SEVERAL q-SERIES RELATED TO RAMANUJAN'S THETA FUNCTIONS

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Abstract. Quite recently, the first author investigated vanishing coefficients of the arithmetic progressions in several q-series expansions. In this paper, we further study the signs of coefficients in two q-series expansions and establish some interlinked identities for several q-series expansions by means of Ramanujan's theta functions. We obtain the 5-dissections of these two q-series and give combinatorial interpretations for these dissections. Moreover, we obtain four q-series identities involving the aforementioned q-series, two of which were proved by Kim and Toh via modular forms.

1. Introduction

Quite recently, Hirschhorn [5] investigated vanishing coefficients of the arithmetic progressions in two q-series expansions. Motivated by the work of Hirschhorn, the first author [7] investigated vanishing coefficients of the arithmetic progressions in following q-series expansions

$$(-q,-q^4;q^5)^2_\infty(q^4,q^6;q^{10})_\infty=\sum_{n=0}^{\infty}g_1(n)q^n, \tag{1.1}$$

$$(-q^2,-q^3;q^5)^2_\infty(q^2,q^8;q^{10})_\infty=\sum_{n=0}^{\infty}h_1(n)q^n. \tag{1.2}$$

Here and in the sequel, we adopt the following standard q-series notation:

$$(a;q)_\infty:=\prod_{n=0}^{\infty}(1-aq^n),$$

$$(a_1,a_2,\ldots,a_m;q)_\infty:=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty,\quad \text{for } |q|<1.$$

In [7, Eqs. (1.3) and (1.4)], the first author proved that for $n\geq 0$,

$$g_1(5n+3)=h_1(5n+1)=0. \tag{1.3}$$

Moreover, the first author conjectured the signs of coefficients in q-series (1.2) are periodic from some n. In this paper, we not only confirm this conjecture, but also establish 5-dissections of (1.1) and (1.2) along with combinatorial interpretations for these dissections.

2010 *Mathematics Subject Classification.* 33D15, 11F33, 30B10.

Key words and phrases. Ramanujan's theta functions; q-series expansions; Jacobi's triple product identity; interlinked identities.

Firstly, we obtain the following 5-dissections of (1.1) and (1.2).

Theorem 1.1. *We have*

$$(-q,-q^4;q^5)_\infty^2(q^4,q^6;q^{10})_\infty = G_0(q^5)+qG_1(q^5)+q^2G_2(q^5)+q^4G_4(q^5),$$

where

$$G_0(q)=\sum_{n=0}^{\infty} g_1(5n)q^n = \frac{1}{(q,q^4;q^5)_\infty^2(q^2,q^8;q^{10})_\infty}, \tag{1.4}$$

$$G_1(q)=\sum_{n=0}^{\infty} g_1(5n+1)q^n = \frac{2}{(q,q^2,q^3,q^4;q^5)_\infty(q^2,q^8;q^{10})_\infty}, \tag{1.5}$$

$$G_2(q)=\sum_{n=0}^{\infty} g_1(5n+2)q^n = \frac{1}{(q,q^4;q^5)_\infty^2(q^4,q^6;q^{10})_\infty}, \tag{1.6}$$

$$G_4(q)=\sum_{n=0}^{\infty} g_1(5n+4)q^n = \frac{1}{(q^2,q^3;q^5)_\infty^2(q^4,q^6;q^{10})_\infty}. \tag{1.7}$$

Theorem 1.2. *We have*

$$(-q^2,-q^3;q^5)_\infty^2(q^2,q^8;q^{10})_\infty = H_0(q^5)+q^2H_2(q^5)+q^3H_3(q^5)+q^4H_4(q^5),$$

where

$$H_0(q)=\sum_{n=0}^{\infty} h_1(5n)q^n = \frac{1}{(q,q^4;q^5)_\infty^2(q^2,q^8;q^{10})_\infty}, \tag{1.8}$$

$$H_2(q)=\sum_{n=0}^{\infty} h_1(5n+2)q^n = \frac{1}{(q^2,q^3;q^5)_\infty^2(q^2,q^8;q^{10})_\infty}, \tag{1.9}$$

$$H_3(q)=\sum_{n=0}^{\infty} h_1(5n+3)q^n = \frac{2}{(q,q^2,q^3,q^4;q^5)_\infty(q^4,q^6;q^{10})_\infty}, \tag{1.10}$$

$$H_4(q)=\sum_{n=0}^{\infty} h_1(5n+4)q^n = \frac{-1}{(q^2,q^3;q^5)_\infty^2(q^4,q^6;q^{10})_\infty}. \tag{1.11}$$

Therefore we get the following combinatorial interpretations.

Corollary 1.3. $g_1(5n)$ *is the number of partitions of* n *into parts which are* ±1, ±2, ±4 (mod 10), *where parts* ±1 *and* ±4 *appear in two flavours,*

$g_1(5n+1)$ *is the twice of number of partitions of* n *into parts which are* ±1, ±2, ±3, ±4 (mod 10), *where parts* ±2 *appear in two flavours,*

$g_1(5n+2)$ *is the number of partitions of* n *into parts which are* ±1, ±4 (mod 10), *where parts* ±1 *parts appear in two flavours and* ±4 *appear in three flavours,*

$g_1(5n+4)$ *is the number of partitions of* n *into parts which are* ±2, ±3, ±4 (mod 10), *where parts* ±2 *and* ±3 *appear in two flavours,*

$h_1(5n)$ *is the number of partitions of* n *into parts which are* ±1, ±2, $\pm4 \pmod{10}$, *where parts* ±1 *and* ±4 *appear in two flavours,*

$h_1(5n+2)$ *is the number of partitions of* n *into parts which are* ±2, $\pm3 \pmod{10}$, *where parts* ±3 *parts appear in two flavours and* ±2 *appear in three flavours,*

$h_1(5n+3)$ *is the twice of number of partitions of* n *into parts which are* ±1, ±2, ±3, $\pm4 \pmod{10}$, *where parts* ±4 *appear in two flavours,*

$-h_1(5n+4)$ *is the number of partitions of* n *into parts which are* ±2, ±3, $\pm4 \pmod{10}$, *where parts* ±2 *and* ±3 *appear in two flavours.*

By Corollary 1.3, we obtain immediately the following inequalities.

