On least distance eigenvalue of uniform hypergraphs

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Abstract

For $k \geq 2$, we determine the connected k-uniform hypergraphs with least distance eigenvalues in $\left(\frac{1-\sqrt{33}}{2}, 0\right)$, the k-uniform hypertrees with least distance eigenvalues in $[-2k, 0)$, and the k-uniform unicyclic hypergraphs with least distance eigenvalues in $\left(-k+1-\sqrt{(k-1)(k-2)}, 0\right)$, respectively, and determine the k-uniform hypergraphs (hypertrees, respectively) with minimum distance spread.

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1 Introduction

Let V be a nonempty finite set, and E a family of nonempty subsets of V. The pair $G = (V, E)$ is called a hypergraph with vertex set $V(G) = V$, and with edge set $E(G) = E$, see [2, 3]. The order of G is the cardinality of $V(G)$. If all edges of G have cardinality k, then G is k-uniform. A 2-uniform hypergraph is an ordinary graph. For $u, v \in V(G)$, if they are contained in some edge of G, then we say that they are adjacent, or v is a neighbor of u. Let $N_G(u)$ be the set of neighbors of u in G.

For $u, v \in V(G)$, a walk from u to v in G is defined to be an alternating sequence of vertices and edges $(v_0, e_1, v_1, \dots, v_{p-1}, e_p, v_p)$ with $v_0 = u$ and $v_p = v$ such that edge e_i contains vertices v_{i-1} and v_i, and $v_{i-1} \neq v_i$ for $i = 1, \dots, p$. The value p is the length of this walk. A path is a walk with all v_i distinct and all e_i distinct. A cycle is a walk containing at least two edges, all e_i are distinct and all v_i are distinct except $v_0 = v_p$. A vertex $u \in V(G)$ is viewed as a path (from u to u) of length 0. If there is a path from u to v for any $u, v \in V(G)$, then we say that G is connected.

A hypertree is a connected hypergraph with no cycles. Note that a k-uniform hypertree with m edges always has order $1 + (k-1)m$, see [3, p. 392]. A unicyclic hypergraph

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is a connected hypergraph with exactly one cycle. Note that a k-uniform unicyclic hypergraph with m edges always has order $(k-1)m$, see [3, p. 393].

Let G be a connected hypergraph with $V(G) = \{v_1, \ldots, v_n\}$. For $u, v \in V(G)$, the distance between u and v is the length of a shortest path from u to v in G, denoted by $d_G(u, v)$. In particular, $d_G(u, u) = 0$. The diameter of G is the maximum distance between all vertex pairs of G. The distance matrix of G is the $n \times n$ matrix $D(G) = (d_G(u, v))_{u,v \in V(G)}$. The eigenvalues of $D(G)$ are called the distance eigenvalues of G. Since $D(G)$ is real and symmetric, the distance eigenvalues of G are real. Let $\rho(G)$ and $\lambda(G)$ be the largest and least distance eigenvalues of G, respectively.

Interest in distance eigenvalues of 2-uniform hypergraphs (ordinary graphs) began during the 1970's with the appearance of the paper by Graham and Pollak [6], in which a relationship was established between the number of negative distance eigenvalues and the addressing problem in data communication systems. Edelberg et al. [4] and Graham and Lovász [5] studied the characteristic polynomial of the distance matrix of graphs. Though the distance eigenvalues of ordinary graphs have been studied to some extent (see the recent survey of Aouchiche and Hansen [1] and referees therein), it still of interest to investigate the largest and the least distance eigenvalues of graphs with particular structures. Sivasubramanian [15] gave a formula for the inverse of a few q-analogs of the distance matrix of 3-uniform hypertrees. The largest distance eigenvalue of uniform hypergraphs has also received attention [12]. Generally, apart from the largest eigenvalue, the least eigenvalue of a symmetric matrix is of most important, see, e.g. [14]. The spread of a real symmetric matrix is the difference between its largest and least eigenvalues, which has applications in combinatorial optimization problems [7]. This quantity has been studied extensively, see, e.g. [16]. For a connected hypergraph G, the distance spread of G is defined as $s(G) = \rho(G) - \lambda(G)$. For some classes of ordinary graphs, it has been studied, see, e.g. [8, 18].

In this paper, we determine the k-uniform hypergraphs with least distance eigenvalues in $\left(\frac{1-\sqrt{33}}{2}, 0\right)$, the k-uniform hypertrees with least distance eigenvalues in $[-2k, 0)$, and the k-uniform unicyclic hypergraphs with least distance eigenvalues in $\left(-k+1-\sqrt{(k-1)(k-2)}, 0\right)$ respectively. Moreover, we determine the k-uniform hypergraphs and hypertrees respectively with minimum distance spread.

2 Preliminaries

For $2 \le k \le n$, the complete k-uniform hypergraph, denoted by K_n^k, is a hypergraph G of order n such that $E(G)$ consists of all k-subsets of $V(G)$.

A set S of vertices of a hypergraph G is a (strongly) independent set of G if any two vertices in S are not adjacent. An independent set S of G is maximal if $S \cup \{u\}$ for any $u \in V(G) \setminus S$ is not an independent set. A hypergraph is s-partite if its vertex set can be partitioned into s parts (called partite sets), each of which is an independent set. A k-uniform hypergraph G is a complete s-partite hypergraph if each choice of k vertices from distinct partite sets forms an edge. Let $K_{n_1,\ldots,n_s}^k$ be the complete s-partite k-uniform hypergraph with partite sets $V_1, \ldots, V_s$ such that $|V_i| = n_i$ for $i = 1, \ldots, s$.

Obviously, $K_n^k = K^k_{\underbrace{1,\dots,1}_{n}}$.

A k-uniform loose path of order n is a hypertree with vertex set $\{v_1, \dots, v_n\}$ and with the set of m edges $e_i = \{v_{i(k-1)+1}, \dots, v_{i(k-1)+k}\}$ for $i = 0, \dots, m-1$, denoted by $P_{n,k}$, where $m = \frac{n-1}{k-1}$.

For a k-uniform hypertree G of order n, if $V(G)$ can be partitioned into $m+1$ subsets $\{u\}, V_1, \dots, V_m$ such that $|V_1| = \dots = |V_m| = k-1$, and $E(G) = \{\{u\} \cup V_i : 1 \le i \le m\}$, then we call G is a hyperstar (with center u), denoted by $S_{n,k}$.

A k-uniform loose cycle of order n is a unicyclic hypergraph with vertex set $\{v_1, \dots, v_n\}$ and with the set of m edges $e_i = \{v_{i(k-1)+1}, \dots, v_{i(k-1)+k}\}$ for $i = 0, \dots, m-1$, denoted by $C_{n,k}$, where $m = \frac{n}{k-1}$, and $v_{(m-1)(k-1)+k} = v_1$.

Let G be a k-uniform hypergraph with $V(G) = \{v_1, \dots, v_n\}$. A column vector $x = (x_{v_1}, \dots, x_{v_n})^\top \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_{v_i}, i.e., $x(v_i) = x_{v_i}$ for $i = 1, \dots, n$. Then λ is a distance eigenvalue with corresponding eigenvector x if and only if $x \neq 0$ and for each $u \in V(G)$, $\lambda x_u = \sum_{v \in V(G)} d_G(u, v) x_v$. Obviously, the distance eigenvalues of G are the roots of $\det(\lambda I_n - D(G)) = 0$, where I_n is the identity matrix of order n.

For a connected k-uniform hypergraph G, if H is a connected k-uniform subhypergraph, and $d_H(u, v) = d_G(u, v)$ for $u, v \in V(H)$, then H is said to be a distance-preserving k-uniform subhypergraph of G. If H is a distance-preserving k-uniform subhypergraph of G with $V(H) = V(G)$, then say that H is a spanning distance-preserving subhypergraph of G.

