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# DECOMPOSITIONS AND BANG-BANG PROPERTIES

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Abstract. We study the bang-bang properties of minimal time and minimal norm control problems (where the target set is the origin of the state space and the controlled system is linear and time-invariant) from a new perspective. More precisely, we study how the bang-bang property of each minimal time (or minimal norm) problem depends on a pair of parameters $(M, y_0)$ (or $(T, y_0)$), where $M > 0$ is a bound of controls and $y_0$ is the initial state (or $T > 0$ is an ending time and $y_0$ is the initial state). The controlled system may have neither the $L^\infty$-null controllability nor the backward uniqueness property.

## Contents



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## 1. **Introduction.**

### 1.1. **Motivation.**

Two kinds of important optimal control problems for linear controlled systems are minimal time control problems and minimal norm control problems. A minimal time control problem is to ask for a control (taking values from a control constraint set which is, in general, a closed and bounded subset in a control space) which drives the corresponding solution of a controlled system from an initial state to a target set in the shortest time, while a minimal norm control problem is to ask for a control which has the minimal norm among all controls that drive the corresponding solutions of a controlled systems from an initial state to a target set at fixed ending time. Several important issues on minimal time (or minimal norm) control problems are as follows: The Pontryagin maximum principle of minimal time (or minimal norm) controls (see, for instance, [8, 19, 22, 24, 46]); The existence of minimal time ( or minimal norm) controls (see, for instance, [3, 23, 34]); Their connections with controllabilities (see, for instance [4, 13, 30]); Numerical analyses on minimal time (or minimal norm) controls (see, for instance, [12, 14, 27, 37, 45]); And the bang-bang property of minimal time (or minimal norm) controls (see, for instance, [6, 18, 19, 22, 25, 26, 28, 31, 33, 36, 40, 42, 43, 44, 47, 49]).

In this paper, we concern the bang-bang properties for these two kinds of problems in the case that both state and control spaces are real Hilbert spaces, controlled systems are linear and time-invariant, target sets are the origin of state spaces, control constraint sets are closed balls in control spaces (centered at the origin) and controls are $L^\infty$ functions. The bang-bang property for a minimal time control problem means that any minimal time control, as a function of time, point-wisely takes its value at the boundary of the control constraint set, while the bang-bang property for a minimal norm control problem means that each minimal norm control, as a function of time, point-wisely takes the minimal norm. The significance of the bang-bang property for minimal time control problems can be explained from the following aspects: (i) Mathematically, the bang-bang property means that each minimizer of a functional (from $[0,\infty)$ to a bounded and closed subset in a Hilbert space) point-wisely takes value on the boundary of this subset. (ii) From application point of view, the bang-bang property means that each minimal time control takes the most advantage of possible control actions. For instance, controls always have bounds which are designed by peoples. The bigger bounds are designed, the more costs peoples pay. If the bang-bang property holds for a minimal time problem, then the designed bound for controls will not be wasted at almost each time. (iii) The bang-bang property is powerful in the studies of minimal time control problems. For instance, in many cases, the uniqueness of minimal time controls follows from this property; in some cases, this property can help people to do more dedicate numerical analyses on minimal time controls (see, for instance, [14, 37]). We can also explain the significance of the bang-bang property for minimal norm control problems from both mathematical and application points of view. In most literatures on the bang-bang property for the minimal time (or minimal norm) control problems, peoples mainly concern about: (i) For a given problem, whether the bang-bang property holds; (ii) Applications of the bang-bang property (see, for instance, [6, 18, 19, 22, 25, 26, 28, 31, 33, 36, 40, 42, 43, 44, 47, 49] and the references therein).

In this paper, we study the bang-bang properties of the minimal time control problems and the minimal norm control problems from a different perspective. The motivation of this study is as follows: Two typical minimal time and minimal norm control problems in the finitely dimensional setting are as follows: Let $\mathbb{R}^n$ and $\mathbb{R}^m$ (with $n, m \in \mathbb{N}^+$) be the state space and the control space. Let $(A, B)$ be a pair of matrices in $\mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$. Given $M > 0$ and $y_0 \in \mathbb{R}^n \setminus \{0\}$, consider the minimal time control problem:

$$(\mathcal{TP})^{M,y_0} \qquad \mathcal{T}(M, y_0) := \{\hat{t} > 0 \ : \ \exists\, u \in \mathbb{U}^M \ \text{ s.t. } \ y(\hat{t}; y_0, u) = 0\}, \tag{1.1}$$

where

$$\mathbb{U}^M := \{u : \ \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^m \ \text{ measurable } \ : \ \|u\|_{L^\infty(\mathbb{R}^+;\mathbb{R}^m)} \le M\}, \tag{1.2}$$

and $y(\cdot; y_0, u)$ is the solution to the equation:

$$y'(t) = Ay(t) + Bu(t),\ t > 0; \ \ y(0) = y_0.$$

Given $y_0 \in \mathbb{R}^n \setminus \{0\}$ and $T \in (0,\infty)$, consider the minimal norm control problem:

$$(\mathcal{NP})^{T,y_0} \qquad \mathcal{N}(T, y_0) := \inf\{\|v\|_{L^\infty(0,T;\mathbb{R}^m)} \ : \ \hat{y}(T; y_0, v) = 0\}, \tag{1.3}$$

where $v \in L^\infty(0, T; \mathbb{R}^m)$ and $\hat{y}(\cdot; y_0, v)$ is the solution to the equation:

$$y'(t) = Ay(t) + Bv(t),\ 0 < t \le T; \ \ y(0) = y_0. \tag{1.4}$$

In the problem $(\mathcal{TP})^{M,y_0}$, $\mathcal{T}(M, y_0)$ is called the minimal time; $\hat{u} \in \mathbb{U}^M$ is called an admissible control if $y(\hat{t}; y_0, \hat{u}) = 0$ for some $\hat{t} \in (0,\infty)$; $u^* \in \mathbb{U}^M$ is called a minimal time control if $y(\mathcal{T}(M, y_0); y_0, u^*) = 0$ and $u^* = 0$ over $(0, \mathcal{T}(M, y_0))$. We say that the problem $(\mathcal{TP})^{M,y_0}$ has the bang-bang property if any minimal time control $u^*$ verifies that $\|u^*(t)\|_{\mathbb{R}^m} = M$ for a.e. $t \in \big(0, \mathcal{T}(M, y_0)\big)$. When $(\mathcal{TP})^{M,y_0}$ has no admissible control, we agree that it does not hold the bang-bang property and $\mathcal{T}(M, y_0) = \infty$. In the problem $(\mathcal{NP})^{T,y_0}$, $\mathcal{N}(T, y_0)$ is called the minimal norm; $\hat{v} \in L^\infty(0, T; \mathbb{R}^m)$ is called an admissible control if $\hat{y}(T; y_0, \hat{v}) = 0$; $v^* \in L^\infty(0, T; \mathbb{R}^m)$ is called a minimal norm control if $\|v^*\|_{L^\infty(0,T;\mathbb{R}^m)} = \mathcal{N}(T, y_0)$ and $\hat{y}(T; y_0, v^*) = 0$. We say that the problem $(\mathcal{NP})^{T,y_0}$ has the bang-bang property if any minimal norm control $v^*$ verifies that $\|v^*(t)\|_{\mathbb{R}^m} = \mathcal{N}(T, y_0)$ for a.e. $t \in (0, T)$.

When $(\mathcal{NP})^{T,y_0}$ has no admissible control, we agree that it does not hold the bang-bang property and $\mathcal{N}(T,y_0)=\infty$.

When $(A,B)$ is fixed in $\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$, the problem $(\mathcal{TP})^{M,y_0}$ depends only on the pair $(M,y_0)$ which belongs to the product space:

$$\mathcal{X}_1 := \big\{(M,y_0)\ :\ 0<M<\infty,\ y_0\in\mathbb{R}^n\setminus\{0\}\big\}; \tag{1.5}$$

and the problem $(\mathcal{NP})^{T,y_0}$ depends only on the pair $(T,y_0)$ which belongs to the space:

$$\mathcal{X}_2 := \big\{(T,y_0)\ :\ 0<T<\infty,\ y_0\in\mathbb{R}^n\setminus\{0\}\big\}.$$

By applying the Kalman controllability decomposition to the pair $(A,B)$ (see, for instance, Lemma 3.3.3 and Lemma 3.3.4 in [38]), we can easily divide the space $\mathcal{X}_1$ into two disjoint parts so that when $(M,y_0)$ is in one part, the corresponding $(\mathcal{TP})^{M,y_0}$ has the bang-bang property; when $(M,y_0)$ is in another part, the corresponding $(\mathcal{TP})^{M,y_0}$ has no admissible control (which implies that it does not hold the bang-bang property). The same can be said about the space $\mathcal{X}_2$. We call such decompositions as the BBP decompositions for $(\mathcal{TP})^{M,y_0}$ and $(\mathcal{NP})^{T,y_0}$, respectively. The exact BBP decompositions for the above two problems are the following (P1) and (P2):

**(P1)**
- When $(M,y_0)\in\mathcal{D}_{bbp}$, $(\mathcal{TP})^{M,y_0}$ has the bang-bang property;
- When $(M,y_0)\in\mathcal{X}_1\setminus\mathcal{D}_{bbp}$, $(\mathcal{TP})^{M,y_0}$ has no admissible control. (1.6)

Here,

$$\mathcal{D}_{bbp} := \big\{(M,y_0)\in(0,\infty)\times(\mathcal{R}\setminus\{0\})\ :\ M>\lim_{T\to\infty}\mathcal{N}(T,y_0)\big\}, \tag{1.7}$$

where

$$\mathcal{R} := \mathcal{B}+A\mathcal{B}+\cdots+A^n\mathcal{B},\ \text{ with }\ \mathcal{B}:=\{Bx\in\mathbb{R}^n\ :\ x\in\mathbb{R}^m\}. \tag{1.8}$$

**(P2)**
- When $(T,y_0)\in\mathcal{X}_{2,1}$, $(\mathcal{NP})^{T,y_0}$ has the bang-bang property;
- When $(T,y_0)\in\mathcal{X}_{2,2}$, $(\mathcal{NP})^{T,y_0}$ has no admissible control, (1.9)

where $\mathcal{X}_{2,1}:=(0,\infty)\times(\mathcal{R}\setminus\{0\})$ and $\mathcal{X}_{2,2}:=(0,\infty)\times(\mathbb{R}^n\setminus\mathcal{R})$. (Notice that both $\mathcal{D}_{bbp}$ and $\mathcal{R}\setminus\{0\}$ are not empty. These are proved in Appendix A, see (8.13) and (8.1).)

The proofs of (P1) and (P2), via the Kalman controllability decomposition, are given in Appendix A. Though the proofs are quite simple, such BBP decompositions seem to be new. (At least we do not find them in any published literature.) *A natural question is how to extend the above-mentioned BBP decompositions to the infinitely dimensional setting where state and control spaces are two real Hilbert spaces, $A$ is a generator of a $C_0$-semigroup on the state space and $B$ is a linear operator from the control space to the state space.* The purpose of this paper is to build up such BBP decompositions in the infinitely dimensional setting. The main difficulty to get such extension is the lack of the Kalman controllability decomposition in the infinitely dimensional setting.

Our first key to overcome this difficulty is to find two properties held by any pair of matrices $(A,B)$ in $\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$ so that they have the following functionalities: (i) With the aid of these properties, we can get the decompositions (P1) and (P2), without using the Kalman controllability decomposition; (ii) These properties can be easily stated in the infinitely dimensional setting. The first one is a kind of unique continuation property from measurable sets for functions: $B^*e^{A^*(T-\cdot)}z$, with $T>0$ and $z\in\mathbb{R}^n$. This property follows immediately from the analyticity of the function $t \to B_1^* e^{A_1^*(T-t)}$, $t \in \mathbb{R}$, in the finitely dimensional setting. In our infinitely dimensional setting, it is the assumption (H2) given in the next subsection. The second property is quite hidden: *For all $t$ and $T$, with $0 < t < T < \infty$, and $u \in L^2(0,T;\mathbb{R}^m)$, with supp $u \subset (0,t)$, there is $v_u \in L^\infty(0,T;\mathbb{R}^m)$, with supp $v \subset (t,T)$, so that $\hat{y}(T;0,u) = \hat{y}(T;0,v_u)$, where $\hat{y}(\cdot;0,u)$ and $\hat{y}(\cdot;0,v_u)$ denote the solutions of (1.4) with the same initial datum $0$ and controls $u$ and $v_u$, respectively.* (Proposition 13 in Appendix B proves that each pair of matrices $(A,B)$ in $\mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$ holds this property.) The assumption (H1) given in the next subsection is exactly the same version of the second property in our finitely dimensional setting. About (H1), two facts are given in order: First, for a pair $(A,B)$ in the finitely dimensional setting, it may happen that the above-mentioned second property holds but $(A,B)$ is not controllable. Second, even in the infinitely dimensional setting, the null controllability of $(A,B)$ implies that the above-mentioned second property (see Proposition 9).

1.2. **Problems and assumptions.** Let us first introduce the minimal time and the minimal norm control problems studied in this paper. Let $X$ be a real Hilbert space (which is our state space), with its inner product $\langle\cdot,\cdot\rangle_X$ and its norm $\|\cdot\|_X$. Let $A : D(A) \subset X \to X$ be a state operator which generates a $C_0$-semigroup $\{S(t)\}_{t\in\mathbb{R}^+}$ on $X$. Write $U$ for another real Hilbert space (which is our control space), with its inner product $\langle\cdot,\cdot\rangle_U$ and its norm $\|\cdot\|_U$. Let $B \in \mathcal{L}(U, X_{-1})$ be a nontrivial control operator (i.e., $B \neq 0$), where $X_{-1} := D(A^*)'$ is the dual of $D(A^*)$ with respect to the pivot space $X$. Throughout this paper, we assume that $B$ is an admissible control operator for $\{S(t)\}_{t\in\mathbb{R}^+}$ (see Section 4.2 in [39]), i.e., for each $\hat{t} \in (0,\infty)$, there is a positive constant $C_1(\hat{t})$, depending on $\hat{t}$, so that

$$\Big\| \int_0^{\hat{t}} S_{-1}(\hat{t}-\tau)Bu(\tau)\,\mathrm{d}\tau \Big\|_X \le C_1(\hat{t})\|u\|_{L^2(0,\hat{t};U)} \text{ for all } u \in L^2_{loc}(\mathbb{R}^+;U), \quad (1.10)$$

where $\{S_{-1}(t)\}_{t\in\mathbb{R}^+}$ denotes the extension of $\{S(t)\}_{t\in\mathbb{R}^+}$ on $X_{-1}$. In the finitely dimensional setting where $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m} \setminus \{0\}$, (1.10) holds automatically.

Two controlled equations studied in this paper are as follows:

$$y'(t) = Ay(t) + Bu(t),\ t > 0;\ \ y(0) = y_0; \quad (1.11)$$

$$y'(t) = Ay(t) + Bv(t),\ 0 < t \le T;\ \ y(0) = y_0. \quad (1.12)$$

Here, $y_0 \in X$, $T > 0$, controls $u$ and $v$ are taken from $L^\infty(\mathbb{R}^+;U)$ and $L^\infty(0,T;U)$, respectively. For each $T > 0$, $y_0 \in X$ and $v \in L^2(0,T;U)$, a solution of the equation (1.12) is defined to be a function $\hat{y}(\cdot;y_0,v) \in C([0,T];X)$ satisfying that when $z \in D(A^*)$,

$$\langle \hat{y}(t;y_0,v), z\rangle_X - \langle y_0, S^*(t)z\rangle_X = \int_0^t \langle v(s), B^*S^*(t-s)z\rangle_U\,\mathrm{d}s,\ \forall\, t \in [0,T]. \quad (1.13)$$

One can easily see from Lemma 2.1 that the definition of $\hat{y}(\cdot;y_0,v)$ is the same as the definition of a solution to (1.12) in [5, Definition 2.36]. Thus, it follows from [5, Theorem 2.37] and Lemma 2.1 that the equation (1.12) is well-posed. For each $y_0 \in X$ and $u \in L^\infty(\mathbb{R}^+;U)$, a solution of the equation (1.11) is defined to be a function $y(\cdot;y_0,u) \in C(\mathbb{R}^+;X)$ so that for each $T > 0$, $y(\cdot;y_0,u)|_{[0,T]}$ (the restriction of $y(\cdot;y_0,u)$ over $[0,T]$) is the solution to (1.12) with $v = u|_{(0,T]}$. Consequently, the system (1.11) is well-posed. Besides, by Proposition 1, one can check the following two facts: First, for each $y_0 \in X$ and $u \in L^\infty(\mathbb{R}^+; U)$, the solution $y(\cdot; y_0, u)$ to the system (1.11) satisfies that

$$y(t; y_0, u) = S(t)y_0 + \int_0^t S_{-1}(t-\tau)Bu(\tau)\,\mathrm{d}\tau, \quad 0 \le t < \infty. \tag{1.14}$$

Second, if for some $y_0 \in X$ and $u \in L^\infty(\mathbb{R}^+; U)$, a function $y(\cdot) \in C(\mathbb{R}^+; X)$ equals to the right hand side of (1.14) point-wisely, then $y(\cdot) = y(\cdot; y_0, u)$ over $\mathbb{R}^+$.

For each pair $(M, y_0) \in (0, \infty) \times (X \setminus \{0\})$, we define a minimal time control problem:

$$(TP)^{M,y_0} \qquad T(M, y_0) := \inf\left\{\hat{t} \in (0,\infty) \ : \ \exists\, u \in \mathcal{U}^M \text{ s.t. } y(\hat{t}; y_0, u) = 0\right\}, \tag{1.15}$$

where

$$\mathcal{U}^M := \left\{u : \mathbb{R}^+ \to U \text{ strongly measurable} \ : \ \|u(t)\|_U \le M \text{ a.e. } t \in \mathbb{R}^+\right\}.$$

In the problem $(TP)^{M,y_0}$, the minimal time, an admissible control and a minimal time control can be defined in the same manners as in $(\mathcal{TP})^{M,y_0}$ (see (1.1)). We say that the problem $(TP)^{M,y_0}$ has the bang-bang property if any minimal time control $u^*$ verifies that $\|u^*(t)\|_U = M$ for a.e. $t \in \big(0, T(M, y_0)\big)$. When $(TP)^{M,y_0}$ has no admissible control, we agree that it does not hold the bang-bang property and $T(M, y_0) = \infty$.

For each pair $(T, y_0) \in (0, \infty) \times (X \setminus \{0\})$, we define a minimal norm control problem:

$$(NP)^{T,y_0} \ N(T, y_0) := \inf\{\|v\|_{L^\infty(0,T;U)} \ : \ v \in L^\infty(0,T;U) \text{ s.t. } \hat{y}(T; y_0, v) = 0\}. \tag{1.16}$$

In the problem $(NP)^{T,y_0}$, the minimal norm, an admissible control and a minimal norm control can be defined in the same ways as in $(\mathcal{NP})^{T,y_0}$ (see (1.3)). We say that the problem $(NP)^{T,y_0}$ has the bang-bang property if any minimal norm control $v^*$ verifies that $\|v^*(t)\|_U = N(T, y_0)$ for a.e. $t \in (0, T)$. When $(NP)^{T,y_0}$ has no admissible control, we agree that it does not hold the bang-bang property and $N(T, y_0) = \infty$.

We say that $(A, B)$ has the $L^\infty$-null controllability if for any $T > 0$ and $y_0 \in X$, there is $v \in L^\infty(0, T; U)$ so that $\hat{y}(T; y_0, v) = 0$. We say that the semigroup $\{S(t)\}_{t\in\mathbb{R}^+}$ has the backward uniqueness property if $S(T)y_0 = 0 \Rightarrow y_0 = 0$. In our infinitely setting, we assume neither the $L^\infty$-null controllability nor the backward uniqueness property. To make up the lack of these properties, we define the following two functions $T^0(\cdot)$ and $T^1(\cdot)$ (which play important roles in our study):

$$T^0(y_0) := \inf\left\{\hat{t} \in \mathbb{R}^+ \ : \ \exists\, u \in L^\infty(\mathbb{R}^+; U) \text{ s.t. } y(\hat{t}; y_0, u) = 0\right\}, \quad y_0 \in X; \tag{1.17}$$

$$T^1(y_0) := \inf\left\{\hat{t} \in \mathbb{R}^+ \ : \ S(\hat{t})y_0 = 0\right\}, \quad y_0 \in X. \tag{1.18}$$

When the set on the right hand side of (1.17) is empty for some $y_0$, we let $T^0(y_0) := \infty$. The same can be said about $T^1(y_0)$.

**Remark 1.** (i) The pair $(A, B)$ has the $L^\infty$-null controllability if and only if for each $y_0 \in X$, $T^0(y_0) = 0$.
(ii) Though many controlled systems, such as internally or boundary controlled heat equations, hold the $L^\infty$-null controllability, there are some controlled systems having no the $L^\infty$-null controllability. Among them, it may happen that $T^0(y_0) \in (0, \infty)$ for some $y_0 \in X$ (see Remark 14).
(iii) The semigroup $\{S(t)\}_{t\in\mathbb{R}^+}$ has the backward uniqueness property if and only if for each $y_0 \in X \setminus \{0\}$, $T^1(y_0) = \infty$.

(iv) Though many semigroups governed by PDEs, such as heat equations and wave equations, hold the backward uniqueness property, there are some semigroups governed by PDEs having no this property. Among them, it may happen that $T^1(y_0) < \infty$ for all $y_0 \in X$. A transport equation over a finite interval is one of such examples.

From (1.16), we see that for each $y_0 \in X \setminus \{0\}$, $T \to N(T, y_0)$ defines a function over $(0, \infty)$. Since the quantities $N(T^0(y_0), y_0)$ and $N(T^1(y_0), y_0)$ will appear frequently, $T^0(\cdot)$ may take values 0 and $\infty$, and $T^1(\cdot)$ may take value $\infty$, it is necessary for us to give definitions for $N(\infty, y_0)$ and $N(0, y_0)$. For this purpose, we notice that for each $y_0 \in X\setminus\{0\}$, $T \to N(T, y_0)$ is a decreasing function from $(0, \infty)$ to $[0, \infty]$. (This can be easily obtained from (1.16), see also (i) of Lemma 3.2 for the detailed proof.) Thus, we can extend this function over $[0, \infty]$ in the following manner:

$$N(\infty, y_0) := \lim_{t\to\infty} N(t, y_0) \;\text{ and }\; N(0, y_0) := \lim_{t\to 0^+} N(t, y_0), \;\; y_0 \in X \setminus \{0\}. \tag{1.19}$$

As mentioned in Subsection 1.1, we impose two assumptions on $(A, B)$ as follows:
(H1) There is $p_0 \in [2, \infty)$ so that $\mathcal{A}_{p_0}(T, \hat{t}) \subset \mathcal{A}_\infty(T, \hat{t})$ for all $T$, $\hat{t}$, with $0 < \hat{t} < T < \infty$, where

$$\mathcal{A}_{p_0}(T, \hat{t}) := \Big\{\hat{y}(T; 0, u) \;:\; u \in L^{p_0}(0, T; U), \;\text{ with }\; u|_{(\hat{t},T)} = 0\Big\};$$
$$\mathcal{A}_\infty(T, \hat{t}) := \Big\{\hat{y}(T; 0, v) \;:\; v \in L^\infty(0, T; U), \;\text{ with }\; v|_{(0,\hat{t})} = 0\Big\}.$$

(H2) If there is $T \in (0, \infty)$, a subset $E \subset (0, T)$ of positive measure and a function $f \in Y_T$ so that $f = 0$ over $E$, then $f \equiv 0$ over $(0, T)$. Here,

$$Y_T := \overline{X}_T^{\|\cdot\|_{L^1(0,T;U)}}, \;\text{ with the }\; L^1(0, T; U)\text{-norm}, \tag{1.20}$$

where

$$X_T := \{B^*S^*(T - \cdot)z|_{(0,T)} \;:\; z \in D(A^*)\}, \;\text{ with the }\; L^1(0, T; U)\text{-norm}. \tag{1.21}$$

**Remark 2.** (i) The assumption (H1) says roughly that the functionality of a control supported on $(0, \hat{t})$ can be replaced by that of a control supported on $(\hat{t}, T)$. The assumption (H2) says, in plain language, that any function in $Y_T$ has some unique continuation property from measurable sets.
(ii) We do not know if every function in $Y_T$ can be expressed as $B^*\varphi$ with $\varphi$ a solution of the adjoint equation over $(0, T)$, even in the case that $B \in \mathcal{L}(U, X)$. However, if $(A, B)$ has the $L^\infty$-null controllability, then the above-mentioned expression holds (see Remark 12).
(iii) Each pair $(A, B)$ in $\mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$ (with $n, m \geq 1$) satisfies both (H1) and (H2) (see Proposition 13 in Appendix B).

Our studies on the BBP decompositions are based on the assumptions (H1) and (H2). However, our main results can be improved, if instead of (H1) and (H2), we impose the following stronger assumptions (H3) and (H4):
(H3) The pair $(A^*, B^*)$ is $L^1$-observable over each interval (or simply $L^1$-observable), i.e., for each $T \in (0, \infty)$, there exists a positive constant $C_1(T)$ so that

$$\|S^*(T)z\|_X \leq C_1(T) \int_0^T \|B^*S^*(T - t)z\|_U \,\mathrm{d}t \;\text{ for all }\; z \in D(A^*).$$

(H4) If $z \in X$ satisfies that $\widetilde{B^*S^*}(T - \cdot)z = 0$ over $E$ for some $T \in (0, \infty)$ and some subset $E \subset (0, T)$ of positive measure, then $\widetilde{B^*S^*}(T - \cdot)z \equiv 0$ over $(0, T)$. Here,

$\widetilde{B^*S^*}(T-\cdot)$ is the natural extension of $B^*S^*(T-\cdot)$ over $X$. (It will be explained in the next remark.)

**Remark 3.** (i) The function $\widetilde{B^*S^*}(T-\cdot)$ in (H4) is defined in the following manner: Since $B\in\mathcal{L}(U,X_{-1})$ is an admissible control operator for $\{S(t)\}_{t\in\mathbb{R}^+}$, it follows from Lemma 2.1 that $B^*$ is an admissible observation operator for $\{S^*(t)\}_{t\in\mathbb{R}^+}$, i.e., for each $T\in(0,\infty)$, there exists a $C(T)>0$ so that

$$\int_0^T \|B^*S^*(T-\tau)z\|_U^2\,\mathrm{d}\tau \le C(T)\|z\|_X^2 \ \text{ for all } \ z\in D(A^*).$$

(Indeed, [39, Theorem 4.4.3] proves that $B\in\mathcal{L}(U,X_{-1})$ is an admissible control operator for $\{S(t)\}_{t\in\mathbb{R}^+}$ if and only if $B^*$ is an admissible observation operator for $\{S^*(t)\}_{t\in\mathbb{R}^+}$ in the case where $X$ and $U$ are complex Hilbert spaces.) Thus, for each $T\in(0,\infty)$, the operator $B^*S^*(T-\cdot):D(A^*)\to L^2(0,T;U)$ can be uniquely extended to a linear bounded operator $\widetilde{B^*S^*}(T-\cdot)$ from $X$ to $L^2(0,T;U)$. More precisely, for each $z\in X$,

$$\widetilde{B^*S^*}(T-\cdot)z := \lim_{n\to\infty} B^*S^*(T-\cdot)z_n \ \text{ in } \ L^2(0,T;U), \tag{1.22}$$

where $\{z_n\}\subset D(A^*)$, with $\lim_{n\to\infty} z_n = z$ in $X$.
(ii) The condition (H3) is an $L^1$-observability estimate for the pair $(A^*,B^*)$, which is equivalent to the $L^\infty$-null controllability for the pair $(A,B)$. (See Proposition 8.)
(iii) The condition (H4) is a kind of unique continuation property of the dual equation over $(0,T)$ for each $T\in(0,\infty)$.
(iv) The condition (H1) can be implied by (H3) (see Proposition 9). However, (H1) may hold when (H3) does not stand. For instance, when $X=\mathbb{R}^n$ and $U=\mathbb{R}^m$ (with $n,m\in\mathbb{N}^+$), any pair $(A,B)\in\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$ satisfies not only the condition (H1) but also the condition (H2) (see Proposition 13 in Appendix B). On the other hand, it is well known that $(A,B)$ is $L^\infty$-null controllable if and only if it is controllable, and the later holds if and only if $(A,B)$ satisfies the Kalman rank condition. Thus any $(A,B)\in\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$ that does not satisfy the Kalman rank condition has the property (H1) but does not hold the property (H3).
(v) The condition (H2) can be derived from (H3) and (H4) (see Proposition 10).

