arXiv:1605.02447v5 [math.PR] 23 Sep 2017

Pointwise Characterizations of Curvature and Second Fundamental Form on Riemannian Manifolds*

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Abstract

Let M be a complete Riemannian manifold possibly with a boundary ∂M. For any C^1-vector field Z, by using gradient/functional inequalities of the (reflecting) diffusion process generated by $L := \Delta + Z$, pointwise characterizations are presented for the Bakry-Emery curvature of L and the second fundamental form of ∂M if exists. These extend and strengthen the recent results derived by A. Naber for the uniform norm $\|\mathrm{Ric}_Z\|_\infty$ on manifolds without boundary. A key point of the present study is to apply the asymptotic formulas for these two tensors found by the first named author, such that the proofs are significantly simplified.

Keywords: Curvature; second fundamental form, diffusion process, path space.

1 Introduction

sect1

Let M be a d-dimensional complete Riemannian manifold possibly with a boundary ∂M. Let $L = \Delta + Z$ for a C^1 vector field Z. We intend to characterize the Bakry-Emery curvature $\mathrm{Ric}_Z := \mathrm{Ric} - \nabla Z$ and the second fundamental form $\mathbb{I}$ of the boundary ∂M using the (reflecting) diffusion process generated by L. When $\partial M = \emptyset$, we set $\mathbb{I} = 0$.

*Supported in part by NNSFC (11771326, 11431014).

There are many equivalent characterizations for the (pointwise or uniform) lower bound of Ric_Z and $\mathbb{I}$ using gradient/functional inequalities of the (Neumann) semigroup generated by L, see e.g. [18] and references within. However, the corresponding upper bound characterizations are still open. It is known that for stochastic analysis on the path space, one needs conditions on the norm of Ric_Z, see [3, 4, 5, 7, 10, 15, 17] and references within. Recently, A. Naber [12, 9] proved that the uniform bounded condition on Ric_Z for $Z = -\nabla f$ is equivalent to some gradient/functional inequalities on the path space, and thus clarified the necessity of bounded conditions used in the above mentioned references. In this paper, we aim to present pointwise characterizations for the norm of Ric_Z and $\mathbb{I}$ when $\partial M \neq \emptyset$, which allow these quantities unbounded on the manifold.

Let $(X_t^x)_{t\geq 0}$ be the (reflecting if ∂M exists) diffusion process generated by $L = \Delta + Z$ on M starting at point x, and let $(U_t^x)_{t\geq 0}$ be the horizontal lift onto the frame bundle $O(M) := \cup_{x\in M} O_x(M)$, where $O_x(M)$ is the set of all orthonormal basis of the tangent space T_xM at point x. It is well known that $(X_t^x, U_t^x)_{t\geq 0}$ can be constructed as the unique solution to the SDEs:

eq1.1
$$
\begin{aligned}
\mathrm{d}X_t^x &= \sqrt{2}\, U_t^x \circ dW_t + Z(X_t^x)\mathrm{d}t + N(X_t^x)\mathrm{d}l_t^x, \quad X_0^x = x,\\
\mathrm{d}U_t^x &= \sqrt{2}\, H_{U_t^x}(U_t^x) \circ \mathrm{d}W_t + H_Z(U_t^x)\mathrm{d}t + H_N(U_t^x)\mathrm{d}l_t^x, \quad U_0^x \in O_x(M),
\end{aligned}
\tag{1.1}
$$

where W_t is the d-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, N is the inward unit normal vector field of ∂M, $H_\cdot : TM \to TO(M)$ is the horizontal lift, $H_u := (H_{ue_i})_{1\leq i\leq d}$ for $u \in O(M)$ and the canonical orthonormal basis $\{e_i\}_{1\leq i\leq d}$ on $\mathbb{R}^d$, and l_t is an adapted increasing process which increases only when $X_t^x \in \partial M$ which is called the local time of X_t^x on ∂M. In the first part of this paper, we assume that the solution is non-explosive, so that the (Neumann) semigroup P_t generated by L is given by

$$
P_t f(x) = \mathbb{E} f(X_t^x), \quad x \in M, f \in \mathscr{B}_b(M), t \geq 0.
$$

For a fixed $T > 0$, consider the path space $W_T(M) := C([0,T]; M)$ and the class of smooth cylindric functions

$$
\mathscr{F}C_T^\infty := \Big\{ F(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_m}) : \ m \geq 1, \ \gamma \in W_T(M),\\
0 < t_1 < t_2 \cdots < t_m \leq T, \ f \in C_0^\infty(M^m) \Big\}.
$$

Let

$$
\mathbb{H}_T = \Big\{ h \in C([0,T]; \mathbb{R}^d) : \quad h(0) = 0, \ \|h\|_{\mathbb{H}_T}^2 := \int_0^T |h_s'|^2 \mathrm{d}s < \infty \Big\}.
$$

For any $F \in \mathscr{F}C_T^\infty$ with $F(\gamma) = f\big(\gamma(t_1), \cdots, \gamma(t_m)\big)$, the Malliavin gradient $DF(X_{[0,T]}^x)$

is an $\mathbb{H}_T$-valued random variable satisfying

MG (1.2)
$$\begin{aligned}\dot D_s F(X^x_{[0,T]}) &:= \frac{\mathrm{d}}{\mathrm{d}s} DF(X^x_{[0,T]})\\ &= \sum_{t_i>s} (U^x_{t_i})^{-1}\nabla_i f\big(X^x_{t_1},\cdots,X^x_{t_m}\big),\quad s\in[0,T],\end{aligned}$$

where ∇_i is the (distributional) gradient operator for the i-th component on M^m, and $P_u:\mathbb{R}^d\to\mathbb{R}^d$ is the projection along $u^{-1}N$, i.e.

$$\langle P_u a,b\rangle := \langle ua,N\rangle\langle ub,N\rangle,\quad a,b\in\mathbb{R}^d, u\in\cup_{x\in\partial M}O_x(M).$$

Note that

For $K\in C(M;[0,\infty))$ and $\sigma\in C(\partial M;[0,\infty))$, we introduce the following random measure $\mu_{x,T}$ on $[0,T]$:

MU (1.3)
$$\mu_{x,T}(\mathrm{d}s) := \mathrm{e}^{\int_0^s K(X^x_r)\mathrm{d}r+\int_0^s\sigma(X^x_r)\mathrm{d}l^x_r}\big\{K(X^x_s)\mathrm{d}s+\sigma(X^x_s)\mathrm{d}l^x_s\big\}.$$

For any $t\in[0,T]$, consider the energy form

$$\mathscr{E}^{K,\sigma}_{t,T}(F,F) = \mathbb{E}\bigg\{\big(1+\mu_{x,T}([t,T])\big)\bigg(\big|\dot D_t F(X^x_{[0,T]})\big|^2+\int_t^T\big|\dot D_s F(X^x_{[0,T]})\big|^2\mu_{x,T}(\mathrm{d}s)\bigg)\bigg\}$$

for $F\in\mathscr{F}C^\infty_T$. Our main result is the following.

T1.1 **Theorem 1.1.** *Let $K\in C(M;[0,\infty))$ and $\sigma\in C(\partial M;[0,\infty))$ be such that*

DE (1.4)
$$\mathbb{E}\mathrm{e}^{(2+\varepsilon)\int_0^T\{K(X^x_s)\mathrm{d}s+\sigma(X^x_s)\mathrm{d}l^x_s\}}<\infty \text{ for some } \varepsilon,T>0.$$

For any $p,q\in[1,2]$, the following statements are equivalent each other:

(1) *For any $x\in M$ and $y\in\partial M$,*

$$\begin{aligned}\|\mathrm{Ric}_Z\|(x) &:= \sup_{X\in T_xM,|X|=1}\big|\mathrm{Ric}(X,X)-\langle\nabla_X Z,X\rangle\big|(x)\le K(x),\\ \|\mathbb{I}\|(y) &:= \sup_{Y\in T_y\partial M,|Y|=1}|\mathbb{I}(Y,Y)|(y)\le\sigma(y).\end{aligned}$$

