On rational functions with more than three branch points

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Abstract: Let Λ be a collection of partitions of a positive integer d of the form

$$(a_1, \cdots, a_p), (b_1, \cdots, b_q), (m_1+1, 1, \cdots, 1), \cdots, (m_l+1, 1, \cdots, 1),$$

where $(m_1, \cdots, m_l)$ is a partition of $p+q-2>0$. We prove that there exists a rational function on the Riemann sphere $\overline{\mathbb{C}}$ with branch data Λ if and only if

$$\max(m_1, \cdots, m_l) < \frac{d}{\mathrm{GCD}(a_1, \cdots, a_p, b_1, \cdots, b_q)}.$$

As an application, we give a new class of branch data which can be realized by Belyi functions on the Riemann sphere.

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1 Introduction

Let X and Y be two compact connected Riemann surfaces, and let $f: X \to Y$ be a holomorphic branched covering of degree d. For each point q in Y, there is a partition $\lambda(q) = (k_1, ..., k_r)$ of d associated to q such that, over a suitable neighborhood of q in Y, f is equivalent to the map

$$\{1, ..., r\} \times \mathbb{D} \to \mathbb{D}, \quad (j, z) \mapsto z^{k_j}, \quad \text{where} \quad \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

with q corresponding to 0 in $\mathbb{D}$. For any partition $\lambda = (k_1, k_2, \cdots, k_r)$ of d, we define its length $\mathrm{Len}(\lambda) = r$. We call the partition λ of d *non-trivial* if $\mathrm{Len}(\lambda) < d$. For the branched covering $f : X \to Y$, We call a point q in Y *a branch point* of f if and only if $\lambda(q)$ is non-trivial, and we call the set of branch points of f *the branch set* of f, denoted by B_f. The collection $\Lambda = \{\lambda(q) : q \in B_f\}$ (with repetitions allowed) is called the *branch data* of f and

$$\nu(f) := \sum_{q \in B(f)} \Big(d - \mathrm{Len}\big(\lambda(q)\big) \Big)$$

the *total branching order* of f. By the Riemann-Hurwitz theorem, we have that

$$\nu(f) = 2g(X) - 2 - d\big(2g(Y) - 2\big)$$

where $g(X)$ (resp. $g(Y)$) denotes the genus of X (resp. Y). Therefore, the total branching order $\nu(f)$ is an even non-negative integer.

The following problem was first proposed by Edmonds-Kulkarni-Stong [4] and we can trace its history to Hurwitz [9].

Realizability Problem. Given a compact connected Riemann surface Y and a collection $\Lambda = \{\lambda_1, \cdots, \lambda_k\}$ of non-trivial partitions of a positive integer d, does there exist another compact connected Riemann surface X together with a branched covering $f : X \to Y$ such that Λ is its branch data? If it does, we call the collection Λ *realizable* or *realized by a branched covering.*

See the classical [2, 4, 6, 7, 8, 10, 11, 13, 14, 22, 23] and the more recent [1, 12, 15, 16, 17, 18, 19, 20, 21, 24] about this problem. Here we only review some necessary background and a small part of known results which are closely related to our discussions in the sequel.

Recall that in order to be realizable, a collection Λ should satisfy the condition that its *total branching order*

$$\nu(\Lambda) := \sum_{j=1}^{k} \big(d - \mathrm{Len}(\lambda_j) \big)$$

is even. We call such a collection *compatible*. It is proved in [10, Theorem 9] and [4, Section 3] that a compatible collection is always realizable if $g(Y) > 0$. Hence, we always assume that Y is the Riemann sphere $\overline{\mathbb{C}}$ in the sequel. It turns out that a compatible collection is not always realizable in this case. We call a compatible collection an *exception* if it is not realizable. Zheng [24] found by computer all the exceptions of degree ≤ 22. Pervova-Petronio [20, 21] used a variety of techniques to give some new infinite series of exceptions, and they used dessins d'enfants to make a theoretical explanation of part of the exceptions given by Zheng [24]. Besides constructing some exceptions, Edmonds-Kulkarni-Stong [4] proposed the so-called *prime degree conjecture*, which says that each compatible collection with prime degree is always realizable, and they reduced it in the same paper to the collections with exactly three partitions. In [18, 19], Pascali-Petronio proved some results which provide strong support to this conjecture.

Characterizing branch data of all rational functions is a very deep and difficult problem, which seems far from being accessible nowadays. Hence, it is meaningful to find reasonably simple, sufficient conditions for a collection to be realizable. Besides the theorems in [18, 19, 20, 21], some of the other known results are as follows: Thom [23] showed that a compatible collection is realizable if one partition in it has length one. Edmonds-Kulkarni-Stong [4, Theorem 5.4] proved that a compatible collection with degree $d \neq 4$ is realizable when its total branching order $\geq 3(d-1)$. In addition, the exceptions with $d=4$ are precisely those with partitions $(2,2),\cdots,(2,2),(3,1)$. Moreover, Boccara [2] obtained a complete determination of the realizability of the collection Λ which consists of the following three partitions of d:

$$(a_1,\cdots,a_p),\,(b_1,\cdots,b_q),\,(m+1,1,\cdots,1).$$

He proved that Λ is realizable if and only if it satisfies one of the following two conditions:

- $\nu(\Lambda)\geq 2d$ is even.
- $\nu(\Lambda)=2d-2$ and $m<\dfrac{d}{\mathrm{GCD}(a_1,\cdots,a_p,b_1,\cdots,b_q)}$. Note that $m=p+q-2$ in this case.

Generalizing the second part of Boccara's result, we show the following

Theorem 1.1 (Main Theorem)**.** *Let* d *and* l *be two positive integers. Consider a collection*

$$\Lambda=\{(a_1,\cdots,a_p),\,(b_1,\cdots,b_q),\,(m_1+1,1,\cdots,1),\cdots,(m_l+1,1,\cdots,1)\}$$

consisting of $l+2$ *partitions of* d *such that* $(m_1,\cdots,m_l)$ *is a partition of* $p+q-2>0$*. Then there exists a rational function on* $\overline{\mathbb{C}}$ *with* Λ *as its branch data if and only if*

$$\max(m_1,\cdots,m_l)<\frac{d}{\mathrm{GCD}(a_1,\cdots,a_p,b_1,\cdots,b_q)}.$$

Remark 1.2. Recently, A. Eremenko [5] applied the main theorem to the investigation of conformal metrics of positive constant curvature on $\overline{\mathbb{C}}$ which have finitely many conical singularities and co-axial holonomy. In particular, he characterized the conical angles of such metrics and used the main theorem in the solution of Question 2, which is the rational case of Question 1, namely the main problem in [5].

We call a rational function on a compact Riemann surface a *Belyi function* if it has at most three branch points. As an application of the main theorem, we have:

Theorem 1.3. *Let* d *and* r *be two positive integers, and*

$$\Lambda=\{(a_1,\cdots,a_p),\,(b_1,\cdots,b_q),\,(c_1+1,\cdots,c_r+1,1,\cdots,1)\}$$

be a collection consisting of partitions of d *such that* $(c_1,\cdots,c_r)$ *is a partition of* $p+q-2>0$*. If*

$$\max(c_1,\cdots,c_r)<\frac{d}{\mathrm{GCD}(a_1,\cdots,a_p,b_1,\cdots,b_q)}$$

then the modified collection

$$\widetilde{\Lambda}:=\{(ra_1,\cdots,ra_p),\,(rb_1,\cdots,rb_q),\,(c_1+1,\cdots,c_r+1,1,\cdots,1)\}$$

of partitions of dr *can be realized by a Belyi function on* $\overline{\mathbb{C}}$*.*

In the remaining of the article, we give proofs of the two theorems. At the very end, we propose a conjecture concerning rational functions on a Riemann surface of positive genus, which is a possible generalization of our main theorem.