For a connected k-uniform hypergraph G with $V_0 \subseteq V(G)$, let $D(G)[V_0]$ be the principal submatrix of $D(G)$ indexed by all the vertices of V_0.

For an $n \times n$ real symmetric matrix M, let $\lambda(M)$ be the least eigenvalue of M. From the interlacing theorem [13, pp. 185–186], we have

Lemma 2.1. *Let N be an $n \times n$ symmetric matrix and M a principal submatrix of N of order m, where $2 \le m \le n$. Then $\lambda(N) \le \lambda(M)$.*

Let $J_{n\times m}$ and $0_{n\times m}$ be the all-one and all-zero $n \times m$ matrices, respectively. Let $1_n = J_{n\times 1}$, $J_n = J_{n\times n}$, and $0_n = 0_{n\times n}$.

3 Least distance eigenvalue

In this section, we study the least distance eigenvalue of a uniform hypergraph, and especially for hypertrees and unicyclic hypergraphs.

Lemma 3.1. *Let G be a connected k-uniform hypergraph with diameter $d \ge 1$, where $k \ge 2$. Then $\lambda(G) \le -d$.*

Proof. Let $u, v \in V(G)$ such that $d_G(u, v) = d$. Then $D(G)[\{u, v\}] = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$. By Lemma 2.1, we have $\lambda(G) \le \lambda(D(G)[\{u, v\}]) = -d$. □

This is actually known [9].

Lemma 3.2. *For $2 \le k \le s \le n-1$, let G be a spanning distance-preserving subhypergraph of a complete s-partite k-uniform hypergraph H of order n. Then $\lambda(G) = -2$.*

Proof. Obviously, $D(G) = D(H)$. Assume that $H = K^k_{n_1,\dots,n_s}$. Then H has partite sets $V_1, \dots, V_s$ such that $|V_i| = n_i$ for $i = 1, \dots, s$. Note that there is no edge of H containing at least two vertices of V_i for $i = 1, \dots, s$, and $E(H)$ contains all k-subsets that have their vertices in some k different partite sets. With respect to the partition $V(H) = V_1 \cup \dots \cup V_s$, we have

$$D(H) = \begin{pmatrix} 2J_{n_1} - 2I_{n_1} & J_{n_1 \times n_2} & \cdots & J_{n_1 \times n_s} \\ J_{n_2 \times n_1} & 2J_{n_2} - 2I_{n_2} & \cdots & J_{n_2 \times n_s} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n_s \times n_1} & J_{n_s \times n_2} & \cdots & 2J_{n_s} - 2I_{n_s} \end{pmatrix}.$$

Observe that the eigenvalues of $D(H) + 2I_n$ are $n_1, \dots, n_s$, and 0 (with multiplicity $n-s$), and thus $\lambda(D(H)) = -2$. □

Since $D(H)$ is just the distance matrix of the complete s-partite graph with partite sizes $n_1, \dots, n_s$, the previous lemma follows also from [10, Lemma 2.5] or [17, Lemma 3.1].

Note that K^k_n is a complete n-partite k-uniform hypergraph.

For an ordinary complete multipartite graph G ($k = 2$), its spanning distance-preserving subgraph must be itself. But this is not true for k-uniform hypergraphs with $k \ge 3$. Consider a 5-uniform hypergraph G with $V(G) = \{1, \dots, 9\}$ and $E(G) = \{e_1, \dots, e_6\}$, where $e_1 = \{1,3,6,7,8\}$, $e_2 = \{1,4,7,8,9\}$, $e_3 = \{1,5,6,7,8\}$, $e_4 = \{2,3,7,8,9\}$, $e_5 = \{2,4,6,7,8\}$, $e_6 = \{2,5,6,7,9\}$. We partition $V(G)$ into $\{1,2\} \cup \{3,4,5\} \cup \{6\} \cup \{7\} \cup \{8\} \cup \{9\}$. Obviously, $D(G) = D(K^5_{2,3,1,1,1,1})$. Thus G is a spanning distance-preserving subhypergraph of $K^5_{2,3,1,1,1,1}$. Obviously, $G \ncong K^5_{2,3,1,1,1,1}$.

Theorem 3.1. *Let G be a connected k-uniform hypergraph of order n, where $2 \le k \le n$. Then*

(i) $\lambda(G) \le -1$ with equality if and only if G is a spanning distance-preserving subhypergraph of K^k_n;

(ii) if G is not a spanning distance-preserving subhypergraph of K^k_n, then $\lambda(G) \le -2$ with equality if and only if G is a spanning distance-preserving subhypergraph of some complete s-partite k-uniform hypergraph of order n with $k \le s \le n-1$;

(iii) if G is not a spanning distance-preserving subhypergraph of any complete s-partite k-uniform hypergraph of order n with $k \le s \le n$, then $\lambda(G) \le \frac{1-\sqrt{33}}{2}$.

Proof. Let d be the diameter of G.

Obviously, if G is a spanning distance-preserving subhypergraph of K^k_n, then $\lambda(G) = \lambda(J_n - I_n) = -1$. If G is not a spanning distance-preserving subhypergraph of K^k_n, then $d \ge 2$, and thus by Lemma 3.1, we have $\lambda(G) \le -2$. Therefore $\lambda(G) = -1$ or $\lambda(G) \le -2$. Now (i) follows.

Suppose that G is not a spanning distance-preserving subhypergraph of K^k_n. Then $d \ge 2$.

If G is a spanning distance-preserving subhypergraph of a complete s-partite k-uniform hypergraph with $k \le s \le n-1$, then by Lemma 3.2, we have $\lambda(G) = -2$.

Suppose that $\lambda(G) = -2$. By Lemma 3.1, $d \leq 2$, and then $d = 2$. Thus any two nonadjacent vertices in G have at least one neighbor in common. Let $u, v \in V(G)$ such that $d_G(u, v) = 2$. Suppose that $w \in N_G(v) \setminus N_G(u)$. Then $d_G(u, w) = 2$ and $d_G(v, w) = 1$. We have

$$D(G)[\{u, v, w\}] = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

By Lemma 2.1, $\lambda(G) \leq \lambda(D(G)[\{u, v, w\}]) = \frac{1-\sqrt{33}}{2} \approx -2.3723 < -2$, a contradiction. Thus $N_G(v) \subseteq N_G(u)$. Similarly, we have $N_G(u) \subseteq N_G(v)$. Then $N_G(u) = N_G(v)$. Thus any two nonadjacent vertices in G have the same neighbors. Since $d = 2$, there are nonadjacent vertices in G. Thus we may choose a maximal independent set V_1 with $|V_1| \geq 2$ such that $N_G(u) = V(G) \setminus V_1$ for any $u \in V_1$. If there are nonadjacent vertices in $V(G) \setminus V_1$ and $V(G) \setminus V_1$ is not an independent set, then we may choose a maximal independent V_2 in $V(G) \setminus V_1$ such that $N_G(u) = V(G) \setminus V_2$ for any $u \in V_2$. It is easily seen that this process can be continued until we reach a maximal independent set V_r such that $V(G) \setminus (V_1 \cup \cdots \cup V_r)$ is an independent set or any two vertices in $V(G) \setminus (V_1 \cup \cdots \cup V_r)$ are adjacent. In the former case, G is an $(r+1)$-partite k-uniform hypergraph with $k \leq r+1 \leq n-1$. In the latter case, let $p = |V(G) \setminus (V_1 \cup \cdots \cup V_r)|$. Then $V(G) \setminus (V_1 \cup \cdots \cup V_r)$ may be partitioned in to p parts each containing a single vertex, and thus G is a $(r+p)$-partite k-uniform hypergraph with $k \leq r+p \leq n-1$. In either case, there is an integer s with $k \leq s \leq n-1$ such that $V(G)$ may be partitioned into s partite sets $V_1, \ldots, V_s$, each partite set is a maximal independent set, and any two vertices in different parts are adjacent. Let $n_i = |V_i|$ for $i = 1, \ldots, s$. Then G is a subhypergraph of $K^k_{n_1,\ldots,n_s}$ and $D(G) = D(K^k_{n_1,\ldots,n_s})$. Thus G is a spanning distance-preserving subhypergraph of $K^k_{n_1,\ldots,n_s}$. This proves (ii).