(2) *For any $f\in C_0^\infty(M)$, $T>0$, and $x\in M$,*

$$\begin{aligned}|\nabla P_T f|^p(x) &\le \mathbb{E}\Big[(1+\mu_{x,T}([0,T]))^p|\nabla f|^p(X^x_T)\Big],\\ \Big|\nabla f(x)-\frac12\nabla P_T f(x)\Big|^q &\le \mathbb{E}\Big[\big(1+\mu_{x,T}([0,T])\big)^{q-1}\\ &\quad\times\bigg(\Big|\nabla f(x)-\frac12 U^x_0(U^x_T)^{-1}\nabla f(X^x_T)\Big|^q+\frac{\mu_{x,T}([0,T])}{2^q}\big|\nabla f(X^x_T)\big|^q\bigg)\Big].\end{aligned}$$

(3) *For any* $F \in \mathscr{F}C_T^\infty, x \in M$ *and* $T > 0$,

$$\big|\nabla_x \mathbb{E} F(X^x_{[0,T]})\big|^q \le \mathbb{E}\bigg[\big(1+\mu_{x,T}([0,T])\big)^{q-1} \times \bigg(\big|\dot{D}_0 F(X^x_{[0,T]})\big|^q + \int_0^T \big|\dot{D}_s F(X^x_{[0,T]})\big|^q \mu_{x,T}(\mathrm{d}s)\bigg)\bigg].$$

(4) *For any* $t_0, t_1 \in [0,T]$ *with* $t_1 > t_0$*, and any* $x \in M$*, the following log-Sobolev inequality holds:*

$$\mathbb{E}\left[\mathbb{E}\big(F^2(X^x_{[0,T]})|\mathscr{F}_{t_1}\big)\log \mathbb{E}(F^2(X^x_{[0,T]})|\mathscr{F}_{t_1})\right] - \mathbb{E}\left[\mathbb{E}\big(F^2(X^x_{[0,T]})|\mathscr{F}_{t_0}\big)\log \mathbb{E}(F^2(X^x_{[0,T]})|\mathscr{F}_{t_0})\right] \le 4\int_{t_0}^{t_1} \mathscr{E}^{K,\sigma}_{s,T}(F,F)\mathrm{d}s, \quad F \in \mathscr{F}C_T^\infty.$$

(5) *For any* $t \in [0,T]$ *and* $x \in M$*, the following Poincaré inequality holds:*

$$\mathbb{E}\Big[\big\{\mathbb{E}(F(X^x_{[0,T]})|\mathscr{F}_t)\big\}^2\Big] - \Big\{\mathbb{E}\big[F(X^x_{[0,T]})\big]\Big\}^2 \le 2\int_0^t \mathscr{E}^{K,\sigma}_{s,T}(F,F)\mathrm{d}s, \quad F \in \mathscr{F}C_T^\infty.$$

Remark 1.1. (1) When $\partial M = \emptyset, Z = -\nabla f$ and K is a constant, it is proved in [12, Theorem 2.1] that $\|\mathrm{Ric}_Z\|_\infty \le K$ is equivalent to each of (3)-(5) with $\sigma = 0$ and a slightly different formulation of $\mathscr{E}^{K,0}_{s,T}$. Comparing with these equivalent statements using references functions on the path space, the statement (2) only depends on reference functions on M and is thus easier to verify.

(2) An important problem in geometry is to identify the Ricci curvature, for instance, to characterize Einstein manifolds where Ric is a constant tensor. According to Theorem 1.1, Ric is identified by ∇Z if and only if all/some of items (2)-(5) hold for $K = 0$.

We will prove this result in the next section. In Section 3, the equivalence of (1), (4) and (5) are proved without condition (1.4) but using the class of truncated cylindrical functions replacing $\mathscr{F}C_T^\infty$.

2 Proof

We first introduce some known results from the monograph [18] which hold under a condition weaker than (1.4).

Let $f \in C_0^\infty(M)$ with $|\nabla f(x)| = 1$ and $\mathrm{Hess}_f(x) = 0$. According to [18, Theorem 3.2.3], if $x \in M \setminus \partial M$ then for any $p > 0$ we have

RIC

$$\begin{aligned}\mathrm{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{t\downarrow 0} \frac{P_t|\nabla f|^p(x) - |\nabla P_t f|^p(x)}{pt} \\ &= \lim_{t\downarrow 0} \frac{1}{t}\bigg(\frac{P_t f^2(x) - (P_t f)^2(x)}{2t} - |\nabla P_t f(x)|^2\bigg);\end{aligned} \tag{2.1}$$

and by [18, Theorem 3.2.3], if $x \in \partial M$ and $\nabla f \in T_x\partial M$ then

II

$$\begin{aligned}\mathbb{I}(\nabla f, \nabla f)(x) &= \lim_{t\downarrow 0} \frac{\sqrt{\pi}}{2p\sqrt{t}}\Big\{P_t|\nabla f|^p(x) - |\nabla P_t f|^p(x)\Big\}\\ &= \lim_{t\downarrow 0}\frac{3\sqrt{\pi}}{8\sqrt{t}}\bigg(\frac{P_tf^2(x) - (P_tf)^2(x)}{2t} - |\nabla P_t f|^2(x)\bigg).\end{aligned}\tag{2.2}$$

We note that in [18, (3.2.9)], $\sqrt{\pi}$ is misprinted as π.

Next, let $\mathrm{Ric}_Z(u)$ for $u \in O(M)$ and $\mathbb{I}(u), P_u$ for $u \in \cup_{x\in\partial M} O_xM$ are matrix-valued functions with

$$\begin{aligned}&\langle P_u a, b\rangle = \langle ua, N\rangle\langle ub, N\rangle,\\ &\langle \mathrm{Ric}_Z(u)a, b\rangle := \mathrm{Ric}_Z(ua, ub),\\ &\langle \mathbb{I}(u)a, b\rangle := \mathbb{I}\big(ua - \langle ua, N\rangle N, ub - \langle ub, N\rangle N\big),\quad a, b\in\mathbb{R}^d.\end{aligned}$$

According to [18, Lemma 4.2.3], for any $F \in \mathscr{F}C_T^\infty$ with $F(\gamma) = f(\gamma_{t_1}, \cdot, \gamma_{t_N}), f \in C_0^\infty(M)$ and $0 \le t_1 < \cdots \le t_N$,

GR

$$(U_0^x)^{-1}\nabla_x\mathbb{E}\big[F(X^x_{[0,T]})\big] = \sum_{i=1}^N \mathbb{E}\big[Q^x_{0,t_i}(U^x_{t_i})^{-1}\nabla_i f(X^x_{t_1}, \cdots, X^x_{t_N})\big],\tag{2.3}$$

where ∇_x denotes the gradient in $x \in M$ and ∇_i is the gradient with respect to the i-th component, and for any $s \ge 0$, $(Q^x_{s,t})_{t\ge s}$ is an adapted right-continuous process on $\mathbb{R}^d \otimes \mathbb{R}^d$ satisfies $Q^x_{s,t}P_{U^x_t} = 0$ if $X^x_t \in \partial M$ and

QQ

$$Q^x_{s,t} = \bigg(I - \int_s^t Q^x_{s,r}\big\{\mathrm{Ric}_Z(U^x_r)\mathrm{d}r + \mathbb{I}(U^x_r)\mathrm{d}l^x_r\big\}\bigg)\Big(I - 1_{\{X^x_t\in\partial M\}}P_{U^x_t}\Big).\tag{2.4}$$

The multiplicative functional $Q^x_{s,t}$ was introduced by Hsu [11] to investigate gradient estimate on P_t. For convenience, let $Q^x_t := Q^x_{0,t}$. In particular, taking $F(\gamma) = f(\gamma_t)$ in (2.3), we obtain

GR2

$$\nabla P_t f(x) = U^x_0\mathbb{E}\big[Q^x_t(U^x_t)^{-1}\nabla f(X^x_t)\big],\quad x\in M, f\in C_0^\infty(M), t\ge 0.\tag{2.5}$$

Finally, for the above $F \in \mathscr{F}C_T^\infty$, let

TTD

$$\tilde{D}_t F(X^x_{[0,T]}) = \sum_{i: t_i > t} Q^x_{t,t_i}U_{t_i}^{-1}\nabla_i f(X^x_{t_1}, \cdots, X^x_{t_N}),\quad t\in[0,T].\tag{2.6}$$

Then [18, Lemma 4.3.2] (see also [17]) implies that

MF

$$\mathbb{E}\big(F(X^x_{[0,T]})\big|\mathscr{F}_t\big) = \mathbb{E}[F(X^x_{[0,T]})] + \sqrt{2}\int_0^t \Big\langle \mathbb{E}(\tilde{D}_s F(X^x_{[0,T]})|\mathscr{F}_s), \mathrm{d}W_s\Big\rangle,\quad t\in[0,T].\tag{2.7}$$

Proof of Theorem 1.1. It is well known that the log-Sobolev inequality in (4) implies the Poincaré inequality in (5), below we prove the theorem by verifying the following implications respectively: (1) $\Rightarrow$ (3) for all $q \geq 1$; (3) $\Rightarrow$ (2) for all $p = q$; (2) for some $p \geq 1$ and $q \in [1,2] \Rightarrow$ (1); (5) $\Rightarrow$ (1); and (1) $\Rightarrow$ (4).