2 Proof of the main theorem

In Subsection 2.1 we prove the necessary part of the main theorem and observe that the sufficient part can be reduced to the case of $\mathrm{GCD}(a_1, \cdots, a_p, b_1, \cdots, b_q) = 1$. Moreover, we recall that by the Riemann existence theorem ([3, Theorem 2, p.49]) the sufficient part is equivalent to the existence of certain permutations in the symmetry group $S_d := S_{\{1,2,\cdots,d\}}$ associated with the collection Λ. For completeness, we give in Subsection 2.2 a proof in our own strategy for the case $l = 1$ of the main theorem which was also proved by Boccara [2]. We prove Case $l \geq 2$ of the main theorem in Subsection 2.3.

2.1 Riemann's existence theorem

At first, we prove the necessary part of the main theorem.

Proof. Suppose that there exists a rational function f on $\overline{\mathbb{C}}$ realizing the branch data Λ. Using suitable Möbius transformations if necessary, we can assume that f has the form

$$f(z) = \frac{(z-z_1)^{a_1} \cdots (z-z_p)^{a_p}}{(z-w_1)^{b_1} \cdots (z-w_q)^{b_q}} \tag{1}$$

where $z_1, \cdots, z_p, w_1, \cdots, w_q$ are $(p+q)$ distinct complex numbers. Let

$$k = \mathrm{GCD}(a_1, \cdots, a_p, b_1, \cdots, b_q).$$

Then we can write f as $f = F^k$ for some rational function on $\overline{\mathbb{C}}$. And F has branch data of the form

$$\{(a_1/k, \cdots, a_p/k), (b_1/k, \cdots, b_q/k), (m_1+1, 1, \cdots, 1), \cdots, (m_l+1, 1, \cdots, 1)\}.$$

Since F has degree d/k, we have $\max(m_1, \cdots, m_l) < d/k$ and complete the proof. $\square$

On the other hand, we claim that *if the collection*

$$\{(a_1/k, \cdots, a_p/k), (b_1/k, \cdots, b_q/k), (m_1+1, 1, \cdots, 1), \cdots, (m_l+1, 1, \cdots, 1)\}$$

is realized by a rational function, then so is

$$\{(a_1, \cdots, a_p), (b_1, \cdots, b_q), (m_1+1, 1, \cdots, 1), \cdots, (m_l+1, 1, \cdots, 1)\}.$$

Actually, by using Riemann's existence theorem and some Möbius transformations, we could assume that there exists a rational function

$$F(z) = \frac{(z-z_1)^{a_1/k} \cdots (z-z_p)^{a_p/k}}{(z-w_1)^{b_1/k} \cdots (z-w_q)^{b_q/k}},$$

where $z_1, \cdots, z_p, w_1, \cdots, w_q$ are $(p+q)$ distinct complex numbers, such that there exist exactly l branch points of F lying in $\overline{\mathbb{C}}\backslash\{0,\infty\}$, say $y_1, \cdots, y_l$, which satisfy that $y_1^k, \cdots, y_l^k$ are mutually distinct and $\lambda(y_j) = (m_j+1, 1, \cdots, 1)$ for $1 \le j \le l$. Since the power function $y \mapsto y^k$ on $\overline{\mathbb{C}}$ does not branch on $\overline{\mathbb{C}}\backslash\{0,\infty\}$, the collection

$$\{(a_1, \cdots, a_p), (b_1, \cdots, b_q), (m_1+1, 1, \cdots, 1), \cdots, (m_l+1, 1, \cdots, 1)\}.$$

is realized by the rational function F^k.

Hence, *in order to show the sufficient part of the main theorem, we may assume* $\mathrm{GCD}(a_1, \cdots, b_q) = 1$. We need to prepare some notions before giving the proof.

Definition 2.1. Let m be a non-negative integer. A vector $\alpha = (a_1, a_2, \cdots, a_{m+2})$ in $\mathbb{Z}^{m+2}$ is called a *residue vector* with $(m+2)$ components if $a_1 + a_2 + \cdots + a_{m+2} = 0$ and $a_1 a_2 \cdots a_{m+2} \neq 0$. Two residue vectors $\alpha = (a_1, \cdots, a_{m+2})$ and $\beta = (b_1, \cdots, b_{m+2})$ are called *equivalent*, denoted by $\alpha \sim \beta$, if there is a nonzero rational number μ and a permutation σ in the symmetry group S_{m+2} such that

$$\mu \cdot \alpha = (\mu a_1, \cdots, \mu a_{m+2}) = \sigma(\beta) := \big(b_{\sigma(1)}, \cdots, b_{\sigma(m+2)}\big).$$

This is an equivalence relation in the set of residue vectors with $(m+2)$ components. The degree of the residue vector α is defined to be

$$\deg\big(a_1, \cdots, a_{m+2}\big) = \frac{\sum_{a_j>0} a_j}{\mathrm{GCD}\big(a_1, \cdots, a_{m+2}\big)}.$$

We call a residue vector $\alpha = (a_1, \cdots, a_{m+2})$ *primitive* if $\mathrm{GCD}(a_1, \cdots, a_{m+2}) = 1$. Clearly, the degree of a primitive residue vector equals the sum of all its positive components. Observe also that the logarithmic differential $d(\log f) = \frac{df}{f}$ of a rational function f in (1) has residues $a_1, \cdots, a_p, -b_1, \cdots, -b_q$, which form a residue vector with degree $d/\mathrm{GCD}\big(a_1, \cdots, a_p, b_1, \cdots, b_q\big)$.

Definition 2.2. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ be a partition of a positive integer n. The *weight* of λ is defined to be

$$\mathrm{wt}(\lambda) = \max(\lambda_1, \cdots, \lambda_l)$$

Use the notions in the main theorem and denote by λ the partition $(m_1, \cdots, m_l)$ of $m = p+q-2 > 0$ and by α the residue vector $(a_1, \cdots, a_p, -b_1, \cdots, -b_q)$. Then the condition in the theorem can be concisely re-expressed as

$$\deg \alpha > \mathrm{wt}(\lambda).$$

By the Riemann existence theorem [3, Theorem 2, p.49], the sufficient part of the main theorem is equivalent to the following

Theorem 2.3. *Under the assumptions of the main theorem, if* $\deg \alpha > \mathrm{wt}(\lambda)$*, then there exist* $(l+2)$ *permutations* $\tau_1, \tau_2, \sigma_1, \cdots, \sigma_l$ *in the symmetry group* $S_d = S_{\{1,2,\cdots,d\}}$ *of* $\{1, 2, \cdots, d\}$ *such that*

- $\tau_1\tau_2\sigma_1\cdots\sigma_l = e$, *where* e *is the unit in* S_d *and permutations are multiplied from right to left;*
- τ_1 *has the type of* $a_1^1 a_2^1 \cdots a_p^1$, τ_2 *of* $b_1^1 b_2^1 \cdots b_q^1$ *and* σ_k *of* $(1+m_k)^1 1^{d-m_k-1}$ *for all* $k = 1, \cdots, l$;
- *The subgroup* $\langle \tau_1, \tau_2, \sigma_1, \cdots, \sigma_l \rangle$ *of* S_d *acts transitively on* $\{1, 2, \cdots, d\}$.

The following lemma will be useful later, which follows from the Riemann existence theorem and the argument in the first three paragraphs of this subsection.

Lemma 2.4. *For each* $m \geq 0$, *proving Theorem 2.3 is equivalent to proving its variant where* α *is primitive. We call the latter the primitive version of Theorem 2.3.*

We shall prove Theorem 2.3 and its primitive version simultaneously by induction on $m = p + q - 2$ in the sequel of this section. The proof will be divided into two parts regarding $l = 1$ or $l > 1$.

