The Euler-Maruyama Method for (Functional) SDEs with Hölder Drift and α-Stable Noise*

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Abstract

Consider the (functional) SDEs in $\mathbb{R}^d$ with Hölder continuous drift driven by α-stable process satisfying **(H1)**. Using Zvonkin type transformation, the convergence rate of Euler-Maruyama method is obtained. The results are new, especially for the functional SDEs with irregular drift.

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1 Introduction

Recently, the convergence rate of Euler-Maruyama (EM for short) method for stochastic differential equations (SDEs for abbreviation) with irregular coefficients has attracted much attention. For instance, by the Meyer-Tanaka formula, [11] revealed the convergence rate in L^1-norm sense for a range of SDEs, where the drift term is Lipschitzian and the diffusion term is Hölder continuous with respect to spatial variable; Adopting the Yamada-Watanabe approximation approach, [3] extended [11] to discuss the strong convergence rate in L^p-norm sense; Using the Yamada-Watanabe approximation approach and heat kernel estimate, [8] studied the strong convergence rate in L^1-norm sense for a class of non-degenerate SDEs, where the bounded drift term satisfies weak monotonicity and is of bounded variation with respect to Gaussian measure and the diffusion term is Hölder continuous. Quite recently, by

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Zvonkin transformation [12], [2] discussed the convergence rate of EM method for the (non-) degenerate SDEs with Dini continuous drift.

We should remark that all the above results focused on the convergence rate of EM method for SDEs driven by Brownian motion. As to the Lévy noise, there are also some results. For example, applying the Zvonkin transformation, [9] obtained the strong convergence rate of EM method with bounded Hölder continuous drift driven by truncated symmetric α-stable process. When the Lévy measure is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$, [7] obtained the convergence rate of EM method.

In this paper, we investigate the convergence rate of EM method for SDEs and functional SDEs (SFDEs) driven by α-stable process with Hölder continuous and bounded drift. We assume (**H1**) holds for the α-stable process which contains the case that the Lévy measure is not absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$. By the Zvonkin transformation, we can change the SDEs with irregular drift to the regular ones, and then we obtain the convergence rate of EM method.

Before moving on, we firstly recall some knowledge on symmetric α-stable process and the Poisson random measure, see [10] for more details. Recall that a $\mathbb{R}^d$-valued Lévy process $L(t)$ is called d-dimensional symmetric α-stable process if the Lévy symbol Ψ has the following representation:

$$\Psi(u) = \int_{\mathbb{R}^d} [1 - \cos\langle u, x\rangle] \nu(\mathrm{d}x).$$

where

$$\nu(D) = \int_S \mu(\mathrm{d}\xi) \int_0^\infty \mathbb{1}_D(r\xi) \frac{\mathrm{d}r}{r^{1+\alpha}}, \quad D \in \mathscr{B}(\mathbb{R}^d),$$

$S = \{x \in \mathbb{R}^d, |x| = 1\}$ and μ is a finite symmetric measure on $(S, \mathscr{B}(S))$, i.e. $\mu(A) = \mu(-A)$, $\forall A \in \mathscr{B}(S)$.

It is easy to see that the Lévy process $L(t)$ has the following two properties:

(**I**) Scaling property: for $t > 0$, let μ_t denotes the law of L_t, $t > 0$, then

$$\mu_t(A) = \mu_1(t^{-\frac{1}{\alpha}} A), \quad A \in \mathscr{B}(\mathbb{R}^d), t > 0.$$

(**II**) For any $\gamma > \alpha$, we have

$$\int_{\{|x| \le 1\}} |x|^\gamma \nu(\mathrm{d}x) < \infty.$$

Refer to [10] for more details. Moreover, the Poisson random measure N associated to L is defined as follows:

$$N([0,t], U) = \sum_{0 \le s \le t} \mathbb{1}_U(\Delta L(s)), \quad U \in \mathscr{B}\left(\mathbb{R}^d \backslash \{0\}\right), t \ge 0.$$

Here $\Delta L(s) = L(s) - L(s-)$ denotes the jump size of L at time $s \ge 0$. The compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}([0,t], U) = N([0,t], U) - t\nu(U), \quad U \in \mathscr{B}\left(\mathbb{R}^d \backslash \{0\}\right), 0 \notin \bar{U}, t \ge 0.$$

It follows from Lévy-Itô decomposition that

$$L(t)=\int_0^t\int_{|x|\le 1} x\tilde{N}(\mathrm{d}s,\mathrm{d}x)+\int_0^t\int_{|x|>1} xN(\mathrm{d}s,\mathrm{d}x),\quad t\ge 0.$$

In addition, for any $T>0$, the predictable σ-algebra $\mathcal{P}$ on $\Omega\times[0,T]$ is generated by all left continuous adapted processes. Letting $U\in\mathscr{B}\left(\mathbb{R}^d\backslash\{0\}\right)$, consider a $\mathcal{P}\times\mathscr{B}(U)$-measurable mapping $F:\Omega\times[0,T]\times U\to\mathbb{R}^d$. If $0\notin\bar{U}$, then

$$\int_0^T\int_U F(\cdot,s,x)N(\mathrm{d}s,\mathrm{d}x)=\sum_{0\le s\le T}F(\cdot,s,\Delta L(s))1_U(\Delta L(s))$$

is a random finite sum. Furthermore, if $\mathbb{E}\int_0^T\int_U|F(\cdot,s,x)|^2\nu(\mathrm{d}x)\mathrm{d}s<\infty$, one can define the stochastic process as

$$Z(t)=\int_0^t\int_U F(\cdot,s,x)\tilde{N}(\mathrm{d}s,\mathrm{d}x),\quad t\in[0,T].$$

