the convergence rate of Euler-Maruyama (EM for short) method for SDEs with irregular coefficients has attracted much attention. For instance, [13, 36] revealed the convergence rate in L^1 and L^p-norm sense for a range of SDEs, where the drift term is Lipschitzian and the diffusion term is Hölder continuous with respect to spatial variable. In addition, by using the Yamada-Watanabe approximation and heat kernel estimate, [26] studied the strong convergence rate for a class of non-degenerate SDEs with bounded drift term satisfies weak monotonicity and is of bounded variation with respect to Gaussian measure and the diffusion term is Hölder continuous.

Quite recently, by Zvonkin transformation [39], the convergence rate of EM method for the SDEs with singular drift are investigated extensively. For instance, [2] discussed the case with Dini continuous drifts; [27] obtained the strong convergence rate of EM method with bounded Hölder continuous drift driven by truncated symmetric α-stable process, see also [22] and [14] for the symmetric α-stable process. As to the distribution dependent SDEs, [37] proved the convergence of the EM scheme under linear growth condition by a discretized version of Krylov's estimate. [3] extended the results of [27] and [13] to the distribution dependent SDEs driven by Brownian motion.

In this paper, we investigate the existence and uniqueness of distribution dependent SDEs with bounded and Hölder continuous drifts, where the noise is α-stable process. Due to the distribution dependence, we adopt an approximation technique by constructing a sequence of classical SDEs and using the Skorohod representation theorem to prove the existence of the weak solution. As to the pathwise uniqueness, we still use the Zvonkin transform which depend on the distribution of one solution to make two solutions be regular ones.

Since the SDE is distribution-dependent, we exploit the stochastic interacting particle systems to approximate it. We will apply a common Zvonkin's transform depending on the distribution of the real solution to make the numerical SDE and interacting particle systems be regular ones, from which the strong convergence rate is obtained.

The paper is organized as follows. In Section 2, we recall some preliminaries on symmetric α-stable process and the Poisson random measure. In Section 3, the existence and uniqueness for the distribution dependent SDEs with Hölder continuous drift driven by α-stable process are established. Finally, by using Zvonkin type transformation, the convergence rate of EM Scheme for SDEs are investigated in Section 4.

2 Some Preparations

2.1 Symmetric α-stable process

Before moving on, we firstly recall some knowledge on symmetric α-stable process and the Poisson random measure (see [14, 19, 20, 28] for more details). Recall that a $\mathbb{R}^d$-valued Lévy process L_t is called d-dimensional symmetric α-stable process if the Lévy symbol Ψ has the following representation:

$$\Psi(u) = \int_{\mathbb{R}^d} [1 - \cos\langle u, x\rangle]\nu(\mathrm{d}x),$$

where

$$\nu(D) = \int_S \mu(\mathrm{d}\xi) \int_0^\infty \mathbb{1}_D(r\xi) \frac{\mathrm{d}r}{r^{1+\alpha}}, \quad D \in \mathscr{B}(\mathbb{R}^d),$$

$\alpha \in (0,2)$, $S = \{x \in \mathbb{R}^d, |x| = 1\}$ and μ is a finite symmetric measure on $(S, \mathscr{B}(S))$, i.e. $\mu(A) = \mu(-A)$, for any $A \in \mathscr{B}(S)$.

The Poisson random measure N associated to L is defined as follows:

$$N([0,t], U) = \sum_{0 \le s \le t} \mathbb{1}_U(\Delta L_s), \quad U \in \mathscr{B}\left(\mathbb{R}^d \backslash \{0\}\right), t \ge 0.$$

Here $\Delta L(s) = L_s - L_{s-}$ denotes the jump size of L at time $s \ge 0$. The compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}([0,t], U) = N([0,t], U) - t\nu(U), \quad U \in \mathscr{B}\left(\mathbb{R}^d \backslash \{0\}\right), 0 \notin \bar{U}, t \ge 0.$$

It follows from the Lévy-Itô decomposition that

$$L_t = \int_0^t \int_{|x| \le 1} x \tilde{N}(\mathrm{d}s, \mathrm{d}x) + \int_0^t \int_{|x| > 1} x N(\mathrm{d}s, \mathrm{d}x), \quad t \ge 0.$$

For convenience, we introduce some notations. Let $\|\cdot\|$ denote the operator norm for a bounded linear operator. For $k \in \mathbb{N}$ and $\beta \in (0,1)$, denote by $C_b^{k+\beta}\left(\mathbb{R}^d\right)$ the set of $\mathbb{R}^d$-valued bounded functions, which have up to k-ordered continuous derivative and the k-th derivative is β Hölder continuous. The norm is

$$\|f\|_{k+\beta} := \sum_{i=0}^k \sup_{x \in \mathbb{R}^d} \|\nabla^i f(x)\| + \sup_{x \ne y} \frac{\|\nabla^k f(x) - \nabla^k f(y)\|}{|x-y|^\beta}, \quad f \in C_b^{k+\beta}\left(\mathbb{R}^d\right).$$

In particular, $C_b^0\left(\mathbb{R}^d\right)$ means the set of $\mathbb{R}^d$-valued bounded functions, equipped the norm $\|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)|$, and we usually denote C_b. Let $T > 0$, for a function f defined on $[0,T] \times \mathbb{R}^d$, let $\|f\|_{T,\infty} = \sup_{t \in [0,T], x \in \mathbb{R}^d} |f(t,x)|$.

2.2 Distribution dependent SDEs

Let $\mathscr{P}$ be the collection of all probability measures on $\mathbb{R}^d$ equipped with weak topology. For $p \ge 1$, if $\mu(|\cdot|^p) := \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) < \infty$, we formulate $\mu \in \mathscr{P}_p$. For $\mu, \bar{\mu} \in \mathscr{P}_p$, $p \ge 1$, the $\mathbb{W}_p$-Wasserstein distance between μ and ν is defined by

$$\mathbb{W}_p(\mu, \bar{\mu}) = \inf_{\pi \in \mathcal{C}(\mu, \bar{\mu})} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}},$$

where $\mathcal{C}(\mu, \bar{\mu})$ stands for the set of all couplings of μ and $\bar{\mu}$. As for a random variable ξ, its law is written by $\mathscr{L}_\xi$, and write $\mathscr{L}_\xi|_{\mathbb{P}}$ as the distribution of ξ under $\mathbb{P}$.

Let $b : \mathbb{R}^d \times \mathscr{P} \to \mathbb{R}^d$ be measurable with respect to the σ-algebra generated by the product topology on $\mathbb{R}^d \times \mathscr{P}$. Consider the following McKean-Vlasov SDE on $\mathbb{R}^d$

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \mathrm{d}L_t, \quad t \ge 0. \tag{2.1}$$

Definition 2.1. A càdlàg adapted process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ is called a (strong) solution of (2.1), if $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t b(X_s, \mathscr{L}_{X_s})\,\mathrm{d}s + L_t, \quad t \geq 0. \tag{2.2}$$

Moreover, if $\mathbb{E}|X_t|^\theta < \infty$ for some $\theta \in [1, \alpha)$ and any $t \geq 0$, then we say (2.1) has a solution in $\mathscr{P}_\theta$. We call the strong uniqueness in $\mathscr{P}_\theta$ for some $\theta \in [1, \alpha)$, if for any $\mathscr{F}_0$-measurable random variable X_0 with $\mathscr{L}_{X_0} \in \mathscr{P}_\theta$, there exists a unique X_t satisfy (2.2) and $\mathbb{E}|X_t|^\theta < \infty$.

(2) A couple $(\tilde{X}_t, \tilde{L}_t)_{t\geq 0}$ is called a weak solution to (2.2), if $\tilde{L}$ is a d-dimensional symmetric α-stable process with respect to a complete filtered probability space $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$, and (2.2) holds for $(\tilde{X}_t, \tilde{L}_t)_{t\geq 0}$ in place of $(X_t, L_t)_{t\geq 0}$.

(3) (2.2) is said to have weak uniqueness in $\mathscr{P}_\theta$ for some $\theta \in [1, \alpha)$, if any two weak solutions of the equation from common initial distribution in $\mathscr{P}_\theta$ are equal in law.