Corollary 1.4. *For any integer* $n \geq 0$,

$$\begin{aligned} g_1(5n) &> 0, \\ g_1(5n+1) &> 0, \\ g_1(5n+2) &> 0, \\ g_1(5n+4) &> 0 \quad (n \neq 1). \end{aligned}$$

Corollary 1.5. *For any integer* $n \geq 0$,

$$\begin{aligned} h_1(5n) &> 0, \\ h_1(5n+2) &> 0 \quad (n \neq 1), \\ h_1(5n+3) &> 0, \\ h_1(5n+4) &< 0 \quad (n \neq 1). \end{aligned}$$

Corollary 1.6. *For any integer* $n \geq 0$,

$$g_1(5n) = h_1(5n), \tag{1.12}$$

$$g_1(5n+4) = -h_1(5n+4). \tag{1.13}$$

Moreover, the first author studied vanishing coefficients in following two general q-series expansions:

$$(-q^r, -q^{t-r}; q^t)_\infty^3 (q^s, q^{2t-s}; q^{2t})_\infty := \sum_{n=0}^{\infty} g_{r,s,t}(n) q^n, \tag{1.14}$$

$$(-q^r, -q^{t-r}; q^t)_\infty (q^s, q^{2t-s}; q^{2t})_\infty^3 := \sum_{n=0}^{\infty} h_{r,s,t}(n) q^n \tag{1.15}$$

where $t \geq 5$ is a prime, r, s are positive integers and $r < t$, $s \neq t$.

Interestingly, we obtain the following identities of q-series expansions (1.14) and (1.15) for $t = 5$, which are parallel to (1.12) and (1.13).

Theorem 1.7. *For any integer* $n \geq 0$,

$$g_{1,2,5}(5n+1) = g_{2,4,5}(5n+2), \tag{1.16}$$

$$g_{1,2,5}(5n+3) = -g_{2,4,5}(5n+4), \tag{1.17}$$

$$g_{1,3,5}(5n) = g_{2,1,5}(5n), \tag{1.18}$$
$$g_{1,3,5}(5n+2) = g_{2,1,5}(5n+2). \tag{1.19}$$
$$h_{1,1,5}(5n) = h_{2,3,5}(5n+2), \tag{1.20}$$
$$h_{1,1,5}(5n+1) = h_{2,3,5}(5n+3), \tag{1.21}$$
$$h_{1,4,5}(5n+1) = h_{2,2,5}(5n), \tag{1.22}$$
$$h_{1,4,5}(5n+2) = -h_{2,2,5}(5n+1). \tag{1.23}$$

Finally, we define the following two q-series expansion

$$(q, q^4; q^5)_\infty^2 (q^4, q^6; q^{10})_\infty = \sum_{n=0}^{\infty} g_2(n) q^n, \tag{1.24}$$
$$(q^2, q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty = \sum_{n=0}^{\infty} h_2(n) q^n. \tag{1.25}$$

We also obtain several q-series identities involving (1.1), (1.2), (1.15), (1.24), and (1.25).

Theorem 1.8. *We have*

$$(-q, -q^4; q^5)_\infty^2 (q^4, q^6; q^{10})_\infty + (-q^2, -q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty = \frac{2(q^{10}; q^{10})_\infty^3}{(q^2; q^2)_\infty (q^5; q^5)_\infty^2} (-q, -q^4; q^5)_\infty (q^4, q^6; q^{10})_\infty^3, \tag{1.26}$$
$$(q, q^4; q^5)_\infty^2 (q^4, q^6; q^{10})_\infty + (q^2, q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty = \frac{2(q; q)_\infty^2 (q^{10}; q^{10})_\infty^4}{(q^2; q^2)_\infty^2 (q^5; q^5)_\infty^4} (-q, -q^4; q^5)_\infty (q^4, q^6; q^{10})_\infty^3, \tag{1.27}$$
$$(-q, -q^4; q^5)_\infty (q^4, q^6; q^{10})_\infty^3 - q(-q^2, -q^3; q^5)_\infty (q^2, q^8; q^{10})_\infty^3 = \frac{(q^2; q^2)_\infty (q^5; q^5)_\infty^2}{(q^{10}; q^{10})_\infty^3} (-q^2, -q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty, \tag{1.28}$$
$$(-q, -q^4; q^5)_\infty (q^4, q^6; q^{10})_\infty^3 + q(-q^2, -q^3; q^5)_\infty (q^2, q^8; q^{10})_\infty^3 = \frac{(q^2; q^2)_\infty^2 (q^5; q^5)_\infty^4}{(q; q)_\infty^2 (q^{10}; q^{10})_\infty^4} (q^2, q^3; q^5)_\infty^2 (q^2, q^8; q^{10})_\infty. \tag{1.29}$$

Remark 1.9. Very recently, Kim and Toh [6, Lemma 3.1] proved the following two q-series identities via modular forms:

$$\frac{(-q^2, -q^3, q^5; q^5)_\infty^2 (q^{10}; q^{10})_\infty}{(q^4, q^6; q^{10})_\infty} + \frac{(-q, -q^4, q^5; q^5)_\infty^2 (q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty} = \frac{2(-q, -q^4, q^5; q^5)_\infty (q^2; q^2)_\infty (q^{10}; q^{10})_\infty^2}{(q^2, q^8; q^{10})_\infty^3 (q^5; q^5)_\infty}, \tag{1.30}$$

$$\frac{(-q^2, -q^3, q^5; q^5)_\infty^2 (q^{10}; q^{10})_\infty}{(q^4, q^6; q^{10})_\infty} + q\frac{(-q^2, -q^3, q^5; q^5)_\infty (q^2; q^2)_\infty (q^{10}; q^{10})_\infty^2}{(q^4, q^6; q^{10})_\infty^3 (q^5; q^5)_\infty}$$

$$= \frac{(-q,-q^4,q^5;q^5)_\infty(q^2;q^2)_\infty(q^{10};q^{10})_\infty^2}{(q^2,q^8;q^{10})_\infty^3(q^5;q^5)_\infty}. \tag{1.31}$$

Interestingly, (1.26) and (1.28) are equivalent to (1.30) and (1.31), respectively.

The rest of this paper is constructed as follows. In Sect. 2, we introduce some necessary notation as well as identities involving theta functions $\varphi(q)$ and $\psi(q)$. In Sect. 3, we prove Theorems 1.1 and 1.2. The proofs of Theorems 1.7 and 1.8 are given in Sect. 4. We conclude in the last section with some remarks to motivate further investigation.

2. Preliminary results

Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1.$$

Basic properties enjoyed by $f(a,b)$ proved in [2, p. 34, Entry 18] include

$$f(a,b) = f(b,a),$$
$$f(1,a) = 2f(a,a^3). \tag{2.1}$$

The function $f(a,b)$ satisfies the well-known Jacobi triple product identity [2, p. 35, Entry 19]:

$$f(a,b) = (-a,-b,ab;ab)_\infty. \tag{2.2}$$

Eq. (2.2) is used frequently and without mention in the sequel.