Notice that we do not assume $0\notin\bar{U}$. The process $Z=(Z(t))$ is a L^2-martingale with a càdlàg modification. Moreover, by [5, Lemma 2.4], we have $\mathbb{E}|Z(t)|^2=\mathbb{E}\int_0^t\int_U|F(\cdot,s,x)|^2\nu(\mathrm{d}x)\mathrm{d}s$. We will use the following L^p-estimates (see [5, Theorem 2.11]): for any $p\ge 2$ and $t\in[0,T]$, there exists a constant $c(p)>0$ such that

$$\mathbb{E}\left[\sup_{0\le s\le t}|Z(s)|^p\right]\le c(p)\mathbb{E}\left[\left(\int_0^t\int_U|F(\cdot,s,x)|^2\nu(\mathrm{d}x)\mathrm{d}s\right)^{\frac{p}{2}}+\int_0^t\int_U|F(\cdot,s,x)|^p\nu(\mathrm{d}x)\mathrm{d}s\right]$$

Write $M(t)=|Z(t)|^2$ and $A(t)=\int_0^t\int_U|F(\cdot,s,x)|^2\nu(\mathrm{d}x)\mathrm{d}s$. When $p\in(0,2)$, by Lemma 4.1 (see section 4), we have

$$\mathbb{E}\left[\sup_{0\le s\le t}|Z(s)|^p\right]\le c(p)\mathbb{E}\left[\left(\int_0^t\int_U|F(\cdot,s,x)|^2\nu(\mathrm{d}x)\mathrm{d}s\right)^{\frac{p}{2}}\right],\quad t\in[0,T]. \tag{1.1}$$

For convenience, we introduce some notations. Denote $\lfloor\cdot\rfloor$ by the floor function, which maps a real number to the greatest preceding integer. Let $\|\cdot\|$ denote the operator norm for a bounded linear operator. For fixed $k\in\mathbb{N}$ and $\beta\in(0,1)$, define set $C_b^\beta\left(\mathbb{R}^d\right)$ and $C_b^{k+\beta}\left(\mathbb{R}^d\right)$ as follows.

(1) Denote $C_b^\beta\left(\mathbb{R}^d\right)$ by the set of $\mathbb{R}^d$-valued bounded functions defined on $\mathbb{R}^d$, which are β Hölder continuous. The norm of $C_b^\beta\left(\mathbb{R}^d\right)$ is

$$\|f\|_\beta:=\sup_{x\in\mathbb{R}^d}|f(x)|+\sup_{x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\beta},\quad f\in C_b^\beta\left(\mathbb{R}^d\right).$$

(2) Denote $C_b^{k+\beta}\left(\mathbb{R}^d\right)$ by the set of $\mathbb{R}^d$-valued bounded functions, which have up to k-ordered continuous derivative and the k-th derivative is β Hölder continuous. The norm is

$$\|f\|_{k+\beta} := \sum_{i=0}^{k} \sup_{x\in\mathbb{R}^d} \|\nabla^i f(x)\| + \sup_{x\neq y} \frac{\|\nabla^k f(x) - \nabla^k f(y)\|}{|x-y|^\beta}, \quad f \in C_b^{k+\beta}\left(\mathbb{R}^d\right).$$

In particular, $C_b^0\left(\mathbb{R}^d\right)$ means the set of $\mathbb{R}^d$-valued bounded functions, equipped the norm $\|f\|_0 := \sup_{x\in\mathbb{R}^d} |f(x)|$, and we usually denote C_b.

Throughout this paper, we assume that

(H1) For fixed $\alpha \geq 1$, there exists a positive constant $C_\alpha > 0$ such that

$$\Psi(u) \geq C_\alpha |u|^\alpha, \quad u \in \mathbb{R}^d.$$

Remark 1.1. *Refer to [10] for more details about this assumption. We know there are two examples satisfying* **(H1)**. *One is when L is a standard α-stable process, i.e. $\Psi(u) = c_\alpha |u|^\alpha$. In this case, ν has density $\frac{C_\alpha}{|x|^{d+\alpha}}$ with respect to the Lebesgue measure in $\mathbb{R}^d$. Moreover the spectral measure μ is the uniform distribution on S. Another example is $\Psi(u) = k_\alpha(\sum_{i=1}^d |u_i|^\alpha)$ and the Lévy measure ν is singular with respect to the Lebesgue measure in $\mathbb{R}^d$. More precisely, ν is concentrated on the union of the coordinates axes, i.e.*

$$\nu(dy) = c_\alpha \sum_{i=1}^{d} \mathbb{1}_{A_i} \frac{1}{|y_i|^{1+\alpha}}$$

where $A_i = \cap_{j\neq i}\{y_j = 0\}$. The spectral measure μ is a linear combination of Dirac measures, i.e. $\mu = \sum_{k=1}^d (\delta_{e_k} + \delta_{-e_k})$, where (e_k) is the canonical basis in $\mathbb{R}^d$.

