Integration by Parts formula for SPDEs with Multiplicative Noise and its Applications*

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Abstract

By using the Malliavin calculus, the Driver-type integration by parts formula is established for the semigroup associated to SPDEs with Multiplicative Noise. Moreover, estimates on the logarithmic derivative of the transition probability measure are obtained. A concrete example to describe evolution of spin systems on discrete lattices is give to illustrate our main result.

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1 Introduction

A significant application of the Malliavin calculus ([7, 8]) is to describe the density of a Wiener functional using the integration by parts formula. In 1997, Driver [3] established the following integration by parts formula for the heat semigroup P_t on a compact Riemannian manifold M:

(1.1) $$P_t(\nabla_Z f) = \mathbb{E}(f(X_t)N_t), \quad f \in C^1(M), Z \in \mathscr{X},$$

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where $\mathscr{X}$ is the set of all smooth vector fields on M, and N_t is a random variable depending on Z and the curvature tensor. This formula has many applications. For example, we are able to characterize the derivative w.r.t. the second variable y of the heat kernel $p_t(x,y)$, moreover, if N_t is exponentially integrable, (1.1) implies the shift Harnack inequality, see [13] for details.

So far, there are many results on the Driver-type integration by parts formula for SDEs or SPDEs. The backward coupling method developed in [13] has been used in [4, 14] for SDEs driven by fractional Brownian motions and SPDEs driven by Wiener processes. Recently, using finite many jumps approximation and Malliavin calculus, [10, 11] obtain integration by parts formulas for SDEs and SPDEs with additive noise driven by subordinated Brownian motion.

However, all the above results are considered in additive noise case. The aim of this paper is to derive the integration by parts formula for SPDEs with multiplicative noise by Malliavin calculus and to derive estimates on the logarithmic derivatives of transition probabilities.

The main difficulty in obtaining the integration by parts formula is to give a representation of N_t in (1.1). Unfortunately, in the multiplicative noise case, the derivative process (Jocobi operator) J_t associated to the solution solves a linear operator-valued SDE instead of an operator-valued random differential equations in the additive noise case. So we develop a Duhamel's formula for the linear SDEs in Lemma 3.3, which is crucial for the representation for the Malliavin direction derivative process $D_{h_k} J_T$ (see (3.28)). Then we can give an explicit representation of N_t.

Let $(\mathbb{H}, \langle,\rangle, |\cdot|)$ be a separable real Hilbert spaces. Consider the following SPDE on $\mathbb{H}$:

$$\mathrm{d}X_t = AX_t\mathrm{d}t + b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t, \quad X_0 = x \in \mathbb{H}, \tag{1.2}$$

where $b : [0,\infty)\times\mathbb{H} \to \mathbb{H}$ and $\sigma : [0,\infty)\times\mathbb{H} \to \mathscr{L}(\mathbb{H})$ are measurable locally bounded (i.e. bounded on bounded sets), where $\mathscr{L}(\mathbb{H})$ is the space of bounded linear operators on $\mathbb{H}$ equipped with the operator norm $\|\cdot\|$. Moreover,

(i) $(A, \mathscr{D}(A))$ is a linear operator generating a C_0-contraction semigroup e^{At} such that $\|\mathrm{e}^{At}\|_{\mathrm{HS}} < \infty$ for any $t > 0$, and

$$\delta_T := \int_0^T \|\mathrm{e}^{At}\|_{\mathrm{HS}}^2 \mathrm{d}t < \infty, \quad T > 0, \tag{1.3}$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm. Let non-decreasing positive sequence $\{\lambda_k\}_{k\geq 1}$ with

$$\lim_{k\to\infty} \lambda_k = \infty$$

be all the spectrum of $-A$ counting by multiples. The corresponding unit eigenvectors are $\{e_k\}_{k\geq1}$, i.e. $Ae_k = -\lambda_k e_k$, $k \geq 1$. W is a cylindrical Brownian motion on $\mathbb{H}$ with respect to a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq0}, \mathbb{P})$, i.e. $W = \sum_{n=1}^{\infty} w^n e_n$, where $\{w^n\}_{n\geq1}$ is a sequence of independent one dimensional Brownian motions with respect to $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq0}, \mathbb{P})$.

(ii) For any $k \geq 1$, let $\sigma^{(k)} := \sigma e_k$. There exists a non-negative non-decreasing function K on $[0, \infty)$ such that

$$\|\nabla b_s(x)\| \vee \left\{\sum_{k=1}^{\infty} \left\|\nabla \sigma_s^{(k)}(x)\right\|^2\right\}^{\frac{1}{2}} \leq K(s), \quad s \geq 0, x \in \mathbb{H}, \tag{1.4}$$

and $\nabla b_s : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ and $\nabla \sigma_s : \mathbb{H} \times \mathbb{H} \to \mathscr{L}_{HS}(\mathbb{H})$ are uniformly continuous on bounded sets.

For $\nabla b_t(x)$ and $\nabla \sigma_t^{(k)}(x)$, we shall define

$$\begin{aligned} \nabla b_t(x)v &= \nabla_v b_t(x) \\ \nabla \sigma_t^{(k)}(x)v &= \nabla_v \sigma_t^{(k)}(x). \end{aligned}$$

Assume (i) and (ii). Then the equation (1.2) has a unique non-explosive mild solution $X_t(x)$, and the associated Markov semigroup P_t is defined as follows:

$$P_t f(x) := \mathbb{E} f(X_t(x)), \quad f \in \mathscr{B}_b(\mathbb{H}), t \geq 0, x \in \mathbb{H}.$$

Since for any $t \geq 0$, $\mathrm{Ker}(\mathrm{e}^{At}) = \{0\}$, the inverse operator $\mathrm{e}^{-At} : \mathrm{Im}(\mathrm{e}^{At}) \to \mathbb{H}$ is well defined.

To establish the integration by parts formula, we also need the following assumptions:

(H1) For any $(t, x) \in [0, \infty) \times \mathbb{H}$, there holds $\nabla b_t(x) : \mathrm{Im}(\mathrm{e}^{At}) \to \mathrm{Im}(\mathrm{e}^{At})$, $\nabla \sigma_t^k(x) : \mathrm{Im}(\mathrm{e}^{At}) \to \mathrm{Im}(\mathrm{e}^{At})$, $k \geq 1$. Let

$$\begin{aligned} B_t(x) &= \mathrm{e}^{-At} \nabla b_t(x) \mathrm{e}^{At}, \\ \Sigma_t^{(k)}(x) &= \mathrm{e}^{-At} \nabla \sigma_t^{(k)}(x) \mathrm{e}^{At}, \quad k \geq 1, t \geq 0, x \in \mathbb{H}. \end{aligned} \tag{1.5}$$

Assume that $B_t(\cdot) : \mathbb{H} \to \mathscr{L}(\mathbb{H})$ is continuously Fréchet differentiable and $\Sigma_t^k(\cdot) : \mathbb{H} \to \mathscr{L}(\mathbb{H})$ is Gâteaux differentiable, with

$$\lim_{y \to x} \sum_{k=1}^{\infty} \left| \left[\nabla_{z_1} \Sigma_t^{(k)}(y) - \nabla_{z_1} \Sigma_t^{(k)}(x)\right] z_2 \right|^2 = 0, \ t > 0, z_1, z_2 \in \mathbb{H},$$

and there exists a positive function K_1 in $L^2_{loc}([0, \infty))$ such that for any $t > 0$, $x \in \mathbb{H}$,

$$\|B_t(x)\| \vee \|\nabla B_t(x)\| \vee \left(\sum_{k=1}^{\infty} \left(\left\|\Sigma_t^{(k)}(x)\right\|_{HS}^2 \vee \left\|\nabla \Sigma_t^{(k)}(x)\right\|^2\right)\right)^{\frac{1}{2}} \leq K_1(t). \tag{1.6}$$

(H2) σ is invertible, and it holds that

$$\left\|\sigma_t^{-1}(x)\right\| \leq \lambda(t), \quad t > 0, x \in \mathbb{H} \tag{1.7}$$

for some strictly positive increasing function λ on $[0, \infty)$.

Remark 1.1. *(H2) is a standard non-degenerate assumption, while (H1) comes from [11], where Σ^k and $\nabla\Sigma^k$ vanish for any $k \geq 1$. (1.6) means that $|\nabla\langle b_t, e_i\rangle|$, $\|\nabla^2\langle b_t, e_i\rangle\|$ should be small enough as i is large enough, and $|\nabla\langle \sigma_t^{(k)}(x), e_i\rangle|$, $\|\nabla^2\langle \sigma_t^{(k)}(x), e_i\rangle\|$ should be small enough as i, k are large enough. For example, if there exist nonnegative sequences $\{\mu_k\}_{k\geq 1}$, $\{\eta_k\}_{k\geq 1}$, $\{\gamma_k\}_{k\geq 1}$ with $\sum_{k\geq 1}(\eta_k^2 + \mu_k^2 + \gamma_k^2) < \infty$ and non-negative function $C_1 \in L^2_{loc}([0,\infty))$ and locally bounded function C_2 on $[0,\infty)$ such that*

$$\begin{aligned}
&|\nabla_{e_j}\langle b_t, e_i\rangle| + |\nabla\nabla_{e_j}\langle b_t, e_i\rangle| \leq C_1(t)\mathrm{e}^{-(\lambda_k-\lambda_j)^+ t}\mu_i,\\
&|\nabla_{e_j}\langle \sigma_t^k, e_i\rangle| + |\nabla\nabla_{e_j}\langle \sigma_t^k, e_i\rangle| \leq C_2(t)\mathrm{e}^{-\lambda_i t}\eta_i\gamma_k \quad t > 0, i \geq 1, k \geq 1, j \geq 1,
\end{aligned}$$

then (1.6) *holds with*

$$K_1^2(t) = \sum_{k\geq 1}(\eta_k^2 + \mu_k^2 + \gamma_k^2)\left(C_1(t) + C_2^2(t)\|\mathrm{e}^{tA}\|_{HS}^2\right).$$