1.3. **Main results.** The main results of this paper concern with the BBP decompositions for $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$. To state them, we notice that the domain $\mathcal{W}$ of the pairs $(T,y_0)$ for $(NP)^{T,y_0}$ and the domain $\mathcal{V}$ of the pairs $(M,y_0)$ for $(TP)^{M,y_0}$ are the following spaces:

$$\mathcal{W}=\big\{(T,y_0)\ :\ 0<T<\infty,\ y_0\in X\setminus\{0\}\big\} \tag{1.23}$$

and

$$\mathcal{V}=\big\{(M,y_0)\ :\ 0<M<\infty,\ y_0\in X\setminus\{0\}\big\}. \tag{1.24}$$

In the domain $\mathcal{W}$, we define the following subsets:

$$\begin{aligned}\mathcal{W}_{1,1} &:= \{(T,y_0)\in\mathcal{W}_1\ :\ T<T^0(y_0)\},\\ \mathcal{W}_{1,2} &:= \{(T,y_0)\in\mathcal{W}_1\ :\ T\ge T^0(y_0)\},\end{aligned} \tag{1.25}$$

where

$$\mathcal{W}_1 := \{(T,y_0)\in\mathcal{W}\ :\ N(T^0(y_0),y_0)=0\}; \tag{1.26}$$

$$\begin{aligned}\mathcal{W}_{2,1} &:= \{(T,y_0)\in\mathcal{W}_2 \;:\; T<T^0(y_0)\},\\ \mathcal{W}_{2,2} &:= \{(T,y_0)\in\mathcal{W}_2 \;:\; T=T^0(y_0)\},\\ \mathcal{W}_{2,3} &:= \{(T,y_0)\in\mathcal{W}_2 \;:\; T^0(y_0)<T<T^1(y_0)\},\\ \mathcal{W}_{2,4} &:= \{(T,y_0)\in\mathcal{W}_2 \;:\; T^0(y_0)<T,\, T\ge T^1(y_0)\},\end{aligned} \tag{1.27}$$

where

$$\mathcal{W}_2 := \{(T,y_0)\in\mathcal{W} \;:\; 0<N(T^0(y_0),y_0)<\infty\}; \tag{1.28}$$

$$\begin{aligned}\mathcal{W}_{3,1} &:= \{(T,y_0)\in\mathcal{W}_3 \;:\; T^0(y_0)<\infty,\, T\le T^0(y_0)\},\\ \mathcal{W}_{3,2} &:= \{(T,y_0)\in\mathcal{W}_3 \;:\; T^0(y_0)<\infty,\, T^0(y_0)<T<T^1(y_0)\},\\ \mathcal{W}_{3,3} &:= \{(T,y_0)\in\mathcal{W}_3 \;:\; T^0(y_0)<\infty,\, T^0(y_0)<T,\, T\ge T^1(y_0)\},\\ \mathcal{W}_{3,4} &:= \{(T,y_0)\in\mathcal{W}_3 \;:\; T^0(y_0)=\infty\},\end{aligned} \tag{1.29}$$

where

$$\mathcal{W}_3 := \{(T,y_0)\in\mathcal{W} \;:\; N(T^0(y_0),y_0)=\infty\}. \tag{1.30}$$

In the domain $\mathcal{V}$, we define the following subsets:

$$\mathcal{V}_1 \;:=\; \{(M,y_0)\in\mathcal{V} \;:\; N(T^0(y_0),y_0)=0\}; \tag{1.31}$$

$$\begin{aligned}\mathcal{V}_{2,1} &:= \{(M,y_0)\in\mathcal{V}_2 \;:\; M\le N(T^1(y_0),y_0)\},\\ \mathcal{V}_{2,2} &:= \{(M,y_0)\in\mathcal{V}_2 \;:\; N(T^1(y_0),y_0)<M<N(T^0(y_0),y_0)\},\\ \mathcal{V}_{2,3} &:= \{(M,y_0)\in\mathcal{V}_2 \;:\; N(T^1(y_0),y_0)<M,\, M=N(T^0(y_0),y_0)\},\\ \mathcal{V}_{2,4} &:= \{(M,y_0)\in\mathcal{V}_2 \;:\; N(T^1(y_0),y_0)<M,\, M>N(T^0(y_0),y_0)\},\end{aligned} \tag{1.32}$$

where

$$\mathcal{V}_2 := \{(M,y_0)\in\mathcal{V} \;:\; 0<N(T^0(y_0),y_0)<\infty\}; \tag{1.33}$$

$$\begin{aligned}\mathcal{V}_{3,1} &:= \{(M,y_0)\in\mathcal{V}_3 \;:\; T^0(y_0)<\infty,\, M\le N(T^1(y_0),y_0)\},\\ \mathcal{V}_{3,2} &:= \{(M,y_0)\in\mathcal{V}_3 \;:\; T^0(y_0)<\infty,\, M>N(T^1(y_0),y_0)\},\\ \mathcal{V}_{3,3} &:= \{(M,y_0)\in\mathcal{V}_3 \;:\; T^0(y_0)=\infty\},\end{aligned} \tag{1.34}$$

where

$$\mathcal{V}_3 := \{(M,y_0)\in\mathcal{V} \;:\; N(T^0(y_0),y_0)=\infty\}. \tag{1.35}$$

The main results of this paper are presented in the following two theorems:

**Theorem 1.1.** *Let $\mathcal{W}$ be given by (1.23). Let $\mathcal{W}_{1,j}$ $(j=1,2)$, $\mathcal{W}_{2,j}$ $(j=1,2,3,4)$, and $\mathcal{W}_{3,j}$ $(j=1,2,3,4)$ be given by (1.25), (1.27) and (1.29), respectively. Then the following conclusions are true:*
*(i) The set $\mathcal{W}$ is the disjoint union of the above mentioned subsets $\mathcal{W}_{i,j}$.*
*(ii) For each $(T,y_0)\in\mathcal{W}_{1,2}\cup\mathcal{W}_{2,4}\cup\mathcal{W}_{3,3}$, $(NP)^{T,y_0}$ has the bang-bang property and the null control is its unique minimal norm control.*
*(iii) Suppose that (H1) and (H2) hold. Then for each $(T,y_0)\in\mathcal{W}_{2,3}\cup\mathcal{W}_{3,2}$, $(NP)^{T,y_0}$ has the bang-bang property and the null control is not a minimal nor-m control to this problem.*
*(iv) For each $(T,y_0)\in\mathcal{W}_{1,1}\cup\mathcal{W}_{2,1}\cup\mathcal{W}_{3,1}\cup\mathcal{W}_{3,4}$, $(NP)^{T,y_0}$ has no admissible control and does not hold the bang-bang property.*
*(v) For each $(T,y_0)\in\mathcal{W}_{2,2}$, $(NP)^{T,y_0}$ has at least one minimal norm control.*

**Theorem 1.2.** *Let $\mathcal{V}$ be given by (1.24). Let $\mathcal{V}_1$, $\mathcal{V}_{2,j}$ ($j = 1, 2, 3, 4$) and $\mathcal{V}_{3,j}$ ($j = 1, 2, 3$) be given by (1.31), (1.32) and (1.34), respectively. Then the following conclusions are true:*
*(i) The set $\mathcal{V}$ is the disjoint union of $\mathcal{V}_1$ and the above mentioned subsets $\mathcal{V}_{i,j}$.*
*(ii) Suppose that (H1) and (H2) hold. Then for each $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$, $(TP)^{M,y_0}$ has the bang-bang property.*
*(iii) Suppose that (H1) holds. Then for each $(M, y_0) \in \mathcal{V}_{2,4}$, $(TP)^{M,y_0}$ has infinitely many different minimal time controls (not including the null control), and does not hold the bang-bang property.*
*(iv) Suppose that (H1) holds. Then for each $(M, y_0) \in \mathcal{V}_1$, $(TP)^{M,y_0}$ has infinitely many different minimal time controls (including the null control), and does not hold the bang-bang property.*
*(v) For each $(M, y_0) \in \mathcal{V}_{3,3}$, $(TP)^{M,y_0}$ has no admissible control and does not hold the bang-bang property. If assume that (H1) holds, then for each $(M, y_0) \in \mathcal{V}_{2,1} \cup \mathcal{V}_{3,1}$, $(TP)^{M,y_0}$ has no admissible control and does not hold the bang-bang property.*
*(vi) For each $(M, y_0) \in \mathcal{V}_{2,3}$, $(TP)^{M,y_0}$ has at least one minimal time control.*

**Remark 4.** To make the BBP decomposition for $(NP)^{T,y_0}$ (i.e., the decomposition of $\mathcal{W}$ given by Theorem 1.1) understood better, a draft is given in Figure 1. We explain it as follows: The abscissa axis denotes the set $X \setminus \{0\}$, while the ordinates axis denotes the set of time variables $T > 0$. Each $p_i$ (with $i = 1, 2, 3, 4$) on the abscissa axis is a "point" of the set $X \setminus \{0\}$.

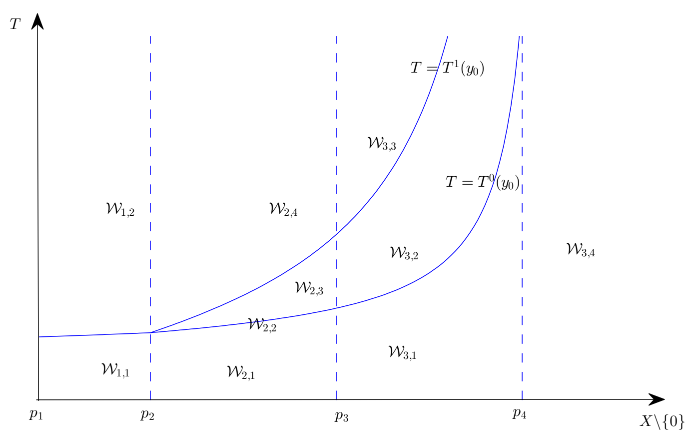


FIGURE 1. The BBP decomposition for $(NP)^{T,y_0}$

In Figure 1, some notations are explained as follows:

- $(p_1, p_2]$ denotes the set: $\{y_0 \in X \setminus \{0\} : N(T^0(y_0), y_0) = 0\}$.
- $(p_2, p_3)$ denotes the set: $\{y_0 \in X \setminus \{0\} : 0 < N(T^0(y_0), y_0) < \infty\}$.
- $[p_3, p_4)$ denotes the set: $\{y_0 \in X \setminus \{0\} : N(T^0(y_0), y_0) = \infty, T^0(y_0) < \infty\}$.
- $[p_4, \infty)$ denotes the set: $\{y_0 \in X \setminus \{0\} : N(T^0(y_0), y_0) = \infty, T^0(y_0) = \infty\}$.
- The two curves above the abscissa axis (from the left to the right) respectively denote the graph of the functions: $y_0 \to T^1(y_0)$, $y_0 \in X \setminus \{0\}$ and $y_0 \to T^0(y_0)$, $y_0 \in X \setminus \{0\}$. These two curves coincide over $(p_1, p_2]$.

Let $\mathcal{W}_{1,j}$ $(j=1,2)$, $\mathcal{W}_{2,j}$ $(j=1,2,3,4)$, and $\mathcal{W}_{3,j}$ $(j=1,2,3,4)$ be given by (1.25), (1.27) and (1.29), respectively. Then we conclude that

- The set $\mathcal{W}_{1,1}$ is the region $\{(T,y_0)\ :\ y_0\in(p_1,p_2],\ 0<T<T^0(y_0)\}$;
- The set $\mathcal{W}_{1,2}$ is the region $\{(T,y_0)\ :\ y_0\in(p_1,p_2],\ T^0(y_0)\le T<\infty\}$;
- The set $\mathcal{W}_{2,1}$ is the region $\{(T,y_0)\ :\ y_0\in(p_2,p_3),\ 0<T<T^0(y_0)\}$;
- The set $\mathcal{W}_{2,2}$ is the region $\{(T,y_0)\ :\ y_0\in(p_2,p_3),\ T=T^0(y_0)\}$;
- The set $\mathcal{W}_{2,3}$ is the region $\{(T,y_0)\ :\ y_0\in(p_2,p_3),\ T^0(y_0)<T<T^1(y_0)\}$;
- The set $\mathcal{W}_{2,4}$ is the region $\{(T,y_0)\ :\ y_0\in(p_2,p_3),\ T^1(y_0)\le T<\infty\}$;
- The set $\mathcal{W}_{3,1}$ is the region $\{(T,y_0)\ :\ y_0\in[p_3,p_4),\ 0<T\le T^0(y_0)\}$;
- The set $\mathcal{W}_{3,2}$ is the region $\{(T,y_0)\ :\ y_0\in[p_3,p_4),\ T^0(y_0)<T<T^1(y_0)\}$;
- The set $\mathcal{W}_{3,3}$ is the region $\{(T,y_0)\ :\ y_0\in[p_3,p_4),\ T^1(y_0)\le T<\infty\}$;
- The set $\mathcal{W}_{3,4}$ is the region $\{(T,y_0)\ :\ y_0\in[p_4,\infty),\ 0<T<\infty\}$;
- When $\{S(t)\}_{t\in\mathbb{R}^+}$ has the backward uniqueness property, we have that $T^1(y_0)=\infty$ for all $y_0\in X\setminus\{0\}$. In this case, the curve: $\{(y_0,T^1(y_0))\ :\ y_0\in X\setminus\{0\}\}$ will not appear in Figure 1; $\mathcal{W}_{1,1}\cup\mathcal{W}_{1,2}\cup\mathcal{W}_{2,4}\cup\mathcal{W}_{3,3}=\emptyset$ (see (iv) of Lemma 3.4).

To make the BBP decomposition for $(TP)^{M,y_0}$ (i.e., the decomposition of $\mathcal{V}$ given by Theorem 1.2) understood better, a draft is given in Figure 2. We explain this figure as follows: The abscissa axis denotes the set $X\setminus\{0\}$, while the ordinates axis denotes the variables $M>0$. Each $p_i$, with $i=1,2,3,4$, on the abscissa axis is a "point" of the set $X\setminus\{0\}$.

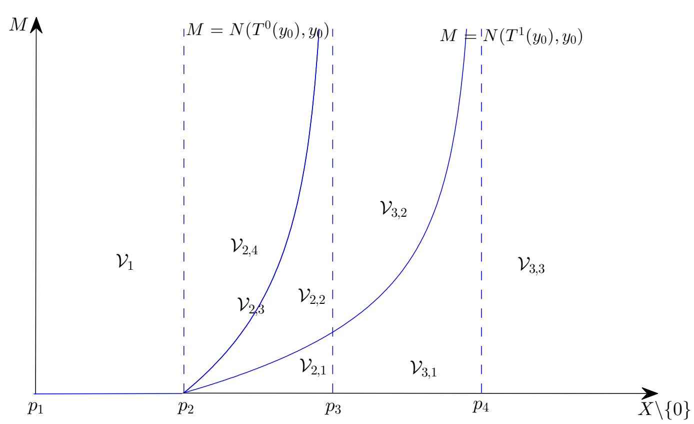


FIGURE 2. The BBP decomposition for $(TP)^{M,y_0}$

In Figure 2, some notations are given in order.

- $(p_1,p_2]$ denotes the set: $\{y_0\in X\setminus\{0\}:\ N(T^0(y_0),y_0)=0\}$.
- $(p_2,p_3)$ denotes the set: $\{y_0\in X\setminus\{0\}:\ 0<N(T^0(y_0),y_0)<\infty\}$.
- $[p_3,p_4)$ denotes the set: $\{y_0\in X\setminus\{0\}:\ N(T^0(y_0),y_0)=\infty,\ T^0(y_0)<\infty\}$.
- $[p_4,\infty)$ denotes the set: $\{y_0\in X\setminus\{0\}:\ N(T^0(y_0),y_0)=\infty,\ T^0(y_0)=\infty\}$.
- The two curves above the abscissa axis (denoted by $F_0$ and $F_1$ from the left to the right) respectively denote the graphs of the functions: $y_0\to N(T^0(y_0),y_0),\ y_0\in X\setminus\{0\}$ and $y_0\to N(T^1(y_0),y_0),\ y_0\in X\setminus\{0\}$. These two curves are identically zero over $(p_1,p_2]$.

Let $\mathcal{V}_1$, $\mathcal{V}_{2,j}$ $(j=1,2,3,4)$, and $\mathcal{V}_{3,j}$ $(j=1,2,3)$ be given by (1.31), (1.32) and (1.34), respectively. Then we have the following conclusions:

- The set $\mathcal{V}_1$ is the region $\{(M,y_0)\ :\ y_0\in(p_1,p_2],\ 0<M<\infty\}$;
- The set $\mathcal{V}_{2,1}$ is the region $\{(M,y_0)\ :\ y_0\in(p_2,p_3),\ 0<M\leq F_1(y_0)\}$;
- The set $\mathcal{V}_{2,2}$ is the region $\{(M,y_0)\ :\ y_0\in(p_2,p_3),\ F_1(y_0)<M<F_0(y_0)\}$;
- The set $\mathcal{V}_{2,3}$ is the region $\{(M,y_0)\ :\ y_0\in(p_2,p_3),\ M=F_0(y_0),\ M\neq F_1(y_0)\}$;
- The set $\mathcal{V}_{2,4}$ is the region $\{(M,y_0)\ :\ y_0\in(p_2,p_3),\ F_0(y_0)<M<\infty\}$;
- The set $\mathcal{V}_{3,1}$ is the region $\{(M,y_0)\ :\ y_0\in[p_3,p_4),\ 0<M\leq F_1(y_0)\}$;
- The set $\mathcal{V}_{3,2}$ is the region $\{(M,y_0)\ :\ y_0\in[p_3,p_4),\ F_1(y_0)<M<\infty\}$;
- The set $\mathcal{V}_{3,3}$ is the region $\{(M,y_0)\ :\ y_0\in[p_4,\infty),\ 0<M<\infty\}$.

**Remark 5.** (i) The decomposition given by Theorem 1.1 is comparable with the decomposition (P2) in Subsection 1.1, except for the part $\mathcal{W}_{2,2}$, which is indeed the following "curve" in the product space $\mathcal{W}$:

$$\gamma_1:=\big\{(T^0(y_0),y_0)\in\mathcal{W}\ :\ 0<N(T^0(y_0),y_0)<\infty\big\}. \tag{1.36}$$

It is a critical curve in the following sense: First, we do not know if it is empty. Second, when $(T,y_0)\in\gamma_1$, we know the corresponding $(NP)^{T,y_0}$ has at least one minimal norm control, but we are not sure if it has the bang-bang property. It deserves to mention that when $(A,B)$ is $L^\infty$-null controllable, this curve is empty (see Theorem 1.3).

(ii) The decomposition given by Theorem 1.2 is comparable with the decomposition (P1) in Subsection 1.1, except for the part $\mathcal{V}_{2,3}$, which is indeed the following "curve" in the product space $\mathcal{V}$:

$$\gamma_2:=\left\{\big(N(T^0(y_0),y_0),y_0\big)\in\mathcal{V}\ :\ \begin{array}{l}0<N(T^0(y_0),y_0)<\infty,\\ N(T^0(y_0),y_0)\neq N(T^1(y_0),y_0)\end{array}\right\}. \tag{1.37}$$

It is a critical curve in the BBP decomposition for $(TP)^{M,y_0}$ in the following sense: First, we do not know if it is empty. Second, when $(M,y_0)\in\gamma_2$, we know the corresponding $(TP)^{M,y_0}$ has at least one minimal time control, but we are not sure if it has the bang-bang property. It deserves to mention that when $(A,B)$ is $L^\infty$-null controllable, this curve is empty (see Theorem 1.3).

**Remark 6.** In the finitely dimensional setting where $(A,B)$ is a pair in $\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$, (with $n,m\in\mathbb{N}^+$), the BBP decompositions for $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$, obtained in Theorem 1.1 and Theorem 1.2, are exactly the same as (P1) and (P2) in Subsection 1.1. This is proved in Appendix C (see Proposition 14).

Under the assumptions (H3) and (H4), the main results obtained in Theorem 1.1 and Theorem 1.2 can be improved as follows:

**Theorem 1.3.** *Let $\mathcal{W}$ and $\mathcal{W}_{3,j}$ $(j=2,3)$ be given by (1.23) and (1.29), respectively. Let $\mathcal{V}$ and $\mathcal{V}_{3,j}$ $(j=1,2)$ be given by (1.24) and (1.34), respectively. Then the following conclusions are true:*
*(i) Suppose that (H3) holds. Then*

$$\mathcal{W}=\mathcal{W}_{3,2}\cup\mathcal{W}_{3,3}\ \ \text{and}\ \ \mathcal{V}=\mathcal{V}_{3,1}\cup\mathcal{V}_{3,2}. \tag{1.38}$$

*In particular,*

$$\gamma_1=\mathcal{W}_{2,2}=\emptyset\ \ \text{and}\ \ \gamma_2=\mathcal{V}_{2,3}=\emptyset. \tag{1.39}$$

*where $\gamma_1$, $\gamma_2$, $\mathcal{W}_{2,2}$ and $\mathcal{V}_{2,3}$ are given respectively by (1.36), (1.37), (1.27) and (1.32).*

*(ii) Suppose that (H3) holds. Then for each $(M, y_0) \in \mathcal{V}_{3,1}$, $(TP)^{M,y_0}$ has no admissible control and does not hold the bang-bang property. If further assume that (H4) holds, then for each $(M, y_0) \in \mathcal{V}_{3,2}$, $(TP)^{M,y_0}$ has the bang-bang property.*
*(iii) For each $(T, y_0) \in \mathcal{W}_{3,3}$, the null control is the unique minimal norm control to $(NP)^{T,y_0}$ and this problem has the bang-bang property. If further assume that (H3) and (H4) hold, then for each $(T, y_0) \in \mathcal{W}_{3,2}$, $(NP)^{T,y_0}$ has the bang-bang property and the null control is not a minimal norm control to this problem.*

1.4. **The ideas to get the main results.** The main difficulty to get the BBP decompositions of $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$ is the lack of the Kalman controllability decomposition. The first key to overcome this difficulty is to find assumptions (H1) and (H2). Then with the aid of functions $T^0(\cdot)$, $T^1(\cdot)$ and $N(\cdot, y_0)$, we get the conclusions (i) in both Theorem 1.1 and Theorem 1.2. In the decomposition of $\mathcal{W}$, the part $\mathcal{W}_{2,2} = \gamma_1$ is a critical curve for us; the studies for the problem $(NP)^{T,y_0}$, with $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, are not easy for us; when $(T, y_0)$ is in the rest parts, it is easy to prove the corresponding conclusions in Theorem 1.1 for $(NP)^{T,y_0}$, through using properties of functions $T^0(\cdot)$, $T^1(\cdot)$ and $N(\cdot, y_0)$. The proof of the corresponding conclusion in Theorem 1.1 for $(NP)^{T,y_0}$, with $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, is mainly based on a maximum principle for $(NP)^{T,y_0}$, as well as (H2). To get the maximum principle, we build up the following affiliated minimal norm problems:

$$(NP)^{y_T} \qquad \|y_T\|_{\mathcal{R}_T} := \inf\big\{\|v\|_{L^\infty(0,T;U)} \ : \ \hat{y}(T; 0, v) = y_T\big\}, \tag{1.40}$$

where $T \in (0, \infty)$ and $y_T$ is in the reachable subspace

$$\mathcal{R}_T := \big\{\hat{y}(T; 0, v) \ : \ v \in L^\infty(0, T; U)\big\}. \tag{1.41}$$

(In the problem $(NP)^{y_T}$, we can define the minimal norm, an admissible control, a minimal norm control and the bang-bang property in the similar manner as in $(NP)^{T,y_0}$ (see (1.16)). By the connection between $(NP)^{y_T}$ and $(NP)^{T,y_0}$ built up in Proposition 3, we realized that the maximum principle for $(NP)^{T,y_0}$ can be derived from a maximum principle for $(NP)^{y_T}$. Though we are not able to get a maximum principle of $(NP)^{y_T}$ for all $y_T \in \mathcal{R}_T$, we get a maximum principle for $(NP)^{y_T}$, with $y_T$ in the subspace:

$$\mathcal{R}^0_T := \big\{\hat{y}(T; 0, v) \ : \ v \in L^\infty(0, T; U), \ \lim_{s\to T} \|v\|_{L^\infty(s,T;U)} = 0\big\}, \ \ T \in (0, \infty). \tag{1.42}$$

More precisely, we obtain that if (H1) holds, then for each $y_T \in \mathcal{R}^0_T \setminus \{0\}$, there exists a vector $f^* \in Y_T \setminus \{0\}$ so that each minimal norm control $v^*$ to $(NP)^{y_T}$ verifies that

$$\langle v^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \le \|y_T\|_{\mathcal{R}_T}} \langle w, f^*(t)\rangle_U \ \ \text{a.e. } t \in (0, T). \tag{1.43}$$

(This is exactly Theorem 5.1.) About (1.43), we would like to mention two facts: First, it is not the standard Pontryagin maximum principle, since we are not sure if the function $f^*$ in (1.43) can be expressed as $B^*\varphi$ with $\varphi$ a solution of the adjoint equation, even in the case that $B \in \mathcal{L}(U, X)$. Second, the proof of (1.43) is the most difficult part in this paper. It is based on two representation theorems (Theorem 2.2 and Theorem 2.6). From (1.43) and the connection between $(NP)^{y_T}$ and $(NP)^{T,y_0}$ built up in Proposition 3, we get the maximum principle for $(NP)^{T,y_0}$, with $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, which along with (H2), yields that when $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, $(NP)^{T,y_0}$ has the bang-bang property.

Regarding the decomposition of $\mathcal{V}$, the part $\mathcal{V}_{2,3} = \gamma_2$ is a critical curve for us; the studies for the problem $(TP)^{M,y_0}$, with $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$, are not easy for us;

when $(M, y_0)$ is in the rest parts, it is easy to prove the corresponding conclusions in Theorem 1.2 for $(TP)^{M,y_0}$, through using properties of functions $T^0(\cdot)$, $T^1(\cdot)$ and $N(\cdot, y_0)$, as well as the assumption (H1). The proof of the corresponding conclusion in Theorem 1.2 for $(TP)^{M,y_0}$, with $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$, is mainly based on a maximum principle for $(TP)^{M,y_0}$, as well as (H2). This maximum principle follows from the above-mentioned maximum principle for $(NP)^{T,y_0}$, with $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, as well as the connection between $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$ built up in Lemma 5.3.

**Remark 7.** The reason to cause curves $\gamma_1$ and $\gamma_2$ to be critical is that in general, we do not know if $(NP)^{y_T}$, with $y_T \in \mathcal{R}_T \setminus \mathcal{R}_T^0$, has the maximum principle (1.43), under the assumption (H1).

1.5. **More about the bang-bang properties.** To the best of our knowledge, there are two ways to derive the bang-bang property for minimal time control problems governed by linear evolution systems, in general. The first one is the use of the $L^\infty$-null controllability from measurable sets. In [6, Section 2.1], H. O. Fattorini studied the minimal time control problem for the abstract system:

$$y'(t) = Ay(t) + u(t),\ t > 0, \tag{1.44}$$

with $A$ generating a $C_0$-semigroup in a Banach space. This corresponds to (1.11) with $U = X$ and $B = Id_X$ (the identity operator on a Banach space $X$). By a constructive method, he proved that the reachable sets of (1.44) have the following property: For any subset $E \subset (0, \infty)$ of positive measure, $\mathcal{R}_{T,E} = \mathcal{R}_T$ for a.e. $T \in E$, where $\mathcal{R}_{T,E} := \{y(T; 0, \chi_E u) : u \in L^\infty(\mathbb{R}^+; U)\}$. From this property, he proved the bang-bang property by a contradiction argument. In [28], V. Mizel and T. Seidman pointed out that the bang-bang property of minimal time control problems for linear time-invariant evolution systems can be derived by the $L^\infty$-null controllability from measurable sets. Indeed, by this controllability and by a translation invariance which holds only for time invariant systems, one can use a contradiction argument to prove the bang-bang property. However, it seems for us that this way does not work for the case where controlled systems are time-varying. In [43], the authors proved the bang-bang property of minimal time control problems for some very special time-varying heat equations. To our best knowledge, how to study the bang-bang property of minimal time control problems for general time-varying systems is still a quite open problem. For studies on the $L^\infty$-null controllability from measurable sets, we would like to mention the literatures [1, 28, 31, 32, 33, 40, 44, 48] and the references therein.

The second way is the use of the Pontryagin maximum principle and the unique continuation property from measurable sets in time. The key is to derive the Pontryagin maximum principle. We would like to mention that the Pontryagin maximum principle may not hold for some cases (see Example 1.4 on Page 132 in [20]). In [6, Chapter 2], H. O. Fattorini studied the Pontryagin maximum principle for both minimal time and minimal norm control problems, with an initial state $\zeta$ and a target state $\bar{y}$, for the system (1.44). He first proved the property that for each $T > 0$, $D(A)$ is continuously embedded into $\mathcal{R}_T$. Then, with the aid of this property, he divided the dual space of $\mathcal{R}_T$ into "the regular part" and "the singular part". After that, he proved that if $\bar{y} - S(T^*)\zeta \in \overline{D(A)}$, then $\bar{y} - S(T^*)\zeta$ and $B_{\mathcal{R}_{T^*}}(0, 1)$ can be separated by a hyperplane (in $\mathcal{R}_{T^*}$), with a regular normal vector. (Here, $T^*$ is the minimal time, $B_{\mathcal{R}_{T^*}}(0, 1)$ is the closed unit ball in $\mathcal{R}_{T^*}$ and the controls for the minimal time control problem are within $L^\infty$-norm not larger than 1.) Finally, with the help of the aforementioned separating property, he obtained the Pontryagin maximum principle. By the second way, one might get the bang-bang property of minimal time control problems for the linear time-varying evolution systems which hold some unique continuation property.

For the minimal norm control problems governed by linear time-varying evolution systems, the $L^\infty$-null controllability from measurable sets implies the bang-bang property. Though the paper [31] proves this only for heat equations with time-varying lower terms, the method in [31] works for general linear time-varying evolution systems.

About studies on minimal time and minimal norm control problems, we would like to mention the literatures [2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 33, 34, 36, 37, 40, 41, 43, 44, 45, 46, 47, 48, 49, 50] and the references therein.

The rest part of this paper is organized as follows: Section 2 studies some properties on the subspaces $\mathcal{R}_T$ and $\mathcal{R}_T^0$. Section 3 shows some properties of functions $N(\cdot, y_0)$, $T^0(\cdot)$ and $T^1(\cdot)$. Section 4 studies the existence of minimal time and minimal norm controls. Section 5 studies maximum principles and bang-bang properties. Section 6 proves the main results. Section 7 gives some applications. Section 8 provides several appendixes.

## 2. **Properties on attainable subspaces.**

In this section, we mainly study the properties on the subspaces $\mathcal{R}_T$ and $\mathcal{R}_T^0$ given by (1.41) and (1.42), respectively. These properties mainly help us to get a maximum principle for the affiliated minimal norm problem $(NP)^{y_T}$, with $y_T \in \mathcal{R}_T^0$. The later is the base in the proofs of (iii) of Theorem 1.1 and (ii) of Theorem 1.2.

### 2.1. **The first representation theorem.**

In this subsection, we will present a representation theorem on the space $Y_T^*$ which is the dual space of $Y_T$ (defined by (1.20)). This theorem was built up for heat equations in [43, (i) of Theorem 1.4]. To prove it, we need the following two results: Proposition 1 and Lemma 2.1. Very similar versions of these two results are given in [5, Section 2.3.1]. For the sake of the completeness of the paper, we give their proofs in Appendix D.