2 The convergence rate of EM Scheme for SDEs

In this section, we consider the following SDEs on $\mathbb{R}^d$ $(d \geq 1)$:

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \mathrm{d}L(t), \quad X(0) = x, \ t \in [0,T], \tag{2.1}$$

where b is a $\mathbb{R}^d \to \mathbb{R}^d$ function and $L(t)$ is a d-dimensional symmetric α-stable process ($\alpha \in (0,2)$) on a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, which satisfies $L(0) = 0$, $\mathbb{P}$-a.s.

For any $m \in \mathbb{N}$, define the EM method as follows:

$$X^m(t) = x + \int_0^t b\left(X^m(\eta_m(s))\right)\mathrm{d}s + L(t), \quad t \in [0,T],$$

where $\eta_m(s) := \lfloor \frac{ms}{T} \rfloor \frac{T}{m}$, $s \in [0,T]$. From now on, we assume m is large enough such that $\frac{T}{m} < 1$. We will give the strong convergence rate of EM method for SDEs (2.1). Besides **(H1)**, we need one more assumption about the function b.

(H2) $b \in C_b^\beta\left(\mathbb{R}^d\right)$ for some $\beta \in (0,1)$ satisfying $2\beta + \alpha > 2$.

In order to transform (2.1) to a SDEs with regular coefficients, for any $\lambda > 0$, consider the following resolvent equation on $\mathbb{R}^d$:

$$\lambda u - \mathscr{L} u - b \nabla u = b, \tag{2.2}$$

where

$$\mathscr{L} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[f(x+y) - f(x) - \langle y, \nabla f(x) \rangle \mathbb{1}_{\{|y| \le 1\}} \right] \nu(\mathrm{d}y), \quad f \in C_c^\infty(\mathbb{R}^d).$$

By [10, Theorem 3.4], we have

Lemma 2.1. *Assume* **(H1)** *and* **(H2)**, *then for any* $\lambda > 0$, *there exists an unique solution* $u = u_\lambda \in C_b^{\alpha+\beta}\left(\mathbb{R}^d\right)$ *to* (2.2). *Moreover, for any* λ_0, *there exists a constant* $c_{\lambda_0} > 0$ *independent of* b *and* u *such that*

$$\lambda \|u\|_0 + \|\nabla u\|_{\alpha+\beta-1} \le c_{\lambda_0} \|b\|_\beta, \quad \lambda > \lambda_0.$$

Finally, we have $\lim_{\lambda \to \infty} \|\nabla u_\lambda\|_0 = 0$.

Lemma 2.2. *Assume* **(H1)** *and* **(H2)**, *then for any* $0 < p < \alpha$,

$$\mathbb{E}\left|X^m(t) - X^m(\eta_m(t))\right|^p \le C(p,\nu) \left(\frac{T}{m}\right)^{\frac{p}{\alpha}}$$

holds for some constant $C(p,\nu)$ *depending on* p *and* ν.

Proof. By the boundedness of b and the scaling property of L, for any $0 < p < \alpha$, noting that $\alpha \ge 1$, we have

$$\begin{aligned} \mathbb{E}\left|X^m(t) - X^m(\eta_m(t))\right|^p &\le \mathbb{E}\left|\int_{\eta_m(t)}^t b(X^m(\eta_m(s)))\mathrm{d}s\right|^p + \mathbb{E}|L(t) - L(\eta_m(t))|^p \\ &\le C(p,\nu)\left[\left(\frac{T}{m}\right)^p + \left(\frac{T}{m}\right)^{\frac{p}{\alpha}}\right] \le C(p,\nu)\left(\frac{T}{m}\right)^{\frac{p}{\alpha}}. \end{aligned}$$

□

The following lemma gives the regularity representation of X^m by Zvonkin transformation. The proof of this lemma refer to [10, Lemma 4.2].

Lemma 2.3. *Let* u *be the unique solution given in Lemma 2.1, then for any* $t \in [0,T]$ *we have*

$$\begin{aligned} \left[X^m(t) + u(X^m(t))\right] &= \left[X(0) + u(X(0))\right] + L(t) + \lambda \int_0^t u\left(X^m(s)\right)\mathrm{d}s \\ &\quad + \int_0^t \left[I + \nabla u(X^m(s))\right]\left[b(X^m(\eta_m(s))) - b(X^m(s))\right]\mathrm{d}s \\ &\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left[u\left(X^m(s-) + x\right) - u\left(X^m(s-)\right)\right]\tilde{N}(\mathrm{d}s, \mathrm{d}x). \end{aligned} \tag{2.3}$$

We state the main result of this section as follows.

Theorem 2.4. *Assume* **(H1)** *and* **(H2)**, *then for any $p>0$ satisfying $p\beta<\alpha$, the estimation*

$$\mathbb{E}\sup_{0\le s\le T}|X(s)-X^m(s)|^p \le C\left(\frac{T}{m}\right)^{\frac{p\beta}{\alpha}}. \tag{2.4}$$

holds for a positive constant $C:=C(p,T,\nu,\lambda,\alpha,\beta)$ depending on T, p, ν, λ, α and β.