Without loss of generality, we may assume the residue vector satisfies the following order assumption:
Order Assumption (OA) $a_1 \leq a_2 \leq \cdots \leq a_p$, $b_1 \geq \cdots \geq b_q$ and $1 \leq p \leq q$.

2.2 Branch data with three partitions

For the completeness of the manuscript, we include in this subsection the proof of the well known case where $l = 1$ of Theorem 2.3 (see [2]), whose strategy we also use while proving in Subsection 2.3 the $l > 1$ case of the theorem.

Here we first make a recall of the case $l = 1$ of Theorem 2.3.

Proposition 2.5 (Case $l = 1$ of Theorem 2.3). *Let* $\alpha = (a_1, \cdots, a_p, -b_1, \cdots, -b_q)$ *be a residue vector and* $\lambda = (m)$ *be a partition of* $m = p + q - 2 > 0$ *such that* $\deg \alpha > \mathrm{wt}(\lambda) = m$. *Then there exist three permutations* τ_1, τ_2, σ_1 *in* S_d *satisfying the following three properties*:

- $\tau_1\tau_2\sigma_1 = e$;
- τ_1 *has the type of* $a_1^1 a_2^1 \cdots a_p^1$, τ_2 *of* $b_1^1 b_2^1 \cdots b_q^1$ *and* σ_1 *of* $(1+m)^1 1^{d-m-1}$;
- *The subgroup* $\langle \tau_1, \tau_2, \sigma_1 \rangle$ *of* S_d *acts transitively on* $\{1, 2, \cdots, d\}$.

Lemma 2.6. *Let* $\alpha = (a_1, \cdots, a_p, -b_1, \cdots, -b_q)$ *be a residue vector with* $\deg \alpha > m = p + q - 2$. *If* $m > 0$, *we have* $a_p > b_q$.

Proof. If $a_p < b_q$, then it follows from the order assumption (OA) that $\sum_{i=1}^{p} a_i < \sum_{j=1}^{q} b_j$, contradicting the definition of residue vector. If $a_p - b_q = 0$, then by OA and $m = p + q - 2 > 0$ we have $p = q \geq 2$ and $a_i = b_j$ for all $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$. Since $p = \deg \alpha > m = p + q - 2 = 2p - 2 = 2q - 2$, we obtain $p = q = 1$, which contradicts $p + q > 2$. □

To prove Proposition 2.5, we need to use the following lemma, where we propose a new concept, called *contraction of a residue vector.*

Lemma 2.7. *Under the assumptions of Proposition 2.5, there exist* $i_0 \in \{1,2,\cdots,p\}$ *and* $j_0 \in \{1,2,\cdots,q\}$ *such that*

$$\widehat{\alpha} = \Big(a_1,\cdots,a_{i_0-1},a_{i_0}-b_{j_0},a_{i_0+1},\cdots,a_p,\ -b_1,\cdots,\widehat{(-b_{j_0})},\cdots,-b_q\Big)$$

is a residue vector with $\deg\widehat{\alpha} > m-1$, *where* $a_{i_0}-b_{j_0}>0$ *and the hat over term* $(-b_{j_0})$ *means that* $(-b_{j_0})$ *is removed. Note that the number of components of* $\widehat{\alpha}$ *is one less than that of* α. *We call* $\widehat{\alpha}$ *a* **contraction** *of* α.

Proof. Without loss of generality, we assume α is primitive. If not, we may replace it by a primitive residue vector equivalent to it.

By OA and Lemma 2.6, we have $q \geq 2$ and $a_p - b_q > 0$, and then we obtain another residue vector

$$\beta_1 := \big(a_1,\cdots,a_{p-1},\,a_p-b_q,\ -b_1,\cdots,-b_{q-1}\big)$$

with $(m-1)$ components. We divide the proof into the following three steps.

Step 1. Assume $q=2$. Then $m=p+q-2=p$ and $\deg\beta_1 \geq p$, so β_1 is a contraction of α. We assume that $q>2$ in the sequel of the proof.

Step 2. Suppose that β_1 is primitive. Then $\deg\beta_1 > (m-1) = (p+q-3)$, so it gives a contraction of α. Indeed, if not, then by OA we have

$$2q-2 \geq p+q-2 > \deg\beta_1 = -b_q + \sum_{i=1}^{p} a_i = -b_q + \sum_{j=1}^{q} b_j = \sum_{j=1}^{q-1} b_j.$$

thus $b_{q-1}=b_q=1$ and $b_{q-2}\leq 2$. Moreover, if $b_{q-2}=2$, then $b_1=\cdots=b_{q-2}=2$. Therefore

$$m+1 = p+q-1 > 1+\sum_{j=1}^{q-1} b_j = \sum_{j=1}^{q} b_j = \sum_{i=1}^{p} a_i = \deg\alpha,$$

which contradicts the assumption on the degree of α. We assume that β_1 is **not** primitive in the left part of the proof.

Step 3. If $\deg\beta_1 > (m-1)$, then we are done. So without loss of generality, suppose $\deg\beta_1 \leq (m-1) = (p+q-3)$. Let $D>1$ be the greatest common divisor of all components of β_1. Then, by OA and the definition of degree, we obtain that

$$2q-2 \geq p+q-2 > \deg\beta_1 = \frac{a_p-b_q}{D} + \sum_{i=1}^{p-1}\frac{a_i}{D} = \sum_{j=1}^{q-1}\frac{b_j}{D}.$$

Hence $b_{q-1}=D$, $D|b_j$ for all $j=1,\cdots,q-2$, $D|a_i$ for all $i=1,\cdots,p-1$, and $D|(a_p-b_q)$. Consequently, $a_p \geq b_q+D > D = b_{q-1}$ and we obtain another residue vector

$$\beta_2 := \big(a_1,\cdots,a_{p-1},\,a_p-b_{q-1},\ -b_1,\cdots,-b_{q-2},-b_q\big)$$

with $(m-1)$ components. If $\deg\beta_2 > (m-1)$, then we are done. So suppose $\deg\beta_2 \le (m-1)$. We divide our discussion into two cases: β_2 is primitive and otherwise.

(a) First, suppose that β_2 is primitive. Since $\deg\beta_2 < m$, we find that $b_q = 1$ and $b_{q-2} \le 2$ by a same argument for β_1 in step 2. As a result $2 \ge b_{q-2} \ge b_{q-1} = D > 1$, and thus $b_{q-2} = b_{q-1} = D = 2$. By the same argument for β_1, we deduce that $b_1 = \cdots = b_{q-1} = D = 2$ and $a_1, \cdots, a_{p-1}$ are even. In particular, $a_1 - b_q = a_1 - 1 > 0$. Hence

$$\widehat{\alpha} := \left(a_1 - b_q, a_2, \cdots, a_p,\ -b_1, \cdots, -b_{q-1}\right)$$

is a primitive residue vector with $(m-1)$ components. Moreover, $\deg\widehat{\alpha} = 2q-2 \ge p+q-2 = m > m-1$ and thus $\widehat{\alpha}$ is a contraction of α.

(b) Second, if β_2 is not primitive. A similar argument as in the β_1 not primitive case gives that $b_q = E$ and $b_{q-2} = E$ or $2E$, where E is the greatest common divisor of the components of β_2. By OA, $E = b_q \le b_{q-1}$. If $D = b_{q-1} = b_q = E$, then $E|(a_p - b_{q-1})$ implying that E divides all the components of α, this contradicts the primitive property of α. Hence, $E = b_q < b_{q-1} \le b_{q-2}$. Also since $b_{q-2} = E$ or $2E$, we have $b_{q-2} = 2E$ and $1 \le b_{q-2}/b_{q-1} < 2$. Recall that $b_{q-1} = D$ and $D|b_{q-2}$, so $D = b_{q-1} = b_{q-2} = 2E = 2b_q$ and $E|a_p$, which gives the same contradiction as above.