The two important special cases of (2.2) are [4, Eqs. (1.5.4) and (1.5.5)]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2},$$
$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{2.3}$$

Lemma 2.1. *We have*

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{2.4}$$
$$4q(q^4;q^4)_\infty(q^{20};q^{20})_\infty = \varphi(q)\varphi(-q^5) - \varphi(-q)\varphi(q^5). \tag{2.5}$$

Proof. Eq. (2.4) follows from [2, p. 40, Entry 25 (i), (ii)] and Eq. (2.5) appears in [2, p. 278]. □

The following lemma is the main ingredient for our proof.

Lemma 2.2. *If $ab = cd$, then*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c},\frac{c}{b}abcd\right)f\left(\frac{b}{d},\frac{d}{b}abcd\right). \tag{2.6}$$

Proof. Equation (2.6) comes from [2, p. 45, Entry 29] and [3, p. 9, Theorem 0.6]. □

Finally, we need the following two identities involving $\varphi(q)$ and $\psi(q)$.

Lemma 2.3. *We have*

$$\varphi(q)-\varphi(q^5)=2q\frac{(q^4,q^6,q^{10},q^{14},q^{16},q^{20};q^{20})_\infty}{(q^3,q^7,q^8,q^{12},q^{13},q^{17};q^{20})_\infty}, \tag{2.7}$$

$$\psi(q^2)-q\psi(q^{10})=\frac{(q,q^9,q^{10},q^{11},q^{19},q^{20};q^{20})_\infty}{(q^2,q^3,q^7,q^{13},q^{17},q^{18};q^{20})_\infty}. \tag{2.8}$$

Proof. Eqs. (2.7) and (2.8) are proved in [4, p. 311, Eqs. (34.1.8) and (34.1.12)]. □

3. Proofs of Theorems 1.1 and 1.2

To obtain (1.4)–(1.7), we first prove two necessary lemmas.

Let $k>0, l\geq 0$ be integers and let $G(q)=\sum_{n=0}^{\infty} g(n)q^n$ be a formal power series. Define an operator $H_{k,l}$ by

$$H_{k,l}\left(G(q)\right):=\sum_{n=0}^{\infty} g(kn+l)q^{kn+l}.$$

Lemma 3.1. *Define*

$$\begin{aligned}M_1(q)&:=f(q^{18},q^{22})^2-q^8f(q^2,q^{38})^2,\\ N_1(q)&:=q^5f(q^{12},q^{28})f(q^2,q^{48})+q^6f(q^8,q^{32})f(q^2,q^{48})\\ &\qquad -qf(q^{12},q^{28})f(q^{18},q^{22})-q^2f(q^8,q^{32})f(q^{18},q^{22}).\end{aligned}$$

Then

$$\varphi(q)M_1(q)+2\psi(q^2)N_1(q)=\varphi(q^5)M_1(q)+2q\psi(q^{10})N_1(q). \tag{3.1}$$

Proof. Putting $(a,b,c,d)=(-q^8,-q^{12},-q^{10},-q^{10})$ in (2.6), we get

$$M_1(q)=f(q^{18},q^{22})^2-q^8f(q^2,q^{38})^2=f(-q^8,-q^{12})f(-q^{10},-q^{10}). \tag{3.2}$$

Similarly, taking $(a,b,c,d)=(-q^5,-q^{15},-q^7,-q^{13})$ in (2.6),

$$qf(q^{12},q^{28})f(q^{18},q^{22})-q^6f(q^8,q^{32})f(q^2,q^{48})=qf(-q^5,-q^{15})f(-q^7,-q^{13}). \tag{3.3}$$

Picking $(a,b,c,d)=(-q^3,-q^{17},-q^5,-q^{15})$ in (2.6),

$$q^2f(q^8,q^{32})f(q^{18},q^{22})-q^5f(q^8,q^{32})f(q^2,q^{48})=q^2f(-q^3,-q^{17})f(-q^5,-q^{15}). \tag{3.4}$$

Finally, taking $(a,b,c,d)=(q,q^9,-q^4,-q^6)$ in (2.6),

$$f(-q^5,-q^{15})f(-q^7,-q^{13})+qf(-q^3,-q^{17})f(-q^5,-q^{15})=f(q,q^9)f(-q^4,-q^6). \tag{3.5}$$

Employing (3.3)–(3.5), we readily obtain

$$N_1(q)=-qf(q,q^9)f(-q^4,-q^6). \tag{3.6}$$

Now, with the help of (2.7), (2.8), (3.2), and (3.6),

$$\begin{aligned}&\left(\varphi(q)-\varphi(q^5)\right)M_1(q)+2\left(\psi(q^2)-q\psi(q^{10})\right)N_1(q)\\=&2q\frac{(q^4,q^6,q^{10},q^{14},q^{16},q^{20};q^{20})_\infty}{(q^3,q^7,q^8,q^{12},q^{13},q^{17};q^{20})_\infty}\times(q^8,q^{10},q^{10},q^{12},q^{20},q^{20};q^{20})_\infty\\&-2q\frac{(q,q^9,q^{10},q^{11},q^{19},q^{20};q^{20})_\infty}{(q^2,q^3,q^7,q^{13},q^{17},q^{18};q^{20})_\infty}\times(-q,q^4,q^6,-q^9,q^{10},q^{10};q^{10})_\infty\\=&\frac{(q^4,q^6,q^{10},q^{10},q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}-\frac{(q^4,q^6,q^{10},q^{10},q^{10};q^{10})_\infty}{(q^3,q^7;q^{10})_\infty}\\=&0,\end{aligned}$$

as desired. □

Lemma 3.2. *We have*

$$\psi(q^2)\varphi(q^5)-q\varphi(q)\psi(q^{10})=(q;q)_\infty(q^5;q^5)_\infty. \tag{3.7}$$

Proof. Firstly, replacing q by $-q^5$ in (2.4), we find that

$$\varphi(-q^5)=\varphi(q^{20})-2q^5\psi(q^{40}). \tag{3.8}$$

Combining (2.4) and (3.8) yields

$$\varphi(q)\varphi(-q^5)=\varphi(q^4)\varphi(q^{20})+2q\psi(q^8)\varphi(q^{20})-2q^5\varphi(q^4)\psi(q^{40})-4q^6\psi(q^8)\psi(q^{40}). \tag{3.9}$$

Replacing q by $-q$ in (3.9),

$$\varphi(-q)\varphi(q^5)=\varphi(q^4)\varphi(q^{20})-2q\psi(q^8)\varphi(q^{20})+2q^5\varphi(q^4)\psi(q^{40})-4q^6\psi(q^8)\psi(q^{40}). \tag{3.10}$$

By (3.9) and (3.10),

$$\varphi(q)\varphi(-q^5)-\varphi(-q)\varphi(q^5)=4q\psi(q^8)\varphi(q^{20})-4q^5\varphi(q^4)\psi(q^{40}). \tag{3.11}$$

Finally, substituting (2.5) into (3.11) and replacing q by $q^{1/4}$, we obtain (3.7). □

Now we turn to prove (1.4).