Remark 2.5. *When $\alpha=2$ and $\beta=1$, the result coincides with the case that the process $L(t)$ is a Brownian motion and b is Lipschitz continuous.*

Remark 2.6. *In the multiplicative noise case, [7, Proposition 1] obtained the convergence rate under condition $p\in(0,\alpha)$. In our result Theorem 2.4, this condition becomes more relaxed, which is $p\in(0,\alpha/\beta)$, $\beta<1$. For this, we need to notice that: in the additive noise case, though $\mathbb{E}|X(t)|^p$ is infinite when $p>\alpha$, $(X(t)-X^m(t))$ still be a bounded process since b is bounded.*

Proof. Combining Lemma 2.3 with [10, (4.4)], we have

$$\begin{aligned}
&\big[X(t)+u(X(t))\big]-\big[X^m(t)+u(X^m(t))\big]\\
&\quad=\int_0^t\lambda\big[u(X(s))-u(X^m(s))\big]\mathrm{d}s-\int_0^t\left[I+\nabla u(X^m(s))\right]\left[b(X^m(\eta_m(s)))-b(X^m(s))\right]\mathrm{d}s\\
&\quad\quad+\int_0^t\int_{\mathbb{R}^d\setminus\{0\}}\left[u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))\right]\tilde{N}(\mathrm{d}s,\mathrm{d}x).
\end{aligned}\tag{2.5}$$

By Lemma 2.1, choosing large enough $\lambda>0$ such that $\|\nabla u\|_0\le 1/3$, it follows from (2.5) that

$$|X(t)-X^m(t)|\le\frac{3}{2}\Lambda_1(t)+\frac{3}{2}\Lambda_2(t)+\frac{3}{2}\Lambda_3(t)+\frac{3}{2}\Lambda_4(t), \tag{2.6}$$

where

$$\begin{aligned}
\Lambda_1(t)&=\left|\int_0^t\int_{|x|>1}\big[u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))\big]\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right|,\\
\Lambda_2(t)&=\int_0^t\lambda|u(X(s))-u(X^m(s))|\mathrm{d}s,\\
\Lambda_3(t)&=\left|\int_0^t\int_{|x|\le1}\big[u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))\big]\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right|,\\
\Lambda_4(t)&=\int_0^t|\left[I+\nabla u(X^m(s))\right]\left[b(X^m(\eta_m(s)))-b(X^m(s))\right]|\mathrm{d}s.
\end{aligned}$$

Firstly, by assumption **(H2)**, Lemma 2.1, Lemma 2.2 and Hölder inequality, we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\Lambda_4^p(s) &\le c_1(p)t^{p-1}\mathbb{E}\int_0^t |\left[I+\nabla u(X^m(s))\right]\left[b(X^m(\eta_m(s)))-b(X^m(s))\right]|^p\mathrm{d}s \\
&\le c_1(p,T,\nu)\left(\frac{T}{m}\right)^{\frac{p\beta}{\alpha}},
\end{aligned}
\tag{2.7}
$$

where $c_1(p,T,\nu)>0$ is a positive constant depending on p,T,ν. Similarly, we have

$$
\mathbb{E}\sup_{0\le s\le t}\Lambda_2^p(s)\le c_2(p,\lambda,T)t^p\mathbb{E}\sup_{0\le s\le t}|X(s)-X^m(s)|^p. \tag{2.8}
$$

Next, we divide two cases to estimate Λ_1 and Λ_3.

Case 1: $0<p<2$. By the boundedness of u and the property of ν, it is easy to show that

$$
\int_0^t\int_{|x|>1}\left|u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))\right|^2\nu(\mathrm{d}x)\mathrm{d}s<\infty.
$$

Since

$$
\begin{aligned}
&|u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))| \\
\le&|u(X(s-)+x)-u(X^m(s-)+x)|+|u(X(s-))-u(X^m(s-))| \\
\le&\frac{2}{3}|X(s-)-X^m(s-)|,
\end{aligned}
$$

combining this with (1.1), we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\Lambda_1^p(s) &\le c_3(p)t^{\frac{p}{2}}\nu(\{|x|>1\})^{\frac{p}{2}}\mathbb{E}\sup_{s\in[0,t]}|X(s)-X^m(s)|^p \\
&= c_3(p,\nu)t^{\frac{p}{2}}\mathbb{E}\sup_{s\in[0,t]}|X(s)-X^m(s)|^p.
\end{aligned}
\tag{2.9}
$$

By [10, Lemma 4.1], assumption **(H2)** and Lemma 2.1, it holds that

$$
\begin{aligned}
&|u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))| \\
&\qquad\le 3^{2-(\alpha+\beta)}2^{\alpha+\beta-1}\|u\|_{\alpha+\beta}|x|^{\alpha+\beta-1}|X(s-)-X^m(s-)|.
\end{aligned}
\tag{2.10}
$$

Since $X(t)-X^m(t)$ is a bounded process, (2.10) and assumption **(H2)** imply that

$$
\mathbb{E}\int_0^t\int_{|x|\le 1}|u(X(s-)+x)-u(X(s-))-u(X^m(s-)+x)+u(X^m(s-))|^2\nu(\mathrm{d}x)\mathrm{d}s<\infty.
$$

It follows from (1.1) that

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le t}\Lambda_3^p(s) &\le c_4(p,\alpha,\beta)\mathbb{E}\left[\int_0^t\int_{|x|\le 1}\|u\|_{\alpha+\beta}^2|X(s)-X^m(s)|^2|x|^{2(\alpha+\beta-1)}\nu(\mathrm{d}x)\mathrm{d}s\right]^{\frac{p}{2}} \\
&\le c_4(p,T,\nu,\alpha,\beta)t^{\frac{p}{2}}\mathbb{E}\sup_{s\in[0,t]}|X(s)-X^m(s)|^p.
\end{aligned}
\tag{2.11}
$$

