IRREDUCIBILITY AND ASYMPTOTICS OF STOCHASTIC BURGERS EQUATION DRIVEN BY α-STABLE PROCESSES

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Abstract

The irreducibility, moderate deviation principle and ψ-uniformly exponential ergodicity with $\psi(x) := 1 + \|x\|_0$ are proved for stochastic Burgers equation driven by the α-stable processes for $\alpha \in (1, 2)$, where the first two are new for the present model, and the last strengthens the exponential ergodicity under total variational norm derived in [7].

Keywords: stochastic Burgers equation; α-stable noises; irreducibility, ψ-uniformly ergodicity, moderate deviation

Mathematics Subject Classification (2000): 60F10, 60H15, 60J75.

1. INTRODUCTION

In [7], the strongly Feller property and exponential ergodicity have been proved for the stochastic Burgers equation driven by rotationally symmetric α-stable processes with $\alpha \in (1, 2)$. In this paper, we prove a stronger ψ-uniformly exponential ergodicity, the irreducibility, and the moderate deviation principle for occupation measures. Before stating our main results, we briefly recall the framework of the study and results derived in [7]. For more research about asymptotics of stochastic systems driven by Lévy processes, we refer the reader to [3, 2, 4, 10, 12, 11, 13, 18, 31, 32, 25, 24, 23, 22, 8, 15, 26, 16, 14, 30, 20, 18, 8].

Let $\mathbb{H}$ be the space of all square integrable functions on the torus $\mathbb{T} = [0, 2\pi)$ with vanishing mean values. Let $Au = -u''$ be the second order differential operator. Then A is a positive self-adjoint operator on $\mathbb{H}$. Let $\lambda_{2k} := \lambda_{2k+1} := k^2$ with $k \in \mathbb{N}$ and

$$e_{2k}(x) := \pi^{-\frac{1}{2}} \cos(kx), \;\; e_{2k+1}(x) := \pi^{-\frac{1}{2}} \sin(kx).$$

It is easy to see that $\{e_k, k \in \mathbb{N}\}$ forms an orthogonal basis of $\mathbb{H}$ and

$$Ae_k = \lambda_k e_k, \; k \in \mathbb{N}.$$

The norm in $\mathbb{H}$ is denoted by $\| \cdot \|_0$.

For $\gamma \in \mathbb{R}$, let $\mathbb{H}^\gamma$ be the domain of the fractional operator $A^{\frac{\gamma}{2}}$:

$$\mathbb{H}^\gamma := A^{-\frac{\gamma}{2}}(\mathbb{H}) = \left\{ \sum_k \lambda_k^{-\frac{\gamma}{2}} a_k e_k : (a_k)_{k\in\mathbb{N}} \subset \mathbb{R}, \sum_k a_k^2 < +\infty \right\}.$$

It is a separable Hilbert space with the inner product

$$\langle u, v \rangle_\gamma := \langle A^{\frac{\gamma}{2}} u, A^{\frac{\gamma}{2}} v \rangle_0 = \sum_k \lambda_k^\gamma \langle u, e_k \rangle_0 \langle v, e_k \rangle_0.$$

For $u \in \mathbb{H}$, let $\|u\|_\gamma = \sqrt{\langle u, u\rangle_\gamma}$ if $u \in \mathbb{H}^\gamma$, and $\|u\|_\gamma = \infty$ otherwise. The C_0-contraction semigroup e^{-tA} generated by $-A$ reads

$$\mathrm{e}^{-tA}u := \sum_k \mathrm{e}^{-t\lambda_k}\langle u, e_k\rangle_0 e_k, \ \ t \geq 0.$$

Obviously,

Semi

$$\|A^\gamma \mathrm{e}^{-tA}u\|_0 \leq \sup_{x>0}(x^\gamma \mathrm{e}^{-x})t^{-\gamma}\|u\|_0 = \gamma^\gamma \mathrm{e}^{-\gamma}t^{-\gamma}\|u\|_0, \ \ \gamma > 0. \tag{1.1}$$

Let $\{W_t^k, t \geq 0\}_{k\in\mathbb{N}}$ be a sequence of independent standard one-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cylindrical Brownian motion on $\mathbb{H}$ is defined by

$$W_t := \sum_k W_t^k e_k.$$

For $\alpha \in (0, 2)$, let S_t be an independent $\alpha/2$-stable subordinator, i.e., an increasing one dimensional Lévy process with Laplace transform

$$\mathbb{E}\mathrm{e}^{-\eta S_t} = \mathrm{e}^{-t|\eta|^{\alpha/2}}, \ \ \eta > 0.$$

The subordinated cylindrical Brownian motion $\{L_t\}_{t\geq 0}$ on $\mathbb{H}$ is defined by

$$L_t := W_{S_t}.$$

Notice that in general L_t does not belong to $\mathbb{H}$.

We are concerned about the following stochastic Burgers equation in the Hilbert space $\mathbb{H}$:

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$$\mathrm{d}X_t \ = \ [-AX_t - B(X_t)]\mathrm{d}t + Q\mathrm{d}L_t, \quad X_0 = x \in \mathbb{H}, \tag{1.2}$$

where $B(u) := B(u, u)$ for the bilinear operator b defined by $B(u, v) := uv'$ for $v \in \mathbb{H}^1$ and $u \in \mathbb{H}$, and $Q \in \mathcal{L}(\mathbb{H})$ is given by

$$Qu : = \sum_{k=1}^\infty \beta_k\langle u, e_k\rangle_0 e_k, \quad u \in \mathbb{H},$$

with $\beta = (\beta_k)_{k\in\mathbb{N}}$ such that there exist some $\delta \in (0, 1)$ and $\frac{3}{2} < \theta' \leq \theta < 2$ satisfying

Assumption

$$\delta\lambda_k^{-\frac{\theta}{2}} \ \leq \ |\beta_k| \ \leq \ \delta^{-1}\lambda_k^{-\frac{\theta'}{2}}, \qquad k \in \mathbb{N}. \tag{1.3}$$

By [21, Lemma 2.1], we have

EE2

$$\langle B(u, v), w\rangle_0 \leq C\|u\|_{\sigma_1}\|v\|_{\sigma_2+1}\|w\|_{\sigma_3}, \ \ \sigma_1 + \sigma_2 + \sigma_3 > 1/2, u, w \in \mathbb{H}, v \in \mathbb{H}^1. \tag{1.4}$$

Moreover, let

ZT0

$$Z_t := \int_0^t \mathrm{e}^{-(t-s)A}Q\mathrm{d}L_s \quad t \geq 0 \tag{1.5}$$

satisfies $Z_\cdot \in \mathcal{D}([0, \infty); \mathbb{H}^1)$ and

*D

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|Z_t\|_1\right] < \infty, \ \ T > 0, \tag{1.6}$$

see e.g. [7, (4.5)]. Recall that for a topology space E, $\mathcal{C}([0, \infty); E)$ (resp. $\mathcal{D}([0, \infty); E)$) stands for the space of the continuous (resp. right continuous with left limits) maps from

$[0,T]$ to E. We further denote by $\mathcal{B}_b(E)$ and $\mathcal{C}_b(E)$ the spaces of bounded measurable functions and bounded continuous functions respectively. The following result is due to [7, Theorem 4.2]. For a σ-finite measure μ on E we denote $\mu(f) = \int_E f \mathrm{d}\mu$, $f \in L^1(\mu)$.

Main

Theorem 1.1 ([7])**.** *Let $\alpha \in (1,2)$ and the assumption* (1.3) *hold for some $\delta \in (0,1)$ and $\frac{3}{2} < \theta' \le \theta < 2$.*

(1) *For any $x \in \mathbb{H}$,* (1.2) *has a unique solution $(X_t^x)_{t\ge 0}$ starting at x, and*
$$X_\cdot^x - Z_\cdot \in \mathcal{C}([0,\infty), \mathbb{H}) \cap \mathcal{C}((0,\infty), \mathbb{H}^1).$$
In particular, $(t,x) \mapsto X_t^x$ is a Markov process on $\mathbb{H}$.

(2) *The Markov semigroup P_t for X_t^x is strongly Feller, and has a unique invariant probability measure μ_0 such that*

EGD
$$\sup_{f \in \mathcal{B}_b(\mathbb{H}), |f| \le 1} |P_t f(x) - \mu_0(f)| \le C(1 + \|x\|_0)\mathrm{e}^{-\gamma t}, \quad t \ge 0, x \in \mathbb{H} \tag{1.7}$$
holds for some constants $C, \gamma > 0$.

In this paper, we prove the following two theorems on the irreducibility, moderate deviation principle of occupation measures for solutions to (1.2), and the ψ-uniformly exponential ergodicity for $\psi(x) := 1 + \|x\|_0$. The first two properties are new for the present model, and the third strengthen the exponential ergodicity (1.7) with $|f| \le \psi$ replacing $|f| \le 1$.

T1.2

Theorem 1.2. *In the situation of Theorem 1.1, for any $x \in \mathbb{H}$, the solution $(X_t^x)_{t\ge 0}$ of* (1.2) *is irreducible in $\mathbb{H}$, i.e.*
$$\mathbb{P}\left(\|X_T^x - a\|_0 < \varepsilon\right) > 0, \quad \varepsilon > 0, T > 0, a \in \mathbb{H}.$$

To state our second result, we recall the notion of moderate deviations (MDP). Let $\mathcal{M}_b(\mathbb{H})$ be the space of signed σ-additive measures of bounded variation on $\mathbb{H}$, equipped with the τ-topology $\tau := \sigma(\mathcal{M}_b(\mathbb{H}), \mathcal{B}_b(\mathbb{H}))$ of convergence against all bounded Borel functions, which is stronger than the usual weak convergence topology $\sigma(\mathcal{M}_b(\mathbb{H}), \mathcal{C}_b(\mathbb{H}))$. We denote $\mathcal{M}_1(\mathbb{H})$ the space of probability measures on $\mathbb{H}$. Given a $\psi : \mathbb{H} \to \mathbb{R}_+$, define
$$\mathcal{B}_\psi := \mathcal{B}_\psi(\mathbb{H}, \mathbb{R}) = \{f \in \mathcal{B}(\mathbb{H}, \mathbb{R}) : |f(x)| \le \psi(x)\}.$$

Let $b(t) : \mathbb{R}^+ \to (0, +\infty)$ be an increasing function verifying

eq: scale
$$\lim_{t\to\infty} b(t) = +\infty, \quad \lim_{t\to\infty} \frac{b(t)}{\sqrt{t}} = 0, \tag{1.8}$$
and let
$$\mathfrak{M}_t := \frac{1}{b(t)\sqrt{t}} \int_0^t (\delta_{X_s} - \mu_0)\mathrm{d}s.$$
To characterize *moderate deviations* of X_t from its *asymptotic limit* μ, one estimates the long time behaviours of

moderate
$$\mathbb{P}_\mu\left(\mathfrak{M}_t \in A\right), \tag{1.9}$$
where $A \in \tau$ is a given domain of deviation, and $\mathbb{P}_\mu$ is the probability measure taken for the system X with initial distribution μ. This problem refers to the central limit theorem for $b(t) = 1$, the large deviation principle (LDP) for $b(t) = \sqrt{t}$, and the moderate deviation principle (MDP) for $b(t)$ satisfying (1.8), see [6]. We say that $\mathbb{P}_\mu\left(\mathfrak{M}_t \in \cdot\right)$ satisfies the MDP

with a rate function I on $\mathcal{M}_1(\mathbb{H})$, if the following three properties hold for any b satisfying (1.8):