**Proposition 1.** *The following equality is valid:*

$$\Big\langle \int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t, z\Big\rangle_X = \int_0^T \langle v(t), B^*S^*(T-t)z\rangle_U\,\mathrm{d}t \tag{2.1}$$

*for all* $T \in (0,\infty)$, $v \in L^\infty(0,T;U)$ *and* $z \in D(A^*)$.

**Lemma 2.1.** *For each* $T \in (0,\infty)$, *there exists a positive constant* $C(T)$ *so that*

$$\int_0^T \|B^*S^*(T-\tau)z\|_U^2\,\mathrm{d}\tau \le C(T)\|z\|_X^2 \ \text{ for all } \ z \in D(A^*). \tag{2.2}$$

**Theorem 2.2.** *For each* $T \in (0,\infty)$, *there is a linear isomorphism* $\Phi_T$ *from* $\mathcal{R}_T$ *to* $Y_T^*$ *so that for all* $y_T \in \mathcal{R}_T$ *and* $f \in Y_T$,

$$\langle y_T, f\rangle_{\mathcal{R}_T,Y_T} := \langle \Phi_T(y_T), f\rangle_{Y_T^*,Y_T} = \int_0^T \langle v(t), f(t)\rangle_U\,\mathrm{d}t, \tag{2.3}$$

*where* $v$ *is any admissible control to* $(NP)^{y_T}$.

*Proof.* Arbitrarily fix a $T \in (0,\infty)$. It follows from (2.2) that

$$B^*S^*(T-\cdot)z \in L^1(0,T;U) \ \text{ for each } \ z \in D(A^*). \tag{2.4}$$

For each $y_T \in \mathcal{R}_T$, define the following set of admissible controls to $(NP)^{y_T}$:

$$\mathcal{U}_{ad}^{y_T} := \{v \in L^\infty(0,T;U) \ : \ \hat{y}(T;0,v) = y_T\}. \tag{2.5}$$

Observe from (1.41) and (2.5) that $\mathcal{U}_{ad}^{y_T} \neq \emptyset$ for each $y_T \in \mathcal{R}_T$, and that $y_T = \hat{y}(T;0,v)$ for each $y_T \in \mathcal{R}_T$ and each $v \in \mathcal{U}_{ad}^{y_T}$. These, along with (1.13) and (2.4), yields that for each $y_T \in \mathcal{R}_T$, $z \in D(A^*)$ and each $v \in \mathcal{U}_{ad}^{y_T}$,

$$\begin{aligned}\langle y_T, z\rangle_X &= \int_0^T \langle v(t), B^*S^*(T-t)z\rangle_U \,\mathrm{d}t \\ &\le \|v(\cdot)\|_{L^\infty(0,T;U)} \|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}.\end{aligned} \tag{2.6}$$

Arbitrarily fix a $y_T \in \mathcal{R}_T$ and then fix a $v_1 \in \mathcal{U}_{ad}^{y_T}$. Then we define a map $\mathcal{F}_{y_T} : X_T \to \mathbb{R}$ in the following manner:

$$\mathcal{F}_{y_T}\big(B^*S^*(T-\cdot)z|_{(0,T)}\big) := \int_0^T \langle v_1(t), B^*S^*(T-t)z\rangle_U \,\mathrm{d}t, \ \forall\, z \in D(A^*), \tag{2.7}$$

where $X_T$ is given by (1.21). Because of the first equality in (2.6), we see from (2.7) that the definition of $\mathcal{F}_{y_T}$ is independent of the choice of $v_1 \in \mathcal{U}_{ad}^{y_T}$. Thus it is well-defined. From (2.7), the inequality in (2.6) and (1.20), we find that $\mathcal{F}_{y_T}$ can be uniquely extended to be an element $\widetilde{\mathcal{F}}_{y_T} \in Y_T^*$. Furthermore, we have that $\|\widetilde{\mathcal{F}}_{y_T}\|_{Y_T^*} \le \|v\|_{L^\infty(0,T;U)}$ for all $v \in \mathcal{U}_{ad}^{y_T}$. Since $y_T \in \mathcal{R}_T$ was arbitrarily fixed, the above estimate, along with (1.40), yields that

$$\|\widetilde{\mathcal{F}}_{y_T}\|_{Y_T^*} \le \inf\{\|v\|_{L^\infty(0,T;U)} \ : \ v \in \mathcal{U}_{ad}^{y_T}\} = \|y_T\|_{\mathcal{R}_T} \ \text{ for all } \ y_T \in \mathcal{R}_T. \tag{2.8}$$

We now define a map $\Phi_T : \ \mathcal{R}_T \longrightarrow Y_T^*$ in the following manner:

$$\Phi_T(y_T) = \widetilde{\mathcal{F}}_{y_T} \ \text{ for each } \ y_T \in \mathcal{R}_T. \tag{2.9}$$

It is clear that $\Phi_T$ is well defined and linear. We claim that $\Phi_T$ is surjective. Arbitrarily take $g \in Y_T^*$. Since $Y_T \subset L^1(0,T;U)$ (see (1.20)), according to the Hahn-Banach theorem, there exists a $\widetilde{g} \in \big(L^1(0,T;U)\big)^*$ so that $\widetilde{g}(\psi) = g(\psi)$ for all $\psi \in Y_T$; and so that $\|\widetilde{g}\|_{\mathcal{L}(L^1(0,T;U);\mathbb{R})} = \|g\|_{Y_T^*}$. Then by the Riesz representation theorem, there is $\widehat{v}$ in $L^\infty(0,T;U)$ so that

$$\int_0^T \langle \widehat{v}(t), B^*S^*(T-t)z\rangle_U \,\mathrm{d}t = g\big(B^*S^*(T-\cdot)z|_{(0,T)}\big) \ \text{ for all } \ z \in D(A^*) \tag{2.10}$$

and so that

$$\|\widehat{v}\|_{L^\infty(0,T;U)} = \|g\|_{Y_T^*}. \tag{2.11}$$

Write $\hat{y}_T := \hat{y}(T;0,\widehat{v})$. Then $\hat{v} \in \mathcal{U}_{ad}^{\hat{y}_T}$ (see (2.5)). This, together with (2.10), (2.7) and (1.20), indicates that $g = \widetilde{\mathcal{F}}_{\hat{y}_T}$ in $Y_T^*$, which, along with (2.9), shows that $\Phi_T$ is surjective.

We now show that $\Phi_T$ is injective. Let $y_T \in \mathcal{R}_T$ satisfy that $\widetilde{\mathcal{F}}_{y_T} = 0$ in $Y_T^*$. Then by (2.7) and (2.6), we find that $\langle y_T, z\rangle_X = 0$ for all $z \in D(A^*)$. Since $D(A^*)$ is dense in $X$, the above yields that $y_T = 0$, which implies that $\Phi_T$ is injective.

We next show that $\Phi_T$ preserves norms. Let $g \in Y_T^*$. Then we have that $g = \widetilde{\mathcal{F}}_{\hat{y}_T}$ in $Y_T^*$, where $\hat{y}_T = \hat{y}(T;0,\hat{v})$, with $\hat{v} \in L^\infty(0,T;U)$ satisfying (2.10) and (2.11). This, along with (1.40) yields that $\|\hat{y}_T\|_{\mathcal{R}_T} \le \|\widetilde{\mathcal{F}}_{\hat{y}_T}\|_{Y_T^*}$. From this and (2.8), we see that $\Phi_T$ preserves norms.

Finally, (2.3) follows from (2.9), (2.7) and (1.20). This ends the proof of this theorem.

□

**Remark 8.** Since $Y_T^*$ is complete, it follows from Theorem 2.2 that the normed space $(\mathcal{R}_T, \|\cdot\|_{\mathcal{R}_T})$ is complete.

2.2. **The second representation theorem.** This subsection mainly presents a representation theorem on $(\mathcal{R}_T^0)^*$, the dual space of the space $\mathcal{R}_T^0$ (defined by (1.42)). This theorem gives an important property of $Y_T$ (which is defined by (1.20)). For this purpose, we need three lemmas.

**Lemma 2.3.** *The following propositions are equivalent:*
$(i)$ *The condition (H1) holds.*
$(ii)$ *There is a $p_1 \in [2,\infty)$ so that for each $T \in (0,\infty)$, each $u \in L^{p_1}(0,T;U)$ and each $t \in (0,T)$, there exists a control $v \in L^\infty(0,T;U)$ satisfying that*

$$\hat{y}(T;0,\chi_{(t,T)}v) = \hat{y}(T;0,\chi_{(0,t)}u) \ \text{ and } \ \|v\|_{L^\infty(0,T;U)} \le C_1\|u\|_{L^{p_1}(0,T;U)}$$

*for some $C_1 := C_1(T,t) > 0$ (independent of $u$).*
$(iii)$ *There is a $p_2 \in (1,2]$ so that when $0 < t < T < \infty$,*

$$\|g\|_{L^{p_2}(0,t;U)} \le C_2\|g\|_{L^1(t,T;U)} \ \text{ for all } \ g \in Y_T$$

*for some $C_2 := C_2(T,t) > 0$ (independent of $g$).*

*Furthermore, when one of the above three propositions is valid, the constants $p_0$ (given in (H1)), $p_1$ and $p_2$ (given in (ii) and (iii), respectively) can be chosen so that $p_0 = p_1 = p_2/(p_2-1)$.*

*Proof.* Our proof is organized by several steps as follows:
*Step 1. To show that (i) ⇒ (ii)*

Suppose that (H1) holds for some $p_0 \in [2,\infty)$. Let $T$ and $t$ satisfy that $0 < t < T < \infty$. Define two maps as follows:

$$L_1 : Y := L^{p_0}(0,T;U) \to X, \quad L_1(u) = \hat{y}(T;0,\chi_{(0,t)}u), \ \ u \in Y;$$

$$L_2 : Z := L^\infty(0,T;U) \to X, \quad L_2(v) = \hat{y}(T;0,\chi_{(t,T)}v), \ \ v \in Z.$$

By (1.14), we have that

$$\hat{y}(T;0,\chi_{(0,t)}u) = \int_0^T S_{-1}(T-\tau)B\chi_{(0,t)}(\tau)u(\tau)\,\mathrm{d}\tau;$$

$$\hat{y}(T;0,\chi_{(t,T)}v) = \int_0^T S_{-1}(T-\tau)B\chi_{(t,T)}(\tau)v(\tau)\,\mathrm{d}\tau.$$

These, together with (1.10), indicate that both $L_1$ and $L_2$ are bounded. Moreover, by (H1), we find that

$$\operatorname{Range} L_1 \subset \operatorname{Range} L_2. \tag{2.12}$$

Let $\pi : \ Z \to \widehat{Z} := Z/\operatorname{Ker} L_2$ be the quotient map. Then $\pi$ is surjective and it stands that

$$\|\pi(v)\|_{\widehat{Z}} = \inf\left\{\|w\|_Z \ : \ w \in v + \operatorname{Ker} L_2\right\} \ \text{ for each } \ v \in Z. \tag{2.13}$$

Define a map $\widehat{L}_2 : \widehat{Z} \to X$ in the following manner:

$$\widehat{L}_2(\pi(v)) = L_2(v), \quad \pi(v) \in \widehat{Z}. \tag{2.14}$$

One can easily check that $\widehat{L}_2$ is well defined, linear and bounded. By (2.12) and (2.14), we see that $\operatorname{Range} L_1 \subset \operatorname{Range} \widehat{L}_2$. Thus, given $u \in Y$, there is a unique $\pi(v_u) \in \widehat{Z}$ so that

$$L_1(u) = \widehat{L}_2(\pi(v_u)). \tag{2.15}$$

We now define another map $\mathcal{T}: Y \to \widehat{Z}$ by

$$\mathcal{T}(u) = \pi(v_u) \ \text{ for each } \ u \in Y. \tag{2.16}$$

One can easily check that $\mathcal{T}$ is well defined and linear. We next use the closed graph theorem to show that $\mathcal{T}$ is bounded. For this purpose, we let $\{u_n\} \subset Y$ satisfy that

$$u_n \to u_0 \quad \text{in } \ Y \ \text{ and } \ \mathcal{T}(u_n) \to h_0 \quad \text{in } \ \widehat{Z}, \quad \text{as } \ n \to \infty.$$

Because $L_1$ and $\widehat{L}_2$ are bounded, we find from (2.16) and (2.15) that

$$\widehat{L}_2 h_0 = \lim_{n\to\infty} \widehat{L}_2(\mathcal{T}(u_n)) = \lim_{n\to\infty} \widehat{L}_2(\pi(v_{u_n})) = \lim_{n\to\infty} L_1(u_n) = L_1 u_0.$$

This, together with (2.16), indicates that $h_0 = \mathcal{T}(u_0)$. Then by the closed graph theorem, we see that $\mathcal{T}$ is bounded. Thus, by (2.16), there exists a $C := C(T,t) > 0$ so that

$$\|\pi(v_u)\|_{\widehat{Z}} = \|\mathcal{T}(u)\|_{\widehat{Z}} \le C\|u\|_Y \ \text{ for each } \ u \in Y. \tag{2.17}$$

Meanwhile, it follows from (2.13) that for each $v \in Z$, there is a $v' \in v + \operatorname{Ker} L_2$ so that $\|v'\|_Z \le 2\|\pi(v)\|_{\widehat{Z}}$. Thus, by (2.15), (2.14) and (2.17), we find that for each $u \in Y$, there is a $v'_u \in Z$ so that $L_1(u) = L_2(v'_u)$ and $\|v'_u\|_Z \le 2C\|u\|_Y$. Hence, by the definitions of $L_1$ and $L_2$, we obtain (ii), with $C_1 = 2C$ and $p_1 = p_0$.

*Step 2. To show that (ii) ⇒ (iii)*

Suppose that (ii) holds for some $p_1 \in [2,\infty)$. Arbitrarily fix $T$ and $t$, with $0 < t < T < \infty$. Then for each $u \in L^{p_1}(0,T;U)$, there is a control $v_u \in L^\infty(0,T;U)$ so that

$$\hat{y}(T;0,\chi_{(0,t)}u) = \hat{y}(T;0,\chi_{(t,T)}v_u) \ \text{ and } \ \|v_u\|_{L^\infty(0,T;U)} \le C_1\|u\|_{L^{p_1}(0,T;U)},$$

where $C_1 := C_1(T,t)$ is given by (ii). These, along with (1.13), yield that for each $z \in D(A^*)$,

$$\begin{aligned}
&\int_0^t \langle B^*S^*(T-\eta)z, u(\eta)\rangle_U \,\mathrm{d}\eta = \int_0^T \left\langle B^*S^*(T-\eta)z, \chi_{(0,t)}(\eta)u(\eta)\right\rangle_U \mathrm{d}\eta \\
= \ & \left\langle z, \hat{y}(T;0,\chi_{(0,t)}u)\right\rangle_X = \left\langle z, \hat{y}(T;0,\chi_{(t,T)}v_u)\right\rangle_X \\
= \ & \int_0^T \left\langle B^*S^*(T-\eta)z, \chi_{(t,T)}(\eta)v_u(\eta)\right\rangle_U \mathrm{d}\eta = \int_t^T \left\langle B^*S^*(T-\eta)z, v_u(\eta)\right\rangle_U \mathrm{d}\eta \\
\le \ & \|B^*S^*(T-\cdot)z\|_{L^1(t,T;U)}\|v_u(\cdot)\|_{L^\infty(t,T;U)} \\
\le \ & C_1\|B^*S^*(T-\cdot)z\|_{L^1(t,T;U)}\|u(\cdot)\|_{L^{p_1}(0,t;U)}.
\end{aligned}$$

Let $p'_1$ be the conjugate index of $p_1$, i.e., $1/p_1 + 1/p'_1 = 1$. Then we find that

$$\|B^*S^*(T-\cdot)z\|_{L^{p'_1}(0,t;U)} \le C_1\|B^*S^*(T-\cdot)z\|_{L^1(t,T;U)} \ \text{ for all } \ z \in D(A^*).$$

The above, as well as (1.20), leads to (iii), with $p_2 = p'_1$ and $C_2 = C_1$.

*Step 3. (iii) ⇒ (i)*

Suppose that (iii) holds for some $p_2 \in (1,2]$. Let $p_2'$ be the conjugate index of $p_2$, i.e., $1/p_2 + 1/p_2' = 1$. Arbitrarily fix $T$ and $t$, with $0 < t < T < \infty$. Define the following subspace of $L^1(t,T;U)$:

$$\mathcal{O} := \big\{ B^*S^*(T-\cdot)z|_{(t,T)} \in L^1(t,T;U) \ : \ z \in D(A^*) \big\}.$$

Let $u \in L^{p_2'}(0,T;U)$. We define a linear map $L_3 : \mathcal{O} \to \mathbb{R}$ by

$$L_3\big(B^*S^*(T-\cdot)z|_{(t,T)}\big) = \int_0^t \langle B^*S^*(T-s)z, u(s)\rangle_U \, \mathrm{d}s, \ \ z \in D(A^*). \tag{2.18}$$

Since $B^*S^*(T-\cdot)z|_{(0,T)} \in Y_T$ for all $z \in D(A^*)$, it follows from (iii) that $L_3$ is well defined. Then by (2.18) and (iii), we find that for each $z \in D(A^*)$,

$$\begin{aligned} \big|L_3\big(B^*S^*(T-\cdot)z|_{(t,T)}\big)\big| &\leq \|B^*S^*(T-\cdot)z\|_{L^{p_2}(0,t;U)} \|u(\cdot)\|_{L^{p_2'}(0,t;U)} \\ &\leq C_2 \|B^*S^*(T-\cdot)z\|_{L^1(t,T;U)} \|u(\cdot)\|_{L^{p_2'}(0,t;U)}, \end{aligned}$$

where $C_2 := C_2(T,t)$ is given by (iii). This implies that $L_3$ is bounded from $\mathcal{O}$ to $\mathbb{R}$. Thus, by the Hahn-Banach theorem, $L_3$ can be extended from $L^1(t,T;U)$ to $\mathbb{R}$ and there exists $g \in \big(L^1(t,T;U)\big)^*$ so that

$$L_3(\psi) = g(\psi) \quad \text{for all } \ \psi \in \mathcal{O}; \ \text{ and } \ \|g\|_{\mathcal{L}(L^1(t,T;U);\mathbb{R})} \leq C_2 \|u\|_{L^{p_2'}(0,t;U)}.$$

Then by the Riesz representation theorem and (2.18), there is $v_u \in L^\infty(t,T;U)$ so that

$$\int_t^T \langle v_u(s), \psi(s)\rangle_U \, \mathrm{d}s = g(\psi) = \int_0^t \langle \psi(s), u(s)\rangle_U \, \mathrm{d}s \ \text{ for all } \ \psi \in \mathcal{O} \tag{2.19}$$

and so that

$$\|v_u\|_{L^\infty(t,T;U)} = \|g\|_{\mathcal{L}(L^1(t,T;U);\mathbb{R})} \leq C_2 \|u\|_{L^{p_2'}(0,t;U)}.$$

Write $\widetilde{v}_u$ for the zero extension of $v_u$ over $(0,T)$. Then we see from (1.13) and (2.19) that for all $z \in D(A^*)$,

$$\begin{aligned} \langle z, \hat{y}(T;0,\chi_{(0,t)}u)\rangle_X &= \int_0^T \big\langle B^*S^*(T-s)z, \chi_{(0,t)}(s)u(s)\big\rangle_U \, \mathrm{d}s \\ = \int_0^t \langle B^*S^*(T-s)z, u(s)\rangle_U \, \mathrm{d}s &= \int_t^T \langle v_u(s), B^*S^*(T-s)z\rangle_U \, \mathrm{d}s \\ = \int_0^T \langle \widetilde{v}_u(s), B^*S^*(T-s)z\rangle_U \, \mathrm{d}s &= \langle z, \hat{y}(T;0,\widetilde{v}_u)\rangle_X. \end{aligned}$$

Since $D(A^*)$ is dense in $X$, the above leads to (H1), with $p_0 = p_2'$.

*Step 4. About the constants $p_0$, $p_1$ and $p_2$*

From the proofs in Step 1-Step 3, we find that $p_0$, $p_1$ and $p_2$ can be chosen so that $p_0 = p_1 = p_2/(p_2-1)$, provided that one of the propositions (i)-(iii) holds.

In summary, we finish the proof of this lemma.

□

**Lemma 2.4.** *Let $T \in (0,\infty)$. The following conclusions are true:*
*(i) If $f \in Y_T$, then $f|_{(0,S)} \in Y_S$ for each $S \in (0,T)$.*
*(ii) Suppose that (H1) holds. If $f \in L^1(0,T;U)$ and $f|_{(0,S)} \in Y_S$ for each $S \in (0,T)$, then $f \in Y_T$.*

*Proof.* (i) Let $f \in Y_T$. Then by (1.20), there exists a subsequence $\{w_n\} \subset D(A^*)$ so that

$$B^*S^*(T-\cdot)w_n \to f(\cdot) \text{ in } L^1(0,T;U). \tag{2.20}$$

Arbitrarily fix an $S \in (0,T)$. Since $S^*(T-S)w_n \in D(A^*)$ for all $n$, by making use of (1.20) again, we find that

$$B^*S^*(T-\cdot)w_n|_{(0,S)} = B^*S^*(S-\cdot)\big(S^*(T-S)w_n\big)|_{(0,S)} \in Y_S.$$

Since $Y_S$ is closed in $L^1(0,S;U)$, the above, as well as (2.20), yields that $f|_{(0,S)} \in Y_S$.

(ii) Suppose that (H1) holds. We organize the proof by the following steps:

*Step 1. To show that for each $s \in (0,\infty)$ and $g^s \in Y_s$, there is a unique function $\widetilde{g}^s$ over $(-1,s)$ so that*

$$\widetilde{g}^s(\tau) = g^s(\tau) \textit{ for all } \tau \in (0,s), \quad \textit{and} \quad \widetilde{g}^s(\cdot-1) \in Y_{s+1} \tag{2.21}$$

Let $0 < s < \infty$ and $g^s \in Y_s$. We first show the existence of such $\widetilde{g}^s$. For this purpose, we define the following subspace:

$$X_s := \big\{g_z(\cdot) \in L^1(0,s;U) \ : \ z \in D(A^*)\big\},$$

where $g_z(\cdot) := B^*S^*(s-\cdot)z$ over $(0,s)$. Then define a map $\mathcal{F}_s : X_s \to Y_{s+1}$ in the following manner: For each $z \in D(A^*)$,

$$(\mathcal{F}_s g_z)(\tau) := B^*S^*(s+1-\tau)z, \ \ \tau \in (0,s+1). \tag{2.22}$$

From (2.22), we find that for each $z \in D(A^*)$,

$$(\mathcal{F}_s g_z)(\tau+1) = g_z(\tau), \ \tau \in (0,s). \tag{2.23}$$

Meanwhile, by (H1) and Lemma 2.3, we have the assertion (iii) of Lemma 2.3, which, together with (2.22), yields that when $z \in D(A^*)$,

$$\begin{aligned} & \big\|\mathcal{F}_s\big(B^*S^*(s-\cdot)z|_{(0,s)}\big)\big\|_{L^1(0,s+1;U)} \\ = & \int_1^{s+1} \|B^*S^*(s+1-\tau)z\|_U \,\mathrm{d}\tau + \int_0^1 \|B^*S^*(s+1-\tau)z\|_U \,\mathrm{d}\tau \\ \le & (1+C_2)\int_1^{s+1} \|B^*S^*(s+1-\tau)z\|_U \,\mathrm{d}\tau = (1+C_2)\|B^*S^*(s-\cdot)z\|_{L^1(0,s;U)} \end{aligned}$$

for some $C_2 > 0$ independent of $z$. (Here we used the time-invariance of the controlled system). Hence, $\mathcal{F}_s$ is linear and bounded from $X_s$ to $Y_{s+1}$. Since $X_s$ is dense in $Y_s$ (see (1.20)), $\mathcal{F}_s$ can be uniquely extended to be a linear and bounded operator $\widetilde{\mathcal{F}_s}$ from $Y_s$ to $Y_{s+1}$. This, along with (2.23), yields that

$$(\widetilde{\mathcal{F}_s} g^s)(\tau+1) = g^s(\tau), \ \tau \in (0,s). \tag{2.24}$$

We now define

$$\widetilde{g}^s(\tau) := (\widetilde{\mathcal{F}_s} g^s)(\tau+1), \ \ \tau \in (-1,s). \tag{2.25}$$

It follows from (2.25) and (2.24) that $\widetilde{g}^s$ satisfies (2.21).

We next show the uniqueness of such $\widetilde{g}^s$. Let $\widehat{g}^s$ be another extension of $g^s$ (over $(-1,s)$) satisfying (2.21). Then we see from (2.21) that

$$(\widetilde{g}^s - \widehat{g}^s)(\tau) = 0 \text{ for all } \tau \in (0,s) \tag{2.26}$$

and

$$(\widetilde{g}^s - \widehat{g}^s)(\cdot-1) \in Y_{s+1}. \tag{2.27}$$

From (2.26), we see that

$$(\widetilde{g}^s - \widehat{g}^s)(\tau - 1) = 0 \ \text{ for all } \ \tau \in (1, s+1). \tag{2.28}$$

By (H1) and Lemma 2.3, we have (iii) of Lemma 2.3. This, along with (2.27), yields that

$$\|(\widetilde{g}^s - \widehat{g}^s)(\cdot - 1)\|_{L^{p_2}(0,1;U)} \leq C_2 \|(\widetilde{g}^s - \widehat{g}^s)(\cdot - 1)\|_{L^1(1,s+1;U)},$$

where $p_2$ and $C_2$ are given by (iii) of Lemma 2.3. This, together with (2.28), implies that $(\widetilde{g}^s - \widehat{g}^s)(\cdot - 1) = 0$ over $(0, s+1)$. Hence, we have that $\widetilde{g}^s(\cdot) = \widehat{g}^s(\cdot)$ over $(-1, s)$. This shows the uniqueness of such $\widetilde{g}^s(\cdot)$ that satisfies (2.21). We call the above $\widetilde{g}^s(\cdot)$ the $Y$-extension of $g^s(\cdot)$.

*Step 2. To show that $f \in Y_T$, when $f \in L^1(0,T;U)$ and $f|_{(0,S)} \in Y_S$ for each $S \in (0,T)$*

Let $f \in L^1(0,T;U)$ satisfy that $f|_{(0,S)} \in Y_S$ for each $S \in (0,T)$. Given $S \in (0,T)$, we write $f_S$ for the $Y$-extension of $f|_{(0,S)}$ over $(-1, S)$ (see the conclusion of Step 1). We claim that

$$f_{S_1} = f_{S_2} \ \text{ over } \ (-1, 0), \ \text{ when } \ 0 < S_1 < S_2 < T.$$

Here is the argument: on one hand, we let

$$\bar{f}(\tau) := f_{S_2}(\tau - 1), \ \tau \in (0, S_2 + 1). \tag{2.29}$$

By (2.29) and the definition of $f_{S_2}$ (see (2.21)), we find that $\bar{f} \in Y_{S_2+1}$. This, as well as (i) in this lemma, yields that

$$\bar{f}|_{(0,S_1+1)} \in Y_{S_1+1}. \tag{2.30}$$

By making use of (2.29) again, we see that $\bar{f}|_{(0,S_1+1)}(\tau) = f_{S_2}|_{(-1,S_1)}(\tau - 1)$ for each $\tau \in (0, S_1 + 1)$. This, along with (2.30), indicates that

$$f_{S_2}|_{(-1,S_1)}(\cdot - 1) \in Y_{S_1+1}. \tag{2.31}$$

Meanwhile, since $f_{S_2} = f$ over $(0, S_2)$, we have that $f_{S_2}|_{(-1,S_1)}(\tau) = f|_{(0,S_1)}(\tau)$ for all $\tau \in (0, S_1)$. This, along with (2.31), indicates that $f_{S_2}|_{(-1,S_1)}(\cdot)$ is the $Y$-extension of $f|_{(0,S_1)}(\cdot)$ over $(-1, S_1)$.

On the other hand, $f_{S_1}$ is also the $Y$-extension of $f|_{(0,S_1)}(\cdot)$ over $(-1, S_1)$. By the uniqueness of the $Y$-extension, we see that $f_{S_1} = f_{S_2}|_{(-1,S_1)}$ over $(-1, S_1)$, which leads to that $f_{S_1} = f_{S_2}$ over $(-1, 0)$. This ends the proof of the above claim.

Now we arbitrarily fix an $S_0 \in (0,T)$. Define a function $\hat{f} : (-1, T) \to U$ by setting

$$\hat{f}(\cdot) = f(\cdot) \ \text{ over } \ (0,T); \ \ \hat{f}(\cdot) = f_{S_0}(\cdot) \ \text{ over } \ (-1, 0]. \tag{2.32}$$

Because of the above-mentioned claim, we find that

$$\hat{f} \ \text{ is independent of the choice of } \ S_0. \tag{2.33}$$

It is clear that $\hat{f} \in L^1(-1,T;U)$. Take a sequence $\{T_k\} \subset (0,T)$ so that $T_k \nearrow T$. Then we see from the first equality in (2.32) that

$$\hat{f}(\cdot + T_k - T)|_{(0,T)} \to \hat{f}(\cdot)\,|_{(0,T)} = f(\cdot) \ \text{ in } \ L^1(0,T;U), \ \text{ as } \ k \to \infty. \tag{2.34}$$

Meanwhile, for each $k$, since $f_{T_k}(\cdot - 1) \in Y_{T_k+1}$ (see (2.21)), by (1.20), there exists a sequence $\{w_{k,n}\} \subset D(A^*)$ so that

$$\int_0^{T_k+1} \|B^* S^*(T_k + 1 - t) w_{k,n} - f_{T_k}(t-1)\|_U \, \mathrm{d}t \to 0, \ \text{ as } \ n \to \infty.$$

Since $f_{T_k} = \hat{f}$ over $(-1, T_k)$ for each $k$ (see (2.33) and (2.32)), the above yields that for all $k$, with $T_k + 1 \geq T$,

$$\int_{T_k+1-T}^{T_k+1} \|B^*S^*(T_k+1-t)w_{k,n} - \hat{f}(t-1)\|_U \,\mathrm{d}t \to 0, \text{ as } n \to \infty,$$

which implies that for all $k$, with $T_k + 1 \geq T$,

$$\int_0^T \|B^*S^*(T-t)w_{k,n} - \hat{f}(t+T_k-T)\|_U \,\mathrm{d}t \to 0, \text{ as } n \to \infty.$$

This, along with (1.20), indicates that

$$\hat{f}(\cdot + T_k - T)|_{(0,T)} \in Y_T \text{ for all } k \text{ with } T_k + 1 \geq T.$$

Since $Y_T$ is closed in $L^1(0,T;U)$, the above, together with (2.34), implies that $f \in Y_T$.