Throughout this paper, we assume that

(H1) For fixed $\alpha \in (1, 2)$, there exists a positive constant $C_\alpha > 0$ such that

$$\Psi(u) \geq C_\alpha |u|^\alpha, \quad u \in \mathbb{R}^d.$$

(H2) $\|b\|_\infty := \sup_{x\in\mathbb{R}^d, \mu\in\mathscr{P}} |b(x, \mu)| < \infty$, and there exists constants $\beta \in (0, 1)$ satisfying $2\beta + \alpha > 2$, $K > 0$ and $\kappa \in [1, \alpha)$ such that

$$|b(x, \mu) - b(y, \bar{\mu})| \leq K(\mathbb{W}_\kappa(\mu, \bar{\mu}) + |x - y|^\beta), \quad \mu, \bar{\mu} \in \mathscr{P}_\kappa, x, y \in \mathbb{R}^d. \tag{2.3}$$

Remark 2.1. *See [14, Remark 1.1] for examples such that* **(H1)** *holds. As to* **(H2)***, we give an example as follows. Let $\tilde{b} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded and satisfy*

$$|\tilde{b}(x_1, y_1) - \tilde{b}(x_2, y_2)| \leq K(|x_1 - x_2|^\beta + |y_1 - y_2|), \quad x_1, y_1, x_2, y_2 \in \mathbb{R}^d$$

for some $K > 0$ and $\beta \in (0, 1)$ with $2\beta + \alpha > 2$. Define

$$b(x, \mu) = \int_{\mathbb{R}^d} \tilde{b}(x, z)\mu(\mathrm{d}z), \quad x \in \mathbb{R}^d, \mu \in \mathscr{P}_1.$$

Then for any $\kappa \in [1, \alpha)$, $\mu, \bar{\mu} \in \mathscr{P}_\kappa$, $x, y \in \mathbb{R}^d$, $\pi \in \mathscr{C}(\mu, \bar{\mu})$, we have

$$\begin{aligned}
|b(x, \mu) - b(y, \bar{\mu})| &= \left| \int_{\mathbb{R}^d} \tilde{b}(x, z)\mu(\mathrm{d}z) - \int_{\mathbb{R}^d} \tilde{b}(y, z')\mu(\mathrm{d}z') \right| \\
&= \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} [\tilde{b}(x, z) - \tilde{b}(y, z')]\pi(\mathrm{d}z, \mathrm{d}z') \right| \\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} K(|x - y|^\beta + |z - z'|)\pi(\mathrm{d}z, \mathrm{d}z') \\
&\leq K|x - y|^\beta + K \int_{\mathbb{R}^d\times\mathbb{R}^d} |z - z'|\pi(\mathrm{d}z, \mathrm{d}z').
\end{aligned}$$

This implies **(H2)***.*

3 Existence and uniqueness

3.1 Weak Solution

We will use the tightness and the Skorohod representation theorem to prove the weak existence. The idea of the proof of the following theorem comes from [38, Proof of Theorem 4.1] (see also [18, Proof of Theorem 4.7] and [15, 25] for the case with Gaussian noise).

Theorem 3.1. *Assume that b is bounded and there exists a constant $K > 0$ such that*

$$|b(x,\mu) - b(x,\bar{\mu})| \le K\mathbb{W}_\kappa(\mu,\bar{\mu}), \quad \mu,\bar{\mu} \in \mathscr{P}_\kappa, x \in \mathbb{R}^d. \tag{3.1}$$

Then for any $\mu_0 \in \mathscr{P}_\kappa$, (2.1) has a weak solution with initial distribution μ_0.

Proof. Let $0 \le \rho \in C_0^\infty(\mathbb{R}^d)$ with support contained in $\{x : |x| \le 1\}$ such that $\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x = 1$. For any $n \ge 1$, let $\rho_n(x) = n^d\rho(nx)$ and define

$$b^n(x,\mu) = \int_{\mathbb{R}^d} b(x',\mu)\rho_n(x-x')\mathrm{d}x', \quad (x,\mu) \in \mathbb{R}^d \times \mathscr{P}. \tag{3.2}$$

Then by (3.1), for any $n \ge 1$, there exists a constant $C_n > 0$ such that

$$|b^n(x,\mu) - b^n(y,\bar{\mu})| \le C_n(|x-y| + \mathbb{W}_\kappa(\mu,\bar{\mu})), \quad (x,\mu),(y,\bar{\mu}) \in \mathbb{R}^d \times \mathscr{P}_\kappa.$$

Moreover, it holds

$$\begin{aligned} |b^n(x,\mu) - b(x,\bar{\mu})| &\le |b^n(x,\mu) - b^n(x,\bar{\mu})| + |b^n(x,\bar{\mu}) - b(x,\bar{\mu})| \\ &\le K\mathbb{W}_\kappa(\mu,\bar{\mu})) + |b^n(x,\bar{\mu}) - b(x,\bar{\mu})|. \end{aligned} \tag{3.3}$$

For any $n \ge 1$, define

$$\mathrm{d}X_t^n = b^n(X_t^n, \mathscr{L}_{X_t^n})\mathrm{d}t + \mathrm{d}L_t, \tag{3.4}$$

with $\mathscr{L}_{X_0^n} = \mu_0$. Then use a distribution iteration method as in the case with Gaussian noise ([34]), it is not difficult to see that (3.4) has a solution $\{X^n\}_{n\ge1}$ on $[0,T]$ with $\mathscr{L}_{X_t^n} \in \mathscr{P}_\kappa$.

Let $\mathbb{D}$ be the space of all $\mathbb{R}^d$-valued càlàg functions on $[0,T]$ equipped with the Skorohod topology such that $\mathbb{D}$ is a Polish space. Set

$$H_s^n = \int_0^s b^n(X_t^n, \mathscr{L}_{X_t^n})\mathrm{d}t, \quad s \in [0,T].$$

Since b is bounded, it is clear that

$$\sup_{s\in[0,T]} |H_s^n| \le T\|b\|_\infty$$

for any $n \ge 1$. Moreover, for any $\varepsilon > 0$ and bounded stopping time τ

$$|H_{t\wedge\tau}^n - H_{t\wedge(\tau+\varepsilon)}^n| \le \varepsilon\|b\|_\infty, \quad t \in [0,T].$$

Thus, $\{H_\cdot^n\}_{n\geq 1}$ in $\mathbb{D}$ is tight, and so does $\{H_\cdot^n, L_\cdot\}_{n\geq 1}$. So there exists a subsequence still denoted by $\{H_\cdot^n, L_\cdot\}_{n\geq 1}$ such that the distribution of $\{H_\cdot^n, L_\cdot\}_{n\geq 1}$ is weakly convergent in $\mathbb{D}\times\mathbb{D}$, which implies weak convergence of the distribution of $\{X_\cdot^n, L_\cdot\}_{n\geq 1}$ in $\mathbb{D}\times\mathbb{D}$. Then, by Skorohod's representation theorem [31], there exists a probability space $(\tilde\Omega, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and $\mathbb{D}\times\mathbb{D}$-valued stochastic processes $\{\tilde X_\cdot^n, \tilde L_\cdot^n)$, $\{\tilde X_\cdot, \tilde L_\cdot)$ such that $\mathscr{L}_{(X_\cdot^n, L_\cdot)}|_{\mathbb{P}} = \mathscr{L}_{(\tilde X_\cdot^n, \tilde L_\cdot^n)}|_{\tilde{\mathbb{P}}}$, and $\tilde{\mathbb{P}}$-a.s. $(\tilde X_\cdot^{(n)}, \tilde L_\cdot^n)$ converges to $(\tilde X_\cdot, \tilde L_\cdot)$ as $n\to\infty$, which implies that for any $t\in[0,T]$, $\mathscr{L}_{\tilde X_t^n}|_{\tilde{\mathbb{P}}}$ weakly converges to $\mathscr{L}_{\tilde X_t}|_{\tilde{\mathbb{P}}}$. In particular, $\tilde L$ is still a symmetric α-stable Lévy process with respect to the complete filtration $\tilde{\mathscr{F}}_t = \overline{\sigma\{\tilde X_s, \tilde L_s, \quad s\leq t\}}^{\tilde{\mathbb{P}}}$ and has the same symbol as L, and

$$\mathrm{d}\tilde X_t^n = b^n(\tilde X_t^n, \mathscr{L}_{\tilde X_t^n})\mathrm{d}t + \mathrm{d}\tilde L_t^n, \quad \tilde X_0^n = \tilde X_0 \tag{3.5}$$

with $\mathscr{L}_{\tilde X_0}|_{\tilde{\mathbb{P}}} = \mathscr{L}_{X_0^n}|_{\mathbb{P}}$. Next, we only need to take limit in (3.5).