For simplicity, below we will write F and f for $F(X^x_{[0,T]})$ and $f(X^x_{t_1}, \cdots, X^x_{t_N})$ respectively.

(a) (1) $\Rightarrow$ (3) for all $q \geq 1$. By (1.2), (2.3) and (2.4) we have

$$\begin{aligned}
U_0^{-1}\nabla_x \mathbb{E}[F] &= \mathbb{E}\Big[\sum_{i=1}^N Q^x_{t_i}(U^x_{t_i})^{-1}\nabla_i f\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^N \Big(I - \int_0^{t_i} Q^x_s \mathrm{Ric}_Z(U_s)\mathrm{d}s - \int_0^{t_i} Q^x_s \mathbb{I}_{U^x_s}\mathrm{d}l^x_s\Big)(U^x_{t_i})^{-1}\nabla_i f\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^N (U^x_{t_i})^{-1}\nabla_i f \\
&\quad - \sum_{i=1}^N \Big(\int_0^{t_i} Q^x_s \mathrm{Ric}_Z(U^x_s)\mathrm{d}s + \int_0^{t_i} Q^x_s \mathbb{I}_{U^x_s}\mathrm{d}l^x_s\Big)(U^x_{t_i})^{-1}\nabla_i f\Big] \\
&= \mathbb{E}\Big[\dot{D}_0 F - \int_0^T \big\{Q^x_s \mathrm{Ric}_Z(U^x_s)\dot{D}_s F\big\}\mathrm{d}s - \int_0^T \big\{Q^x_s \mathbb{I}(U^x_s)\dot{D}_s F\big\}\mathrm{d}l^x_s\Big].
\end{aligned}$$

By [18, Theorem 3.2.1], we have

QQ2
$$\|Q^x_s\| \leq \exp\left[\int_0^s K(X_r)\mathrm{d}r + \int_0^s \sigma(X_r)\mathrm{d}l^x_r\right]. \tag{2.8}$$

Combining these with (1), (1.3), and using Hölder's inequality twice, we obtain

$$\begin{aligned}
\big|\nabla_x \mathbb{E}[F]\big|^q &\leq \Big\{\mathbb{E}|\dot{D}_0 F| + \mathbb{E}\int_0^T |\dot{D}_s F|\mu_{x,T}(\mathrm{d}s)\Big\}^q \\
&\leq \mathbb{E}\Big\{|\dot{D}_0 F| + \int_0^T |\dot{D}_s F|\mu_{x,T}(\mathrm{d}s)\Big\}^q \\
&\leq \mathbb{E}\Big\{\Big(|\dot{D}_0 F|^q + \frac{\big(\int_0^T |\dot{D}_s F(X^x_{[0,T]})|\mu_{x,T}(\mathrm{d}s)\big)^q}{\{\mu_{x,T}([0,T])\}^{q-1}}\Big)\big(1 + \mu_{x,T}([0,T])\big)^{q-1}\Big\} \\
&\leq \mathbb{E}\Big\{\Big(|\dot{D}_0 F|^q + \int_0^T |\dot{D}_s F(X^x_{[0,T]})|^q \mu_{x,T}(\mathrm{d}s)\Big)\big(1 + \mu_{x,T}([0,T])\big)^{q-1}\Big\}.
\end{aligned}$$

Thus, the inequality in (3) holds.

(b) (3) $\Rightarrow$ (2) for all $p = q$. Take $F(\gamma) = f(\gamma_T)$. Then $\mathbb{E}F(X^x_{[0,T]}) = P_T f(x)$ and by (1.2), $|\dot{D}_s F| \le |\nabla f(X_T)|$ for $s \in [0,T]$. So, the first inequality in (2) with $p = q$ follows from (3) immediately. Similarly, by taking $F(\gamma) = f(\gamma_0) - \frac{1}{2} f(\gamma_T)$, we have $\mathbb{E}F = f(x) - \frac{1}{2} P_T f(x)$ and

$$
\begin{aligned}
|\dot{D}_0 F| &= \Big|\nabla f(x) - \frac{1}{2} U^x_0 (U^x_T)^{-1} \nabla f(X^x_T)\Big|, \\
|\dot{D}_s F| &\le \frac{1}{2} |\nabla f(X^x_T)|, \quad s \in (0,T].
\end{aligned}
$$

Then the second inequality in (2) is implied by (3).

(c) (2) for some $p \ge 1$ and $q \in [1,2] \Rightarrow$ (1). Let $x \in M \setminus \partial M$. There exists $r > 0$ such that $B(x,r) := \{y \in M : \rho(x,y) \le r\} \subset M \setminus \partial M$, where ρ is the Riemannian distance. Let $\tau_r = \inf\{t \ge 0 : \rho(x, X^x_t) \ge r\}$. By [18, Lemma 3.1.1] (see also [2, Lemma 2.3]), there exists a constant $c > 0$ such that

LO (2.9)
$$\mathbb{P}(\tau_r \le T) \le \mathrm{e}^{-c/T}, \quad T \in (0,1].$$

Then $\mathbb{P}(l^x_T > 0) \le \mathrm{e}^{-c/T}$ so that for each $n \ge 1$

LI (2.10)
$$\lim_{T \to 0} T^{-n} l^x_T = 0, \quad \mathbb{P} - a.s..$$

Combining this with (1.3) we obtain

MUL (2.11)
$$\lim_{T \to 0} \frac{\mu_{x,T}([0,T])}{T} = K(x).$$

Therefore, by the dominated convergence theorem due to (1.4), the first inequality in (2) and (2.1) yield

X1 (2.12)
$$
\begin{aligned}
-\mathrm{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{T \to 0} \frac{|\nabla P_T f|^p(x) - P_T |\nabla f|^p(x)}{pT} \\
&\le \lim_{T \to 0} \frac{\mathbb{E}\big\{[(1 + \mu_{x,T}([0,T]))^p - 1] |\nabla f|^p(X^x_T)\big\}}{pT} = K(x),
\end{aligned}
$$

where $f \in C_0^\infty(M)$ with $\mathrm{Hess}_f(x) = 0$ and $|\nabla f(x)| = 1$. This implies $\mathrm{Ric}_Z(X,X) \ge -K(x)$ for any $X \in T_x M$ with $|X| = 1$.