On one hand, according to [7], we find that

$$\begin{aligned}\sum_{n=0}^{\infty}g_1(n)q^n=&\frac{\varphi(q^5)}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\left(S_1-q^4S_2+q^2S_3-q^6S_4\right)\\&+\frac{2\psi(q^{10})}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\left(qS_5-q^5S_6+q^4S_7-q^8S_8\right),\end{aligned} \tag{3.12}$$

where

$$S_1=\sum_{m,n=-\infty}^{\infty}q^{20m^2+2m+20n^2+6n},\quad S_2=\sum_{m,n=-\infty}^{\infty}q^{20m^2+18m+20n^2+6n},$$

$$S_3=\sum_{m,n=-\infty}^{\infty}q^{20m^2+2m+20n^2+14n},\quad S_4=\sum_{m,n=-\infty}^{\infty}q^{20m^2+18m+20n^2+14n},$$

$$S_5=\sum_{m,n=-\infty}^{\infty}q^{20m^2+2m+20n^2+4n},\quad S_6=\sum_{m,n=-\infty}^{\infty}q^{20m^2+18m+20n^2+4n},$$
$$S_7=\sum_{m,n=-\infty}^{\infty}q^{20m^2+2m+20n^2+16n},\quad S_8=\sum_{m,n=-\infty}^{\infty}q^{20m^2+18m+20n^2+16n}.$$

In S_1, if $2m+6n\equiv 0 \pmod 5$, then $2m+n\equiv 0 \pmod 5$. Equivalently, $m-2n\equiv 0 \pmod 5$. Assume $2m+n=5r$ and $m-2n=-5s$, it follows that $m=2r-s$ and $n=r+2s$. Therefore

$$H_{5,0}(S_1)=\sum_{r,s=-\infty}^{\infty}q^{100r^2+10r+100s^2+10s}=f(q^{90},q^{110})^2. \tag{3.13}$$

Similarly, we obtain

$$H_{5,0}(q^4S_2)=q^{20}f(q^{10},q^{190})f(q^{90},q^{110}), \tag{3.14}$$
$$H_{5,0}(q^2S_3)=q^{20}f(q^{10},q^{190})f(q^{90},q^{110}), \tag{3.15}$$
$$H_{5,0}(q^6S_4)=q^{40}f(q^{10},q^{190})^2, \tag{3.16}$$
$$H_{5,0}(qS_5)=q^{25}f(q^{60},q^{140})f(q^{10},q^{190}), \tag{3.17}$$
$$H_{5,0}(q^5S_6)=q^{5}f(q^{60},q^{140})f(q^{90},q^{110}), \tag{3.18}$$
$$H_{5,0}(q^4S_7)=q^{30}f(q^{40},q^{160})f(q^{10},q^{190}), \tag{3.19}$$
$$H_{5,0}(q^8S_8)=q^{10}f(q^{40},q^{160})f(q^{90},q^{110}). \tag{3.20}$$

Picking out the term involving q^{5n} in (3.12), applying (3.13)–(3.20) and replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}g_1(5n)q^n=&\frac{\varphi(q)}{(q;q)_\infty^2(q^2;q^2)_\infty}\left(f(q^{18},q^{22})^2-q^8f(q^2,q^{38})^2\right)\\&+\frac{2\psi(q^2)}{(q;q)_\infty^2(q^2;q^2)_\infty}\left(q^5f(q^{12},q^{28})f(q^2,q^{38})+q^6f(q^8,q^{32})f(q^2,q^{38})\right.\\&\left.-qf(q^{12},q^{28})f(q^{18},q^{22})-q^2f(q^8,q^{32})f(q^{18},q^{22})\right).\end{aligned} \tag{3.21}$$

On the other hand,

$$\begin{aligned}\frac{1}{(q,q^4;q^5)_\infty^2}=\frac{(q^2,q^3,q^5;q^5)_\infty^2}{(q;q)_\infty^2}&=\frac{1}{(q;q)_\infty^2}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{(5m^2+m)/2+(5n^2+n)/2}\\&=\frac{1}{(q;q)_\infty^2}\left(\sum_{r,s=-\infty}^{\infty}q^{(5(r+s)^2+(r+s))/2+(5(r-s)^2+(r-s))/2}\right.\\&\left.\quad-\sum_{r,s=-\infty}^{\infty}q^{(5(r+s-1)^2+(r+s-1))/2+(5(r-s)^2+(r-s))/2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(q;q)_\infty^2}\left(\sum_{m,n=-\infty}^{\infty} q^{5m^2+m+5n^2} - \sum_{m,n=-\infty}^{\infty} q^{5m^2+5m+5n^2+4n}\right)\\
&= \frac{\varphi(q^5)}{(q;q)_\infty^2}\sum_{m=-\infty}^{\infty} q^{5m^2+m} - \frac{2q^2\psi(q^{10})}{(q;q)_\infty^2}\sum_{m=-\infty}^{\infty} q^{5m^2+4m}\\
&= \frac{\varphi(q^5)}{(q;q)_\infty^2}\left(\sum_{m=-\infty}^{\infty} q^{20m^2+2m} + q^4\sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right)\\
&\quad - \frac{2\psi(q^{10})}{(q;q)_\infty^2}\left(q^2\sum_{m=-\infty}^{\infty} q^{20m^2+8m} + q^3\sum_{m=-\infty}^{\infty} q^{20m^2+12m}\right). \qquad (3.22)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{1}{(q^2,q^8;q^{10})_\infty} &= \frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q^2;q^2)_\infty} = \frac{1}{(q^2;q^2)_\infty}\sum_{m=-\infty}^{\infty}(-1)^m q^{5m^2+m}\\
&= \frac{1}{(q^2;q^2)_\infty}\left(\sum_{m=-\infty}^{\infty} q^{20m^2+2m} - q^4\sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right). \qquad (3.23)
\end{aligned}
$$

