Order Preservation for Path-Distribution Dependent SDEs*

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Abstract

Sufficient and necessary conditions are presented for the order preservation of path-distribution dependent SDEs. Differently from the corresponding study of distribution independent SDEs, to investigate the necessity of order preservation for the present model we need to construct a family of probability spaces in terms of the ordered pair of initial distributions.

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1 Introduction

The order preservation of stochastic processes is a crucial property for one to compare a complicated process with simpler ones, and a result to ensure this property is called "comparison theorem" in the literature. There are two different type order preservations, one is in the distribution (weak) sense and the other is in the pathwise (strong) sense, where the latter implies the former. The weak order preservation has been investigated for diffusion-jump Markov processes in [2, 14] and references within, as well as a class of super

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processes in [12]. There are also plentiful results on the strong order preservation, see, for instance, [1, 3, 6, 8, 9, 10, 11, 15, 16] and references within for comparison theorems on forward/backward SDEs (stochastic differential equations), with jumps and/or with memory. Recently, sufficient and necessary conditions have been derived in [5] for the order preservation of SDEs with memory.

On the other hand, to characterize the non-linear Fokker-Planck equations, path-distribution dependent SDEs have been investigated in [4], see also [13] and references within for distribution-dependent SDEs without memory. The aim of this paper is to present sufficient and necessary conditions of the order preservations for path-distribution dependent SDEs.

Let $r_0 \geq 0$ be a constant and $d \geq 1$ be a natural number. The path space $\mathscr{C} = C([-r_0, 0]; \mathbb{R}^d)$ is Polish under the uniform norm $\|\cdot\|_\infty$. For any continuous map $f : [-r_0, \infty) \to \mathbb{R}^d$ and $t \geq 0$, let $f_t \in \mathscr{C}$ be such that $f_t(\theta) = f(\theta + t)$ for $\theta \in [-r_0, 0]$. We call $(f_t)_{t\geq 0}$ the segment of $(f(t))_{t\geq -r_0}$. Next, let $\mathscr{P}(\mathscr{C})$ be the set of probability measures on $\mathscr{C}$ equipped with the weak topology. Finally, let $W(t)$ be an m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$.

We consider the following Distribution-dependent SDEs with memory:

$$\begin{cases} \mathrm{d}X(t) = b(t, X_t, \mathscr{L}_{X_t})\,\mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\,\mathrm{d}W(t), \\ \mathrm{d}\bar{X}(t) = \bar{b}(t, \bar{X}_t, \mathscr{L}_{\bar{X}_t})\,\mathrm{d}t + \bar{\sigma}(t, \bar{X}_t, \mathscr{L}_{\bar{X}_t})\,\mathrm{d}W(t), \end{cases} \tag{1.1}$$

where

$$b, \bar{b} : [0, \infty) \times \mathscr{C} \times \mathscr{P}(\mathscr{C}) \to \mathbb{R}^d; \quad \sigma, \bar{\sigma} : [0, \infty) \times \mathscr{C} \times \mathscr{P}(\mathscr{C}) \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable.

For any $s \geq 0$ and $\mathscr{F}_s$-measurable $\mathscr{C}$-valued random variables $\xi, \bar{\xi}$, a solution to (1.1) for $t \geq s$ with $(X_s, \bar{X}_s) = (\xi, \bar{\xi})$ is a continuous adapted process $(X(t), \bar{X}(t))_{t\geq s}$ such that for all $t \geq s$,

$$\begin{aligned} X(t) &= \xi(0) + \int_s^t b(r, X_r, \mathscr{L}_{X_r})\mathrm{d}r + \int_s^t \sigma(r, X_r, \mathscr{L}_{X_r})\mathrm{d}W(r), \\ \bar{X}(t) &= \bar{\xi}(0) + \int_s^t \bar{b}(r, \bar{X}_r, \mathscr{L}_{\bar{X}_r})\mathrm{d}r + \int_s^t \bar{\sigma}(r, \bar{X}_r, \mathscr{L}_{\bar{X}_r})\mathrm{d}W(r), \end{aligned}$$

where $(X_t, \bar{X}_t)_{t\geq s}$ is the segment process of $(X(t), \bar{X}(t))_{t\geq s-r_0}$ with $(X_s, \bar{X}_s) = (\xi, \bar{\xi})$.

Following the line of [4], we consider the class of probability measures of finite second moment:

$$\mathscr{P}_2(\mathscr{C}) = \left\{ \nu \in \mathscr{P}(\mathscr{C}) : \nu(\|\cdot\|_\infty^2) := \int_{\mathscr{C}} \|\xi\|_\infty^2 \nu(d\xi) < \infty \right\}.$$

It is a Polish space under the Wasserstein distance

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathbf{C}(\mu_1, \mu_2)} \left(\int_{\mathscr{C}\times\mathscr{C}} \|\xi - \eta\|_\infty^2 \pi(d\xi, d\eta) \right)^{\frac{1}{2}}, \quad \mu_1, \mu_2 \in \mathscr{P}_2(\mathscr{C}),$$

where $\mathbf{C}(\mu_1,\mu_2)$ is the set of all couplings for μ_1 and μ_2; that is, $\pi\in\mathbf{C}(\mu_1,\mu_2)$ if it is a probability measure on $\mathscr{C}^2$ such that

$$\pi(\cdot\times\mathscr{C})=\mu_1,\quad \pi(\mathscr{C}\times\cdot)=\mu_2.$$

To investigate the order preservation, we make the following assumptions.