Now suppose that G is not a spanning distance-preserving subhypergraph of any complete s-partite k-uniform hypergraph with $k \leq s \leq n$. Then either $d \geq 3$ or $d = 2$ and there is a pair of nonadjacent vertices such that they do not have the same neighbors. By above argument, $\lambda(G) \leq \frac{1-\sqrt{33}}{2}$. This proves (iii). □

Note that the above result may be stated using the language of ordinary graphs, see [17].

By Theorem 3.1, we have

Corollary 3.1. *If G is a k-uniform hypergraph of order n, where $2 \leq k \leq n$, then $\lambda(G) \in \left(\frac{1-\sqrt{33}}{2}, 0\right)$ if and only if G is a spanning distance-preserving subhypergraph of some complete s-partite k-uniform hypergraph of order n with $k \leq s \leq n$.*

Lemma 3.3. *For integers n, k with $2 \leq k \leq n$, we have*

(i) $\lambda(P_{3k-2,k}) = -k - \sqrt{k^2 - k}$;

(ii) $\lambda(S_{n,k}) = -k$ if $\frac{n-1}{k-1} \geq 2$.

Proof. For $k = 2$, the result in (i) follows from direct calculation, and the result in (ii) follows from Lemma 3.2.

Suppose that $k \geq 3$.

First we prove (i). Let $E(P_{3k-2,k}) = \{e_1, e_2, e_3\}$ with $e_1 \cap e_2 = \{u\}$ and $e_2 \cap e_3 = \{v\}$. Let $\lambda = \lambda(P_{3k-2,k})$. Let x be an eigenvector of $D(P_{3k-2,k})$ corresponding to λ. For $w \in e_1 \setminus \{u\}$, we have

$$\lambda x_w = \sum_{z \in e_1 \setminus \{w\}} x_z + 2 \sum_{z \in e_2 \setminus \{u\}} x_z + 3 \sum_{z \in e_3 \setminus \{v\}} x_z.$$

Thus for $w, w' \in e_1 \setminus \{u\}$ with $w \neq w'$, we have $(\lambda + 1)(x_w - x_{w'}) = 0$. By Lemma 3.1, we have $\lambda \leq -3$, and thus $x_w = x_{w'}$. Therefore, the entry of x at each vertex of $e_1 \setminus \{u\}$ is the same, which is denoted by a. Similarly, the entry of x at each vertex of $e_2 \setminus \{u, v\}$ is the same, which is denoted by b, and the entry of x at each vertex of $e_3 \setminus \{v\}$ is the same, which is denoted by c. Then

$$\begin{aligned}
\lambda a &= (k-2)a + x_u + 2(k-2)b + 2x_v + 3(k-1)c, \\
\lambda x_u &= (k-1)a + (k-2)b + x_v + 2(k-1)c, \\
\lambda b &= 2(k-1)a + x_u + (k-3)b + x_v + 2(k-1)c, \\
\lambda x_v &= 2(k-1)a + x_u + (k-2)b + (k-1)c, \\
\lambda c &= 3(k-1)a + 2x_u + 2(k-2)b + x_v + (k-2)c.
\end{aligned}$$

We view these equations as a homogeneous linear system in the five variables a, x_u, b, x_v, and c. Thus λ is the least root of $g_1(t) = 0$, where $g_1(t) = (t^2 + 2kt + k)f_1(t)$ and $f_1(t) = t^3 - t^2(5k-7) - t(4k^2 - k - 4) - 3k^2 + 3k$.

Let t_1 and t_2 be the roots of

$$f_1'(t) = 3t^2 - 2(5k-7)t - 4k^2 + k + 4 = 0,$$

where $t_1 \leq t_2$. Then $-k - \sqrt{k^2 - k} < t_{1,2} = \frac{5k-7 \pm \sqrt{37k^2 - 73k + 37}}{3}$. Noting that $f_1(-k - \sqrt{k^2 - k}) = -2(k-1)(5k^2 - 4k + (5k-2)\sqrt{k^2 - k}) < 0$, $f_1(t_1) > 0$ and $f_1(t_2) < 0$, the least root of $f_1(t) = 0$ is more than $-k - \sqrt{k^2 - k}$. Thus $\lambda = -k - \sqrt{k^2 - k}$.

Now we prove (ii). Let $m = \frac{n-1}{k-1}$. We partition $V(S_{n,k})$ into $\{u\} \cup V_1 \cup \cdots \cup V_m$ such that $|V_1| = \cdots = |V_m| = k - 1$, and $E(S_{n,k}) = \{\{u\} \cup V_i : 1 \leq i \leq m\}$. Then with respect to this partition, we have

$$D(S_{n,k}) = \begin{pmatrix} 0 & 1_{k-1}^\top & 1_{k-1}^\top & \cdots & 1_{k-1}^\top \\ 1_{k-1} & J_{k-1} - I_{k-1} & 2J_{k-1} & \cdots & 2J_{k-1} \\ 1_{k-1} & 2J_{k-1} & J_{k-1} - I_{k-1} & \cdots & 2J_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{k-1} & 2J_{k-1} & 2J_{k-1} & \cdots & J_{k-1} - I_{k-1} \end{pmatrix}.$$

Let $b_i = 2 + (i-1)(k-1)$ for $1 \leq i \leq m$.

To calculate $\det(tI_n - D(S_{n,k}))$, first we subtract the b_i-th row with $1 \leq i \leq m$ from the $(b_i + 1)$-th, $\ldots$, $(b_i + k - 2)$-th rows, respectively, to obtain

$$\det(tI_n - D(S_{n,k})) = \det \begin{pmatrix} t & -1_{k-1}^\top & -1_{k-1}^\top & \cdots & -1_{k-1}^\top \\ B & A & C & \cdots & C \\ B & C & A & \cdots & C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & C & C & \cdots & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} t & -1_{k-2}^{\top} \\ -(1+t)\cdot 1_{k-2} & (1+t)I_{k-2} \end{pmatrix}, B = \begin{pmatrix} -1 \\ 0_{(k-2)\times 1} \end{pmatrix}$$

and

$$C = \begin{pmatrix} -2 & -2\cdot 1_{k-2}^{\top} \\ 0_{(k-2)\times 1} & 0_{k-2} \end{pmatrix}.$$