Combining (3.22) and (3.23) yields

$$
\begin{aligned}
&\frac{1}{(q,q^4;q^5)_\infty^2(q^2,q^8;q^{10})_\infty}\\
=&\Bigg(\frac{\varphi(q^5)}{(q;q)_\infty^2}\left(\sum_{m=-\infty}^{\infty} q^{20m^2+2m} + q^4\sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right)\\
&\quad - \frac{2\psi(q^{10})}{(q;q)_\infty^2}\left(q^2\sum_{m=-\infty}^{\infty} q^{20m^2+8m} + q^3\sum_{m=-\infty}^{\infty} q^{20m^2+12m}\right)\Bigg)\\
&\quad \times \frac{1}{(q^2;q^2)_\infty}\left(\sum_{m=-\infty}^{\infty} q^{20m^2+2m} - q^4\sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right)\\
=&\frac{\varphi(q^5)}{(q;q)_\infty^2(q^2;q^2)_\infty}\left(f(q^{18},q^{22})^2 - q^8 f(q^2,q^{38})^2\right)\\
&\quad + \frac{2\psi(q^{10})}{(q;q)_\infty^2(q^2;q^2)_\infty}\big(q^6 f(q^{12},q^{28})f(q^2,q^{38}) + q^7 f(q^8,q^{32})f(q^2,q^{38})\\
&\quad - q^2 f(q^{12},q^{28})f(q^{18},q^{22}) - q^3 f(q^8,q^{32})f(q^{18},q^{22})\big). \qquad (3.24)
\end{aligned}
$$

Equation (1.4) follows from (3.1), (3.21), and (3.24).
Next we are ready to prove (1.5).

Following the same line of proving (3.21), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} g_1(5n+1)q^n =& \frac{2\varphi(q)}{(q;q)_\infty^2(q^2;q^2)_\infty}\left(q^5 f(q^{10},q^{30})f(q^2,q^{38}) - qf(q^{10},q^{30})f(q^{18},q^{22})\right)\\ &+ \frac{2\psi(q^2)}{(q;q)_\infty^2(q^2;q^2)_\infty}\left(f(q^{20},q^{20})f(q^{18},q^{22}) + 2q^5 f(q^{40},q^{120})f(q^{18},q^{22})\right.\\ &\left.- q^4 f(q^{20},q^{20})f(q^2,q^{38}) - 2q^9 f(q^{40},q^{120})f(q^2,q^{38})\right)\\ :=& \frac{2\varphi(q)}{(q;q)_\infty^2(q^2;q^2)_\infty}M_2(q) + \frac{2\psi(q^2)}{(q;q)_\infty^2(q^2;q^2)_\infty}N_2(q), \end{aligned}\tag{3.25}$$

say.

Taking $(a,b,c,d) = (-q^4,-q^{16},-q^6,-q^{14})$ in (2.6),

$$f(q^{10},q^{30})f(q^{18},q^{22}) - q^4 f(q^{10},q^{30})f(q^2,q^{38}) = f(-q^4,-q^{16})f(-q^6,-q^{14}),$$

which yields

$$M_2(q) = -qf(-q^4,-q^{16})f(-q^6,-q^{14}).\tag{3.26}$$

In view of (2.1) and (2.3),

$$\begin{aligned}N_2(q) =& f(q^{20},q^{20})f(q^{18},q^{22}) - 2q^9 f(q^{40},q^{120})f(q^2,q^{38})\\ &+ 2q^5 f(q^{40},q^{120})f(q^{18},q^{22}) - q^4 f(q^{20},q^{20})f(q^2,q^{38})\\ =& f(q^{20},q^{20})f(q^{18},q^{22}) - q^9 f(1,q^{40})f(q^2,q^{38})\\ &+ q^5 f(1,q^{40})f(q^{18},q^{22}) - q^4 f(q^{20},q^{20})f(q^2,q^{38}).\end{aligned}\tag{3.27}$$

Similarly, putting $(a,b,c,d) = (-q^9,-q^{11},-q^{11},-q^9)$ in (2.6),

$$f(q^{20},q^{20})f(q^{18},q^{22}) - q^9 f(1,q^{40})f(q^2,q^{38}) = f(-q^9,-q^{11})^2.\tag{3.28}$$

Picking $(a,b,c,d) = (-q,-q^{19},-q^{19},-q)$ in (2.6),

$$f(q^{20},q^{20})f(q^2,q^{38}) - qf(1,q^{40})f(q^{18},q^{22}) = f(-q,-q^{19})^2.\tag{3.29}$$

Taking $(a,b,c,d) = (-q^4,-q^6,q^5,q^5)$ in (2.6),

$$f(-q^9,-q^{11})^2 - q^4 f(-q,-q^{19})^2 = f(q^5,q^5)f(-q^4,-q^6) = \varphi(q^5)f(-q^4,-q^6).\tag{3.30}$$

With the help of (3.27)–(3.30),

$$N_2(q) = \varphi(q^5)f(-q^4,-q^6).\tag{3.31}$$

Substituting (3.26) and (3.31) into (3.25),

$$\begin{aligned}\sum_{n=0}^{\infty} g_1(5n+1)q^n =& \frac{2\psi(q^2)\varphi(q^5)f(-q^4,-q^6)}{(q;q)_\infty^2(q^2;q^2)_\infty} - \frac{2q\varphi(q)f(-q^4,-q^{16})f(-q^6,-q^{14})}{(q;q)_\infty^2(q^2;q^2)_\infty}\\ =& \frac{2\psi(q^2)\varphi(q^5)f(-q^4,-q^6)}{(q;q)_\infty^2(q^2;q^2)_\infty} - \frac{2q\varphi(q)\psi(q^{10})f(-q^4,-q^6)}{(q;q)_\infty^2(q^2;q^2)_\infty}.\end{aligned}\tag{3.32}$$

On the other hand,

$$\begin{aligned}\frac{2}{(q,q^2,q^3,q^4;q^5)_\infty(q^2,q^8;q^{10})_\infty} &= \frac{2f(-q^2,-q^3)f(-q,-q^4)f(-q^4,-q^6)}{(q;q)_\infty^2(q^2;q^2)_\infty}\\ &= \frac{2(q;q)_\infty(q^5;q^5)_\infty f(-q^4,-q^6)}{(q;q)_\infty^2(q^2;q^2)_\infty}.\end{aligned} \tag{3.33}$$

In light of (3.7), (3.32), and (3.33), we obtain (1.5).
The proofs of (1.6) and (1.7) are similar to that of (1.4).
The proof of Theorem 1.2 is similar to Theorem 1.1.