(H1) (Continuity) There exists an increasing function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ such that for any $t\ge 0;\xi,\eta\in\mathscr{C};\mu,\nu\in\mathscr{P}_2(\mathscr{C})$,

$$\begin{aligned}&|b(t,\xi,\mu)-b(t,\eta,\nu)|^2+|\bar b(t,\xi,\mu)-\bar b(t,\eta,\nu)|^2+\|\sigma(t,\xi,\mu)-\sigma(t,\eta,\nu)\|_{HS}^2\\&+\|\bar\sigma(t,\xi,\mu)-\bar\sigma(t,\eta,\nu)\|_{HS}^2\le\alpha(t)\big(\|\xi-\eta\|_\infty^2+\mathbb{W}_2(\mu,\nu)^2\big).\end{aligned}$$

(H2) (Growth) There exists an increasing function $K:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|b(t,0,\delta_0)|^2+|\bar b(t,0,\delta_0)|^2+\|\sigma(t,0,\delta_0)\|_{HS}^2+\|\bar\sigma(t,0,\delta_0)\|_{HS}^2\le K(t),\quad t\ge 0,$$

where δ_0 is the Dirac measure at point $0\in\mathscr{C}$.

It is easy to see that these two conditions imply assumptions $(H1)$-$(H3)$ in [4], so that by [4, Theorem 3.1], for any $s\ge 0$ and $\mathscr{F}_s$-measurable $\mathscr{C}$-valued random variables $\xi,\bar\xi$ with finite second moment, the equation (1.1) has a unique solution $\{X(s,\xi;t),\bar X(s,\bar\xi;t)\}_{t\ge s}$ with $X_s=\xi$ and $\bar X_s=\bar\xi$. Moreover, the segment process $\{X(s,\xi)_t,\bar X(s,\bar\xi)_t\}_{t\ge s}$ satisfies

$$\mathbb{E}\sup_{t\in[s,T]}\big(\|X(s,\xi)_t\|_\infty^2+\|\bar X(s,\bar\xi)_t\|_\infty^2\big)<\infty,\quad T\in[s,\infty).\tag{1.2}$$

To characterize the order-preservation for solutions of (1.1), we introduce the partial-order on $\mathscr{C}$. For $x=(x^1,\cdots,x^d)$ and $y=(y^1,\cdots,y^d)\in\mathbb{R}^d$, we write $x\le y$ if $x^i\le y^i$ holds for all $1\le i\le d$. Similarly, for $\xi=(\xi^1,\cdots,\xi^d)$ and $\eta=(\eta^1,\cdots,\eta^d)\in\mathscr{C}$, we write $\xi\le\eta$ if $\xi^i(\theta)\le\eta^i(\theta)$ holds for all $\theta\in[-r_0,0]$ and $1\le i\le d$. A function f on $\mathscr{C}$ is called increasing if $f(\xi)\le f(\eta)$ for $\xi\le\eta$. Moreover, for any $\xi_1,\xi_2\in\mathscr{C}$, $\xi_1\wedge\xi_2\in\mathscr{C}$ is defined by

$$(\xi_1\wedge\xi_2)^i:=\min\{\xi_1^i,\xi_2^i\},\quad 1\le i\le d.$$

For two probability measures $\mu,\nu\in\mathscr{P}(\mathscr{C})$, we write $\mu\le\nu$ if $\mu(f)\le\nu(f)$ holds for any increasing function $f\in C_b(\mathscr{C})$. According to [7, Theorem 5], $\mu\le\nu$ if and only if there exists $\pi\in\mathbf{C}(\mu,\nu)$ such that $\pi(\{(\xi,\eta)\in\mathscr{C}^2:\xi\le\eta\})=1$.

Definition 1.1. The stochastic differential system (1.1) is called order-preserving, if for any $s\ge 0$ and $\xi,\bar\xi\in L^2(\Omega\to\mathscr{C},\mathscr{F}_s,\mathbb{P})$ with $\mathbb{P}(\xi\le\bar\xi)=1$,

$$\mathbb{P}\big(X(s,\xi;t)\le\bar X(s,\bar\xi;t),\ t\ge s\big)=1.$$

We first present the following sufficient conditions for the order preservation, which reduce back to the corresponding ones in [5] when the system is distribution-independent.

Theorem 1.1. *Assume* **(H1)** *and* **(H2)***. The system* (1.1) *is order-preserving provided the following two conditions are satisfied:*

(1) *For any* $1 \le i \le d$*,* $\mu, \nu \in \mathscr{P}_2(\mathscr{C})$ *with* $\mu \le \nu$*,* $\xi, \eta \in \mathscr{C}$ *with* $\xi \le \eta$ *and* $\xi^i(0) = \eta^i(0)$*,*

$$b^i(t, \xi, \mu) \le \bar{b}^i(t, \eta, \nu), \quad a.e.\ t \ge 0.$$

(2) *For a.e.* $t \ge 0$ *it holds:* $\sigma(t, \cdot, \cdot) = \bar{\sigma}(t, \cdot, \cdot)$ *and* $\sigma^{ij}(t, \xi, \mu) = \sigma^{ij}(t, \eta, \nu)$ *for any* $1 \le i \le d$*,* $1 \le j \le m$*,* $\mu, \nu \in \mathscr{P}_2(\mathscr{C})$ *and* $\xi, \eta \in \mathscr{C}$ *with* $\xi^i(0) = \eta^i(0)$*.*

Condition (2) means that for a.e. $t \ge 0$, $\sigma(t, \xi, \mu) = \bar{\sigma}(t, \xi, \mu)$ and the dependence of $\sigma^{ij}(t, \xi, \mu)$ on (ξ, μ) is only via $\xi^i(0)$.