In summary, we complete the proof of this lemma.

□

**Lemma 2.5.** *Let $T \in (0,\infty)$. If $f \in L^1(0,T;U)$ satisfies that*

$$\int_0^T \langle f(t), u(t)\rangle_U \,\mathrm{d}t = 0 \text{ for all } u \in \{v \in L^\infty(0,T;U) : \hat{y}(T;0,v) = 0\}, \quad (2.35)$$

*then $f \in Y_T$.*

*Proof.* By contradiction, we suppose that for some $T \in (0,\infty)$, there were a function $f \in L^1(0,T;U)$, with the property (2.35), so that $f \notin Y_T$. Then, by the Hahn-Banach theorem, we could find a function $\widehat{u} \in L^\infty(0,T;U)$ so that

$$0 = \int_0^T \langle g(t), \widehat{u}(t)\rangle_U \,\mathrm{d}t < \int_0^T \langle f(t), \widehat{u}(t)\rangle_U \,\mathrm{d}t \text{ for each } g \in Y_T. \quad (2.36)$$

(Here, we noticed that $Y_T$ is a closed subspace of $L^1(0,T;U)$.) From Theorem 2.2 and the first assertion in (2.36), we find that $\hat{y}(T;0,\widehat{u}) = 0$, which, along with (2.35) and the second assertion in (2.36), leads to a contradiction. This ends the proof.

□

The following result is a representation theorem on $(\mathcal{R}_T^0)^*$, which plays an important role in our study.

**Theorem 2.6.** *Suppose that (H1) holds. Then for each $T \in (0,\infty)$, there is a linear isomorphism $\Psi_T$ from $Y_T$ to $(\mathcal{R}_T^0)^*$ so that for all $g \in Y_T$ and $y_T \in \mathcal{R}_T^0$,*

$$\langle g, y_T\rangle_{Y_T,\mathcal{R}_T^0} := \langle \Psi_T(g), y_T\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0} = \int_0^T \langle g(t), v(t)\rangle_U \,\mathrm{d}t, \quad (2.37)$$

*where $v$ is any admissible control to $(NP)^{y_T}$.*

*Proof.* Let $0 < T < \infty$. Recall that $\mathcal{R}_T^0$, with the norm $\|\cdot\|_{\mathcal{R}_T}$, is a subspace of $\mathcal{R}_T$ (see (1.41) and (1.42)). According to Theorem 2.2, each $g \in Y_T$ defines a linear bounded functional $\widehat{\mathcal{F}}_g$ over $\mathcal{R}_T^0$ (i.e., $\widehat{\mathcal{F}}_g \in (\mathcal{R}_T^0)^*$), via

$$\widehat{\mathcal{F}}_g(y_T) := \langle g, y_T\rangle_{Y_T,\mathcal{R}_T}, \quad y_T \in \mathcal{R}_T^0, \quad (2.38)$$

where $\langle\cdot,\cdot\rangle_{Y_T,\mathcal{R}_T}$ is given by (2.3). Then we define a map $\Psi_T$ from $Y_T$ to $(\mathcal{R}_T^0)^*$ by

$$\Psi_T(g) := \widehat{\mathcal{F}}_g, \quad g \in Y_T. \quad (2.39)$$

One can easily check that $\Psi_T$ is linear. The rest of the proof is organized by three steps.

*Step 1. To show that $\|g\|_{Y_T} = \|\Psi_T(g)\|_{\mathcal{R}_T}$ for all $g \in \mathcal{R}_T^0$*

Let $g \in Y_T$ be given. On one hand, from (2.38), we see that

$$\|\widehat{\mathcal{F}}_g\|_{(\mathcal{R}_T^0)^*} = \sup_{y_T \in B_{\mathcal{R}_T^0}(0,1)} \langle g, y_T \rangle_{Y_T, \mathcal{R}_T} \leq \|g\|_{Y_T}, \tag{2.40}$$

where $B_{\mathcal{R}_T^0}(0,1)$ is the closed unit ball in $\mathcal{R}_T^0$. On the other hand, we arbitrarily fix $S \in (0,T)$. Then according to the Hahn-Banach theorem, there is a control $\hat{u}_S \in L^\infty(0,S;U)$ so that

$$\|g\|_{L^1(0,S;U)} = \langle g, \hat{u}_S \rangle_{L^1(0,S;U), L^\infty(0,S;U)} \text{ and } \|\hat{u}_S\|_{L^\infty(0,S;U)} = 1. \tag{2.41}$$

Write $\widetilde{u}_S$ for the zero extension of $\hat{u}_S$ over $(0,T)$. Then it follows from (1.42) that $\hat{y}(T;0,\widetilde{u}_S) \in \mathcal{R}_T^0$. Now, by (2.41), (2.3), (2.38) and (1.40), one can directly check that

$$\begin{aligned} \|g\|_{L^1(0,S;U)} &= \langle g, \widetilde{u}_S \rangle_{L^1(0,T;U), L^\infty(0,T;U)} = \langle g, \hat{y}(T;0,\widetilde{u}_S) \rangle_{Y_T, \mathcal{R}_T} \\ &= \widehat{\mathcal{F}}_g\big(\hat{y}(T;0,\widetilde{u}_S)\big) \leq \|\widehat{\mathcal{F}}_g\|_{(\mathcal{R}_T^0)^*} \|\hat{y}(T;0,\widetilde{u}_S)\|_{\mathcal{R}_T} \\ &\leq \|\widehat{\mathcal{F}}_g\|_{(\mathcal{R}_T^0)^*} \|\widetilde{u}_S\|_{L^\infty(0,S;U)} = \|\widehat{\mathcal{F}}_g\|_{(\mathcal{R}_T^0)^*}, \end{aligned}$$

which yields that $\|g\|_{Y_T} = \|g\|_{L^1(0,T;U)} \leq \|\widehat{\mathcal{F}}_g\|_{(\mathcal{R}_T^0)^*}$ (since $S$ was arbitrarily taken from $(0,T)$). This, along with (2.40), leads to that $\|g\|_{Y_T} = \|\Psi_T(g)\|_{\mathcal{R}_T}$.

*Step 2. To show that $\Psi_T$ is surjective*

Let $\hat{f} \in (\mathcal{R}_T^0)^*$. We aim to find a $\hat{g} \in Y_T$ so that

$$\hat{f} = \Psi_T(\hat{g}) \text{ in } (\mathcal{R}_T^0)^*. \tag{2.42}$$

In what follows, for each $u \in L^\infty(0,S;U)$, with $S \in (0,T)$, we denote by $\widetilde{u}$ the zero extension of $u$ over $(0,T)$. Then it follows from (1.42) that $\hat{y}(T;0,\widetilde{u}) \in \mathcal{R}_T^0$. We define, for each $S \in (0,T)$, a map $G_{\hat{f},S}$ from $L^\infty(0,S;U)$ to $\mathbb{R}$ by setting

$$\mathcal{G}_{\hat{f},S}(u) := \langle \hat{f}, \hat{y}(T;0,\widetilde{u}) \rangle_{(\mathcal{R}_T^0)^*, \mathcal{R}_T^0} \text{ for each } u \in L^\infty(0,S;U). \tag{2.43}$$

From (2.43), we see that for each $S \in (0,T)$,

$$|\mathcal{G}_{\hat{f},S}(u)| \leq \|\hat{f}\|_{(\mathcal{R}_T^0)^*} \|\hat{y}(T;0,\widetilde{u})\|_{\mathcal{R}_T} \text{ for each } u \in L^\infty(0,S;U). \tag{2.44}$$

Arbitrarily fix an $S \in (0,T)$. By (H1) and Lemma 2.3, we have (ii) of Lemma 2.3. Thus, there exists a $C_1(T,S) > 0$ so that for each $u \in L^\infty(0,S;U)$, there is a control $\hat{v}_u \in L^\infty(0,T;U)$ verifying that

$$\hat{y}(T;0,\widetilde{u}) = \hat{y}(T;0,\chi_{(S,T)}\hat{v}_u) \text{ and } \|\hat{v}_u\|_{L^\infty(0,T;U)} \leq C_1(T,S)\|\widetilde{u}\|_{L^{p_1}(0,T;U)} \tag{2.45}$$

for some $p_1 \in [2,\infty)$. From the first assertion in (2.45) and (1.40), we find that

$$\|\hat{y}(T;0,\widetilde{u})\|_{\mathcal{R}_T} \leq \|\hat{v}_u\|_{L^\infty(0,T;U)},$$

which, together with the second assertion in (2.45), indicates that

$$\|\hat{y}(T;0,\widetilde{u})\|_{\mathcal{R}_T} \leq C_1(T,S)\|\widetilde{u}\|_{L^{p_1}(0,S;U)} \text{ for all } u \in L^\infty(0,S;U).$$

This, as well as (2.44), yields that for each $S \in (0,T)$,

$$|\mathcal{G}_{\hat{f},S}(u)| \leq C_1(T,S)\|\hat{f}\|_{(\mathcal{R}_T^0)^*}\|u\|_{L^{p_1}(0,S;U)} \text{ for all } u \in L^\infty(0,S;U). \tag{2.46}$$

By (2.46) and the Hahn-Banach theorem, we can uniquely extend $\mathcal{G}_{\hat{f},S}$ to be an element in $\big(L^{p_1}(0,S;U)\big)^*$, denoted in the same manner, so that

$$|\mathcal{G}_{\hat{f},S}(u)| \leq C_1(T,S)\|\hat{f}\|_{(\mathcal{R}_T^0)^*}\|u\|_{L^{p_1}(0,S;U)} \ \text{ for all } \ u\in L^{p_1}(0,S;U). \tag{2.47}$$

From (2.47), using the Riesz representation theorem, we find that for each $S\in(0,T)$, there exists a $g_S\in L^{p_1'}(0,S;U)$, with $1/p_1+1/p_1'=1$, so that

$$\mathcal{G}_{\hat{f},S}(u)=\int_0^S\langle g_S(t),u(t)\rangle_U\,\mathrm{d}t \ \text{ for all } \ u\in L^{p_1}(0,S;U). \tag{2.48}$$

Next, arbitrarily fix an $S\in(0,T)$. Then take $v\in L^\infty(0,S;U)$ so that $\hat{y}(T;0,\widetilde{v})=0$. (Here, $\widetilde{v}$ is the zero extension of $v$ over $(0,T)$.) By (2.48) and (2.43), we see that

$$\int_0^S\langle g_S(t),v(t)\rangle_U\,\mathrm{d}t=\mathcal{G}_{\hat{f},S}(v)=0.$$

This, along with Lemma 2.5, yields that

$$g_S\in Y_S \quad \text{for each} \quad S\in(0,T). \tag{2.49}$$

Meanwhile, from (2.48), (2.44) and (1.40), one can easily check that for each $u\in L^\infty(0,S;U)$,

$$\int_0^S\langle g_S(t),u(t)\rangle_U\,\mathrm{d}t\leq\|\hat{f}\|_{(\mathcal{R}_T^0)^*}\|\hat{y}(T;0,\widetilde{u})\|_{\mathcal{R}_T}\leq\|\hat{f}\|_{(\mathcal{R}_T^0)^*}\|u\|_{L^\infty(0,S;U)}.$$

This, together with (2.49), implies that

$$\|g_S\|_{Y_S}=\|g_S\|_{L^1(0,S;U)}\leq\|\hat{f}\|_{(\mathcal{R}_T^0)^*} \ \text{ for all } \ S\in(0,T). \tag{2.50}$$

We now define a function $\hat{g}:(0,T)\to U$ in the following manner: For each $S\in(0,T)$,

$$\hat{g}(t):=g_S(t) \ \text{ for all } \ t\in(0,S). \tag{2.51}$$

The map $\hat{g}$ is well defined on $(0,T)$. In fact, when $0<S_1<S_2<T$, it follows from (2.48) and (2.43) that for each $u\in L^\infty(0,S_1;U)$,

$$\begin{aligned}\int_0^{S_1}\langle g_{S_1}(t),u(t)\rangle_U\,\mathrm{d}t &= \mathcal{G}_{\hat{f},S_1}(u)=\langle\hat{f},\hat{y}(T;0,\widetilde{u})\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0}=\mathcal{G}_{\hat{f},S_2}(\widetilde{u}|_{(0,S_2)})\\ &= \int_0^{S_2}\langle g_{S_2}(t),\widetilde{u}(t)\rangle_U\,\mathrm{d}t=\int_0^{S_1}\langle g_{S_2}(t),u(t)\rangle_U\,\mathrm{d}t,\end{aligned}$$

which implies that $g_{S_1}(\cdot)=g_{S_2}(\cdot)$ over $(0,S_1)$. So one can check from (2.51) that the function $\hat{g}$ is well defined. By (2.51) and (2.50), we see that

$$\|\hat{g}\|_{L^1(0,T;U)}\leq\|\hat{f}\|_{(\mathcal{R}_T^0)^*}. \tag{2.52}$$

Since (H1) was assumed, from (2.52), (2.51), (2.49) and (ii) of Lemma 2.4, we find that

$$\hat{g}\in Y_T \ \text{ and } \ \|\hat{g}\|_{Y_T}\leq\|\hat{f}\|_{(\mathcal{R}_T^0)^*}.$$

By (2.43), (2.48) and (2.51), we deduce that for each $S\in(0,T)$,

$$\langle\hat{f},\hat{y}(T;0,\widetilde{u})\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0}=\int_0^T\langle\hat{g}(t),\widetilde{u}(t)\rangle_U\mathrm{d}t \ \text{ for all } \ u\in L^\infty(0,S;U). \tag{2.53}$$

Now, for each $y_T \in \mathcal{R}_T^0$, it follows by (1.42) that there is an $u_{y_T} \in L^\infty(0,T;U)$ so that

$$y_T = \hat{y}(T;0,u_{y_T}) \text{ and } \lim_{S\to T} \|u_{y_T}\|_{L^\infty(S,T;U)} = 0.$$

From these and (1.40), one can check that when $S$ goes to $T$,

$$\big\|\hat{y}(T;0,\chi_{(0,S)}u_{y_T}) - y_T\big\|_{\mathcal{R}_T} = \|\hat{y}(T;0,\chi_{(S,T)}u_{y_T})\|_{\mathcal{R}_T} \le \|u_{y_T}\|_{L^\infty(S,T;U)} \to 0,$$

which implies that

$$\hat{y}(T;0,\chi_{(0,S)}u_{y_T}) \to y_T \text{ in } \mathcal{R}_T, \text{ as } S \to T. \tag{2.54}$$

Notice that $\hat{y}(T;0,\chi_{(0,S)}u_{y_T}) \in \mathcal{R}_T^0$ and $\hat{g} \in Y_T$. Thus, from (2.54), (2.53) and (2.3), using the dominated convergence theorem, we find that for each $y_T \in \mathcal{R}_T^0$,

$$\begin{aligned}\langle \hat{f}, y_T\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0} &= \lim_{S\to T} \big\langle \hat{f}, \hat{y}(T;0,\chi_{(0,S)}u_{y_T})\big\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0} \\ &= \lim_{S\to T} \int_0^T \big\langle \hat{g}(t), \chi_{(0,S)}(t)u_{y_T}(t)\big\rangle_U \,\mathrm{d}t \\ &= \int_0^T \langle \hat{g}(t), u_{y_T}(t)\rangle_U \,\mathrm{d}t = \langle \hat{g}, y_T\rangle_{Y_T,\mathcal{R}_T}.\end{aligned}$$

This, along with (2.38), yields that

$$\langle \hat{f}, y_T\rangle_{(\mathcal{R}_T^0)^*,\mathcal{R}_T^0} = \widehat{\mathcal{F}}_{\hat{g}}(y_T) \text{ for all } y_T \in \mathcal{R}_T^0, \text{ i.e., } \hat{f} = \widehat{\mathcal{F}}_{\hat{g}} \text{ in } (\mathcal{R}_T^0)^*,$$

which, together with (2.39), leads to (2.42). So $\Psi_T$ is surjective.

*Step 3. To show the second equality in (2.37)*

The second equality in (2.37) follows from (2.39), (2.38) and (2.3) (in Theorem 2.2).

In summary, we finish the proof of this theorem. □

**Remark 9.** We do not know whether $\mathcal{R}_T^0$ is a closed subspace of $\mathcal{R}_T$ in general.

**Corollary 1.** *Suppose that (H1) holds. Then for each $T \in (0,\infty)$, $B_{Y_T}$ (the closed unit ball in $Y_T$) is compact in the topology $\sigma(Y_T, \mathcal{R}_T^0)$ (i.e., weak star compact).*

*Proof.* By Theorem 2.6, we have that $Y_T = (\mathcal{R}_T^0)^*$. Then by the Banach-Alaoglu theorem, $B_{Y_T}$ is compact in the topology $\sigma(Y_T, \mathcal{R}_T^0)$. This ends the proof. □

2.3. **Further studies on attainable subspaces.** The following Lemma presents the non-triviality of the subspaces $Y_T$ and $\mathcal{R}_T^0$, with $T \in (0,\infty)$. (Consequently, $\mathcal{R}_T$ is also non trivial.) Here, we will use the assumption that the control operator $B$ is non-trivial.

**Lemma 2.7.** *Let $0 < T < \infty$. Then the sets $Y_T \setminus \{0\}$ and $\mathcal{R}_T^0 \setminus \{0\}$ are nonempty.*

*Proof.* Arbitrarily fix a $T \in (0,\infty)$. We first show that $Y_T \setminus \{0\} \neq \emptyset$. Seeking for a contradiction, we suppose that $Y_T \setminus \{0\} = \emptyset$. Since $X_T \subset Y_T$ (see (1.20)), we could derive from (1.21) that for each $z \in D(A^*)$, $B^*S^*(T-\cdot)z = 0$ over $(0,T)$. Since $\{S^*(t)|_{D(A^*)}\}_{t\in\mathbb{R}^+}$ is a $C_0$-semigroup on $D(A^*)$ and $B^* \in \mathcal{L}(D(A^*),U)$, the above yields that for each $t \in [0,T]$ and each $z \in D(A^*)$, $B^*S^*(T-t)z = 0$. Taking $t = T$ in above equality leads to that $B^*z = 0$ for all $z \in D(A^*)$, i.e., $B^* = 0$, which contradicts the assumption that $B \neq 0$. Thus we have proved that $Y_T \setminus \{0\} \neq \emptyset$.

We next verify that the set $\mathcal{R}_T^0 \setminus \{0\}$ is nonempty. By contradiction, suppose that it was not true. Then we would have that

$$\mathcal{R}_T^0 \setminus \{0\} = \emptyset, \text{ i.e., } \mathcal{R}_T^0 = \{0\}. \tag{2.55}$$

Arbitrarily fix an $\varepsilon \in (0,T)$. We find from (1.42) that $\hat{y}(T;0,\widetilde{v}) \in \mathcal{R}_T^0$ for all $v \in L^\infty(0,\varepsilon;U)$, where $\widetilde{v}$ denotes the zero extension of $v$ over $(0,T)$. This, together with (2.55) and (1.13), yields that for all $z \in D(A^*)$ and $v \in L^\infty(0,\varepsilon;U)$,

$$\int_0^T \langle B^*S^*(T-t)z, \widetilde{v}(t)\rangle_U \,\mathrm{d}t = \langle z, \hat{y}(T;0,\widetilde{v})\rangle_X = 0.$$

From the above, we find that for each $z \in D(A^*)$, $B^*S^*(T-\cdot)z = 0$ over $(0,\varepsilon)$. Since $\varepsilon$ was arbitrarily taken from $(0,T)$, the above indicates that $B^*S^*(T-\cdot)z = 0$ over $(0,T)$, for each $z \in D(A^*)$. From this and (1.21), we find that $X_T = \{0\}$, which, along with (1.20), indicates that $Y_T = \{0\}$. This leads to a contradiction, since we have proved that $Y_T \setminus \{0\} \neq \emptyset$. Therefore, $\mathcal{R}_T^0 \setminus \{0\} \neq \emptyset$. Thus, we end the proof of this lemma.

□

The next result presents an expression on the norm $\|\cdot\|_{\mathcal{R}_T}$.

**Proposition 2.** *Let $0 < T < \infty$. Write*

$$\hat{Z}_T := \{z \in D(A^*) \ : \ B^*S^*(T-\cdot)z \neq 0 \ \text{ in } \ L^1(0,T;U)\}.$$

*Then it stands that*

$$\|y_T\|_{\mathcal{R}_T} = \sup_{z\in\hat{Z}_T} \frac{\langle y_T, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}} \ \text{ for all } \ y_T \in \mathcal{R}_T. \tag{2.56}$$

*Proof.* Arbitrarily fix $T \in (0,\infty)$. First of all, we notice that $\hat{Z}_T \neq \emptyset$. Indeed, if it was not true, then by (1.21), we would have that $X_T = \{0\}$, which, along with (1.20), yields that $Y_T = \{0\}$. This contradicts Lemma 2.7. So we have proved that $\hat{Z}_T \neq \emptyset$.

Recall (2.9) for the linear isomorphism $\Phi_T$ from $\mathcal{R}_T$ to $Y_T^*$. It is clear that

$$\|y_T\|_{\mathcal{R}_T} = \|\Phi_T(y_T)\|_{Y_T^*} = \|\widetilde{\mathcal{F}}_{y_T}\|_{Y_T^*} = \sup_{f\in Y_T\setminus\{0\}} \frac{\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T}}{\|f\|_{Y_T}}, \ \forall\, y_T \in \mathcal{R}_T. \tag{2.57}$$

We claim that for each $y_T \in \mathcal{R}_T$,

$$\sup_{f\in Y_T\setminus\{0\}} \frac{\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T}}{\|f\|_{Y_T}} = \sup_{z\in\hat{Z}_T} \frac{\langle y_T, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}}. \tag{2.58}$$

To this end, we arbitrarily take $y_T \in \mathcal{R}_T$ and then fix $v \in \mathcal{U}_{ad}^{y_T}$. (Since $y_T \in \mathcal{R}_T$, it follows by (2.5) that $\mathcal{U}_{ad}^{y_T} \neq \emptyset$.) On one hand, given $f \in Y_T \setminus \{0\}$, it follows by (1.20) that there is a sequence $\{z_n\}$ in $D(A^*)$ so that

$$B^*S^*(T-\cdot)z_n \to f(\cdot) \ \text{ in } \ L^1(0,T;U). \tag{2.59}$$

Since $f \neq 0$, we see from (2.59) that when $n$ is large enough,

$$B^*S^*(T-\cdot)z_n \neq 0 \ \text{ in } \ L^1(0,T;U), \ \text{ i.e., } \ z_n \in \hat{Z}_T. \tag{2.60}$$

From (2.59), the definition of $\widetilde{\mathcal{F}}_{y_T}$ (see (2.7)) and the first equality in (2.6), we find that

$$\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T} = \lim_{n\to\infty} \int_0^T \langle v(t), B^*S^*(T-t)z_n\rangle_U \,\mathrm{d}t = \lim_{n\to\infty} \langle y_T, z_n\rangle_X.$$

This, together with (2.59) and (2.60), yields that

$$\frac{\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T}}{\|f\|_{Y_T}} = \lim_{n\to\infty} \frac{\langle y_T, z_n\rangle_X}{\|B^*S^*(T-\cdot)z_n\|_{L^1(0,T;U)}} \leq \sup_{z\in\hat{Z}_T} \frac{\langle y_T, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}}.$$

Since $f$ was arbitrarily taken from $Y_T\setminus\{0\}$, the above leads to

$$\sup_{f\in Y_T\setminus\{0\}} \frac{\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T}}{\|f\|_{Y_T}} \leq \sup_{z\in\hat{Z}_T} \frac{\langle y_T, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}}. \tag{2.61}$$

On the other hand, let $z\in\hat{Z}_T$ be arbitrarily fixed. It is clear that

$$B^*S^*(T-\cdot)z \in Y_T\setminus\{0\}. \tag{2.62}$$

Moreover, it follows from the first equality in (2.6) and (2.7) that

$$\langle y_T, z\rangle_X = \int_0^T \langle v(t), B^*S^*(T-t)z\rangle_U\,\mathrm{d}t = \widetilde{\mathcal{F}}_{y_T}\big(B^*S^*(T-\cdot)z|_{(0,T)}\big). \tag{2.63}$$

By (2.63) and (2.62). we find that

$$\frac{\langle y_T, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}} = \frac{\widetilde{\mathcal{F}}_{y_T}\big(B^*S^*(T-\cdot)z|_{(0,T)}\big)}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}} \leq \sup_{f\in Y_T\setminus\{0\}} \frac{\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T}}{\|f\|_{Y_T}}.$$

Since $z$ was arbitrarily taken from $\hat{Z}_T$, the above leads to that

$$\sup_{z\in\hat{Z}_T} \frac{\langle y_T, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}} \leq \sup_{f\in Y_T\setminus\{0\}} \frac{\langle \widetilde{\mathcal{F}}_{y_T}, f\rangle_{Y_T^*,Y_T}}{\|f\|_{Y_T}}. \tag{2.64}$$

Finally, (2.58) follows from (2.61) and (2.64). This, along with (2.57), proves (2.56). We end the proof of this proposition.

□

The following proposition is about the relation between $(NP)^{y_T}$ and $(NP)^{T,y_0}$ with $y_T = -S(T)y_0$.

**Proposition 3.** *Let $y_0\in X$ and $T\in(0,\infty)$ satisfy that $-S(T)y_0\in\mathcal{R}_T$. Then the following conclusions are valid:*
*(i) Any admissible control to $(NP)^{y_T}$ (with $y_T := -S(T)y_0$) is an admissible control to $(NP)^{T,y_0}$. And the reverse is also true.*
*(ii) $\|-S(T)y_0\|_{\mathcal{R}_T} = N(T,y_0)$.*
*(iii) Any minimal norm control to $(NP)^{y_T}$ (with $y_T = -S(T)y_0$) is a minimal norm control to $(NP)^{T,y_0}$. And the reverse is also true.*

*Proof.* (i) Let $\hat{v}$ be an admissible control to $(NP)^{y_T}$, with $y_T := -S(T)y_0$. Then it follows from (1.40) that $\hat{y}(T;0,\hat{v}) = -S(T)y_0$, which yields that $\hat{y}(T;y_0,\hat{v}) = 0$. This, along with (1.16), implies that $\hat{v}$ is an admissible control to $(NP)^{T,y_0}$.

Conversely, if $\widetilde{v}$ is an admissible control to $(NP)^{T,y_0}$, then by (1.16), we see that $\hat{y}(T;y_0,\widetilde{v}) = 0$, which yields that $\hat{y}(T;0,\widetilde{v}) = -S(T)y_0$. This, along with (1.40), indicates that $\widetilde{v}$ is an admissible control to $(NP)^{y_T}$, with $y_T = -S(T)y_0$.

(ii) By (1.16) and (1.40), one can directly check that $N(T,y_0) = \|-S(T)y_0\|_{\mathcal{R}_T}$.

(iii) Let $v^*$ be a minimal norm control to $(NP)^{y_T}$, with $y_T = -S(T)y_0$. Then by (i) of this proposition, $v^*$ is an admissible control to $(NP)^{T,y_0}$, i.e.,

$$\hat{y}(T;y_0,v^*) = 0. \tag{2.65}$$

Meanwhile, by the optimality of $v^*$, we have that $\|v^*\|_{L^\infty(0,T;U)} = \|-S(T)y_0\|_{\mathcal{R}_T}$, which, along with (ii) of this proposition, shows that

$$\|v^*\|_{L^\infty(0,T;U)} = N(T, y_0). \tag{2.66}$$

By (2.65) and (2.66), we see that $v^*$ is a minimal norm control to $(NP)^{T,y_0}$.

Similarly, we can show the reverse. Thus, we finish the proof of this proposition. □

**Corollary 2.** *Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < \infty$. Write*

$$\hat{Z}_T := \big\{z \in D(A^*) \ : \ B^*S^*(T-\cdot)z \neq 0 \ \text{ in } \ L^1(0,T;U)\big\}, \ 0 < T < \infty.$$

*Then for each $T \in \big(T^0(y_0), \infty\big)$,*

$$N(T, y_0) = \sup_{z \in \hat{Z}_T} \frac{\langle S(T)y_0, z\rangle_X}{\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)}} < \infty. \tag{2.67}$$

*Proof.* Arbitrarily fix $T \in \big(T^0(y_0), \infty\big)$. At the start of the proof of Proposition 2, we already proved that $\hat{Z}_s \neq \emptyset$ for each $s \in (0, \infty)$. Since $T > T^0(y_0)$, by (1.17), there exists a control $u \in L^\infty(0,T;U)$ so that $\hat{y}(T; y_0, u) = 0$. This, along with (1.41), yields that $-S(T)y_0 = \hat{y}(T; 0, u) \in \mathcal{R}_T$, which, together with (ii) of Proposition 3 and Proposition 2, leads to (2.67). We end the proof. □

The property on $\mathcal{R}_T^0$ presented in the following Proposition 4 plays another important role in the studies of a maximum principle for $(NP)^{y_T}$, with $y_T \in \mathcal{R}_T^0$. In what follows, we denote by $B_{\mathcal{R}_T^0}$ and $B_{\mathcal{R}_T}$ the closed unit balls in $\mathcal{R}_T^0$ and $\mathcal{R}_T$, respectively.