Case 2: $p \geq 2$. By (1.1), we get

$$
\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq t} \Lambda_1^p(s) &\leq c_3(p) t^{\frac{p}{2}} \nu(\{|x|>1\})^{\frac{p}{2}} \mathbb{E} \sup_{s\in[0,t]} |X(s)-X^m(s)|^p \\
&\quad + c_3(p) t \nu(\{|x|>1\}) \mathbb{E} \sup_{s\in[0,t]} |X(s)-X^m(s)|^p \\
&= c_3(p,\nu) t \mathbb{E} \sup_{s\in[0,t]} |X(s)-X^m(s)|^p,
\end{aligned} \tag{2.12}
$$

and

$$
\begin{aligned}
\mathbb{E} \sup_{0\leq s\leq t} \Lambda_3^p(s) &\leq c_4(p,\alpha,\beta) \mathbb{E} \left[\int_0^t \int_{|x|\leq 1} \|u\|_{\alpha+\beta}^2 |X(s)-X^m(s)|^2 |x|^{2(\alpha+\beta-1)} \nu(\mathrm{d}x)\mathrm{d}s \right]^{\frac{p}{2}} \\
&\quad + c_4(p,\alpha,\beta) \mathbb{E} \int_0^t \int_{|x|\leq 1} \|u\|_{\alpha+\beta}^p |X(s)-X^m(s)|^p |x|^{p(\alpha+\beta-1)} \nu(\mathrm{d}x)\mathrm{d}s \\
&\leq c_4(p,T,\nu,\alpha,\beta) t \mathbb{E} \sup_{s\in[0,t]} |X(s)-X^m(s)|^p.
\end{aligned} \tag{2.13}
$$

Combining formulas (2.6)–(2.13), we get

$$
\mathbb{E} \sup_{0\leq s\leq t} |X(s)-X^m(s)|^p \leq c_1(p,T,\nu) \left(\frac{T}{m}\right)^{\frac{p\beta}{\alpha}} + \tilde{c} t^{\frac{p}{2}\wedge 1} \mathbb{E} \sup_{0\leq s\leq t} |X(s)-X^m(s)|^p.
$$

where $\tilde{c}$ is a constant depending on $p, T, \nu, \lambda, \alpha, \beta$. Taking $t_0 := [2c(p,T,\nu,\lambda,\alpha,\beta)]^{-\left(\frac{p}{2}\wedge 1\right)}$, we have

$$
\mathbb{E} \sup_{0\leq s\leq t_0} |X(s)-X^m(s)|^p \leq c(p,T,\nu) \left(\frac{T}{m}\right)^{\frac{p\beta}{\alpha}}.
$$

Finally, by recursion, it is easy to show that

$$
\mathbb{E} \sup_{0\leq s\leq T} |X(s)-X^m(s)|^p \leq c(p,T,\nu) \left(\lfloor T/t_0 \rfloor + 1\right) \left(\frac{T}{m}\right)^{\frac{p\beta}{\alpha}},
$$

which completes the proof of Theorem 2.4. □

3 The convergence rate of EM Scheme for SFDEs

Fix $r > 0$, denote $\mathscr{D}$ by the set of all $\mathbb{R}^d$-valued càdlàg functions on $[-r, 0]$ equipped with the uniform norm $\|\xi\|_\infty := \sup_{s\in[-r,0]} |\xi(s)|$. For any $\mathbb{R}^d$-valued càdlàg function f on $[-r,\infty)$, define $f_t \in \mathscr{D}$ as $f_t(\theta) = f(t+\theta)$, $\theta \in [-r, 0]$.

In this section, we consider the following functional SDEs on $\mathbb{R}^d$ $(d \geq 1)$:

$$
\mathrm{d}X(t) = b(X(t))\mathrm{d}t + B(X_t)\mathrm{d}t + \mathrm{d}L(t), \quad X_0 = \xi, \tag{3.1}
$$

where $b:\mathbb{R}^d\to\mathbb{R}^d$ and $B:\mathscr{D}\to\mathbb{R}^d$ are measurable functions, and $L(t)$ is the $\mathbb{R}^d$-valued Lévy process introduced in Section 1. Next, we introduce the EM method for SFDEs (3.1), refer to [1] for more details. For some integers $M>T$ and $N>r$, define $\Delta\in(0,1)$ as

$$\Delta:=\frac{r}{N}=\frac{T}{M}.$$

For any integers $k\in[-N,\infty)$ and $i\in[-N,-1]$, write

$$\begin{aligned}&\bar{y}(k\Delta)=\xi(k\Delta),\quad -N\le k\le 0;\\&\bar{y}((k+1)\Delta)=\bar{y}(k\Delta)+b(\bar{y}(k\Delta))\Delta+B(\bar{y}_{k\Delta})\Delta+L((k+1)\Delta)-L(k\Delta),\quad k\ge 0,\end{aligned}\tag{3.2}$$

where $\bar{y}_{k\Delta}$ is a $\mathscr{D}$-valued random variable defined by

$$\bar{y}_{k\Delta}(s)=\frac{(i+1)\Delta-s}{\Delta}\bar{y}((k+i)\Delta)+\frac{s-i\Delta}{\Delta}\bar{y}((k+i+1)\Delta),\quad s\in[i\Delta,(i+1)\Delta].\tag{3.3}$$