On the other hand, the next result shows that these conditions are also necessary if all coefficients are continuous on $[0, \infty) \times \mathscr{C} \times \mathscr{P}_2(\mathscr{C})$, so that [5, Theorem 1.2] is covered when the system is distribution-independent.

Theorem 1.2. *Assume* **(H1)***,* **(H2)** *and that* (1.1) *is order-preserving for any complete filtered probability space* $(\Omega, \{\mathscr{F}_t\}_{t \ge 0}, \mathbb{P})$ *and* m*-dimensional Brownian motion* $W(t)$ *thereon. Then for any* $1 \le i \le d$*,* $\mu, \nu \in \mathscr{P}_2(\mathscr{C})$ *with* $\mu \le \nu$*, and* $\xi, \eta \in \mathscr{C}$ *with* $\xi \le \eta$ *and* $\xi^i(0) = \eta^i(0)$*, the following assertions hold:*

(1′) $b^i(t, \xi, \mu) \le \bar{b}^i(t, \eta, \nu)$ *if* b^i *and* $\bar{b}^i$ *are continuous at points* (t, ξ, μ) *and* (t, η, ν) *respectively.*

(2′) *For any* $1 \le j \le m$*,* $\sigma^{ij}(t, \xi, \mu) = \bar{\sigma}^{ij}(t, \eta, \nu)$ *if* σ^{ij} *and* $\bar{\sigma}^{ij}$ *are continuous at points* (t, ξ, μ) *and* (t, η, ν) *respectively.*

Consequently, when $b, \bar{b}, \sigma$ *and* $\bar{\sigma}$ *are continuous on* $[0, \infty) \times \mathscr{C} \times \mathscr{P}_2(\mathscr{C})$*, conditions* (1) *and* (2) *hold.*

These two theorems will be proved in Section 2 and Section 3 respectively.

2 Proof of Theorem 1.1

Assume **(H1)** and **(H2)**, and let conditions (1) and (2) hold. For any $T > t_0 \ge 0$ and $\xi, \bar{\xi} \in L^2(\Omega \to \mathscr{C}, \mathscr{F}_{t_0}, \mathbb{P})$ with $\mathbb{P}(\xi \le \bar{\xi}) = 1$, it suffices to prove

$$\mathbb{E} \sup_{t \in [t_0, T]} (X^i(t_0, \xi; t) - \bar{X}^i(t_0, \bar{\xi}; t))^+ = 0, \quad 1 \le i \le d, \tag{2.1}$$

where $s^+ := \max\{0, s\}$. For simplicity, in the following we denote $X(t) = X(t_0, \xi; t)$ and $\bar{X}(t) = \bar{X}(t_0, \bar{\xi}; t)$ for $t \geq t_0 - r_0$. Then

$$X(t) = \xi(t - t_0), \quad \bar{X}(t) = \bar{\xi}(t - t_0), \quad t \in [t_0 - r_0, t_0].$$

To prove (2.1) using Itô's formula, we take the following C^2-approximation of s^+ as in the proof of [5, Theorem 1.1]. For any $n \geq 1$, let $\psi_n : \mathbb{R} \to [0, \infty)$ be constructed as follows: $\psi_n(s) = \psi_n'(s) = 0$ for $s \in (-\infty, 0]$, and

$$\psi_n''(s) = \begin{cases} 4n^2 s, & s \in [0, \frac{1}{2n}], \\ -4n^2(s - \frac{1}{n}), & s \in [\frac{1}{2n}, \frac{1}{n}], \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$0 \leq \psi_n' \leq 1_{(0,\infty)}, \text{ and as } n \uparrow \infty: \ 0 \leq \psi_n(s) \uparrow s^+, \quad s\psi_n''(s) \leq 1_{(0,\frac{1}{n})}(s) \downarrow 0. \tag{2.2}$$

Let

$$\tau_k = \inf\big\{t \geq t_0 : |X(t) - X(t) \wedge \bar{X}(t)| \geq k\big\}, \quad k \geq 1.$$

Since

$$\psi_n(X^i(t_0) - \bar{X}^i(t_0)) = \psi_n(\xi^i(0) - \bar{\xi}^i(0)) = 0,$$

and due to (2) $\sigma(t, \cdot, \cdot) = \bar{\sigma}(t, \cdot, \cdot)$ for a.e. $t \geq 0$, by Itô's formula we obtain

$$\begin{aligned} &\psi_n(X^i(t \wedge \tau_k) - \bar{X}^i(t \wedge \tau_k))^2 \\ &= M_i(t \wedge \tau_k) + 2\int_{t_0}^{t\wedge\tau_k} (b^i(s, X_s, \mathscr{L}_{X_s}) - \bar{b}^i(s, \bar{X}_s, \mathscr{L}_{\bar{X}_s}))\{\psi_n\psi_n'\}(X^i(s) - \bar{X}^i(s))\mathrm{d}s \\ &\quad + \sum_{j=1}^m \int_{t_0}^{t\wedge\tau_k} (\sigma^{ij}(s, X_s, \mathscr{L}_{X_s}) - \sigma^{ij}(s, \bar{X}_s, \mathscr{L}_{\bar{X}_s}))^2\{\psi_n\psi_n'' + \psi_n'^2\}(X^i(s) - \bar{X}^i(s))\mathrm{d}s \end{aligned} \tag{2.3}$$