In fact, for any $y, z \in \mathbb{H}$, $t > 0$

$$\begin{aligned}
\|B_t(x)y\|^2 &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathrm{e}^{2(\lambda_i-\lambda_j)t}\left|\nabla_{e_j}\langle b_t, e_i\rangle\right|^2|\langle y, e_j\rangle|^2\\
&\leq C_1^2(t)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mathrm{e}^{2(\lambda_i-\lambda_j)t}\mathrm{e}^{-2(\lambda_i-\lambda_j)^+ t}\mu_i^2|\langle y, e_j\rangle|^2\\
&\leq C_1^2(t)\left(\sum_{i=1}^{\infty}\mu_i^2\right)|y|^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=1}^{\infty}\left\|\Sigma_t^{(k)}(x)\right\|_{HS}^2 &= \sum_{k=1,i=1,j=1}^{\infty} e^{-2(\lambda_j-\lambda_i)t}\nabla_{e_j}|\langle\sigma(x)^{(k)}, e_i\rangle|^2\\
&\leq C_2^2(t)\sum_{k=1,i=1,j=1}^{\infty} e^{-2(\lambda_j-\lambda_i)t}\mathrm{e}^{-2\lambda_i t}\eta_i^2\gamma_k^2\\
&\leq C_2^2(t)||\mathrm{e}^{tA}||_{HS}^2\sum_{i=1}^{\infty}\eta_i^2\sum_{k=1}^{\infty}\gamma_k^2.
\end{aligned}$$

$|\nabla B_t(x)|$ and $\sum_{k=1}^{\infty}\left\|\nabla\Sigma_t^{(k)}(x)\right\|$ can be estimated similarly. To illustrate (H1) and our main result, a concrete example is presented in Section 2.

Finally, we introduce a notation which will be used throughout this paper:

$$[\Sigma_t(x)v]e_k := \Sigma_t^{(k)}(x)v, \ x, v \in \mathbb{H}.$$

2 Main results

To state our main results, for any $s \geq 0$, we introduce $\mathscr{L}(\mathbb{H})$-valued processes $(J_{s,t})_{t\geq s}$ and $(J^A_{s,t})_{t\geq s}$, which solve the following operator-valued SDEs respectively:

$$\mathrm{d}J_{s,t} = B_t(X_t)J_{s,t}\mathrm{d}t + \sum_{k=1}^{\infty} \Sigma_t^{(k)}(X_t)J_{s,t}\mathrm{d}w_t^k, \quad J_{s,s} = I \tag{2.1}$$

$$\mathrm{d}J^A_{s,t} = (A + \nabla b_t(X_t))J^A_{s,t}\mathrm{d}t + \sum_{k=1}^{\infty} \nabla\sigma_t^{(k)}(X_t)J^A_{s,t}\mathrm{d}w_t^k, \quad J^A_{s,s} = I. \tag{2.2}$$

According to (H1) and Lemma 3.1 below, (2.1) and (2.2) are well defined. Denote $J_t = J_{0,t}$ and $J^A_t = J^A_{0,t}$. Since the inverse of J^A_t is usually an unbounded operator in infinite dimension, we shall use an auxiliary process J_t^{-1} and use the relationship between J_t and J^A_t(see (3.16) in Remark 3.1) to construct h such that "D_hX_t" equals to some vector in $\mathbb{H}$ (see details in the proof of Theorem 2.1). By (H1) and Lemma 3.1 below, J_t is invertible with

$$\mathrm{d}J_t^{-1} = -J_t^{-1}\left\{B_t(X_t) - \sum_{k=1}^{\infty}\left(\Sigma_t^{(k)}(X_t)\right)^2\right\}\mathrm{d}t - \sum_{k=1}^{\infty} J_t^{-1}\Sigma_t^{(k)}(X_t)\mathrm{d}w_t^k, \quad J_0^{-1} = I. \tag{2.3}$$

Remark 2.1. *Since $\mathscr{L}(\mathbb{H})$ with operator norm is not a UMD Banach space in infinite dimension space, see [9], to ensure the stochastic integration in (2.1) make sense, we assume that $\Sigma_t^k(x) \in \mathscr{L}_{HS}(\mathbb{H})$ and satisfies (1.6).*

The main result is the following.

Theorem 2.1. *Assume (H1) and (H2), then the integration formula by parts holds, i.e.*

$$P_T(\nabla_{\mathrm{e}^{AT}v}f) = \frac{1}{T}\mathbb{E}\{f(X_T)M_T^v\}, \quad v \in \mathbb{H}, f \in C_b^1(\mathbb{H}) \tag{2.4}$$

where

$$\begin{aligned} M_T^v = &\left\langle \int_0^T \left[\sigma_t^{-1}(X_t)J_t^A\right]^* \mathrm{d}W_t, J_T^{-1}v\right\rangle + \int_0^T t\mathrm{Tr}\left\{\mathrm{e}^{tA}\left[(\nabla.B_t)(X_t)J_tJ_T^{-1}v\right]\right\}\mathrm{d}t \\ &+ \left\langle \sum_{j=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\{\mathrm{e}^{-t\lambda_k}J_t^*\left(\nabla_{e_k}\Sigma_t^{(j)}\right)^*(X_t)e_k\right\}\mathrm{d}w_t^j, J_T^{-1}v\right\rangle \\ &+ \int_0^T \mathrm{Tr}\left\{\mathrm{e}^{tA}\left[\Sigma_t(X_t)J_tJ_T^{-1}v\right]\sigma_t^{-1}(X_t)\right\}\mathrm{d}t \\ &- \int_0^T t\mathrm{Tr}\left\{\mathrm{e}^{tA}\sum_{j=1}^{\infty}\Sigma_t^{(j)}(X_t)\left[\left(\nabla.\Sigma_t^{(j)}\right)(X_t)J_tJ_T^{-1}v\right]\right\}\mathrm{d}t. \end{aligned} \tag{2.5}$$

Remark 2.2. *Every term in (2.5) is well defined by (1.3), (H1), (H2), and Lemma 3.2. This result extends [13, Theorem 5.1] where σ only depends on time, see also [12, Theorem 3.2.4(1)]. Unlike [11], the integrands of stochastic integrations in M_T^v here is adapted.*

To illustrate Theorem 2.1, we give an example on countable systems of stochastic differential equations, which can be used to describe evolution of spin systems on discrete lattices, see for instance [2, 6].

Example 2.2. *Let $\mathbb{Z}$ be the set of all integers, $d \in \mathbb{N}^+$, $k_0 \in \mathbb{N}$, $\lambda_0 > 0$ and $\mathbb{H} = l^2(\mathbb{Z}^d)$. For $\gamma = (\gamma^1, \cdots, \gamma^d) \in \mathbb{Z}^d$, set $|\gamma| = \sum_{j=1}^d |\gamma^j|$. Let $\{\lambda_\gamma\}_{\gamma \in \mathbb{Z}^d}$ be a positive sequence with*

$$\lambda_{\gamma_1} \begin{cases} = \lambda_{\gamma_2}, & |\gamma_1| = |\gamma_2|, \\ > \lambda_{\gamma_2}, & |\gamma_1| > |\gamma_2|, \end{cases}$$

and $\sum_{\gamma \in \mathbb{Z}^d} \lambda_\gamma^{-1} < \infty$. Let $(Ax)_\gamma = -\lambda_\gamma x_\gamma, \gamma \in \mathbb{Z}^d$, $x \in l^2(\mathbb{Z}^d)$. For each $\gamma \in \mathbb{Z}^d$, let

$$\Gamma_\gamma = \{\eta \in \mathbb{Z}^d \mid |\gamma| \leq |\eta| \leq |\gamma| + k_0\},$$

and let g_γ and f_γ be functions defined on $\mathbb{R}^{\Gamma_\gamma}$ such that $f_\gamma \geq \lambda_0$, and there are positive constants β_γ and $\bar{\beta}_\gamma$ such that $\sum_{\gamma \in \mathbb{Z}^d} \left(\beta_\gamma^2 + \bar{\beta}_\gamma^2\right) < \infty$ and

$$\sup_{x \in \mathbb{R}^{\Gamma_\gamma}} \left(|\nabla g_\gamma(x)|^2 + |\nabla f_\gamma(x)|^2\right) \leq \beta_\gamma^2,$$
$$\sup_{x \in \mathbb{R}^{\Gamma_\gamma}} \left(||\nabla\nabla g_\gamma(x)||_{HS}^2 + ||\nabla\nabla f_\gamma(x)||_{HS}^2\right) \leq \bar{\beta}_\gamma^2.$$

Define $b : \mathbb{H} \to \mathbb{R}^{\mathbb{Z}^d}$ and $\sigma : \mathbb{H} \to \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ as follows: for any $\gamma, \eta \in \mathbb{Z}^d, x \in \mathbb{H}$,

$$b_\gamma(x) = g_\gamma(\pi_{\Gamma_\gamma}(x)),$$

and

$$\sigma_{\gamma\eta}(x) = \begin{cases} f_\gamma(\pi_{\Gamma_\gamma}(x)), & \gamma = \eta, \\ 0, & \gamma \neq \eta, \end{cases}$$

where π_{Γ_γ} is a natural projection from $\mathbb{H}$ to $\mathbb{R}^{\Gamma_\gamma}$ with $(\pi_{\Gamma_\gamma}(x))_\eta = x_\eta$, $\eta \in \Gamma_\gamma$. Then the equation (1.2) *of $X_t \in \mathbb{H}$ satisfies*

$$\mathrm{d}X_{\gamma,t} = -\lambda_\gamma X_{\gamma,t}\mathrm{d}t + b_\gamma(X_t)\mathrm{d}t + \sigma_{\gamma\gamma}(X_t)\mathrm{d}W_t^\gamma, \ \gamma \in \mathbb{Z}^d.$$

It is a routine mechanical task to check the conditions of Theorem 2.1, so we omit it.