(a1) for any $a \geq 0$, $\{\nu \in \mathcal{M}_1(\mathbb{H}); I(\nu) \leq a\}$ is compact in $(\mathcal{M}_1(\mathbb{H}), \tau)$;

(a2) (the upper bound) for any closed set F in $(\mathcal{M}_1(\mathbb{H}), \tau)$,

$$\limsup_{T\to\infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in F) \leq -\inf_F I;$$

(a3) (the lower bound) for any open set G in $(\mathcal{M}_1(\mathbb{H}), \tau)$,

$$\liminf_{T\to\infty} \frac{1}{b^2(T)} \log \mathbb{P}_\mu(\mathfrak{M}_T \in G) \geq -\inf_G I.$$

t:MDP **Theorem 1.3.** *In the situation of Theorem 1.1, let $\psi(x) = 1 + \|x\|_0$. Then the following statements hold.*

(1) *The Markov semigroup P_t associated with* (1.2) *has a unique invariant measure μ_0 with $\mu_0(\|\cdot\|_0) := \int_{\mathbb{H}} \|x\|_0 \mu_0(\mathrm{d}x) < \infty$ and*

$$\sup_{f\in\mathcal{B}_\psi} |P_t f(x) - \mu_0(f)| \leq C\mathrm{e}^{-\gamma t}(1 + \|x\|_0), \quad x \in \mathbb{H}, t \geq 0$$

holds for some constants $C, \gamma > 0$.

(2) *For any initial distribution μ with $\mu(\|\cdot\|_0) < +\infty$ and any measurable function f with $\|f\psi^{-1}\|_\infty := \sup_{\mathbb{H}} |f\psi^{-1}| < \infty$, the limit*

$$\sigma^2(f) := \lim_{t\to\infty} \frac{1}{t} \mathbb{E}^{\mu_0} \left(\int_0^t (f(X_s) - \mu_0(f))\mathrm{d}s \right)^2$$

exists. Moreover, the family $\{\mathbb{P}_\mu(\mathfrak{M}_t \in \cdot) : t \geq 0\}$ satisfies the MDP with rate function

$$I(\mu) := \sup\left\{ \mu(f) - \frac{1}{2}\sigma^2(f) : \ f \in \mathcal{B}_b(\mathbb{H}) \right\}.$$

To prove the irreducibility by a standard argument developed in [17] for SDEs driven by cylindrical α-stable process, one needs to solve a control problem for an associated deterministic system, and establish a maximum inequality. Unlike the cylindrical α-stable process where component processes are independent, the rotationally α-stable process we are considering has strong correlations between any two components, which leads to essential difficulty to follow the line of [17]. To overcome the difficulty, we propose a new procedure including the following three steps: taking a sample path of $\alpha/2$-stable subordinator ℓ, solving a new control problem by mollifying ℓ as in [32], and proving the irreducibility by showing that for the stochastic systems driven by W_{ℓ_t}. With these preparations, Theorems 1.2 and 1.3 will be proved in Sections 4 and 5 respectively.

2. A CONTROL PROBLEM FOR THE ASSOCIATED DETERMINISTIC SYSTEM

Consider the path space of the subordinator S_t ([33]):

$$\mathcal{S} \ = \ \{\ell : [0,\infty) \to [0,\infty); \ell \text{ is strictly increasing, right continuous and has left limit}\}.$$

For any $\ell \in \mathcal{S}$, the set of jumps

$$\mathcal{J}(\ell) \ := \ \{t \geq 0 : \ell_{t-} \neq \ell_t\}$$

is at most countable. Let

$$\gamma_t = \inf\{s \geq 0 : \ell_s \geq t\}, \ \ t \geq 0.$$

Consider the following deterministic system in $\mathbb{H}$:

e:xtl-0
$$\mathrm{d}x_t^\ell + \left[Ax_t^\ell + B\left(x_t^\ell\right)\right]\mathrm{d}t = Q\mathrm{d}u_{\ell_t}, \quad x_0^\ell = x_0, \tag{2.1}$$

where $u : [0, \infty) \to \mathbb{H}$ is the controller to be chosen later. Let

ZT2
$$z_t^\ell = \int_0^t \mathrm{e}^{-A(t-s)} Q\mathrm{d}u_{\ell_s}, \ \ y_t^\ell = x_t^\ell - z_t^\ell, \ \ t \geq 0. \tag{2.2}$$

Then

e:ytlEqn
$$\frac{\mathrm{d}y_t^\ell}{\mathrm{d}t} + Ay_t^\ell + B(y_t^\ell + z_t^\ell) = 0, \quad x_0^\ell = x_0. \tag{2.3}$$

Define

e:Tea
$$t_\varepsilon(a, T) \ = \ \sup\left\{t < \frac{T}{2} : \ \|\mathrm{e}^{-At}a - a\|_0 < \frac{\varepsilon}{2}\right\}, \ \ T > 0, \varepsilon > 0, a \in \mathbb{H}. \tag{2.4}$$

It is easy to see that $t_\varepsilon(a, T) \in (0, T/2]$. For notational simplicity, we often write $t_\varepsilon = t_\varepsilon(a, T)$. The main result in this section is the following.

p:AppCon
Proposition 2.1. *Let $\ell \in \mathcal{S}$ and $x_0 \in \mathbb{H}^1$. For any $\varepsilon > 0$, $T > 0$ and $a \in \mathbb{H}$, there exist $u \in \mathcal{C}([0, \ell_T]; \mathbb{H}^2)$ with bounded total variation and $x^\ell \in D([0, T]; \mathbb{H}^1)$ solving* (2.1) *such that*

$$\| \, x_T^\ell - a\|_0 \ \leq \ \varepsilon, \qquad T \notin \mathcal{J}(\ell).$$

Moreover,

$$\|z_t^\ell\|_2 \ \leq \ C_T(1 + \|\mathrm{e}^{-At_\varepsilon}a\|_6^2 + \| \, x_{t_\varepsilon}\|_6^2), \quad 0 \leq t \leq T,$$

where t_ε is defined by (2.4) *and x_{t_ε} is determined by* (2.1) *with $u_{\ell_t} = 0$ for $t \in [0, t_\varepsilon]$.*

To prove this result, we regularize $\ell \in \mathcal{S}$ by

$$\ell_t^\delta = \frac{1}{\delta}\int_0^\delta \ell_{t+r}\mathrm{d}r, \qquad t \geq 0, \delta > 0,$$

and prove the assertion for ℓ_t^δ replacing ℓ. It is clear that ℓ_t^δ is strictly increasing and continuous. Let γ_t^δ be the inverse of ℓ_t^δ.

l:Gam2Del
Lemma 2.2. *For all $\delta > 0$, we have*

$$\gamma_t^\delta \leq \gamma_t \leq \gamma_t^\delta + \delta, \quad \forall \ t \geq 0.$$

Proof. Denote $t_0 = \gamma_t$ and $t_1 = \gamma_t^\delta$, it is easy to see $\ell_{t_1}^\delta = t$ and $\ell_{t_0} \geq t$. Observe $\ell_{t_0}^\delta = \frac{1}{\delta}\int_0^\delta \ell_{t_0+r}\mathrm{d}r > t$ since $\ell_{t_0+r} > t$ for $r > 0$. If $t_0 < t_1$, then $t < \ell_{t_0}^\delta < \ell_{t_1}^\delta = t$. Contradiction. If $t_0 > t_1 + \delta$, we have $\ell_{t_1+\delta} < t$, otherwise $t_0 \leq t_1 + \delta$. Consequently, $\ell_{t_1}^\delta = \frac{1}{\delta}\int_0^\delta \ell_{t_1+r}\mathrm{d}r < t$ since $\ell_{t_1+r} < t$ for all $r \in [0, \delta]$, but $\ell_{t_1}^\delta = t$, contradiction. Hence, $t_0 \in [t_1, t_1 + \delta]$. □

l:Con-1
Lemma 2.3. *For any $T > 0, \varepsilon > 0, \delta > 0, a \in \mathbb{H}$, let $t_\varepsilon = t_\varepsilon(a, T)$ be defined by* (2.4) *and take*

e:xtl-2
$$u_t := 1_{[\ell_{t_\varepsilon}^\delta, \ell_T^\delta]}(t)Q^{-1}F(\gamma_t^\delta), \ \ t \in [0, \ell_T^\delta], \tag{2.5}$$

where γ^δ_t is the inverse function of ℓ^δ_t and

e:Ft

$$F(t) := x^{\ell^\delta}_t - x^{\ell^\delta}_{t_\varepsilon} + \int_{t_\varepsilon}^t A x^{\ell^\delta}_s \mathrm{d}s + \int_{t_\varepsilon}^t B(x^{\ell^\delta}_s)\mathrm{d}s, \quad t \in [t_\varepsilon, T]\,. \tag{2.6}$$

Then $u \in \mathcal{C}(\left[0, \ell^\delta_T\right]; \mathbb{H}^2)$ and $F \in \mathcal{C}([t_\varepsilon, T]; \mathbb{H}^4)$ with

e:FReg-1

$$\|F(t)\|_4 \le C_T(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x^{\ell^\delta}_{t_\varepsilon}\|_6^2) < \infty, \quad t \in [t_\varepsilon, T]\,, \tag{2.7}$$

e:FReg-2

$$\|F(t_1) - F(t_2)\|_4 \le C_T(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x^{\ell^\delta}_{t_\varepsilon}\|_6^2)|t_1 - t_2|, \quad t_1, t_2 \in [t_\varepsilon, T]\,. \tag{2.8}$$

Moreover, let $x^{\ell^\delta} \in \mathcal{C}\left([0, T]; \mathbb{H}^1\right)$ solve the system (2.1) *with ℓ^δ replacing ℓ. Then*

$$\|\, x^{\ell^\delta}_T - a\|_0 < \varepsilon/2.$$

Proof. We first observe that $x^{\ell^\delta}_t$ has the representation

e:xtl

$$x^{\ell^\delta}_t = \mathrm{e}^{-At} x_0 + \int_0^t \mathrm{e}^{-A(t-s)} B(x^{\ell^\delta}_s)\mathrm{d}s, \quad 0 \le t \le t_\varepsilon, \tag{2.9}$$

e:xtl-1

$$x^{\ell^\delta}_t = \frac{t - t_\varepsilon}{T - t_\varepsilon}\mathrm{e}^{-At_\varepsilon} a + \frac{T - t}{T - t_\varepsilon} x^{\ell^\delta}_{t_\varepsilon}, \quad t_\varepsilon \le t \le T. \tag{2.10}$$

Indeed, by (2.5), $u_t = 0$ for all $t \in \left[0, \ell^\delta_{t_\varepsilon}\right]$, the system (2.1) is a deterministic Burgers equation, which admits a unique solution $x^{\ell^\delta} \in \mathcal{C}\left([0, t_\varepsilon]; \mathbb{H}^1\right)$ given by (2.9). On the other hand, for $t \in [t_\varepsilon, T]$, substituting $x^{\ell^\delta}_t$ with the form (2.10) into the left hand of the system (2.1), we obtain

$$Q u_{\ell^\delta_t} = F(t), \quad t \in [t_\varepsilon, T]\,,$$

where $F(t)$ is defined by (2.6). Taking

$$u_t = Q^{-1} F(\gamma^\delta_t), \quad t \in \left[\ell^\delta_{t_\varepsilon}, \ell^\delta_T\right],$$

we immediately obtain that (x, u) solves the system (2.1) for $t \in [t_\varepsilon, T]$.