For any $n\geq m\geq 1$, we have

$$\int_0^s |b^n(\tilde X_t^n, \mathscr{L}_{\tilde X_t^n}) - b(\tilde X_t, \mathscr{L}_{\tilde X_t})|\,\mathrm{d}t \leq I_1(s) + I_2(s) + I_3(s),$$

where

$$\begin{aligned} I_1(s) &:= \int_0^s |b^n(\tilde X_t^n, \mathscr{L}_{\tilde X_t^n}) - b^m(\tilde X_t^n, \mathscr{L}_{\tilde X_t})|\,\mathrm{d}t,\\ I_2(s) &:= \int_0^s |b^m(\tilde X_t^n, \mathscr{L}_{\tilde X_t}) - b^m(\tilde X_t, \mathscr{L}_{\tilde X_t})|\,\mathrm{d}t,\\ I_3(s) &:= \int_0^s |b^m(\tilde X_t, \mathscr{L}_{\tilde X_t}) - b(\tilde X_t, \mathscr{L}_{\tilde X_t})|\,\mathrm{d}t. \end{aligned}$$

Below we estimate these $I_i(s)$ respectively. For simplicity, let $\tilde\mu_t = \mathscr{L}_{\tilde X_t}$ and $\tilde\mu_t^n = \mathscr{L}_{\tilde X_t^n}$.

Firstly, since $\|b^n\|_\infty \leq \|b\|_\infty$, applying Krylov's estimate in [38, Theorem 3.1] and Chebyshev's inequality, we arrive at for any $p > \frac{d}{\alpha}\vee 1$ and $q > \frac{p\alpha}{p\alpha - d}$,

$$\begin{aligned} \tilde{\mathbb{P}}(\sup_{s\in[0,T]} I_1(s) \geq \frac{\varepsilon}{3}) &\leq \frac{9}{\varepsilon^2}\tilde{\mathbb{E}}\int_0^T 1_{\{|\tilde X_t^n|\leq R\}}|b^n(\tilde X_t^n, \tilde\mu_t^n) - b^m(\tilde X_t^n, \tilde\mu_t)|^2\,\mathrm{d}t\\ &\quad + \frac{9}{\varepsilon^2}\tilde{\mathbb{E}}\int_0^T 1_{\{|\tilde X_t^n|>R\}}|b^n(\tilde X_t^n, \tilde\mu_t^n) - b^m(\tilde X_t^n, \tilde\mu_t)|^2\,\mathrm{d}t\\ &\leq \frac{C}{\varepsilon^2}\left(\int_0^T\left(\int_{|x|\leq R}|b^n(x,\tilde\mu_t^n) - b^m(x,\tilde\mu_t)|^{2p}\mathrm{d}x\right)^{q/p}\mathrm{d}t\right)^{\frac{1}{q}}\\ &\quad + \frac{C}{\varepsilon^2}\int_0^T \tilde{\mathbb{P}}(|\tilde X_t^n| > R)\mathrm{d}t. \end{aligned}$$

Since $\tilde X_t^n$ converges to $\tilde X_t$ in probability, it is clear

$$\lim_{n\to\infty}\mathbb{W}_\kappa(\tilde\mu_t^n, \mu_t) = 0,$$

and

$$\lim_{n\to\infty}\tilde{\mathbb{P}}(|\tilde{X}_t^n|>R)\le\tilde{\mathbb{P}}(|\tilde{X}_t|\ge\frac{R}{2}).$$

Then it follows from the definition of b^n and (3.3) that

$$\lim_{n\to\infty}|b^n(x,\tilde{\mu}_t^n)-b(x,\tilde{\mu}_t)|=0,\quad a.e.\quad x\in\mathbb{R}^d.$$

So, we may apply the dominated convergence theorem to derive

$$\begin{aligned}&\limsup_{n\to\infty}\tilde{\mathbb{P}}(\sup_{s\in[0,T]}I_1(s)\ge\frac{\varepsilon}{3})\\ &\le\frac{C}{\varepsilon^2}\left(\int_0^T\left(\int_{|x|\le R}|b(x,\tilde{\mu}_t)-b^m(x,\tilde{\mu}_t)|^{2p}\mathrm{d}x\right)^{q/p}\mathrm{d}t\right)^{\frac{1}{q}}\\ &+\frac{C}{\varepsilon^2}\int_0^T\tilde{\mathbb{P}}(|\tilde{X}_t|\ge R)\mathrm{d}t.\end{aligned}\tag{3.6}$$

Since b^m is bounded and continuous, it follows that

$$\limsup_{n\to\infty}\tilde{\mathbb{P}}\Big(\sup_{s\in[0,T]}I_2(s)\ge\frac{\varepsilon}{3}\Big)\le\limsup_{n\to\infty}\frac{3}{\varepsilon}\mathbb{E}\int_0^T|b^m(\tilde{X}_t^n,\mathscr{L}_{\tilde{X}_t})-b^m(\tilde{X}_t,\mathscr{L}_{\tilde{X}_t})|\,\mathrm{d}t=0.$$

Finally, since $\tilde{X}_t^n\to\tilde{X}_t$ in probability, the Krylov's estimate in [38, Theorem 3.1] also holds for $\tilde{X}$ replacing $\tilde{X}^n$. Therefore, inequality (3.6) holds for I_3 replacing I_1. In conclusion, we arrive at

$$\begin{aligned}&\limsup_{n\to\infty}\tilde{\mathbb{P}}\Big(\sup_{s\in[0,T]}\int_0^s|b^n(\tilde{X}_t^n,\mathscr{L}_{\tilde{X}_t^n})-b(\tilde{X}_t,\mathscr{L}_{\tilde{X}_t})|\,\mathrm{d}t\ge\varepsilon\Big)\\ &\le\limsup_{n\to\infty}\sum_{i=1}^3\tilde{\mathbb{P}}\Big(\sup_{s\in[0,T]}I_i(s)\ge\frac{\varepsilon}{3}\Big)\\ &\le\frac{C}{\varepsilon^2}\left(\int_0^T\left(\int_{|x|\le R}|b(x,\tilde{\mu}_t)-b^m(x,\tilde{\mu}_t)|^{2p}\mathrm{d}x\right)^{q/p}\mathrm{d}t\right)^{\frac{1}{q}}\\ &+\frac{C}{\varepsilon^2}\int_0^T\tilde{\mathbb{P}}(|\tilde{X}_t|\ge R)\mathrm{d}t\end{aligned}$$

for any $m>0$ and $R>0$. Then letting first $m\to\infty$ and then $R\to\infty$, we obtain from the dominated convergence theorem that

$$\limsup_{n\to\infty}\tilde{\mathbb{P}}\Big(\sup_{s\in[0,T]}\int_0^s|b^n(\tilde{X}_t^n,\mathscr{L}_{\tilde{X}_t^n})-b(\tilde{X}_t,\mathscr{L}_{\tilde{X}_t})|\,\mathrm{d}t\ge\varepsilon\Big)=0.$$

Finally, letting n go to infinity in (3.5), we have

$$\mathrm{d}\tilde{X}_t=b(\tilde{X}_t,\mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}t+\mathrm{d}\tilde{L}_t.\tag{3.7}$$

Thus, $(\tilde{X},\tilde{L})$ is a weak solution to (2.1). □

Remark 3.2. *Since the Krylov's estimate also holds under some integrable condition on the drift in [38], the existence of weak solution can be proved when b satisfies some integrable condition. We only consider the bounded measurable drift in this paper since the convergence rate of Euler-Maruyama method in Section 4 cannot be obtained under integrable condition.*

Theorem 3.3. *Assume* **(H1)**-**(H2)**. *Then* (2.1) *has weak uniqueness in* $\mathscr{P}_\kappa$.