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.3. *Assume (H1) and (H2). Then for any $p \in (1, \infty]$, it holds that*

$$|P_T(\nabla_{\mathrm{e}^{AT}v} f)| \leq \left\{\Gamma_{T, \frac{p}{p-1} \vee 2, A}\right\}^{\frac{p-1}{p} \wedge \frac{1}{2}} \frac{|v|}{T} (P_T|f|^p)^{\frac{1}{p}}, \ f \in C_b^1(\mathbb{H}), \ v \in \mathbb{H},$$

where $p = \infty$ means $\frac{p}{p-1} = 1$ and $(P_T|f|^p)^{\frac{1}{p}} = \sup_{x \in \mathbb{H}} |f|(x)$, and

$$\Gamma_{T,q,A} = C(q, T, K_1, K_2) \left\{ \lambda^q(T) \delta_T^{\frac{q}{2}} + (T^q + T^{\frac{q}{2}} + \lambda^q(T)) \delta_T^q \right\}$$

for δ_T defined in (1.3) *and some constant $C(q, T, K_1, K_2) \geq 0$ depending on $q \geq 2$, T, K_1, K_2.*

Basing on the integration by parts formula, we can study the regularity of transition probability measure of P_T. A finite measure μ on $\mathbb{H}$ is called weak Fomin differentiable along a vector $v \in \mathbb{H}$, if there is a finite signed measure $\partial_v \mu$ on $\mathbb{H}$ such that

$$\int_{\mathbb{H}} f(x)\partial_v\mu(\mathrm{d}x) = -\int_{\mathbb{H}} \nabla_v f(x)\mu(\mathrm{d}x),\ f \in C_b^1(\mathbb{H}).$$

When $\mathbb{H} = \mathbb{R}^d$, we may take $A = 0$ and so that Theorem 2.1 with $J^A = J$ covers the result in [13, Theorem 2.1]. In this case, according to [13], the integration by parts formula implies that P_T has a density $p_T(x, y)$ with respect to the Lebesgue measure, which is differentiable in y with

$$\nabla_v \log p_T(x,\cdot)(y) = -\frac{1}{T}\mathbb{E}\left(M_T^v|X_T(x) = y\right), \quad x, v \in \mathbb{R}^d. \tag{2.6}$$

If $\partial_v\mu \ll \mu$, then we can define the logarithmic derivative of μ along v by the Radon-Nikodym derivative $\frac{\mathrm{d}\partial_v\mu}{\mathrm{d}\mu}$. Then, we obtain a corollary for the logarithmic derivative of the transition probability measure $p_T(x, dy)$ of P_T from Theorem 2.1 and Corollary 2.3 directly. Moreover, it is clear that for $\mathbb{H} = \mathbb{R}^d$ and $A = 0$

$$\frac{\mathrm{d}\partial_v p_T(x,\cdot)}{\mathrm{d}p_T(x,\cdot)}(y) = \nabla_v \log p_T(x,\cdot)(y),\ p_T(x,\mathrm{d}y)\text{-}a.s.$$

Corollary 2.4. *Assume (H1) and (H2), $v \in \mathbb{H}$, $T > 0$. Then the transition probability $p_T(x,\mathrm{d}y)$ of P_T is weak Fomin differentiable along $e^{TA}v$ with logarithmic derivative*

$$\left(\frac{\mathrm{d}\partial_{e^{TA}v} p_T(x,\cdot)}{\mathrm{d}p_T(x,\cdot)}\right)(y) = -\frac{1}{T}\mathbb{E}\left(M_T^v|X_T(x) = y\right),\ x, y \in \mathbb{H},$$

and

$$\int_{\mathbb{H}} \left|\frac{\mathrm{d}\partial_{e^{TA}v} p_T(x,\cdot)}{\mathrm{d}p_T(x,\cdot)}\right|^p (y) p_T(x,\mathrm{d}y) \le \frac{(\Gamma_{T,p\vee 2,A})^{\frac{p}{p\vee 2}}|v|^p}{T^p}.$$

Particularly, if furthermore, $\mathbb{H} = \mathbb{R}^d$ and $A = 0$, then for any $p > 1$, $T > 0$, it holds that

$$\int_{\mathbb{R}^d} |\nabla_v \log p_T(x,\cdot)|^p(y) p_T(x,y)\mathrm{d}y \le \frac{|v|^p}{T^p}\left\{\Gamma_{T,p\vee 2,0}\right\}^{\frac{p}{p\vee 2}}, \quad x \in \mathbb{R}^d.$$

Remark 2.3. (2.1) *and* (2.3) *mean that M_T^v has the form as* $\exp(X)$ *with a Gaussian random variable X. This implies that $\mathbb{E}\left(\exp(\delta|M_T^v|)\right) = \infty$ for any $\delta > 0$. Thus, it can not yield the shift Harnack inequality with power by Young's inequality from* (2.4) *as in [13].*

The remainder of the paper is organized as follows. In Section 3, we give some important lemmas and prove them. The proofs of Theorem 2.1 and corollaries are put in Section 4.

3 Proof of Lemmas

To get the existence and uniqueness of (2.1) and (2.2), we consider the following slightly general operator-valued SDEs:

$$\mathrm{d}G_t = AG_t + F_tG_t\mathrm{d}t + \sum_{k=1}^{\infty} R_t^k G_t \mathrm{d}w_t^k, \tag{3.1}$$

$$\mathrm{d}g_t = f_t g_t \mathrm{d}t + \sum_{k=1}^{\infty} r_t^k g_t \mathrm{d}w_t^k, \tag{3.2}$$

with A defined as above, F, f, $\{R^k\}_{k\geq 1}, \{r^k\}_{k\geq 1}$ are $\mathscr{L}(\mathbb{H})$-valued progressive strong measurable processes, G_0 and g_0 are strong measurable $\mathscr{L}(\mathbb{H})$-valued random variables and $\mathbb{E}||G_0||^2 + \mathbb{E}||g_0||^2 < \infty$. Then

Lemma 3.1. (1) *If there exists a positive function K_3 on $(0,\infty)$ with*

$$\int_0^t K_3^2(s)\mathrm{d}s < \infty,\ t > 0 \tag{3.3}$$

such that

$$\|\mathrm{e}^{rA}F_t\| + \left(\sum_{k=1}^{\infty} \left\|\mathrm{e}^{rA}R_t^k\right\|_{HS}^2\right)^{1/2} \leq K_3(r), \quad r > 0, t \geq 0, \tag{3.4}$$

then (3.1) *has a unique solution $\{G_t\}_{t\geq 0}$ in $\mathscr{L}(\mathbb{H})$. If furthermore,*

$$||\mathrm{e}^{rA}F_t||_{HS} \leq K_3(r), \quad r > 0, t \geq 0, \tag{3.5}$$

then for $t > 0$, $G_t \in \mathscr{L}_{HS}(\mathbb{H})$ and $\mathbb{E}||G_t||_{HS}^2 < \infty$.

(2) *If there exists a positive function K_4 on $(0,\infty)$ with*

$$\int_0^t K_4^2(s)\mathrm{d}s < \infty,\ t > 0 \tag{3.6}$$

such that

$$||f_t|| + \left(\sum_{k=1}^{\infty} \left|\left|r_t^k\right|\right|_{HS}^2\right)^{1/2} \leq K_4(t),\ t > 0, \tag{3.7}$$

then (3.2) *has a unique solution $\{g_t\}_{t\geq 0}$ in $\mathscr{L}(\mathbb{H})$ which is invertible, and its inverse g_t^{-1} satisfies*

$$\mathrm{d}g_t^{-1} = -g_t^{-1}\left(f_t - \sum_{k=1}^{\infty}\left(r_t^k\right)^2\right)\mathrm{d}t - g_t^{-1}\sum_{k=1}^{\infty} r_t^k \mathrm{d}w_t^k, \quad g_0^{-1} = I. \tag{3.8}$$

Proof. (1) We shall consider the following form of (3.1):

$$G_t = \mathrm{e}^{tA} G_0 + \int_0^t \mathrm{e}^{(t-s)A} F_s G_s \mathrm{d}s + \sum_{k=1}^{\infty} \int_0^t \mathrm{e}^{(t-s)A} R_s^k G_s \mathrm{d}w_s^k, \ t > 0. \tag{3.9}$$

Since for $\mathscr{L}(\mathbb{H})$-valued progressive strong measurable process $\{G_s\}_{s\geq 0}$ with for all $t \geq 0$, $\sup_{s\in[0,t]} \mathbb{E}||G_s||^2 < \infty$ and

$$\mathbb{P}\left(\int_0^t ||G_s||^2 \mathrm{d}s < \infty, \ t \geq 0\right) = 1,$$

(3.3) and (3.4) imply that $\mathbb{P}$-a.s.

$$\int_0^t |e^{(t-s)A} F_s G_s x|^2 \mathrm{d}s < \infty, \ t \geq 0, \ x \in \mathbb{H}, \ \mathbb{P}\text{-a.s.},$$

$$\mathbb{E} \int_0^t \sum_{k=1}^{\infty} \left\|\mathrm{e}^{(s-r)A} R_r^k G_r\right\|_{HS}^2 \mathrm{d}s < \infty.$$

So $\int_0^t \mathrm{e}^{(t-s)A} F_s G_s \mathrm{d}s$ defines a strong measurable adapted process and the stochastic integral can be defined in the Hilbert space $\mathscr{L}_{HS}(\mathbb{H})$. Hence the right hand side of (3.9) defines a $\mathscr{L}(\mathbb{H})$-valued strong measurable process.