Next, since $x^{\ell^\delta}_T = \mathrm{e}^{-At_\varepsilon} a$ and $\|\mathrm{e}^{-At_\varepsilon} a - a\|_0 \le \varepsilon/2$, we have $\|\, x^{\ell^\delta}_T - a\|_0 \le \varepsilon/2$. It remains to verify the claimed properties of u and F. By the regularity of Burgers equation (see the detailed proof below) and $\mathrm{e}^{-At_\varepsilon}$ respectively, $x^{\ell^\delta}_{t_\varepsilon} \in \mathbb{H}^6$ and $e^{-At_\varepsilon} a \in \mathbb{H}^6$. For all $t \in [t_\varepsilon, T]$, we have

$$\|x^{\ell^\delta}_t\|_4 \le \|\mathrm{e}^{-At_\varepsilon} a\|_6 + \|x^{\ell^\delta}_{t_\varepsilon}\|_6^2,$$

$$\|B(x^{\ell^\delta}_t)\|_4 \;\le\; C\|x^{\ell^\delta}_t\|_6^2 \;\le\; C\left(\|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x^{\ell^\delta}_{t_\varepsilon}\|_6^2\right),$$

$$\|A x^{\ell^\delta}_t\|_4 \;\le\; C\left(\|\mathrm{e}^{-At_\varepsilon} a\|_6 + \|x^{\ell^\delta}_{t_\varepsilon}\|_6\right) \;\le\; C\left(1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}\|_6^2\right),$$

where the second inequality is by [21, Lemma 2.1]. Combining the above inequalities, we immediately get (2.7) and (2.8), as desired. Therefore, $F \in \mathcal{C}([t_\varepsilon, T]; \mathbb{H}^4)$, which, together with the assumption of Q and (2.5), yields $u \in \mathcal{C}(\left[0, \ell^\delta_T\right]; \mathbb{H}^2)$.

Finally, it is easy to see that $\|x^{\ell^\delta}_{t_\varepsilon}\|_6 < \infty$. Below we present a proof for completeness. Noting that $x^{\ell^\delta}_t \in \mathbb{H}^1$ for all $t \in [0, t_\varepsilon]$, letting $t_1 = t_\varepsilon/3, t_2 = 2t_\varepsilon/3$, $t_3 = t_\varepsilon$ and taking $\delta \in (0, \frac{1}{4})$, we have

$$
\begin{aligned}
\|x^{\ell^\delta}_t\|_2 \;&\le\; \|\mathrm{e}^{-At}x_0\|_2 + \int_0^t \|A^{1-\delta}\mathrm{e}^{-A(t-s)}\| \|B(x^{\ell^\delta}_s)\|_{2\delta}\mathrm{d}s \\
&\le\; Ct^{-\frac{1}{2}}\|x_0\|_1 + C\int_0^t (t-s)^{-1+\delta}\|x^{\ell^\delta}_s\|_1^2\mathrm{d}s \qquad (2.11)\\
&\le\; C\left(t^{-\frac{1}{2}}\|x_0\|_1 + t^\delta \sup_{0\le t\le t_3}\|x^{\ell^\delta}_s\|_1^2\right),\ \ t \in (0, t_3],
\end{aligned}
$$

where the last inequality is by (1.1) and (1.4). Now taking $x^{\ell^\delta}_{t_1}$ as the initial data, we obtain

$$
\begin{aligned}
\|x^{\ell^\delta}_t\|_4 \;&\le\; \|\mathrm{e}^{-A(t-t_1)}x^{\ell^\delta}_{t_1}\|_4 + \int_{t_1}^t \|A^{1-\delta}\mathrm{e}^{-A(t-t_1-s)}\| \|B(x^{\ell^\delta}_s)\|_{2+2\delta}\mathrm{d}s \\
&\le\; C(t-t_1)^{-1}\|x^{\ell^\delta}_{t_1}\|_2 + C\int_{t_1}^t (t-s)^{-1+\delta}\|x^{\ell^\delta}_s\|_2^2\mathrm{d}s \qquad (2.12)\\
&\le\; C\left((t-t_1)^{-1}\|x^{\ell^\delta}_{t_1}\|_2 + (t-t_1)^\delta \sup_{t_1\le t\le t_3}\|x^{\ell^\delta}_s\|_2^2\right),\ \ t \in (t_1, t_3].
\end{aligned}
$$

Similarly, taking $x^{\ell^\delta}_{t_2}$ as the initial data we get

$$
\|x^{\ell^\delta}_t\|_6 \;\le\; C\left((t-t_2)^{-1}\|x^{\ell^\delta}_{t_1}\|_4 + (t-t_2)^\delta \sup_{t_2\le t\le t_3}\|x^{\ell^\delta}_s\|_4^2\right),\ \ t \in (t_2, t_3]. \qquad (2.13)
$$

This completes the proof. □

Lemma 2.4. *For all* $t > 0$*, let*

$$
z^\ell_t = \int_0^t \mathrm{e}^{-A(t-s)}Q\mathrm{d}u_{\ell_s}, \quad z^{\ell^\delta}_t = \int_0^t \mathrm{e}^{-A(t-s)}Q\mathrm{d}u_{\ell^\delta_s}.
$$

Then

e:DelZSmall
$$
\|z^{\ell^\delta}_t - z^\ell_t\|_2 \;\le\; C_T(1 + \|\mathrm{e}^{-At_\varepsilon}a\|_6^2 + \|x_{t_\varepsilon}\|_6^2)\delta, \quad t \in [0, T] \setminus \mathcal{J}(\ell). \qquad (2.14)
$$

Proof. By (2.5), we have $u_t = 0$ for all $0 \le t \le \ell^\delta_{t_\varepsilon}$. Since $\ell_t \le \ell^\delta_t$,

e:Delzt
$$
z^\ell_t = z^{\ell^\delta}_t = 0, \quad t \in [0, t_\varepsilon]. \qquad (2.15)
$$

Using integration by parts, we get

$$
z^\ell_t \;=\; Qu_{\ell_t} - \int_0^t A\mathrm{e}^{-A(t-s)}Qu_{\ell_s}\mathrm{d}s. \qquad (2.16)
$$

It is easy to see by (2.5) and (2.7) that for all $0 \le t \le T$,

$$
\|Qu_{\ell_t}\|_2 \;=\; \|F(\gamma^\delta_{\ell_t})\|_2 \;\le\; \sup_{0\le t\le T}\|F(\gamma^\delta_{\ell_t})\|_2 \;\le\; C_T(1 + \|\mathrm{e}^{-At_\mathrm{e}}a\|_6^2 + \|x^{\ell^\delta}_{t_\mathrm{e}}\|_6^2),
$$

and that for all $0 \le t \le T$ and $0 \le s \le t$,

$$
\begin{aligned}
\|A\mathrm{e}^{-A(t-s)}Qu_{\ell_s}\|_2 \;&=\; \|\mathrm{e}^{-A(t-s)}Qu_{\ell_s}\|_4 \;\le\; \|Qu_{\ell_s}\|_4 \;=\; \|F(\gamma^\delta_{\ell_s})\|_4 \qquad (2.17)\\
&\le\; C_T(1 + \|\mathrm{e}^{-At_\mathrm{e}}a\|_6^2 + \|x^{\ell^\delta}_{t_\varepsilon}\|_6^2).
\end{aligned}
$$

Hence,

$$\|z_t^\ell\|_2 \ \le\ C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2),\quad 0\le t\le T.$$

Similarly,

$$\|\ z_t^{\ell^\delta}\|_2 \le C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2),\quad 0\le t\le T.$$

Using integration by parts again, we further get

$$z_t^{\ell^\delta}-z_t^\ell = Q(u_{\ell_t^\delta}-u_{\ell_t})-\int_0^t A\mathrm{e}^{-A(t-s)}Q(u_{\ell_s^\delta}-u_{\ell_s})\mathrm{d}s$$

which, together with (2.5) and (2.8), yields

$$\begin{aligned}\|z_t^{\ell^\delta}-z_t^\ell\|_2 \ &\le\ \|F(\gamma^\delta_{\ell_t^\delta})-F(\gamma^\delta_{\ell_t})\|_2+\int_0^t\|Q(u_{\ell_s^\delta}-u_{\ell_s})\|_4\mathrm{d}s\\ &\le\ \|F(\gamma^\delta_{\ell_t^\delta})-F(\gamma^\delta_{\ell_t})\|_2+\int_0^t\|F(\gamma^\delta_{\ell_s^\delta})-F(\gamma^\delta_{\ell_s})\|_4\mathrm{d}s\\ &\le\ C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\left[|\gamma^\delta_{\ell_t^\delta}-\gamma^\delta_{\ell_t}|+\int_0^t|\gamma^\delta_{\ell_s^\delta}-\gamma^\delta_{\ell_s}|\mathrm{d}s\right]\\ &=\ C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\left[|t-\gamma^\delta_{\ell_t}|+\int_0^t|s-\gamma^\delta_{\ell_s}|\mathrm{d}s\right],\end{aligned}$$

where the last equality is by $\gamma^\delta_{\ell_t^\delta}=t$ for all $t\ge 0$. By the definition of $\gamma_\cdot$, if $t\notin\mathcal{J}(\ell)$, i.e. t is a continuous point of ℓ, we have $\gamma_{\ell_t}=t$. Therefore, by Lemma 2.2, we have

$$|t-\gamma^\delta_{\ell_t}|\ \le\ |t-\gamma_{\ell_t}|+\left|\gamma^\delta_{\ell_t}-\gamma_{\ell_t}\right|\ \le\ |t-\gamma_{\ell_t}|+\delta\ \le\ \delta,\qquad t\in[0,T]\setminus\mathcal{J}(\ell).$$

Since $\ell_\cdot$ has at most countably infinite jump points, Lebesgue measure of $\mathcal{J}(\ell)$ is zero. Thus,

$$\int_0^t|s-\gamma^\delta_{\ell_s}|\mathrm{d}s\ \le\ T\delta,\qquad t\in[0,T]$$

and

$$\|z_t^{\ell^\delta}-z_t^\ell\|_2\ \le\ C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^2+\|x_{t_\varepsilon}\|_6^2)\delta,\qquad t\in[0,T]\setminus\mathcal{J}(\ell).$$

□

We are now at the position to prove Proposition 2.1.