Next, we prove that the second inequality in (2) implies $\mathrm{Ric}_Z \le K$. By Hölder's inequality, the second inequality in (2) for some $q \in [1,2]$ implies the same inequality for $q = 2$:

$$
\begin{aligned}
&\Big|\nabla f(x) - \frac{1}{2} \nabla P_T f(x)\Big|^2 \\
&\le \mathbb{E}\left[(1 + \mu_{x,T}([0,T]))\Big(\Big|\nabla f(x) - \frac{1}{2} U^x_0 (U^x_T)^{-1} \nabla f(X^x_T)\Big|^2 + \frac{\mu_{x,T}([0,T])}{4} |\nabla f(X^x_T)|^2\Big)\right].
\end{aligned}
$$

Then

$$\frac{|\nabla P_T f(x)|^2 - P_T|\nabla f(x)|^2}{4T} \le \frac{1}{T}\mathbb{E}\Big\{\big\langle \nabla f(x), \nabla P_T f(x) - \mathbb{E}[U_0^x (U_T^x)^{-1}\nabla f(X_T^x)\big\rangle$$

X2 (2.13)
$$+ \mu_{x,T}([0,T])\Big|\nabla f(x) - \frac{1}{2}U_0^x (U_T^x)^{-1}\nabla f(X_T^x)\Big|^2$$

$$+ \frac{(1+\mu_{x,T}([0,T]))\mu_{x,T}([0,T])}{4}|\nabla f(X_T^x)|^2\Big\}.$$

Combining this with (2.1) and (2.11), we arrive at

$$\begin{aligned}&-\frac{1}{2}\mathrm{Ric}_Z(\nabla f, \nabla f)(x)\\ &\le \frac{1}{2}K(x)|\nabla f(x)|^2 + \limsup_{T\to 0}\frac{1}{T}\mathbb{E}\Big\langle \nabla f(x), \nabla P_T f(x) - \mathbb{E}[U_0^x (U_T^x)^{-1}\nabla f(X_T^x)]\Big\rangle.\end{aligned}$$

Since by (2.5), (2.4) and (2.10) we have

$$\begin{aligned}&\big\langle \nabla f(x), \nabla P_T f(x) - \mathbb{E}[U_0^x (U_T^x)^{-1}\nabla f(X_T^x)]\big\rangle\\ &= -\int_0^T \big\langle \nabla f(x), U_0^x \mathrm{Ric}_Z(U_r^x)(U_T^x)^{-1}\nabla f(X_T^x)\big\rangle \mathrm{d}r = -T\mathrm{Ric}_Z(\nabla f, \nabla f)(x) + \mathrm{o}(T)\end{aligned}$$

for small $T > 0$, this implies $\mathrm{Ric}_Z(\nabla f, \nabla f)(x) \le K(x)$.

On the other hand, to prove the desired bound on $\|\mathbb{I}\|$, we let $x \in \partial M$, $f \in C_0^\infty(M)$ with $\langle \nabla f, N\rangle(x) = 0, |\nabla f(x)| = 1$ and $\mathrm{Hess}_f(x) = 0$. By [18, Lemma 3.1.2],

$$\mathbb{E}\mathrm{e}^{\lambda l^x_{T\wedge\tau_1}} < \infty, \quad \mathbb{E}l^x_{T\wedge\tau_1} = \frac{2\sqrt{T}}{\sqrt{\pi}} + \mathrm{O}(T^{3/2})$$

for all $\lambda > 0$ and small $T > 0$. Combining this with (1.3), (1.4), and (2.9), we obtain

LOC (2.14)
$$\lim_{T\to 0}\frac{\mathbb{E}\mu_{x,T}([0,T])}{\sqrt{T}} = \frac{2\sigma(x)}{\sqrt{\pi}}, \quad \lim_{T\to 0}\frac{[\mathbb{E}\mu_{x,T}([0,T])]^2}{\sqrt{T}} = 0.$$

Then repeating the above argument with (2.2) replacing (2.1), we prove

$$|\mathbb{I}(\nabla f, \nabla f)(x)| \le \sigma(x).$$

Indeed, by (2.2) and (2.14), instead of (2.12) we have

$$-\mathbb{I}(\nabla f, \nabla f)(x) \le \frac{\sqrt{\pi}}{2}\lim_{T\to\infty}\frac{|\nabla P_T f|^p(x) - P_T|\nabla f|^p(x)}{p\sqrt{T}} = \sigma(x),$$

while multiplying (2.13) by $\sqrt{T}$ and letting $T \to \infty$ leads to

$$-\frac{1}{\sqrt{\pi}}\mathbb{I}(\nabla f, \nabla f)(x) \le \frac{\sigma(x)}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}\mathbb{I}(\nabla f, \nabla f)(x).$$

(d) (5) $\Rightarrow$ (1). Let $F(\gamma) = f(\gamma_T)$. Then (5) implies

55 (2.15)
$$P_T f^2(x) - (P_T f(x))^2 \le 2\int_0^T \mathbb{E}\big[(1+\mu_{x,T}([s,T]))^2 |\nabla f(X_T^x)|^2\big] \mathrm{d}s. \tag{2.15}$$

For f in (2.1), combining this with (2.1) and (2.11) we obtain

$$\begin{aligned}
\mathrm{Ric}_Z(\nabla f, \nabla f)(x) &= \lim_{T\to 0} \frac{1}{T}\bigg(\frac{P_T f^2(x) - (P_T f)^2(x)}{2T} - |\nabla P_T f|^2\bigg)\\
&\le \lim_{T\to 0} \frac{1}{T}\bigg\{\frac{1}{T}\int_0^T \Big\{\mathbb{E}\big[(1+\mu([s,T]))^2|\nabla f(X_T^x)|^2\big] - |\nabla P_T f(x)|^2\Big)\mathrm{d}s\bigg\}\\
&= \lim_{T\to 0} \frac{1}{T}\bigg\{P_T|\nabla f|^2(x) - |\nabla P_T f|^2(x) + \frac{2|\nabla f|^2(x)}{T}\int_0^T (T-s)K(x)\mathrm{d}s\bigg\}\\
&= 2\mathrm{Ric}_Z(\nabla f, \nabla f)(x) + K(x)|\nabla f|^2(x).
\end{aligned}$$

This implies $\mathrm{Ric}_Z(\nabla f, \nabla f)(x) \ge -K(x)|\nabla f(x)|^2$. Next, for f in (2.2), combining (2.15) with (2.2) and (2.14), we obtain

$$\begin{aligned}
&\mathbb{I}(\nabla f, \nabla f)(x) = \lim_{T\to 0} \frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{P_T f^2(x) - (P_T f)^2(x)}{2T} - |\nabla P_T f(x)|^2\bigg)\\
&\le \lim_{T\to 0} \frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg\{\frac{1}{T}\int_0^T \Big\{\mathbb{E}\big[(1+\mu([s,T]))^2|\nabla f(X_T^x)|^2\big] - |\nabla P_T f(x)|^2\Big)\mathrm{d}s\bigg\}\\
&= \lim_{T\to 0} \frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg\{P_T|\nabla f|^2(x) - |\nabla P_T f|^2(x) + \frac{2|\nabla f(x)|^2}{T}\int_0^T \frac{2\sigma(x)(\sqrt{T}-\sqrt{s})}{\sqrt{\pi}}\mathrm{d}s + \mathrm{o}\big(\sqrt{T}\big)\bigg\}\\
&= \frac{3}{2}\mathbb{I}(\nabla f, \nabla f)(x) + \frac{1}{2}\sigma(x).
\end{aligned}$$

Hence, $\mathbb{I}(\nabla f, \nabla f)(x) \ge -\sigma(x)|\nabla f(x)|^2$.

On the other hand, to prove the upper bound estimates, we take $F(\gamma) = f(\gamma_\varepsilon) - \frac{1}{2}f(\gamma_T)$ for $\varepsilon \in (0,T)$. By (1.2),

$$|\dot{D}_t F| = \Big|\nabla f(X_\varepsilon) - \frac{1}{2}U_\varepsilon^x (U_T^x)^{-1}\nabla f(X_T^x)\Big| 1_{[0,\varepsilon)}(t) + \frac{1}{2}|\nabla f(X_T^x)| 1_{[\varepsilon,T]}(t).$$

Then (5) implies

56 (2.16)
$$\begin{aligned}
I_\varepsilon &:= \mathbb{E}\Big[f(X_\varepsilon^x) - \frac{1}{2}\mathbb{E}(f(X_T^x)|\mathscr{F}_\varepsilon)\Big]^2 - \Big(P_\varepsilon f(x) - \frac{1}{2}P_T f(x)\Big)^2\\
&\le 2\varepsilon \mathbb{E}\bigg\{(1+\mu_{x,T}([0,T]))\bigg(\Big|\nabla f(X_\varepsilon^x) - \frac{1}{2}U_\varepsilon^x (U_T^x)^{-1}\nabla f(X_T^x)\Big|^2\\
&\qquad + \frac{\mu_{x,T}([0,T])|\nabla f(X_T^x)|^2}{4}\bigg\} + c\varepsilon^2 =: J_\varepsilon, \quad \varepsilon \in (0,T)
\end{aligned} \tag{2.16}$$

for some constant $c > 0$. Obviously,

57 (2.17)
$$\lim_{\varepsilon\to0}\frac{J_\varepsilon}{\varepsilon}=\mathbb{E}\bigg\{(1+\mu_{x,T}([0,T]))\Big(\Big|\nabla f(x)-\frac{1}{2}U_0^x(U_T^x)^{-1}\nabla f(X_T^x)\Big|^2 + \frac{\mu_{x,T}([0,T])}{4}|\nabla f|^2(X_T^x)\Big)\bigg\}.$$