Next for $1 \leq i \leq m$, we add the (b_i+1)-th, $\ldots, (b_i+k-2)$-th columns to the b_i-th column, to obtain

$$\det(tI_n - D(S_{n,k})) = \det \begin{pmatrix} t & P & P & \cdots & P \\ B & A^* & C^* & \cdots & C^* \\ B & C^* & A^* & \cdots & C^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & C^* & C^* & \cdots & A^* \end{pmatrix} = (t+1)^{(k-2)m} \cdot \det(M),$$

where $P = \begin{pmatrix} -k+1 & -1_{k-2}^{\top} \end{pmatrix}$,

$$A^* = \begin{pmatrix} t-k+2 & -1_{k-2}^{\top} \\ 0_{(k-2)\times 1} & (1+t)I_{k-2} \end{pmatrix}, C^* = \begin{pmatrix} -2(k-1) & -2\cdot 1_{k-2}^{\top} \\ 0_{(k-2)\times 1} & 0_{k-2} \end{pmatrix},$$

and

$$M = \begin{pmatrix} t & -k+1 & -(k-1)\cdot 1_{m-1}^{\top} \\ -1 & t-k+2 & -2(k-1)\cdot 1_{m-1}^{\top} \\ -1_{m-1} & -2(k-1)\cdot 1_{m-1} & -2(k-1)J_{m-1} + (k+t)I_{m-1} \end{pmatrix}.$$

To calculate $\det(M)$, by subtracting the 2nd row from the 3th, $\ldots, (m+1)$-th rows of M, respectively, we have

$$\det(M) = \det \begin{pmatrix} t & -k+1 & -(k-1)\cdot 1_{m-1}^{\top} \\ -1 & t-k+2 & -2(k-1)\cdot 1_{m-1}^{\top} \\ 0_{(m-1)\times 1} & (-k-t)\cdot 1_{m-1} & (k+t)I_{m-1} \end{pmatrix},$$

and then by adding each of last $m-1$ columns to the 2nd column for above determinant, we have

$$\det(M) = \det \begin{pmatrix} t & -m(k-1) & -(k-1)\cdot 1_{m-1}^{\top} \\ -1 & t-k+2-2(m-1)(k-1) & -2(k-1)\cdot 1_{m-1}^{\top} \\ 0_{(m-1)\times 1} & 0_{(m-1)\times 1} & (k+t)I_{m-1} \end{pmatrix}.$$

Therefore

$$\begin{aligned} \det(tI_n - D(S_{n,k})) &= (t+1)^{(k-2)m} \det(M) \\ &= (t+1)^{(k-2)m} \cdot (t+k)^{m-1} \\ &\quad \cdot \det \begin{pmatrix} t & -m(k-1) \\ -1 & t-k+2-2(m-1)(k-1) \end{pmatrix} \\ &= (t+1)^{(k-2)m} \cdot (t+k)^{m-1} \cdot \det \begin{pmatrix} t & -(n-1) \\ -1 & t-2n+k+2 \end{pmatrix} \\ &= (t+1)^{(k-2)m} \cdot (t+k)^{m-1} \cdot (t^2 - (2n-k-2)t - n + 1). \end{aligned}$$

Then the distance eigenvalues of $S_{n,k}$ are -1 (with multiplicity $(k-2)m$), $-k$ (with multiplicity $m-1$), and $\frac{2n+k-2\pm\sqrt{4n^2-4nk+k^2+4k-4n}}{2}$ (>-1). Thus $\lambda(S_{n,k})=-k$ if $m\geq 2$. $\square$

For integers k,n,a with $2\leq k\leq n$ and $1\leq a\leq\left\lfloor\frac{n-k}{2k-2}\right\rfloor$, let $D_{n,k,a}$ be the k-uniform hypergraph obtained from vertex-disjoint hyperstars $S_{a(k-1)+1,k}$ with center u and $S_{n-k-a(k-1)+1,k}$ with center v by adding $k-2$ new vertices $w_1,\ldots,w_{k-2}$ and a new edge $\{u,v,w_1,\ldots,w_{k-2}\}$.

For integers $k,n,a_1,\ldots,a_k$ with $2\leq k\leq n$, $a_1\geq\cdots\geq a_k\geq 0$ and $\sum_{i=1}^k a_i+1=\frac{n-1}{k-1}$, let $D_k(n;a_1,\ldots,a_k)$ be the k-uniform hypergraph obtained from $S_{k,k}$ by attaching a_i pendant edges at v_i with $V(S_{k,k})=\{v_1,\ldots,v_k\}$. Obviously, $D_{4,2,1}=D_2(4;1,1)$.

Lemma 3.4. *For $k\geq 2$, we have*

(i) $\lambda(P_{4k-3,k})<-2k$;

(ii) $\lambda(D_{4k-3,k,1})<-2k$ if $k\geq 3$;

(iii) $\lambda\left(D_k(k^2;\underbrace{1,\ldots,1}_{k})\right)=-k-\sqrt{k^2-k}$.

Proof. For $k=2$, the results in (i) and (iii) follow from direct calculation.

Suppose that $k\geq 3$.

First we prove (i). Let $G_1=P_{4k-3,k}$, and $E(G_1)=\{e_1,e_2,e_3,e_4\}$ with $e_1\cap e_2=\{u\}$, $e_2\cap e_3=\{v\}$ and $e_3\cap e_4=\{w\}$. Let $\lambda=\lambda(G_1)$. Let x be an eigenvector of $D(G_1)$ corresponding to λ. As in the proof of Lemma 3.3 (i), the entry of x at each vertex of $e_1\setminus\{u\}$ is the same, which is denoted by x_1, the entry of x at each vertex of $e_2\setminus\{u,v\}$ is the same, which is denoted by x_2, the entry of x at each vertex of $e_3\setminus\{v,w\}$ is the same, which is denoted by x_3, and the entry of x at each vertex of $e_4\setminus\{w\}$ is the same, which is denoted by x_4. Then

$$\begin{aligned}
\lambda x_1 &= (k-2)x_1+x_u+2(k-2)x_2+2x_v+3(k-2)x_3+3x_w+4(k-1)x_4,\\
\lambda x_u &= (k-1)x_1+(k-2)x_2+x_v+2(k-2)x_3+2x_w+3(k-1)x_4,\\
\lambda x_2 &= 2(k-1)x_1+x_u+(k-3)x_2+x_v+2(k-2)x_3+2x_w+3(k-1)x_4,\\
\lambda x_v &= 2(k-1)x_1+x_u+(k-2)x_2+(k-2)x_3+x_w+2(k-1)x_4,\\
\lambda x_3 &= 3(k-1)x_1+2x_u+2(k-2)x_2+x_v+(k-3)x_3+x_w+2(k-1)x_4,\\
\lambda x_w &= 3(k-1)x_1+2x_u+2(k-2)x_2+x_v+(k-2)x_3+(k-1)x_4,\\
\lambda x_4 &= 4(k-1)x_1+3x_u+3(k-2)x_2+2x_v+2(k-2)x_3+x_w+(k-2)x_4.
\end{aligned}$$

We view these equations as a homogeneous linear system in the seven variables x_1, x_u, x_2, x_v, x_3, x_w and x_4. Thus λ is the least root of $g_2(t)=0$, where

$$\begin{aligned}
g_2(t) &= (t^3+t^2(4k-1)+t(2k^2+k)+k^2)\\
&\quad\cdot(t^4-t^3(8k-11)-t^2(10k^2-3k-10)-t(13k^2-12k-2)-4k^2+4k).
\end{aligned}$$

Note that $g_2(-2k)=2k^4(4k-5)(20k^2-25k+6)>0$. Therefore the least root of $g_2(t)=0$ is less than $-2k$, i.e., $\lambda<-2k$.