Proof of Proposition 2.1. Let $\delta>0$ be small enough to be chosen. By Lemma 2.3, the equation

:xldelta-0 (2.18)
$$\mathrm{d}x_t^{\ell^\delta}+\left[Ax_t^{\ell^\delta}+B(x_t^{\ell^\delta})\right]\mathrm{d}t=Q\mathrm{d}u_{\ell_t^\delta},\quad x_0^{\ell^\delta}=x_0$$

is solved by $u\in\mathcal{C}(\left[0,\ell_T^\delta\right];\mathbb{H}^2)$ and $x^{\ell^\delta}\in\mathcal{C}([0,T];\mathbb{H}^1)$, which have the forms (2.6)-(2.9) and

$$\|\ x_T^{\ell^\delta}-a\|_0\le\varepsilon/2.$$

We will compare Eq. (2.18) with the following equation:

e:xldelta (2.19)
$$\mathrm{d}x_t^\ell+\left[Ax_t^\ell+B(x_t^\ell)\right]\mathrm{d}t=Q\mathrm{d}u_{\ell_t},\quad x_0=x_0.$$

Denote $y_t^\ell = x_t^\ell - z_t^\ell$ and $y_t^{\ell^\delta} = x_t^{\ell^\delta} - z_t^{\ell^\delta}$. Then

$$\frac{\mathrm{d}y_t^{\ell^\delta}}{\mathrm{d}t} + Ay_t^{\ell^\delta} + B(x_t^{\ell^\delta}) = 0, \quad y_0^{\ell^\delta} = x_0,$$
$$\frac{\mathrm{d}y_t^{\ell}}{\mathrm{d}t} + Ay_t^{\ell} + B(x_t^{\ell}) = 0, \quad y_0^{\ell} = x_0.$$

By (2.15), we have

$$y_t^{\ell^\delta} - y_t^\ell = 0, \quad t \in [0, t_\varepsilon].$$

Write $\Delta y_t^\ell = y_t^\ell - y_t^{\ell^\delta}$, $\Delta x_t^\ell = x_t^\ell - x_t^{\ell^\delta}$ and $\Delta z_t^\ell = z_t^\ell - z_t^{\ell^\delta}$ for $t \in [t_\varepsilon, T]$. Then

e:DelY (2.20)
$$\|\Delta y_t^\ell\|_0^2 + 2\int_{t_\varepsilon}^t \|\Delta y_t^\ell\|_1^2 \mathrm{d}s \le 2\left|\int_{t_\varepsilon}^t \langle \Delta y_t^\ell, B(x_s^{\ell^\delta}) - B(x_s^\ell)\rangle_0 \mathrm{d}s\right|.$$

Noting that

$$\begin{aligned} B(x_s^\ell) - B(x_s^{\ell^\delta}) =\ & B(x_s^\ell, \Delta x_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell^\delta}) \\ =\ & B(\Delta x_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell^\delta}) + B(x_s^{\ell^\delta}, \Delta x_s^\ell) \\ =\ & B(\Delta y_s^\ell) + B(\Delta z_s^\ell) + B(\Delta y_s^\ell, \Delta z_s^\ell) + B(\Delta z_s^\ell, \Delta y_s^\ell) + B(\Delta x_s^\ell, x_s^{\ell^\delta}) + B(x_s^{\ell^\delta}, \Delta x_s^\ell), \end{aligned}$$

and that $\langle x, B(x,x)\rangle_0 = 0$ for $x \in \mathbb{H}^1$, we obtain

$$\begin{aligned} |\langle \Delta y_s^\ell, B(x_s^\ell) - B(x_s^{\ell^\delta})\rangle_0| \ \le\ & \|\Delta y_s^\ell\|_0 \Big[\|B(\Delta z_s^\ell)\|_0 + \|B(\Delta y_s^\ell, \Delta z_s^\ell)\|_0 + \|B(\Delta z_s^\ell, \Delta y_s^\ell)\|_0 \\ & + \|B(\Delta x_s^\ell, x_s^{\ell^\delta})\|_0 + \|B(x_s^{\ell^\delta}, \Delta x_s^\ell)\|_0 \Big]. \end{aligned}$$

Combining this with (1.4) and the inequality $2ab \le a^2 + b^2$ for $a \ge 0$ and $b \ge 0$, we arrive at

$$\begin{aligned} |\langle \Delta y_s^\ell, B(x_s^\ell) - B(x_s^{\ell^\delta})\rangle_0| \ \le\ & C\|\Delta y_s^\ell\|_0 \Big[\|\Delta z_s^\ell\|_1^2 + \|\Delta y_s^\ell\|_1 \|\Delta z_s^\ell\|_1 + \|\Delta x_s^\ell\|_1 \|x_s^{\ell^\delta}\|_1 \Big] \\ \le\ & C\|\Delta y_s^\ell\|_0 \Big[\|\Delta z_s^\ell\|_1^2 + \|\Delta y_s^\ell\|_1 \|\Delta z_s^\ell\|_1 + \|\Delta y_s^\ell\|_1 \|x_s^{\ell^\delta}\|_1 + \|\Delta z_s^\ell\|_1 \|x_s^{\ell^\delta}\|_1 \Big] \\ \le\ & \|\Delta y_s^\ell\|_1^2 + C\|\Delta y_s^\ell\|_0^2 \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2 \right) + C\|\Delta z_s^\ell\|_1^2. \end{aligned}$$

This, together with (2.20) and (2.14), implies

$$\begin{aligned} \|\Delta y_t^\ell\|_0^2 \ \le\ & C\int_{t_\varepsilon}^t \|\Delta y_s^\ell\|_0^2 \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2 \right) \mathrm{d}s + C\int_{t_\varepsilon}^t \|\Delta z_s^\ell\|_1^2 \mathrm{d}s \\ \le\ & C\int_{t_\mathrm{e}}^t \|\Delta y_s^\ell\|_0^2 \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2 \right) \mathrm{d}s + C_T(1 + \|\,\mathrm{e}^{-At_\varepsilon} a\|_6^4 + \|x_{t_\varepsilon}\|_6^4)\delta^2, \ \ t \in [t_\varepsilon, T]. \end{aligned}$$

By Gronwall's inequality, we obtain

$$\|\Delta y_T^\ell\|_0^2 \ \le\ C_T \exp\left[C\int_{t_\varepsilon}^T \left(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2 \right) \mathrm{d}s \right] (1 + \|\mathrm{e}^{-At_\varepsilon} a\|_6^2 + \|x_{t_\varepsilon}\|_6^2)\delta^2.$$

On the other hand, (2.10) implies

$$\|x_t^{\ell^\delta}\|_1 \ \le\ \|\mathrm{e}^{-At_\varepsilon} a\|_1 + \|x_{t_\varepsilon}^{\ell^\delta}\|_1 \ \le\ C\left(\|\mathrm{e}^{-At_\varepsilon} a\|_6 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6 \right), \quad t \in [t_\varepsilon, T],$$

which, together with (2.14), leads to

$$\int_{t_\varepsilon}^{T} \Big(\|\Delta z_s^\ell\|_1^2 + \|x_s^{\ell^\delta}\|_1^2\Big)\,\mathrm{d}s \ \le\ C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^4 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^4).$$

Hence,

$$\|\Delta y_T^\ell\|_0^2 \ \le\ C_T \exp\Big[C_T(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^4 + \|x_{t_\varepsilon}^{\ell^\delta}\|_6^4)\Big]\,(1+\|\mathrm{e}^{-At_\varepsilon}a\|_6^4 + \|x_{t_\varepsilon}\|_6^4)\delta^2.$$

Combining this with (2.14), as long as $\delta > 0$ is chosen to be sufficiently small we obtain

$$\|\Delta x_T^\ell\|_0^2 \ \le\ 2\|\Delta y_T^\ell\|_0^2 + 2\|\Delta z_T^\ell\|_0^2 \ \le\ \frac{\varepsilon^2}{4}, \qquad T \notin \mathcal{J}(\ell).$$

Therefore, it follows from Lemma 2.3 that

$$\|x_T^\ell - a\|_0 \ \le\ \|\Delta x_T^\ell\|_0 + \|x_T^{\ell^\delta} - a\|_0 \ \le\ \varepsilon, \qquad T \notin \mathcal{J}(\ell).$$

The proof is then complete. □

3. ESTIMATE OF CONVOLUTIONS

For $\ell \in \mathcal{S}$, $T > 0$ and $u \in \mathcal{C}([0,\ell_T])$, let z_t^ℓ be given in (2.2), and define

ZT1 (3.1)
$$Z_t^\ell := \int_0^t \mathrm{e}^{-(t-s)A} Q\mathrm{d}W_{\ell_s} \quad t \ge 0.$$

Lemma 3.1. *For any $T > 0$, $\gamma \in \big[1, \theta' - \frac{1}{2}\big)$ and $p \ge 1$, there exists a constant $C > 0$ such that*

e:ZtpG (3.2)
$$\mathbb{E}\Big[\sup_{0\le t\le T} \|\, Z_t^\ell\|_\gamma^p\Big] \le C\ell_T^{p/2},\ \ \ell \in \mathcal{S}.$$

Proof. Using integration by parts, we have

$$Z_t^\ell = \int_0^t \mathrm{e}^{-A(t-s)} Q\mathrm{d}W_{\ell_s} = QW_{\ell_t} + \int_0^t A\mathrm{e}^{-A(t-s)} QW_{\ell_s}\mathrm{d}s.$$

By (1.3) and the martingale inequality, we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\|QW_{\ell_t}\|_\gamma^p &\le \mathbb{E}\sup_{0\le t\le \ell_T}\|QW_t\|_\gamma^p \\ &\le C_{\gamma,\theta'}\mathbb{E}\sup_{0\le t\le \ell_T}\|W_t\|_{\gamma-\theta'}^p \\ &\le C_{\gamma,\theta',p}\mathbb{E}\|W_{\ell_T}\|_{\gamma-\theta'}^p \le C_{\gamma,\theta',p}\ell_T^{p/2}.\end{aligned}$$