In order to define $\bar{y}_{-\Delta}$, we set $\bar{y}(-(N+1)\Delta)=\xi(-N\Delta)$. Thus we give a discrete-time approximation $\{\bar{y}(k\Delta)\}_{k\ge0}$. Following this, we give a continuous-time approximation $y(t)$ by setting $y(t)=\xi(t)$ when $t\in[-r,0]$. For $t\in[0,T]$, define

$$y(t)=\xi(0)+\int_0^t(b(\bar{y}_s(0))+B(\bar{y}_s))\mathrm{d}s+L(t),\tag{3.4}$$

where $\bar{y}_t:=\bar{y}_{\lfloor\frac{t}{\Delta}\rfloor\Delta}$. It is easy to see that $\bar{y}(k\Delta)=y(k\Delta)$ for $-N\le k\le M$. Combining this and formulas (3.2)–(3.4), we have the following properties:

(N1) $\|\bar{y}_{k\Delta}\|_\infty=\sup_{-N\le i\le 0}|\bar{y}((k+i)\Delta)|,\quad -1\le k\le M.$

(N2) $\|\bar{y}_{k\Delta}\|_\infty\le\|y_{k\Delta}\|_\infty,\quad -1\le k\le M.$

(N3) $\|\bar{y}_t\|_\infty=\|\bar{y}_{\lfloor\frac{t}{\Delta}\rfloor\Delta}\|_\infty\le\|y_{\lfloor\frac{t}{\Delta}\rfloor\Delta}\|_\infty\le\sup_{-r\le s\le t}|y(s)|,\quad t\in[0,T].$

In this section, we study the convergence rate of EM method for SFDEs (3.1). Before moving further, we give the assumptions in this model:

(H2') $b\in C_b^\beta\left(\mathbb{R}^d\right)$ for some $\beta\in(0,1)$ satisfying $\alpha+\beta=2$.

(H3) B is a bounded and Lipschitz continuous function, i.e. there exists a left-continuous nondecreasing function $\rho:[-r,0]\to[0,\infty)$ such that for all $\xi,\eta\in\mathscr{D}$,

$$|B(\xi)-B(\eta)|^2\le\int_{[-r,0]}|\xi(s)-\eta(s)|^2\mathrm{d}\rho(s).$$

(H4) For some $p\in(0,\alpha)$, there exists a constant $\gamma>0$ such that

$$|\xi(s)-\xi(t)|^p\le\gamma|s-t|,\quad\forall s,t\in[-r,0].$$

Noticing that the assumption **(H3)** implies [4, **(H2)**]. Under assumptions **(H1)**, **(H2')** and **(H3)**, it is easy to prove that (3.1) has an unique strong solution by using the result of [4, Theorem 3.2]. Moreover, for any $t \in [0,T]$, we have

$$
\begin{aligned}
[X(t) + u(X(t))] = [X(0) + u(X(0))] + L(t) \\
+ \lambda \int_0^t u(X(s))\mathrm{d}s + \int_0^t (I + \nabla u(X(s)))B(X_s)\mathrm{d}s \\
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X(s-) + x) - u(X(s-))]\tilde{N}(\mathrm{d}s, \mathrm{d}x).
\end{aligned}
\tag{3.5}
$$

By (2.2), Lemma 2.1 and Itô formula, we give a regularity representation of $y(t)$ by Zvonkin transformation. For more details of the proof, refer to [10, Lemma 4.2] and [4, Theorem 3.2].

Lemma 3.1. *Let u be in Lemma 2.1, then*

$$
\begin{aligned}
[y(t) + u(y(t))] = [y(0) + u(y(0))] + L(t) \\
+ \lambda \int_0^t u(y(s))\mathrm{d}s + \int_0^t (I + \nabla u(y(s)))B(\bar{y}_s)\mathrm{d}s \\
+ \int_0^t (I + \nabla u(y(s)))(b(\bar{y}_s(0)) - b(y(s)))\mathrm{d}s \\
+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(y(s-) + x) - u(y(s-))]\tilde{N}(\mathrm{d}s, \mathrm{d}x), \quad t \in [0,T].
\end{aligned}
\tag{3.6}
$$

The following lemma is useful in the proof of the main result in this section. The proof is similar to the one of [1, Lemma 3.2], for convenience, we show it in detail.

Lemma 3.2. *Assume* **(H2')**, **(H3)** *and* **(H4)**, *then for $p \in (0, \alpha)$ introduced in* **(H4)** *and $t \in [0,T]$*

$$
\mathbb{E}|y(t+s) + \bar{y}_t(s)|^p \le C(p,\gamma)\Delta^{\frac{p}{\alpha}}, \quad s \in [-r,0]. \tag{3.7}
$$