4. Proofs of Theorems 1.7 and 1.8

We only prove (1.16), and (1.17)–(1.23) can be proved similarly.
From [7], we have the following representation for $\sum_{n=0}^{\infty} g_{1,2,5}(n)q^n$:

$$\begin{aligned}\sum_{n=0}^{\infty} g_{1,2,5}(n)q^n &= \frac{(q^{10};q^{10})_\infty^4(q^{80};q^{80})_\infty^5 f(q,q^4)}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^2(q^{40};q^{40})_\infty^2(q^{160};q^{160})_\infty^2}\\
&\quad\times\left(\sum_{n=-\infty}^{\infty} q^{40n^2+12n} - q^4\sum_{n=-\infty}^{\infty} q^{40n^2+28n}\right)\\
&\quad+\frac{2(q^{10};q^{10})_\infty^4(q^{160};q^{160})_\infty^2 f(q,q^4)}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^2(q^{80};q^{80})_\infty}\left(q^{14}\sum_{n=-\infty}^{\infty} q^{40n^2+28n} - q^{10}\sum_{n=-\infty}^{\infty} q^{40n^2+12n}\right)\\
&\quad+\frac{2(q^{20};q^{20})_\infty^2 f(q^{30},q^{50})f(q,q^4)}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\left(q\sum_{n=-\infty}^{\infty} q^{40n^2+2n} - q^{10}\sum_{n=-\infty}^{\infty} q^{40n^2+38n}\right.\\
&\quad\left.+q^4\sum_{n=-\infty}^{\infty} q^{40n^2+22n} - q^3\sum_{n=-\infty}^{\infty} q^{40n^2+18n}\right)\\
&\quad+\frac{2(q^{20};q^{20})_\infty^2 f(q^{10},q^{70})f(q,q^4)}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\left(q^{15}\sum_{n=-\infty}^{\infty} q^{40n^2+38n} - q^9\sum_{n=-\infty}^{\infty} q^{40n^2+22n}\right.\\
&\quad\left.+q^8\sum_{n=-\infty}^{\infty} q^{40n^2+18n} - q^6\sum_{n=-\infty}^{\infty} q^{40n^2+2n}\right).\end{aligned} \tag{4.1}$$

With the aid of (3.22) and (3.23),

$$\begin{aligned}(-q^2,-q^3;q^5)_\infty^2 &= \frac{(-q^2,-q^3,q^5;q^5)_\infty^2}{(q^5;q^5)_\infty^2}\\
&= \left(\frac{(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty^2}\sum_{m=-\infty}^{\infty} q^{20m^2+2m} + \frac{q^4(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty^2}\sum_{m=-\infty}^{\infty} q^{20m^2+18m}\right)\end{aligned}$$

$$+\left(\frac{2q^2(q^{20};q^{20})_\infty^2}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\sum_{m=-\infty}^{\infty}q^{20m^2+8m}+\frac{2q^3(q^{20};q^{20})_\infty^2}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\sum_{m=-\infty}^{\infty}q^{20m^2+12m}\right) \tag{4.2}$$

and

$$(q^4,q^6;q^{10})_\infty=\frac{1}{(q^{10};q^{10})_\infty}\left(\sum_{m=-\infty}^{\infty}q^{20m^2+2m}-q^4\sum_{m=-\infty}^{\infty}q^{20m^2+18m}\right). \tag{4.3}$$

Combining (4.2) and (4.3) as well as following the similar strategy of proving (3.22), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}g_{2,4,5}(n)q^n &= \frac{f(q^2,q^3)}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty}\left(\sum_{m=-\infty}^{\infty}q^{20m^2+2m}-q^4\sum_{m=-\infty}^{\infty}q^{20m^2+18m}\right)\\
&\quad\times\left(\frac{(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty^2}\sum_{n=-\infty}^{\infty}q^{20n^2+2n}+\frac{q^4(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty^2}\sum_{n=-\infty}^{\infty}q^{20n^2+18n}\right.\\
&\quad\left.+\frac{2q^2(q^{20};q^{20})_\infty^2}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}q^{20n^2+8n}+\frac{2q^3(q^{20};q^{20})_\infty^2}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\sum_{n=-\infty}^{\infty}q^{20n^2+12n}\right)\\
&=\frac{(q^{10};q^{10})_\infty^4(q^{80};q^{80})_\infty^5f(q^2,q^3)}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^2(q^{40};q^{40})_\infty^2(q^{160};q^{160})_\infty^2}\left(\sum_{n=-\infty}^{\infty}q^{40n^2+4n}-q^8\sum_{n=-\infty}^{\infty}q^{40n^2+36n}\right)\\
&\quad+\frac{2(q^{10};q^{10})_\infty^4(q^{160};q^{160})_\infty^2f(q^2,q^3)}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^2(q^{80};q^{80})_\infty}\left(q^{18}\sum_{n=-\infty}^{\infty}q^{40n^2+36n}-q^{10}\sum_{n=-\infty}^{\infty}q^{40n^2+4n}\right)\\
&\quad+\frac{2(q^{20};q^{20})_\infty^2f(q^{30},q^{50})f(q^2,q^3)}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\left(q^2\sum_{n=-\infty}^{\infty}q^{40n^2+6n}-q^9\sum_{n=-\infty}^{\infty}q^{40n^2+34n}\right.\\
&\quad\left.+q^3\sum_{n=-\infty}^{\infty}q^{40n^2+14n}-q^6\sum_{n=-\infty}^{\infty}q^{40n^2+26n}\right)\\
&\quad+\frac{2(q^{20};q^{20})_\infty^2f(q^{10},q^{70})f(q^2,q^3)}{(q^5;q^5)_\infty^3(q^{10};q^{10})_\infty^2}\left(q^{14}\sum_{n=-\infty}^{\infty}q^{40n^2+34n}-q^8\sum_{n=-\infty}^{\infty}q^{40n^2+14n}\right.\\
&\quad\left.+q^{11}\sum_{n=-\infty}^{\infty}q^{40n^2+26n}-q^7\sum_{n=-\infty}^{\infty}q^{40n^2+6n}\right).
\end{aligned}$$

Define

$$W_1:=f(q,q^4)\sum_{n=-\infty}^{\infty}q^{40n^2+12n}-q^4f(q,q^4)\sum_{n=-\infty}^{\infty}q^{40n^2+28n},$$

$$W_2 := q^{14} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+28n} - q^{10} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+12n},$$

$$W_3 := q f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+2n} - q^{10} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+38n},$$

$$W_4 := q^{4} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+22n} - q^{3} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+18n},$$

$$W_5 := q^{15} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+38n} - q^{9} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+22n},$$

$$W_6 := q^{8} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+18n} - q^{6} f(q,q^4) \sum_{n=-\infty}^{\infty} q^{40n^2+2n},$$

$$T_1 := f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+4n} - q^{8} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+36n},$$

$$T_2 := q^{18} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+36n} - q^{10} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+4n},$$

$$T_3 := q^{2} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+6n} - q^{9} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+34n},$$

$$T_4 := q^{3} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+14n} - q^{6} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+26n},$$

$$T_5 := q^{14} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+34n} - q^{8} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+14n},$$

$$T_6 := q^{11} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+26n} - q^{7} f(q^2,q^3) \sum_{n=-\infty}^{\infty} q^{40n^2+6n}.$$

Next, we prove that

$$H_{5,1}(W_i) = H_{5,2}(T_i) \quad \text{for } 1 \leq i \leq 6.$$

We only prove the case $H_{5,1}(W_1) = H_{5,2}(T_1)$ here because the proofs of remaining cases are similar.