By Minkowski inequality and Hölder inequality,

$$\begin{aligned} \sup_{s\in[0,t]} \mathbb{E} \left\| \int_0^s \mathrm{e}^{(s-r)A} F_r G_r \mathrm{d}r \right\|^2 &\leq \sup_{s\in[0,t]} \left\{ \int_0^s \left[\mathbb{E} \left\|\mathrm{e}^{(s-r)A} F_r G_r\right\|^2 \right]^{\frac{1}{2}} \mathrm{d}r \right)^2 \\ &\leq \left(\int_0^t K_3(r) \mathrm{d}r \right)^2 \left(\sup_{r\in[0,t]} \mathbb{E}\|G_r\|^2 \right) \\ &= t \int_0^t K_3^2(r) \mathrm{d}r \left(\sup_{r\in[0,t]} \mathbb{E}\|G_r\|^2 \right). \end{aligned} \tag{3.10}$$

Itô's isometric formula yields that

$$\begin{aligned} \sup_{s\in[0,t]} \mathbb{E} \left\| \sum_{k=1}^{\infty} \int_0^s \mathrm{e}^{(s-r)A} R_r^k G_r \mathrm{d}w_r^k \right\|^2 &\leq \sup_{s\in[0,t]} \mathbb{E} \left\| \sum_{k=1}^{\infty} \int_0^s \mathrm{e}^{(s-r)A} R_r^k G_r \mathrm{d}w_r^k \right\|_{HS}^2 \\ &= \sup_{s\in[0,t]} \int_0^s \mathbb{E} \sum_{k=1}^{\infty} \left\|\mathrm{e}^{(s-r)A} R_r^k G_r\right\|_{HS}^2 \mathrm{d}r \\ &\leq \int_0^t K_3^2(r) \mathrm{d}r \left(\sup_{r\in[0,t]} \mathbb{E}\|G_r\|^2 \right). \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) with the fixed point theorem, we obtain existence and uniqueness of solutions to (3.1) satisfying $\sup_{s\in[0,t]} \mathbb{E}||G_s||^2 < \infty$, $t \geq 0$ and

$$\mathbb{P}\left(\int_0^t ||G_s||^2 \mathrm{d}s < \infty, \ t \geq 0\right) = 1.$$

Moreover, from (3.9), Gronwall's lemma implies that there exist nonnegative constants c_1, c_2 such that

$$\sup_{s\in[0,t]} \mathbb{E}\|G_s\|^2 \le c_1 \mathrm{e}^{c_2 t}. \tag{3.12}$$

Next, if furthermore, (3.5) holds, then by (3.4), we get

$$\mathbb{E}\left\|\int_0^s \mathrm{e}^{(s-r)A} F_r G_r \mathrm{d}r\right\|_{HS}^2 \le \left(\sup_{r\in[0,s]} \mathbb{E}\|G_r\|^2\right) s\int_0^s K_3^2(r)\mathrm{d}r.$$

Thus from (3.9), (3.11) and (3.12), it holds that

$$\begin{aligned}\mathbb{E}\|G_t\|_{HS}^2 &\le 3\|\mathrm{e}^{tA}\|_{HS}^2 + 3\left(\sup_{s\in[0,t]} \mathbb{E}\|G_s\|^2\right) t\int_0^t K_3^2(s)\mathrm{d}s \\ &\le 3\|\mathrm{e}^{tA}\|_{HS}^2 + 3c_1\mathrm{e}^{c_2 t} t\int_0^t K_3^2(s)\mathrm{d}s,\ t>0.\end{aligned} \tag{3.13}$$

(2) Similarly, from (3.6), (3.7), applying Minkowski inequality and the fixed point theorem, it is easy to derive the existence and uniqueness of solutions to (3.2). Denote the solution by g_t.

$\mathbb{H}$ is separable, so $\left(\sum_{k=1}^\infty \left(r_t^k\right)^2\right)^*$ is also a strong measurable process. Note that

$$\left\|\left[\sum_{k=1}^\infty \left(r_t^k\right)^2\right]^*\right\| \le \sum_{k=1}^\infty \left\|r_t^k\right\|^2 \le K_4^2(t).$$

Then, as (3.10), we have

$$\sup_{s\in[0,t]} \mathbb{E}\left\|\int_0^s \left(\sum_{k=1}^\infty \left(r_t^k\right)^2\right)^* U_r \mathrm{d}r\right\|^2 \le \left(\int_0^t K_4^2(r)\mathrm{d}r\right)^2 \left(\sup_{r\in[0,t]} \mathbb{E}\|U_r\|^2\right).$$

Thus, repeating the above argument again, the operator-valued SDE

$$\mathrm{d}U_t = -\left(f_t - \sum_{k=1}^\infty \left(r_t^k\right)^2\right)^* U_t \mathrm{d}t - \sum_{k=1}^\infty \left(r_t^k\right)^* U_t \mathrm{d}w_t^k,\quad U_0 = I \tag{3.14}$$

has a unique solution $U_t \in \mathscr{L}(\mathbb{H})$. For all $u, v \in \mathbb{H}$, by Itô's formula, it is easy to see that

$$\mathrm{d}\langle g_t u, U_t v\rangle = 0.$$

Thus $U_t^* g_t = U_0^* J_0 = I$. That means g_t is invertible with $g_t^{-1} = U_t^*$, and g_t^{-1} satisfies the left action equation (3.8).

□

Remark 3.1. *According to (H1),* (2.1)*,* (2.2) *and Lemma 3.1,* $\{J_t\}_{t\geq 0}$ *and* $\{J_t^A\}_{t\geq 0}$ *are strong measurable* $\mathscr{L}(\mathbb{H})$*-value processes, and* $J_t^A \in \mathscr{L}_{HS}(\mathbb{H})$*,* $t > 0$*,* $\mathbb{P}$*-a.s. Moreover, fixing* $s \geq 0$*,* (2.1) *implies that for any* $t \geq s$*,*

$$J_{s,t}J_s = J_s + \int_s^t B_r(X_r)(J_{s,r}J_s)\mathrm{d}r + \int_s^t \sum_{k=1}^{\infty} \Sigma_r^{(k)}(X_r)(J_{s,r}J_s)\mathrm{d}w_r^k,$$

which means $\{J_{s,t}J_s\}_{t\geq s}$ *is a solution to the equation:*

$$\mathrm{d}\Gamma_t = B_t(X_t)\Gamma_t\mathrm{d}t + \sum_{k=1}^{\infty} \Sigma_t^{(k)}(X_t)\Gamma_t\mathrm{d}w_t^k, \quad \Gamma_s = J_s, \ t \geq s. \tag{3.15}$$

Combining the definition of J_t *and* (2.1)*,*

$$\begin{aligned} J_t &= J_0 + \int_0^t B_r(X_r)J_r\mathrm{d}r + \int_0^t \sum_{k=1}^{\infty} \Sigma_r^{(k)}(X_r)J_r\mathrm{d}w_r^k \\ &= J_s + \int_s^t B_r(X_r)J_r\mathrm{d}r + \int_s^t \sum_{k=1}^{\infty} \Sigma_r^{(k)}(X_r)J_r\mathrm{d}w_r^k, \end{aligned}$$

thus, $\{J_t\}_{t\geq s}$ *is also a solution to* (3.15)*. By Lemma 3.1, we have* $\mathbb{P}$*-a.s.* $J_t = J_{s,t}J_s$ *due to the uniqueness of* (3.15)*. Similarly,* $\mathbb{P}$*-a.s.* $J_t^A = J_{s,t}^A J_s^A$*. On the other hand,* (2.1) *yields that*

$$\begin{aligned} e^{tA}J_t &= e^{At} + \int_0^t e^{tA}B_s(X_s)J_s\mathrm{d}s + \int_0^t e^{tA}\sum_{k=1}^{\infty} \Sigma_s^{(k)}(X_s)J_s\mathrm{d}w_s^k \\ &= e^{At} + \int_0^t e^{(t-s)A}\nabla b_s(X_s)\left(e^{sA}J_s\right)\mathrm{d}s + \int_0^t \sum_{k=1}^{\infty} e^{(t-s)A}\nabla\sigma_s^{(k)}(X_s)\left(e^{sA}J_s^A\right)\mathrm{d}w_s^k. \end{aligned}$$

Again, by the uniqueness of (2.2)*, for any* $t \geq 0$*,* $\mathbb{P}$*-a.s.* $J_t^A = e^{tA}J_t$*. As a consequence, for any* $t \geq s \geq 0$*,* $\mathbb{P}$*-a.s.*

$$J_t = J_{s,t}J_s, \quad J_t^A = J_{s,t}^A J_s^A, \quad J_t^A = e^{tA}J_t. \tag{3.16}$$

Next, we shall give some estimate of the norm of operator J_t and J_t^{-1}.