□

We shall prove Proposition 2.5 by induction on m. To this end, we need two lemmas.

Lemma 2.8. *Let σ and τ be two permutations in S_d for some positive integer d such that σ is a cycle of length greater than 1 and τ has form $\nu_1\nu_2\cdots\nu_r$, where ν_i's are mutually disjoint cycles of length d_i for $i = 1,2,\cdots,r$ and $d_1 + d_2 + \cdots + d_r = d$. We call ν_j's* **cycle factors** *of τ. Then the following two conditions are equivalent*:

(i) The subgroup $\langle\sigma,\tau\rangle$ of S_d acts transitively on the set $\{1,2,\cdots,d\}$.

(ii) Each cycle factor ν_i of τ intersects the cycle σ in the sense that the subset of $\{1,2,\cdots,d\}$ associated with ν_i intersects that associated with the cycle σ.

Proof. (i) $\Rightarrow$ (ii) We prove the intersection by contradiction. Suppose that there exists a cycle factor ν_i of τ not intersecting σ. Then ν_i is a cycle factor of $\tau\sigma$. Therefore, the subset associated with ν_i forms an orbit under the action of the subgroup $\langle\tau,\sigma\rangle$ of S_d on the set $\{1,2,\cdots,d\}$. Since this action is transitive, the cycle ν_i has length d. Hence, ν_i must intersect σ, contradiction!

(ii) $\Rightarrow$ (i) For each $1 \le i \le r$, we choose a number x_i lying in both ν_i and σ. Consider the action of the subgroup $\langle\tau,\sigma\rangle$ on the set $\{1,\cdots,d\}$. Then $x_1,\cdots,x_r$ lie in the same orbit of this action by assumption. Moreover, for each $1 \le i \le r$, x_i belongs to the same orbit with any other numbers in the cycle ν_i, and the numbers in $\nu_1,\cdots,\nu_r$ exhaust all the numbers in $\{1,\cdots,d\}$. Therefore, the action has a single orbit. □

Remark 2.9. *We consider a more general case of the part of (i)⇒(ii) in the above lemma by replacing σ by $\sigma_1, \sigma_2, \cdots, \sigma_l$, all of which are cycles in S_d of length greater than 1. Suppose that the subgroup $\langle \sigma_1, \sigma_2, \cdots, \sigma_l, \tau \rangle$ of S_d acts transitively on the set $\{1, 2, \cdots, d\}$. Then it follows from a similar argument as in the preceding proof that each cycle factor of τ must intersect some σ_k for $1 \le k \le l$. However, for $l \ge 2$, the converse fails in general. For example, the subgroup of S_4 generated by $(12), (34)$ and $(12)(34)$ acts on the set $\{1, 2, 3, 4\}$ with the two orbits of $\{1, 2\}$ and $\{3, 4\}$.*

Lemma 2.10. *Under the assumptions of Proposition 2.5, suppose that there exist permutations τ_1, τ_2, σ_1 in S_d such that $\tau_1\tau_2\sigma_1 = e$ and they have types of $a_1^1 a_2^1 \cdots a_p^1$, $b_1^1 b_2^1 \cdots b_q^1$ and $(1+m)^1 1^{d-m-1}$, respectively. Then the subgroup $\langle \tau_1, \tau_2, \sigma_1 \rangle$ acts transitively on the set $\{1, 2, \cdots, d\}$ if and only if each cycle factor of τ_1 and τ_2 intersects the $(m+1)$-cycle σ_1.*

Proof. Since $\tau_1\tau_2\sigma_1 = e$, the following three subgroups coincide with each other:

$$\langle \tau_1, \sigma_1 \rangle = \langle \tau_1, \tau_2, \sigma_1 \rangle = \langle \tau_2, \sigma_1 \rangle.$$

The result follows from Lemma 2.8. □

Now we give the proof of Proposition 2.5.

Proof. We argue by induction on $m = p + q - 2 \ge 0$. It holds trivially as $m = 0$, which is equivalent to $p = q = 1$. Assume $p + q > 2$ in what follows. By Lemma 2.7, there exists a contraction $\widehat{\alpha}$ of α. Without loss of generality, we assume that the contraction $\widehat{\alpha}$ has the form

$$\widehat{\alpha} = \left(a_1, \cdots, a_{p-1}, a_p - b_q,\ -b_1, \cdots, -b_{q-1}\right).$$

By the induction hypothesis, there exist in S_{d-b_q} a permutation $\tau_2 = \nu_1\nu_2 \cdots \nu_{q-1}$ of type $b_1^1 b_2^1 \cdots b_{q-1}^1$ and a cycle σ_1 of length $(p+q-2)$ such that $\tau_2\sigma_1 = \nu_1\nu_2 \cdots \nu_{q-1}\sigma_1$ has type of $a_1^1 \cdots a_{p-1}^1 (a_p - b_q)^1$ and the subgroup generated by τ_2 and σ_1 acts transitively on the set $\{1, 2, \cdots, d - b_p\}$, where ν_j is a cycle factor of length b_j of τ_2 for all $j = 1, 2, \cdots, q-1$. In addition, by the induction hypothesis, $\tau_2\sigma_1$ has the form

$$\nu_1\nu_2 \cdots \nu_{q-1}\sigma_1 = \mu_1\mu_2 \cdots \mu_p$$

where μ_i's are mutually disjoint cycles for $1 \le i \le p$, the length of μ_p is $(a_p - b_q)$ and μ_k has the length a_k for $1 \le k \le p-1$.

By Lemma 2.10, we can choose an integer $1 \le x \le (d - b_q)$ lying in both μ_p and σ_1. Choose in S_d a cycle ν_q of length b_q such that ν_q does not intersect ν_j for all $j = 1, \cdots, q-1$, for example $\nu_q = (d, d-1, \cdots, d - b_q + 1)$ and pick an integer y in ν_q. Then $\nu_1\nu_2 \cdots \nu_{q-1}\nu_q$ has type $b_1^1 b_2^1 \cdots b_q^1$ and $\widetilde{\sigma_1} := \sigma_1(x, y)$ is a cycle of length $(p+q-1)$. The subgroup $\langle \nu_1\nu_2 \cdots \nu_{q-1}\nu_q, \widetilde{\sigma_1} \rangle$ of S_d acts transitively on the set $\{1, 2, \cdots, d\}$ by Lemma 2.10. We also observe that $\tilde{\mu}_p := \nu_q \mu_p (x, y)$ is a cycle of length a_p since ν_q does not intersect μ_p and x and y lie in μ_p and ν_q, respectively. Moreover, since ν_q does not intersect μ_i for all $1 \le i \le p-1$, $\tilde{\mu}_p$ does not intersect μ_i for all $1 \le i \le p-1$. So we have

$$\begin{aligned}(\nu_1\nu_2\cdots\nu_{q-1}\nu_q)\,\widetilde{\sigma_1} &= \nu_q(\nu_1\nu_2\cdots\nu_{q-1}\sigma_1)(x,y) = \nu_q(\mu_1\mu_2\cdots\mu_p)(x,y)\\ &= (\mu_1\mu_2\cdots\mu_{p-1})(\nu_q\mu_p(x,y)) = \mu_1\mu_2\cdots\mu_{p-1}\tilde{\mu}_p.\end{aligned}$$

Hence, we see that the three permutations $(\mu_1\mu_2\cdots\mu_{p-1}\tilde{\mu}_p)^{-1}$, $\nu_1\nu_2\cdots\nu_{q-1}\nu_q$ and $\widetilde{\sigma_1}$ in S_d satisfy the three properties listed in Proposition 2.5. □

2.3 Branch data with more than three partitions

We prove Case $l \ge 2$ of Theorem 2.3. At first we deal with residue vectors with components only ± 1.