Proof. Let $(X_t)_{t\geq 0}$ solve (2.1) with $\mathscr{L}_{X_0} = \mu_0$, and let $(\tilde{X}_t, \tilde{L}_t)$ on $(\tilde{\Omega}, \{\tilde{\mathscr{F}}_t\}_{t\geq 0}, \tilde{\mathbb{P}})$ be a weak solution of (2.1) such that $\mathscr{L}_{X_0}|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}_0}|_{\tilde{\mathbb{P}}} = \mu_0$, i.e. $\tilde{X}_t$ solves

$$\mathrm{d}\tilde{X}_t = b(\tilde{X}_t, \mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}t + \mathrm{d}\tilde{L}_t, \quad \mathscr{L}_{\tilde{X}_0} = \mu_0. \tag{3.8}$$

We aim to prove $\mathscr{L}_X|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}}|_{\tilde{\mathbb{P}}}$. Let $\mu_t = \mathscr{L}_{X_t}|_{\mathbb{P}}$ and

$$\bar{b}_t(x) = b(x, \mu_t), \quad x \in \mathbb{R}^d.$$

According to [28], the stochastic differential equation

$$\mathrm{d}\bar{X}_t = \bar{b}_t(\bar{X}_t)\mathrm{d}t + \mathrm{d}\tilde{L}_t, \quad \bar{X}_0 = \tilde{X}_0 \tag{3.9}$$

has a unique solution under **(H1)**-**(H2)**. According to Yamada–Watanabe [17], it also satisfies weak uniqueness. Noting that

$$\mathrm{d}X_t = \bar{b}_t(X_t)\mathrm{d}t + \mathrm{d}L_t, \quad \mathscr{L}_{X_0}|_{\mathbb{P}} = \mathscr{L}_{\tilde{X}_0}|_{\tilde{\mathbb{P}}},$$

the weak uniqueness of (3.9) implies

$$\mathscr{L}_{\bar{X}}|_{\tilde{\mathbb{P}}} = \mathscr{L}_X|_{\mathbb{P}}. \tag{3.10}$$

So, (3.9) reduces to

$$\mathrm{d}\bar{X}_t = b(\bar{X}_t, \mathscr{L}_{\bar{X}_t}|_{\tilde{\mathbb{P}}})\mathrm{d}t + \mathrm{d}\tilde{L}_t, \quad \bar{X}_0 = \tilde{X}_0.$$

By the strong uniqueness of (2.1) according to Theorem 3.6 below, we obtain $\bar{X} = \tilde{X}$. Therefore, (3.10) implies $\mathscr{L}_{\tilde{X}}|_{\tilde{\mathbb{P}}} = \mathscr{L}_X|_{\mathbb{P}}$ as wanted. □

3.2 Strong Solution

The next lemma characterize the relationship between the existence of weak and strong solution (see [15, 16]).

Lemma 3.4. *Let* $(\bar{\Omega}, \{\bar{\mathscr{F}}_t\}_{t\geq 0}, \bar{\mathbb{P}})$ *and* $(\bar{X}_t, L_t)$ *be a weak solution to* (2.1) *with* $\mu_t := \mathscr{L}_{\bar{X}_t}|_{\bar{\mathbb{P}}}$. *If the SDE*

$$\mathrm{d}X_t = b(X_t, \mu_t)\,\mathrm{d}t + \mathrm{d}L_t, \quad 0 \leq t \leq T \tag{3.11}$$

has a unique strong solution X_t *up to life time with* $\mathscr{L}_{X_0} = \mu_0$, *then* (2.1) *has a strong solution.*

Proof. Since $\mu_t = \mathscr{L}_{\bar{X}_t}|_{\bar{\mathbb{P}}}$, $\bar{X}_t$ is a weak solution to (3.11). By Yamada-Watanabe principle, the strong uniqueness of (3.11) implies the weak uniqueness, so that X_t is nonexplosive with $\mathscr{L}_{X_t} = \mu_t, t \geq 0$. Therefore, X_t is a strong solution to (2.1). $\square$

Remark 3.5. *According to [28],* (3.11) *has a unique strong solution under* **(H1)**-**(H2)***. This together with Lemma 3.4 and Theorem 3.1 implies that* (2.1) *has a strong solution.*

Theorem 3.6. *Assume* **(H1)**-**(H2)***. Let X and Y be two solutions to* (2.1) *in $\mathscr{P}_\kappa$ with $X_0 = Y_0$. Then $\mathbb{P}$-a.s. $X = Y$.*

Proof. Let $\mu_t = \mathscr{L}_{X_t}, \bar{\mu}_t = \mathscr{L}_{Y_t}, t \in [0,T]$. Then $\mu_0 = \bar{\mu}_0$. Let

$$b_t^\mu(x) = b(x, \mu_t), \quad b_t^{\bar{\mu}}(x) = b(x, \bar{\mu}_t), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Then it holds

$$\begin{aligned} \mathrm{d}X_t &= b_t^\mu(X_t)\,\mathrm{d}t + \mathrm{d}L_t, \\ \mathrm{d}Y_t &= b_t^{\bar{\mu}}(Y_t)\mathrm{d}t + \mathrm{d}L_t. \end{aligned} \tag{3.12}$$

For $\lambda > 0$, consider the following PDE for $u^{\lambda,\mu} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$:

$$\partial_t u_t^{\lambda,\mu} + \mathscr{L} u_t^{\lambda,\mu} + \nabla_{b_t^\mu} u_t^{\lambda,\mu} + b_t^\mu = \lambda u_t^{\lambda,\mu}, \quad u_T^{\lambda,\mu} = 0, \tag{3.13}$$

where

$$\mathscr{L} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[f(x+y) - f(x) - \langle y, \nabla f(x) \rangle \mathbb{1}_{\{|y| \leq 1\}} \right] \nu(\mathrm{d}y), \quad f \in C_c^\infty(\mathbb{R}^d). \tag{3.14}$$

According to [28, Theorem 3.4], for $\lambda > 0$ large enough, (3.13) has a unique solution $u^{\lambda,\mu} \in C^1([0,T], C_b^{\alpha+\beta}(\mathbb{R}^d; \mathbb{R}^d))$ with

$$\|\nabla u^{\lambda,\mu}\|_{T,\infty} \leq \frac{1}{2}, \tag{3.15}$$

and

$$\lambda \|u^{\lambda,\mu}\|_{T,\infty} + \sup_{t \in [0,T]} \|\nabla u_t^{\lambda,\mu}\|_{\alpha+\beta-1} \leq C \|b\|_\beta. \tag{3.16}$$

Let $\theta_t^{\lambda,\mu}(x) = x + u_t^{\lambda,\mu}(x)$. By (3.12), (3.13), and using the Itô formula, we derive

$$\begin{aligned} \mathrm{d}\theta_t^{\lambda,\mu}(X_t) &= \lambda u_t^{\lambda,\mu}(X_t)\mathrm{d}t + \mathrm{d}L_t \\ &\quad + \int_{\mathbb{R}^d \setminus \{0\}} \left[u_t^{\lambda,\mu}(X_{t-} + x) - u_t^{\lambda,\mu}(X_{t-}) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}x), \\ \mathrm{d}\theta_t^{\lambda,\mu}(Y_t) &= \{\lambda u_t^{\lambda,\mu}(Y_t) + \nabla \theta_t^{\lambda,\mu}(b_t^{\bar{\mu}} - b_t^\mu)(Y_t)\}\mathrm{d}t + \mathrm{d}L_t \\ &\quad + \int_{\mathbb{R}^d \setminus \{0\}} \left[u_t^{\lambda,\mu}(Y_{t-} + x) - u_t^{\lambda,\mu}(Y_{t-}) \right] \tilde{N}(\mathrm{d}t, \mathrm{d}x). \end{aligned} \tag{3.17}$$