for any $k, n \geq 1$, $1 \leq i \leq d$ and $t \geq t_0$, where

$$M_i(t) := 2\sum_{j=1}^m \int_{t_0}^t (\sigma^{ij}(s, X_s, \mathscr{L}_{X_s}) - \sigma^{ij}(s, \bar{X}_s, \mathscr{L}_{\bar{X}_s}))\{\psi_n\psi_n'\}(X^i(s) - \bar{X}^i(s))\mathrm{d}W^j(s).$$

Noting that $0 \leq \psi_n'(X^i(s) - \bar{X}^i(s)) \leq 1_{\{X^i(s) > \bar{X}^i(s)\}}$ and when $X^i(s) > \bar{X}^i(s)$ one has $(X_s \wedge \bar{X}_s)^i(0) = (\bar{X}_s)^i(0)$, it follows from (1) that for a.e. $s \in [t_0, T]$,

$$[b^i(s, X_s \wedge \bar{X}_s, \mathscr{L}_{X_s \wedge \bar{X}_s}) - \bar{b}^i(s, \bar{X}_s, \mathscr{L}_{\bar{X}_s})]\psi_n\psi_n'(X^i(s) - \bar{X}^i(s)) \leq 0, \quad n \geq 1.$$

Combining this with **(H1)** and $0 \le \psi_n' \le 1$, we obtain

$$
\begin{aligned}
&2\int_{t_0}^{t\wedge\tau_k} [b^i(s,X_s,\mathscr{L}_{X_s}) - \bar b^i(s,\bar X_s,\mathscr{L}_{\bar X_s})]\{\psi_n\psi_n'\}(X^i(s)-\bar X^i(s))\mathrm{d}s\\
&= 2\int_{t_0}^{t\wedge\tau_k} \Big\{[b^i(s,X_s,\mathscr{L}_{X_s}) - b^i(s,X_s\wedge\bar X_s,\mathscr{L}_{X_s\wedge\bar X_s})]\{\psi_n\psi_n'\}(X^i(s)-\bar X^i(s))\\
&\quad + [b^i(s,X_s\wedge\bar X_s,\mathscr{L}_{X_s\wedge\bar X_s}) - \bar b^i(s,\bar X_s,\mathscr{L}_{\bar X_s})]\{\psi_n\psi_n'\}(X^i(s)-\bar X^i(s))\Big\}\mathrm{d}s\\
&\le 2\int_{t_0}^{t\wedge\tau_k} [b^i(s,X_s,\mathscr{L}_{X_s}) - b^i(s,X_s\wedge\bar X_s,\mathscr{L}_{X_s\wedge\bar X_s})]\{\psi_n\psi_n'\}(X^i(s)-\bar X^i(s))\mathrm{d}s\\
&\le \int_{t_0}^{t\wedge\tau_k} [|b^i(s,X_s,\mathscr{L}_{X_s}) - b^i(s,X_s\wedge\bar X_s,\mathscr{L}_{X_s\wedge\bar X_s})|^2 + \psi_n(X^i(s)-\bar X^i(s))^2]\mathrm{d}s\\
&\le \int_{t_0}^{t\wedge\tau_k} \alpha(T)\Big[\|X_s - X_s\wedge\bar X_s\|_\infty^2 + \mathbb{W}_2(\mathscr{L}_{X_{s\wedge\tau_k}},\mathscr{L}_{X_{s\wedge\tau_k}\wedge\bar X_{s\wedge\tau_k}})^2\Big]\mathrm{d}s\\
&\quad + \int_{t_0}^{t\wedge\tau_k} \psi_n(X^i(s)-\bar X^i(s))^2\mathrm{d}s,\quad n\ge 1, t\in[t_0,T].
\end{aligned}
\tag{2.4}
$$

Next, by (2), for a.e. $s\in[t_0,T]$, $\sigma^{ij}(s,X_s,\mathscr{L}_{X_s}) = \bar\sigma^{ij}(s,X_s,\mathscr{L}_{X_s})$ depends only on $X^i(s)$. So, (2.2) and **(H1)** yield

$$
\begin{aligned}
&\sum_{j=1}^m \int_{t_0}^{t\wedge\tau_k} |\sigma^{ij}(s,X_s,\mathscr{L}_{X_s}) - \sigma^{ij}(s,\bar X_s,\mathscr{L}_{\bar X_s})|^2\{\psi_n\psi_n'' + \psi_n'^2\}(X^i(s)-\bar X^i(s))\mathrm{d}s\\
&\le \int_{t_0}^{t\wedge\tau_k} (1_{\{X^i(s)-\bar X^i(s)\in(0,\frac{1}{n})\}} + 1)\alpha(T)\{(X^i(s)-\bar X^i(s))^+\}^2\mathrm{d}s\\
&\le 2\alpha(T)\int_{t_0}^{t\wedge\tau_k} \{(X^i(s)-\bar X^i(s))^+\}^2\mathrm{d}s,\quad n\ge 1, t\in[t_0,T],
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^m \int_{t_0}^{t\wedge\tau_k} \big|\sigma^{ij}(s,X_s,\mathscr{L}_{X_s}) - \sigma^{ij}(s,\bar X_s,\mathscr{L}_{\bar X_s})\big|^2\big|(\psi_n\psi_n')(X^i(s)-\bar X^i(s))\big|^2\mathrm{d}s\\
&\le \alpha(T)\int_{t_0}^{t\wedge\tau_k} \{(X^i(s)-\bar X^i(s))^+\}^2\psi_n(X^i(s)-\bar X^i(s))^2\mathrm{d}s,\quad n\ge 1, t\in[t_0,T].
\end{aligned}
\tag{2.6}
$$