Lemma 3.2. *Assume (H1). Then for any* $x \in \mathbb{H}$*,* $t \geq 0$*,* $p \geq 2$*, it holds that*

$$\sup_{s\in[0,t]} \mathbb{E}\|J_s\|^p \leq 3^{p-1}\exp\left\{3^{p-1}\left(t^{\frac{p}{2}}+1\right)\left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{\frac{p}{2}}\right\} \tag{3.17}$$

$$\sup_{s\in[0,t]} \mathbb{E}\|J_s^{-1}\|^p \leq 4^{p-1}\exp\left\{4^{p-1}\left[\left(t^{\frac{p}{2}}+1\right)\left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{p}\right]\right\}. \tag{3.18}$$

Proof. By Burkerholder-Davis-Gundy inequality and Hölder inequality, it follows from (1.6) that

$$\begin{aligned}
\mathbb{E}\,\|J_t\|^p &\le 3^{p-1} + 3^{p-1}\mathbb{E}\left\|\int_0^t B_s(X_s)J_s\mathrm{d}s\right\|^p + 3^{p-1}\mathbb{E}\left\|\int_0^t \sum_{k=1}^\infty \Sigma_s^{(k)}(X_s)J_s\mathrm{d}w_s^k\right\|_{HS}^p \\
&\le 3^{p-1} + 3^{p-1}t^{\frac{p-2}{2}}\left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{\frac{p}{2}}\int_0^t \mathbb{E}||J_s||^p\mathrm{d}s \\
&\quad + 3^{p-1}\mathbb{E}\left(\int_0^t \sum_{k=1}^\infty \|\Sigma_s^{(k)}(X_s)\|_{HS}^2\|J_s\|^2\mathrm{d}s\right)^{\frac{p}{2}} \\
&\le 3^{p-1} + 3^{p-1}t^{\frac{p-2}{2}}\left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{\frac{p}{2}}\int_0^t \mathbb{E}||J_s||^p\mathrm{d}s \\
&\quad + 3^{p-1}\left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{\frac{p-2}{2}}\int_0^t K_1^2(s)\mathbb{E}\|J_s\|^p\mathrm{d}s.
\end{aligned}$$

Applying Gronwall inequality, we obtain (3.17).

Noting that

$$\left\|\sum_{k=1}^\infty \left(\Sigma_t^{(k)}(x)\right)^2\right\| \le \sum_{k=1}^\infty \left\|\Sigma_t^{(k)}(x)\right\|^2 \le K_1^2(t), \quad t>0, x\in\mathbb{H},$$

we have

$$\mathbb{E}\left\|\int_0^t \sum_{k=1}^\infty \left(\Sigma_t^{(k)}(x)\right)^2 J_s\mathrm{d}s\right\|^p \le \left(\int_0^t K_1^2(s)\mathrm{d}s\right)^{p-1}\int_0^t K_1^2(s)\mathbb{E}||J_s||^p\mathrm{d}s.$$

So, we obtain (3.18) similarly to (3.17). □

Next, we introduce a Duhamel's formula for the solution of a class of semi-linear $\mathscr{L}(\mathbb{H})$-valued SDEs.

Lemma 3.3. *Let f_t, $\{r_t^k\}_{k\ge1}$ satisfy the condition of Lemma 3.1 (2), and let a_t, $\{l_t^k\}_{k\ge1}$ be $\mathscr{L}(\mathbb{H})$-valued progressive strong measurable processes with*

$$\int_0^t \mathbb{E}\left(\|a_s\|^2 + \sum_{k=1}^\infty \|l_s^k\|_{HS}^2\right)\mathrm{d}s < \infty, \quad t\ge 0. \tag{3.19}$$

Then $\mathscr{L}(\mathbb{H})$-valued SDE

$$\mathrm{d}Y_t = a_t\mathrm{d}t + f_tY_t\mathrm{d}t + \sum_{k=1}^\infty r_t^kY_t\mathrm{d}w_t^k + \sum_{k=1}^\infty l_t^k\mathrm{d}w_t^k, \tag{3.20}$$

starting from a $\mathscr{F}_0$-measurable $\mathscr{L}(\mathbb{H})$-valued random variable Y_0 with $\mathbb{E}\|Y_0\|^2 < \infty$, has a unique solution Y_t, and

$$Y_t = g_t \left\{ Y_0 + \int_0^t g_s^{-1} a_s \mathrm{d}s + \int_0^t g_s^{-1} \sum_{k=1}^{\infty} l_s^k \mathrm{d}w_s^k - \int_0^t g_s^{-1} \sum_{k=1}^{\infty} f_s^k l_s^k \mathrm{d}s \right\}, \quad t \geq 0, \tag{3.21}$$

where g_t and g_t^{-1} are the solutions of (3.2) *and* (3.8) *respectively.*

Proof. The existence and uniqueness of the solution of (3.20) are easy to obtain by (3.6), (3.7), (3.19) and fixed point theorem. Since the proof is similar to the that of Lemma 3.1 (2), we omit here. Let $\{g_t^{-1}\}_{t\geq 0}$ be the solution of (3.8). Then (3.8) and Itô's formula yield

$$\begin{aligned} \mathrm{d}\left(g_t^{-1} Y_t\right) &= g_t^{-1} \left[-f_t \mathrm{d}t - \sum_{k=1}^{\infty} r_t^k \mathrm{d}w_t^k + \sum_{k=1}^{\infty} \left(r_t^k\right)^2 \mathrm{d}t \right] Y_t \\ &\quad + g_t^{-1} \left[a_t \mathrm{d}t + f_t Y_t \mathrm{d}t + \sum_{k=1}^{\infty} r_t^k Y_t \mathrm{d}w_t^k + \sum_{k=1}^{\infty} l_t^k \mathrm{d}w_t^k \right] \\ &\quad - g_t^{-1} \sum_{k=1}^{\infty} \left(r_t^k\right)^2 Y_t \mathrm{d}t - g_t^{-1} \sum_{k=1}^{\infty} r_t^k l_t^k \mathrm{d}t \\ &= g_t^{-1} a_t \mathrm{d}t + g_t^{-1} \sum_{k=1}^{\infty} l_t^k \mathrm{d}w_t^k - g_t^{-1} \sum_{k=1}^{\infty} r_t^k l_t^k \mathrm{d}t. \end{aligned}$$

Thus (3.21) holds. □

3.1 Proof of Theorem 2.1 and Corollary 2.3

To make the procedure more clear, we shall start with some explanations on the key ideas of the proof. The proof basises on the integration by parts formula of the Malliavin gradient operator, see for instance [8, 10, 13]. Let $(D, \mathscr{D}(D))$ be the Malliavin gradient operator, and $(D^*, \mathscr{D}(D^*))$ be its adjoint operator (i.e. the Malliavin divergence operator). Fix $T > 0$. Let $\bar{h}$ be a function from $[0, T] \times \Omega$ to $\mathbb{H}$, and let $D_{\bar{h}} X_T$ be the Malliavin derivative of X_T along $\bar{h}$. If $D_{\bar{h}} X_T = e^{TA} v$, then

$$P_T \nabla_{e^{TA}v} f(x) = \mathbb{E} \nabla_{e^{TA}v} f(X_T) = \mathbb{E} \nabla_{D_{\bar{h}} X_T} f(X_T) = \mathbb{E} D_{\bar{h}}(f(X_T)) = \mathbb{E} f(X_T) D^*(\bar{h}).$$

If $\bar{h}$ is adapted, then $D^*(\bar{h})$ is an Itô integral and the integration by parts formula follows. However, in the situation of stochastic equations with multiplicative noise, $\bar{h}$ is usually not an adapted process. Formally, we construct $\bar{h}$ as follows. $D_{\bar{h}} X_T$ satisfies the following equation

$$\begin{aligned} \mathrm{d}D_{\bar{h}} X_t &= (A + \nabla b_t(X_t)) D_{\bar{h}} X_t \mathrm{d}t \\ &\quad + \sum_{j=1}^{\infty} \nabla \sigma_t^{(j)}(X_t) D_{\bar{h}} X_t \mathrm{d}w_t^j + \sigma_t(X_t) \mathrm{d}\bar{h}(t), \quad D_{\bar{h}} X_0 = 0, \end{aligned}$$

and then we can write it in the integral form

$$D_{\bar{h}}X_T = J_T^A \int_0^T (J_s^A)^{-1}\sigma_t(X_t)\bar{h}'(t)\mathrm{d}t.$$

Letting $\bar{h}'(t) = \frac{1}{T}\sigma_t^{-1}(X_t)J_t^A J_T^{-1}v$, formally, we have

$$D_{\bar{h}}X_T = \frac{1}{T}J_T^A \int_0^T (J_s^A)^{-1}J_s^A J_T^{-1}v\mathrm{d}t = e^{TA}v,$$

where we use $J_T^A = e^{AT}J_T$(see (3.16) in Remark 3.1). To avoid the trouble caused by the non-adaptedness of $\bar{h}$, we shall rewrite $\bar{h}$:

$$\bar{h}'(t) = \frac{1}{T}\sigma_t^{-1}(X_t)J_t^A J_T^{-1}v = \frac{1}{T}\sum_{k=1}^{\infty}\langle J_T^{-1}v, e_k\rangle\sigma_t^{-1}(X_t)J_t^A e_k \equiv \frac{1}{T}\sum_{k=1}^{\infty}F_k h_k'(t).$$

Then h_k is adapted, $D_{\bar{h}}X_T = \frac{1}{T}\sum_{k=1}^{\infty}F_k D_{h_k}X_T$, and by the chain rule

$$\begin{aligned}\mathbb{E}\nabla_{e^{TA}v}f(X_T) &= \frac{1}{T}\sum_{k=1}^{\infty}\mathbb{E}F_k\nabla_{D_{h_k}X_T}f(X_T)\\ &= \frac{1}{T}\sum_{k=1}^{\infty}\mathbb{E}\left[D_{h_k}(F_k f(X_T)) - f(X_T)D_{h_k}(F_k)\right].\end{aligned}\tag{3.22}$$

What we shall do is to make these all rigorous, and prove that

$$\sum_{k=1}^{\infty}\mathbb{E}\left[D_{h_k}(F_k f(X_T)) - f(X_T)D_{h_k}(F_k)\right] = \mathbb{E}f(X_T)\sum_k\left(F_k D^*(h_k) - D_{h_k}(F_k)\right),$$

and give a representation to the right hand side of the equality above.