Noting that [19, Theorem 2.11] together with (3.15) and (3.16) implies

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[0,T]}\left|\int_0^s\int_{\mathbb{R}^d\setminus\{0\}}\left[u_t^{\lambda,\mu}\left(Y_{t-}+x\right)-u_t^{\lambda,\mu}\left(Y_{t-}\right)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}x)\right|^2\\
&\le c\mathbb{E}\int_0^T\int_{\mathbb{R}^d\setminus\{0\}}\left|u_t^{\lambda,\mu}\left(Y_{t-}+x\right)-u_t^{\lambda,\mu}\left(Y_{t-}\right)\right|^2\nu(\mathrm{d}x)\mathrm{d}t\\
&\le c\mathbb{E}\int_0^T\int_{\{x:0<|x|\le 1\}}\left|u_t^{\lambda,\mu}\left(Y_{t-}+x\right)-u_t^{\lambda,\mu}\left(Y_{t-}\right)\right|^2\nu(\mathrm{d}x)\mathrm{d}t\\
&+c\mathbb{E}\int_0^T\int_{\{x:|x|>1\}}\left|u_t^{\lambda,\mu}\left(Y_{t-}+x\right)-u_t^{\lambda,\mu}\left(Y_{t-}\right)\right|^2\nu(\mathrm{d}x)\mathrm{d}t\\
&\le cT\|\nabla u^{\lambda,\mu}\|_{T,\infty}^2\int_{\{x:0<|x|\le 1\}}|x|^2\nu(\mathrm{d}x)+cT\|u^{\lambda,\mu}\|_{T,\infty}^2\nu(\{x:|x|>1\})<\infty.
\end{aligned}
$$

So, (3.17) is well defined and we have

$$|\theta_t^{\lambda,\mu}(X_t)-\theta_t^{\lambda,\mu}(Y_t)|\le\sum_{i=1}^3\Lambda_i(t), \tag{3.18}$$

where

$$
\begin{aligned}
\Lambda_1(t)&=\left|\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}\left[u_s^{\lambda,\mu}\left(X_{s-}+x\right)-u_s^{\lambda,\mu}\left(X_{s-}\right)-u_s^{\lambda,\mu}\left(Y_{s-}+x\right)+u_t^{\lambda,\mu}\left(Y_{s-}\right)\right]\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right|,\\
\Lambda_2(t)&=\int_0^t\lambda|u_s^{\lambda,\mu}(X_s)-u_s^{\lambda,\mu}(Y_s)|\mathrm{d}s,\\
\Lambda_3(t)&=\int_0^t|\nabla\theta_s^{\lambda,\mu}(b_s^{\bar{\mu}}-b_s^{\mu})(Y_s)|\mathrm{d}s.
\end{aligned}
$$

Firstly, by **(H2)**, (3.15) and Hölder inequality, for any $p\ge\kappa$, we obtain

$$\mathbb{E}\sup_{0\le s\le t}\Lambda_3^p(s)\le c_1(p,T)\int_0^t\mathbb{W}_\kappa(\bar{\mu}_s,\mu_s)^p\mathrm{d}s\le c_1(p,T)\int_0^t\mathbb{E}\sup_{0\le s\le r}|X_s-Y_s|^p\mathrm{d}r. \tag{3.19}$$

Similarly, we have

$$\mathbb{E}\sup_{0\le s\le t}\Lambda_2^p(s)\le c_2(p,\lambda,T)\int_0^t\mathbb{E}\sup_{0\le s\le r}|X_s-Y_s|^p\mathrm{d}r. \tag{3.20}$$

Finally, by [14], for any $p\ge\kappa$, we have

$$\mathbb{E}\sup_{0\le s\le t}\Lambda_1^p(s)\le c_3(p,T,\nu,\alpha,\beta)\int_0^t\mathbb{E}\sup_{s\in[0,r]}|X_s-Y_s|^p\,\mathrm{d}r. \tag{3.21}$$

Combining formulas (3.18)–(3.21) and (3.15), we get

$$\mathbb{E}\sup_{0\le s\le t}|X_s-Y_s|^p\le C\int_0^t\mathbb{E}\sup_{s\in[0,r]}|X_s-Y_s|^p\,\mathrm{d}r.$$

Since $X-Y$ is a bounded process, Gronwall's inequality implies that $\mathbb{P}$-a.s. $X_t = Y_t$ for all $t \in [0,T]$. □

Remark 3.7. *Although the PDE considered in [28] is elliptic, the PDE* (3.13) *is parabolic, we can obtain* (3.15) *and* (3.16) *by the same method in [28] since both the elliptic and parabolic PDEs have similar probability representation.*

4 The convergence rate of EM Scheme for SDEs

In this section, we exploit the stochastic interacting particle systems to approximate (2.1). Let $N \geq 1$ be an integer and $(X_0^i, L_t^i)_{1\leq i\leq N}$ be i.i.d. copies of (X_0, L_t). Consider the following stochastic non-interacting particle systems

$$\mathrm{d}X_t^i = b(X_t^i, \mu_t^i)\mathrm{d}t + \mathrm{d}L_t^i, \quad t \geq 0, \quad i \in \mathcal{S}_N := \{1, \cdots, N\} \tag{4.1}$$

with $\mu_t^i := \mathscr{L}_{X_t^i}$. By the weak uniqueness, we have $\mu_t = \mu_t^i, i \in \mathcal{S}_N$. Let δ_x be Dirac's delta measure centered at the point $x \in \mathbb{R}^d$ and $\tilde{\mu}_t^N$ be the empirical distribution associated with $X_t^1, \cdots, X_t^N$, i.e.,

$$\tilde{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}. \tag{4.2}$$

Moreover, the stochastic N-interacting particle systems is defined:

$$\mathrm{d}X_t^{i,N} = b(X_t^{i,N}, \hat{\mu}_t^N)\mathrm{d}t + \mathrm{d}L_t^i,\ t \geq 0,\ X_0^{i,N} = X_0^i,\ i \in \mathcal{S}_N, \tag{4.3}$$

where $\hat{\mu}_t^N$ means the empirical distribution corresponding to $X_t^{1,N}, \cdots, X_t^{N,N}$, namely,

$$\hat{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}.$$

We remark that particles $(X^i)_{i\in\mathcal{S}_N}$ are mutually independent and that particles $(X^{i,N})_{i\in\mathcal{S}_N}$ are interacting and are not independent.

Let $\lfloor a \rfloor$ be the integer part of $a \geq 0$. To discretize (4.3) in time, we introduce the continuous time EM scheme defined as below: for any $\delta \in (0, \mathrm{e}^{-1})$,

$$\mathrm{d}X_t^{\delta,i,N} = b(X_{t_\delta}^{\delta,i,N}, \hat{\mu}_{t_\delta}^{\delta,N})\mathrm{d}t + \mathrm{d}L_t^i, \quad t \geq 0, \quad X_0^{\delta,i,N} = X_0^{i,N}, \tag{4.4}$$

where $t_\delta := \lfloor t/\delta \rfloor \delta$ and

$$\hat{\mu}_{k\delta}^{\delta,N} := \frac{1}{N}\sum_{j=1}^N \delta_{X_{k\delta}^{\delta,j,N}}, \qquad k \geq 0.$$

The following result states that the continuous time EM scheme corresponding to s- tochastic interacting particle systems converges strongly to the non-interacting particle sys- tem whenever the particle number goes to infinity and the stepsize approaches to zero and moreover provides the convergence rate.

Theorem 4.1. *Assume* **(H1)**-**(H2)** *and suppose further* $\mathscr{L}_{X_0} \in \mathscr{P}_p$ *for some* $p \in [\kappa, \alpha)$. *Then, for any* $T > 0$ *and* $q \in (p, \alpha)$, *there exists a constant* $C > 0$ *depending on* p, d, q, T *and* $\sup_{t\in[0,T]} \mathbb{E}|X_t^i|^q$ *such that*

$$\sup_{i\in\mathcal{S}_N} \mathbb{E}\Big(\sup_{0\le t\le T} |X_t^i - X_t^{\delta,i,N}|^p \Big) \le C \begin{cases} \delta^{\frac{p\beta}{\alpha}} + N^{-\frac{1}{2}} + N^{\frac{p}{q}-1}, & p > \frac{d}{2}, \quad q \neq 2p, \\ \delta^{\frac{p\beta}{\alpha}} + N^{-\frac{1}{2}} \log(1+N) + N^{\frac{p}{q}-1}, & p = \frac{d}{2}, \quad q \neq 2p \\ \delta^{\frac{p\beta}{\alpha}} + N^{-\frac{2}{d}} + N^{\frac{p}{q}-1}, & p \in (0, \frac{d}{2}), \quad q \neq \frac{d}{d-p}. \end{cases}$$

Firstly, under **(H1)**-**(H2)**, the stochastic N-interacting particle systems (4.3) are strongly wellposed, see Lemma 4.2 below.