By the Burkholder-Davis-Gundy inequality, we deduce from (2.6) that

$$
\begin{aligned}
\mathbb{E}\sup_{s\in[t_0,t]} M_i(s\wedge\tau_k) \le & C_1\mathbb{E}\bigg(\int_{t_0}^{t\wedge\tau_k}\{(X^i(s)-\bar X^i(s))^+\}^2\psi_n(X^i(s)-\bar X^i(s))^2\mathrm{d}s\bigg)^{\frac12}\\
\le & 2C_1^2\mathbb{E}\int_{t_0}^{t\wedge\tau_k}\|X_s-X_s\wedge\bar X_s\|_\infty^2\mathrm{d}s\\
& +\frac18\mathbb{E}\sup_{s\in[t_0,t\wedge\tau_k]}\psi_n(X^i(s)-\bar X^i(s))^2,\quad n\ge1, t\in[t_0,T]
\end{aligned}\tag{2.7}
$$

holds for some constant $C_1>0$.

Now, let

$$\phi_k(s)=\sup_{r\in[t_0-r_0,s\wedge\tau_k]}|X(r)-X(r)\wedge\bar X(r)|^2,\quad s\in[t_0,T].$$

By substituting (2.4), (2.5) and (2.7) into (2.3) and noting that $X_{t_0}\le\bar X_{t_0}$ and

$$\mathbb{W}_2(\mathscr{L}_{X_{s\wedge\tau_k}},\mathscr{L}_{X_{s\wedge\tau_k}\wedge\bar X_{s\wedge\tau_k}})^2\le\mathbb{E}\|X_{s\wedge\tau_k}-X_{s\wedge\tau_k}\wedge\bar X_{s\wedge\tau_k}\|_\infty^2,$$

we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[t_0-r_0,t\wedge\tau_k]}\psi_n(X^i(r)-\bar X^i(r))^2=\mathbb{E}\sup_{r\in[t_0,t\wedge\tau_k]}\psi_n(X^i(r)-\bar X^i(r))^2\\
&\le C_2\mathbb{E}\int_{t_0}^{t\wedge\tau_k}\|X_{s\wedge\tau_k}-X_{s\wedge\tau_k}\wedge\bar X_{s\wedge\tau_k}\|_\infty^2\mathrm{d}s+\frac18\mathbb{E}\sup_{s\in[t_0,t\wedge\tau_k]}\psi_n(X^i(s)-\bar X^i(s))^2,
\end{aligned}\tag{2.8}
$$

for some constants $C_2>0$ and all $k,n\ge1, t\in[t_0,T], 1\le i\le d$. Therefore, there exists a constant $C>0$ such that for any $n,k\ge1$ and $t\in[t_0,T]$,

$$\sum_{i=1}^d\mathbb{E}\sup_{r\in[t_0-r_0,t\wedge\tau_k]}\psi_n(X^i(r)-\bar X^i(r))^2\le C\int_{t_0}^t\mathbb{E}\phi_k(s)\mathrm{d}s,\quad k,n\ge1.$$

Letting $n\uparrow\infty$, we arrive at

$$\mathbb{E}\phi_k(t)\le C\int_{t_0}^t\mathbb{E}\phi_k(s)\mathrm{d}s,\quad t\in[t_0,T],k\ge1.$$

By the definition of ϕ_k and τ_k, $\mathbb{E}\phi_k(t)$ is locally bounded in $t\ge0$. So, Gronwall's inequality implies

$$\mathbb{E}\phi_k(T)=0,\quad k\ge1.$$

Letting $k\uparrow\infty$ we prove (2.1).

3 Proof of Theorem 1.2

We first observe that when $b, \bar{b}, \sigma, \bar{\sigma}$ are continuous on $[0,\infty) \times \mathscr{C} \times \mathscr{P}_2(\mathscr{C})$, $(1')$ and $(2')$ imply (1) and (2). Obviously, $(1')$ implies (1). It suffices to prove (2).

Firstly, taking $\xi = \eta$ and $\mu = \nu$, by the continuity of σ and $\bar{\sigma}$, $(2')$ implies $\sigma = \bar{\sigma}$.