**Proposition 4.** *For each $T \in (0, \infty)$, it holds that $B_{\mathcal{R}_T} = \overline{B}_{\mathcal{R}_T^0}^{\sigma(\mathcal{R}_T, Y_T)}$. Here, the set $\overline{B}_{\mathcal{R}_T^0}^{\sigma(\mathcal{R}_T, Y_T)}$ is the closure of $B_{\mathcal{R}_T^0}$ in the space $\mathcal{R}_T$, under the topology $\sigma(\mathcal{R}_T, Y_T)$.*

*Proof.* Let $0 < T < \infty$. We first prove that

$$B_{\mathcal{R}_T} \subset \overline{B}_{\mathcal{R}_T^0}^{\sigma(\mathcal{R}_T, Y_T)}. \tag{2.68}$$

Let $y_T \in B_{\mathcal{R}_T}$. From (1.40), there exists a sequence $\{v_k\}$ so that for all $k \in \mathbb{N}^+$,

$$y_T = \hat{y}(T; 0, v_k) \ \text{ and } \ \|y_T\|_{\mathcal{R}_T} \le \|v_k\|_{L^\infty(0,T;U)} \le \|y_T\|_{\mathcal{R}_T} + 1/k. \tag{2.69}$$

For each $k \in \mathbb{N}^+$, we set

$$\lambda_k := \frac{\|y_T\|_{\mathcal{R}_T}}{\|y_T\|_{\mathcal{R}_T} + 1/k} \ \text{ and } \ u_k := \chi_{(0, T-1/k)} \lambda_k v_k. \tag{2.70}$$

It is clear that

$$\|u_k\|_{L^\infty(0,T;U)} \le \|y_T\|_{\mathcal{R}_T} \le 1 \ \text{ for all } \ k \in \mathbb{N}^+. \tag{2.71}$$

From (1.42), (2.70), (1.40) and (2.71), we can easily check that

$$\hat{y}(T; 0, u_k) \in B_{\mathcal{R}_T^0} \ \text{ for all } \ k \in \mathbb{N}^+. \tag{2.72}$$

Meanwhile, from (2.69), (2.3) and (2.70), we find that for each $f \in Y_T$,

$$\begin{aligned} \big\langle \hat{y}(T; 0, u_k) - y_T, f \big\rangle_{\mathcal{R}_T, Y_T} &= \big\langle \hat{y}(T; 0, u_k - v_k), f \big\rangle_{\mathcal{R}_T, Y_T} \\ &= \int_0^T \big\langle u_k(t) - v_k(t), f(t) \big\rangle_U \, \mathrm{d}t \to 0, \ \text{ as } \ k \to \infty. \end{aligned}$$

This, along with (2.72), yields that $y_T \in \overline{B}_{\mathcal{R}_T^0}^{\sigma(\mathcal{R}_T, Y_T)}$. Since $y_T$ was arbitrarily taken from $B_{\mathcal{R}_T}$, the above leads (2.68).

We next show that

$$B_{\mathcal{R}_T} \supseteq \overline{B}_{\mathcal{R}_T^0}^{\sigma(\mathcal{R}_T, Y_T)}. \tag{2.73}$$

For this purpose, we let $y_T \in \mathcal{R}_T$ and $\{y_n\} \subset B_{\mathcal{R}_T^0}$ so that

$$y_n \to y_T \ \text{ in the topology } \ \sigma(\mathcal{R}_T, Y_T), \ \text{ as } \ n \to \infty.$$

Since $\mathcal{R}_T = Y_T^*$ (see Theorem 2.2), we find that

$$y_n \to y_T \ \text{ in the weak star topology}, \ \text{ as } \ n \to \infty.$$

Hence,

$$\|y_T\|_{\mathcal{R}_T} \leq \liminf_{n\to\infty} \|y_n\|_{\mathcal{R}_T} \leq 1,$$

which yields that $y_T \in B_{\mathcal{R}_T}$. This proves (2.73).

Finally, it follows from (2.68) and (2.73) that $B_{\mathcal{R}_T} = \overline{B}_{\mathcal{R}_T^0}^{\sigma(\mathcal{R}_T, Y_T)}$. This ends the proof.

□

The following lemma mainly shows that the reachable subspaces $\mathcal{R}_T$ and $\mathcal{R}_T^0$ are independent of $T \in (0, \infty)$, provided that the condition (H1) holds.

**Proposition 5.** *Suppose that (H1) holds. Let $0 < T_1 < T_2 < \infty$. Then the following conclusions are valid:*
*(i) The spaces $\mathcal{R}_{T_1}$ and $\mathcal{R}_{T_2}$ are same, and the norms $\|\cdot\|_{\mathcal{R}_{T_1}}$ and $\|\cdot\|_{\mathcal{R}_{T_2}}$ are equivalent.*
*(ii) The spaces $\mathcal{R}_{T_1}^0$ and $\mathcal{R}_{T_2}^0$ are same.*

*Proof.* Suppose that (H1) holds. Arbitrarily fix $0 < T_1 < T_2 < \infty$. We will prove the conclusions (i)-(ii) one by one.

(i) Arbitrarily fix $y_{T_1} \in \mathcal{R}_{T_1}$ and $k \in \mathbb{N}^+$. Then by (1.41) and (1.40), there exists a control $u_{y_{T_1}} \in L^\infty(0, T_1; U)$ so that

$$y_{T_1} = \hat{y}(T_1; 0, u_{y_{T_1}}) \ \text{ and } \ \|u_{y_{T_1}}\|_{L^\infty(0,T_1;U)} \leq \|y_{T_1}\|_{\mathcal{R}_{T_1}} + 1/k. \tag{2.74}$$

Define another control $\hat{u}_{y_{T_1}}$ by setting

$$\hat{u}_{y_{T_1}}(t) = \begin{cases} 0, & t \in (0, T_2 - T_1], \\ u_{y_{T_1}}(t - T_2 + T_1), & t \in (T_2 - T_1, T_2). \end{cases} \tag{2.75}$$

Then from (1.14), the first equality in (2.74) and (2.75), one can easily check that $y_{T_1} = \hat{y}(T_2; 0, \hat{u}_{y_{T_1}})$, which, along with (1.41), (1.40), (2.75) and the second inequality in (2.74), yields that $y_{T_1} \in \mathcal{R}_{T_2}$ and $\|y_{T_1}\|_{\mathcal{R}_{T_2}} \leq \|y_{T_1}\|_{\mathcal{R}_{T_1}} + 1/k$. Since $k$ was arbitrarily taken from $\mathbb{N}^+$, the above implies that for each $y_{T_1} \in \mathcal{R}_{T_1}$,

$$y_{T_1} \in \mathcal{R}_{T_2} \ \text{ and } \ \|y_{T_1}\|_{\mathcal{R}_{T_2}} \leq \|y_{T_1}\|_{\mathcal{R}_{T_1}}. \tag{2.76}$$

Conversely, arbitrarily fix $y_{T_2} \in \mathcal{R}_{T_2}$ and $k \in \mathbb{N}^+$. Then by (1.41) and (1.40), there exists a control $u_{y_{T_2}} \in L^\infty(0, T_2; U)$ so that

$$y_{T_2} = \hat{y}(T_2; 0, u_{y_{T_2}}) \ \text{ and } \ \|u_{y_{T_2}}\|_{L^\infty(0,T_2;U)} \leq \|y_{T_2}\|_{\mathcal{R}_{T_2}} + 1/k. \tag{2.77}$$

By (H1), we can apply Lemma 2.3 to get the conclusion (ii) of Lemma 2.3 with some $p_1 \in [2, \infty)$. Because $\chi_{(0,T_2-T_1)} u_{y_{T_2}} \in L^{p_1}(0, T_2; U)$, it follows from (ii) of Lemma

2.3 (where $T = T_2$ and $t = T_2 - T_1$) that there exists a control $\hat{v} \in L^\infty(0, T_2; U)$ so that

$$\hat{y}(T_2; 0, \chi_{(0,T_2-T_1)} u_{y_{T_2}}) = \hat{y}(T_2; 0, \chi_{(T_2-T_1,T_2)} \hat{v}) \tag{2.78}$$

and

$$\|\hat{v}\|_{L^\infty(0,T_2;U)} \leq C_1 \|u_{y_{T_2}}\|_{L^{p_1}(0,T_2;U)} \leq C_1 (T_2)^{1/p_1} \|u_{y_{T_2}}\|_{L^\infty(0,T_2;U)}, \tag{2.79}$$

where $C_1 := C_1(T_2, T_2 - T_1)$ is given by (ii) of Lemma 2.3. Define a control $\widetilde{v}(\cdot)$ by setting

$$\widetilde{v}(t) := u_{y_{T_2}}(t + T_2 - T_1) + \hat{v}(t + T_2 - T_1),\ t \in (0, T_1).$$

Then, by the first assertion in (2.77) and (2.78), one can directly check that $y_{T_2} = \hat{y}(T_1; 0, \widetilde{v})$, which, together with (1.41), (1.40), (2.79) and the inequality in (2.77), indicates that

$$y_{T_2} \in \mathcal{R}_{T_1} \ \text{ and } \ \|y_{T_2}\|_{\mathcal{R}_{T_1}} \leq (1 + C_1(T_2)^{1/p_1})\big(\|y_{T_1}\|_{\mathcal{R}_{T_2}} + 1/k\big).$$

Since $k$ was arbitrarily taken from $\mathbb{N}^+$, the above implies that for each $y_{T_2} \in \mathcal{R}_{T_2}$,

$$y_{T_2} \in \mathcal{R}_{T_1} \ \text{ and } \ \|y_{T_2}\|_{\mathcal{R}_{T_1}} \leq (1 + C_1(T_2)^{1/p_1})\|y_{T_2}\|_{\mathcal{R}_{T_2}}. \tag{2.80}$$

Now, the conclusion (i) follows from (2.76) and (2.80).

(ii) By a very similar way as that used in the proof of the conclusion (i), we can show that $\mathcal{R}^0_{T_1} = \mathcal{R}^0_{T_2}$.

In summary, we end the proof of this proposition. □

3. **Properties of several functions.** This section presents some properties on functions $N(\cdot, y_0)$ (with $y_0 \in X \setminus \{0\}$), $T^0(\cdot)$ and $T^1(\cdot)$, which are defined by (1.16), (1.17) and (1.18), respectively. The decompositions of $\mathcal{W}$ and $\mathcal{V}$ (given in (i) of Theorem 1.1 and (i) of Theorem 1.2, respectively) are based on these properties. We begin with the following Lemma 3.1. Since the exactly same result as that in this lemma was not found by us in literatures (but the proof for the similar result to Lemma 3.1 can be found in, for instance, [7, Lemma 1.1]), we give its proof in Appendix E, for the sake of the completeness of the paper.

**Lemma 3.1.** *Let $\{T_n\}_{n=1}^\infty \subset [0, \infty)$ and $\{u_n\}_{n=1}^\infty \subset L^2(\mathbb{R}^+; U)$ satisfy that*

$$T_n \to \widehat{T} \ \text{ and } \ u_n \to \hat{u} \ \text{ weakly in } \ L^2(\mathbb{R}^+; U), \ \text{ as } \ n \to \infty \tag{3.1}$$

*for some $\widehat{T} \in [0, \infty)$ and $\hat{u} \in L^2(\mathbb{R}^+; U)$. Then for each $y_0 \in X$,*

$$y(T_n; y_0, u_n) \to y(\widehat{T}; y_0, \hat{u}) \text{ weakly in } X, \ \text{ as } \ n \to \infty. \tag{3.2}$$

The next lemma concerns the monotonicity of the function $N(\cdot, y_0)$.

**Lemma 3.2.** *Let $y_0 \in X \setminus \{0\}$. Then the following conclusions are true:*
*(i) The function $N(\cdot, y_0)$ is decreasing from $(0, \infty)$ to $[0, \infty]$.*
*(ii) The function $N(\cdot, y_0)$, when extended over $[0, \infty]$ via (1.19), is decreasing from $[0, \infty]$ to $[0, \infty]$.*

*Proof.* (i) We first show that $N(\cdot, y_0)$ is decreasing over $(0, \infty)$. For this purpose, let $T_1$ and $T_2$ satisfy that $0 < T_1 < T_2 < \infty$. There are only two possibilities on $N(T_1, y_0)$: either $N(T_1, y_0) = \infty$ or $N(T_1, y_0) < \infty$.

In the case that $N(T_1, y_0) = \infty$, it is obvious that $N(T_1, y_0) \geq N(T_2, y_0)$. In the case that $N(T_1, y_0) < \infty$, we arbitrarily fix a $\varepsilon > 0$. It follows from (1.16) that there exists a control $v_\varepsilon$ so that $\hat{y}(T_1; y_0, v_\varepsilon) = 0$ and $\|v_\varepsilon\|_{L^\infty(0,T_1;U)} \leq N(T_1, y_0) + \varepsilon$. Write $\widetilde{v}_\varepsilon$ for the zero extension of $v_\varepsilon$ over $(0, T_2)$. Then from the above, we find that

$$\hat{y}(T_2; y_0, \widetilde{v}_\varepsilon) = 0 \ \text{ and } \ \|\widetilde{v}_\varepsilon\|_{L^\infty(0,T_2;U)} = \|v_\varepsilon\|_{L^\infty(0,T_1;U)} \leq N(T_1, y_0) + \varepsilon. \tag{3.3}$$

From the first equality in (3.3), it follows that $\widetilde{v}_\varepsilon$ is an admissible control to $(NP)^{T_2,y_0}$. This, along with the optimality of $N(T_2, y_0)$ and the second assertion in (3.3), yields that

$$N(T_2, y_0) \leq \|\widetilde{v}_\varepsilon\|_{L^\infty(0,T_2;U)} \leq N(T_1, y_0) + \varepsilon.$$

Since $\varepsilon$ was arbitrarily taken, the above leads to the following inequality in this case: $N(T_1, y_0) \geq N(T_2, y_0)$. Hence, the function $N(\cdot, y_0)$ is decreasing over $(0, \infty)$.

Next, by (1.16), we see that $0 \leq N(T, y_0) \leq \infty$ for all $T \in (0, \infty)$. Thus, the conclusion (i) of this lemma has been proved.

(ii) The conclusion (ii) follows from the conclusion (i) of this lemma and (1.19).

In summary, we end the proof of this lemma. □

The following two lemmas concern with some relations among $N(\cdot, y_0)$, $T^0(\cdot)$ and $T^1(\cdot)$.

**Lemma 3.3.** *Let $y_0 \in X \backslash \{0\}$. Then the following conclusions are true:*
*(i) $T^0(y_0) \leq T^1(y_0)$. (ii) $T^1(y_0) > 0$. (iii) $N(T, y_0) > 0$ for all $T \in \big(0, T^1(y_0)\big)$.*
*(iv) $N(0, y_0) = \infty$. (v) If $T^1(y_0) < \infty$, then $N(T, y_0) = 0$ for all $T \in \big[T^1(y_0), \infty\big]$.*
*(vi) $N(T^1(y_0), y_0) = N(\infty, y_0)$.*

*Proof.* (i) There are only two possibilities on $T^0(y_0)$: either $T^0(y_0) = 0$ or $T^0(y_0) > 0$. In the case that $T^0(y_0) = 0$, it is clear that $T^0(y_0) \leq T^1(y_0)$. In the case when $T^0(y_0) > 0$, we assume, by contradiction, that $T^0(y_0) > T^1(y_0)$. Fix a $T \in \big(T^1(y_0), T^0(y_0)\big)$. Then by (1.17), we would have that for all $u \in L^\infty(0, T; U)$, $\hat{y}(T; y_0, u) \neq 0$; and by (1.18), we would have that $\hat{y}(T; y_0, 0) = S(T)y_0 = 0$. These lead to a contradiction. Hence, $T^0(y_0) \leq T^1(y_0)$.

(ii) By contradiction, suppose that $T^1(y_0) = 0$. Then by (1.18), we could have that for each $\hat{t} > 0$, $S(\hat{t})y_0 = 0$, which yields that $y_0 = \lim_{t \to 0^+} S(t)y_0 = 0$. This leads to a contradiction, since we assumed that $y_0 \in X \setminus \{0\}$. Hence, $T^1(y_0) > 0$.

(iii) By contradiction, suppose that $N(T_0, y_0) = 0$ for some $T_0 \in \big(0, T^1(y_0)\big)$. Then by (1.16), there would be a sequence $\{v_n\}$ in $L^\infty(0, T_0; U)$ so that

$$\hat{y}(T_0; y_0, v_n) = 0 \ \text{ for all } \ n \in \mathbb{N}^+; \ \text{ and } \ \|v_n\|_{L^\infty(0,T_0;U)} \to 0, \ \text{ as } \ n \to \infty.$$

From these and Lemma 3.1, we find that $S(T_0)y_0 = \hat{y}(T_0; y_0, 0) = 0$. From the above and (1.18), we see that $T^1(y_0) \leq T_0$, which leads to a contradiction, since $T_0 \in \big(0, T^1(y_0)\big)$. Hence, $N(T, y_0) > 0$ for all $T \in \big(0, T^1(y_0)\big)$.

(iv) By contradiction, suppose that $N(0, y_0) < \infty$. Then by (ii) of Lemma 3.2, we could find a sequence $\{T_n\} \subset \mathbb{R}^+$ so that

$$T_n \searrow 0, \ \text{ as } \ n \to \infty \tag{3.4}$$

and

$$N(T_n, y_0) \leq N(0, y_0) < \infty \ \text{ for all } \ n \in \mathbb{N}^+. \tag{3.5}$$

By (3.5) and (1.16), we see that for each $n \in \mathbb{N}^+$, $(NP)^{T_n,y_0}$ has an admissible control $u_n$ so that $\|u_n\|_{L^\infty(0,T_n;U)} \leq N(0, y_0) + 1$. Write $\widetilde{u}_n$ for the zero extension of $u_n$ over $\mathbb{R}^+$, $n \in \mathbb{N}^+$. Then we have that

$$y(T_n; y_0, \widetilde{u}_n) = 0 \ \text{ for all } \ n \in \mathbb{N}^+ \tag{3.6}$$

and

$$\|\widetilde{u}_n\|_{L^\infty(\mathbb{R}^+;U)} = \|u_n\|_{L^\infty(0,T_n;U)} \leq N(0,y_0)+1 \text{ for all } n\in\mathbb{N}^+. \tag{3.7}$$

From (3.4) and (3.7), we see that

$$\chi_{(0,T_n)}\widetilde{u}_n \to 0 \text{ strongly in } L^2(\mathbb{R}^+;U) \text{ as } n\to\infty. \tag{3.8}$$

From (3.4), (3.8) and Lemma 3.1, we find that

$$y(T_n;y_0,\chi_{(0,T_n)}\widetilde{u}_n) \to y(0;y_0,0)=y_0 \quad \text{weakly in} \quad X, \quad \text{as} \quad n\to\infty.$$

This, along with (3.6), yields that $y_0=0$, which leads to a contradiction, since it was assumed that $y_0\in X\setminus\{0\}$. So we have proved that $N(0,y_0)=\infty$.

(v) Assume that $T^1(y_0)<\infty$. We first claim that

$$N(T,y_0)=0 \text{ for each } T\in\big[T^1(y_0),\infty\big). \tag{3.9}$$

By contradiction, we suppose that $N(T_1,y_0)\neq 0$ for some $T_1\in[T^1(y_0),\infty)$. Then we would have that $\hat{y}(T_1;y_0,0)\neq 0$, i.e., $S(T_1)y_0\neq 0$. By the continuity of the function $t\to S(t)y_0$ at $T_1$, there is a $\delta>0$ so that $S(T_1+\delta)y_0\neq 0$, which implies that for each $t\in[0,T_1+\delta]$, $S(t)y_0\neq 0$. This, together with (1.18), implies that

$$T_1+\delta\leq T^1(y_0). \tag{3.10}$$

However, we had that $T_1\geq T^1(y_0)$ and $\delta>0$. These contradict (3.10). So (3.9) is proved.

Next, we see from the first equality in (1.19) and (3.9) that $N(\infty,y_0)=0$. This, together with (3.9), proves the conclusion (v).

(vi) There are only two possibilities on $T^1(y_0)$: either $T^1(y_0)=\infty$ or $T^1(y_0)<\infty$. In the case when $T^1(y_0)=\infty$, it is clear that $N(T^1(y_0),y_0)=N(\infty,y_0)$. In the case that $T^1(y_0)<\infty$, we see from (v) in this lemma that

$$N(T^1(y_0),y_0)=0=N(\infty,y_0).$$

This implies that $N(T^1(y_0),y_0)=N(\infty,y_0)$ in this case.

In summary, we end the proof of this lemma. □

**Lemma 3.4.** *Let $y_0\in X\setminus\{0\}$. Then the following conclusions are true:*
*(i) If $N(T^0(y_0),y_0)=\infty$, then either $T^0(y_0)<T^1(y_0)$ or $T^0(y_0)=T^1(y_0)=\infty$.*
*(ii) If $T^0(y_0)=\infty$, then $N(T^0(y_0),y_0)=\infty$.*
*(iii) If $0<N(T^0(y_0),y_0)<\infty$, then $T^0(y_0)<T^1(y_0)$.*
*(iv) $N(T^0(y_0),y_0)=0$ if and only if $T^0(y_0)=T^1(y_0)<\infty$.*
*(v) If $T^0(y_0)<\infty$, then $N(T^1(y_0),y_0)<\infty$.*

*Proof.* (i) By contradiction, we suppose that the conclusion (i) was not true. Then, by (i) of Lemma 3.3, we would have that

$$N(T^0(y_0),y_0)=\infty \text{ and } T^0(y_0)=T^1(y_0)<\infty. \tag{3.11}$$

The second conclusion in (3.11), along with (v) of Lemma 3.3, yields that

$$N(T^0(y_0),y_0)=N(T^1(y_0),y_0)=0.$$

This contradicts the first equality in (3.11). So the conclusion (i) is true.

(ii) Assume that $T^0(y_0)=\infty$. Then we find from (1.17) that when $T\in(0,\infty)$, $\hat{y}(T;y_0,u)\neq 0$ for all $u\in L^\infty(0,T;U)$. Thus, for each $T\in(0,\infty)$, $(NP)^{T,y_0}$ has no admissible control. So we have that $N(T, y_0) = \infty$ for all $T \in (0, \infty)$. Since $T^0(y_0) = \infty$, the above, as well as the first equality in (1.19), indicates that

$$N(T^0(y_0), y_0) = N(\infty, y_0) = \lim_{T\to\infty} N(T, y_0) = \infty.$$

This ends the proof of the conclusion (ii).

(iii) Assume that $0 < N(T^0(y_0), y_0) < \infty$. Suppose, by contradiction, that the conclusion (iii) was not true. Then, by (i) of Lemma 3.3, we would have that

$$0 < N(T^0(y_0), y_0) < \infty \ \text{ and } \ T^0(y_0) = T^1(y_0). \tag{3.12}$$

These, along with (ii) of this lemma, yield that $T^1(y_0) = T^0(y_0) < \infty$. Then by (v) of Lemma 3.3, we see that $N(T^0(y_0), y_0) = N(T^1(y_0), y_0) = 0$, which contradicts the first conclusion in (3.12). Hence, the conclusion (iii) is true.

(iv) We first show that

$$T^0(y_0) = T^1(y_0) < \infty \Rightarrow N(T^0(y_0), y_0) = 0. \tag{3.13}$$

Suppose that the assertion on left side of (3.13) holds. Then by (v) of Lemma 3.3, we see that $N(T^0(y_0), y_0) = N(T^1(y_0), y_0) = 0$, which leads to the equality on the right side of (3.13).

We next show that

$$N(T^0(y_0), y_0) = 0 \Rightarrow T^0(y_0) = T^1(y_0). \tag{3.14}$$

By contradiction, we suppose that (3.14) did not hold. Then by (i) of Lemma 3.3, we would have that

$$N(T^0(y_0), y_0) = 0 \ \text{ and } \ \ T^0(y_0) < T^1(y_0). \tag{3.15}$$

In the case that $T^0(y_0) = 0$, we find from (iv) of Lemma 3.3 that $N(T^0(y_0), y_0) = \infty$, which contradicts the first equality in (3.15). In the case that $T^0(y_0) > 0$, we see from the second inequality of (3.15) and (iii) of Lemma 3.3 that $N(T^0(y_0), y_0) > 0$, which contradicts the first equality in (3.15). Hence, (3.14) is true.

Finally, the conclusion (iv) follows from (3.13) and (3.14).

(v) Assume that $T^0(y_0) < \infty$. There are only two possibilities on $T^1(y_0)$: either $T^1(y_0) < \infty$ or $T^1(y_0) = \infty$. In the first the case that $T^1(y_0) < \infty$, we can apply the conclusion (v) of Lemma 3.3 to find that $N(T^1(y_0), y_0) = 0 < \infty$. Hence, the conclusion (v) holds in the first case. We now consider the second case that $T^1(y_0) = \infty$. Because $T^0(y_0) < \infty$, we can take $\hat{t} \in \big(T^0(y_0), \infty\big)$. Then by (1.17), we find that $\hat{y}(\hat{t}; y_0, \hat{u}) = 0$ for some $\hat{u} \in L^\infty(0, \hat{t}; U)$. This shows that $\hat{u}$ is an admissible control to $(NP)^{\hat{t}, y_0}$, from which, we see that

$$N(\hat{t}, y_0) < \infty. \tag{3.16}$$

Because $T^1(y_0) = \infty$, it follows from (ii) of Lemma 3.2 and (3.16) that

$$N(T^1(y_0), y_0) = N(\infty, y_0) \le N(\hat{t}, y_0) < \infty.$$

Hence, the conclusion (v) of this Lemma holds in the second case.

In summary, we finish the proof of this lemma.

□

**Remark 10.** (i) Let $y_0 \in X \setminus \{0\}$. From the above lemma, we have the following two observations: (a) $T^0(y_0) < T^1(y_0)$ if and only if either $0 < N(T^0(y_0), y_0) < \infty$ or $N(T^0(y_0), y_0) = \infty$ and $T^0(y_0) < \infty$; (b) $T^0(y_0) = T^1(y_0)$ if and only if either $N(T^0(y_0), y_0) = 0$ or $N(T^0(y_0), y_0) = \infty$ and $T^0(y_0) = \infty$.

(ii) From the above two observations and the definitions of $\mathcal{W}_{2,3}$, $\mathcal{W}_{3,2}$, $\mathcal{V}_{2,2}$ and $\mathcal{V}_{3,2}$ (see (1.27), (1.29), (1.32) and (1.34), respectively), one can easily find that

$$\mathcal{W}_{2,3} \cup \mathcal{W}_{3,2} = \{(T, y_0) \in \mathcal{W} \ : \ T^0(y_0) < T < T^1(y_0)\}$$

and

$$\mathcal{V}_{2,2} \cup \mathcal{V}_{3,2} = \{(M, y_0) \in \mathcal{V} \ : \ N(T^1(y_0), y_0) < M < N(T^0(y_0), y_0)\}.$$

The next Proposition 6 presents the strict monotonicity and the continuity for the function $N(\cdot, y_0)$ over $\big(T^0(y_0), T^1(y_0)\big)$. These properties will help us to build up a connection between minimal time control problems and minimal norm control problems. This connection plays an important role in the studies of the maximum principle for $(TP)^{M,y_0}$. We would like to mention what follows: The properties in Proposition 6 was proved in [46] for the internally controlled heat equation, with the aid of the bang-bang property and the $L^\infty$-null controllability. Here, we have neither the bang-bang property nor the $L^\infty$-null controllability. We prove it under a weaker condition (H1).