Notice that

$$f(q^2,q^3) = \sum_{m=-\infty}^{\infty} q^{(5m^2+m)/2}$$

$$= \sum_{m=-\infty}^{\infty} q^{10m^2+m} + q^2 \sum_{m=-\infty}^{\infty} q^{10m^2+9m}$$
$$= \sum_{m=-\infty}^{\infty} q^{40m^2+2m} + q^9 \sum_{m=-\infty}^{\infty} q^{40m^2+38m} + q^2 \sum_{m=-\infty}^{\infty} q^{40m^2+18m} + q^3 \sum_{m=-\infty}^{\infty} q^{40m^2+22m}$$

and

$$f(q,q^4) = \sum_{m=-\infty}^{\infty} q^{40m^2+6m} + q^7 \sum_{m=-\infty}^{\infty} q^{40m^2+34m} + q \sum_{m=-\infty}^{\infty} q^{40m^2+14m} + q^4 \sum_{m=-\infty}^{\infty} q^{40m^2+26m}.$$

Therefore,

$$S_1 = P_1 + P_2 + P_3 + P_4 - P_5 - P_6 - P_7 - P_8,$$
$$T_1 = Q_1 + Q_2 + Q_3 + Q_4 - Q_5 - Q_6 - Q_7 - Q_8,$$

where

$$P_1 = \sum_{m,n=-\infty}^{\infty} q^{40m^2+6m+40n^2+12n}, \ P_2 = q^7 \sum_{m,n=-\infty}^{\infty} q^{40m^2+34m+40n^2+12n},$$
$$P_3 = q \sum_{m,n=-\infty}^{\infty} q^{40m^2+14m+40n^2+12n}, \ P_4 = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+26m+40n^2+12n},$$
$$P_5 = q^4 \sum_{m,n=-\infty}^{\infty} q^{40m^2+6m+40n^2+28n}, \ P_6 = q^{11} \sum_{m,n=-\infty}^{\infty} q^{40m^2+34m+40n^2+28n},$$
$$P_7 = q^5 \sum_{m,n=-\infty}^{\infty} q^{40m^{14}+14m+40n^2+28n}, \ P_8 = q^8 \sum_{m,n=-\infty}^{\infty} q^{40m^2+26m+40n^2+28n},$$
$$Q_1 = q^9 \sum_{m,n=-\infty}^{\infty} q^{40m^2+38m+40n^2+4n}, \ Q_2 = \sum_{m,n=-\infty}^{\infty} q^{40m^2+2m+40n^2+4n},$$
$$Q_3 = q^2 \sum_{m,n=-\infty}^{\infty} q^{40m^2+18m+40n^2+4n}, \ Q_4 = q^3 \sum_{m,n=-\infty}^{\infty} q^{40m^2+22m+40n^2+4n},$$
$$Q_5 = q^{17} \sum_{m,n=-\infty}^{\infty} q^{40m^2+38m+40n^2+36n}, \ Q_6 = q^8 \sum_{m,n=-\infty}^{\infty} q^{40m^2+2m+40n^2+36n},$$
$$Q_7 = q^{10} \sum_{m,n=-\infty}^{\infty} q^{40m^2+18m+40n^2+36n}, \ Q_8 = q^{11} \sum_{m,n=-\infty}^{\infty} q^{40m^2+22m+40n^2+36n}.$$

Following the similar strategy of proving (3.13), we deduce that

$$H_{5,1}(P_i) = H_{5,2}(Q_i) \quad \text{for } 1 \le i \le 8.$$

This establishes (1.16).

Finally, we are ready to prove (1.26)–(1.29).

It follows easily from (2.2) that

$$f(-q^{2s},-q^{2t})=\frac{(q^{2s+2t};q^{2s+2t})_\infty}{(q^{s+t};q^{s+t})_\infty^2}f(q^s,q^t)f(-q^s,-q^t),\quad s,t\in\mathbb{N}_+, \tag{4.4}$$

$$f(q,q^4)f(q^2,q^3)=\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty}, \tag{4.5}$$

$$\begin{aligned}f(-q^2,-q^3)f(-q^4,-q^6)&=(q^5;q^5)_\infty(q^2,q^3,q^4,q^6,q^7,q^8,q^{10};q^{10})_\infty\\&=\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty}f(-q^3,-q^7),\end{aligned} \tag{4.6}$$

$$\begin{aligned}f(-q,-q^4)f(-q^2,-q^8)&=(q^5;q^5)_\infty(q,q^2,q^4,q^6,q^8,q^9,q^{10};q^{10})_\infty\\&=\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty}{(q^{10};q^{10})_\infty}f(-q,-q^9).\end{aligned} \tag{4.7}$$

On one hand, according to (4.4) and (4.5), we obtain

$$\begin{aligned}&(-q,-q^4;q^5)_\infty^2(q^4,q^6;q^{10})_\infty+(-q^2,-q^3;q^5)_\infty^2(q^2,q^8;q^{10})_\infty\\=&\frac{1}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}\Big(f(q,q^4)^2f(-q^4,-q^6)+f(q^2,q^3)^2f(-q^2,-q^8)\Big)\\=&\frac{1}{(q^5;q^5)_\infty^4}f(q,q^4)f(q^2,q^3)\Big(f(q,q^4)f(-q^2,-q^3)+f(-q,-q^4)f(q^2,q^3)\Big)\\=&\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty}\Big(f(q,q^4)f(-q^2,-q^3)+f(-q,-q^4)f(q^2,q^3)\Big).\end{aligned} \tag{4.8}$$

Taking $(a,b,c,d)=(-q,-q^4,q^2,q^3)$ in (2.6),

$$f(-q,-q^4)f(q^2,q^3)=f(-q^3,-q^7)f(-q^4,-q^6)-qf(-q,-q^9)f(-q^2,-q^8). \tag{4.9}$$

Picking $(a,b,c,d)=(q,q^4,-q^2,-q^3)$ in (2.6),

$$f(q,q^4)f(-q^2,-q^3)=f(-q^3,-q^7)f(-q^4,-q^6)+qf(-q,-q^9)f(-q^2,-q^8). \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8),

$$\begin{aligned}&(-q,-q^4;q^5)_\infty^2(q^4,q^6;q^{10})_\infty+(-q^2,-q^3;q^5)_\infty^2(q^2,q^8;q^{10})_\infty\\&\qquad=\frac{2(q^2;q^2)_\infty}{(q;q)_\infty(q^5;q^5)_\infty(q^{10};q^{10})_\infty}f(-q^3,-q^7)f(-q^4,-q^6).\end{aligned}$$

On the other hand, with the help of (4.5) and (4.6),

$$\begin{aligned}&\frac{2(q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty(q^5;q^5)_\infty^2}(-q,-q^4;q^5)_\infty(q^4,q^6;q^{10})_\infty^3\\=&\frac{2}{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}f(q,q^4)f(-q^4,-q^6)^3\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}\Big(\left(f(q,q^4)f(q^2,q^3)\right)\left(f(-q^2,-q^3)f(-q^4,-q^6)\right) f(-q^4,-q^6)\Big)\\
&= \frac{2(q^2;q^2)_\infty}{(q;q)_\infty (q^5;q^5)_\infty (q^{10};q^{10})_\infty} f(-q^3,-q^7)f(-q^4,-q^6).
\end{aligned}
$$

This establishes (1.26).