In general, for $1 \le i \le d$, $\mu, \nu \in \mathscr{P}_2(\mathscr{C})$, and $\xi, \eta \in \mathscr{C}$ with $\xi^i(0) = \eta^i(0)$, we take

$$(\mu \wedge \nu)(\cdot) = (\mu \times \nu)\big(\{(\xi_1, \xi_2) \in \mathscr{C}^2 : \xi_1 \wedge \xi_2 \in \cdot\}\big).$$

Then $\mu \wedge \nu \in \mathscr{P}_2(\mathscr{C})$ and $\mu \wedge \nu \le \mu,\ \mu \wedge \nu \le \nu$. Moreover, $\xi \wedge \eta \le \xi,\ \xi \wedge \eta \le \eta$ with $\xi^i(0) = (\xi \wedge \eta)^i(0) = \eta^i(0)$. So, applying $(2')$ twice we obtain

$$\sigma^{ij}(t, \xi, \mu) = \sigma^{ij}(t, \xi \wedge \eta, \mu \wedge \nu) = \sigma^{ij}(t, \eta, \nu).$$

Since $\sigma = \bar{\sigma}$, this implies (2).

Now, let $t_0 \ge 0$ $1 \le i \le d$, $\mu, \nu \in \mathscr{P}_2(\mathscr{C})$ with $\mu \le \nu$, and $\xi, \eta \in \mathscr{C}$ with $\xi \le \eta$ and $\xi^i(0) = \eta^i(0)$. To prove $(1')$ and $(2')$ for $t = t_0$, we construct a family of complete filtration probability spaces $(\Omega, \{\mathscr{F}_t^\varepsilon\}_{t\ge 0}, \mathbb{P}^\varepsilon)_{\varepsilon\in[0,1)}$, m-dimensional Brownian motion $W(t)$, and initial random variables $X_{t_0} \le \bar{X}_{t_0}$ as follows.

Firstly, since $\mu \le \nu$, by [7, Theorem 5] we may take $\pi_0 \in \mathbf{C}(\mu, \nu)$ such that

$$\pi_0(\{(\xi_1, \xi_2) \in \mathscr{C}^2 : \xi_1 \le \xi_2\}) = 1. \tag{3.1}$$

For any $\varepsilon \in [0, 1)$, let

$$\pi_\varepsilon = (1 - \varepsilon)\pi_0 + \varepsilon \delta_{(\xi,\eta)}, \tag{3.2}$$

where $\delta_{(\xi,\eta)}$ is the Dirac measure at point (ξ, η). Let $\mathbb{P}_0$ be the standard Wiener measure on $\Omega_0 := C([0,\infty) \to \mathbb{R}^m)$, and let $\mathscr{F}_{0,t}$ be the completion of $\sigma(\omega_0 \mapsto \omega_0(s) : s \le t)$ with respect to the Wiener measure. Then the coordinate process $\{W_0(t)\}(\omega_0) := \omega_0(t),\ \omega_0 \in \Omega_0, t \ge 0$ is an m-dimensional Brwonian motion on the filtered probability space $(\Omega_0, \{\mathscr{F}_{0,t}\}_{t\ge 0}, \mathbb{P}_0)$.

Next, for any $\varepsilon \in [0, 1)$, let $\Omega = \Omega_0 \times \mathscr{C}^2,\ \mathbb{P}^\varepsilon = \mathbb{P}_0 \times \pi_\varepsilon$ and $\mathscr{F}_t^\varepsilon$ be the completion of $\mathscr{F}_{0,t} \times \mathscr{B}(\mathscr{C}^2)$ under the probability measure $\mathbb{P}^\varepsilon$. Then the process

$$\{W(t)\}(\omega) := \{W_0(t)\}(\omega_0) = \omega_0(t), \quad t \ge 0, \omega = (\omega_0; \xi_1, \xi_2) \in \Omega = \Omega_0 \times \mathscr{C}^2$$

is an m-dimensional Brownian motion on the complete probability space $(\Omega, \{\mathscr{F}_t^\varepsilon\}_{t\ge 0}, \mathbb{P}^\varepsilon)$.

Finally, let

$$X_{t_0}(\omega) := \xi_1, \quad \bar{X}_{t_0}(\omega) := \xi_2, \quad \omega = (\omega_0; \xi_1, \xi_2) \in \Omega = \Omega_0 \times \mathscr{C}^2.$$

They are $\mathscr{F}_{t_0}^\varepsilon$-measurable random variables with

$$\mathscr{L}_{X_{t_0}}|_{\mathbb{P}^\varepsilon} = \mu_\varepsilon := \pi_\varepsilon(\cdot \times \mathscr{C}), \quad \mathscr{L}_{\bar{X}_{t_0}}|_{\mathbb{P}^\varepsilon} = \nu_\varepsilon := \pi_\varepsilon(\mathscr{C} \times \cdot), \tag{3.3}$$

where $\mathscr{L}_X|_{\mathbb{P}^\varepsilon}$ denotes the distribution of a random variable X under probability $\mathbb{P}^\varepsilon$. By $\xi \le \eta$ and (3.1), (3.2), we have

$$\mathbb{P}^\varepsilon(X_{t_0} \le \bar{X}_{t_0}) = \pi_\varepsilon\big(\{(\xi_1,\xi_2) \in \mathscr{C}^2 : \ \xi_1 \le \xi_2\}\big) = 1, \quad \varepsilon \in [0,1). \tag{3.4}$$

So, letting $(X_t, \bar{X}_t)_{t \ge t_0}$ be the segment process of the solution to (1.1) with initial value $(X_{t_0}, \bar{X}_{t_0})$, the order preservation implies

$$\mathbb{P}^\varepsilon\big(X_t \le \bar{X}_t, \ t \ge t_0\big) = 1, \quad \varepsilon \in [0,1). \tag{3.5}$$

Let $\mathbb{E}^\varepsilon$ be the expectation for $\mathbb{P}^\varepsilon$. With the above preparations, we are able to prove (1′) and (2′) as follows.