Lemma 4.2. *Assume that* **(H1)** *and* **(H2)** *hold. Then for any* $\mathscr{F}_0$*-measurable random variable* X_0 *with* $\mathscr{L}_{X_0} \in \mathscr{P}_p$ *for some* $p \in (0, \alpha)$, (4.3) *admits a strong solution satisfying*

$$\sup_{i\in\mathcal{S}_N} \mathbb{E}|X_t^{i,N}|^p < \infty, \quad t > 0.$$

Proof. For $x := (x_1, \cdots, x_N)^* \in (\mathbb{R}^d)^N$, $x_i \in \mathbb{R}^d$, set

$$\tilde{\mu}_x^N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \quad \hat{b}(x) := (b(x_1, \tilde{\mu}_x^N), \cdots, b(x_N, \tilde{\mu}_x^N))^*, \quad \hat{L}_t := (L_t^1, \cdots, L_t^N)^*.$$

Obviously, $(\hat{L}_t)_{t\ge0}$ is an Nd-dimensional Lévy process. Then, (4.3) can be reformulated as

$$\mathrm{d}X_t = \hat{b}(X_t)\mathrm{d}t + \mathrm{d}\hat{L}_t, \quad t \ge 0. \tag{4.5}$$

Firstly, $\hat{L}_t$ has symbol $\tilde{\Psi}(u) = \sum_{i=1}^N \Psi(u_i)$ for $u = (u_1, \cdots, u_N)^* \in (\mathbb{R}^d)^N$. Clearly, (**H1**) holds for $\tilde{\Psi}(u)$ for some constant $C(N, \alpha) > 0$ since Ψ satisfies (**H1**) and the inequality

$$C_\alpha \sum_{i=1}^N |u_i|^\alpha \ge C(N, \alpha) \left(\sum_{i=1}^N |u_i|^2 \right)^{\frac{\alpha}{2}}$$

holds for some constant $C(N, \alpha) > 0$. By (**H2**), a straightforward calculation shows that

$$|\hat{b}(x)| \le C_N, \quad x \in (\mathbb{R}^d)^N \tag{4.6}$$

for some constant $C_N > 0$. Observe that

$$\frac{1}{N} \sum_{j=1}^N (\delta_{x_j} \times \delta_{y_j}) \in \mathcal{C}(\tilde{\mu}_x^N, \tilde{\mu}_y^N), \qquad x_j, y_j \in \mathbb{R}, \tag{4.7}$$

so that we have

$$\mathbb{W}_\kappa(\tilde{\mu}_x^N, \tilde{\mu}_y^N) \le \left(\frac{1}{N} \sum_{j=1}^N |x_j - y_j|^\kappa \right)^{\frac{1}{\kappa}}. \tag{4.8}$$

This together with (**H2**) and Hölder inequality implies that

$$|\hat{b}(x)-\hat{b}(x')|\le \hat{C}_N\{|x-x'|+|x-x'|^\beta\}, \tag{4.9}$$

for some constant $\hat{C}_N>0$. Thus, according to [28], (4.3) has a unique strong solution. Finally, the estimate follows from the fact $\mathbb{E}|L_t|^p<\infty$ for any $p\in(0,\alpha)$ and the boundedness of b. □

4.1 Proof of Theorem 4.1

The proof of Theorem 4.1 is based on two lemmas below, where the first one is concerned with propagation of chaos for McKean-Vlasov SDEs with irregular drift coefficients. We state it as follows.

Lemma 4.3. *Under the assumptions of Theorem 4.1, then for any $T>0$ and $q\in(p,\alpha)$, there exists a constant $C>0$ depending on p,d,q,T and $\sup_{t\in[0,T]}\mathbb{E}|X_t^i|^q$ such that*

$$\sup_{i\in\mathcal{S}_N}\mathbb{E}\Big(\sup_{0\le t\le T}|X_t^i-X_t^{i,N}|^p\Big)\le C\begin{cases}N^{-\frac12}+N^{\frac{p}{q}-1}, & p>\frac{d}{2},\quad q\neq 2p,\\ N^{-\frac12}\log(1+N)+N^{\frac{p}{q}-1}, & p=\frac{d}{2},\quad q\neq 2p\\ N^{-\frac{2}{d}}+N^{\frac{p}{q}-1}, & p\in(0,\frac{d}{2}),\quad q\neq\frac{d}{d-p}\end{cases}$$

Proof. For any $i\in\mathcal{S}_N$ and $x\in\mathbb{R}^d$, let $b_t^{\mu^i}(x)=b(x,\mu_t^i)$ and $b_t^{\hat{\mu}^N}=b(x,\hat{\mu}_t^N)$. Then, (4.1) and (4.3) can be rewritten respectively as

$$\begin{aligned}\mathrm{d}X_t^i&=b_t^{\mu^i}(X_t^i)\mathrm{d}t+\mathrm{d}L_t^i,\\ \mathrm{d}X_t^{i,N}&=b_t^{\hat{\mu}^N}(X_t^{i,N})\mathrm{d}t+\mathrm{d}L_t^i.\end{aligned}$$

For $\lambda>0$, consider the following PDE for $u^{\lambda,\mu^i}:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$:

$$\partial_t u_t^{\lambda,\mu^i}+\mathscr{L}u_t^{\lambda,\mu^i}+\nabla_{b_t^{\mu^i}}u_t^{\lambda,\mu^i}+b_t^{\mu^i}=\lambda u_t^{\lambda,\mu^i},\quad u_T^{\lambda,\mu^i}=0, \tag{4.10}$$

where $\mathscr{L}$ is defined in (3.14). Since $\mu^i=\mu$ for any $i\in\mathcal{S}_N$, there exists large enough $\lambda>0$ independent of i, such that (4.10) has a unique solution $u^{\lambda,\mu^i}\in C^1([0,T],C_b^{\alpha+\beta}(\mathbb{R}^d,\mathbb{R}^d))$, which is equal to $u^{\lambda,\mu}$. Moreover, (3.15) and (3.16) hold.

Applying Itô's formula to $\theta_t^{\lambda,\mu^i}(x):=x+u_t^{\lambda,\mu^i}(x), x\in\mathbb{R}^d$ yields

$$\begin{aligned}\mathrm{d}\theta_t^{\lambda,\mu^i}(X_t^i)&=\lambda u_t^{\lambda,\mu^i}(X_t^i)\mathrm{d}t+\mathrm{d}L_t^i\\ &\quad+\int_{\mathbb{R}^d\setminus\{0\}}\Big[u_t^{\lambda,\mu^i}\left(X_{t-}^i+x\right)-u_t^{\lambda,\mu^i}\left(X_{t-}^i\right)\Big]\tilde{N}(\mathrm{d}t,\mathrm{d}x),\\ \mathrm{d}\theta_t^{\lambda,\mu^i}(X_t^{i,N})&=\{\lambda u_t^{\lambda,\mu^i}(X_t^{i,N})+\mathrm{d}L_t^i+\nabla\theta_t^{\lambda,\mu^i}(b_t^{\hat{\mu}^N}-b_t^{\mu^i})(X_t^{i,N})\}\mathrm{d}t\\ &\quad+\int_{\mathbb{R}^d\setminus\{0\}}\Big[u_t^{\lambda,\mu^i}\left(X_{t-}^{i,N}+x\right)-u_t^{\lambda,\mu^i}\left(X_{t-}^{i,N}\right)\Big]\tilde{N}(\mathrm{d}t,\mathrm{d}x).\end{aligned} \tag{4.11}$$