Now we prove (ii). Let $G_2=D_{4k-3,k,1}$, and $E(G_2)=\{e_1,e_2,e_3,e_4\}$ with $e_1\cap e_2\cap e_4=\{u\}$ and $e_2\cap e_3=\{v\}$. Let $\lambda'=\lambda(G_2)$. Let x be an eigenvector of $D(G_2)$ corresponding to λ'. As in the proof of Lemma 3.3 (i), the entry of x at each vertex of $e_1 \setminus \{u\}$ is the same, which is denoted by x_1, the entry of x at each vertex of $e_2 \setminus \{u, v\}$ is the same, which is denoted by x_2, the entry of x at each vertex of $e_3 \setminus \{v\}$ is the same, which is denoted by x_3, and the entry of x at each vertex of $e_4 \setminus \{u\}$ is the same, which is denoted by x_4. Then

$$\begin{aligned}
\lambda' x_1 &= (k-2)x_1 + x_u + 2(k-2)x_2 + 2x_v + 3(k-1)x_3 + 2(k-1)x_4, \\
\lambda' x_4 &= 2(k-1)x_1 + x_u + 2(k-2)x_2 + 2x_v + 3(k-1)x_3 + (k-2)x_4,
\end{aligned}$$

and thus $(\lambda' + k)(x_1 - x_4) = 0$. Since $P_{3k-2,k}$ is a distance-preserving k-uniform sub-hypergraph of G_2, we have by Lemma 2.1 and Lemma 3.3 (i) that $\lambda' \le \lambda(P_{3k-2,k}) = -k - \sqrt{k^2 - k}$. Thus $x_1 = x_4$.

For G_2, we have

$$\begin{aligned}
\lambda' x_1 &= (3k-4)x_1 + x_u + 2(k-2)x_2 + 2x_v + 3(k-1)x_3, \\
\lambda' x_u &= (2k-2)x_1 + (k-2)x_2 + x_v + 2(k-1)x_3, \\
\lambda' x_2 &= (4k-4)x_1 + x_u + (k-3)x_2 + x_v + 2(k-1)x_3, \\
\lambda' x_v &= (4k-4)x_1 + x_u + (k-2)x_2 + (k-1)x_3, \\
\lambda' x_3 &= (6k-6)x_1 + 2x_u + 2(k-2)x_2 + x_v + (k-2)x_3.
\end{aligned}$$

We view these equations as a homogeneous linear system in the five variables x_1, x_u, x_2, x_v and x_3. Thus λ' is the least root of $g_3(t) = 0$, where

$$\begin{aligned}
g_3(t) &= t^5 - t^4(5k-9) - t^3(23k^2 - 27k - 2) - t^2(13k^3 + 7k^2 - 26k + 2) \\
&\quad - t(15k^3 - 12k^2 - 4k) - 4k^3 + 4k^2.
\end{aligned}$$

Note that $g_3(-2k) = 2k^2(10k^3 - 35k^2 + 30k - 6) > 0$. Therefore the least root of $g_3(t) = 0$ is less than $-2k$, i.e., $\lambda' < -2k$.

Finally we prove (iii). Let $G_3 = D_k(k^2; \underbrace{1, \dots, 1}_{k})$, and $E(G_3) = \{e_1, \dots, e_k, e\}$ with $e = \{v_1, \dots, v_k\}$ and $e_i \cap e = \{v_i\}$ for $1 \le i \le k$. We partition $V(G_3)$ into $(e_1 \setminus \{v_1\}) \cup \dots \cup (e_k \setminus \{v_k\}) \cup e$. Then with respect to this partition, we have

$$D(G_3) = \begin{pmatrix}
J_{k-1} - I_{k-1} & 3J_{k-1} & \cdots & 3J_{k-1} & A_1 \\
3J_{k-1} & J_{k-1} - I_{k-1} & \cdots & 3J_{k-1} & A_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
3J_{k-1} & 3J_{k-1} & \cdots & J_{k-1} - I_{k-1} & A_k \\
A_1^\top & A_2^\top & \cdots & A_k^\top & J_k - I_k
\end{pmatrix},$$

where, for $1 \le i \le k$, A_i is the matrix obtained from $2J_{(k-1)\times k}$ by subtracting 1 from each entry of i-th column.

Let $s_i = 1 + (i-1)(k-1)$ for $1 \le i \le k$.

To calculate $\det(tI_n - D(G_3))$, first we subtract the s_i-th row with $1 \le i \le k$ from the $(s_i + 1)$-th, $\dots$, $(s_i + k - 2)$-th rows, respectively, to obtain

$$\det(tI_n - D(G_3)) = \det \begin{pmatrix}
B & C & \cdots & C & A_1^* \\
C & B & \cdots & C & A_2^* \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C & C & \cdots & B & A_k^* \\
-A_1^\top & -A_2^\top & \cdots & -A_k^\top & (t+1)I_k - J_k
\end{pmatrix},$$

where

$$B = \begin{pmatrix} t & -1_{k-2}^{\top} \\ -(t+1)\cdot 1_{k-2} & (t+1)I_{k-2} \end{pmatrix}, C = \begin{pmatrix} -3\cdot 1_{k-1}^{\top} \\ 0_{(k-2)\times(k-1)} \end{pmatrix}, A_j^* = \begin{pmatrix} B_j \\ 0_{(k-2)\times k} \end{pmatrix},$$

and B_j is the matrix obtained from $-2\times 1_k^{\top}$ by adding 1 from the j-th entry for $1 \le j \le k$.

Next for $1 \le i \le k$, we add the (s_i+1)-th, $\ldots, (s_i+k-2)$-th columns to the s_i-th column, to obtain

$$\det(tI_n - D(G_3)) = \det \begin{pmatrix} B^* & C^* & \cdots & C^* & A_1^* \\ C^* & B^* & \cdots & C^* & A_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C^* & C^* & \cdots & B^* & A_k^* \\ P_1 & P_2 & \cdots & P_k & (t+1)I_k - J_k \end{pmatrix} = (t+1)^{k(k-2)}\cdot \det(M),$$

where

$$B^* = \begin{pmatrix} t-k+2 & -1_{k-2}^{\top} \\ 0_{(k-2)\times 1} & (t+1)I_{k-2} \end{pmatrix}, C^* = \begin{pmatrix} -3(k-1) & -3\cdot 1_{k-2}^{\top} \\ 0_{(k-2)\times 1} & 0_{k-2} \end{pmatrix},$$

$$M = \begin{pmatrix} (t+2k-1)I_k - (3k-3)J_k & I_k - 2J_k \\ (k-1)I_k - (2k-2)J_k & (t+1)I_k - J_k \end{pmatrix},$$

and P_j is the matrix obtained from $-A_j^{\top}$ by adding 2nd, $\ldots, (k-1)$-th columns to the first column for $1 \le j \le k$.

To calculate $\det(M)$, by subtracting the first row from the 2nd, $\ldots, k$-th rows, respectively, and subtracting the $(k+1)$-th row from the $(k+2)$-th, $\ldots, 2k$-th rows, respectively, to obtain

$$\det(M) = \det \begin{pmatrix} t-k+2 & -(3k-3)\cdot 1_{k-1}^{\top} & -1 & -2\cdot 1_{k-1}^{\top} \\ (-t-2k+1)\cdot 1_{k-1} & (t+2k-1)I_{k-1} & -1_{k-1} & I_{k-1} \\ -k+1 & -(2k-2)\cdot 1_{k-1}^{\top} & t & -1_{k-1}^{\top} \\ -(k-1)\cdot 1_{k-1} & (k-1)I_{k-1} & -(t+1)\cdot 1_{k-1} & (t+1)I_{k-1} \end{pmatrix},$$

and then by adding the 2nd, $\ldots, k$-th rows from the first row, and adding the $(k+2)$-th, $\ldots, (2k)$-th rows to the $(k+1)$-th row, we have

$$\det(M) = \det \begin{pmatrix} t-3k^2+5k-1 & -(3k-3)\cdot 1_{k-1}^{\top} & -2k+1 & -2\cdot 1_{k-1}^{\top} \\ 0_{(k-1)\times 1} & (t+2k-1)I_{k-1} & 0_{(k-1)\times 1} & I_{k-1} \\ -(k-1)(2k-1) & -(2k-2)\cdot 1_{k-1}^{\top} & t-k+1 & -1_{k-1}^{\top} \\ 0_{(k-1)\times 1} & (k-1)I_{k-1} & 0_{(k-1)\times 1} & (t+1)I_{k-1} \end{pmatrix}.$$