For $\gamma' \in (\gamma, \theta' - \frac{1}{2})$, (2.1) implies

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t A\mathrm{e}^{-A(t-s)}QW_{\ell_s}\mathrm{d}s\right\|_\gamma^p &\le \mathbb{E}\sup_{0\le t\le T}\left(\int_0^t \|A\mathrm{e}^{-A(t-s)}QW_{\ell_s}\|_\gamma \mathrm{d}s\right)^p \\ &= \mathbb{E}\sup_{0\le t\le T}\left(\int_0^t \|A^{1+\gamma-\gamma'}\mathrm{e}^{-A(t-s)}QA^{\gamma'-\gamma}W_{\ell_s}\|_\gamma \mathrm{d}s\right)^p \\ &\le C_{\gamma,\gamma'}\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t (t-s)^{-1-\gamma+\gamma'}\|QA^{\gamma'-\gamma}W_{\ell_s}\|_\gamma \mathrm{d}s\right)^p \\ &\le C_{\gamma,\gamma',\theta'}\mathbb{E}\sup_{0\le t\le T}\left(\int_0^t (t-s)^{-1-\gamma+\gamma'}\|W_{\ell_s}\|_{\gamma'-\theta'} \mathrm{d}s\right)^p.\end{aligned}$$

Since

$$\begin{aligned}\int_0^t (t-s)^{-1-\gamma+\gamma'}\|W_{\ell_s}\|_{\gamma'-\theta'}\mathrm{d}s &\le \sup_{0\le t\le T}\|W_{\ell_s}\|_{\gamma'-\theta'}\int_0^t (t-s)^{-1+\gamma+\gamma'}\mathrm{d}s \\ &\le C_{\gamma,\gamma',T}\sup_{0\le t\le T}\|W_{\ell_s}\|_{\gamma'-\theta'},\end{aligned}$$

by the same argument as the above we get

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t A\mathrm{e}^{-A(t-s)}QW_{\ell_s}\mathrm{d}s\right\|_\gamma^p \le C_{\gamma,\gamma',\theta',p,T}\ell_T^{p/2}.$$

Collecting the above inequalities, we obtain the desired estimate. □

Lemma 3.2. *For any $\ell \in \mathcal{S}$, $T > 0$ and $\varepsilon > 0$,*

$$\mathbb{P}\left(\sup_{0\le t\le T}\|Z_t^\ell - z_t^\ell\|_1 \le \varepsilon\right) > 0.$$

Proof. For any $N \in \mathbb{N}$, let $\mathcal{H}_N = \mathrm{span}\{e_i : i \le N\}$ and let $\mathcal{H}^N$ be its orthogonal complementary. Let $\Pi_N : \mathbb{H} \to \mathcal{H}_N$ and $\Pi^N : \mathbb{H} \to \mathcal{H}^N$ to be the corresponding orthogonal projections. We have

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T}\|Z_t^\ell - z_t^\ell\|_1 \le \varepsilon\right) \\ &\ge \mathbb{P}\left(\sup_{0\le t\le T}\|\Pi_N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}, \sup_{0\le t\le T}\|\Pi^N(Z_t^\ell - z_t^\ell\|_1 \le \frac{\varepsilon}{2}\right) \\ &= \mathbb{P}\left(\sup_{0\le t\le T}\|\Pi_N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}\right)\mathbb{P}\left(\sup_{0\le t\le T}\|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}\right),\end{aligned}$$

where the last inequality follows from the independence of $\Pi_N Z_t^\ell$ and $\Pi^N Z_t^\ell$. Below, we estimate these two probabilities respectively.

For the first one, using integration by parts, we get

$$Z_t^\ell - z_t^\ell = Q(W_{\ell_t} - u_{\ell_t}) + \int_0^t A\mathrm{e}^{-A(t-s)}Q(W_{\ell_s} - u_{\ell_s})\mathrm{d}s.$$

Obviously, there exist a constant $C_N > 0$ such that

$$\|\,\Pi_N\left[Q(W_{\ell_t} - u_{\ell_t})\right]\|_1 \le C_N\|\Pi_N\left[W_{\ell_t} - u_{\ell_t}\right]\|_0,$$

and

$$\left\|\Pi_N \int_0^t A\mathrm{e}^{-A(t-s)} Q(W_{\ell_s} - u_{\ell_s})\mathrm{d}s\right\|_1 \le \int_0^t \left\|\Pi_N \int_0^t A\mathrm{e}^{-A(t-s)} Q(W_{\ell_s} - u_{\ell_s})\right\|_1 \mathrm{d}s$$
$$\le C_N \int_0^t \|\Pi_N [W_{\ell_s} - u_{\ell_s}]\|_0 \mathrm{d}s$$
$$\le TC_N \sup_{0\le t\le \ell_T} \|\Pi_N [W_t - u_t]\|_0.$$

Hence,

$$\sup_{0\le t\le T} \|\Pi^N(Z_t^\ell - z_t^\ell\|_1 \le TC_N \sup_{0\le t\le T} \|\Pi_N [W_{\ell_t} - u_{\ell_t}]\|_0$$
$$\le TC_N \sup_{0\le t\le \ell_T} \|\Pi_N [W_t - u_t]\|_0.$$

It is clear $(\Pi_N W_t)_{t\ge 0}$ and $(\Pi_N u_t)_{t\ge 0}$ can be identified with an N dimensional standard Wiener process and a continuous function in $\mathcal{C}([0,\infty);\mathbb{R}^N)$. Since the support of a Brownian motion is the whole continuous function space, we have

$$\mathbb{P}\left(\sup_{0\le t\le \ell_T} \|\Pi_N (W_t - u_t)\|_0 \le \delta\right) > 0, \ \ \delta > 0.$$

Therefore,

e:PLow
$$\mathbb{P}\left(\sup_{0\le t\le T} \|\Pi_N(Z_t^\ell - z_t^\ell)\|_1 \le \frac{\varepsilon}{2}\right) > 0. \tag{3.3}$$

On the other hand, by (3.2) with $\gamma \in (1, \theta' - \frac{1}{2})$, Chebyshev's inequality and the spectral inequality $\|\Pi^N x\|_1 \le \lambda_N^{\gamma-1}\|x\|_\gamma$ for $x \in \mathbb{H}^\gamma$, we have

$$\mathbb{P}\left(\sup_{0\le t\le T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 \ge \frac{\varepsilon}{2}\right) \le \mathbb{P}\left(\sup_{0\le t\le T} \|\,(Z_t^\ell - z_t^\ell)\|_\gamma \ge \frac{\varepsilon}{2}\lambda_N^{\gamma-1}\right)$$
$$\le \frac{2\mathbb{E}\left[\sup_{0\le t\le T}\|\,Z_t^\ell\|_\gamma\right] + 2\sup_{0\le t\le T}\|z_t^\ell\|_\gamma}{\varepsilon\lambda_N^{\gamma-1}}.$$

From the previous inequality and (3.2), choose a sufficiently large N, we get

$$\mathbb{P}\left(\sup_{0\le t\le T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 \ge \frac{\varepsilon}{2}\right) < 1,$$

equivalently,

e:PHigh
$$\mathbb{P}\left(\sup_{0\le t\le T} \|\Pi^N(Z_t^\ell - z_t^\ell)\|_1 < \frac{\varepsilon}{2}\right) > 0. \tag{3.4}$$

Combining (3.3) and (3.4), we finish the proof. □

4. Proof of Theorem 1.2

For $\ell \in \mathcal{S}$, let Z_t^ℓ be in (3.1), and let X_t^ℓ solve

XT1
$$\mathrm{d}X_t^\ell = [-AX_t^\ell - B(X_t^\ell)]\mathrm{d}t + Q\mathrm{d}W_{\ell_t}, \quad X_0^\ell = x_0 \in \mathbb{H}. \tag{4.1}$$

Then $Y_t^\ell := X_t^\ell - Z_t^\ell$ satisfies

e:YtlEqn
$$\frac{\mathrm{d}Y_t^\ell}{\mathrm{d}t} + AY_t^\ell + B(Y_t^\ell + Z_t^\ell) = 0, \quad Y_0^\ell = x_0. \tag{4.2}$$

Proof of Theorem 1.2. Since $S_\cdot \in \mathcal{S}$ a.s., it suffices to show that for each $\ell \in \mathcal{S}$,

4.3
$$\mathbb{P}(\|X_T^\ell - a\|_0 \le \varepsilon) > 0. \tag{4.3}$$

Since $X_t^\ell \in \mathbb{H}^1$ for $t > 0$, by the Markov property, we may and do assume that $x_0 \in \mathbb{H}^1$. Below, we prove (4.3) for $x_0 \in \mathbb{H}^1$.

By Proposition 2.1, there exist $u \in \mathcal{C}([0,T]\,;\mathbb{H}^4)$ with bounded total variation and $x^\ell \in \mathcal{D}([0,T]\,;\mathbb{H}^1)$ solving

$$\mathrm{d}x_t^\ell + \left[Ax_t^\ell + B(x_t^\ell)\right]\mathrm{d}t = Q\mathrm{d}u_{\ell_t}, \quad x_0^\ell = x_0,$$

such that

$$\|x_T^\ell - a\|_0 \le \varepsilon/2, \quad T \notin \mathcal{J}(\ell).$$

So, when $T \notin \mathcal{J}(\ell)$ we have

e:Irr-1
$$\begin{aligned}
\mathbb{P}(\|X_T^\ell - a\|_0 \le \varepsilon) \ge\ & \mathbb{P}\bigg(\| X_T^\ell - x_T^\ell\|_0 \le \frac{\varepsilon}{2}, \|x_T^\ell - a\|_0 \le \frac{\varepsilon}{2}\bigg) \\
= \mathbb{P}\bigg(\| X_T^\ell - x_T^\ell\|_0 \le \frac{\varepsilon}{2}\bigg) &\ge \mathbb{P}\bigg(\| Y_T^\ell - y_T^\ell\|_0 \le \frac{\varepsilon}{4}, \| Z_T^\ell - z_T^\ell\|_0 \le \frac{\varepsilon}{4}\bigg) \\
\ge \mathbb{P}\bigg(\| Y_T^\ell - y_T^\ell\|_0 \le \frac{\varepsilon}{4}, &\sup_{0\le t\le T} \| Z_t^\ell - z_t^\ell\|_0 \le \varepsilon'\bigg), \quad \varepsilon' \in (0, \varepsilon/4),
\end{aligned} \tag{4.4}$$

where $z_t^\ell = \int_0^t e^{-A(t-s)}Q\mathrm{d}u_{\ell_s}$ and y_t^ℓ is in (2.2).