Proof of (1′). Let $b^i, \bar{b}^i$ be continuous at points (t_0,ξ,μ) and (t_0,η,ν) respectively. We intend to prove $b^i(t_0,\xi,\mu) \le \bar{b}^i(t_0,\eta,\nu)$. Otherwise, there exists a constant $c_0 > 0$ such that

$$b^i(t_0,\xi,\mu) \ge c_0 + \bar{b}^i(t_0,\eta,\nu). \tag{3.6}$$

Let $\mu_\varepsilon, \nu_\varepsilon$ be in (3.3). Obviously, $\{\mu_\varepsilon,\nu_\varepsilon\}_{\varepsilon\in[0,1)}$ are bounded in $\mathscr{P}_2(\mathscr{C})$ and, as $\varepsilon \to 0$, $\mu_\varepsilon \to \mu, \ \nu_\varepsilon \to \nu$ weakly. Consequently,

$$\lim_{\varepsilon\downarrow 0}\{\mathbb{W}_2(\mu_\varepsilon,\mu) + \mathbb{W}_2(\nu_\varepsilon,\nu)\} = 0.$$

Combining this with **(H1)** and (3.6), there exists $\varepsilon \in (0,1)$ such that

$$b^i(t_0,\xi,\mu_\varepsilon) \ge \frac{1}{2}c_0 + \bar{b}^i(t_0,\eta,\nu_\varepsilon) > \bar{b}^i(t_0,\eta,\nu_\varepsilon). \tag{3.7}$$

Now, consider the event

$$A := \{X_{t_0} = \xi, \bar{X}_{t_0} = \eta\} \in \mathscr{F}^\varepsilon_{t_0}. \tag{3.8}$$

Then

$$\mathbb{P}^\varepsilon(A) \ge \varepsilon\delta_{(\xi,\eta)}(\{(\xi,\eta)\}) = \varepsilon > 0. \tag{3.9}$$

By (1.1) and (3.5), for any $s \ge 0$, $\mathbb{P}^\varepsilon$-a.s.

$$\begin{aligned} 0 \ge X^i(t_0+s) - \bar{X}^i(t_0+s) &= X^i_{t_0}(0) - \bar{X}^i_{t_0}(0) \\ &+ \int_{t_0}^{t_0+s} b^i(r,X_r,\mathscr{L}_{X_r}|_{\mathbb{P}^\varepsilon})\,\mathrm{d}r - \int_{t_0}^{t_0+s} \bar{b}^i(r,\bar{X}_r,\mathscr{L}_{\bar{X}_r}|_{\mathbb{P}^\varepsilon})\,\mathrm{d}r \\ &+ \sum_{j=1}^m \int_{t_0}^{t_0+s} \sigma^{ij}(r,X_r,\mathscr{L}_{X_r}|_{\mathbb{P}^\varepsilon})\,\mathrm{d}W^j(r) - \int_{t_0}^{t_0+s} \bar{\sigma}^{ij}(r,\bar{X}_r,\mathscr{L}_{\bar{X}_r}|_{\mathbb{P}^\varepsilon})\,\mathrm{d}W^j(r). \end{aligned} \tag{3.10}$$

Since on A we have $X^i_{t_0}(0) - \bar X^i_{t_0}(0) = 0$, and A is $\mathscr{F}^\varepsilon_{t_0}$-measurable, by multiplying (3.10) with $\frac{1_A}{s}$ and taking conditional expectation with respect to $\mathscr{F}^\varepsilon_{t_0}$, we obtain $\mathbb{P}^\varepsilon$-a.s.

$$\mathbb{E}^\varepsilon\left(\frac{1_A}{s}\int_{t_0}^{t_0+s} b^i(r, X_r, \mathscr{L}_{X_r}|_{\mathbb{P}^\varepsilon})\,\mathrm{d}r\middle|\mathscr{F}^\varepsilon_{t_0}\right) \le \mathbb{E}^\varepsilon\left(\frac{1_A}{s}\int_{t_0}^{t_0+s} \bar b^i(r, \bar X_r, \mathscr{L}_{\bar X_r}|_{\mathbb{P}^\varepsilon})\,\mathrm{d}r\middle|\mathscr{F}^\varepsilon_{t_0}\right),\quad s>0.$$

Combining this with the fact that b^i and $\bar b^i$ are continuous at points (t_0, ξ, μ) and (t_0, η, ν) respectively, and using **(H1)**, **(H2)**, (1.2) and the continuity of the solution, taking $s \downarrow 0$ we derive $\mathbb{P}^\varepsilon$-a.s.

$$\mathbb{E}^\varepsilon\big(1_A b^i(t_0, X_{t_0}, \mathscr{L}_{X_{t_0}}|_{\mathbb{P}^\varepsilon})\big|\mathscr{F}^\varepsilon_{t_0}\big) \le \mathbb{E}^\varepsilon\big(1_A \bar b^i(t_0, \bar X_{t_0}, \mathscr{L}_{\bar X_{t_0}}|_{\mathbb{P}^\varepsilon})\big|\mathscr{F}^\varepsilon_{t_0}\big).$$