Now we add $\frac{-k+1}{t+1}$ times of the $(k+i)$-th column with $2 \le i \le k$ to the i-th column, to obtain

$$\begin{aligned} \det(M) &= (t+1)^{k-1}\left(t+2k-1+\frac{-k+1}{t+1}\right)^{k-1} \begin{pmatrix} t-3k^2+5k-1 & -2k+1 \\ -(k-1)(2k-1) & t-k+1 \end{pmatrix} \\ &= (t^2+2kt+k)^{k-1}\cdot(t^2-(3k^2-4k)t-k^3+k). \end{aligned}$$

Thus

$$\det(tI_n - D(G_3)) = (t+1)^{k(k-2)} \cdot (t^2 + 2kt + k)^{k-1} \cdot (t^2 - (3k^2 - 4k)t - k^3 + k).$$

Then distance eigenvalues of G_3 are -1 (with multiplicity $k(k-2)$), $-k \pm \sqrt{k^2-k}$ (with multiplicity $k-1$), and $\frac{3k^2-4k\pm\sqrt{9k^4-20k^3+16k^2-4k}}{2}$ $\left(> -k - \sqrt{k^2-k}\right)$. Thus $\lambda(G_3) = -k - \sqrt{k^2-k}$. □

Theorem 3.2. *Let G be a k-uniform hypertree of order n with $2 \le k \le n$. Then $\lambda(G) \in [-2k, 0)$ if and only if $G \cong S_{n,2}$, $D_{4,2,1}$, or $D_{5,2,1}$ when $k = 2$, and $G \cong S_{n,k}$ or $D_k(n; \underbrace{1, \dots, 1}_{\frac{n-1}{k-1}-1}, 0, \dots, 0)$ when $k \ge 3$.*

Proof. By Lemma 3.3 (ii), $\lambda(S_{n,k}) = -k > -2k$. By direct calculation, $\lambda(D_{4,2,1}) = -2-\sqrt{2} > -4$ and $\lambda(D_{5,2,1}) \approx -3.867 > -4$. For $k \ge 3$, note that $P_{3k-2,k}$ is a distance-preserving k-uniform subhypergraph of $D_k(n; \underbrace{1, \dots, 1}_{\frac{n-1}{k-1}-1}, 0, \dots, 0)$, and $D_k(n; \underbrace{1, \dots, 1}_{\frac{n-1}{k-1}-1}, 0, \dots, 0)$ is a distance-preserving k-uniform subhypergraph of $D_k(n; \underbrace{1, \dots, 1}_{k})$. By Lemma 3.3 (i), Lemma 3.4 (iii) and Lemma 2.1, we have

$$\begin{aligned} -k - \sqrt{k^2-2} &= \lambda(P_{3k-2,k}) \\ &\ge \lambda\left(D_k(n; \underbrace{1, \dots, 1}_{\frac{n-1}{k-1}-1}, 0, \dots, 0)\right) \\ &\ge \lambda\left(D_k(n; \underbrace{1, \dots, 1}_{k})\right) \\ &= -k - \sqrt{k^2-2}. \end{aligned}$$

Thus $\lambda\left(D_k(n; \underbrace{1, \dots, 1}_{\frac{n-1}{k-1}-1}, 0, \dots, 0)\right) = -k - \sqrt{k^2-2} > -2k.$

Suppose that $\lambda(G) \ge -2k$. If the diameter of G is at least four, then since $P_{4k-3,k}$ is a distance-preserving k-uniform subhypergraph of G, we have by Lemma 2.1 and Lemma 3.4 (i) that $\lambda(G) \le \lambda(P_{3k-2,k}) < -2k$, a contradiction. Thus the diameter of G is at most three.

If the diameter is at most two, then it is obvious that $G \cong S_{n,k}$.

Suppose that the diameter of G is three. Suppose that $k = 2$. Then $G \cong D_{n,2,a}$ for some $1 \le a \le \left\lfloor \frac{n-2}{2} \right\rfloor$. By direct calculation, $\lambda(D_{6,2,1}) \approx -4.1409 < -4$ and $\lambda(D_{6,2,2}) \approx -4.5616 < -4$. By Lemma 2.1, G can contain neither $D_{6,2,1}$ nor $D_{6,2,2}$ as a subgraph. Thus $G \cong D_{4,2,1}$, or $D_{5,2,1}$. Now suppose that $k \ge 3$. Then $G \cong D_k(n; a_1, \dots, a_k)$, where $a_1 \ge \dots \ge a_k \ge 0$, $a_2 \ge 1$ and $\sum_{i=1}^{k} a_i + 1 = \frac{n-1}{k-1}$. Suppose that $a_1 \ge 2$. Then $D_{4k-3,k,1}$ is a distance-preserving k-uniform subhypergraph of G. By Lemma 2.1 and

Lemma 3.4 (ii), we have $\lambda(G) \leq \lambda(D_{4k-3,k,1}) < -2k$, a contradiction. Thus $a_1 = 1$, implying that $G \cong D_k(n; \underbrace{1, \ldots, 1}_{\frac{n-1}{k-1}-1}, 0, \ldots, 0)$. $\square$

By the proof of Theorem 3.2, and Lemmas 2.1, 3.3 and 3.4, we have

Corollary 3.2. *(i) For $k \geq 2$, there dose not exist a k-uniform hypertree G with $\lambda(G) \in \big[-2k, -k-\sqrt{k^2-k}\big) \cup \big(-k-\sqrt{k^2-k}, -k\big) \cup (-k,-1) \cup (-1,0)$ except $D_{5,2,1}$ when $k = 2$;*

(ii) If G is a k-uniform hypertree of order n with $2 \leq k \leq n-1$, then $\lambda(G) = -k$ if and only if $G \cong S_{n,k}$;

(iii) If G is a k-uniform hypertree of order n with $2 \leq k \leq n-1$, then $\lambda(G) = -k-\sqrt{k^2-k}$ if and only if $G \cong D_k(n; \underbrace{1, \ldots, 1}_{\frac{n-1}{k-1}}, 0, \ldots, 0)$ with $\frac{n-1}{k-1} \geq 2$.

Lemma 3.5. *For $k \geq 2$, $\lambda(C_{3k-3,k}) = \frac{-k-\sqrt{k^2-4}}{2}$.*

Proof. The case $k = 2$ is trivial. Suppose that $k \geq 3$. Let $E(C_{3k-3,k}) = \{e_1, e_2, e_3\}$ with $e_1 \cap e_2 = \{u\}$, $e_2 \cap e_3 = \{v\}$, and $e_1 \cap e_3 = \{w\}$. Let x be an eigenvector of $D(C_{3k-3,k})$ corresponding to $\lambda(C_{3k-3,k})$. As in the proof of Lemma 3.3 (i), the entry of x at each vertex of $e_1 \setminus \{u, w\}$ is the same, which is denoted by x_1, the entry of x at each vertex of $e_2 \setminus \{u, v\}$ is the same, which is denoted by x_2, the entry of x at each vertex of $e_3 \setminus \{v, w\}$ is the same, which is denoted by x_3. Then

$$\begin{aligned}
\lambda x_u &= x_v + x_w + (k-2)x_1 + (k-2)x_2 + 2(k-2)x_3, \\
\lambda x_v &= x_u + x_w + 2(k-2)x_1 + (k-2)x_2 + (k-2)x_3, \\
\lambda x_w &= x_u + x_v + (k-2)x_1 + 2(k-2)x_2 + (k-2)x_3, \\
\lambda x_1 &= x_u + 2x_v + x_w + (k-3)x_1 + 2(k-2)x_2 + 2(k-2)x_3, \\
\lambda x_2 &= x_u + x_v + 2x_w + 2(k-2)x_1 + (k-3)x_2 + 2(k-2)x_3, \\
\lambda x_3 &= 2x_u + x_v + x_w + 2(k-2)x_1 + 2(k-2)x_2 + (k-3)x_3.
\end{aligned}$$