Proposition 2.11. *Let* D *be a positive integer. Assume* $\alpha = (\underbrace{1,1,\cdots,1}_{D+1},\underbrace{-1,-1,\cdots,-1}_{D+1})$. *Then for each partition* $\lambda = (m_1, m_2, \cdots, m_l)$ *of* $2D$ *such that* $\mathrm{wt}(\lambda) < \deg\alpha = D+1$, *there exist* l *permutations* $\sigma_1, \cdots, \sigma_l$ *in* S_{D+1} *such that the following properties hold*

- $\sigma_1 \cdots \sigma_l = e$;
- σ_j*'s are cycles of length* (m_j+1);
- $\langle \sigma_1, \cdots, \sigma_l \rangle$ *acts transitively on the set* $\{1, \cdots, D+1\}$.

Proof. It is easy to see that $l \ge 2$. We divide the proof by considering three cases.

Case 1 If $l = 2$, we know that $m_1 = m_2 = D$ since $2D = m_1 + m_2$ and $m_1, m_2 \le D$. Then we are done by choosing

$$\sigma_1 = (1, 2, \cdots, D+1), \quad \sigma_2 = \sigma_1^{-1}.$$

Case 2 If $l = 3$. Since $m_1, m_2, m_3 \le D$ and $m_1 + m_2 + m_3 = 2D$, We have $m_1 + m_2 \ge D$. Choosing

$$\begin{aligned}&\sigma_1 = (1, 2, \cdots, m_1+1) \quad \text{and}\\ &\sigma_2 = (1, \underbrace{m_1+1, m_1, m_1-1, \cdots, m_1+m_3-D+2}_{m_1+m_2-D}, \underbrace{m_1+2, m_1+3, \cdots, D+1}_{D-m_1}),\end{aligned}$$

we obtain

$$\sigma_1\sigma_2 = \begin{cases} (\underbrace{m_1+2, m_1+3, \cdots, D+1}_{D-m_1}, \underbrace{2, 3, \cdots, m_1+m_3-D+2}_{m_1+m_3-D+1}), & \text{if} \quad m_3 < D,\\ (\underbrace{m_1+2, m_1+3, \cdots, D+1}_{D-m_1}, \underbrace{2, 3, \cdots, m_1+1, 1}_{m_1+1}), & \text{if} \quad m_3 = D.\end{cases}$$

Then the permutations σ_1, σ_2 and $(\sigma_1\sigma_2)^{-1}$ satisfy the three properties.

Case 3 Suppose $l > 3$. Since $m_1, \cdots, m_l \le D$, we can choose $1 < r \le l$ such that $m_1 + m_2 + \cdots + m_{r-1} \le D$ and $m_1 + m_2 + \cdots + m_r > D$.

Subcase 3.1 Suppose that $r < l$. Choosing

$$\begin{aligned} \sigma_1 &= (1, 2, \cdots, m_1 + 1), \\ \sigma_2 &= (m_1 + 1, m_1 + 2, \cdots, m_1 + m_2 + 1), \\ &\cdots \\ \sigma_{r-1} &= (m_1 + \cdots + m_{r-2} + 1, \cdots, m_1 + \cdots + m_{r-1} + 1), \end{aligned}$$

we obtain

$$\tau_1 := \sigma_1 \sigma_2 \cdots \sigma_{r-1} = (1, 2, 3, \cdots, m_1 + \cdots + m_{r-1} + 1)$$

By Case 2, there exist two cycles τ_2 and τ_3 which have length $1 + m_r$ and $m_{r+1} + \cdots + m_l + 1 < D + 1$, respectively, such that $\tau_1\tau_2\tau_3 = e$. As the construction of $\sigma_1, \cdots, \sigma_{r-1}$, we can find $\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_l$ directly such that σ_j has the type of $(1 + m_j)^1 1^{D-m_j}$ for $r + 1 \le j \le l$ and $\sigma_{r+1} \cdots \sigma_l = \tau_3$. Therefore the l cycles of $\sigma_1, \cdots, \sigma_{r-1}$, $\sigma_r := \tau_2$, $\sigma_{r+1}, \cdots, \sigma_l$ satisfy the three properties.

Subcase 3.2 Suppose $r = l > 3$. Since $m_1 + \cdots + m_l = 2D$ and $\max\left(m_1, \cdots, m_l\right) \le D$, we have $m_1 + \cdots + m_{l-1} = m_l = D$. Then the problem can be reduced to Case 1 by a similar argument as above.

□

To complete the proof of Theorem 2.3, we need the following lemma and its two corollaries.

Lemma 2.12. *Let Γ be a subgroup of $S_d = S_{\{1,2,\cdots,d\}}$ for some integer $d > 1$ and $\theta \in S_d$ a cycle of length greater than 1. Assume that the subgroup G generated by Γ and θ acts transitively on the set $\{1, 2, \cdots, d\}$. Then, for each number $1 \le x \le d$ not contained in θ, the Γ-orbit Γx of x intersects θ.*

Proof. We argue by contradiction. Suppose that the orbit Γx does not intersect θ. Take an arbitrary permutation ξ in G. We can express it as

$$\xi = \pi_1 \pi_2 \cdots \pi_s$$

where either $\pi_i = \theta$ or $\pi_i \in \Gamma$. Let ξ' be the permutation obtained from the product $\pi_1\pi_2 \cdots \pi_s$ by removing all those π_i's satisfying $\pi_i = \theta$. Then $\xi' \in \Gamma$. Since each number not contained in θ is a fixed point of θ, by the hypothesis of the contradiction argument, we find that $\xi(x) = \xi'(x)$ is not contained in θ. Since $\xi \in G$ has been chosen arbitrarily, the orbit Gx does not intersect θ, which contradicts that G acts transitively on $\{1, 2, \cdots, d\}$. □

As an application of the above lemma, we have

Corollary 2.13. *Let $\gamma_1,\gamma_2,\cdots,\gamma_l,\theta$ be $(l+1)$ permutations in S_{d-1} for some integer $d>2$ and $\theta=(x_1,x_2,\cdots,x_n)$ a cycle in S_{d-1} of length $n>1$. Suppose that the subgroup $\langle\gamma_1,\gamma_2,\cdots,\gamma_l,\theta\rangle$ acts transitively on the set $\{1,2,\cdots,d-1\}$. Then so does the subgroup $\langle\gamma_1,\gamma_2,\cdots,\gamma_l,\widetilde{\theta}\rangle$ on $\{1,2,\cdots,d\}$, where $\widetilde{\theta}:=(x_1,\cdots,x_n,d)$ is a cycle of length $(n+1)$ in S_d.*

Proof. By Lemma 2.12, we can see that for each number $1\le x\le d-1$ not contained in θ, there exists $\gamma\in\Gamma:=\langle\gamma_1,\cdots,\gamma_l\rangle$ such that $\gamma(x)$ is contained in θ. Hence, the action of $\langle\gamma_1,\gamma_2,\cdots,\gamma_l,\widetilde{\theta}\rangle$ on $\{1,2\cdots,d\}$ has only one orbit. □

Similarly, we obtain

Corollary 2.14. *Let $\gamma_1,\gamma_2,\cdots,\gamma_l,\theta$ be permutations in S_d for some integer $d>2$ and $\theta=(x_1,x_2,\cdots,x_n)$ a cycle of length $1<n<d$. Suppose that the subgroup $\langle\gamma_1,\gamma_2,\cdots,\gamma_l,\theta\rangle$ acts transitively on the set $\{1,2,\cdots,d\}$. Then so does the subgroup $\langle\gamma_1,\gamma_2,\cdots,\gamma_l,\widetilde{\theta}\rangle$ of S_d on the set $\{1,2,\cdots,d\}$, where $\widetilde{\theta}=(x_1,\cdots,x_n,y)\in S_d$ is a cycle of length $(n+1)$ with $y\in\{1,2,\cdots,d\}\backslash\{x_1,\cdots,x_n\}$.*

Now we arrive at proving the case $l\ge 2$ of Theorem 2.3.