**Proposition 6.** *Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then the following conclusions are true:*
*(i) The function $N(\cdot, y_0)$ is continuous and strictly decreasing from $\big(T^0(y_0), T^1(y_0)\big)$ onto $\big(N(T^1(y_0), y_0), N(T^0(y_0), y_0)\big)$.*
*(ii) When $T \in \big(T^0(y_0), T^1(y_0)\big)$,*

$$N(t_1, y_0) > N(T, y_0) > N(t_2, y_0) \ \text{ for all } \ t_1, \, t_2 \ \text{ with } \ 0 \le t_1 < T < t_2 \le \infty. \tag{3.17}$$

*Proof.* (i) Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that $T^0(y_0) < T^1(y_0)$. From (iii) of Lemma 3.3 and Corollary 2, we see that

$$0 < N(T, y_0) < \infty \text{ for all } T \in \big(T^0(y_0), T^1(y_0)\big). \tag{3.18}$$

We organize the rest of the proof by the following three steps:

*Step 1. To show that the function $N(\cdot, y_0)$ is strictly decreasing over $\big(T^0(y_0), T^1(y_0)\big)$*

Arbitrarily fix two numbers $T_1$ and $T_2$ so that $T^0(y_0) < T_1 < T_2 < T^1(y_0)$. Because (H1) holds, we can apply Lemma 2.3 to get the conclusion (ii) of Lemma 2.3. Let $p_1 \in [2, \infty)$ and $C_1 := C_1(T_2, T_1)$ be given by (ii) of Lemma 2.3. Then by (3.18), there is a $\delta > 0$ so that

$$\lambda := \frac{2\delta}{N(T_1, y_0) + \delta} \in (0, 1) \ \text{ and } \ C_1 \lambda T_1^{1/p_1} \le \frac{N(T_1, y_0) - \delta}{N(T_1, y_0) + \delta}. \tag{3.19}$$

Meanwhile, by (3.18), we have that $N(T_1, y_0) < \infty$. This, along with (1.16), yields that there exists an admissible control $v_1$ to $(NP)^{T_1, y_0}$ so that

$$\hat{y}(T_1; y_0, v_1) = 0 \text{ and } \|v_1\|_{L^\infty(0,T_1;U)} \le N(T_1, y_0) + \delta. \tag{3.20}$$

Write $\widetilde{v}_1$ for the zero extension of $v_1$ over $(0, T_2)$. According to (ii) of Lemma 2.3, there is a control $v_2 \in L^\infty(0, T_2; U)$ so that

$$\hat{y}(T_2; 0, \chi_{(0,T_1)} \lambda \widetilde{v}_1) = \hat{y}(T_2; 0, \chi_{(T_1,T_2)} v_2) \tag{3.21}$$

and so that

$$\|v_2\|_{L^\infty(0,T_2;U)} \le C_1 \|\lambda \widetilde{v}_1\|_{L^{p_1}(0,T_2;U)} \le C_1 \lambda T_1^{1/p_1} \|v_1\|_{L^\infty(0,T_1;U)}. \tag{3.22}$$

We now define another control:

$$v_3(t) := \chi_{(0,T_1)}(t)(1 - \lambda)\widetilde{v}_1(t) + \chi_{(T_1,T_2)}(t) v_2(t), \ t \in (0, T_2). \tag{3.23}$$

From (3.23), (3.21) and the first equality in (3.20), one can check that $\hat{y}(T_2; y_0, v_3) = S(T_2 - T_1)\hat{y}(T_1; y_0, v_1) = 0$, which implies that $v_3$ is an admissible control to $(NP)^{T_2,y_0}$. This, together with the definition of $N(T_2, y_0)$ (see (1.16)) and (3.23), implies that

$$N(T_2, y_0) \leq \|v_3\|_{L^\infty(0,T_2;U)} \leq \max\big\{(1-\lambda)\|v_1\|_{L^\infty(0,T_1;U)}, \|v_2\|_{L^\infty(0,T_2;U)}\big\}.$$

From this, (3.20), (3.22) and (3.19), after some simple computations, we deduce that

$$\begin{aligned} N(T_2, y_0) &\leq \max\Big\{(1-\lambda)\big(N(T_1, y_0) + \delta\big), C_1\lambda T_1^{1/p_1}\big(N(T_1, y_0) + \delta\big)\Big\} \\ &= N(T_1, y_0) - \delta < N(T_1, y_0). \end{aligned}$$

So $N(\cdot, y_0)$ is strictly decreasing over $\big(T^0(y_0), T^1(y_0)\big)$.
*Step 2. To show that*

$$N(T, y_0) \leq \liminf_{t\in\mathcal{A},\, t\to T} N(T, y_0) \ \ \textit{for all}\ \ T \in \mathcal{A} := \big[T^0(y_0), T^1(y_0)\big) \tag{3.24}$$

Arbitrarily fix a $T_0 \in \big[T^0(y_0), T^1(y_0)\big)$. Then arbitrarily take a sequence:

$$\{T_n\}_{n=1}^\infty \subset \big(T^0(y_0), T^1(y_0)\big), \ \text{ with } \lim_{n\to\infty} T_n = T_0. \tag{3.25}$$

To show (3.24), it suffices to prove that

$$N(T_0, y_0) \leq \liminf_{n\to\infty} N(T_n, y_0). \tag{3.26}$$

By contradiction, we suppose that $\liminf_{n\to\infty} N(T_n, y_0) < N(T_0, y_0)$. Then there would be a subsequence $\{T_{n_k}\}_{k=1}^\infty$ of $\{T_n\}_{n=1}^\infty$ so that

$$\lim_{k\to\infty} N(T_{n_k}, y_0) = \liminf_{n\to\infty} N(T_n, y_0) < N(T_0, y_0). \tag{3.27}$$

Thus there is a positive constant $C$ so that

$$N(T_{n_k}, y_0) < C < \infty \quad \text{for all } \ k \geq 1. \tag{3.28}$$

It is clear that $0 < T_{n_k} < \infty$ (see (3.25)) for each $k \in \mathbb{N}^+$. This, along with (1.16) and (3.28), yields that for each $k \in \mathbb{N}^+$, there is a control $u_{n_k} \in L^\infty(0, T_{n_k}; U)$ so that

$$\hat{y}(T_{n_k}; y_0, u_{n_k}) = 0 \ \text{ and } \ \|u_{n_k}\|_{L^\infty(0,T_{n_k};U)} < N(T_{n_k}, y_0) + 1/k. \tag{3.29}$$

For each $k \in \mathbb{N}^+$, we let $\widetilde{u}_{n_k}$ be the zero extension of $u_{n_k}$ over $\mathbb{R}^+$. From (3.29) and (3.28), it follows that $\{\widetilde{u}_{n_k}\}_{k=1}^\infty$ is bounded in $L^\infty(\mathbb{R}^+; U)$. Then there is a subsequence $\{\widetilde{u}_{n_{k_l}}\}_{l=1}^\infty$ of $\{\widetilde{u}_{n_k}\}_{k=1}^\infty$ and a control $v_0 \in L^\infty(\mathbb{R}^+; U)$ so that

$$\widetilde{u}_{n_{k_l}} \to v_0 \ \text{ weakly star in } \ L^\infty(\mathbb{R}^+; U), \ \text{ as } \ l \to \infty, \tag{3.30}$$

which implies that

$$\widetilde{u}_{n_{k_l}} \to v_0 \ \text{ weakly in } \ L^2(\mathbb{R}^+; U), \ \text{ as } \ l \to \infty.$$

Because $\lim_{l\to\infty} T_{n_{k_l}} = T_0$, the above convergence, together with Lemma 3.1, yields that

$$y(T_{n_{k_l}}; y_0, \widetilde{u}_{n_{k_l}}) \to y(T_0; y_0, v_0) \ \text{ weakly in } \ X, \ \text{ as } \ l \to \infty,$$

which, along with the first equality in (3.29), implies that

$$y(T_0; y_0, v_0) = 0. \tag{3.31}$$

Since $y_0 \in X \setminus \{0\}$ and $T_0 < T^1(y_0)$, the equality (3.31) indicates that $0 < T_0 < \infty$. Therefore, the problem $(NP)^{T_0,y_0}$ makes sense. From (3.31), we know that $v_0|_{(0,T_0)}$

is an admissible control to $(NP)^{T_0,y_0}$. This, along with (1.16), (3.30) and the second inequality in (3.29), yields that

$$N(T_0,y_0)\leq\|v_0\|_{L^\infty(0,T_0;U)}\leq\liminf_{l\to\infty}\|\widetilde{u}_{n_{k_l}}\|_{L^\infty(\mathbb{R}^+;U)}\leq\liminf_{l\to\infty}N(T_{n_{k_l}},y_0),$$

which contradicts (3.27). Thus, (3.26) is true. This ends the proof of (3.24).

*Step 3. To show that*

$$N(T,y_0)\geq\limsup_{t\in\mathcal{B},\,t\to T}N(T,y_0)\ \ \textit{for all}\ \ T\in\mathcal{B}:=\big(T^0(y_0),T^1(y_0)\big]\tag{3.32}$$

Arbitrarily fix a $T_0\in\big(T^0(y_0),T^1(y_0)\big]$. We aim to show that (3.32) holds for $T=T_0$. There are only two possibilities on $T_0$: either $T_0=\infty$ or $T_0<\infty$. In the case that $T_0=\infty$, (3.32), with $T=T_0$, follows directly from the first equality in (1.19).

The key of this step is to prove that

$$N(T_0,y_0)\geq\limsup_{t\to T_0}N(t,y_0),\ \ \text{when}\ \ T_0<\infty.\tag{3.33}$$

To this end, we arbitrarily take $\{T_n\}_{n=1}^\infty$ in $\big(T^0(y_0),T^1(y_0)\big)$ so that $\lim_{n\to\infty}T_n=T_0<\infty$. According to Corollary 2, there is a sequence $\{z_n\}_{n=1}^\infty\subset D(A^*)$ so that for each $n\geq1$,

$$\|B^*S^*(T_n-\cdot)z_n\|_{L^1(0,T_n;U)}=1\tag{3.34}$$

and

$$N(T_n,y_0)-1/n\leq\langle S(T_n)y_0,z_n\rangle_X\leq N(T_n,y_0).\tag{3.35}$$

Arbitrarily fix a sequence:

$$\{t_k\}_{k=1}^\infty\subset\big(T^0(y_0),T_0\big)\ \ \text{with}\ \ t_k\nearrow T_0.\tag{3.36}$$

The rest of the proof of this step is divided into three parts as follows:

*Part 3.1. To prove that there is a subsequence $\{n_l\}_{l=1}^\infty$ in $\mathbb{N}^+$ and a function $g\in B_{Y_{T_0}}$ so that for each $k\in\mathbb{N}^+$,*

$$B^*S^*(T_{n_l}-\cdot)z_{n_l}\to g\ \ \textit{weakly in}\ \ L^1(0,t_k;U),\ \ \textit{as}\ \ l\to\infty\tag{3.37}$$

For each $n$, we define a function $\psi_n$ over $(0,T_0)$ in the following manner:

$$\psi_n(t)=\begin{cases}B^*S^*(T_n-t)z_n, & t\in\big(0,\min\{T_n,T_0\}\big),\\ 0, & t\in\big[\min\{T_n,T_0\},T_0\big).\end{cases}$$

For each $k\in\mathbb{N}^+$, since $t_k<T_0$ (see (3.36)) and $\lim_{n\to\infty}T_n=T_0$, we see that there is $N(k)\in\mathbb{N}^+$ so that $t_k<\min\{T_n,T_0\}$, when $n\geq N(k)$. Since $z_n\in D(A^*)$ for all $n$, we have that for each $k\in\mathbb{N}^+$, $S^*(T_n-t_k)z_n\in D(A^*)$, when $n\geq N(k)$. Then by (1.20), we find that when $k\in\mathbb{N}^+$ and $n\geq N(k)$,

$$\psi_n\mid_{(0,t_k)}=B^*S^*(T_n-\cdot)z_n\mid_{(0,t_k)}=B^*S^*(t_k-\cdot)\big(S^*(T_n-t_k)z_n\big)\mid_{(0,t_k)}\in Y_{t_k}.\tag{3.38}$$

This, along with (3.34), yields that for each $k\in\mathbb{N}^+$, $\psi_n\mid_{(0,t_k)}\in B_{Y_{t_k}}$, when $n\geq N(k)$. From this, (H1) and Corollary 1 (with $T=t_k$), we see that for each $k\in\mathbb{N}^+$, there is a function $g_k\in B_{Y_{t_k}}$ and a subsequence $\{\psi_{k_n}\}_{n=1}^\infty$ so that

$$\{\psi_{k_n}\}_{n=1}^\infty\subset\{\psi_{(k-1)_n}\}_{n=1}^\infty\subset\{\psi_n\}_{n=1}^\infty,\ \ \text{with}\ \ \{\psi_{0_n}\}_{n=1}^\infty:=\{\psi_n\}_{n=1}^\infty,$$

and so that

$$\psi_{k_n}\mid_{(0,t_k)}\to g_k\ \text{in the topology}\ \sigma(Y_{t_k},\mathcal{R}^0_{t_k}),\ \ \text{as}\ n\to\infty.$$

From these and the diagonal law, the subsequence $\{\psi_{n_n}\}_{n=1}^{\infty}$ of $\{\psi_n\}_{n=1}^{\infty}$ satisfies that for each $k \in \mathbb{N}^+$,

$$\psi_{n_n}|_{(0,t_k)} \to g_k \text{ in the topology } \sigma(Y_{t_k}, \mathcal{R}^0_{t_k}), \text{ as } n \to \infty. \tag{3.39}$$

Arbitrarily fix a $k \in \mathbb{N}^+$ and then arbitrarily take $u_k \in \mathcal{U}_k$ where

$$\mathcal{U}_k := \Big\{ u \in L^\infty(0,t_k;U) \ : \ \lim_{s \to t_k} \|u\|_{L^\infty(s,t_k;U)} = 0 \Big\}. \tag{3.40}$$

By (1.42), we have that $\hat{y}(t_k;0,u_k) \in \mathcal{R}^0_{t_k}$. This, along with (3.39), yields that for each $k \in \mathbb{N}^+$,

$$\langle \psi_{n_n}, \hat{y}(t_k;0,u_k)\rangle_{Y_{t_k},\mathcal{R}^0_{t_k}} \to \langle g_k, \hat{y}(t_k;0,u_k)\rangle_{Y_{t_k},\mathcal{R}^0_{t_k}}, \text{ as } n \to \infty. \tag{3.41}$$

Since $u_k$ is an admissible control to $(NP)^{y_{t_k}}$, with $y_{t_k} := \hat{y}(t_k;0,u_k)$, we can use Theorem 2.6 and (3.41) to get that for each $k$ and each $u_k \in \mathcal{U}_k$,

$$\int_0^{t_k} \langle \psi_{n_n}(t), u_k(t)\rangle_U \,\mathrm{d}t \to \int_0^{t_k} \langle g_k(t), u_k(t)\rangle_U \,\mathrm{d}t, \text{ as } n \to \infty. \tag{3.42}$$

We next claim that

$$g_j = g_{j'} \text{ over } [0,t_j] \text{ for all } j,\, j' \in \mathbb{N}^+ \text{ with } j < j'. \tag{3.43}$$

For this purpose, we arbitrarily fix $j, j' \in \mathbb{N}^+$ so that $j < j'$. Let $u_j \in \mathcal{U}_j$. Write $\widetilde{u}_j$ for the zero extension of $u_j$ over $(0,t_{j'})$. It follows from (3.40) that $\widetilde{u}_j \in \mathcal{U}_{j'}$. This, along with (3.42), indicates that

$$\begin{aligned} &\int_0^{t_j} \langle g_j(t), u_j(t)\rangle_U \,\mathrm{d}t = \lim_{n\to\infty} \int_0^{t_j} \langle \psi_{n_n}(t), u_j(t)\rangle_U \,\mathrm{d}t \\ = \ &\lim_{n\to\infty} \int_0^{t_{j'}} \langle \psi_{n_n}(t), \widetilde{u}_j(t)\rangle_U \,\mathrm{d}t = \int_0^{t_{j'}} \langle g_{j'}(t), \widetilde{u}_j(t)\rangle_U \,\mathrm{d}t = \int_0^{t_j} \langle g_{j'}(t), u_j(t)\rangle_U \,\mathrm{d}t. \end{aligned}$$

Since $u_j$ was arbitrarily taken from $\mathcal{U}_j$ (see (3.40)), the above leads to (3.43).

Now, define $g(\cdot) : (0,T_0) \to U$ by

$$g(t) := g_k(t), \ t \in (0,t_k], \text{ for each } k \in \mathbb{N}^+. \tag{3.44}$$

From (3.43), we see that $g$ is well defined. By (3.42) and (3.44), we find that for each $k \in \mathbb{N}^+$ and each $u_{k+1} \in \mathcal{U}_{k+1}$,

$$\int_0^{t_{k+1}} \langle \psi_{n_n}(t), u_{k+1}(t)\rangle_U \,\mathrm{d}t \to \int_0^{t_{k+1}} \langle g(t), u_{k+1}(t)\rangle_U \,\mathrm{d}t, \text{ as } n \to \infty. \tag{3.45}$$

Given a $v_k \in L^\infty(0,t_k;U)$, let $\widetilde{v}_k$ be the zero extension of $v_k$ over $(0,t_{k+1})$. Then $\widetilde{v}_k \in \mathcal{U}_{k+1}$. Replacing $u_{k+1}$ by $\widetilde{v}_k$ in (3.45), we obtain that for each $k \in \mathbb{N}^+$ and each $v_k \in L^\infty(0,t_k;U)$,

$$\int_0^{t_k} \langle \psi_{n_n}(t), v_k(t)\rangle_U \,\mathrm{d}t \to \int_0^{t_k} \langle g(t), v_k(t)\rangle_U \,\mathrm{d}t, \text{ as } n \to \infty,$$

from which, it follows that for each $k \in \mathbb{N}^+$,

$$\psi_{n_n} \to g \text{ weakly in } L^1(0,t_k;U), \text{ as } n \to \infty. \tag{3.46}$$

We now prove that $g \in B_{Y_{T_0}}$. Indeed, since $g_k \in B_{Y_{t_k}}$ for each $k \in \mathbb{N}^+$, by (3.44) and (i) of Lemma 2.4, we deduce that $g|_{(0,s)} \in Y_s$ for all $s \in (0,T_0)$ and that $\|g\|_{L^1(0,T_0;U)} \leq 1$. From these, as well as (H1) and (ii) of Lemma 2.4, we see that $g \in B_{Y_{T_0}}$. This, together with (3.46), leads to the conclusion of Part 3.1.

*Part 3.2. To show that the subsequence $\{n_l\}_{l=1}^{\infty}$, obtained in Part 3.1, satisfies that*

$$\langle S(T_{n_l})y_0, z_{n_l}\rangle_X \to \langle S(T_0)y_0, g\rangle_{\mathcal{R}_{T_0},Y_{T_0}}, \quad \text{as } l \to \infty \tag{3.47}$$

Recall (3.36) for $\{t_k\}_{k=1}^{\infty}$. Since $t_1 > T^0(y_0)$, we see from (1.17) that there is an $u_1 \in L^\infty(\mathbb{R}^+; U)$ so that $0 = y(t_1; y_0, \chi_{(0,t_1)}u_1)$, from which, it follows from (1.14) that for each $T \geq t_1$,

$$0 = \hat{y}(T; y_0, \chi_{(0,t_1)}u_1|_{(0,T)}) = S(T)y_0 + \int_0^T S_{-1}(T-\tau)B\chi_{(0,t_1)}(\tau)u_1(\tau)\,\mathrm{d}\tau. \tag{3.48}$$

Because $\lim_{l\to\infty} T_{n_l} = T_0 > t_1$, there exists an $N_0 > 0$ so that $T_{n_l} \geq t_1$ for all $l \geq N_0$. This, along with (3.48) (with $T = T_{n_l}$) and (1.13), yields that for each $l \geq N_0$,

$$\langle S(T_{n_l})y_0, z_{n_l}\rangle_X = -\int_0^{t_1} \left\langle \chi_{(0,t_1)}(\tau)u_1(\tau), B^*S^*(T_{n_l}-\tau)z_{n_l}\right\rangle_U \mathrm{d}\tau,$$

which, together with (3.37) (where $k = 2$), implies that

$$\lim_{l\to\infty} \langle S(T_{n_l})y_0, z_{n_l}\rangle_X = -\int_0^{t_1} \left\langle \chi_{(0,t_1)}(\tau)u_1(\tau), g(\tau)\right\rangle_U \mathrm{d}\tau. \tag{3.49}$$

Meanwhile, since $T_0 > t_1$, it follows by (1.41) and (3.48) (where $T = T_0$) that

$$S(T_0)y_0 \in \mathcal{R}_{T_0}. \tag{3.50}$$

By (3.48), we know that $-\chi_{(0,t_1)}u_1|_{(0,T_0)}$ is an admissible control to $(NP)^{y_{T_0}}$, with $y_{T_0} := S(T_0)y_0$. Thus, it follows from (3.50) and (2.3) that

$$\langle S(T_0)y_0, g\rangle_{\mathcal{R}_{T_0},Y_{T_0}} = -\int_0^{t_1} \left\langle \chi_{(0,t_1)}(\tau)u_1(\tau), g(\tau)\right\rangle_U \mathrm{d}\tau.$$

This, along with (3.49), yields (3.47).

*Part 3.3. To show (3.33)*

It is clear that $T^0(y_0) < T_0 < \infty$. Then by (3.50) and (ii) of Proposition 3, we see that

$$N(T_0, y_0) = \|-S(T_0)y_0\|_{\mathcal{R}_{T_0}}. \tag{3.51}$$

From (3.50) and (2.3), we find that

$$\|-S(T_0)y_0\|_{\mathcal{R}_{T_0}}\|g\|_{Y_{T_0}} \geq \langle S(T_0)y_0, g\rangle_{\mathcal{R}_{T_0},Y_{T_0}}.$$

This, along with (3.51), implies that

$$N(T_0, y_0)\|g\|_{Y_{T_0}} \geq \langle S(T_0)y_0, g\rangle_{\mathcal{R}_{T_0},Y_{T_0}}. \tag{3.52}$$

Since $g \in B_{Y_{T_0}}$ (see Part 3.1), we have that $\|g\|_{Y_{T_0}} \leq 1$. This, as well as (3.52) and (3.47), yields that

$$N(T_0, y_0) \geq N(T_0, y_0)\|g\|_{Y_{T_0}} \geq \lim_{l\to\infty} \langle S(T_{n_l})y_0, z_{n_l}\rangle_X. \tag{3.53}$$

From (3.53) and (3.35), we obtain that $N(T_0, y_0) \geq \lim_{l\to\infty} N(T_{n_l}, y_0)$. Since the function $N(\cdot, y_0)$ is decreasing (see (ii) of Lemma 3.2), the above leads to (3.33) (in the case that $T_0 < \infty$).

In summary, we conclude that (3.32) holds. This ends the proof of Step 3.

Now, from Lemma 3.2 and the conclusions in Step 2 and Step 3, we see that $N(\cdot, y_0)$ is continuous from $\big(T^0(y_0), T^1(y_0)\big)$ onto $\big(N(T^1(y_0), y_0), N(T^0(y_0), y_0)\big)$. This, along with the conclusion in Step 1, proves the conclusion (i) of Proposition 6.

(ii) Fix a $y_0 \in X \setminus \{0\}$ so that $T^0(y_0) < T^1(y_0)$. Let $T \in \big(T^0(y_0), T^1(y_0)\big)$ and $0 \le s_1 < T < s_2 \le \infty$. Choose two numbers $s_1'$ and $s_2'$ so that

$$s_1',\, s_2' \in \big(T^0(y_0), T^1(y_0)\big) \ \text{ and } \ s_1 < s_1' < T < s_2' < s_2. \tag{3.54}$$

Because $N(\cdot, y_0)$ is strictly decreasing over $\big(T^0(y_0), T^1(y_0)\big)$ (see the conclusion (i) in this proposition), it follows from (3.54) that

$$N(s_1', y_0) > N(T, y_0) > N(s_2', y_0). \tag{3.55}$$

Since $N(\cdot, y_0)$ is decreasing over $[0, \infty]$ (see (ii) of Lemma 3.2), it follows by (3.54) and (3.55) that

$$N(s_1, y_0) \ge N(s_1', y_0) > N(T, y_0) > N(s_2', y_0) \ge N(s_2, y_0),$$

which leads to (3.17). The conclusion (ii) is proved.

In summary, we finish the proof of Proposition 6.

□

**Corollary 3.** *Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then the following conclusions are valid:*
*(i) When $M \in \big(N(T^1(y_0), y_0), N(T^0(y_0), y_0)\big)$,*

$$T^0(y_0) < T(M, y_0) < T^1(y_0) \ \text{ and } \ M = N(T(M, y_0), y_0). \tag{3.56}$$

*(ii) When $T \in \big(T^0(y_0), T^1(y_0)\big)$,*

$$N(T^1(y_0), y_0) < N(T, y_0) < N(T^0(y_0), y_0) \ \text{ and } \ T = T(N(T, y_0), y_0). \tag{3.57}$$

*Proof.* (i) Let $y_0 \in X \setminus \{0\}$, with $T^0(y_0) < T^1(y_0)$. Then by (H1), we can apply (i) of Proposition 6 to see that $N(T^1(y_0), y_0) < N(T^0(y_0), y_0)$. Let

$$M \in \big(N(T^1(y_0), y_0), N(T^0(y_0), y_0)\big). \tag{3.58}$$

According to (i) of Proposition 6, there is $\hat{T}$ so that

$$T^0(y_0) < \hat{T} < T^1(y_0) \ \text{ and } \ M = N(\hat{T}, y_0). \tag{3.59}$$

To prove (3.56), it suffices to show that

$$\hat{T} = T(M, y_0). \tag{3.60}$$

By contradiction, suppose that (3.60) were not true. Then we would have that either $\hat{T} < T(M, y_0)$ or $\hat{T} > T(M, y_0)$. In the case that $\hat{T} < T(M, y_0)$, we first observe from (3.59) and (3.58) that $N(\hat{T}, y_0) = M < N(T^0(y_0), y_0) \le \infty$. Thus, it follows from (1.16) that for each $n \ge 1$, there is a control $v_n$ so that

$$\|v_n\|_{L^\infty(0,\hat{T};U)} \le N(\hat{T}, y_0) + 1/n < \infty \tag{3.61}$$

and

$$\hat{y}(\hat{T}; y_0, v_n) = 0. \tag{3.62}$$

Write $\widetilde{v}_n$ for the zero extension of $v_n$ over $\mathbb{R}^+$, $n \in \mathbb{N}^+$. From (3.61), we see that on a subsequence of $\{\widetilde{v}_n\}_{n=1}^\infty$, still denoted in the same manner,

$$\widetilde{v}_n \to v_0 \ \text{ weakly star in } \ L^\infty(\mathbb{R}^+; U), \ \text{ as } \ n \to \infty. \tag{3.63}$$

It is clear that $\widetilde{v}_n$ converges to $v_0$ weakly in $L^2(\mathbb{R}^+; U)$. Then by Lemma 3.1 and (3.62), we find that

$$y(\hat{T}; y_0, v_0) = 0. \tag{3.64}$$

Meanwhile, from (3.63), (3.61) and (3.59), we have that

$$\|v_0\|_{L^\infty(\mathbb{R}^+;U)} \leq \liminf_{n\to\infty} \|\widetilde{v}_n\|_{L^\infty(\mathbb{R}^+;U)} \leq N(\hat{T}, y_0) = M. \tag{3.65}$$

From (3.64) and (3.65), we see that $v_0$ is an admissible control to $(TP)^{M,y_0}$. Then by (1.15), we see that $\hat{T} \geq T(M, y_0)$, which leads to a contradiction, since we are in the case that $\hat{T} < T(M, y_0)$.

In the case when $\hat{T} > T(M, y_0)$, we have that $T(M, y_0) < \infty$. This, along with (1.15), yields that for each $n \geq 1$, there is a control $u_n \in \mathcal{U}^M$ and a number $T_n$ so that

$$T(M, y_0) \leq T_n \leq T(M, y_0) + 1/n < \infty; \tag{3.66}$$

$$\|u_n\|_{L^\infty(\mathbb{R}^+;U)} \leq M \;\text{ and }\; y(T_n; y_0, u_n) = 0. \tag{3.67}$$

Since $y_0 \in X \setminus \{0\}$, these imply that $0 < T_n < \infty$ for all $n \geq 1$. From this and the second equality in (3.67), it follows that for each $n$, $u_n|_{(0,T_n)}$ is an admissible control to $(NP)^{T_n,y_0}$. This, along with the first inequality in (3.67) and the definition of $N(T_n, y_0)$ (see (1.16)), yields that for each $n$, $M \geq \|u_n\|_{L^\infty(\mathbb{R}^+;U)} \geq N(T_n, y_0)$, which, together with the second equality in (3.59), implies that

$$N(\hat{T}, y_0) \geq N(T_n, y_0) \;\text{ for each }\; n. \tag{3.68}$$

Since (H1) holds and $\hat{T} \in \big(T^0(y_0), T^1(y_0)\big)$, we see from (3.17) and (3.68) that for each $n \in \mathbb{N}^+$, $T_n \geq \hat{T}$ which, together with (3.66), indicates that $T(M, y_0) \geq \hat{T}$. This leads to a contradiction, because we are in the case that $\hat{T} > T(M, y_0)$. Thus, the conclusion (i) of this corollary is true.

(ii) Let $y_0 \in X \setminus \{0\}$, with $T^0(y_0) < T^1(y_0)$. Arbitrarily fix $T \in (T^0(y_0), T^1(y_0)$. Since (H1) holds, we can use the conclusion (i) of Proposition 6 to see the $T$ satisfies the first inequality in (3.57). Then by this and (3.56) (where $M = N(T, y_0)$), we find that

$$T^0(y_0) < T(N(T, y_0), y_0) < T^1(y_0) \;\text{ and }\; N(T, y_0) = N\big(T(N(T, y_0), y_0), y_0\big). \tag{3.69}$$

Since $N(\cdot, y_0)$ is strictly decreasing over $\big(T^0(y_0), T^1(y_0)\big)$ (see (i) of Proposition 6) and because $T \in \big(T^0(y_0), T^1(y_0)\big)$, it follows from (3.69) that $T$ satisfies the second equality in (3.57).

In summary, we finish the proof of this corollary. □

We can have the following property on $T(M, y_0)$, without assuming (H1). (Compare it with the conclusion (i) of Corollary 3.)

**Proposition 7.** *Let $y_0 \in X \setminus \{0\}$. Then $T^0(y_0) \leq T(M, y_0) \leq T^1(y_0)$ for each $M \in (0, \infty)$.*

*Proof.* Let $y_0 \in X \setminus \{0\}$ and $M \in (0, \infty)$. We first show that

$$T(M, y_0) \geq T^0(y_0). \tag{3.70}$$

By contradiction, suppose that $T(M, y_0) < T^0(y_0)$. Then by (1.15), there would be $\hat{t} \in \big[T(M, y_0), T^0(y_0)\big)$ and $u_1 \in \mathcal{U}^M$ so that $y(\hat{t}; y_0, u_1) = 0$. This contradicts the definition of $T^0(y_0)$ (see (1.17)). So we have proved (3.70).

We next show that

$$T(M, y_0) \leq T^1(y_0). \tag{3.71}$$

By contradiction, suppose that $T^1(y_0) < T(M, y_0)$. Then by (ii) of Lemma 3.3, we would have that $0 < T^1(y_0) < \infty$. By this and (1.16), we find that the problem

$(NP)^{T^1(y_0),y_0}$ makes sense. Since $T^1(y_0) < \infty$, it follows from (v) of Lemma 3.3 that $N(T^1(y_0), y_0) = 0$. From this and (1.16), we see that there exists a control $v_1$ to $(NP)^{T^1(y_0),y_0}$ so that

$$\hat{y}(T^1(y_0); y_0, v_1) = 0 \ \text{ and } \ \|v_1\|_{L^\infty(0,T^1(y_0);U)} < M. \tag{3.72}$$

Let $\widetilde{v}_1$ be the zero extension of $v_1$ over $\mathbb{R}^+$. Then from (3.72), it follows that

$$y(T^1(y_0); y_0, \widetilde{v}_1) = 0 \ \text{ and } \ \|\widetilde{v}_1\|_{L^\infty(\mathbb{R}^+;U)} < M. \tag{3.73}$$

From (3.73), we see that $\widetilde{v}_1$ is an admissible control to $(TP)^{M,y_0}$. Then, from the first equation in (3.73) and (1.15), we see that $T(M, y_0) \le T^1(y_0)$, which leads to a contradiction. Hence, (3.71) is true.

Finally, by (3.70) and (3.71), we end the proof of Proposition 7.