According to (4.4)–(4.7) and (4.9),

$$
\begin{aligned}
&(-q,-q^4;q^5)_\infty (q^4,q^6;q^{10})_\infty^3 - q(-q^2,-q^3;q^5)_\infty (q^2,q^8;q^{10})_\infty^3\\
&= \frac{1}{(q^5;q^5)_\infty (q^{10};q^{10})_\infty^3}\Big(f(q,q^4)f(-q^4,-q^6)^3 - qf(q^2,q^3)f(-q^2,-q^8)^3\Big)\\
&= \frac{1}{(q^5;q^5)_\infty^3 (q^{10};q^{10})_\infty^2} f(q,q^4)f(q^2,q^3)\Big(\left(f(-q^2,-q^3)f(-q^4,-q^6)\right) f(-q^4,-q^6)\\
&\qquad - q\left(f(-q,-q^4)f(-q^2,-q^8)\right) f(-q^2,-q^8)\Big)\\
&= \frac{(q^2;q^2)_\infty^2 (q^5;q^5)_\infty}{(q;q)_\infty (q^{10};q^{10})_\infty^4} f(-q,-q^4)f(q^2,q^3).
\end{aligned}
$$

Also, using (4.4),

$$
\begin{aligned}
&\frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^2}{(q^{10};q^{10})_\infty^3}(-q^2,-q^3;q^5)_\infty^2 (q^2,q^8;q^{10})_\infty\\
&= \frac{(q^2;q^2)_\infty}{(q^{10};q^{10})_\infty^4} f(q^2,q^3)^2 f(-q^2,-q^8)\\
&= \frac{(q^2;q^2)_\infty}{(q^5;q^5)_\infty^2 (q^{10};q^{10})_\infty^3}\left(f(q,q^4)f(q^2,q^3)\right) f(-q,-q^4)f(q^2,q^3)\\
&= \frac{(q^2;q^2)_\infty^2 (q^5;q^5)_\infty}{(q;q)_\infty (q^{10};q^{10})_\infty^4} f(-q,-q^4)f(q^2,q^3).
\end{aligned}
$$

This proves (1.28).

The proofs of (1.27) and (1.29) are similar to those of (1.26) and (1.28), respectively.

5. Final remarks

We close this paper with some remarks.

1) Following the same line of proving (1.16)–(1.19), we can also prove

$$a_{1,1,7}(7n+1) = a_{3,3,7}(7n+3), \tag{5.1}$$

$$a_{1,6,7}(7n+6) = -a_{2,2,7}(7n+6), \tag{5.2}$$

$$a_{4,6,11}(11n+5) = -a_{5,2,11}(11n+4), \tag{5.3}$$

$$a_{4,6,11}(11n+7) = a_{5,2,11}(11n+6). \tag{5.4}$$

There are other identities similar to (5.1)–(5.4) for $t = 11$. Therefore it is natural to ask whether or not there exist some identities between $a_{r,s,t}(n)$ and $b_{r,s,t}(n)$ for arbitrary prime t, which parallel to (5.1)–(5.4).

2) Following the similar method of proving (1.3) in [7], we can also obtain
$$g_2(5n+3) = h_2(5n+1) = 0, \tag{5.5}$$
which is parallel to (1.3).
Eqs. (1.27) and (5.5) imply
$$h_2(5n+3) \equiv 0 \pmod 2.$$
Furthermore, there are some results similar to (1.16)–(1.23) in another types of q-series expansions. Relating to (1.15) and (1.26), define
$$(-q,-q^4;q^5)_\infty^2(q^4,q^6;q^{10})_\infty^2(q^2,q^8;q^{10})_\infty = \sum_{n=0}^{\infty} \widehat{g_1}(n)q^n,$$
$$(-q^2,-q^3;q^5)_\infty^2(q^2,q^8;q^{10})_\infty^2(q^4,q^6;q^{10})_\infty = \sum_{n=0}^{\infty} \widehat{h_1}(n)q^n.$$
Following the similar strategy of proving (1.16), we can also obtain
$$\widehat{g_1}(5n) = \widehat{h_1}(5n),$$
$$\widehat{g_1}(5n+2) = \widehat{h_1}(5n+1) = 0,$$
$$\widehat{g_1}(5n+3) = -\widehat{h_1}(5n+3).$$
Of course, we can also obtain similar results for (1.27)–(1.29).

3) We also learn from Nayandeep Deka Baruah and Mandeep Kaur [1] that they have provided new proofs of (1.18)–(1.21). Their proofs rely highly on two known q-identities [4, Eqs. (40.1.1) and (41.1.5)] involving Ramanujan's continued fractions.

4) Finally, with the help of computer, the signs of coefficients in q-series (1.24) and (1.25) appear to be periodic.

Conjecture 5.1. *For any integer $n \geq 0$,*
$$g_2(5n) > 0, \tag{5.6}$$
$$g_2(5n+1) < 0, \tag{5.7}$$
$$g_2(5n+2) > 0, \tag{5.8}$$
$$g_2(5n+4) < 0, \tag{5.9}$$
$$h_2(5n) > 0, \tag{5.10}$$
$$h_2(5n+2) < 0, \tag{5.11}$$
$$h_2(5n+3) < 0, \tag{5.12}$$
$$h_3(5n+4) > 0. \tag{5.13}$$

It would be interesting to find an elementary proof of (5.6)–(5.13).

Acknowledgement

The authors are indebted to Shishuo Fu for his helpful comments on a preliminary version of this paper. The authors also would like to thank the anonymous referee for his/her careful reading and helpful comments on an earlier version of the paper. The first author was supported by the National Natural Science Foundation of China (No. 11501061) and the Fundamental Research Funds for the Central Universities (No. 2018CDXYST0024). The second author was supported by the National Natural Science Foundation of China (No. 11571143) and Jiangsu National Funds for Distinguished Young Scientists (No. BK20180044).

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