We view these equations as a homogeneous linear system in the six variables x_u, x_v, x_w, x_1, x_2 and x_3. Thus $\lambda(C_{3k-3,k})$ is the least root of $g(t) = 0$, where

$$g(t) = (t^2 - (5k-9)t - 6k + 10)(t^2 + kt + 1)^2.$$

Note that the roots of $g(t) = 0$ are $\frac{5k-9\pm\sqrt{25k^2-66k+41}}{2}$ and $\frac{-k\pm\sqrt{k^2-4}}{2}$ (with multiplicity 2). It follows that $\lambda(C_{3k-3,k}) = \frac{-k-\sqrt{k^2-4}}{2}$. $\square$

For $k \geq 3$ and $a = \frac{n}{k-1} - 2$, let $E(C_{2k-2,k}) = \{e_1, e_2\}$ with $e_1 \cap e_2 = \{u, v\}$, and let $F_{n,k,a}$ be the hypergraph obtained from $C_{2k-2,k}$ by adding $ak - a$ new vertices $u_1, \ldots, u_{ak-a}$ and a new edges $\{u, u_{i(k-1)+1}, \ldots, u_{i(k-1)+k-1}\}$, where $i = 0, \ldots, a-1$. Obviously, if $a = 0$, then $F_{n,k,a} \cong C_{2k-2,k}$.

Lemma 3.6. *For integers n, k, a with $3 \leq k \leq n$ and $a = \frac{n}{k-1} - 2 \geq 0$, we have*

(i) $\lambda(F_{n,k,0}) = -k+1$;

(ii) $\lambda(F_{n,k,a}) \in \Big(-k+1-\sqrt{(k-1)(k-2)}, -k\Big)$ if $a \geq 1$.

Proof. Let $E(F_{n,k,a}) = \{e_1, \dots, e_{a+2}\}$ with $e_1 \cap e_2 = \{u, v\}$. We partition $V(F_{n,k,a})$ into $\{u\} \cup \{v\} \cup (e_1 \setminus \{u, v\}) \cup (e_2 \setminus \{u, v\}) \cup (e_3 \setminus \{u\}) \cup \cdots \cup (e_{a+2} \setminus \{u\})$. With respect to this partition, we have

$$D(F_{n,k,a}) = \begin{pmatrix} 0 & 1 & \mathbf{1}_{k-2}^\top & \mathbf{1}_{k-2}^\top & \mathbf{1}_{k-1}^\top & \cdots & \mathbf{1}_{k-1}^\top \\ 1 & 0 & \mathbf{1}_{k-2}^\top & \mathbf{1}_{k-2}^\top & 2\cdot\mathbf{1}_{(k-1)}^\top & \cdots & 2\cdot\mathbf{1}_{k-1}^\top \\ \mathbf{1}_{k-2} & \mathbf{1}_{k-2} & J_{k-2}-I_{k-2} & 2J_{k-2} & 2J_{(k-2)\times(k-1)} & \cdots & 2J_{(k-2)\times(k-1)} \\ \mathbf{1}_{k-2} & \mathbf{1}_{k-2} & 2J_{k-2} & J_{k-2}-I_{k-2} & 2J_{(k-2)\times(k-1)} & \cdots & 2J_{(k-2)\times(k-1)} \\ \mathbf{1}_{k-1} & 2\cdot\mathbf{1}_{k-1} & 2J_{(k-1)\times(k-2)} & 2J_{(k-1)\times(k-2)} & J_{k-1}-I_{k-1} & \cdots & 2J_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{k-1} & 2\cdot\mathbf{1}_{k-1} & 2J_{(k-1)\times(k-2)} & 2J_{(k-1)\times(k-2)} & 2J_{k-1} & \cdots & J_{k-1}-I_{k-1} \end{pmatrix}.$$

If $a = 0$, then as in the proof of Lemma 3.3 (ii), we have

$$\begin{aligned} \det(tI_n - D(F_{n,k,a})) &= (t+1)^{2(k-3)+1} \cdot \det \begin{pmatrix} t-1 & -(k-2) & -(k-2) \\ -2 & t-k+3 & -2(k-2) \\ -2 & -2(k-2) & t-k+3 \end{pmatrix} \\ &= (t+1)^{2(k-3)+1} \cdot (t+k-1) \cdot (t^2 - (3k-6)t - k + 1), \end{aligned}$$

and thus the distance eigenvalues of $F_{n,k,a}$ are -1 (with multiplicity $2(k-3)+1$), $-k+1$ and $\frac{3k-6\pm\sqrt{9k^2-32k+32}}{2}$ $(> -k+1)$. Thus $\lambda(F_{n,k,a}) = -k+1$. This is (i).

Suppose that $a \geq 1$. As in the proof of Lemma 3.3, we have

$$\begin{aligned} \det(tI_n - D(F_{n,k,a})) &= (t+1)^{2(k-3)+a(k-2)} \cdot (t+k-1) \cdot (t+k)^{a-1} \\ &\quad \cdot \det \begin{pmatrix} t & -1 & -2(k-2) & -(k-1)a \\ -1 & t & -2(k-2) & -2(k-1)a \\ -1 & -1 & t-3k+7 & -2(k-1)a \\ -1 & -2 & -4(k-2) & t-2(k-1)(a-1)-k+2 \end{pmatrix} \\ &= (t+1)^{2(k-3)+a(k-2)} \cdot (t+k-1) \cdot (t+k)^{a-1} h(t), \end{aligned}$$

where

$$\begin{aligned} h(t) &= t^4 + t^3(-2ak - 2k + 2a + 7) + t^2(-3k^2 - 2ak^2 - ak + 3k + 7 + 3a) \\ &\quad + t(-4k^2 - ak^2 - 4ak + 6k + 5a + 1) - k^2 + k - 2ak + 2a. \end{aligned}$$

Thus the distance eigenvalues of $F_{n,k,a}$ are -1 (with multiplicity $2(k-3) + a(k-2)$), $-k+1$, $-k$ (with multiplicity $a-1$ for $a \geq 2$), and the roots of $h(t) = 0$. Let $\lambda = \lambda(F_{n,k,a})$. Since $h(-k) = -a(k-1)(k-2)(2k-1) < 0$, we have $\lambda < -k$, and thus λ is the least root of $h(t) = 0$. Note also that $\rho(D(F_{n,k,a})$ is the largest root of $h(t) = 0$. Let $t_1 \leq t_2 \leq t_3 \leq t_4$ be the roots of $h(t) = 0$, where $t_1 = \lambda$ and $t_4 = \rho(F_{n,k,a})$. Then $t_1 + t_2 + t_3 + t_4 = 2ak + 2k - 2a - 7$.

Note that $t_4 \leq 2ak + 3k - 2a - 5$ (which is the maximum row sum of $D(F_{n,k,a})$).