On the other hand, we have

IV (2.18)
$$\frac{I_\varepsilon}{\varepsilon}=\frac{P_\varepsilon f^2-(P_\varepsilon f)^2}{\varepsilon}+\frac{1}{4\varepsilon}\mathbb{E}\Big[\big\{\mathbb{E}\big(f(X_T^x)|\mathscr{F}_\varepsilon\big)\big\}^2-(P_Tf)^2(x)\Big] + \frac{\mathbb{E}[f(X_T^x)\{P_\varepsilon f(x)-f(X_\varepsilon^x)\}]}{\varepsilon}.$$

Let $f\in C_0^\infty(M)$ satisfy the Neumann boundary condition, we have

IV2 (2.19)
$$\lim_{\varepsilon\to0}\frac{P_\varepsilon f^2-(P_\varepsilon f)^2}{\varepsilon}=2|\nabla f|^2(x).$$

Next, (2.6) and (2.7) yield

*0 (2.20)
$$\mathbb{E}(f(X_T^x)|\mathscr{F}_\varepsilon)=P_Tf(x)+\sqrt{2}\int_0^\varepsilon\big\langle\mathbb{E}\big(Q_{s,T}^x(U_T^x)^{-1}\nabla f(X_T^x)\big|\mathscr{F}_s\big),\mathrm{d}W_s\big\rangle.$$

Then
$$\mathbb{E}[\mathbb{E}(f(X_T^x)|\mathscr{F}_\varepsilon)]^2=(P_Tf)^2+2\int_0^\varepsilon\mathbb{E}|Q_{0,T}^x(U_T^x)^{-1}\nabla f(X_T^x)|^2\mathrm{d}s.$$

This together with (2.5) leads to

IV3 (2.21)
$$\lim_{\varepsilon\to0}\frac{1}{4\varepsilon}\mathbb{E}\Big[\big\{\mathbb{E}\big(f(X_T^x)|\mathscr{F}_\varepsilon\big)\big\}^2-(P_Tf)^2(x)\Big] = \frac{1}{2}\Big|\mathbb{E}\big[Q_{0,T}^x(U_T^x)^{-1}\nabla f(X_T^x)\big]\Big|^2=\frac{1}{2}|\nabla P_Tf(x)|^2.$$

Finally, by Itô's formula we have

$$\begin{aligned}P_\varepsilon f(x)-f(X_\varepsilon^x)&=P_\varepsilon f(x)-f(x)-\int_0^\varepsilon Lf(X_s^x)\mathrm{d}s-\sqrt{2}\int_0^\varepsilon\langle\nabla f(X_s^x),U_s^x\mathrm{d}W_s\rangle\\&=\mathrm{o}(\varepsilon)-\sqrt{2}\int_0^\varepsilon\langle\nabla f(X_s^x),U_s^x\mathrm{d}W_s\rangle.\end{aligned}$$

Combining this with (2.20) and (2.5), we arrive at

$$\lim_{\varepsilon\to0}\frac{\mathbb{E}[f(X_T^x)\{P_\varepsilon f(x)-f(X_\varepsilon^x)\}]}{\varepsilon}=-2\langle\nabla f(x),\nabla P_tf(x)\rangle.$$

Substituting this and (2.19)-(2.21) into (2.18), we obtain

$$\lim_{\varepsilon\to 0}\frac{I_\varepsilon}{\varepsilon} = 2\Big|\nabla f(x) - \frac{1}{2}\nabla P_T f(x)\Big|^2.$$

Combining this with (2.16) and (2.17), we prove the second inequality in (2) for $q = 2$, which implies $\mathrm{Ric}_Z \le K$ and $\mathbb{I} \le \sigma$ as shown in step **(c)**.

(e) (1) $\Rightarrow$ (4). According to (2.7),

eq2.27

$$G_t := \mathbb{E}(F^2|\mathscr{F}_t) = \mathbb{E}(F^2) + \sqrt{2}\int_0^t \big\langle \mathbb{E}(\tilde{D}_s F^2|\mathscr{F}_s), \mathrm{d}W_s\big\rangle, \quad t\in[0,T]. \tag{2.22}$$

By Itô's formula,

eq2.28

$$\begin{aligned} \mathrm{d}(G_t\log G_t) &= (1+\log G_t)\mathrm{d}G_t + \frac{|\mathbb{E}(\tilde{D}_s F^2|\mathscr{F}_s)|^2}{G_t}\mathrm{d}t \\ &\le (1+\log G_t)\mathrm{d}G_t + 4\mathbb{E}(|\tilde{D}_s F|^2|\mathscr{F}_s)\mathrm{d}t. \end{aligned} \tag{2.23}$$

Then

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$$\mathbb{E}[G_{t_1}\log G_{t_1}] - \mathbb{E}[G_{t_0}\log G_{t_0}] \le 4\int_{t_0}^{t_1}\mathbb{E}|\tilde{D}_s F|^2\mathrm{d}s. \tag{2.24}$$

By (2.6) we have

$$\begin{aligned} \tilde{D}_s F &= \sum_{i=1}^N 1_{\{s<t_i\}} Q^x_{s,t_i}(U^x_{t_i})^{-1}\nabla_i f \\ &= \sum_{i=1}^N 1_{\{s<t_i\}}\bigg(I - \int_s^{t_i} Q^x_{s,t}\big\{\mathrm{Ric}_V(U^x_t)\mathrm{d}t + \mathbb{I}_{U^x_t}\mathrm{d}l^x_t\big\}\bigg)\Big(I - 1_{\{X^x_{t_i}\in\partial M\}}P_{U^x_{t_i}}\Big)(U^x_{t_i})^{-1}\nabla_i f \\ &= \dot{D}_0 F - \int_s^T Q^x_{s,t}\big\{\mathrm{Ric}_Z(U^x_t)\mathrm{d}t + \mathbb{I}(U^x_t)\mathrm{d}l^x_t\big\}. \end{aligned}$$

Combining this with (1), (2.8) and (2.11), and using the Schwarz inequality, we prove

eq2.25

$$|\tilde{D}_s F|^2 \le (1+\mu_{x,T}([s,T]))\bigg(|\dot{D}_0 F|^2 + \int_s^T |\dot{D}_s F|^2\mu_{x,T}(\mathrm{d}s)\bigg). \tag{2.25}$$

This together with (2.24) implies the log-Sobolev inequality in (4).

□

3 Extension of Theorem 1.1

In this section, we aim to drop the condition (1.4) in Theorem 1.1 and allow the (reflecting) diffusion process generated by L to be explosive. The idea is to make a conformal change of metric such that the condition (1.4) holds on the new Riemannian manifold. Since both Ric_Z and $\mathbb{I}$ are local quantity, they doe not change at x if the new metric coincides with the original one around point x.