□

4. **Existence of minimal time and minimal norm controls.** In this section, we present the existence of minimal time and minimal norm controls for $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$, and the non-existence of admissible controls for $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$ for all possible cases. These properties play import roles in the proofs of Theorem 1.1 and Theorem 1.2. We also study the existence of minimal norm controls for affiliated minimal norm problems $(NP)^{y_T}$, with $y_T \in \mathcal{R}_T$ (given by (1.40) and (1.41)). Such existence will be used in the studies of a maximum principle for $(NP)^{y_T}$, with $y_0 \in \mathcal{R}_T^0$ (given by (1.42)). The later is the base of the studies of maximum principles, as well as the bang-bang properties for $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$. The first theorem in this section concerns with the existence of minimal norm controls to the problem $(NP)^{y_T}$.

**Theorem 4.1.** *Let $T \in (0, \infty)$. The following conclusions are true:*
*(i) For each $y_T \in \mathcal{R}_T$, $(NP)^{y_T}$ has at least one minimal norm control.*
*(ii) The null control is the unique minimal norm control to $(NP)^{y_T}$, with $y_T = 0$ in $\mathcal{R}_T$.*

*Proof.* Arbitrarily fix a $T \in (0, \infty)$. We are going to show the conclusions (i)-(ii) one by one.

(i) Let $y_T \in \mathcal{R}_T$ be arbitrarily given. According to the definitions of the problem $(NP)^{y_T}$ and the subspace $\mathcal{R}_T$ (see (1.40) and (1.41)), $(NP)^{y_T}$ has at least one admissible control. Thus there is a minimization sequence $\{v_n\}_{n=1}^\infty \subset L^\infty(0, T; U)$ for $(NP)^{y_T}$ so that

$$\hat{y}(T; 0, v_n) = y_T \ \text{ for all } \ n \in \mathbb{N}^+ \tag{4.1}$$

and

$$\|v_n\|_{L^\infty(0,T;U)} \le \|y_T\|_{\mathcal{R}_T} + 1/n \ \text{ for all } \ n \in \mathbb{N}^+. \tag{4.2}$$

From (4.2), we find that there is a subsequence of $\{v_n\}_{n=1}^\infty$, denoted in the same manner, and a control $v_0 \in L^\infty(0, T; U)$ so that

$$v_n \to v_0 \ \text{ weakly star in } \ L^\infty(0, T; U), \ \text{ as } \ n \to \infty. \tag{4.3}$$

From (4.3), Lemma 3.1 and (4.1), we see that

$$\hat{y}(T; 0, v_0) = y_T. \tag{4.4}$$

This, along with (1.40), (4.3) and (4.2), yields that

$$\|y_T\|_{\mathcal{R}_T} \le \|v_0\|_{L^\infty(0,T;U)} \le \liminf_{n\to\infty} \|v_n\|_{L^\infty(0,T;U)} \le \|y_T\|_{\mathcal{R}_T},$$

from which, it follows that

$$\|y_T\|_{\mathcal{R}_T} = \|v_0\|_{L^\infty(0,T;U)}. \tag{4.5}$$

By (4.4) and (4.5), we find that $v_0$ is a minimal norm control to $(NP)^{y_T}$. This ends the proof of the conclusion (i).

(ii) By (1.14), we see that $\hat{y}(T;0,0) = 0$. Meanwhile, since $\|\cdot\|_{\mathcal{R}_T}$ is a norm (see (1.40)), we find that $\|0\|_{\mathcal{R}_T} = 0$. Therefore, we see that when $y_T = 0$, the null control is a minimal norm control to $(NP)^{y_T}$ and that the minimal norm of $(NP)^{y_T}$ is 0. The latter shows that $(NP)^{y_T}$, with $y_T = 0$, has no non-zero minimal norm control. Thus, the null control is the unique minimal norm control to $(NP)^{y_T}$, with $y_T = 0$.

In summary, we complete the proof of this theorem.

□

We now present the following lemma which will be used in the studies on the existence of minimal norm controls to $(NP)^{T,y_0}$ and minimal time controls to $(TP)^{M,y_0}$.

**Lemma 4.2.** *Let $y_0 \in X \setminus \{0\}$, $T \in (0,\infty)$ and $M \in (0,\infty)$. Then the following conclusions are true:*
*(i) If $(NP)^{T,y_0}$ has an admissible control, then it has at least one minimal norm control.*
*(ii) If $(TP)^{M,y_0}$ has an admissible control, then it has at least one minimal time control.*
*(iii) If $N(T,y_0) < \infty$, then $(NP)^{T,y_0}$ has at least one minimal norm control.*
*(iv) If $N(T,y_0) = 0$, then the null control is the unique minimal norm control to $(NP)^{T,y_0}$.*
*(v) If $N(T,y_0) = \infty$, then $(NP)^{T,y_0}$ has no admissible control.*

*Proof.* (i) Suppose that $(NP)^{T,y_0}$ has an admissible control. Then it has a minimization sequence $\{v_n\}_{n=1}^\infty \subset L^\infty(0,T;U)$ so that

$$\hat{y}(T;y_0,v_n) = 0 \ \text{ for all } \ n \in \mathbb{N}^+ \tag{4.6}$$

and

$$\|v_n\|_{L^\infty(0,T;U)} \le N(T,y_0) + 1/n \ \text{ for all } \ n \in \mathbb{N}^+. \tag{4.7}$$

By (4.7), we see that there is a subsequence of $\{v_n\}_{n=1}^\infty$, denoted in the same manner, and a control $v_0 \in L^\infty(0,T;U)$ so that

$$v_n \to v_0 \ \text{ weakly star in } \ L^\infty(0,T;U), \ \text{ as } \ n \to \infty. \tag{4.8}$$

From (4.8), Lemma 3.1 and (4.6), we find that

$$\hat{y}(T;y_0,v_0) = 0. \tag{4.9}$$

This, together with (1.16), (4.8) and (4.7), yields that

$$N(T,y_0) \le \|v_0\|_{L^\infty(0,T;U)} \le \liminf_{n\to\infty} \|v_n\|_{L^\infty(0,T;U)} \le N(T,y_0).$$

Hence, we have that

$$N(T,y_0) = \|v_0\|_{L^\infty(0,T;U)}. \tag{4.10}$$

By (4.9) and (4.10), we find that $v_0$ is a minimal norm control to $(NP)^{T,y_0}$.

(ii) Suppose that $(TP)^{M,y_0}$ has an admissible control. Then there are two sequences $\{u_n\}_{n=1}^\infty \subset L^\infty(\mathbb{R}^+;U)$ and $\{T_n\}_{n=1}^\infty \subset \mathbb{R}^+$ so that

$$y(T_n;y_0,u_n) = 0 \ \text{ for all } \ n \in \mathbb{N}^+, \tag{4.11}$$

$$T_n \searrow T(M,y_0), \ \text{ as } \ n \to \infty \tag{4.12}$$

and

$$\|u_n\|_{L^\infty(\mathbb{R}^+;U)} \leq M \ \text{ for all } \ n \in \mathbb{N}^+. \tag{4.13}$$

By (4.13), we see that there are a subsequence of $\{u_n\}_{n=1}^\infty$, still denoted in the same manner, and an $u_0 \in L^\infty(\mathbb{R}^+;U)$ so that

$$u_n \to u_0 \ \text{ weakly star in } \ L^\infty(\mathbb{R}^+;U), \ \text{ as } \ n \to \infty. \tag{4.14}$$

From (4.12), (4.14), Lemma 3.1 and (4.11), it follows that

$$y\big(T(M,y_0);y_0,u_0\big) = 0. \tag{4.15}$$

Meanwhile, it follows from (4.14) and (4.13) that

$$\|u_0\|_{L^\infty(\mathbb{R}^+;U)} \leq \liminf_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R}^+;U)} \leq M. \tag{4.16}$$

By (4.15) and (4.16), we see that $u_0$ is a minimal time control to $(TP)^{M,y_0}$.

(iii) Suppose that $N(T,y_0) < \infty$. Then it follows by (1.16) that $(NP)^{T,y_0}$ has an admissible control. Thus, by (i) of this lemma, we find that $(NP)^{T,y_0}$ has at least one minimal norm control.

(iv) Suppose that $N(T,y_0) = 0$. On one hand, by the conclusion (iii) in this lemma, we see that $(NP)^{T,y_0}$ has at least one minimal norm control. On the other hand, if $v^*$ is a minimal norm control to $(NP)^{T,y_0}$, then we have that

$$\|v^*\|_{L^\infty(0,T;U)} = N(T,y_0) = 0,$$

which yields that $v^* = 0$. Hence, the null control is the unique minimal norm control to $(NP)^{T,y_0}$.

(v) Assume that $N(T,y_0) = \infty$. By contradiction, suppose that $(NP)^{T,y_0}$ had an admissible control $v^* \in L^\infty(0,T;U)$. Then, by (1.16), we would have that

$$\infty = N(T,y_0) \leq \|v^*\|_{L^\infty(0,T;U)} < \infty.$$

This leads to a contradiction. Hence, $(NP)^{T,y_0}$ has no admissible control.

In summary, we finish the proof of this lemma.

□

The next theorem concerns with the existence of minimal norm controls to the problem $(NP)^{T,y_0}$, in the case when $T^0(y_0) < \infty$.

**Theorem 4.3.** *Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < \infty$. Then the following conclusions are true:*
*(i) If $T^0(y_0) < T < \infty$, then $(NP)^{T,y_0}$ has at least one minimal norm control.*
*(ii) If $T^0(y_0) > 0$ and $0 < T < T^0(y_0)$, then $(NP)^{T,y_0}$ has no admissible control.*
*(iii) If $0 < N(T^0(y_0),y_0) < \infty$, then*

$$T^0(y_0) > 0 \tag{4.17}$$

*and $(NP)^{T^0(y_0),y_0}$ has at least one minimal norm control.*
*(iv) If $N(T^0(y_0),y_0) = 0$, then (4.17) holds and the null control is the unique minimal norm control to the problem $(NP)^{T^0(y_0),y_0}$.*
*(v) If $N(T^0(y_0),y_0) = \infty$ and $T^0(y_0) > 0$, then $(NP)^{T^0(y_0),y_0}$ has no admissible control.*
*(vi) If $T^0(y_0) = 0$, then the problem $(NP)^{T^0(y_0),y_0}$ does not make sense.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that

$$T^0(y_0) < \infty. \tag{4.18}$$

(i) Suppose that

$$T^0(y_0) < T < \infty. \tag{4.19}$$

Then by (1.17) and (4.19), there are a $\hat{t} \in \big(T^0(y_0), T\big)$ and an $\hat{u} \in L^\infty(0, \hat{t}; U)$ so that

$$\hat{y}(\hat{t}; y_0, \hat{u}) = 0. \tag{4.20}$$

Extend $\hat{u}$ over $(0, T)$ by setting it to be zero over $[\hat{t}, T)$. Denote the extension in the same manner. Then we see from (4.20) that $\hat{y}(T; y_0, \hat{u}) = 0$, from which, it follows that $\hat{u}$ is an admissible control to $(NP)^{T,y_0}$. This, along with (i) of Lemma 4.2, yields that $(NP)^{T,y_0}$ has at least one minimal norm control.

(ii) Suppose that

$$T^0(y_0) > 0 \ \text{ and } \ 0 < T < T^0(y_0). \tag{4.21}$$

Then it follows from (1.17) and (4.21) that $(NP)^{T,y_0}$ has no admissible control.

(iii) Assume that

$$0 < N(T^0(y_0), y_0) < \infty. \tag{4.22}$$

We first show (4.17). By contradiction, suppose that (4.17) were not true. Then we would have that $T^0(y_0) = 0$. This, along with (iv) in Lemma 3.3, yields that

$$N(T^0(y_0), y_0) = N(0, y_0) = \infty,$$

which contradicts (4.22). Hence, we have proved (4.17). Next, it follows from (4.17) and (4.18) that $0 < T^0(y_0) < \infty$. This shows that the problem $(NP)^{T^0(y_0),y_0}$ makes sense. (Notice that in the definition of $(NP)^{T,y_0}$, it is required that $0 < T < \infty$, see (1.16).) Finally, by (4.22), we can apply (iii) of Lemma 4.2 to find that $(NP)^{T^0(y_0),y_0}$ has at least one minimal norm control.

(iv) Suppose that

$$N(T^0(y_0), y_0) = 0. \tag{4.23}$$

We first show that (4.17) stands in this case. By contradiction, suppose that (4.17) were not true. Then we would have that $T^0(y_0) = 0$. This, together with (iv) in Lemma 3.3, indicates that

$$N(T^0(y_0), y_0) = N(0, y_0) = \infty,$$

which contradicts (4.23). So (4.17) in this case. Next, by (4.17) and (4.18), we see that $0 < T^0(y_0) < \infty$. Hence, the problem $(NP)^{T^0(y_0),y_0}$ makes sense. Finally, by (4.23), we can apply (iv) of Lemma 4.2 to find that the null control is the unique minimal norm control to $(NP)^{T^0(y_0),y_0}$.

(v) Suppose that

$$N(T^0(y_0), y_0) = \infty \ \text{ and } \ T^0(y_0) > 0. \tag{4.24}$$

Then it follows from the second inequality in (4.24) and (4.18) that $0 < T^0(y_0) < \infty$. Hence, the problem $(NP)^{T^0(y_0),y_0}$ makes sense. Finally, by the first equality in (4.24), we can apply (v) of Lemma 4.2 to find that $(NP)^{T^0(y_0),y_0}$ has no admissible control.

(vi) Suppose that $T^0(y_0) = 0$. Then the problem $(NP)^{T^0(y_0),y_0}$ does not make sense, since in the definition of $(NP)^{T,y_0}$, it is required that $T \in (0, \infty)$ (see (1.16)).

In summary, we finish the proof of Theorem 4.3.

□

The following theorem concerns with the existence of minimal time controls to $(TP)^{M,y_0}$ and minimal norm controls to $(NP)^{T,y_0}$, in the case that $T^0(y_0)=\infty$.

**Theorem 4.4.** *Let $y_0 \in X\setminus\{0\}$ satisfy that $T^0(y_0)=\infty$. Then the following conclusions are true:*
*(i) For each $M\in(0,\infty)$, $(TP)^{M,y_0}$ does not have any admissible control.*
*(ii) For each $T\in(0,\infty)$, $(NP)^{T,y_0}$ does not have any admissible control.*

*Proof.* Arbitrarily fix a $y_0\in X\setminus\{0\}$ so that $T^0(y_0)=\infty$. First of all, since $T^0(y_0)=\infty$, it follows from (1.17) that for each $T\in(0,\infty)$,

$$\hat{y}(T;y_0,u)\neq 0 \ \text{ for all } \ u\in L^\infty(0,T;U). \tag{4.25}$$

We next show the conclusions (i)-(ii) one by one.

(i) By contradiction, suppose that for some $\hat{M}\in(0,\infty)$, $(TP)^{\hat{M},y_0}$ had an admissible control $\hat{u}$. Then we would have that $y(\hat{t};y_0,\hat{u})=0$ for some $\hat{t}\in(0,\infty)$, which contradicts (4.25). So for each $M\in(0,\infty)$, $(TP)^{M,y_0}$ has no admissible control.

(ii) By contradiction, we suppose that for some $\hat{T}\in(0,\infty)$, $(NP)^{\hat{T},y_0}$ had an admissible control $\hat{v}$. Then we would have that $\hat{y}(\hat{T};y_0,\hat{v})=0$, which contradicts (4.25). So for each $T\in(0,\infty)$, $(NP)^{T,y_0}$ has no admissible control.

Thus we complete the proof of this theorem.

□

The following theorem concerns with the existence of minimal time controls to $(TP)^{M,y_0}$, in the case when $T^0(y_0)<\infty$.

**Theorem 4.5.** *Let $y_0\in X\setminus\{0\}$ satisfy that $T^0(y_0)<\infty$. Then it holds that*

$$N(T^1(y_0),y_0)<\infty. \tag{4.26}$$

*Furthermore, the following conclusions are true:*

*(i) If $N(T^1(y_0),y_0)<M<\infty$, then $(TP)^{M,y_0}$ has at least one minimal time control.*

*(ii) If $N(T^1(y_0),y_0)>0$ and $0<M<N(T^1(y_0),y_0)$, then $(TP)^{M,y_0}$ has no admissible control.*

*(iii) Suppose that (H1) holds. If $M_0:=N(T^1(y_0),y_0)>0$, then $(TP)^{M_0,y_0}$ has no admissible control.*

*Proof.* Arbitrarily fix a $y_0\in X\setminus\{0\}$ so that $T^0(y_0)<\infty$. Then (4.26) follows from (v) of Lemma 3.4. Next, we are going to show conclusions (i)-(iii) one by one.

(i) Let $M\in(N(T^1(y_0),y_0),\infty)$. Then by (vi) of Lemma 3.3, we see that

$$\infty>M>N(T^1(y_0),y_0)=N(\infty,y_0). \tag{4.27}$$

Since $T^0(y_0)<\infty$, it follows from (4.27) and the first equality in (1.19) that there is a number $T_1$ so that

$$T^0(y_0)<T_1<\infty \ \text{ and } \ N(T_1,y_0)<M<\infty. \tag{4.28}$$

By the first conclusion in (4.28), we can apply (i) of Theorem 4.3 to find that $(NP)^{T_1,y_0}$ has a minimal norm control $v^*$. Hence we have that

$$\hat{y}(T_1;y_0,v^*)=0 \ \text{ and } \ \|v^*\|_{L^\infty(0,T_1;U)}=N(T_1,y_0). \tag{4.29}$$

Write $\widetilde{v}^*$ for the zero extension of $v^*$ over $\mathbb{R}^+$. Then it follows from (4.29) and (4.28) that

$$y(T_1;y_0,\widetilde{v}^*)=0 \ \text{ and } \ \|\widetilde{v}^*\|_{L^\infty(\mathbb{R}^+;U)}=N(T_1,y_0)<M<\infty.$$

These imply that $\widetilde{v}^*$ is an admissible control to $(TP)^{M,y_0}$. Then by (ii) of Lemma 4.2, we find that $(TP)^{M,y_0}$ has at least one minimal time control.

(ii) Assume that

$$N(T^1(y_0), y_0) > 0 \text{ and } 0 < M < N(T^1(y_0), y_0). \tag{4.30}$$

We aim to show that $(TP)^{M,y_0}$ has no admissible control. By contradiction, suppose that $(TP)^{M,y_0}$ had an admissible control. Then according to (ii) of Lemma 4.2, $(TP)^{M,y_0}$ would have a minimal time control $u_1^*$. Hence, it holds that

$$T(M, y_0) < \infty, \ \|u_1^*\|_{L^\infty(\mathbb{R}^+;U)} \le M \quad \text{and} \ y(T(M, y_0); y_0, u_1^*) = 0. \tag{4.31}$$

Since $y_0 \in X \setminus \{0\}$, from the third and the first conclusions in (4.31), we see that

$$0 < T(M, y_0) < \infty. \tag{4.32}$$

Write $\hat{u}_1^*$ for the restriction of $u_1^*$ over $\big(0, T(M, y_0)\big)$. Then it follows from (4.31) that

$$\|\hat{u}_1^*\|_{L^\infty(0,T(M,y_0);U)} \le M \tag{4.33}$$

and

$$\hat{y}(T(M, y_0); y_0, \hat{u}_1^*) = 0. \tag{4.34}$$

By (4.32), the problem $(NP)^{T(M,y_0),y_0}$ makes sense (see (1.16)). Then by (4.34), we find that $\hat{u}_1^*$ is an admissible control to $(NP)^{T(M,y_0),y_0}$. This, along with the definition of $N(T(M, y_0), y_0)$ (see (1.16)) and (4.33), yields that

$$N(T(M, y_0), y_0) \le \|\hat{u}_1^*\|_{L^\infty(0,T(M,y_0);U)} \le M,$$

which, together with the second inequality in (4.30), indicates that

$$N(T(M, y_0), y_0) < N(T^1(y_0), y_0). \tag{4.35}$$

From (4.35), (ii) of Lemma 3.2 and the first inequality in (4.31), it follows that

$$T^1(y_0) < T(M, y_0) < \infty. \tag{4.36}$$

By (4.36), we can apply (v) of Lemma 3.3 to get that $N(T^1(y_0), y_0) = 0$, which contradicts the first inequality in (4.30). Hence, $(TP)^{M,y_0}$ has no admissible control in this case.

(iii) Suppose that (H1) holds. And assume that

$$M_0 := N(T^1(y_0), y_0) > 0. \tag{4.37}$$

Then by (4.37) and (4.26), it follows that $0 < M_0 < \infty$. Hence, the problem $(TP)^{M_0,y_0}$ makes sense. (It is required that $0 < M_0 < \infty$ in the definition of $(TP)^{M_0,y_0}$, see (1.15).)

We aim to show that $(TP)^{M_0,y_0}$ has no admissible control. By contradiction, suppose that it had an admissible control. Then we could apply (ii) of Lemma 4.2 to get a minimal time control $u_2^*$ for $(TP)^{M_0,y_0}$. Hence, we have that

$$T(M_0, y_0) < \infty, \ \|u_2^*\|_{L^\infty(\mathbb{R}^+;U)} \le M_0 \ \text{and} \ y(T(M_0, y_0); y_0, u_2^*) = 0. \tag{4.38}$$

Since $y_0 \in X \setminus \{0\}$, from the third and the first assertions in (4.38), we see that

$$0 < T(M_0, y_0) < \infty. \tag{4.39}$$

Write $\hat{u}_2^*$ for the restriction of $u_2^*$ over $\big(0, T(M_0, y_0)\big)$. Then it follows from (4.38) that

$$\|\hat{u}_2^*\|_{L^\infty(0,T(M_0,y_0);U)} \le M_0 \tag{4.40}$$

and

$$\hat{y}(T(M_0, y_0); y_0, \hat{u}_2^*) = 0. \tag{4.41}$$

By (4.39), the problem $(NP)^{T(M_0,y_0),y_0}$ makes sense. Then from (4.41), we find that $\hat{u}_2^*$ is an admissible control to $(NP)^{T(M_0,y_0),y_0}$. This, along with the definition of $N(T(M_0, y_0), y_0)$ (see (1.16)), (4.40) and (4.37), yields that

$$N(T(M_0, y_0), y_0) \leq \|\hat{u}_2^*\|_{L^\infty(0,T(M_0,y_0);U)} \leq M_0 = N(T^1(y_0), y_0). \tag{4.42}$$

By (4.42) and (vi) of Lemma 3.3, we find that

$$N(T(M_0, y_0), y_0) \leq N(T^1(y_0), y_0) = N(\infty, y_0). \tag{4.43}$$

Next, we will use (4.43) to prove that $T^1(y_0) < \infty$. When this is proved, we can apply (v) of Lemma 3.3 to get that $N(T^1(y_0), y_0) = 0$, which contradicts (4.37). Hence, $(TP)^{M_0,y_0}$ has no admissible control in this case.

The remainder is to show that $T^1(y_0) < \infty$. By contradiction, suppose that it were not true. Then we would have that $T^1(y_0) = \infty$. Since we are in the case that $T^0(y_0) < \infty$, it holds that

$$T^0(y_0) < \infty = T^1(y_0). \tag{4.44}$$

By the first inequality in (4.38) and (4.44), we can find a number $\widehat{T}$ so that

$$\max\{T^0(y_0), T(M_0, y_0)\} < \widehat{T} < \infty. \tag{4.45}$$

Meanwhile, by (H1) and (4.44), we can apply (i) of Proposition 6 to find that $N(\cdot, y_0)$ is strictly decreasing over $\big(T^0(y_0), \infty\big)$. This, together with (4.45) and the first equality in (1.19), yields that

$$N(\widehat{T}, y_0) > N(\infty, y_0). \tag{4.46}$$

Since $N(\cdot, y_0)$ is decreasing over $[0, \infty]$ (see (ii) of Lemma 3.2), we find from the first inequality in (4.45) and (4.46) that

$$N(T(M_0, y_0), y_0) \geq N(\widehat{T}, y_0) > N(\infty, y_0).$$

This contradicts (4.43). Hence, we have proved that $T^1(y_0) < \infty$. This ends the proof of the conclusion (iii) of this theorem.

In summary, we complete the proof of this theorem.

□

Theorem 4.3, Theorem 4.4 and Theorem 4.5 contain results on the existence of minimal time controls and minimal norm controls and the non-existence of admissible controls of $(TP)^{M,y_0}$ and $(NP)^{T,y_0}$ for all possible cases. In order to use them in the proofs of our BBP decomposition theorems better, we need several corollaries as follows.

**Corollary 4.** *Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then the following conclusions are true:*
*(i) If $T^0(y_0) > 0$ and $0 < T < T^0(y_0)$, then $(NP)^{T,y_0}$ has no admissible control.*
*(ii) If $T^1(y_0) < \infty$ and $T^1(y_0) \leq T < \infty$, then the null control is the unique minimal norm control to $(NP)^{T,y_0}$.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that $T^0(y_0) < T^1(y_0)$. Then, we have that

$$T^0(y_0) < \infty \ \text{ and } \ T^1(y_0) > 0. \tag{4.47}$$

We will prove the conclusions (i)-(ii) one by one.

(i) Suppose that

$$T^0(y_0) > 0 \ \text{ and } \ 0 < T < T^0(y_0). \tag{4.48}$$

Then we see that $T \in (0, \infty)$. Thus, the problem $(NP)^{T,y_0}$ makes sense. Furthermore, since $T^0(y_0) < \infty$ (see (4.47)), by (4.48), we can apply (ii) of Theorem 4.3 to find that $(NP)^{T,y_0}$ has no admissible control.

(ii) Suppose that

$$T^1(y_0) < \infty \ \text{ and } \ T^1(y_0) \leq T < \infty. \tag{4.49}$$

By (4.49) and (v) of Lemma 3.3, we find that

$$N(T, y_0) = 0. \tag{4.50}$$

Meanwhile, from (4.49) and the second inequality in (4.47), it follows that $T \in (0, \infty)$. Hence, we find from (iv) of Lemma 4.2 and (4.50) that the null control is the unique minimal norm control to $(NP)^{T,y_0}$.

In summary, we finish the proof of this corollary.

□

**Corollary 5.** *Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) = T^1(y_0)$. Then it holds that $T^0(y_0) > 0$. Furthermore, the following conclusions are true:*
*(i) If $0 < T < T^0(y_0)$, then $(NP)^{T,y_0}$ has no admissible control.*
*(ii) If $T^0(y_0) < \infty$ and $T^0(y_0) \leq T < \infty$, then the null control is the unique minimal norm control to $(NP)^{T,y_0}$.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that $T^0(y_0) = T^1(y_0)$. Then by (ii) of Lemma 3.3, we have that

$$T^0(y_0) > 0. \tag{4.51}$$

Next, we will show the conclusions (i)-(ii) one by one.

(i) Suppose that

$$0 < T < T^0(y_0). \tag{4.52}$$

In the case that $T^0(y_0) < \infty$, by (4.51) and (4.52), we can apply (ii) of Theorem 4.3 to find that $(NP)^{T,y_0}$ has no admissible control in this situation. In the case that $T^0(y_0) = \infty$, we can apply (ii) of Theorem 4.4 to find that $(NP)^{T,y_0}$ has no admissible control in this situation. Hence, $(NP)^{T,y_0}$ has no admissible control.

(ii) Suppose that

$$T^0(y_0) < \infty \ \text{ and } \ T^0(y_0) \leq T < \infty. \tag{4.53}$$

Since we are in the case that $T^0(y_0) = T^1(y_0)$, it follows from (4.53) that $T^1(y_0) \leq T < \infty$. Then by (v) of Lemma 3.3, we find that

$$N(T, y_0) = 0. \tag{4.54}$$

Meanwhile, it follows from (4.51) and (4.53) that $0 < T < \infty$. By this and (4.54), we can apply (iv) of Lemma 4.2 to see that the null control is the unique minimal norm control to $(NP)^{T,y_0}$.

In summary, we end the proof of this corollary.

□

**Corollary 6.** *Let $y_0 \in X \setminus \{0\}$ satisfy that*

$$T^0(y_0) < T^1(y_0) \ \text{ and } \ N(T^0(y_0), y_0) < \infty. \tag{4.55}$$

*Then it holds that*

$$0 < N(T^0(y_0), y_0). \tag{4.56}$$

*Furthermore, the following conclusions are true:*
*(i) It holds that*

$$T(M, y_0) = T^0(y_0) \in (0, \infty) \ \text{ for each } \ M \in \big[N(T^0(y_0), y_0), \infty\big). \tag{4.57}$$

*(ii) For each* $M \in \big[N(T^0(y_0), y_0), \infty\big)$, $(TP)^{M,y_0}$ *has a minimal time control* $u^*$ *so that* $u^*|_{(0,T^0(y_0))}$ *(the restriction of* $u^*$ *over* $(0, T^0(y_0))$*) is a minimal norm control to* $(NP)^{T^0(y_0),y_0}$*.*
*(iii) For each* $M \in \big[N(T^0(y_0), y_0), \infty\big)$*, the null control is not a minimal time control to* $(TP)^{M,y_0}$*.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ satisfying (4.55). We first prove (4.56). By contradiction, suppose that it were not true. Then we would have that $N(T^0(y_0), y_0) = 0$. By this and (iv) of Lemma 3.4, we find that $T^0(y_0) = T^1(y_0) < \infty$, which contradicts the first inequality in (4.55). So (4.56) stands.

Next, we are going to show conclusions (i)-(iii) one by one.

(i) We first show that

$$0 < T^0(y_0) < \infty. \tag{4.58}$$

Indeed, by the first inequality in (4.55), we see that $T^0(y_0) < \infty$. Then by the second inequality in (4.55) and by (iii) and (iv) of Theorem 4.3, we find that $T^0(y_0) > 0$. Hence, (4.58) stands.