Proof of Theorem 2.1

From now on, we fix $T > 0$.

(1) We shall give a rigorous proof of (3.22). Let

$$h_k(t) = \int_0^t \sigma_s^{-1}(X_s)J_s^A e_k\mathrm{d}s, \quad F_k = \left\langle J_T^{-1}v, e_k\right\rangle, \quad k \geq 1, t \in [0, T].\tag{3.23}$$

Then according to [1, Theorem A.2], from (1.2) and (ii), we have

$$\begin{aligned}\mathrm{d}D_{h_k}X_t &= (A + \nabla b_t(X_t))D_{h_k}X_t\mathrm{d}t\\ &\quad + \sum_{j=1}^{\infty}\nabla\sigma_t^{(j)}(X_t)D_{h_k}X_t\mathrm{d}w_t^j + \sigma_t(X_t)\mathrm{d}h_k(t), \quad D_h X_0 = 0,\end{aligned}\tag{3.24}$$

and $J_t^A e_k$ satisfies the following equation

$$\mathrm{d}J_t^A e_k = (A + \nabla b_t(X_t))J_t^A e_k \mathrm{d}t + \sum_{j=1}^{\infty} \nabla \sigma_t^{(j)}(X_t) J_t^A e_k \mathrm{d}w_t^j.$$

Let $v \in \mathbb{H}$. Then

$$\begin{aligned}
&\int_0^t \langle e^{(t-s)A}\nabla b_s(X_s)(sJ_s^A e_k), v\rangle \mathrm{d}s + \sum_j \int_0^t s\langle e^{(t-s)A}(\nabla \sigma_s^j(X_s))(J_s^A e_k), v\rangle \mathrm{d}W_s^j \\
&= t\int_0^t \langle e^{(t-s)A}\nabla b_s(X_s)J_s^A e_k, v\rangle \mathrm{d}s - \int_0^t \int_0^r \langle e^{(t-s)A}\nabla b_s(X_s)(sJ_s^A e_k), v\rangle \mathrm{d}s\mathrm{d}r \\
&\quad + t\sum_j \int_0^t s\langle e^{(t-s)A}(\nabla \sigma_s^j(X_s))(J_s^A e_k), v\rangle \mathrm{d}W_s^j \\
&\quad - \sum_j \int_0^t \int_0^r \langle e^{(t-s)A}(\nabla \sigma_s^j(X_s))(J_s^A e_k), v\rangle \mathrm{d}W_s^j \mathrm{d}r \\
&= t\langle J_t^A, v\rangle - \int_0^t \langle e^{(t-r)A} J_r^A, v\rangle \mathrm{d}r.
\end{aligned}$$

Thus $tJ_t^A e_k$ is a mild solution of (3.24). By pathwise uniqueness of (3.24),

$$D_{h_k} X_t = tJ_t^A e_k,\ t \in [0,T]. \tag{3.25}$$

Hölder inequality and (3.17), (3.18) yield that

$$\begin{aligned}
\mathbb{E}\sum_{k=1}^{\infty} |F_k D_{h_k} X_T| &\le T\left\{\mathbb{E}\sum_{k=1}^{\infty} \left|\langle J_T^{-1}v, e_k\rangle\right|^2\right\}^{\frac{1}{2}} \left\{\mathbb{E}\sum_{k=1}^{\infty} \left|J_T^A e_k\right|^2\right\}^{\frac{1}{2}} \\
&\le Tv\left\{\mathbb{E}\left\|J_T^{-1}\right\|^2\right\}^{\frac{1}{2}} \left\{\mathbb{E}\left\|J_T^A\right\|_{HS}^2\right\}^{\frac{1}{2}} < \infty.
\end{aligned}$$

Hence $\sum_{k=1}^{\infty} F_k D_{h_k} X_T$ converges in $\mathbb{H}$ $\mathbb{P}$-a.s. Moreover, (3.16) implies that

$$\sum_{k=1}^{\infty} F_k D_{h_k} X_T = T\sum_{k=1}^{\infty} \langle J_T^{-1}v, e_k\rangle J_T^A e_k = TJ_T^A J_T^{-1} v = Te^{AT}v.$$

And Fubini theorem implies that

$$\begin{aligned}
\mathbb{E}(\nabla_{\mathrm{e}^{AT}v} f)(X_T) &= \frac{1}{T}\sum_{k=1}^{\infty} \mathbb{E}F_k D_{h_k}(f(X_T)) \\
&= \frac{1}{T}\sum_{k=1}^{\infty} \mathbb{E}f(X_T)\left(F_k D^* h_k - D_{h_k} F_k\right).
\end{aligned} \tag{3.26}$$

(2) We shall give a representation of the last term of (3.26). Noting that h_k is adapted with $\mathbb{E}\int_0^T |h_k'(t)|^2 \mathrm{d}t < \infty$, we obtain

$$\begin{aligned} D^*(h_k) &= \int_0^T \langle h_k'(s), \mathrm{d}W_s \rangle \\ &= \int_0^T \left\langle \sigma_s^{-1}(X_s) J_s^A e_k, \mathrm{d}W_s \right\rangle \\ &= \left\langle e_k, \int_0^T \left[\sigma_s^{-1}(X_s) J_s^A\right]^* \mathrm{d}W_s \right\rangle, \quad k \geq 1. \end{aligned}$$

Since (3.13) and $J_T^{-1} v \in \mathbb{H}$ a.s. hold, we have

$$\begin{aligned} \sum_{k=1}^\infty F_k D^*(h_k) &= \sum_{k=1}^\infty \langle J_T^{-1} v, e_k \rangle \left\langle e_k, \int_0^T \left[\sigma_s^{-1}(X_s) J_s^A\right]^* \mathrm{d}W_s \right\rangle \\ &= \left\langle \int_0^T \left[\sigma_s^{-1}(X_s) J_s^A\right]^* \mathrm{d}W_s, J_T^{-1} v \right\rangle. \end{aligned} \tag{3.27}$$

From (2.1) and (H1), for all $u \in \mathbb{H}$

$$\begin{aligned} \mathrm{d}D_{h_k} J_t u = {} & B_t(X_t) D_{h_k} J_t u \mathrm{d}t + \sum_{j=1}^\infty \Sigma_t^{(j)}(X_t) D_{h_k} J_t u \mathrm{d}w_t^j \\ & + \left(\nabla_{D_{h_k} X_t} B_t\right)(X_t) J_t u \mathrm{d}t + \sum_{j=1}^\infty \left(\nabla_{D_{h_k} X_t} \Sigma_t^{(j)}\right)(X_t) J_t u \mathrm{d}w_t^j \\ & + \sum_{j=1}^\infty \Sigma_t^{(j)}(X_t) J_t u \mathrm{d}h_k^j(t), \quad D_{h_k} J_0 u = 0, \end{aligned}$$

where $h_k^j := \langle h_k, e_j \rangle$, $j \geq 1$. By Lemma 3.3, we obtain

$$\begin{aligned} D_{h_k} J_T u = {} & J_T \int_0^T J_t^{-1} \left(\nabla_{D_{h_k} X_t} B_t\right)(X_t) J_t u \mathrm{d}t \\ & + J_T \int_0^T J_t^{-1} \sum_{j=1}^\infty \left(\nabla_{D_{h_k} X_t} \Sigma_t^{(j)}\right)(X_t) J_t u \mathrm{d}w_t^j \\ & + J_T \int_0^T J_t^{-1} \sum_{j=1}^\infty \left\langle \sigma_t^{-1}(X_t) J_t^A e_k, e_j \right\rangle \Sigma_t^{(j)}(X_t) J_t u \mathrm{d}t \\ & - J_T \int_0^T J_t^{-1} \sum_{j=1}^\infty \Sigma_t^{(j)}(X_t) \left(\nabla_{D_{h_k} X_t} \Sigma_t^{(j)}\right)(X_t) J_t u \mathrm{d}t. \end{aligned} \tag{3.28}$$