Let (M, g) be a Riemannian manifold with boundary, and let N be the inward pointing unit normal vector field of ∂M. Let $\phi \in C_0^\infty(M)$ be non-negative with non-empty $M_\phi := \{\phi > 0\}$. Then, M_ϕ is a complete Riemannian manifold under the metric $g_\phi := \phi^{-2} g$. Let $\nabla^\phi, \Delta^\phi, \mathrm{Ric}^\phi$ and $\mathbb{I}^\phi$ be the associated Laplacian, gradient, Ricci curvature and the second fundamental form of ∂M_ϕ. By e.g. [6, Theorem 1.159 d)],

$$\nabla_X^\phi Y = \nabla_X Y - \langle X, \nabla \log \phi \rangle Y - \langle Y, \nabla \log \phi \rangle X + \langle X, Y \rangle \nabla \log \phi.$$

Moreover, according to [18, Theorem 1.2.4] and the proof of [18, Theorem 1.2.5], we have

$$\begin{aligned} &\mathrm{Ric}_\phi = \mathrm{Ric} + (d-2)\phi^{-1}\mathrm{Hess}_\phi + (\phi^{-1}\Delta\phi - (d-3)|\nabla \log \phi|)g, \\ &\mathbb{I}^\phi = \phi^{-1}\mathbb{I} + (N \log \phi) g. \end{aligned}$$

Noting that $|X| = 1$ if and only if $g_\phi(\phi X, \phi X) = 1$, we obtain

$$\|\mathbb{I}_g\|_\infty = \sup_{X \in T\partial M_\phi, |X|=1} |\mathbb{I}_\phi(\phi X, \phi X)| < \infty,$$

and for $\mathrm{Ric}_{\phi Z}^\phi$ the curvature of $L^\phi := \Delta^\phi + \phi Z$,

$$\|\mathrm{Ric}_{\phi Z}^\phi\|_\infty = \sup_{X \in TM_\phi, |X|=1} |\mathrm{Ric}^\phi(\phi X, \phi X) - g_\phi(\nabla_{\phi X}(\phi Z), \phi X)| < \infty.$$

Therefore, Theorem 1.1 applies to L^ϕ on the manifold M_ϕ. In particular, by taking ϕ such that $\phi = 1$ around a point x, we have $\mathrm{Ric}_Z = \mathrm{Ric}^\phi$ and $\mathbb{I} = \mathbb{I}^\phi$ at point x, so that in this way we characterize these two quantities at x. To this end, we will take $\phi = \ell(\rho_x)$, where ρ_x is the Riemannian distance to x and $\ell \in C_0^\infty(\mathbb{R})$ is such that $0 \le \ell \le 1$, $\ell(s) = 1$ for $s \le r$ and $\ell(s) = 0$ for $s \ge 2r$ for some constant $r > 0$ with compact $B_{2r}(x) := \{\rho_x \le 2r\}$.

Obviously, before exiting the ball $B_r(x)$ the diffusion process generated by L coincides with that generated by L^ϕ. So, to use the original diffusion process in place of the new one, we will take references functions which vanishes as soon as the diffusion exits this ball. To this end, we will make truncation of cylindrical functions in terms of the uniform distance

$$\tilde{\rho_x}(\gamma) := \sup_{t \in [0,1]} \rho(\gamma(t), x).$$

To make the manifold M_ϕ complete, let $\delta : M \to (0,\infty)$ be a smooth function such that $B_R(x)$ is compact for any $R \le \delta_x$. Consider the class of truncated cylindrical functions

e1 (3.1) $$\mathscr{F}C^\infty_{T,loc} := \Big\{F\ell(\tilde\rho_x) : F \in \mathscr{F}C^\infty_T,\ x \in M,\ \ell \in C_0^\infty(\mathbb{R}),\ \mathrm{supp}\ell \subset [0,\delta_x)\Big\}.$$

To define $\mathscr{E}^{K,\sigma}_{t,T}(\tilde F,\tilde F)$ for $\tilde F = F\ell(\tilde\rho_x) \in \mathscr{F}C^\infty_{T,loc}$, we take $\phi \in C_0^\infty(M)$ such that $0 \le \phi \le 1$, $\phi = 1$ for $\ell(\rho_x) > 0$, and $\phi = 0$ for $\rho_x \ge \delta_x$. Then M_ϕ is complete with bounded $\mathrm{Ric}^\phi_{\phi Z}$ and $\mathbb{I}^\phi$. Let $X^{x,\phi}_{[0,T]}$ be the (reflecting) diffusion process generated by L^ϕ. Similarly to the proof of [4, Lemma 2.1] for the case without boundary, we see that $|\dot D_s\tilde F(X^{x,\phi}_{[0,T]})|$ is well defined and bounded for $s \in [0,T]$. Noting that $\tilde F$ is supported on $\{\ell(\tilde\rho_x) > 0\} \subset W_T(M^\phi)$ and $X^{x,\phi}_{[0,T]} = X^x_{[0,T]}$ if $\ell(\tilde\rho_x(X^{x,\phi}_{[0,T]})) > 0$ (see (3.4) below), we conclude that $|\dot D_s\tilde F(X^x_{[0,T]})| = |\dot D_s\tilde F(X^{x,\phi}_{[0,T]})|$ is well defined and bounded in $s \in [0,T]$ as well, which does not depend on the choice of ϕ. Again since $\tilde F$ is supported on $\{\ell(\tilde\rho_x) > 0\} \subset W_T(M^\phi)$ and M^ϕ is relatively compact in M, we have

$$\mathscr{E}^{K,\sigma}_{t,T}(\tilde F,\tilde F) := \mathbb{E}\bigg\{\big(1+\mu_{x,T}([t,T])\big)\bigg(|\dot D_t\tilde F(X^x_{[0,T]})|^2+\int_t^T |\dot D_s\tilde F(X^x_{[0,T]})|^2\mu_{x,T}(\mathrm{d}s)\bigg)\bigg\} < \infty.$$

T3.1 **Theorem 3.1.** *Let $K \in C(M;[0,\infty))$ and $\sigma \in C(\partial M;[0,\infty))$. The following statements are equivalent each other:*

(1) *For any $x \in M$ and $y \in \partial M$,*

$$\|\mathrm{Ric}_Z\|(x) := \sup_{X\in T_xM,|X|=1} |\mathrm{Ric}(X,X) - \langle\nabla_X Z, X\rangle|(x) \le K(x),$$
$$\|\mathbb{I}\|(y) := \sup_{Y\in T_y\partial M,|Y|=1} |\mathbb{I}(Y,Y)|(y) \le \sigma(y).$$

(2) *For any $t_0,t_1 \in [0,T]$ with $t_1 > t_0$, and any $x \in M$, the following log-Sobolev inequality holds:*

$$\mathbb{E}\left[\mathbb{E}\big(F^2(X^x_{[0,T]})|\mathscr{F}_{t_1}\big)\log\mathbb{E}(F^2(X^x_{[0,T]})|\mathscr{F}_{t_1})\right]$$
$$-\,\mathbb{E}\left[\mathbb{E}\big(F^2(X^x_{[0,T]})|\mathscr{F}_{t_0}\big)\log\mathbb{E}(F^2(X^x_{[0,T]})|\mathscr{F}_{t_0})\right] \le 4\int_{t_0}^{t_1} \mathscr{E}^{K,\sigma}_{s,T}(F,F)\mathrm{d}s,\quad F \in \mathscr{F}C^\infty_{T,loc}.$$

(3) *For any $t \in [0,T]$ and $x \in M$, the following Poincaré inequality holds:*

$$\mathbb{E}\Big[\big\{\mathbb{E}(F(X^x_{[0,T]})|\mathscr{F}_t)\big\}^2\Big] - \Big\{\mathbb{E}\big[F(X_{[0,T]})\big]\Big\}^2 \le 2\int_0^t \mathscr{E}^{K,\sigma}_{s,T}(F,F)\mathrm{d}s,\quad F \in \mathscr{F}C^\infty_{T,loc}.$$

Proof. Since (2) $\Rightarrow$ (3) is well known, we only prove (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(a) (1) $\Rightarrow$ (2). Fix $x \in M$. For any $\tilde{F} := F\ell(\tilde{\rho}_x) \in \mathscr{F}C^\infty_{T,loc}$, there exists $R \in (0, \delta_x)$ such that $\mathrm{supp}(\ell(\tilde{\rho}_x)) \subset B_R(x) := \{y \in M : \rho(x,y) \le R\}$. Let $\phi_R \in C_0^\infty(M)$ such that $\phi_R|_{B_R(x)} = 1$ and $0 \le \phi_R \le 1$. We consider the following Riemannian metric on the manifold $M_R := \{y \in M : \ \phi_R(y) > 0\}$:

$$g_R := \phi_R^{-2} g.$$

As explained above that (M_R, g_R) is a complete Riemannian manifold with

eq3.1
$$K_R := \sup_{M_R} \|\mathrm{Ric}_Z^R\|_\infty < \infty, \quad \sigma_R := \sup_{M_R} \|\mathbb{I}^R\|_\infty < \infty. \tag{3.2}$$

We consider the SDE (1.1) on M,

eq3.2
$$\begin{cases} \mathrm{d}U_t^x = \sqrt{2}\, H_{U_t^x}(U_t^x) \circ \mathrm{d}W_t + H_Z(U_t^x)\mathrm{d}t + H_N(U_t^x)\mathrm{d}l_t^x, \\ U_0 = u_0. \end{cases} \tag{3.3}$$

Then $X_t := \pi(U_t)$ is the (reflecting if ∂M exists) diffusion process on M generated by $L = \Delta + Z$.