For simplicity, set $\Lambda_t^{\lambda,i,N} = \theta_t^{\lambda,\mu^i}(X_t^i) - \theta_t^{\lambda,\mu^i}(X_t^{i,N})$. We have

$$
\begin{aligned}
|\Lambda_t^{\lambda,i,N}| \le{}& \lambda \int_0^t |u_s^{\lambda,\mu^i}(X_s^i) - u_s^{\lambda,\mu^i}(X_s^{i,N})|\mathrm{d}s + \int_0^t |(\nabla\theta_s^{\lambda,\mu^i}(b_s^{\hat{\mu}^N} - b_s^{\mu^i}))(X_s^{i,N})|\mathrm{d}s \\
&+ \left| \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} \bigg(\Big[u_s^{\lambda,\mu^i}\left(X_{s-}^i + x\right) - u_s^{\lambda,\mu^i}\left(X_{s-}^i\right)\Big] \right. \\
&\left. - \Big[u_s^{\lambda,\mu^i}\left(X_{s-}^{i,N} + x\right) - u^{\lambda,\mu^i}\left(X_{s-}^{i,N}\right)\Big] \bigg) \tilde{N}(\mathrm{d}s, \mathrm{d}x) \right| \\
=:{}& I_{1,i}(t) + I_{2,i}(t) + I_{3,i}(t).
\end{aligned}
$$

Completely the same with (3.21), we have

$$
\mathbb{E} \sup_{s\in[0,t]} |I_{3,i}(s)|^p \le \int_0^t \mathbb{E} \sup_{s\in[0,r]} |X_s^i - X_s^{i,N}|^p \mathrm{d}r.
$$

Next, by assumption **(H2)** and Hölder inequality, for any $p \ge \kappa$, we obtain

$$
\begin{aligned}
\mathbb{E} \sup_{0\le s\le t} |I_{2,i}(s)|^p &\le C_2 \mathbb{E} \int_0^t \mathbb{W}_\kappa(\hat{\mu}_s^N, \tilde{\mu}_s^N)^p + \mathbb{W}_\kappa(\tilde{\mu}_s^N, \mu_s^i)^p \mathrm{d}s \\
&\le C_2 \int_0^t \{\mathbb{E} \sup_{s\in[0,r]} |X_s^i - X_s^{i,N}|^p + \mathbb{E}\mathbb{W}_\kappa(\tilde{\mu}_r^N, \mu_r^i)^p\}\mathrm{d}r \\
&\le C_2 \int_0^t \{\mathbb{E} \sup_{s\in[0,r]} |X_s^i - X_s^{i,N}|^p + \mathbb{E}\mathbb{W}_p(\tilde{\mu}_r^N, \mu_r^i)^p\}\mathrm{d}r
\end{aligned} \tag{4.12}
$$

Similarly, we have

$$
\mathbb{E} \sup_{0\le s\le t} |I_{1,i}(s)|^p \le c_2(p,\lambda,T) \int_0^t \mathbb{E} \sup_{0\le s\le r} |X_s^i - X_s^{i,N}|^p \mathrm{d}r. \tag{4.13}
$$

Thus, we find that for some constant $C_{2,\lambda} > 0$,

$$
\mathbb{E}\Big(\sup_{0\le s\le t} |\Lambda_s^{\lambda,i,N}|^p \Big) \le C_{2,\lambda} \int_0^t \{\mathbb{E} \sup_{0\le s\le r} |X_s^i - X_s^{i,N}|^p + \mathbb{E}\mathbb{W}_p(\tilde{\mu}_r^N, \mu_r^i)^p\}\mathrm{d}r.
$$

Set $Z_t^{i,N} = X_t^i - X_t^{i,N}$ for convenience. This, together with the facts that $|Z_t^{i,N}|^p \le 2^p |\Lambda_t^{\lambda,i,N}|^p$ due to (3.15), leads to

$$
\mathbb{E}\Big(\sup_{0\le s\le t} |Z_s^{i,N}|^p \Big) \le C_{3,\lambda} \int_0^t \{\mathbb{E} \sup_{0\le s\le r} |X_s^i - X_s^{i,N}|^p + \mathbb{E}\mathbb{W}_p(\tilde{\mu}_r^N, \mu_r^i)^p\}\mathrm{d}r
$$

for some constant $C_{3,\lambda} > 0$. On the other hand, according to [12, Theorem 1], for any $q \in (p, \alpha)$,

$$
\sup_{0\le t\le T} \mathbb{E}\mathbb{W}_p(\tilde{\mu}_t^N, \mu_t^i)^p \le C_4 \begin{cases} N^{-\frac{1}{2}} + N^{\frac{p}{q}-1}, & p > \frac{d}{2}, \quad q \ne 2p, \\ N^{-\frac{1}{2}} \log(1+N) + N^{\frac{p}{q}-1}, & p = \frac{d}{2}, \quad q \ne 2p \\ N^{-\frac{2}{d}} + N^{\frac{p}{q}-1}, & p \in (0, \frac{d}{2}), \quad q \ne \frac{d}{d-p} \end{cases} \tag{4.14}
$$

holds for some constant $C_4 > 0$ depending on p, d, q and $\sup_{t\in[0,T]} \mu_t^i(|\cdot|^q)$. Hence, due to the boundedness of $X^i - X^{i,N}$, the desired assertion follows from Gronwall's inequality. $\square$

Remark 4.4. *Noting that for $q \geq \alpha$, $\mu_t^i(|\cdot|^q) = \infty$, the condition in [7, Theorem 5.8] does not hold. So we adopt [12, Theorem 1] in place of [7, Theorem 5.8] used in [3].*

The next lemma gives the estimate for $|X_t^{\delta,i,N} - X_{t_\delta}^{\delta,i,N}|$, which is useful in the sequel.

Lemma 4.5. *Assume* **(H1)** *and* **(H2)***, then for any $0 < p < \alpha$, $t \in [0, T]$,*

$$\sup_{i\in\mathcal{S}_N} \mathbb{E}\left|X_t^{\delta,i,N} - X_{t_\delta}^{\delta,i,N}\right|^p \leq C(p,\nu)\delta^{\frac{p}{\alpha}}$$

holds for some constant $C(p,\nu)$ depending on p and ν.

Proof. The result follows immediately from (4.4), the boundedness of b, the scaling property of L and $\mathbb{E}|L_t|^p < \infty$ for $p \in (0, \alpha)$. $\square$

Lemma 4.6. *Under the assumptions of Theorem 4.1, then for any $T > 0$ and $q \in (p, \alpha)$, there exists a constant $C > 0$ depending on p, d, q, T and $\sup_{t\in[0,T]} \mathbb{E}|X_t^i|^q$ such that*

$$\sup_{i\in\mathcal{S}_N} \mathbb{E}\Big(\sup_{0\leq t\leq T} |X_t^{i,N} - X_t^{\delta,i,N}|^p\Big) \leq C \begin{cases} \delta^{\frac{p\beta}{\alpha}} + N^{-\frac{1}{2}} + N^{\frac{p}{q}-1}, & p > \frac{d}{2},\ q \neq 2p, \\ \delta^{\frac{p\beta}{\alpha}} + N^{-\frac{1}{2}}\log(1+N) + N^{\frac{p}{q}-1}, & p = \frac{d}{2},\ q \neq 2p, \\ \delta^{\frac{p\beta}{\alpha}} + N^{-\frac{2}{d}} + N^{\frac{p}{q}-1}, & p \in (0, \frac{d}{2}),\ q \neq \frac{d}{d-p}. \end{cases}$$

Proof. For $x \in \mathbb{R}^d$ and $i \in \mathcal{S}_N$, let $b_{k\delta}^{\hat{\mu}^{\delta,N}}(x) = b(x, \hat{\mu}_{k\delta}^{\delta,N})$ so that (4.4) can be reformulated as

$$\mathrm{d}X_t^{\delta,i,N} = b_{t_\delta}^{\hat{\mu}^{\delta,N}}(X_{t_\delta}^{\delta,i,N})\mathrm{d}t + \mathrm{d}L_t^i.$$