Since

$$\langle D_{h_k}(J_T^{-1} v), e_k \rangle = -\langle D_{h_k}((J_T^{-1})^* e_k), v \rangle = -\langle J_T^* D_{h_k}((J_T^{-1})^* e_k), J_T^{-1} v \rangle$$

$$= -\sum_{j=1}^{\infty}\langle D_{h_k}((J_T^{-1})^* e_k), J_T e_j\rangle\langle J_T^{-1}v, e_j\rangle$$
$$= \sum_{j=1}^{\infty}\langle e_k, J_T^{-1}D_{h_k}(J_T e_j)\rangle\langle J_T^{-1}v, e_j\rangle,$$

combining this with (3.25) and (3.28), we get

$$\begin{aligned}\sum_{k=1}^{\infty} D_{h_k}F_k =& -\sum_{k=1}^{\infty}\int_0^T \left\langle J_t^{-1}\left(\nabla_{tJ_t^A e_k}B_t\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t \\ &-\lim_{n\to\infty}\sum_{k=1}^{n}\left\langle\left[\int_0^T J_t^{-1}\sum_{j=1}^{\infty}\left(\nabla_{tJ_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_t\mathrm{d}w_t^j\right]J_T^{-1}v, e_k\right\rangle \\ &-\sum_{k=1}^{\infty}\int_0^T\sum_{j=1}^{\infty}\langle\sigma_t^{-1}(X_t)J_t^A e_k, e_j\rangle\left\langle J_t^{-1}\Sigma_t^{(j)}(X_t)J_tJ_T^{-1}v, e_k\right\rangle\mathrm{d}t \\ &+\sum_{k=1}^{\infty}\int_0^T\left\langle J_t^{-1}\sum_{j=1}^{\infty}\Sigma_t^{(j)}(X_t)\left(\nabla_{tJ_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle\mathrm{d}t.\end{aligned} \tag{3.29}$$

For the second term. Let π_n be the orthogonal projection from $\mathbb{H}$ to $\mathrm{span}\{e_1,\cdots,e_n\}$. Then

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{j=1}^{\infty}\left\langle\left[\int_0^T J_t^{-1}\left(\nabla_{tJ_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_t\mathrm{d}w_t^j\right]J_T^{-1}v, e_k\right\rangle \\ &=\sum_{k=1}^{n}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left\langle\left[\int_0^T \pi_nJ_t^{-1}\left(\nabla_{tJ_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_t\mathrm{d}w_t^j\right]e_l, e_k\right\rangle\left\langle J_T^{-1}v, e_l\right\rangle \\ &=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\int_0^T t\sum_{k=1}^{n}\left\langle\pi_nJ_t^{-1}\left(\nabla_{J_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_te_l, e_k\right\rangle\mathrm{d}w_t^j\langle J_T^{-1}v, e_l\rangle \\ &=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\langle\pi_nJ_t^{-1}\left(\nabla_{J_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_te_l, e_k\right\rangle\mathrm{d}w_t^j\langle J_T^{-1}v, e_l\rangle \\ &=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\int_0^T t\mathrm{Tr}\left\{\pi_nJ_t^{-1}\left(\nabla_{J_t^A\cdot}\Sigma_t^{(j)}\right)(X_t)J_te_l\right\}\mathrm{d}w_t^j\langle J_T^{-1}v, e_l\rangle \\ &=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\int_0^T t\mathrm{Tr}\left\{J_t^A\pi_nJ_t^{-1}\left[\left(\nabla_{\cdot}\Sigma_t^{(j)}\right)(X_t)J_te_l\right]\right\}\mathrm{d}w_t^j\langle J_T^{-1}v, e_l\rangle \\ &=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left\langle J_t\pi_nJ_t^{-1}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_te_l\right], e_k\right\rangle\right\}\mathrm{d}w_t^j\langle J_T^{-1}v, e_l\rangle \\ &=\sum_{j=1}^{\infty}\left\langle\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_nJ_t^*e_k\right\}\mathrm{d}w_t^j, J_T^{-1}v\right\rangle.\end{aligned}$$

Since

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{j=m}^{m+p}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_n J_t^* e_k\right\}\mathrm{d}w_t^j\right|^2\\
&=\sum_{j=m}^{m+p}\mathbb{E}\int_0^T t^2\left|\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_n J_t^* e_k\right\}\right|^2\mathrm{d}t\\
&\leq\int_0^T t^2\left(\sum_{k=1}^{\infty}e^{-t\lambda_k}\right)^2\mathbb{E}\left(\sum_{j=m}^{\infty}\left|\left|\left(\nabla\Sigma_t^{(j)}\right)(X_t)\right|\right|^2||J_t||^4||J_t^{-1}||^2\right)\mathrm{d}t,
\end{aligned}
$$

by Lemma 3.2 and dominated convergence theorem, we have

$$
\lim_{m\to\infty}\sup_{p\geq 0}\mathbb{E}\left|\sum_{j=m}^{m+p}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_n J_t^* e_k\right\}\mathrm{d}w_t^j\right|^2=0,
$$

thus

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_n J_t^* e_k\right\}\mathrm{d}w_t^j\\
&=\lim_{m\to\infty}\sum_{j=1}^{m}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_n J_t^* e_k\right\}\mathrm{d}w_t^j
\end{aligned}
$$

holds in $L^2(\mathbb{P})$.

Since

$$
\begin{aligned}
&\mathbb{E}\left|\sum_{j=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*(I-\pi_n)J_t^* e_k\right\}\mathrm{d}w_t^j\right|^2\\
&=\mathbb{E}\int_0^T t^2\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*(I-\pi_n)J_t^* e_k\right\}\right|^2\mathrm{d}t\\
&\leq\mathbb{E}\int_0^T t^2\left(\sum_{k=1}^{\infty}e^{-t\lambda_k}|(I-\pi_n)J_t^* e_k|\right)^2\sum_{j=1}^{\infty}\left|\left|\left[\left(\nabla\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\right|\right|^2\mathrm{d}t,
\end{aligned}
$$

by (1.6), Lemma 3.2 and dominated convergence theorem, it holds in $L^2(\mathbb{P})$ that

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{j=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}\left[\left(\nabla_{e_k}\Sigma_t^{(j)}\right)(X_t)J_t\right]^*\left(J_t^{-1}\right)^*\pi_n J_t^* e_k\right\}\mathrm{d}w_t^j\\
&=\sum_{j=1}^{\infty}\int_0^T t\sum_{k=1}^{\infty}\left\{e^{-t\lambda_k}J_t^*\left(\nabla_{e_k}\Sigma_t^{(j)}\right)^*(X_t)e_k\right\}\mathrm{d}w_t^j. \qquad (3.30)
\end{aligned}
$$

Next, by (3.16),

$$tJ_t^A e_k = te^{tA} J_t e_k = te^{\frac{t}{2}A}(e^{\frac{t}{2}A}J_t)e_k.$$

Then by (3.34) and Lemma 3.2, it is clear that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}\int_0^T \left|\left\langle J_t^{-1}\left(\nabla_{tJ_t^A e_k}B_t\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle\right| \mathrm{d}t\\
&\le \left(\mathbb{E}\int_0^T t\left|\left|e^{\frac{t}{2}A}J_t\right|\right|_{HS}^2 \mathrm{d}t\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^T t\left|\left|J_t^{-1}\left(\nabla_{e^{\frac{t}{2}A}}B_t\right)(X_t)J_tJ_T^{-1}v\right|\right|_{HS}^2 \mathrm{d}t\right)^{\frac{1}{2}}\\
&\le \sup_{t\in(0,T]} t||e^{\frac{t}{2}A}||_{HS}^2\left(\mathbb{E}\int_0^T ||J_t||^2\mathrm{d}t\right)^{\frac{1}{2}}\mathbb{E}\int_0^T K_1^2(t)||J_t^{-1}||^2||J_tJ_T^{-1}v||^2\mathrm{d}t < \infty.
\end{aligned}$$

So, by Fubini theorem and (3.16), it holds that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}f(X_T)\int_0^T \left\langle J_t^{-1}\left(\nabla_{tJ_t^A e_k}B_t\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t\\
&= \mathbb{E}f(X_T)\int_0^T \sum_{k=1}^{\infty}\left\langle J_t^{-1}\left(\nabla_{tJ_t^A e_k}B_t\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t \qquad (3.31)\\
&= \mathbb{E}f(X_T)\int_0^T t\mathrm{Tr}\left\{\mathrm{e}^{tA}\left[(\nabla_. B_t)(X_t)J_tJ_T^{-1}v\right]\right\}\mathrm{d}t.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{E}f(X_T)\Big\{-\sum_{j=1}^{\infty}\left\langle\sigma_t^{-1}(X_t)J_t^Ae_k, e_j\right\rangle\left\langle J_t^{-1}\Sigma_t^{(j)}(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t\\
&\qquad + \int_0^T \left\langle J_t^{-1}\sum_{j=1}^{\infty}\Sigma_t^{(j)}(X_t)\left(\nabla_{tJ_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t\Big\}\\
&= \mathbb{E}f(X_T)\Big\{-\int_0^T\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left\langle\sigma_t^{-1}(X_t)J_t^Ae_k, e_j\right\rangle\left\langle J_t^{-1}\Sigma_t^{(j)}(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t\\
&\qquad + \int_0^T\sum_{k=1}^{\infty}\left\langle J_t^{-1}\sum_{j=1}^{\infty}\Sigma_t^{(j)}(X_t)\left(\nabla_{tJ_t^A e_k}\Sigma_t^{(j)}\right)(X_t)J_tJ_T^{-1}v, e_k\right\rangle \mathrm{d}t\Big\}\\
&= \mathbb{E}f(X_T)\Big\{-\int_0^T \mathrm{Tr}\left\{\mathrm{e}^{tA}\left[\Sigma_t(X_t)J_tJ_T^{-1}v\right]\sigma_t^{-1}(X_t)\right\}\mathrm{d}t \qquad (3.32)\\
&\qquad + \int_0^T t\mathrm{Tr}\left(\mathrm{e}^{tA}\sum_{j=1}^{\infty}\Sigma_t^{(j)}(X_t)\left[\left(\nabla\Sigma_t^{(j)}\right)(X_t)J_tJ_T^{-1}v\right]\right)\mathrm{d}t\Big\}.
\end{aligned}$$

Thus (3.26), (3.27), (3.30)-(3.32) and dominated convergence theorem imply that

$$\mathbb{E}(\nabla_{\mathrm{e}^{AT}v}f)(X_T) = \frac{1}{T}\mathbb{E}f(X_T)\left(\sum_{k=1}^{\infty}F_kD^*h_k - \sum_{k=1}^{\infty}D_{h_k}F_k\right)$$

and (2.4) holds.