Proof. By Lemma 2.4 we could also assume that the residue vector $\alpha=(a_1,\cdots,a_p,-b_1,\cdots,-b_q)$ is primitive so that $\deg\alpha=d=a_1+\cdots+a_p=b_1+\cdots+b_q$.

Part I Suppose $d\ge m+1=p+q-1$. By Proposition 2.5 there exist τ_1,τ_2, σ such that
(1) $\tau_1\tau_2\sigma=e$;
(2) τ_1 has type of $a_1^1a_2^1\cdots a_p^1$, τ_2 of $b_1^1b_2^1\cdots b_q^1$, σ of $(1+m)^1 1^{d-m-1}$;
(3) the subgroup $\langle\tau_1,\tau_2,\sigma\rangle$ acts transitively on $\{1,2,\cdots,d\}$.
Assume that $\sigma=(1,2,\cdots,m+1)$ for simplicity of notion. We are done by choosing

$$\begin{aligned}
&\sigma_1=(1,2,\cdots,m_1+1),\\
&\sigma_2=(m_1+1,m_1+2,\cdots,m_1+m_2+1),\\
&\cdots\cdots\\
&\sigma_l=(m_1+\cdots+m_{l-1}+1,m_1+\cdots+m_{l-1}+2,\cdots,m_1+\cdots+m_l+1),
\end{aligned}$$

Part II Suppose $d=\deg\alpha\le m=p+q-2$. We first reduce the problem to the two cases that $l=2$ and $l=3$, then we prove these two cases by using the contraction argument and the induction argument similar as the proof of Proposition 2.5. The details given as follows form the left part of this section.

By OA, we have that $d=\sum_{j=1}^{q}b_j\ge q\ge\frac{p+q}{2}$. Since $d=\deg\alpha>\mathrm{wt}(\lambda)=\max\left(m_1,\cdots,m_l\right)$, the partition λ of $m=(p+q-2)$ has at least two components, i.e. $l>1$.

At first we show that

Claim 1: *the problem can be reduced to the two cases where the partitions of m have two and three components, respectively.*

Proof of Claim 1

Suppose that Part II holds for each partition of m which has three components and has weight less than d. Then we shall prove that so does Part II for each partition $(m_1,\cdots,m_l)$ of m such that $l>3$ and its weight is less than d. To this end, since $m_1,\cdots,m_l<d$, we can choose $1<r\le l$ such that

$$m_1+\cdots+m_{r-1}<d \quad \text{and} \quad m_1+\cdots+m_r\ge d.$$

We shall define a new partition, called λ', with three components as following.

• Suppose $r<l$. Then we consider the partition $\lambda':=(m_1',m_2',m_3')$ of m, where m_1',m_2',m_3' are defined by

$$m_1':=m_1+\cdots+m_{r-1}<d, \quad m_2':=m_r<d, \quad m_3':=m_{r+1}+\cdots+m_l.$$

Moreover, since $d\ge\frac{p+q}{2}$, we observe that

$$m_3'=p+q-2-(m_1'+m_2')<(p+q-d)-1\le\frac{p+q}{2}-1\le d-1.$$

• Suppose $r=l>3$. Then we choose the partition $\lambda':=\left(m_1+\cdots+m_{l-2},m_{l-1},m_l\right)$ of m, which has weight less than d.

Since the partition λ' has weight less than d, and we have assumed the validity of Case $l=3$ for α and λ', we can find in S_d the following five permutations $\tau_1,\tau_2,\sigma_1',\sigma_2',\sigma_3'$ which satisfy the three properties. By a similar construction as Case 3 in the proof of Proposition 2.11, we know that the proposition holds for the partition $(m_1,\cdots,m_l)$. Therefore, we have justified the claim.

We always assume $l=2$ or 3 in the left part of the proof.

Recall that OA states that $1\le p\le q, a_1\le a_2\le\cdots\le a_p, b_1\ge b_2\ge\cdots\ge b_q$. Without loss of generality, we further assume $m_1\le m_2\le\cdots\le m_l$ for the partition λ. We call these two assumptions OA in the sequel by an abuse of notation.

By OA, we can see that if $a_p=1$, then $a_1=a_2=\cdots=a_p=b_1=b_2=\cdots=b_q=1$ and we are done by Proposition 2.11. We may assume that $a_p>1$ in the left part of the proof. Since $d\le m$, we have $\sum_{j=1}^{q}b_j\le p+q-2\le 2q-2$. Then, we have $b_{q-1}=b_q=1$. Since $m_1\le\cdots\le m_l$ and $a_p>1$, $\deg\alpha>\mathrm{wt}(\lambda)$ and $l=2$ or 3, we observe that m_l is always greater than 1 except when $\alpha=(1,2,-1,-1,-1)$ and $\lambda=(1,1,1)$, for which Part II holds trivially. We may assume that $m_l>1$ in the left part of the proof. In order to do induction on m, we choose the partition

$$\lambda_1:=\left(m_1,\cdots,m_{l-1},(m_l-1)\right)$$

of $(m-1)$ and the residue vector

$$\widehat{\alpha}=\left(a_1,a_2,\cdots,a_{p-1},a_p-b_q,-b_1,-b_2,\cdots,-b_{q-1}\right)$$

with $(p+q-1)=(m+1)$ components. Then, since $b_{q-1}=b_q=1$, $\widehat{\alpha}$ is primitive and

$$\deg\widehat{\alpha}=-1+\deg\alpha.$$

Then we make the following

CLAIM 2: $\mathrm{wt}(\lambda_1) < \deg\widehat{\alpha}$. Hence we may think of $(\widehat{\alpha}, \lambda_1)$ as a contraction of (α, λ) and we shall use the former to do induction argument.

PROOF OF CLAIM 2

(i) If $m_1 \leq \cdots \leq m_{l-1} = m_l$, then $\mathrm{wt}(\lambda_1) = m_{l-1} = m_l$ and $2m_l \leq m = p+q-2 \leq 2q-2$, i.e. $m_l \leq q-1$. If $\deg\widehat{\alpha} > q-1$, then we are done. Assume $\deg\widehat{\alpha} = q-1$, which implies that $b_1 = \cdots = b_q = 1$ and $p < q$ since $a_p > 1$. Then $2m_l \leq m = p+q-2 < 2q-2$ and $m_l < q-1$. Hence $\deg\widehat{\alpha} = q-1 > m_l = \mathrm{wt}(\lambda_1)$.

(ii) If $m_{l-1} < m_l$, then $\mathrm{wt}(\lambda_1) = m_l - 1 = \mathrm{wt}(\lambda) - 1 < \deg\alpha - 1 = \deg\widehat{\alpha}$. The claim is proved.

Then in the left part of the proof we use the induction on m to prove that Part II of Theorem 2.3 holds provided that l equals either 2 or 3. We observe that the initial case of $m = 2$ holds trivially, where $\alpha = (1,1,-1,-1)$ and $\lambda = (1,1)$.