Let u^{λ,μ^i} be the solution to (4.10). Again applying Itô's formula to $\theta_t^{\lambda,\mu^i}(x) = x + u_t^{\lambda,\mu^i}(x)$ gives that

$$\begin{aligned} \mathrm{d}\theta_t^{\lambda,\mu^i}(X_t^{\delta,i,N}) = \Big\{&\lambda u_t^{\lambda,\mu^i}(X_t^{\delta,i,N}) + \nabla\theta_t^{\lambda,\mu^i}(X_t^{\delta,i,N})(b_{t_\delta}^{\hat{\mu}^{\delta,N}}(X_{t_\delta}^{\delta,i,N}) - b_t^{\mu^i}(X_t^{\delta,i,N})) + \mathrm{d}L_t^i \\ &\int_{\mathbb{R}^d\setminus\{0\}} \Big[u_t^{\lambda,\mu^i}\Big(X_{t-}^{\delta,i,N} + x\Big) - u_t^{\lambda,\mu^i}\Big(X_{t-}^{\delta,i,N}\Big)\Big]\tilde{N}(\mathrm{d}t, \mathrm{d}x). \end{aligned} \tag{4.15}$$

Set

$$\Theta_t^{\lambda,i,N} := \theta_t^{\lambda,\mu^i}(X_t^{i,N}) - \theta_t^{\lambda,\mu^i}(X_t^{\delta,i,N}), \qquad Z_t^{\delta,i,N} := X_t^{i,N} - X_t^{\delta,i,N}.$$

Then, for any $p \in [\kappa, \alpha)$, from (4.15) and the second SDE in (4.11), we deduce from Hölder's inequality that

$$\begin{aligned} &\mathbb{E}\Big(\sup_{0\leq s\leq t} |\Theta_s^{\lambda,i,N}|^p\Big) \\ &\leq C_{\lambda,p,T}\Big\{\int_0^t \mathbb{E}\left|u_s^{\lambda,\mu^i}(X_s^{\delta,i,N}) - u_s^{\lambda,\mu^i}(X_s^{i,N})\right|^p \mathrm{d}s \end{aligned}$$

$$+\int_0^t \mathbb{E}\left|\nabla\theta_s^{\lambda,\mu^i}(b_s^{\hat\mu^N}-b_s^{\mu^i})(X_s^{i,N})-\nabla\theta_s^{\lambda,\mu^i}(X_s^{\delta,i,N})(b_{s_\delta}^{\hat\mu^{\delta,N}}(X_{s_\delta}^{\delta,i,N})-b_s^{\mu^i}(X_s^{\delta,i,N}))\right|^p\mathrm{d}s$$

$$+\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r\int_{\mathbb{R}^d\setminus\{0\}}\left(\left[u_s^{\lambda,\mu^i}\left(X_{s-}^{\delta,i,N}+x\right)-u_s^{\lambda,\mu^i}\left(X_{s-}^{\delta,i,N}\right)\right]\right.\right.$$

$$\left.\left.-\left[u_s^{\lambda,\mu^i}\left(X_{s-}^{i,N}+x\right)-u^{\lambda,\mu^i}\left(X_{s-}^{i,N}\right)\right]\right)\tilde N(\mathrm{d}s,\mathrm{d}x)\right|^p\Bigg\}$$

$$=:C_{\lambda,p,T}\{J_1(t)+J_2(t)+J_3(t)\}$$

for some constant $C_{\lambda,p,T}>0$. In what follows, we intend to estimate $J_i(t), i=1,2,3$, one-by-one. Owing to (3.15) and (3.21), there exists a constant $c_1>0$ such that

$$J_1(t)+J_3(t)\le c_1\int_0^t\mathbb{E}\sup_{s\in[0,r]}|Z_s^{\delta,i,N}|^p\mathrm{d}r. \tag{4.16}$$

It remains to estimate $J_2(t)$. By (3.15), we arrive at

$$\begin{aligned}J_2(t)&\le c_2\int_0^t\{\mathbb{E}\mathbb{W}_\kappa(\mu_s^i,\hat\mu_s^N)^p+\mathbb{E}|X_s^{\delta,i,N}-X_{s_\delta}^{\delta,i,N}|^{p\beta}+\mathbb{E}\mathbb{W}_\kappa(\mu_s^i,\hat\mu_{s_\delta}^{\delta,N})^p\}\mathrm{d}s\\&\le c_3\int_0^t\{\delta^{\frac{p\beta}{\alpha}}+\mathbb{E}\mathbb{W}_\kappa(\mu_s^i,\tilde\mu_s^N)^p+\mathbb{E}\mathbb{W}_\kappa(\tilde\mu_s^N,\hat\mu_s^N)^p+\mathbb{E}\mathbb{W}_\kappa(\tilde\mu_s^N,\hat\mu_{s_\delta}^{\delta,N})^p\}\mathrm{d}s\end{aligned} \tag{4.17}$$

for some constants $c_2,c_3>0$, where we have used Lemma 4.5. On the other hand, similarly to (4.8), we obtain from Lemma 4.5

$$\begin{aligned}&\mathbb{E}\mathbb{W}_\kappa(\tilde\mu_t^N,\hat\mu_t^N)^p+\mathbb{E}\mathbb{W}_\kappa(\tilde\mu_t^N,\hat\mu_{t_\delta}^{\delta,N})^p\\&\le\frac{1}{N}\sum_{j=1}^N\{\mathbb{E}|X_t^j-X_t^{j,N}|^p+\mathbb{E}|X_t^j-X_{t_\delta}^{\delta,j,N}|^p\}\\&\le C_{1,T}\delta^{\frac{p}{\alpha}}+\mathbb{E}|X_t^i-X_t^{i,N}|^p+c(p)\mathbb{E}|X_t^{i,N}-X_t^{\delta,i,N}|^p\end{aligned} \tag{4.18}$$

for some $C_{1,T}>0$, where in the last display we used the facts that $(X^j-X^{j,N})_{j\in\mathcal{S}_N}$ and $(X^j-X^{\delta,j,N})_{j\in\mathcal{S}_N}$ are identically distributed. Then, plugging (4.18) back into (4.17) gives that

$$J_2(t)\le C_{2,T}\int_0^t\{\delta^{\frac{p\beta}{\alpha}}+\mathbb{E}\mathbb{W}_p(\mu_s^i,\tilde\mu_s^N)^p+\mathbb{E}|X_s^i-X_s^{i,N}|^p+\mathbb{E}\sup_{r\in[0,s]}|Z_r^{\delta,i,N}|^p\}\mathrm{d}s \tag{4.19}$$

for some constant $C_{2,T}>0$. Now, combining (4.16), (4.19), we arrive at

$$\mathbb{E}\Big(\sup_{0\le s\le t}|\Theta_s^{\lambda,i,N}|^p\Big)\le C_{4,T}\int_0^t\{\delta^{\frac{p\beta}{\alpha}}+\mathbb{E}\mathbb{W}_p(\mu_s^i,\tilde\mu_s^N)^p+\mathbb{E}|X_s^i-X_s^{i,N}|^p+\mathbb{E}\sup_{r\in[0,s]}|Z_r^{\delta,i,N}|^p\}\mathrm{d}s$$

for some constant $C_{4,T}>0$. This, together with $|Z_t^{\delta,i,N}|^p\le 2^p|\Theta_t^{\lambda,i,N}|^p$ due to (3.15), yields

$$\mathbb{E}\Big(\sup_{0\le s\le t}|Z_s^{\delta,i,N}|^p\Big)\le C_{5,T}\int_0^t\{\delta^{\frac{p\beta}{\alpha}}+\mathbb{E}\mathbb{W}_p(\mu_s^i,\tilde\mu_s^N)^p+\mathbb{E}|X_s^i-X_s^{i,N}|^p+\mathbb{E}\sup_{r\in[0,s]}|Z_r^{\delta,i,N}|^p\}\mathrm{d}s$$

for some constant $C_{5,T>0}$. Consequently, the desired assertion holds true by applying Gronwall's inequality and employing Lemma 4.3 and (4.14).

□

Proof of Theorem 4.1. Theorem 4.1 immediately follows from Lemma 4.3 and Lemma 4.6. □

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