Proof. Fix $t \in [0,T]$, $s \in [-r,0]$, let $k_t = \lfloor \frac{t}{\Delta} \rfloor \Delta$ and $k_s = \lfloor \frac{|s|}{\Delta} \rfloor \Delta$. For convenience, we write $v = t + s$. Then it is easy to see that $0 \le v - k_v \le \Delta$. By (3.3), it is clear that

$$
\bar{y}_t(s) = \bar{y}_{k_t}(s) = \bar{y}(k_v) + \frac{s - k_s}{\Delta}(\bar{y}(k_v + \Delta) - \bar{y}(k_v)),
$$

which implies that

$$
\mathbb{E}|y(t+s) + \bar{y}_t(s)|^p \le c(p)\mathbb{E}|y(v) - \bar{y}(k_v)|^p + c(p)\mathbb{E}|\bar{y}(k_v + \Delta) - \bar{y}(k_v)|^p. \tag{3.8}
$$

If $k_v \le -\Delta$, assumption **(H4)** implies that

$$
\mathbb{E}|\bar{y}(k_v + \Delta) - \bar{y}(k_v)|^p \le \gamma\Delta. \tag{3.9}
$$

If $k_v \geq 0$, since b and B are bounded, using (3.2) we have

$$\mathbb{E}|\bar{y}(k_v + \Delta) - \bar{y}(k_v)|^p \leq c\Delta^{\frac{p}{\alpha}}. \tag{3.10}$$

Next, we divide three cases to estimate the first term on the right-hand side of (3.8).

Case1: $k_v \geq 0$. By (3.4) we have

$$\begin{aligned}\mathbb{E}|y(v) - \bar{y}(k_v)|^p &\leq C(p)\left(\left|\int_{k_v}^{v}(b(\bar{y}_s(0)) + B(\bar{y}_s))\mathrm{d}s\right|^p + |L(v) - L(k_v)|^p\right)\\ &\leq C(p)\Delta^{\frac{p}{\alpha}}.\end{aligned} \tag{3.11}$$

Case2: $k_v \leq -\Delta$ and $v > 0$. In this case $v \leq \Delta$, by assumption **(H4)** and (3.4), we have

$$\begin{aligned}\mathbb{E}|y(v) - \bar{y}(k_v)|^p &\leq C'(p)\mathbb{E}|y(v) - \bar{y}(0)|^p + C(p)\mathbb{E}|y(0) - \bar{y}(k_v)|^p\\ &\leq C'(p)\left(\left|\int_{0}^{v}(b(\bar{y}_s(0)) + B(\bar{y}_s))\mathrm{d}s\right|^p + |L(v)|^p\right) + C(p)\gamma\Delta\\ &\leq C'(p)\Delta^{\frac{p}{\alpha}}.\end{aligned} \tag{3.12}$$

Case3: $k_v \leq -\Delta$ and $v \leq 0$. By assumption **(H4)**, we have

$$\mathbb{E}|y(v) - \bar{y}(k_v)|^p \leq 2\gamma\Delta. \tag{3.13}$$

Finally, combining formulas (3.8)–(3.13), we obtain (3.7). □

Theorem 3.3. *Assume* **(H1)**, **(H2')**, **(H3)** *and* **(H4)**, *then we have*

$$\mathbb{E}\sup_{0\leq s\leq T}|X(s) - y(s)|^p \leq c\Delta^{\frac{p\beta}{\alpha}} \tag{3.14}$$

where $p \in (0, \alpha)$ is given in assumption **(H4)** *and c is a positive constant depending on $p, T, \nu, \lambda, \alpha, \beta, \gamma$.*

Proof. Combining (3.5) and (3.6), we have

$$\begin{aligned}&[X(t) + u(X(t))] - [y(t) + u(y(t))]\\ =&\int_0^t \lambda[u(X(s)) - u(y(s))]\mathrm{d}s - \int_0^t [I + \nabla u(y(s))][b(\bar{y}_s(0)) - b(y(s))]\mathrm{d}s\\ &+ \int_0^t\int_{\mathbb{R}^d\setminus\{0\}}[u(X(s-) + x) - u(X(s-)) - u(y(s-) + x) + u(y(s-))]\tilde{N}(\mathrm{d}s, \mathrm{d}x)\\ &+ \int_0^t[(I + \nabla u(X(s)))B(X_s) - (I + \nabla u(y(s)))B(\bar{y}_s)]\mathrm{d}s.\end{aligned} \tag{3.15}$$

By Lemma 2.1, choosing large enough $\lambda > 0$ such that $\|\nabla u\|_0 \leq 1/3$, it follows from (3.15) that

$$|X(t) - y(t)| \leq \frac{3}{2}\Gamma_1(t) + \frac{3}{2}\Gamma_2(t) + \frac{3}{2}\Gamma_3(t) + \frac{3}{2}\Gamma_4(t) + \frac{3}{2}\Gamma_5(t), \tag{3.16}$$

where

$$\Gamma_1(t) = \left| \int_0^t \int_{|x|>1} [u(X(s-)+x) - u(X(s-)) - u(y(s-)+x) + u(y(s-))] \tilde{N}(\mathrm{d}s, \mathrm{d}x) \right|,$$

$$\Gamma_2(t) = \int_0^t \lambda |u(X(s)) - u(y(s))| \mathrm{d}s,$$

$$\Gamma_3(t) = \left| \int_0^t \int_{|x|\le 1} [u(X(s-)+x) - u(X(s-)) - u(y(s-)+x) + u(y(s-))] \tilde{N}(\mathrm{d}s, \mathrm{d}x) \right|,$$

$$\Gamma_4(t) = \int_0^t [I + \nabla u(y(s))][b(\bar{y}_s(0)) - b(y(s))] \mathrm{d}s,$$

$$\Gamma_5(t) = \int_0^t [(I + \nabla u(X(s)))B(X_s) - (I + \nabla u(y(s)))B(\bar{y}_s)] \mathrm{d}s.$$