This together with (3.8) and (3.3) leads to $\mathbb{P}^\varepsilon$-a.s.

$$b^i(t_0, \xi, \mu_\varepsilon)1_A \le \bar b^i(t_0, \eta, \nu_\varepsilon)1_A,$$

which is impossible according to (3.7) and (3.9). Therefore, $b^i(t_0, \xi, \mu) \le \bar b^i(t_0, \eta, \nu)$ has to be true. $\square$

Proof of (2′). Let σ^{ij} and $\bar\sigma^{ij}$ be continuous at points (t_0, ξ, μ) and (t_0, η, ν) respectively. If $\sigma^{ij}(t_0, \xi, \mu) \ne \bar\sigma^{ij}(t_0, \eta, \nu)$, by **(H1)**, there exist constants $c_1 > 0$ and $\varepsilon \in (0, 1)$ such that

$$|\sigma^{ij}(t_0, \xi, \mu_\varepsilon) - \bar\sigma^{ij}(t_0, \eta, \nu_\varepsilon)|^2 \ge 2c_1 > 0. \tag{3.11}$$

Let

$$\begin{aligned}
&\tau = \inf\big\{t \ge t_0 : |\sigma^{ij}(t_0, \xi, \mu_\varepsilon) - \bar\sigma^{ij}(t_0, \eta, \nu_\varepsilon)|^2 \le c_1\big\},\\
&\tau_n = \tau \wedge \inf\big\{t \ge t_0 : |b^i(t, X_t, \mathscr{L}_{X_t}|_{\mathbb{P}^\varepsilon}) - \bar b^i(t, \bar X_t, \mathscr{L}_{\bar X_t}|_{\mathbb{P}^\varepsilon})|^2 \ge n\big\},\quad n \ge 1.
\end{aligned}$$

Let $g_n(s) = \mathrm{e}^{ns} - 1$. Then $g_n \in C_b^2((-\infty, 0])$. By the order preservation we have $X^i_t \le \bar X^i_t,\ t \ge t_0$. So, applying Itô's formula, we obtain $\mathbb{P}^\varepsilon$-a.s.

$$\begin{aligned}
&0 \ge \mathbb{E}^\varepsilon\big(g_n((X^i - \bar X^i)(t\wedge\tau_n))|\mathscr{F}^\varepsilon_{t_0}\big) = g_n((X^i - \bar X^i)(t_0))\\
&+ \mathbb{E}^\varepsilon\left(\sum_{j=1}^m \int_{t_0}^{t\wedge\tau_n} g_n'((X^i - \bar X^i)(s))\big(\sigma^{ij}(s, X_s, \mathscr{L}_{X_s}|_{\mathbb{P}^\varepsilon}) - \bar\sigma^{ij}(s, \bar X_s, \mathscr{L}_{\bar X_s}|_{\mathbb{P}^\varepsilon})\big)\mathrm{d}W^j(s)\middle|\mathscr{F}^\varepsilon_{t_0}\right)\\
&+ \mathbb{E}^\varepsilon\left(\int_{t_0}^{t\wedge\tau_n}\left\{g_n'((X^i - \bar X^i)(s))\big(b^i(s, X_s, \mathscr{L}_{X_s}|_{\mathbb{P}^\varepsilon}) - \bar b^i(s, \bar X_s, \mathscr{L}_{\bar X_s}|_{\mathbb{P}^\varepsilon})\big)\right.\right.\\
&\left.\left.+\frac{g_n''(X^i(s) - \bar X^i(s))}{2}\sum_{j=1}^m\left|\sigma^{ij}(s, X_s, \mathscr{L}_{X_s}|_{\mathbb{P}^\varepsilon}) - \bar\sigma^{ij}(s, \bar X_s, \mathscr{L}_{\bar X_s}|_{\mathbb{P}^\varepsilon})\right|^2\right\}\mathrm{d}s\middle|\mathscr{F}^\varepsilon_{t_0}\right)\\
&\ge g_n((X^i - \bar X^i)(t_0)) + \Big(\frac{n^2c_1}{2} - n^{3/2}\Big)\mathbb{E}^\varepsilon(t\wedge\tau_n - t_0|\mathscr{F}^\varepsilon_{t_0}),\quad n \ge 1.
\end{aligned}$$

By (3.8) and $\xi^i(0) = \eta^i(0)$, this implies

$$1_A\Big(\frac{n^2c_1}{2} - n^{3/2}\Big)\mathbb{E}^\varepsilon(t\wedge\tau_n - t_0|\mathscr{F}^\varepsilon_{t_0}) \le -1_A g_n((X^i - \bar{X}^i)(t_0)) = -1_A g_n(0) = 0$$

for all $n \ge 1$ and $t > t_0$. Therefore,

$$1_A\mathbb{E}^\varepsilon(t\wedge\tau_n - t_0|\mathscr{F}^\varepsilon_{t_0}) = 0, \quad \sqrt{n} > \frac{2}{c_1}, t > t_0. \tag{3.12}$$

But by (**H1**), (3.11) and the continuity of the solution, on the set A we have

$$\lim_{n\to\infty}\tau_n = \tau > t_0.$$

So, (3.12) implies $\mathbb{P}^\varepsilon(A) = 0$, which contradicts (3.9). Hence, $\sigma^{ij}(t_0,\xi,\mu) = \bar{\sigma}^{ij}(t_0,\eta,\nu)$. $\square$

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