Let $\lambda^* = -k + 1 - \sqrt{(k-1)(k-2)}$. Then

$$h(\lambda^*) = \lambda^* p(k) - 5k^3 + 11ak^2 - 2ak^3 + 28k^2 - 49k - 18ak + 26 + 9a,$$

where $p(k) = -10k^3 + 23ak^2 - 4ak^3 + 58k^2 - 107k - 40ak + 62 + 21a$. Note that $p(k)$ is decreasing for $k \geq 3$. We have $p(k) \leq p(3) = -7 < 0$. If $k = 3$, then $h(\lambda^*) = 10+7\sqrt{2} > 0$. If $k \geq 4$, then $\lambda^* p(k) > (-k)p(k)$, and thus

$$\begin{aligned} h(\lambda^*) &> (-k) \cdot p(k) - 5k^3 + 11ak^2 - 2ak^3 + 28k^2 - 49k - 18ak + 26 + 9a \\ &= (10k^2 + 4ak^2 - 25ak - 63k + 38a + 98)k^2 \\ &\quad +(37k + 13ak - 111 - 39a)k + 26 + 9a \\ &> 0. \end{aligned}$$

It follows that $h(\lambda^*) > 0$ for $k \geq 3$. Thus either $\lambda^* < t_1$ or $t_2 < \lambda^* < t_3$. Suppose that $t_2 < \lambda^* < t_3$. Since $\lambda^* < -k$ and $h(-k) < 0$, we have $t_1 \leq t_2 < \lambda^* < t_3 < -k < t_4$. Thus

$$\begin{aligned} t_4 &= 2ak + 2k - 2a - 7 - t_1 - t_2 - t_3 \\ &> 2ak + 2k - 2a - 7 - 2\lambda^* - (-k) \\ &= 2ak + 5k - 2a + 2\sqrt{(k-1)(k-2)} - 9 \\ &> 2ak + 3k - 2a - 5, \end{aligned}$$

a contradiction. Thus $\lambda^* < t_1 = \lambda$. Therefore $\lambda \in \left(-k + 1 - \sqrt{(k-1)(k-2)}, -k\right)$. This proves (ii). $\square$

If G is an ordinary unicyclic graph of order $n \geq 3$, then by Theorem 3.1, $\lambda(G) \leq -1$ with equality if and only if $G \cong C_{3,2}$, see [11]. By Corollary 3.1, there is no k-uniform hypergraph G with $\lambda(G) \in (-1, 0)$ for $k \geq 2$.

Theorem 3.3. *Let G be a k-uniform unicyclic hypergraph of order n, where $3 \leq k \leq n$. Then $\lambda(G) \in \left(-k + 1 - \sqrt{(k-1)(k-2)}\,, 0\right)$ if and only if $G \cong C_{3k-3,k}$, or $F_{n,k,a}$ with $a = \frac{n}{k-1} - 2 \geq 0$.*

Proof. Suppose that $\lambda(G) > -k+1-\sqrt{(k-1)(k-2)}$. By Lemma 3.3 (i), $\lambda(P_{3k-5,k-1}) = -k + 1 - \sqrt{(k-1)(k-2)}$. Suppose that the diameter of G is at least three. Then $D(P_{3k-5,k-1})$ is a principal matrix of $D(G)$. By Lemma 2.1, $\lambda(G) \leq \lambda(D(P_{3k-5,k-1}))$, a contradiction. Thus the diameter of G is two, which implies that the cycle length of G is at most three.

If the length of the cycle in G is three, then since the diameter of G is two, there is no vertex lying outside the unique cycle, and thus $G \cong C_{3k-3,k}$. Suppose that the cycle length of G is two. If there is no vertex lying outside the unique cycle, then $G \cong F_{n,k,a}$ with $a = \frac{n}{k-1} - 2 = 0$. Otherwise, since the diameter of G is two, all those vertices lying outside the unique cycle are adjacent to a common vertex of degree two of the unique cycle, and thus $G \cong F_{n,k,a}$ with $a = \frac{n}{k-1} - 2 \geq 1$.

If $G \cong C_{3k-3,k}$ or $F_{n,k,a}$ with $a = \frac{n}{k-1} - 2 \geq 0$, then by Lemmas 3.5 and 3.6, we have $\lambda(G) > -k + 1 - \sqrt{(k-1)(k-2)}$. $\square$

By the proof of Theorem 3.3 and Lemmas 3.5 and 3.6, we have

Corollary 3.3. *For $k \geq 3$, there does not exist a k-uniform unicyclic hypergraph G with $\lambda(G) \in \left(-k + 1 - \sqrt{(k-1)(k-2)}, -k\right) \cup \left(-k, \frac{-k-\sqrt{k^2-4}}{2}\right) \cup \left(\frac{-k-\sqrt{k^2-4}}{2}, 0\right)$.*

4 Distance spread

The following lemma is an immediate consequence of Perron-Frobenius Theorem.

Lemma 4.1. *Let G be a connected k-uniform hypergraph with $u, v \in V(G)$, and u is not adjacent with v. Let $e \subseteq V(G)$ with $u, v \in e$ and $|e| = k$. Then $\rho(G) > \rho(G + e)$.*

Lemma 4.2. *Let G be a connected k-uniform hypergraph of order n, where $2 \leq k \leq n$. Then $\rho(G) \geq n - 1$ with equality if and only if G is a spanning distance-preserving subhypergraph of K_n^k.*

Proof. Let G be a k-uniform hypergraph with minimum distance spectral radius among connected hypergraphs of order n. Suppose that the diameter of G is at least 2. Then there are $u, v \in V(G)$ such that u is not adjacent to v. Let e be a k-subset of $V(G)$ containing u and v. Obviously, $e \notin E(G)$. By Lemma 4.1, we have $\rho(G) > \rho(G + e)$, a contradiction. Thus the diameter of G is one. Therefore, $D(G) = J_n - I_n$, implying that G is a spanning distance-preserving subhypergraph of K_n^k with distance spectral radius $n - 1$ (the greatest eigenvalue of $J_n - I_n$). $\square$

Theorem 4.1. *Let G be a connected k-uniform hypergraph on n vertices, where $2 \leq k \leq n$. Then $s(G) \geq n$ with equality if and only if G is a spanning distance-preserving subhypergraph of K_n^k.*

Proof. By Lemma 4.2 and Theorem 3.1 (i), it is easily seen that $s(G) \geq n - 1$.

Suppose that G is not a spanning distance-preserving subhypergraph of K_n^k. By Lemma 4.2, $\rho(G) > n-1$. By Theorem 3.1 (i), $\lambda(G) < -1$. Thus $s(G) = \rho(G) - \lambda(G) > n = \rho(H) - \lambda(H) = s(H)$ for a spanning distance-preserving subhypergraph H of K_n^k. $\square$

Recall that we prove the following result in [12].

Lemma 4.3. *Let T be a k-uniform hypertree on n vertices, where $2 \leq k \leq n$. Then $\rho(T) \geq \rho(S_{n,k})$ with equality if and only if $T \cong S_{n,k}$.*

Theorem 4.2. *Let T be a k-uniform hypertree on n vertices, where $2 \leq k \leq n$. Then $s(T) \geq s(S_{n,k})$ with equality if and only if $T \cong S_{n,k}$.*

Proof. Suppose that $T \ncong S_{n,k}$. Then $\frac{n-1}{k-1} \geq 2$. By Lemma 4.2, $\rho(T) > \rho(S_{n,k})$. By Lemma 2.1 and Lemma 3.3, $\lambda(T) \leq \lambda(P_{3k-2,k}) = -k - \sqrt{k^2 - k} < -k = \lambda(S_{n,k})$. Thus $s(T) = \rho(T) - \lambda(T) > \rho(S_{n,k}) - \lambda(S_{n,k})$. $\square$

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