□

Proof of Corollary 2.3

(1) For simplicity, letting

$$
\begin{aligned}
\Theta_1 &= \left\langle \int_0^T \left[\sigma_t^{-1}(X_t) J_t^A\right]^* \mathrm{d}W_t, J_T^{-1} v \right\rangle; \\
\Theta_2 &= \int_0^T t \mathrm{Tr}\left\{ e^{tA} \left[(\nabla . B_t)(X_t) J_t J_T^{-1} v\right]\right\} \mathrm{d}t; \\
\Theta_3 &= \left\langle \sum_{j=1}^\infty \int_0^T t \sum_{k=1}^\infty \left\{ e^{-t\lambda_k} J_t^* \left(\nabla_{e_k} \Sigma_t^{(j)}\right)^* (X_t) e_k \right\} \mathrm{d}w_t^j, J_T^{-1} v \right\rangle; \\
\Theta_4 &= \int_0^T \mathrm{Tr}\left\{ e^{tA} \left[\Sigma_t(X_t) J_t J_T^{-1} v\right] \sigma_t^{-1}(X_t) \right\} \mathrm{d}t; \\
\Theta_5 &= -\int_0^T t \mathrm{Tr}\left\{ e^{tA} \sum_{j=1}^\infty \Sigma_t^{(j)}(X_t) \left[\left(\nabla . \Sigma_t^{(j)}\right)(X_t) J_t J_T^{-1} v\right]\right\} \mathrm{d}t,
\end{aligned}
$$

we have $M_T^v = \sum_{i=1}^5 \Theta_i$. For any $q \geq 2$, by Burkerholder-Davis-Gundy inequality, Minkowski inequality, (H2) and Lemma 3.2, we have

$$
\mathbb{E}|\Theta_1|^q \leq C(q)\lambda^q(T)|v|^q \left[\int_0^T \|\mathrm{e}^{At}\|_{\mathrm{HS}}^2 \mathrm{d}t\right]^{\frac{q}{2}} \left[\sup_{t\in[0,T]} \mathbb{E}\|J_t\|^{2q}\right]^{\frac{1}{2}} \left\{\mathbb{E}\left|J_T^{-1}\right|^{2q}\right\}^{\frac{1}{2}}. \tag{3.33}
$$

Noticing that A is a negative definite self-adjoint operator, it is easy to see that

$$
\sup_{s\in(0,T]} s||\mathrm{e}^{sA}||_{HS}^2 \leq \sup_{s\in(0,T]} \int_0^s ||\mathrm{e}^{rA}||_{HS}^2 \mathrm{d}r = \int_0^T ||\mathrm{e}^{rA}||_{HS}^2 \mathrm{d}r = \delta_T < \infty. \tag{3.34}
$$

Then Minkowski inequality, (H1) and Lemma 3.2 yield that

$$
\begin{aligned}
\mathbb{E}|\Theta_2|^q &\leq |v|^q \left(\sup_{s\in(0,T]} s||e^{\frac{s}{2}A}||_{HS}^2\right)^q \left(\int_0^T K_1(s)^2 \mathrm{d}s\right)^q \mathbb{E}\left(\int_0^T ||J_s||^2 \mathrm{d}s\right)^{\frac{q}{2}} ||J_T^{-1}||^q \\
&\leq 2^q T^{\frac{1}{2}} \delta_{T/2}^q |v|^q \left(\int_0^T K_1(s)^2 \mathrm{d}s\right)^q \left(\sup_{t\in[0,T]} \mathbb{E}||J_t||^{2q}\right)^{1/2} \left(\mathbb{E}||J_T^{-1}||^{2q}\right)^{1/2}. \qquad (3.35)
\end{aligned}
$$

Again by Burkerholder-Davis-Gundy inequality, Minkowski inequality, (H1) and Lemma 3.2, it holds that

$$
\mathbb{E}|\Theta_3|^q \leq \left(\mathbb{E}\left|\sum_{j=1}^\infty \int_0^T t \sum_{k=1}^\infty \left\{J_t^* (\nabla_{e_k} \Sigma_t^{(j)})^* (X_t) e^{tA} e_k\right\} \mathrm{d}w_t^j\right|^{2q}\right)^{\frac{1}{2}} \left(\mathbb{E}|J_T^{-1} v|^{2q}\right)^{\frac{1}{2}}
$$

$$\leq C(q)\left(\mathbb{E}\left(\sum_{j=1}^{\infty}\int_0^T t^2\left(\sum_{k=1}^{\infty}e^{-\lambda_k t}\right)^2||J_t^*\nabla\Sigma_t^{(j)}(X_t)||^2\mathrm{d}t\right)^q\right)^{\frac{1}{2}}\left(\mathbb{E}|J_T^{-1}v|^{2q}\right)^{\frac{1}{2}}$$

$$\leq C(q)\delta_{T/2}^q\left(\mathbb{E}\left(\int_0^T K_1^2(t)||J_t||^2\mathrm{d}t\right)^q\right)^{\frac{1}{2}}\left(\mathbb{E}|J_T^{-1}v|^{2q}\right)^{\frac{1}{2}} \tag{3.36}$$

$$\leq C(q)\delta_{T/2}^q\left(\int_0^T K_1^2(t)\mathrm{d}t\right)^q\left(\sup_{t\in[0,T]}\mathbb{E}||J_t||^{2q}\right)^{\frac{1}{2}}\left(\mathbb{E}|J_T^{-1}v|^{2q}\right)^{\frac{1}{2}}.$$

Since

$$||\Sigma_t(X_t)J_tJ_T^{-1}v||_{HS}^2=\sum_{j=1}^{\infty}|\Sigma_t^{(j)}(X_t)J_tJ_T^{-1}v|^2\leq K_1^2(t)|J_tJ_T^{-1}v|^2,$$

it is easy to see that

$$\mathbb{E}|\Theta_4|^q\leq\mathbb{E}\left(\int_0^T||e^{tA}||_{HS}||\Sigma_t(X_t)J_tJ_T^{-1}v||_{HS}\mathrm{d}t\right)^q$$

$$\leq\left(\int_0^T||e^{tA}||_{HS}^2\mathrm{d}t\right)^{\frac{q}{2}}\left(\mathbb{E}\left(\int_0^T K_1^2(t)||J_t||^2\mathrm{d}t\right)^q\right)^{\frac{1}{2}}\left(\mathbb{E}|J_T^{-1}v|^{2q}\right)^{\frac{1}{2}} \tag{3.37}$$

$$\leq\delta_T^{\frac{q}{2}}\left(\int_0^T K_1^2(t)\mathrm{d}t\right)^{\frac{q}{2}}\left(\sup_{t\in[0,T]}||J_t||^{2q}\right)^{\frac{1}{2}}\left(\mathbb{E}|J_T^{-1}v|^{2q}\right)^{\frac{1}{2}}$$

and

$$\mathbb{E}|\Theta_5|^q\leq\mathbb{E}\left(\int_0^T t||e^{tA}||_{HS}\sum_{j=1}^{\infty}||\Sigma_t^{(j)}(X_t)||_{HS}||\nabla\Sigma_t^{(j)}(X_t)J_tJ_T^{-1}v||\mathrm{d}t\right)^q$$

$$\leq(T\delta_T)^{\frac{q}{2}}\mathbb{E}\left(\int_0^T K_1^2(t)||J_t||\mathrm{d}t\right)^q|J_T^{-1}v|^q \tag{3.38}$$

$$\leq(T\delta_T)^{\frac{q}{2}}\left(\int_0^T K_1^2(t)\mathrm{d}t\right)^{\frac{q}{2}}\left(\mathbb{E}\sup_{t\in[0,T]}||J_t||^{2q}\right)^{\frac{1}{2}}\left(\mathbb{E}|J_T^{-1}v|^{2q}\right).$$

Let $\Gamma_{T,q,A}$ be in Corollary 2.3 for $T>0, q\geq 2$. Combining (3.17), (3.18), (3.33)-(3.38), for any $q\geq 2$, it holds that

$$(\mathbb{E}|M_T^v|^q)^{\frac{1}{q}}\leq\left\{5^{q-1}\sum_{i=1}^{5}\mathbb{E}|\Theta_i|^q\right\}^{\frac{1}{q}}|v|\leq\{\Gamma_{T,q,A}\}^{\frac{1}{q}}|v|. \tag{3.39}$$

On the other hand, Jensen inequality yields that

$$(\mathbb{E}|M_T^v|^q)^{\frac{1}{q}}\leq\left(\mathbb{E}|M_T^v|^2\right)^{\frac{1}{2}} \tag{3.40}$$

for any $1 < q < 2$. Combining (3.39) and (3.40), it follows from (2.4) and Hölder inequality that for any $p > 1$,

$$\begin{aligned}|P_T(\nabla_{e^{AT}v} f)| &\le \frac{1}{T}(P_T|f|^p)^{\frac{1}{p}} \left(\mathbb{E}|M_T^v|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\ &\le \{\Gamma_{T,[\frac{p}{p-1}]\vee 2,A}\}^{[\frac{p-1}{p}]\wedge\frac{1}{2}} \frac{|v|}{T}(P_T|f|^p)^{\frac{1}{p}}, \quad f \in C_b^1(\mathbb{H}).\end{aligned}$$

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