Similarly, let $\{H_{i,R}\}_{i=1}^n$ and $H_{\phi_R Z,R}$ be the orthonormal basis of horizontal vector fields and horizontal lift of $\phi_R Z$ under the metric g_R. Since $g_R = g$ and $\phi_R = 1$ on $B_R(x)$, for $u \in O(M_R)$ with $\pi u \in B_R(x)$ we have $H_{i,R}(u) = H_i(u)$ and $H_{\phi Z,R}(u) = H_Z(u)$. For W_t and u_0 in (3.3), we consider the following SDE on the manifold M_R:

$$\begin{cases} \mathrm{d}U_{t,R} = \displaystyle\sum_{i=1}^n H_{i,R}(U_{t,R}) \circ \mathrm{d}W_t^i + H_{\phi_R Z,R}(U_t^x)\mathrm{d}t + H_N(U_t^x)\mathrm{d}l_{R,t}^x, \\ U_{0,R} = u_0. \end{cases}$$

Then $X_\cdot^{x,R} := \pi(U_{\cdot,R})$ is the (reflecting if ∂M_R exists) diffusion process on M_R generated by $L_R := \Delta_R + \phi_R Z$, where Δ_R is the Laplacian on M_R. Obviously,

eq3.3
$$U_{t,R} = U_t, \ l_{R,t}^x = l_t^x \ \text{ for} t \le \tau_R := \inf\{t \ge 0 : X_t \notin B_R(x)\}. \tag{3.4}$$

Denote by $\mathbb{P}_{R,x}^T$ the distribution of the process $X_{[0,T]}^{x,R}$. By [18] and (2.24), we have the damped logarithmic Sobolev inequality holds

eq3.4
$$\mathbb{E}[G_{t_1} \log G_{t_1}] - \mathbb{E}[G_{t_0} \log G_{t_0}] \le 4\tilde{\mathscr{E}}_R^{t_1,t_0}(G,G), \quad G \in \mathscr{F}C_T^\infty, \tag{3.5}$$

where $G_t := \mathbb{E}(G^2(X_{[0,T]}^{x,R})|\mathscr{F}_t)$ and

$$\tilde{\mathscr{E}}_R^{t_1,t_0}(H,G) = \int_{W_x^T(M_R)} \int_{t_0}^{t_1} \langle \tilde{D}_s^R F, \tilde{D}_s^R G \rangle \mathrm{d}s \mathrm{d}\mathbb{P}_{R,x}^T.$$

According to [18], the form $(\tilde{\mathscr{E}}_R^{t_1,t_0}, \mathscr{F}C_T^\infty)$ is closable in $L^2(\mathbb{P}_{R,x}^T)$. Let $(\tilde{\mathscr{E}}_R^{t_1,t_0}, \mathscr{D}(\tilde{\mathscr{E}}_R^{t_1,t_0}))$ be its closure. Let ρ^R be the Riemannian distance on M_R and

$$\tilde{\rho_x}^R(\gamma) := \sup_{t\in[0,1]} \rho^R(\gamma(t), x), \quad \gamma \in W_x^T(M_R).$$

We have $\tilde{\rho_x}^R(\gamma) = \tilde{\rho_x}(\gamma)$ for each $\gamma \in W_x^T(M_R) \subseteq W_x^T(M)$ satisfying $\rho_x^R(\gamma) \le R$. Then [4, Lemma 2.1] implies that $\ell(\tilde{\rho}_x)$ is in $\mathscr{D}(\tilde{\mathscr{E}}_{\mathbb{P}_{R,x}^T})$, and so is $\tilde{F} := F\ell(\tilde{\rho}_x)$. Combining this with (3.4) and (3.5), we get

eq3.6

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{E}\big(\tilde{F}^2(X_{[0,T]}^x)|\mathscr{F}_{t_1}\big)\log\mathbb{E}(\tilde{F}^2(X_{[0,T]}^x)|\mathscr{F}_{t_1})\Big]\\
&\quad - \mathbb{E}\Big[\mathbb{E}\big(\tilde{F}^2(X_{[0,T]}^x)|\mathscr{F}_{t_0}\big)\log\mathbb{E}(\tilde{F}^2(X_{[0,T]}^x)|\mathscr{F}_{t_0})\Big]\\
&= \mathbb{E}\Big[\mathbb{E}\big(\tilde{F}^2(X_{[0,T]}^{x,R})|\mathscr{F}_{t_1}\big)\log\mathbb{E}(\tilde{F}^2(X_{[0,T]}^{x,R})|\mathscr{F}_{t_1})\Big]\\
&\quad - \mathbb{E}\Big[\mathbb{E}\big(\tilde{F}^2(X_{[0,T]}^{x,R})|\mathscr{F}_{t_0}\big)\log\mathbb{E}(\tilde{F}^2(X_{[0,T]}^{x,R})|\mathscr{F}_{t_0})\Big]\\
&\le 4\int_{W_x^T(M_R)}\int_{t_0}^{t_1}\langle \tilde{D}_s^R\tilde{F}, \tilde{D}_s^R\tilde{F}\rangle \mathrm{d}s\mathrm{d}\mathbb{P}_{R,x}^T = 4\int_{W_x^T(M)}\int_{t_0}^{t_1}\langle \tilde{D}_s\tilde{F}, \tilde{D}_s\tilde{F}\rangle \mathrm{d}s\mathrm{d}\mathbb{P}_x^T.
\end{aligned} \tag{3.6}$$

Combining this with (2.25), we prove (2).

(a) (3) $\Rightarrow$ (1). We first prove the lower bound estimates. When $x \in M \setminus \partial M$, there exists $r \in (0, \frac{1}{2}\delta_x)$ such that $B_{2r}(x) \subset M \setminus \partial M$. Let $\Phi = \ell(\tilde{\rho}_x)$, where $\ell \in C_0^\infty(\mathbb{R})$ such that $0 \le \ell \le 1$, $\ell(s) = 1$ for $s \le r$ and $\ell(s) = 0$ for $s \ge 2r$. Let $\tau_s = \inf\{t \ge 0 : \rho(x, X_t^x) \ge s\}$ for $s > 0$. Consider $\tilde{F}(\gamma) = (\Phi F)(\gamma) = \Phi(\gamma) f(\gamma_T)$ for f in (2.1). Then (3) and (2.9) imply

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$$\begin{aligned}
&\mathbb{E}\Big[(F\Phi)^2(X_{[0,T]}^x)\Big] - \Big\{\mathbb{E}\big[(F\Phi)(X_{[0,T]})\big]\Big\}^2 \le 2\int_0^T \mathscr{E}_{t,T}^{K,\sigma}(\tilde{F}, \tilde{F})\mathrm{d}t\\
&= 2\int_0^T \mathbb{E}\Big\{\big(1+\mu_{x,T}([t,T])\big)\Big(|\dot{D}_t\tilde{F}(X_{[0,T]}^x)|^2 + \int_t^T |\dot{D}_s\tilde{F}(X_{[0,T]}^x)|^2\mu_{x,T}(\mathrm{d}s)\Big)\Big\}\mathrm{d}t\\
&\le 2\int_0^T \mathbb{E}\Big[1_{\{\tau_{2r}>T\}}\big(1+\mu_{x,T}([t,T])\big)^2|\nabla f(X_T^x)|^2\Big]\mathrm{d}t + C\mathbb{P}(\tau_r \le T)\\
&= 2\int_0^T \mathbb{E}\Big[1_{\{\tau_{2r}>T\}}\big(1+\mu_{x,T}([t,T])\big)^2|\nabla f(X_T^x)|^2\Big]\mathrm{d}t + \mathrm{o}(T^3),
\end{aligned} \tag{3.7}$$

where $C > 0$ is a constant depending on f and Φ. On the other hand, by (2.1) and