Write $\Delta Y_t^\ell = Y_t^\ell - y_t^\ell, \Delta X_t^\ell = X_t^\ell - x_t^\ell$ and $\Delta Z_t^\ell = Z_t^\ell - z_t^\ell$. Then (2.3) and (4.2) yield

$$\frac{\mathrm{d}\Delta Y_t^\ell}{\mathrm{d}t} + A\Delta Y_t^\ell + B(X_t^\ell) - B(x_t^\ell) = 0, \quad \Delta Y_0^\ell = 0,$$

which clearly implies

$$\|\Delta Y_t^\ell\|_0^2 + 2\int_0^t \|\Delta Y_t^\ell\|_1^2 \mathrm{d}s \le 2\int_0^t |\langle \Delta Y_s^\ell, B(X_s^\ell) - B(x_s^\ell)\rangle_0| \mathrm{d}s.$$

Since $\langle x, B(x,x)\rangle_0 = 0$ for $x \in \mathbb{H}^1$, we have

$$\begin{aligned}
&|\langle \Delta Y_s^\ell, B(X_s^\ell) - B(x_s^\ell)\rangle_0| \\
&= \langle \Delta Y_s^\ell, B(\Delta X_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(\Delta X_s^\ell, x_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(x_s^\ell, \Delta X_s^\ell)\rangle_0 \\
&= \langle \Delta Y_s^\ell, B(\Delta Y_s^\ell, \Delta Z_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(\Delta Z_s^\ell, \Delta Y_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(\Delta Z_s^\ell, \Delta Z_s^\ell)\rangle_0 \\
&\quad + \langle \Delta Y_s^\ell, B(\Delta X_s^\ell, x_s^\ell)\rangle_0 + \langle \Delta Y_s^\ell, B(x_s^\ell, \Delta X_s^\ell)\rangle_0,
\end{aligned}$$

which, together with (1.4) and the inequality $2ab \le a^2+b^2$ for $a,b\ge 0$, implies

$$\begin{aligned}
&|\langle Y_s^\ell, B(X_s^\ell)-B(x_s^\ell)\rangle_0|\\
&\le C(\|\Delta Y_s^\ell\|_0\|\Delta Y_s^\ell\|_1\|\Delta Z_s^\ell\|_1+\|\Delta Y_s^\ell\|_0\|\Delta Z_s^\ell\|_1^2+\|x_s^\ell\|_1\|\Delta Y_s^\ell\|_0\|\Delta X_s^\ell\|_1)\\
&\le C(\|\Delta Z_s^\ell\|_1^2+\|x_s^\ell\|_1^2)\|\Delta Y_s^\ell\|_0^2+C\|\Delta Z_s^\ell\|_1^2+\left(\frac{1}{2}\|\Delta Y_s^\ell\|_1^2+\frac{1}{4}\|\Delta X_s^\ell\|_1^2\right)\\
&\le C(\|\Delta Z_s^\ell\|_1^2+\|x_s^\ell\|_1^2)\|\Delta Y_s^\ell\|_0^2+\|\Delta Y_s^\ell\|_1^2+C\|\Delta Z_s^\ell\|_1^2
\end{aligned}$$

for some constant $C>0$. Hence,

$$\begin{aligned}
&\|\Delta Y_t^\ell\|^2 \le C\int_0^t(\|\Delta Z_s^\ell\|_1^2+\|x_s^\ell\|_1^2)\|\Delta Y_s^\ell\|_0^2\mathrm{d}s+C\int_0^t\|\Delta Z_s^\ell\|_1^2\mathrm{d}s\\
&\le C(\sup_{0\le t\le T}\|\Delta Z_t^\ell\|_1^2+\sup_{0\le t\le T}\|x_t^\ell\|_1^2)\int_0^t\|\Delta Y_s^\ell\|_0^2\mathrm{d}s+CT\sup_{0\le t\le T}\|\Delta Z_t^\ell\|_1^2,\quad 0\le t\le T.
\end{aligned}$$

When $\sup_{0\le t\le T}\|\Delta Z_t^\ell\|_0\le\varepsilon'$, we have

$$\|\Delta Y_t^\ell\|^2\le C((\varepsilon')^2+\sup_{0\le t\le T}\|x_t^\ell\|_1^2)\int_0^t\|\Delta Y_s^\ell\|_0^2\mathrm{d}s+CT(\varepsilon')^2.$$

By Gronwall's inequality,

$$\|\Delta Y_T^\ell\|^2\ \le\ CT\exp\left[C(\varepsilon'+\sup_{0\le t\le T}\|x_t^\ell\|_1)T\right](\varepsilon')^2,\ \text{ if }\sup_{0\le t\le T}\|\Delta Z_t^\ell\|_0\le\varepsilon'.$$

Since $\sup_{0\le t\le T}\|x_t^\ell\|_1<\infty$, when ε' is sufficiently this implies

$$\|\Delta Y_T^\ell\|_0\le\frac{\varepsilon}{4},\ \text{ if }\sup_{0\le t\le T}\|\Delta Z_t^\ell\|_0\le\varepsilon'.$$

Hence, for small enough $\varepsilon'>0$,

$$\mathbb{P}\left(\|\,Y_T^\ell-y_T^\ell\|_0\le\frac{\varepsilon}{4},\sup_{0\le t\le T}\|\,Z_T^\ell-z_T^\ell\|_0\le\varepsilon'\right)=\mathbb{P}\left(\|\,Z_T^\ell-z_T^\ell\|_0\le\varepsilon'\right)>0.$$

This and (4.4) yield that (4.3) holds for $T\notin\mathcal{J}(\ell)$. Since X_t^ℓ is right continuous and the set $[0,\infty)\setminus\mathcal{J}(\ell)$ is dense, (4.3) holds for all $T>0$. Then the proof is finished. □

5. ψ-UNIFORMLY EXPONENTIAL ERGODICITY AND MODERATE DEVIATION

5.1. **Galerkin approximation.** Recall that $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$. For any $m\in\mathbb{N}$, let $\mathcal{H}_m:=\mathrm{span}\{e_k:k\le m\}$ with orthogonal projection $\Pi_m:\mathbb{H}\to\mathcal{H}_m$. Then the Galerkin approximation of (1.2) reads

e:GalEqn (5.1)
$$\mathrm{d}\tilde X_t^m+[A\tilde X_t^m+B^m(\tilde X_t^m)]\mathrm{d}t=Q\mathrm{d}L_t^m,\quad \tilde X_0^m=x^m,$$

where $x^m=\Pi_m x$, $B^m(x)=\Pi_m[B(x)]$ for $x\in\mathbb{H}$, and $L_t^m=\Pi_m L_t=W_{S_t}^m$ with W_t^m being an m-dimensional standard Brownian motion.

Since the Lévy measure of W_{S_t} can not be approximated by those of $W_{S_t}^m$, the approximation procedure in [26] does not apply. Alternatively, we show that $\Delta X_t^m=\tilde X_t^m-X_t^m$ converges to zero. The advantage of this new procedure is that the approximation of W_{S_t} is avoided.

Theorem 5.1. *For all $t > 0$, $\mathbb{P}$-a.s.*

$$\lim_{m\to\infty} \|\tilde X_t^m - X_t\|_1 \;=\; 0. \tag{5.2}$$

Proof. Let X_t solve (1.2) with $X_0 = x$, and denote $X_t^m = \Pi_m X_t$. Then

e:GalEqn-1
$$\mathrm{d}X_t^m + [AX_t^m + B^m(X_t)]\mathrm{d}t = Q\mathrm{d}L_t^m, \quad X_0^m = x^m. \tag{5.3}$$

By (1.6) and Theorem 1.1,

$$\lim_{m\to\infty} \|X_t^m - X_t\|_1 = 0, \qquad t > 0.$$

Combining this with Lemma 5.2 below, we finish the proof. □

l:DelXm
Lemma 5.2. *Let $\Delta X_t^m = \tilde X_t^m - X_t^m$. Then $\mathbb{P}$-a.s.*

$$\lim_{m\to\infty} \|\Delta X_t^m\|_1 = 0, \qquad t \ge 0.$$

Proof. (1) We first prove that for some constant $C > 0$,

RPP
$$\sup_{0\le t\le T, m\in\mathbb{N}} \|\tilde X_t^m\|_0^2 \le A_T, \ \ T > 0, m \in \mathbb{N}, \tag{5.4}$$

holds for

$$A_T := 2\exp\left(C\int_0^T (1 + \|Z_s\|_1^2)\mathrm{d}s\right)\left[\|x\|_0^2 + T\sup_{0\le t\le T} |Z_t\|_1^4\right] + 2\sup_{0\le t\le T} \|Z_t\|_1^2.$$

For $\ell \in \mathcal{S}$, let

$$Z_t^{m,\ell} \;=\; \int_0^t \mathrm{e}^{-A(t-s)} Q\mathrm{d}W_{\ell_s}^m.$$

Then

$$\|Z_t^{m,\ell}\|_\gamma \;\le\; \|Z_t^\ell\|_\gamma, \quad \gamma \in \mathbb{R}.$$

By (3.2) with $\gamma = 1$, we have $\mathbb{P}$-a.s.

e:SupZm
$$\sup_{0\le t\le T, m\in\mathbb{N}} \|Z_t^{m,\ell}\|_0 \;\le\; \sup_{0\le t\le T, m\in\mathbb{N}} \|Z_t^{m,\ell}\|_1 \;\le\; \sup_{0\le t\le T} \|Z_t^\ell\|_1 \;<\; \infty. \tag{5.5}$$

It is easy to see that $\tilde Y_t^{m,\ell} := \tilde X_t^{m,\ell} - Z_t^{m,\ell}$ solves the equation

e:GalEqnY
$$\partial_t \tilde Y_t^{m,\ell} + A\tilde Y_t^{m,\ell} + B^m(\tilde Y_t^{m,\ell} + Z_t^{m,\ell}) = 0, \quad \tilde X_0^{m,\ell} = x^m. \tag{5.6}$$

Applying the chain role to $\|\tilde Y_t^{m,\ell}\|_0^2$ gives

YPP
$$\|\tilde Y_t^{m,\ell}\|_0^2 + 2\int_0^t \|\tilde Y_s^{m,\ell}\|_1^2 \mathrm{d}s \;=\; \|x^m\|_0^2 + 2\int_0^t \langle \tilde Y_s^{m,\ell}, B^m(\tilde Y_s^{m,\ell} + Z_s^{m,\ell})\rangle \mathrm{d}s. \tag{5.7}$$

Letting $\tilde B^m(x,y) = B^m(x,y) + B^m(y,x)$, the relation $\langle \tilde Y_s^{m,\ell}, B^m(\tilde Y_s^{m,\ell})\rangle = 0$ implies

$$\begin{aligned}
&\big|\langle \tilde Y_s^{m,\ell}, B^m(\tilde Y_s^{m,\ell} + Z_s^{m,\ell})\rangle\big| \\
&= \big|\ \langle \tilde Y_s^{m,\ell}, \tilde B^m(\tilde Y_s^{m,\ell}, Z_s^{m,\ell}) + B^m(Z_s^{m,\ell})\rangle\big| \\
&\le\ C\|\tilde Y_s^{m,\ell}\|_0 \|\tilde Y_s^{m,\ell}\|_1 \|Z_s^{m,\ell}\|_1 + C\|\tilde Y_s^{m,\ell}\|_0 \|Z_s^{m,\ell}\|_1^2 \\
&\le\ C(1 + \|Z_s^{m,\ell}\|_1^2)\|\tilde Y_s^{m,\ell}\|_0^2 + \|\tilde Y_s^{m,\ell}\|_1^2 + \|Z_s^{m,\ell}\|_1^4 \\
&\le\ C(1 + \|Z_s^\ell\|_1^2)\|\tilde Y_s^{m,\ell}\|_0^2 + \|\tilde Y_s^{m,\ell}\|_1^2 + \|Z_s^\ell\|_1^4,
\end{aligned}$$

for some constant $C > 0$ independent of m and T. Combining this with (5.7) and $\|x^m\|_0 \leq \|x\|_0$, we arrive at

$$\|\tilde{Y}_t^{m,\ell}\|_0^2 \leq \|x\|_0^2 + C\int_0^t \left(1 + \|Z_s^\ell\|_1^2\right)\|\tilde{Y}_s^{m,\ell}\|_0^2 \mathrm{d}s + \int_0^t \|Z_s^\ell\|_1^4 \mathrm{d}s.$$

By Gronwall's lemma this implies

$$\|\tilde{Y}_t^{m,\ell}\|_0^2 \leq \exp\left(C\int_0^t (1 + \|Z_s^\ell\|_1^2)\mathrm{d}s\right)\|x\|_0^2 + \int_0^t \exp\left[C\int_s^t (1 + \|Z_r^\ell\|_1^2)\mathrm{d}r\right]|Z_s^\ell\|_1^4 \mathrm{d}s,$$

so that (5.4) holds.