– Suppose $l = 2$. We recall our setting as follows. Take a primitive residue vector

$$\alpha = (a_1, \cdots, a_p, -b_1, \cdots, -b_q)$$

and a partition $\lambda = (m_1, m_2)$ of $m = p+q-2 \geq 3$ such that $d = \deg\alpha \leq m$, and α and λ satisfy OA. Then we have $2 \leq m_2 = \mathrm{wt}(\lambda) < d$ and $b_{q-1} = b_q = 1$. Then, by claim 2, we could take another primitive residue vector

$$\widehat{\alpha} = (a_1, a_2, \cdots, a_{p-1}, a_p - b_q, -b_1, -b_2, \cdots, -b_{q-1})$$

and another partition $\lambda_1 = (m_1, m_2 - 1)$ such that $\mathrm{wt}(\lambda_1) < \deg\widehat{\alpha} = d-1$. By the induction hypothesis, there exist in $S_{d-1} = S_{\{1,2,\cdots,d-1\}}$ a permutation of type $b_1^1 \cdots b_{q-1}^1$, called τ_2, and two cycles σ_1, σ_2 of length $(1+m_1)$, m_2, respectively, such that the subgroup $\langle\sigma_1, \sigma_2, \tau_2\rangle$ of S_{d-1} acts transitively on $\{1, 2, \cdots, d-1\}$ and the permutation

$$\tau_1 := \tau_2\sigma_1\sigma_2$$

has the type of $a_1^1 \cdots a_{p-1}^1 (a_p - b_q)^1$. We re-express τ_1 by $\tau_1 = \mu_1 \cdots \mu_p$ such that μ_j's are its cycle factors and μ_j has length a_j for $1 \leq j < p$ and μ_p has length $(a_p - b_q)$. Since $b_q = 1$, we can think of τ_2 as a permutation in S_d with the type of $b_1^1 \cdots b_{q-1}^1 b_q^1$. By Remark 2.9, the cycle factor μ_p intersects either σ_1 or σ_2. We shall divide the left part of the proof of Case $l = 2$ into the following two steps.

Step 2.1 Suppose that σ_2 intersects μ_p. Then we pick a number x in both σ_2 and μ_p and define $\widetilde{\sigma}_2 := \sigma_2(x, d)$ and $\widetilde{\mu}_p := \mu_p(x, d)$. Then, since $b_q = 1$, $\widetilde{\sigma}_2$ and $\widetilde{\mu}_p$ are cycles in S_d of length $(1+m_2)$ and a_p, respectively. Moreover, we have

$$\tau_2\sigma_1\widetilde{\sigma}_2 = \mu_1\mu_2 \cdots \mu_{p-1}\widetilde{\mu}_p.$$

Since $\langle\tau_2,\sigma_1,\sigma_2\rangle$ acts transitively on the set $\{1,2,\cdots,d-1\}$, the action of $\langle\tau_2,\sigma_1,\widetilde{\sigma}_2\rangle$ on the set $\{1,2,\cdots,d\}$ has a single orbit by Corollary 2.13. Therefore, the following four permutations $\left(\mu_1\mu_2\cdots\mu_{p-1}\widetilde{\mu}_p\right)^{-1}, \tau_2, \sigma_1, \widetilde{\sigma}_2$ satisfy the three properties.

Step 2.2 Suppose that σ_2 does not intersect μ_p. Then σ_1 intersects μ_p. Choose a number x in both σ_1 and μ_p. Then

$$\begin{aligned}\tau_2\sigma_1\sigma_2(x,d) &= \mu_1\cdots\mu_p(x,d)\\ &= \mu_1\cdots\mu_{p-1}\widetilde{\mu}_p \quad \text{where} \quad \widetilde{\mu}_p := \mu_p\,(x,d)\\ &=: \widetilde{\tau}_1\end{aligned}$$

where $\widetilde{\tau}_1$ has the type of $a_1^1\cdots a_p^1$. Meanwhile, since σ_2 does not intersect (x,d), we have

$$\widetilde{\tau}_1 = \tau_2\sigma_1\sigma_2(x,d) = \tau_2\sigma_1(x,d)\sigma_2 =: \tau_2\sigma_1'\sigma_2,$$

where $\sigma_1' := \sigma_1\,(x,d)$ is a cycle of length (m_1+2). Observe that σ_1' intersects σ_2. For, otherwise, σ_1 does not intersect σ_2 and there are (m_1+m_2+1) different numbers appearing in both σ_1 and σ_2, lying in S_{d-1}. Hence, we have

$$(d-1) \geq 1+m_1+m_2 = 1+m \geq 1+d,$$

contradiction!

Take the smallest positive integer s such that $y := \left(\sigma_1'\right)^{-s}(x)$ is contained in σ_2. Since x is not contained in σ_2, by the minimal property of s, $\left(\sigma_1'\right)(y)$ is not contained in σ_2. We can rewrite σ_1' as

$$\sigma_1' = (x_1,x_2,\cdots,x_{m_1},y,\sigma_1'(y)) = \widetilde{\sigma}_1\left(y,\sigma_1'(y)\right) \quad \text{with} \quad \widetilde{\sigma}_1 := (x_1,\cdots,x_{m_1},y).$$

Then we have

$$\sigma_1'\sigma_2 = \widetilde{\sigma}_1\left(y,\sigma_1'(y)\right)\sigma_2 = \widetilde{\sigma}_1\widetilde{\sigma}_2 \quad \text{where} \quad \widetilde{\sigma}_2 := \left(y,\sigma_1'(y)\right)\sigma_2,$$

and

$$\widetilde{\tau}_1 = \tau_2\sigma_1'\sigma_2 = \tau_2\widetilde{\sigma}_1\widetilde{\sigma}_2,$$

where $\widetilde{\tau}_1$ and τ_2 have types of $a_1^1\cdots a_p^1$ and $b_1^1\cdots b_{q-1}^1 b_q^1$, respectively, and the two cycles $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ have lengths of $1+m_1$ and $1+m_2$, respectively. Since $\langle\tau_2,\widetilde{\tau}_1,\sigma_2\rangle = \langle\tau_2,\sigma_1',\sigma_2\rangle$ acts transitively on the set $\{1,2,\cdots,d\}$ by Corollary 2.13, so does $\langle\tau_2,\widetilde{\tau}_1,\widetilde{\sigma}_2\rangle$ on the same set by Corollary 2.14. Therefore, the following four permutations $\left(\widetilde{\tau}_1\right)^{-1}, \tau_2, \widetilde{\sigma}_1, \widetilde{\sigma}_2$ satisfy the three properties.

– Suppose $l=3$. We may further assume that $m_1+m_2 \geq d$. Otherwise, replacing λ by $\lambda' = (m_1+m_2, m_3)$ and using the similar reduction argument as above, we can reduce the problem to the known case of $l=2$.

By the induction hypothesis, there exist in S_{d-1} a permutation τ_2 of type $b_1^1 \cdots b_{q-1}^1$ and three cycles $\sigma_1, \sigma_2, \sigma_3$ of length $1+m_1, 1+m_2, m_3$, respectively, such that the subgroup $\langle \sigma_1, \sigma_2, \sigma_3, \tau_2 \rangle$ acts transitively on the set $\{1, 2, \cdots, d-1\}$ and

$$\tau_2\sigma_1\sigma_2\sigma_3 = \tau_1 = \mu_1 \cdots \mu_p,$$

where $\tau_1 = \mu_1 \cdots \mu_p$ has type $a_1^1 \cdots a_{p-1}^1 (a_p - b_q)^1$, μ_j's are the cycle factors of τ_1 and μ_j has length a_j for $1 \le j < p$ and μ_p has length $(a_p - b_q) = (a_p - 1)$. Since $m_1 + m_2 \ge d$, by OA, any two of the three cycles $\sigma_1, \sigma_2, \sigma_3$ in S_{d-1} intersect since $1 + m_i + m_j > d > d-1$ for $1 \le i \ne j \le 3$. We divide the left part of the proof into the following three steps.

Step 3.1 If μ_p intersects σ_3, we are done by a similar argument as in step 2.1.

Step 3.2 If μ_p intersects σ_2 but does not intersect σ_3, the proof goes through as Step 2.2 since σ_2 intersects σ_3.