(2.9), we have

$$\begin{aligned}
&\lim_{T\to 0}\frac{1}{T}\bigg(\frac{\mathbb{E}[F^2\Phi^2(X^x_{[0,T]})]-\big\{\mathbb{E}\big[F\Phi(X_{[0,T]})\big]\big\}^2}{2T}-|\nabla P_T f|^2\bigg)\\
&=\lim_{T\to 0}\frac{1}{T}\bigg(\frac{P_T f^2(x)-(P_T f)^2(x)}{2T}-|\nabla P_T f|^2\bigg)\\
&=\mathrm{Ric}_Z(\nabla f,\nabla f)(x).
\end{aligned}$$

Since $l^x_s=0$ for $s\le\tau_{2r}$, these two estimates together with (2.9) and (1.3) lead to

$$\begin{aligned}
&\mathrm{Ric}_Z(\nabla f,\nabla f)(x)=\lim_{T\to 0}\frac{1}{T}\bigg(\frac{\mathbb{E}[(F\Phi)^2(X^x_{[0,T]})]-\big\{\mathbb{E}\big[(F\Phi)(X_{[0,T]})\big]\big\}^2}{2T}-|\nabla P_T f|^2\bigg)\\
&\le\lim_{T\to 0}\frac{1}{T}\bigg\{\frac{1}{T}\int_0^T\Big\{\mathbb{E}\big[1_{\{\tau_{2r}>T\}}(1+\mu([s,T]))^2|\nabla f(X^x_T)|^2\big]-|\nabla P_T f(x)|^2\Big)\mathrm{d}s\bigg\}\\
&\le\lim_{T\to 0}\bigg(\frac{P_T|\nabla f|^2(x)-|\nabla P_T f|^2(x)}{T}+\frac{\int_0^T\mathbb{E}\{1_{\{\tau_{2r}>T\}}[(1+\mu([s,T]))^2-1]|\nabla f(X^x_T)|^2\}\mathrm{d}s}{T^2}\bigg)\\
&=2\mathrm{Ric}_Z(\nabla f,\nabla f)(x)+K(x)|\nabla f|^2(x).
\end{aligned}$$

Therefore, $\mathrm{Ric}_Z(\nabla f,\nabla f)(x)\ge -K(x)|\nabla f(x)|^2$.

Next, let $x\in\partial M$. For f in (2.2), by (2.9) we have

eq3.7 (3.8)
$$\begin{aligned}
&\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{\mathbb{E}[(F\Phi)^2(X^x_{[0,T]})]-\big\{\mathbb{E}\big[(F\Phi)(X_{[0,T]})\big]\big\}^2}{2T}-|\nabla P_T f|^2\bigg)\\
&=\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{P_T f^2(x)-(P_T f)^2(x)}{2T}-|\nabla P_T f|^2\bigg)\\
&=\mathbb{I}(\nabla f,\nabla f)(x).
\end{aligned}$$

Combining this with (3.7) and (2.14), we obtain

$$\begin{aligned}
&\mathbb{I}(\nabla f,\nabla f)(x)=\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\frac{\mathbb{E}[(F\Phi)^2(X^x_{[0,T]})]-\big\{\mathbb{E}\big[(F\Phi)(X_{[0,T]})\big]\big\}^2}{2T}-|\nabla P_T f(x)|^2\bigg)\\
&\le\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg(\int_0^T\frac{\mathbb{E}\big\{1_{\{\tau_{2r}>T\}}\big(1+\mu_{x,T}([t,T])\big)^2|\nabla F(X^x_T)|^2\big\}}{T}\mathrm{d}t-|\nabla P_T f(x)|^2\bigg)\\
&=\lim_{T\to 0}\frac{3\sqrt{\pi}}{8\sqrt{T}}\bigg\{P_T|\nabla f|^2(x)-|\nabla P_T f|^2(x)+\frac{2|\nabla f(x)|^2}{T}\int_0^T\frac{2\sigma(x)(\sqrt{T}-\sqrt{s})}{\sqrt{\pi}}\mathrm{d}s\bigg\}\\
&=\frac{3}{2}\mathbb{I}(\nabla f,\nabla f)(x)+\frac{1}{2}\sigma(x).
\end{aligned}$$

Therefore, $\mathbb{I}(\nabla f,\nabla f)(x)\ge -\sigma(x)|\nabla f(x)|^2$.

To prove the upper bound estimates, we take $F(\gamma) = f(\gamma_\varepsilon) - \frac{1}{2} f(\gamma_T)$ for $\varepsilon \in (0, T)$. By (1.2),

$$|\dot{D}_t F| = \Big|\nabla f(X_\varepsilon) - \frac{1}{2} U_\varepsilon^x (U_T^x)^{-1} \nabla f(X_T^x)\Big| 1_{[0,\varepsilon)}(t) + \frac{1}{2} |\nabla f(X_T^x)| 1_{[\varepsilon, T]}(t).$$

Moreover, by (3) and (2.9), we may find a constant $C > 0$ depending on f and Φ such that for any $\varepsilon, T \in (0, 1)$,

eq3.8

$$\begin{aligned} I_\varepsilon := \mathbb{E}\Big[\mathbb{E}\Big(\Phi(X_{[0,T]}^x) f(X_\varepsilon^x) - \frac{1}{2}\Phi(X_{[0,T]}^x) f(X_T^x)\Big|\mathscr{F}_\varepsilon\Big)\Big]^2 \\ - \Big[\mathbb{E}\Big(\Phi(X_{[0,T]}^x) f(X_\varepsilon^x) - \frac{1}{2}\Phi(X_{[0,T]}^x) f(X_T^x)\Big)\Big]^2 \\ \le 2\int_0^\varepsilon \mathbb{E}\Big\{\big(1 + \mu_{x,T}([t, T])\big) |\Phi(X_{[0,T]}^x) \dot{D}_t F|^2 \\ + \int_t^T |\Phi(X_{[0,T]}^x) \dot{D}_s F|^2 \mu_{x,T}(\mathrm{d}s)\Big)\Big\} \mathrm{d}t + C\varepsilon T^4. \end{aligned} \tag{3.9}$$

Then

eq3.9

$$\begin{aligned} \limsup_{\varepsilon \to 0} \frac{I_\varepsilon}{\varepsilon} \le \mathbb{E}\Big\{\Phi(X_{[0,T]}^x)(1 + \mu_{x,T}([0, T]))\Big(\Big|\nabla f(x) - \frac{1}{2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x)\Big|^2 \\ + \frac{\Phi(X_{[0,T]}^x)\mu_{x,T}([0, T])}{4} |\nabla f|^2(X_T^x)\Big)\Big\} + \mathrm{o}(T^3) \end{aligned} \tag{3.10}$$

for small $T > 0$. On the other hand, according to (d) of proof in Theorem 1.1, we have

eq3.10

$$\begin{aligned} \frac{I_\varepsilon}{\varepsilon} =& \frac{P_\varepsilon f^2 - (P_\varepsilon f)^2}{\varepsilon} + \frac{1}{4\varepsilon}\mathbb{E}\Big[\big\{\mathbb{E}\big(f(X_T^x)|\mathscr{F}_\varepsilon\big)\big\}^2 - (P_T f)^2(x)\Big] \\ &+ \frac{\mathbb{E}[f(X_T^x)\{P_\varepsilon f(x) - f(X_\varepsilon^x)\}]}{\varepsilon} + o(T^3 \\ &= 2\Big|\nabla f(x) - \frac{1}{2}\nabla P_T f(x)\Big|^2 + o(T^3). \end{aligned} \tag{3.11}$$

Combining this with (3.10), we arrive at

eq3.11

$$\begin{aligned} & 2\Big|\nabla f(x) - \frac{1}{2}\nabla P_T f(x)\Big|^2 \\ & \le \mathbb{E}\Big\{\Phi(X_{[0,T]}^x)(1 + \mu_{x,T}([0, T]))\Big(\Big|\nabla f(x) - \frac{1}{2} U_0^x (U_T^x)^{-1} \nabla f(X_T^x)\Big|^2 \\ & + \frac{\Phi(X_{[0,T]}^x)\mu_{x,T}([0, T])}{4} |\nabla f|^2(X_T^x)\Big)\Big\} + o(T^3) \end{aligned} \tag{3.12}$$

With this estimate, we may repeat the last part in the proof of (2) $\Rightarrow$ (1) of Theorem 1.1 to derive the desired upper bound estimates on Ric_Z and $\mathbb{I}$ at point x. $\square$

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