(2) By the equations (5.1) and (5.3), we have

$$\partial_t \Delta X_t^m + AX_t^m + B^m(\tilde{X}_t^m) - B^m(X_t) = 0, \quad \Delta X_0^m = 0.$$

Then there exists a constant $C > 0$ such that

YPP2

$$\begin{aligned} \|\Delta X_t^m\|_0 &\leq \int_0^t \|\mathrm{e}^{-(t-s)}\left[B_m(\tilde{X}_s^m) - B_m(X_s)\right]\|_0 \mathrm{d}s \\ &= \int_0^t \|\mathrm{e}^{-(t-s)}\left[B(\tilde{X}_s^m) - B(X_s)\right]\|_0 \mathrm{d}s \\ &\leq C\int_0^t (t-s)^{-\frac{5}{6}}\|B(\tilde{X}_s^m) - B(X_s)\|_{-\frac{5}{3}} \mathrm{d}s \end{aligned} \tag{5.8}$$

Since $B(x) = B(x^m + (x - x^m))$ for $x \in \mathbb{H}^1$, it follows that

$$B(\tilde{X}_s^m) - B(X_s) = B(\tilde{X}_s^m) - B(X_s^m) - \tilde{B}(X_s^m, X_s - X_s^m) - B(X_s - X_s^m),$$

where $\tilde{B}(x,y) = B(x,y) + B(y,x)$ for $x, y \in \mathbb{H}^1$. Applying Eq. (1.4) with $\sigma_1 = \frac{5}{3}, \sigma_2 = -1, \sigma_3 = 0$, we obtain

$$\begin{aligned} \|B(\tilde{X}_s^m) - B(X_s^m)\|_{-\frac{5}{3}} &\leq \|B(\Delta X_s^m, \tilde{X}_s^m)\|_{-\frac{5}{3}} + \|B(X_s^m, \Delta X_s^m)\|_{-\frac{5}{3}} \\ &\leq \|\Delta X_s^m\|_0 \|\tilde{X}_s^m\|_0 + \|\Delta X_s^m\|_0 \|\|X_s^m\|_0 \\ &\leq \left(\sqrt{A_T} + \sup_{0\leq t\leq T}\|X_t\|_0\right)\|\Delta X_s^m\|_0. \end{aligned}$$

Combining this with (5.8) gives

$$\begin{aligned} \|\Delta X_t^m\|_0^2 \leq\ & C\int_0^t (t-s)^{-\frac{5}{6}}\left(\sqrt{A_T} + \sup_{0\leq t\leq T}\|X_t\|_0\right)\|\Delta X_s^m\|_0 \mathrm{d}s \\ & + C\int_0^t (t-s)^{-\frac{5}{6}}\left(\|X_s\|_0\|X_s - X_s^m\|_0 + \|X_s - X_s^m\|_0^2\right)\mathrm{d}s. \end{aligned}$$

Noting that

$$\|\Delta X_t^m\|_0 \leq \|X_t^m\|_0 + \|\tilde{X}_t^m\|_0 \leq \sup_{0\leq t\leq T}\|X_t\|_0 + \sqrt{A_T} < \infty, \ \ t \in [0,T],$$

by Fatou's lemma we get

$$\limsup_{m\to\infty}\|\Delta X_t^m\|_0^2 \leq C\int_0^t (t-s)^{-\frac{5}{6}}\left(\sqrt{A_T} + \sup_{0\leq t\leq T}\|X_t\|_0\right)\limsup_{m\to\infty}\|\Delta X_s^m\|_0 \mathrm{d}s, \quad 0 \leq t \leq T,$$

so that by Gronwall's inequality,

$$\limsup_{m\to\infty} \|\Delta X_t^m\|_0 \;=\; 0, \qquad t\in[0,T].$$

□

5.2. **ψ-uniformly exponential ergodicity and moderate deviation.** We will use the following exponential ergodicity result in [9].

t:DMT95 **Theorem 5.3** (Theorem 5.2 (b), [9]). *Let $(X_t)_{t\ge 0}$ be an irreducible and aperiodic Markov process on a Polish space E with Markov semigroup P_t, and let $\psi \ge 1$ be a measurable function on E. If*

$$P_t\psi(x) \;\le\; \lambda(t)\psi(x) + b1_{\mathcal{K}}(x), \quad t\in(0,T], x\in E$$

holds for some constants $T, b>0$, a measurable petite set $\mathcal{K}$ on E, and a bounded function λ on $[0,T]$ with $\lambda(T)<1$, then X_t is ψ-uniformly ergodic, i.e., there exist constants $C,\gamma>0$ such that

e:PsiExp (5.9)
$$\sup_{|f|\le\psi} |P_t f(x) - \mu_0(f)| \;\le\; C\mathrm{e}^{-\gamma t}\psi(x), \qquad t>0.$$

Proof of Theorem 1.3 (1). Since $1+\|\cdot\|_0$ is comparable with $\sqrt{M+\|\cdot\|_0^2}$ for any $M\ge 1$, we will take $\psi(x) = \sqrt{M+\|x\|_0^2}$ instead of $1+\|x\|_0$ for $M>1$ large enough to be determined.

(1) We first observe that it suffices to find out a constant $C>0$ such that

:StaIntEst (5.10)
$$\begin{aligned}&\left|\int_{\mathcal{H}^m} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\le 1})\nu_m(\mathrm{d}y)\right|\\ &\le\; C\Big(1+\frac{1}{\sqrt{M}}\Big), \;\; x^m\in\mathcal{H}^m, \;\; x^m\in\mathcal{H}_m := \mathrm{span}\{e_i : i\le m\}.\end{aligned}$$

Let $\mathcal{L}^m$ be the generator of $\tilde{X}_t^m$ given by (5.1). Since $\langle x^m, B_m(x^m)\rangle = 0$, it is easy to see that

$$\begin{aligned}\mathcal{L}^m\psi(x^m) \;&=\; -\langle Ax^m + B_m(x^m), \nabla\psi(x^m)\rangle_0\\ &\quad + \int_{\mathcal{H}^m} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\le 1})\nu_m(\mathrm{d}y)\\ &=\; -\frac{\|x^m\|_1^2}{\psi(x^m)} + \int_{\mathcal{H}^m} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy,\nabla\psi(x^m)\rangle_0 1_{\|y\|_0\le 1})\nu_m(\mathrm{d}y).\end{aligned}$$

where the last equality is by $\langle x^m, B_m(x^m)\rangle = 0$. Let $\mathcal{K}_m = \{x^m\in\mathcal{H}^m : \|x^m\|_1 \le M\}$. By (5.10) and (5.2), we have

$$\begin{aligned}\mathcal{L}^m\psi(x^m) \;&\le\; -\frac{\|x^m\|_1^2}{\psi(x^m)} + C\Big(1+\frac{1}{\sqrt{M}}\Big)\\ &\le\; -\frac{\|x^m\|_1^2+M}{\psi(x^m)} + \frac{M}{\psi(x^m)} + C\Big(1+\frac{1}{\sqrt{M}}\Big)\\ &\le\; -\psi(x^m)+\sqrt{M}+C\Big(1+\frac{1}{\sqrt{M}}\Big), \;\; x^m\in\mathcal{K}_m.\end{aligned}$$

On the other hand, if $x^m \notin \mathcal{K}_m$, then $\|x^m\|_1 \geq M$ and thus,

$$
\begin{aligned}
\mathcal{L}^m\psi(x^m) &\leq -\frac{\|x^m\|_1^2}{\psi(x^m)} + C(1+\frac{1}{\sqrt{M}}) \\
&\leq -\frac{\frac{1}{2}(M+\|x^m\|_1^2)}{\psi(x^m)} + C(1+\frac{1}{\sqrt{M}}) \\
&\leq -\frac{1}{2}\psi(x^m) + C(1+\frac{1}{\sqrt{M}}) \\
&\leq -\frac{1}{4}\psi(x^m),
\end{aligned} \tag{5.11}
$$

as long as we choose $M > 1$ sufficiently large. In conclusion, when $M > 1$ is large enough, there exists a constant $b > 0$ such that

$$
\mathcal{L}^m\psi(x^m) \leq -\frac{1}{4}\psi(x^m) + b1_{\mathcal{K}_m}(x^m), \quad m \geq 1.
$$

By [9, Theorem 5.1 (d)], this implies

$$
\mathbb{E}[\psi(\tilde{X}_t^m)] \leq \mathrm{e}^{-t/4}\psi(x^m) + b1_{\mathcal{K}_m}(x^m), \quad t \geq 0.
$$

(Note the b in the previous two relations may be different.) Since $\lim_{m\to\infty}\|x^m - x\|_0 = 0$ and $\lim_{m\to\infty}\|\tilde{X}_t^m - X_t\|_1 = 0$ a.s. for $t > 0$, by letting $m \to \infty$ we obtain

$$
\mathbb{E}[\psi(X_t)] \leq \mathrm{e}^{-t/4}\psi(x) + b1_{\mathcal{K}}(x), \quad t \geq 0,
$$

where $\mathcal{K} := \{x \in \mathbb{H} : \|x\|_1 \leq M\}$ is a compact (hence petite) set in $\mathbb{H}$. By Theorem 5.3, we prove the ψ-uniformly exponential ergodicity of X_t.