Firstly, by assumption **(H2)**, Lemma 2.1 and Hölder inequality, we have

(3.17)

$$\begin{aligned}
\mathbb{E} \sup_{0\le s\le t} \Gamma_5^p(s) &\le c_5(p) t^{p-1} \mathbb{E} \int_0^t |(I + \nabla u(X(s)))B(X_s) - (I + \nabla u(y(s)))B(\bar{y}_s)|^p \mathrm{d}s \\
&\le c_5(p) t^{p-1} \mathbb{E} \int_0^t |(I + \nabla u(X(s)))B(X_s) - (I + \nabla u(X(s)))B(y_s)|^p \mathrm{d}s \\
&\quad + c_5(p) t^{p-1} \mathbb{E} \int_0^t |(I + \nabla u(X(s)))B(y_s) - (I + \nabla u(X(s)))B(\bar{y}_s)|^p \mathrm{d}s \\
&\quad + c_5(p) t^{p-1} \mathbb{E} \int_0^t |(I + \nabla u(X(s)))B(\bar{y}_s) - (I + \nabla u(y(s)))B(\bar{y}_s)|^p \mathrm{d}s \\
&\le c_5(p) t^{p-1} \int_0^t \mathbb{E} \sup_{0\le q\le s} |X(q) - y(q)|^p \mathrm{d}s \\
&\quad + c_5(p) t^{p-1} \mathbb{E} \int_0^t \int_{[-r,0]} |y_s(v) - \bar{y}_s(v)|^p \mathrm{d}\rho(v) \mathrm{d}s \\
&\le c_5(p,T) t^p \mathbb{E} \sup_{0\le s\le t} |X(s) - y(s)|^p + c_5(p,T) \Delta^{\frac{p}{\alpha}}
\end{aligned}$$

where $c_5(p,T)$ is a positive constant. Similarly, we have

$$\mathbb{E} \sup_{0\le s\le t} \Gamma_2^p(s) \le c_2(p,\lambda) t^p \mathbb{E} \sup_{0\le s\le t} |X(s) - y(s)|^p. \tag{3.18}$$

Next, by (2.9), we obtain

$$\mathbb{E} \sup_{0\le s\le t} \Gamma_1^p(s) \le c_1(p,T,\nu) t^{\frac{p}{2}} \mathbb{E} \sup_{0\le s\le t} |X(s) - y(s)|^p, \tag{3.19}$$

and by (2.11), it holds that

$$\mathbb{E} \sup_{0\le s\le t} \Gamma_3^p(s) \le c_3(p,T,\nu,\alpha,\beta) t^{\frac{p}{2}} \mathbb{E} \sup_{0\le s\le t} |X(s) - y(s)|^p. \tag{3.20}$$

Fix the constant $p \in (0, \alpha)$ given in assumption **(H4)**. Finally, by assumption **(H2)**, Lemma 2.1, Lemma 2.2, Hölder inequality and Jensen inequality,

$$\begin{aligned}\mathbb{E} \sup_{0 \le s \le t} \Gamma_4^p(s) &\le c_4(p) t^{p-1} \mathbb{E} \int_0^t |(I + \nabla u(X_s^m))(b(\bar{y}_s(0)) - b(y(s))|^p \mathrm{d}s \\ &\le c_4(p, T, \nu) \Delta^{\frac{p\beta}{\alpha}}\end{aligned} \tag{3.21}$$

holds for a constant $c_4(p, T, \nu) > 0$.

Combining formulas (3.16)–(3.21), we get

$$\mathbb{E} \sup_{0 \le s \le t} |X(s) - y(s)|^p \le c(p, T, \nu) \Delta^{\frac{p\beta}{\alpha}} + c(p, T, \nu, \lambda, \alpha, \beta, \gamma) t^{\frac{p}{2}} \mathbb{E} \sup_{0 \le s \le t} |X(s) - y(s)|^p.$$

Taking $t_0 := [2c(p, T, \nu, \lambda, \alpha, \beta, \gamma)]^{-\frac{p}{2}}$. Since functions b and B are bounded, $(X(t) - y(t))$ is a bounded process, and then

$$\mathbb{E} \sup_{0 \le s \le t_0} |X(s) - y(s)|^p \le c(p, T, \nu) \Delta^{\frac{p\beta}{\alpha}}.$$

Finally, by recursion, it is easy to see that

$$\mathbb{E} \sup_{0 \le s \le T} |X(s) - y(s)|^p \le c(p, T, \nu) \left(\lfloor T/t_0 \rfloor + 1 \right) \Delta^{\frac{p\beta}{\alpha}}.$$

□

4 Appendix

Lemma 4.1. [Lenglart's inequality [6]] *Let $M(t)$ be a nonnegative càdlàg process and $A(t)$ be an increasing predictable process on some probability space. If for any finite stopping time τ, it holds that*

$$\mathbb{E} M(\tau) \le \mathbb{E} A(\tau),$$

then for any $p \in (0, 1)$ and stopping time τ, the following inequality holds:

$$\mathbb{E} \sup_{0 \le s \le \tau} M(s)^p \le \frac{2 - p}{1 - p} \mathbb{E}[A(\tau)]^p.$$

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