Step 3.3 Suppose that neither σ_2 nor σ_3 intersects μ_p. Then, by Remark 2.9, μ_p intersects σ_1 since $\langle \tau_1, \sigma_1, \sigma_2, \sigma_3 \rangle$ acts transitively on the set $\{1, 2, \cdots, d-1\}$. Choosing a number x in both μ_p and σ_1 and denoting $\widetilde{\mu}_p := \mu_p(x, d)$ and $\sigma_1' := \sigma_1(x, d)$, we obtain

$$\tau_2\sigma_1'\sigma_2\sigma_3 = \tau_2\sigma_1\sigma_2\sigma_3(x, d) = \mu_1 \cdots \mu_{p-1}\widetilde{\mu}_p =: \widetilde{\tau}_1$$

where σ_1' is a cycle of length $(m_1 + 2)$, $\widetilde{\mu}_p$ is a cycle factor of $\widetilde{\tau}_1$, and $\widetilde{\tau}_1$ has the type of $a_1^1 \cdots a_{p-1}^1 a_p^1$.

Step 3.3.A Suppose that σ_2 contains a number which is not contained in σ_3. Then we could find in S_d the three cycles of $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3$ with length $1+m_1, 1+m_2, 1+m_3$, respectively, such that

$$\tau_2\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_3 = \widetilde{\tau}_1$$

and $\langle \tau_2, \widetilde{\tau}_1, \widetilde{\sigma}_1, \widetilde{\sigma}_3 \rangle$ acts transitively on $\{1, 2, \cdots, d\}$. Indeed, rewrite σ_1' as

$$\sigma_1' = \widetilde{\sigma}_1(b_1, z_1)$$

where b_1 lies in the intersection of σ_1 and σ_2, the number z_1 is contained in σ_1' but not in σ_2, and $\widetilde{\sigma}_1$ is a cycle of length $(1+m_1)$. Choose $\sigma_2' := (b_1, z_1)\sigma_2$ and rewrite it as

$$\sigma_2' = \widetilde{\sigma}_2(b_2, z_2),$$

where b_2 lies in both σ_2' and σ_3, the number z_2 is contained in σ_2' but not in σ_3, $\widetilde{\sigma}_2$ is a $(1+m_2)$-cycle. Hence $\widetilde{\sigma}_3 := (b_2, z_2)\sigma_3$ is a $(1+m_3)$-cycle and the equality $\tau_2\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_3 = \widetilde{\tau}_1$ holds. Then all the following subgroups

$$\langle \tau_2, \sigma_1', \sigma_2, \sigma_3 \rangle = \langle \tau_2, \widetilde{\tau}_1, \sigma_2, \sigma_3 \rangle, \quad \langle \tau_2, \widetilde{\tau}_1, \sigma_2', \sigma_3 \rangle = \langle \tau_2, \widetilde{\tau}_1, \widetilde{\sigma}_1, \sigma_3 \rangle$$

and $\langle \tau_2, \widetilde{\tau}_1, \widetilde{\sigma}_1, \widetilde{\sigma}_3 \rangle$ of S_d act transitively on $\{1, 2, \cdots, d\}$ by Corollaries 2.13 and 2.14. Therefore, the following permutations

$$\left(\widetilde{\tau}_1\right)^{-1}, \tau_2, \widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3$$

satisfy the three properties.

Step 3.3.B Suppose that every number in the cycle σ_2 is contained in σ_3. Rewrite σ_1' as

$$\sigma_1' = \widetilde{\sigma}_1(b, z)$$

where $\widetilde{\sigma}_1$ is a $(1+m_1)$-cycle, b lies in both σ_1' and σ_3, z is not contained in σ_3 and then it is not contained in σ_2, either.

- Suppose that b is not contained in σ_2. Then

$$\begin{aligned}\tau_2\sigma_1'\sigma_2\sigma_3 &= \tau_2\widetilde{\sigma}_1(b, z)\sigma_2\sigma_3 \\ &= \tau_2\widetilde{\sigma}_1\sigma_2((b, z)\sigma_3) \\ &= \tau_2\widetilde{\sigma}_1\sigma_2\widetilde{\sigma}_3,\end{aligned}$$

where $\widetilde{\sigma}_3 := (b, z)\sigma_3$ is a $(1+m_3)$-cylce. By Corollary 2.14, the subgroup $\langle \tau_2, \widetilde{\tau}_1, \sigma_2, \widetilde{\sigma}_3 \rangle$ acts transitively on the set $\{1, 2, \cdots, d\}$ since the subgroup $\langle \tau_2, \widetilde{\tau}_1, \sigma_2, \sigma_3 \rangle = \langle \tau_2, \sigma_1', \sigma_2, \sigma_3 \rangle$ has the same property. Therefore, the following five permutations $\left(\widetilde{\tau}_1\right)^{-1}, \tau_2, \widetilde{\sigma}_1, \sigma_2, \widetilde{\sigma}_3$ satisfy the three properties.

- Suppose that b is contained in σ_2. Then we have

$$\tau_2\sigma_1'\sigma_2\sigma_3 = \tau_2\widetilde{\sigma}_1((b, z)\sigma_2)\sigma_3 = \tau_2\widetilde{\sigma}_1\sigma_2'\sigma_3 \quad \text{where} \quad \sigma_2' := (b, z)\sigma_2.$$

Then applying a similar argument in Step 2.2 to σ_2' and σ_3, we obtain the two cycles $\widetilde{\sigma}_2$ and $\widetilde{\sigma}_3$ with length of $(1+m_2), (1+m_3)$, respectively, such that

$$\sigma_2'\sigma_3 = \widetilde{\sigma}_2\widetilde{\sigma}_3 \quad \text{and} \quad \tau_2\widetilde{\sigma}_1\widetilde{\sigma}_2\widetilde{\sigma}_3 = \widetilde{\tau}_1.$$

Moreover, by Corollaries 2.13 and 2.14, the action of $\langle \tau_2, \widetilde{\tau}_1, \widetilde{\sigma}_1, \widetilde{\sigma}_3 \rangle$ on the set $\{1, 2, \cdots, d\}$ is transitive. Therefore, the following five permutations $\left(\widetilde{\tau}_1\right)^{-1}, \tau_2, \widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3$ satisfy the three properties.

□

3 Proof of Theorem 1.3

We prove Theorem 1.3 in this subsection.

Proof. Consider the collection

$$\Lambda^* = \{(a_1, \cdots, a_p), (b_1, \cdots, b_q), (c_1+1, 1, \cdots, 1), \cdots, (c_r+1, 1, \cdots, 1)\}$$

of degree d and with total branching order $2d-2$. By Theorem 2.3 and the Riemann existence theorem, there exists a rational function f of degree d on the Riemann sphere such that

(i) f branches over $0, \infty, \zeta, \cdots, \zeta^r$, where $\zeta = \exp\left(2\pi\sqrt{-1}/r\right)$.

(ii) The partitions of the above branch points coincide with $(a_1, \cdots, a_p)$, $(b_1, \cdots, b_q)$, $(c_1 + 1, 1, \cdots, 1), \cdots, (c_r + 1, 1, \cdots, 1)$, respectively.

Then f^r is the Belyi function as desired. □

4 A conjecture

In order to generalize the first part of Boccara's result, we make the following

Conjecture *Let* d, g *and* l *be three positive integers. Suppose that the collection* Λ *consists of* $l+2$ *partitions of* d *and has form*

$$\Lambda = \{(a_1, \cdots, a_p), (b_1, \cdots, b_q), (m_1 + 1, 1, \cdots, 1), \cdots, (m_l + 1, 1, \cdots, 1)\},$$

where $(m_1, \cdots, m_l)$ *is a partition of* $p+q-2+2g$*. Then there always exists a rational function on some compact Riemann surface of genus* g *with branch data* Λ.

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