(2) It remains to prove (5.10). Obviously,

YPP4

$$
\begin{aligned}
&\left|\int_{\mathcal{H}^m}(\psi(x^m+Qy) - \psi(x^m) - \langle Qy, \nabla\psi(x^m)\rangle_0 1_{\|y\|_0\leq 1})\nu_m(\mathrm{d}y)\right| \\
&\leq \left|\int_{\|y\|_0\leq 1}(\psi(x^m+Qy) - \psi(x^m) - \langle Qy, \nabla\psi(x^m)\rangle_0)\nu_m(\mathrm{d}y)\right| \\
&\quad + \left|\int_{\|y\|_0> 1}(\psi(x^m+Qy) - \psi(x^m))\nu_m(\mathrm{d}y)\right|
\end{aligned} \tag{5.12}
$$

By Taylor's expansion,

$$
\begin{aligned}
&|\psi(x^m+Qy) - \psi(x^m) - \langle Qy, \nabla\psi(x^m)\rangle_0| \\
&\leq \sup_{\theta\in[0,1]}\left|\frac{\|y\|_0^2}{\psi(x^m+\theta Qy)} - \frac{|\langle y, x^m+\theta Qy\rangle_0|^2}{\psi^3(x^m+\theta Qy)}\right| \leq \frac{2}{\sqrt{M}}\|y\|_0^2.
\end{aligned}
$$

Since ν_m has a density $\frac{C_m}{\|y\|_0^{m+\alpha}}$ for $y \in \mathcal{H}_m$ with $C_m = \frac{\alpha 2^\alpha \Gamma(\frac{m}{2}+\frac{\alpha}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{2-\alpha}{2})}$, we have

$$
\begin{aligned}
&\left|\int_{\|y\|_0\leq 1}(\psi(x^m+Qy) - \psi(x^m) - \langle Qy, \nabla\psi(x^m)\rangle_0)\nu_m(\mathrm{d}y)\right| \\
&\leq \frac{2}{\sqrt{M}}\int_{\|y\|_0\leq 1}\|y\|_0^2\frac{C_m}{\|y\|_0^{m+\alpha}}\mathrm{d}y = \frac{2C_m}{\sqrt{M}}\int_0^1\int_{\mathbb{S}_{m-1}} r^{1-\alpha}\mathrm{d}r\mathrm{d}\sigma_{m-1} = \frac{2C_m|\mathbb{S}_{m-1}|}{(2-\alpha)\sqrt{M}},
\end{aligned}
$$

where $|\mathbb{S}_{m-1}| = \frac{2(\pi)^{m/2}}{\Gamma(m/2)}$ is the volume of $\mathbb{S}_{m-1}$. Moreover,

$$\begin{aligned} C_m|\mathbb{S}_{m-1}| &= \frac{\alpha 2^\alpha \Gamma\left(\frac{m}{2}+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)}\frac{2\pi^{m/2}}{\Gamma(m/2)} \leq \frac{\alpha 2^\alpha \Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)}\frac{2\pi^{m/2}}{\Gamma(m/2)} \\ &= \frac{\alpha 2^\alpha \frac{m}{2}\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)}\frac{2\pi^{m/2}}{\Gamma(m/2)} \leq \sup_{m\geq 1} \frac{\alpha 2^\alpha m \pi^{m/2}}{\Gamma\left(\frac{2-\alpha}{2}\right)\Gamma\left(\frac{m}{2}\right)} =: C' < \infty. \end{aligned}$$

Hence,

$$\left|\int_{\|y\|_0\leq 1} (\psi(x^m+Qy)-\psi(x^m)-\langle Qy, \nabla\psi(x^m)\rangle_0)\nu_m(\mathrm{d}y)\right| \leq \frac{C'}{\sqrt{M}}.$$

Similarly, there exist constants $C_Q > 0$ such that

$$\begin{aligned} &\left|\int_{\|y\|_0>1} (\psi(x^m+Qy)-\psi(x^m))\nu_m(\mathrm{d}y)\right| \\ &\leq \left|\int_{\|y\|_0>1} \frac{|\langle x^m+\theta Qy, Qy\rangle_0|}{\psi(x^m+\theta Qy)}\nu_m(\mathrm{d}y)\right| \leq \left|\int_{\|y\|_0>1} \|Qy\|_0\nu_m(\mathrm{d}y)\right| \\ &\leq C_Q\left|\int_{\|y\|_0>1} \|y\|_0\nu_m(\mathrm{d}y)\right| \leq \sup_{m\geq 1} C_Q\int_1^\infty\int_{\mathbb{S}_{m-1}} \frac{C_m}{r^\alpha}\mathrm{d}r\mathrm{d}\sigma_{m-1} < \infty. \end{aligned}$$

Therefore, (5.10) holds for some constant $C > 0$. □

Proof of Theorem 1.3 (2). We follow the argument in [27, p. 429-431]. Without loss of generality, we assume $C\mathrm{e}^{-\gamma} = \rho < 1$ in (5.9), otherwise one can choose t sufficiently large so that $C\mathrm{e}^{-\gamma t} < 1$. Given $f \in \mathcal{B}_b(\mathbb{H})$, consider the following Feynman-Kac formula

$$P_t^{\lambda f}g(x) = \mathbb{E}\left[\exp\left(\lambda\int_0^t f(X_s^x)\mathrm{d}s\right)g(X_t^x)\right], \quad g \in \mathcal{B}_\psi.$$

For any $\delta > 0$ and $|\lambda| \leq \delta$, we have

$$\|P_t^{\lambda f}g\|_\psi \leq \mathrm{e}^{\delta\|f\|t}\|g\|_\psi.$$

So, $\lambda \to P_1^{\lambda f}g \in \mathcal{B}_\psi$ is holomorphic for all $|\lambda| < \delta$.

When $\lambda = 0$, $P_1 g = \mathbb{E}[g(X_1^x)]$ with $g \in \mathcal{B}_\psi$. By the exponential ergodicity result (5.9), we get that 1 is an isolated simple spectrum of P_1 and the constant function is the corresponding eigenfunction. Denote $\mathcal{P}_0$ be the projection with respect to the eigenvalue 1, which is defined by

$$\mathcal{P}_0 g = \mu_0(g), \quad g \in \mathcal{B}_\psi.$$

The spectrum of the $P_1(I-\mathcal{P}_0)$ has a spectrum radius less than ρ from (5.9).

By Kato's holomorphic perturbation theorem, for any $r \in \left(\rho, \frac{1+\rho}{2}\right)$, there exist some $\tilde{\delta} \in (0,\delta)$ such that for all $D_{\tilde{\delta}} = \{\lambda \in \mathbb{C} : |\lambda| \leq \tilde{\delta}\}$ the operator $P_1^{\lambda f}$ acting on $\mathcal{B}_\psi$ has the following properties: (1) $P_1^{\lambda f}$ has a single simple eigenvalue $\sigma(\lambda)$ with the largest modulus of the spectrum, moreover, there exists some number $c \in (\frac{1}{2}, 1)$ such that $|\sigma(\lambda)| \geq c$; (2) $\mathcal{P}_\lambda$ is the projection of $P_1^{\lambda f}$ corresponding to $\sigma(\lambda)$, $\lambda \in D_{\tilde{\delta}} \to \mathcal{P}_\lambda \in \mathcal{L}(\mathcal{B}_\psi)$ is holomorphic and $\|\mathcal{P}_\lambda 1 - \mathcal{P}_0 1\|_\psi \leq \varepsilon$ with some sufficiently small $\varepsilon \in (0,1)$; (3) the spectral radius of $P_1^{\lambda f}(I-\mathcal{P}_\lambda)$ is strictly less than r.

By (3), the following relation holds

$$N := \sup_{z\in S(\frac{1}{r}),\lambda\in D_{\tilde\delta}} \|(I - zP_1^{\lambda f}(I-\mathcal{P}_\lambda))^{-1}\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} < \infty,$$

where $S(1/r) = \{z\in\mathbb{C} : |z| = \frac{1}{r}\}$.

By Cauchy integral we have

$$\begin{aligned}(P_1^{\lambda f}(I-\mathcal{P}_\lambda))^n &= \frac{1}{n!}\frac{\partial^n}{\partial^n z}(I - zP_1^{\lambda f}(I-\mathcal{P}_\lambda))^{-1}|_{z=0}\\ &= \frac{1}{2\pi i}\int_{S(\frac{1}{r})}\frac{(I-zP^{\lambda f}(I-\mathcal{P}_\lambda))^{-1}}{z^{n+1}}\mathrm{d}z,\end{aligned}$$

from which we get

$$\|P_n^{\lambda f} - \sigma(\lambda)^n\mathcal{P}_\lambda\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} = \|(P_1^{\lambda f}(I-\mathcal{P}_\lambda))^n\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} \le Nr^n.$$

Since $\|P_t^{\lambda f}\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} \le e^{\lambda\|f\|}$ for $0\le t\le 1$, by a standard argument and the semigroup property of $P_t^{\lambda f}$, we have

e:RtCon (5.13)
$$\|P_t^{\lambda f} - \exp\left(t\log\sigma(\lambda)\right)\mathcal{P}_\lambda\|_{\mathcal{B}_\psi\to\mathcal{B}_\psi} \le Cr^t. \tag{5.13}$$

For any probability measure μ with $\mu(\psi)<\infty$, by (5.13), for all large t so that $Cr^t<1$, $\log\int_{\mathbb{H}} P_t^{\lambda f}1\mathrm{d}\mu$ are holomorphic on $D_{\tilde\delta}$. Moreover, by the inequality in (2),

$$\lim_{t\to\infty}\sup_{|\lambda|<\tilde\delta}\sup_{\mu:\mu(\psi)<\infty}\left|\frac{1}{t}\log\int_{\mathbb{H}}P_t^{\lambda f}1\mathrm{d}\mu - \log\sigma(\lambda)\right| = 0.$$

By Cauchy's theorem for holomorphic function, for any $\varepsilon\in(0,\tilde\delta)$ we have

$$\lim_{t\to\infty}\sup_{|\lambda|<\varepsilon}\sup_{\mu:\mu(\psi)<\infty}\left|\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\frac{1}{t}\log\int_{\mathbb{H}}P_t^{\lambda f}1\mathrm{d}\mu - \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\log\sigma(\lambda)\right| = 0, \quad k\in\mathbb{N}.$$

By the C^2-regularity criterion in [27, Theorem 1.2], we have

$$\lim_{t\to\infty}\sup_{\mu:\mu(\psi)<\infty}\left|\frac{1}{b^2(t)}\log\mathbb{E}^\mu\exp\left(b^2(t)\mathfrak{M}_t(f)\right) - \frac{1}{2}\sigma^2(f)\right| = 0,$$

where $\mathfrak{M}_t(f) := \frac{1}{b(t)\sqrt{t}}\left(\int_0^t f(X_s)\mathrm{d}s - \mu_0(f)\right)$ with $b(t)\to\infty$ and $\frac{b(t)}{\sqrt{t}}\to 0$ as $t\to\infty$, and

$$\sigma^2(f) = \lim_{t\to\infty}\left(\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\frac{1}{t}\log\int_{\mathbb{H}}P_t^{\lambda f}1\mathrm{d}\mu\right)|_{\lambda=0} = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}^{\mu_0}\left(\int_0^t(f(X_s)-\mu_0(f))\mathrm{d}s\right)^2.$$

By [6, Chapter 6], we immediately obtain the MDP result in the theorem. □

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