We next show (4.57). From (4.58), we see that the problem $(NP)^{T^0(y_0),y_0}$ makes sense. Since $T^0(y_0) < \infty$ (see (4.58)), by the second inequality in (4.55), we can apply (iii) and (iv) of Theorem 4.3 to find that $(NP)^{T^0(y_0),y_0}$ has a minimal norm control $v^*$. From this, we have that

$$\hat{y}(T^0(y_0); y_0, v^*) = 0 \quad \text{and} \quad \|v^*\|_{L^\infty(0,T^0(y_0);U)} = N(T^0(y_0), y_0). \tag{4.59}$$

Write $\hat{v}^*$ for the zero extension of $v^*$ over $\mathbb{R}^+$. Then by (4.59), it follows that

$$y(T^0(y_0); y_0, \hat{v}^*) = 0 \tag{4.60}$$

and

$$\|\hat{v}^*\|_{L^\infty(\mathbb{R}^+;U)} = N(T^0(y_0), y_0) \le M \ \text{ for each } \ M \in \big[N(T^0(y_0), y_0), \infty\big). \tag{4.61}$$

Arbitrarily fix $M \in \big[N(T^0(y_0), y_0), \infty\big)$. It follows from (4.56) that $0 < M < \infty$. So the problem $(TP)^{M,y_0}$ makes sense. (In the definition of $(TP)^{M,y_0}$, it is required that $M \in (0, \infty)$, see (1.15).) Since $0 < T^0(y_0) < \infty$ (see (4.58)), from (4.60) and (4.61), it follows that $\hat{v}^*$ is an admissible control to $(TP)^{M,y_0}$. This, along with (1.15) and (4.60), indicates that

$$T(M, y_0) \le T^0(y_0). \tag{4.62}$$

Meanwhile, since $M \in \big[N(T^0(y_0), y_0), \infty\big)$, it follows from Proposition 7 that

$$T(M, y_0) \ge T^0(y_0). \tag{4.63}$$

By (4.62) and (4.63), we see that $T(M, y_0) = T^0(y_0)$. This, along with (4.58), leads to (4.57).

(ii) Arbitrarily fix an $M \in \big[N(T^0(y_0), y_0), \infty\big)$. Let $v^*$ and $\hat{v}^*$ be given in the proof of the conclusion (i) of this corollary (see (4.59) and (4.60), respectively). Write $u^* := \hat{v}^*$. It is clear that

$$u^*|_{(0,T^0(y_0))} = v^*. \tag{4.64}$$

Then by (4.57), (4.60) and (4.61), we see that

$$y(T(M, y_0); y_0, u^*) = 0 \ \text{ and } \ \|u^*\|_{L^\infty(\mathbb{R}^+;U)} \leq M.$$

These yield that $u^*$ is a minimal time control to $(TP)^{M,y_0}$. Meanwhile, it follows by (4.59) and (4.64) that $u^*|_{(0,T^0(y_0))}$ is a minimal norm control to $(NP)^{T^0(y_0),y_0}$. Hence, in this case, $(TP)^{M,y_0}$ has a minimal time control whose restriction over $\big(0, T^0(y_0)\big)$ is a minimal norm control to $(NP)^{T^0(y_0),y_0}$.

(iii) By contradiction, suppose that the null control were a minimal time control to $(TP)^{M_0,y_0}$ for some $M_0 \in \big[N(T^0(y_0), y_0), \infty\big)$. Then by (4.57), we would have that

$$S\big(T^0(y_0)\big)y_0 = y(T^0(y_0); y_0, 0) = y(T(M_0, y_0); y_0, 0) = 0.$$

This, along with (1.18), implies that $T^1(y_0) \leq T^0(y_0)$, which contradicts the first equality in (4.55). Hence, the conclusion (iii) is true.

In summary, we finish the proof of this corollary.

□

**Corollary 7.** *Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$ satisfy that*

$$T^0(y_0) < T^1(y_0) \ \text{ and } \ N(T^1(y_0), y_0) > 0. \tag{4.65}$$

*Then the following conclusions are true:*
*(i) It holds that*

$$N(T^1(y_0), y_0) < \infty. \tag{4.66}$$

*(ii) For each $M \in \big(0, N(T^1(y_0), y_0)\big]$, $(TP)^{M,y_0}$ has no admissible control.*

*Proof.* Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$ satisfy (4.65). We will show the conclusions (i)-(ii) one by one.

(i) We observe from the first inequality in (4.65) that $T^0(y_0) < \infty$. Then (4.66) follows from (4.26).

(ii) Arbitrarily fix an $M$ so that

$$0 < M \leq N(T^1(y_0), y_0). \tag{4.67}$$

By (4.66) and (4.67), we see that $M \in (0, \infty)$. Thus the problem $(TP)^{M,y_0}$ makes sense. Then, by (H1) and (4.67), we can apply (ii) and (iii) of Theorem 4.5 to find that $(TP)^{M,y_0}$ has no admissible control.

Thus, we finish the proof of this corollary.

□

**Corollary 8.** *Suppose that*

$$T^0(y_0) = T^1(y_0) = \infty. \tag{4.68}$$

*Then for each $M \in (0, \infty)$, $(TP)^{M,y_0}$ does not have any admissible control.*

*Proof.* Suppose that (4.68) holds. Then we can apply (i) of Theorem 4.4 to find that for each $M \in (0,\infty)$, $(TP)^{M,y_0}$ has no admissible control. This ends the proof this corollary.

□

**Corollary 9.** *Let $y_0 \in X \setminus \{0\}$ satisfy that*

$$T^0(y_0) = T^1(y_0) < \infty. \tag{4.69}$$

*Then the following conclusions are true:*
*(i) It holds that*

$$T(M, y_0) = T^0(y_0) \in (0,\infty) \text{ for all } M \in (0,\infty). \tag{4.70}$$

*(ii) For each $M \in (0,\infty)$, the null control is a minimal time control to $(TP)^{M,y_0}$.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that (4.69) holds. We now show the conclusions (i)-(ii) one by one.

(i) By (ii) of Lemma 3.3 and the first inequality in (4.69), we have that $T^0(y_0) = T^1(y_0) > 0$. This, together with the second inequality in (4.69), yields that

$$0 < T^0(y_0) < \infty. \tag{4.71}$$

Meanwhile, by Proposition 7 , we find that

$$T^0(y_0) \le T(M, y_0) \le T^1(y_0) \text{ for all } M \in (0,\infty).$$

From the above and the first equality in (4.69), we find that $T(M, y_0) = T^0(y_0)$ for all $M \in (0,\infty)$, which, along with (4.71), leads to (4.70).

(ii) Because of (4.69), we can apply (iv) of Lemma 3.4 to get that $N(T^0(y_0), y_0) = 0$. Since $T^0(y_0) < \infty$ (see (4.69)), the above, along with (iv) of Theorem 4.3, implies that the null control is the unique minimal norm control to $(NP)^{T^0(y_0),y_0}$. Thus, we have that $y(T^0(y_0); y_0, 0) = 0$, which, together with (4.70), shows that $y(T(M, y_0); y_0, 0) = 0$ for all $M \in (0,\infty)$. Form this, we see that the null control is a minimal time control to each $(TP)^{M,y_0}$ with $M \in (0,\infty)$.

In summary, we finish the proof of this corollary.

□

## 5. **Maximum principles and bang-bang properties.**

In this section, we derive maximum principles for $(NP)^{T,y_0}$, with $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, and $(TP)^{M,y_0}$, with $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$, under the assumption (H1). Here, $\mathcal{W}_{2,3}$, $\mathcal{W}_{3,2}$, $\mathcal{V}_{2,2}$ and $\mathcal{V}_{3,2}$ are given by (1.27), (1.29), (1.32) and (1.34), respectively. Then we prove the bang-bang properties for these problems under assumptions (H1) and (H2). The key to obtain the above-mentioned results is a maximum principle for affiliated minimal norm problem $(NP)^{y_T}$, with $y_T \in \mathcal{R}^0_T$. Recall (1.40) for the definitions of $(NP)^{y_T}$ and $\|y_T\|_{\mathcal{R}_T}$; (1.41) for the definition of $\mathcal{R}_T$; (1.42) for the definition of $\mathcal{R}^0_T$; (1.17) for the definition of $T^0(y_0)$; (1.18) for the definition of $T^1(y_0)$; and (1.19) for the definitions of $N(0, y_0)$ and $N(\infty, y_0)$.

5.1. **Maximum principle for affiliated problem.** This subsection presents a maximum principle of $(NP)^{y_T}$, with $y_T \in \mathcal{R}_T^0 \setminus \{0\}$. Write $B_{\mathcal{R}_T}(0, \|y_T\|_{\mathcal{R}_T})$ and $B_{\mathcal{R}_T^0}(0, \|y_T\|_{\mathcal{R}_T})$ for the closed balls in $\mathcal{R}_T$ and $\mathcal{R}_T^0$, centered at the origin and of radius $\|y_T\|_{\mathcal{R}_T}$, respectively. The way to build up the maximum principle of $(NP)^{y_T}$, with $y_T \in \mathcal{R}_T^0 \setminus \{0\}$, is as follows: First, with the aid of Theorem 2.6, we use the Hahn-Banach separation theorem to separate $y_T$ from $B_{\mathcal{R}_T^0}(0, \|y_T\|_{\mathcal{R}_T})$ in the space $\mathcal{R}_T^0$ by a hyperplane with a normal vector $f^* \in Y_T$. Then, with the help of Theorem 2.2, Theorem 2.6, and Proposition 4, we prove that the above-mentioned $f^*$ also separates $y_T$ from $B_{\mathcal{R}_T}(0, \|y_T\|_{\mathcal{R}_T})$ in the space $\mathcal{R}_T$. Finally, we apply Theorem 2.2 to the aforementioned separation in $\mathcal{R}_T$ to get the maximum principle for $(NP)^{y_T}$.

**Theorem 5.1.** *Suppose that (H1) holds. Let $T \in (0, \infty)$. Then for each $y_T \in \mathcal{R}_T^0 \setminus \{0\}$, there is an $f^* \in Y_T \setminus \{0\}$ so that each minimal norm control $v^*$ to $(NP)^{y_T}$ verifies that*

$$\langle v^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \le \|y_T\|_{\mathcal{R}_T}} \langle w, f^*(t)\rangle_U \text{ for a.e. } t \in (0,T). \tag{5.1}$$

*Proof.* First of all, we notice that $\mathcal{R}_T^0 \setminus \{0\} \neq \emptyset$ for all $T \in (0,\infty)$ (see Lemma 2.7). Arbitrarily fix a $T \in (0, \infty)$ and then fix a $y_T \in \mathcal{R}_T^0 \setminus \{0\}$. We organize the proof by several steps.

*Step 1. To find a vector $f^* \in Y_T \setminus \{0\}$ separating $y_T$ from $B_T^0(0, \|y_T\|_{\mathcal{R}_T})$ in $\mathcal{R}_T^0$ in the sense that*

$$\max_{z_T \in B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T})} \langle f^*, z_T\rangle_{Y_T, \mathcal{R}_T^0} = \langle f^*, y_T\rangle_{Y_T,\mathcal{R}_T^0} \tag{5.2}$$

Since $y_T \neq 0$ in $\mathcal{R}_T^0$, $B_{\mathcal{R}_T^0}(0, \|y_T\|_{\mathcal{R}_T})$ is a non-degenerating closed ball in $\mathcal{R}_T^0$. Thus, we can apply the Hahn-Banach separation theorem in the space $\mathcal{R}_T^0$ to find a vector $\eta_0 \in (\mathcal{R}_T^0)^* \setminus \{0\}$ so that

$$\langle \eta_0, z_T\rangle_{(\mathcal{R}_T^0)^*, \mathcal{R}_T^0} \le \langle \eta_0, y_T\rangle_{(\mathcal{R}_T^0)^*, \mathcal{R}_T^0} \text{ for each } z_T \in B_{\mathcal{R}_T^0}(0, \|y_T\|_{\mathcal{R}_T}).$$

Since $y_T \in B_{\mathcal{R}_T^0}(0, \|y_T\|_{\mathcal{R}_T})$, the above yields that

$$\max_{z_T \in B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T})} \langle \eta_0, z_T\rangle_{(\mathcal{R}_T^0)^*, \mathcal{R}_T^0} = \langle \eta_0, y_T\rangle_{(\mathcal{R}_T^0)^*, \mathcal{R}_T^0}. \tag{5.3}$$

Meanwhile, because (H1) holds, we can apply Theorem 2.6 to find a vector $f^* \in Y_T$ so that

$$\langle f^*, z_T\rangle_{Y_T, \mathcal{R}_T^0} = \langle \eta_0, z_T\rangle_{(\mathcal{R}_T^0)^*, \mathcal{R}_T^0} \text{ for all } z_T \in \mathcal{R}_T^0; \text{ and } \|f^*\|_{Y_T} = \|\eta_0\|_{(\mathcal{R}_T^0)^*}. \tag{5.4}$$

Now, (5.2) follows from (5.3) and (5.4). Besides, since $\eta_0 \neq 0$ in $(\mathcal{R}_T^0)^*$, it follows from the second equality in (5.4) that $f^* \neq 0$ in $Y_T$.

*Step 2. To show that $f^*$ given in Step 1 also separates $y_T$ from $B_{\mathcal{R}_T}(0, \|y_T\|_{\mathcal{R}_T})$ in $\mathcal{R}_T$ in the sense that*

$$\sup_{z_T \in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T, Y_T} = \langle y_T, f^*\rangle_{\mathcal{R}_T, Y_T} \tag{5.5}$$

We first claim that

$$\langle f^*, z_T\rangle_{Y_T, \mathcal{R}_T^0} = \langle z_T, f^*\rangle_{\mathcal{R}_T, Y_T} \text{ for all } z_T \in \mathcal{R}_T^0. \tag{5.6}$$

In fact, for each $z_T \in \mathcal{R}_T^0$, it follows from (i) of Theorem 4.1 that $(NP)^{z_T}$ has a minimal norm control $v_{z_T}$. Then by Theorem 2.2 and Theorem 2.6 (more precisely, by (2.3) and (2.37)), we have that

$$\langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T} = \int_0^T \langle v_{z_T}(t), f^*(t)\rangle_U \,\mathrm{d}t \;\text{ and }\; \langle f^*, z_T\rangle_{Y_T,\mathcal{R}_T^0} = \int_0^T \langle f^*(t), v_{z_T}(t)\rangle_U \,\mathrm{d}t.$$

These lead to (5.6).

We next claim that

$$\sup_{z_T\in B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T} = \sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T}. \tag{5.7}$$

Indeed, on one hand, since

$$B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T}) \subseteq B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T}),$$

we have that

$$\sup_{z_T\in B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T} \le \sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T}. \tag{5.8}$$

On the other hand, it follows from Proposition 4 that for each $z_T \in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})$, there is a sequence $\{z_{T,n}\}_{n=1}^\infty$ in $B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T})$ so that

$$z_{T,n} \to z_T \;\text{ in }\; \sigma(\mathcal{R}_T, Y_T), \;\text{ as }\; n\to\infty,$$

which yields that

$$\langle z_{T,n}, f^*\rangle_{\mathcal{R}_T,Y_T} \to \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T}, \;\text{ as }\; n\to\infty.$$

From this, one can easily check that

$$\sup_{z_T\in B_{\mathcal{R}_T^0}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T} \ge \sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T}. \tag{5.9}$$

By (5.8) and (5.9), (5.7) follows at once.

Finally, (5.5) follows from (5.2), (5.6) and (5.7) at once.

*Step 3. To derive from (5.5) that*

$$\sup_{\|v\|_{L^\infty(0,T;U)}\le\|y_T\|_{\mathcal{R}_T}} \int_0^T \langle v(t), f^*(t)\rangle_U \,\mathrm{d}t = \int_0^T \langle v^*(t), f^*(t)\rangle_U \,\mathrm{d}t, \tag{5.10}$$

*for any minimal norm control $v^*$ to $(NP)^{y_T}$*

First, according to Theorem 2.2 (more precisely, see (2.3)), any minimal norm control $v^*$ to $(NP)^{y_T}$ (the existence of $v^*$ is guaranteed by Theorem 4.1) satisfies that

$$\langle y_T, f^*\rangle_{\mathcal{R}_T,Y_T} = \int_0^T \langle v^*(t), f^*(t)\rangle_U \,\mathrm{d}t. \tag{5.11}$$

We next claim that

$$\sup_{\|v\|_{L^\infty(0,T;U)}\le\|y_T\|_{\mathcal{R}_T}} \int_0^T \langle v(t), f^*(t)\rangle_U \,\mathrm{d}t = \sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T}. \tag{5.12}$$

In fact, on one hand, arbitrarily fix a $v \in L^\infty(0,T;U)$ so that $\|v\|_{L^\infty(0,T;U)} \le \|y_T\|_{\mathcal{R}_T}$. Then we find from (1.40) that

$$\|\hat{y}(T;0,v)\|_{\mathcal{R}_T} \le \|v\|_{L^\infty(0,T;U)} \le \|y_T\|_{\mathcal{R}_T}. \tag{5.13}$$

Meanwhile, since the above-mentioned $v$ is an admissible control to the problem $(NP)^{z_T}$, with $z_T := \hat{y}(T;0,v)$, we see from Theorem 2.2 (more precisely, from (2.3)) that

$$\int_0^T \langle v(t), f^*(t)\rangle_U \, \mathrm{d}t = \langle \hat{y}(T;0,v), f^*\rangle_{\mathcal{R}_T,Y_T}. \tag{5.14}$$

From (5.14) and (5.13), it follows that

$$\int_0^T \langle v(t), f^*(t)\rangle_U \, \mathrm{d}t = \langle \hat{y}(T;0,v), f^*\rangle_{\mathcal{R}_T,Y_T} \leq \sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T},$$

which leads to that

$$\sup_{\|v\|_{L^\infty(0,T;U)}\leq\|y_T\|_{\mathcal{R}_T}} \int_0^T \langle v(t), f^*(t)\rangle_U \, \mathrm{d}t \leq \sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T}. \tag{5.15}$$

On the other hand, arbitrarily fix a $z_T \in B_{\mathcal{R}_T}(0, \|y_T\|_{\mathcal{R}_T})$. According to Theorem 4.1, $(NP)^{z_T}$ has a minimal norm control $v^*_{z_T}$ satisfying that

$$z_T = \hat{y}(T;0,v^*_{z_T}) \text{ and } \|v^*_{z_T}\|_{L^\infty(0,T;U)} = \|z_T\|_{\mathcal{R}_T} \leq \|y_T\|_{\mathcal{R}_T}.$$

Then, by (2.3), we find that

$$\langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T} = \int_0^T \langle v_{z_T}(t), f^*(t)\rangle_U \, \mathrm{d}t \leq \sup_{\|v\|_{L^\infty(0,T;U)}\leq\|y_T\|_{\mathcal{R}_T}} \int_0^T \langle v(t), f^*(t)\rangle_U \, \mathrm{d}t.$$

From this, we see that

$$\sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})} \langle z_T, f^*\rangle_{\mathcal{R}_T,Y_T} \leq \sup_{\|v\|_{L^\infty(0,T;U)}\leq\|y_T\|_{\mathcal{R}_T}} \int_0^T \langle v(t), f^*(t)\rangle_U \, \mathrm{d}t. \tag{5.16}$$

By (5.15) and (5.16), we obtain (5.12).

Finally, (5.10) follows from (5.5), (5.11) and (5.12) at once.

*Step 4. To get (5.1) by dropping the integral in (5.10)*

Arbitrarily fix a minimal norm control $v^*$ to $(NP)^{y_T}$. Since $f^* \in L^1(0,T;U)$ and $y_T \neq 0$ in $\mathcal{R}_T$, we have that

$$\|f^*\|_{L^1(0,T;U)} = \sup_{\|v\|_{L^\infty(0,T;U)\leq\|y_T\|_{\mathcal{R}_T}}} \frac{\langle f^*, v\rangle_{L^1(0,T;U),L^\infty(0,T;U)}}{\|y_T\|_{\mathcal{R}_T}},$$

which, together with (5.10), yields that

$$\int_0^T \|y_T\|_{\mathcal{R}_T}\|f^*(t)\|_U \, \mathrm{d}t = \int_0^T \langle v^*(t), f^*(t)\rangle_U \, \mathrm{d}t. \tag{5.17}$$

Meanwhile, since $v^*$ is a minimal norm control to $(NP)^{y_T}$, $\|v^*\|_{L^\infty(0,T;U)} = \|y_T\|_{\mathcal{R}_T}$. This yields that $\|v^*(t)\|_U \leq \|y_T\|_{\mathcal{R}_T}$ for a.e. $t \in (0,T)$. Hence, we have that

$$\langle v^*(t), f^*(t)\rangle_U \leq \|y_T\|_{\mathcal{R}_T}\|f^*(t)\|_U \text{ for a.e. } t \in (0,T). \tag{5.18}$$

From (5.18) and (5.17), we find that

$$\langle v^*(t), f^*(t)\rangle_U = \|y_T\|_{\mathcal{R}_T}\|f^*(t)\|_U \text{ for a.e. } t \in (0,T). \tag{5.19}$$

Meanwhile, we have that

$$\|y_T\|_{\mathcal{R}_T}\|f^*(t)\|_U = \max_{\|w\|_U\leq\|y_T\|_{\mathcal{R}_T}} \langle w, f^*(t)\rangle_U \text{ for a.e. } t \in (0,T). \tag{5.20}$$

From (5.19) and (5.20), we are led to (5.1).

In summary, we finish the proof of this theorem.

□

**Remark 11.** (i) We would like to mention that (5.1) is not a standard Pontryagin maximum principle, since we are not sure if $f^*$ can be expressed as $B^*\varphi$ with $\varphi$ a solution of the adjoint equation over $(0,T)$, even in the case that $B\in\mathcal{L}(U,X)$.
(ii) It is natural to ask if we can directly apply the Hahn-Banach separation theorem to separate $\{y_T\}$ from $B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})$ in the state space $X$? By our understanding, the answer seems to be negative in general. However, if we have that

$$B_{\mathcal{R}_T}\big(0,\|y_T\|_{\mathcal{R}_T}\big)^o\neq\emptyset, \tag{5.21}$$

where $B_{\mathcal{R}_T}\big(0,\|y_T\|_{\mathcal{R}_T}\big)^o$ is the interior of the set $B_{\mathcal{R}_T}\big(0,\|y_T\|_{\mathcal{R}_T}\big)$ in the space:

$$\widetilde{X}:=\overline{\text{span}\,\mathcal{R}_T}^{\|\cdot\|_X},\ \ \text{with the norm}\ \ \|\cdot\|_X,$$

then the answer to the above question is positive. Indeed, we first notice that $\widetilde{X}$ is a closed subspace of $X$. Next, since $\{y_T\}$ lies at the boundary of $B_{\mathcal{R}_T}\big(0,\|y_T\|_{\mathcal{R}_T}\big)$, by the assumption (5.21), we can apply the Hahn-Banach separation theorem in the space $\widetilde{X}$ to separate $\{y_T\}$ from $B_{\mathcal{R}_T}\big(0,\|y_T\|_{\mathcal{R}_T}\big)$ via a normal vector $\eta^*\in X\setminus\{0\}$, i.e.,

$$\langle z_T,\eta^*\rangle_X\leq\langle y_T,\eta^*\rangle_X\ \ \text{for all}\ \ z_T\in B_{\mathcal{R}_T}\big(0,\|y_T\|_{\mathcal{R}_T}\big). \tag{5.22}$$

Meanwhile, from the first assertion in (2.6), (2.3) and (1.22), one can easily check that

$$\langle z_T,\eta\rangle_X=\langle z_T,\widetilde{B^*S^*}(T-\cdot)\eta\rangle_{\mathcal{R}_T,Y_T}\ \ \text{for all}\ \ z_T\in\mathcal{R}_T\ \ \text{and}\ \ \eta\in X.$$

This, along with (5.22), yields that

$$\sup_{z_T\in B_{\mathcal{R}_T}(0,\|y_T\|_{\mathcal{R}_T})}\langle z_T,f^*\rangle_{\mathcal{R}_T,Y_T}=\langle y_T,f^*\rangle_{\mathcal{R}_T,Y_T},$$

where $f^*(\cdot):=\widetilde{B^*S^*}(T-\cdot)\eta^*$. Then by the similar arguments as those used in (5.5)-(5.20), we can obtain the standard Pontryagin maximum principle.

Unfortunately, the condition (5.21) does not hold in general. In fact, consider the inclusion map $i_{\mathcal{R}_T}:(\mathcal{R}_T,\|\cdot\|_{\mathcal{R}_T})\hookrightarrow\widetilde{X}(\subset X)$. If (5.21) holds, then one can easily show that this map is surjective. By the open mapping theorem, we find that $i_{\mathcal{R}_T}$ is isomorphic from $(\mathcal{R}_T,\|\cdot\|_{\mathcal{R}_T})$ to $(\widetilde{X},\|\cdot\|_X)$. Hence, $\mathcal{R}_T(=\widetilde{X})$ is closed in $X$ and norms $\|\cdot\|_{\mathcal{R}_T}$ and $\|\cdot\|_X$ are equivalent. However, these fail for general controlled system $(A,B)$, such as the internally controlled heat equations. (It is well known that the reachable subspace at time $T$ for the internally controlled heat equations over $\Omega\times(0,T)$ is not closed in $L^2(\Omega)$, where $\Omega\subset\mathbb{R}^n$ is an open bounded domain of $C^2$.)

5.2. **Maximum principles for minimal norm and time controls.** We first present a maximum principle for $(NP)^{T,y_0}$, with $(T,y_0)\in\mathcal{W}_{2,3}\cup\mathcal{W}_{3,2}$ in next Theorem 5.2. We would like to mention two facts as follows: First, it is not obvious, at the first sight, that the region of pairs $(T,y_0)$ described in Theorem 5.2, is the same as $\mathcal{W}_{2,3}\cup\mathcal{W}_{3,2}$. However, from (ii) of Remark 10, we know that they are the same. Second, the proof of Theorem 5.2 is based on Theorem 5.1 and the connection between $(NP)^{y_T}$ and $(NP)^{T,y_0}$ built up in Proposition 3.

**Theorem 5.2.** *Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then for each $T \in \big(T^0(y_0), T^1(y_0)\big)$, there is an $f^* \in Y_T \setminus \{0\}$ so that every minimal norm control $v^*$ to $(NP)^{T,y_0}$ satisfies that*

$$\langle v^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \le N(T,y_0)} \langle w, f^*(t)\rangle_U \ \text{ for a.e. } t \in (0,T). \tag{5.23}$$

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that $T^0(y_0) < T^1(y_0)$, and then fix a $T \in \big(T^0(y_0), T^1(y_0)\big)$. Write

$$\hat{y}_T := -S(T)y_0. \tag{5.24}$$

First, we claim that

$$\hat{y}_T \in \mathcal{R}_T^0 \setminus \{0\}. \tag{5.25}$$

In fact, since $T > T^0(y_0)$, it follows from (1.17) that there is a $\hat{t} \in \big[T^0(y_0), T\big)$ so that $\hat{y}(\hat{t}; y_0, \hat{v}) = 0$ for some $\hat{v} \in L^\infty(0, \hat{t}; U)$. Write $\widetilde{v}$ for the zero extension of $\hat{v}$ over $(0,T)$. It is clear that $\hat{y}(T; y_0, \widetilde{v}) = 0$ and $\lim_{s\to T} \|\widetilde{v}\|_{L^\infty(s,T;U)} = 0$. These, together with (1.14), (1.42) and (1.41), yield that

$$-S(T)y_0 = \int_0^T S_{-1}(T-t)B\widetilde{v}(t)\,\mathrm{d}t = \hat{y}(T; 0, \widetilde{v}) \in \mathcal{R}_T^0 \subset \mathcal{R}_T. \tag{5.26}$$

By (5.24) and (5.26), we can apply (ii) of Proposition 3 to get that

$$\|\hat{y}_T\|_{\mathcal{R}_T} = \|-S(T)y_0\|_{\mathcal{R}_T} = N(T, y_0). \tag{5.27}$$

Meanwhile, since $T \in \big(T^0(y_0), T^1(y_0)\big) \subseteq \big(0, T^1(y_0)\big)$, we can apply (iii) of Lemma 3.3 to find that

$$N(T, y_0) > 0. \tag{5.28}$$

From (5.27) and (5.28), we obtain that $S(T)y_0 \neq 0$ in $\mathcal{R}_T$, which along with (5.26), leads to (5.25).

Next, by (H1) and (5.25), we can apply Theorem 5.1 (where $y_T = \hat{y}_T$ is given by (5.24)) to find an $f^* \in Y_T \setminus \{0\}$ so that for each minimal norm control $\hat{v}^*$ to $(NP)^{\hat{y}_T}$,

$$\langle \hat{v}^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \le \|\hat{y}_T\|_{\mathcal{R}_T}} \langle w, f^*(t)\rangle_U \ \text{ for a.e. } t \in (0,T). \tag{5.29}$$

Finally, we arbitrarily fix a minimal norm control $v^*$ to $(NP)^{T,y_0}$. (The existence of $v^*$ is guaranteed by (i) of Theorem 4.3, since $T \in (T^0(y_0), T^1(y_0))$.) Because of (5.26), we can apply (iii) of Proposition 3 to see that $v^*$ is also a minimal norm control to $(NP)^{\hat{y}_T}$. This, together with (5.29) and (5.27), indicates that $v^*$ satisfies (5.23) with $f^*$ given by (5.29). This ends the proof of this theorem. □

To get the maximum principle for $(TP)^{M,y_0}$, we need the following lemma.

**Lemma 5.3.** *Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$, with $T^0(y_0) < T^1(y_0)$. Then it holds that*

$$N(T^1(y_0), y_0) < N(T^0(y_0), y_0). \tag{5.30}$$

*Furthermore, the following conclusions are true:*

*(i) If $M \in \big(N(T^1(y_0), y_0), N(T^0(y_0), y_0)\big)$ and $u^*$ is a minimal time control to $(TP)^{M,y_0}$, then $u^*|_{(0,T(M,y_0))}$ (the restriction of $u^*$ over $(0, T(M,y_0))$) is a minimal norm control to $(NP)^{T(M,y_0),y_0}$.*

*(ii) If $T \in \big(T^0(y_0), T^1(y_0)\big)$ and $v^*$ is a minimal norm control to $(NP)^{T,y_0}$, then the zero extension of $v^*$ over $\mathbb{R}^+$ is a minimal time control to $(TP)^{N(T,y_0),y_0}$.*

*Proof.* Since (H1) holds, we can apply (i) of Proposition 6 to get (5.30). Next we will prove the conclusions (i)-(ii) one by one.

(i) Arbitrarily fix an $M$ so that

$$N(T^1(y_0), y_0) < M < N(T^0(y_0), y_0). \tag{5.31}$$

Suppose that $u^*$ is a minimal time control to $(TP)^{M,y_0}$. (Since $T^0(y_0) < T^1(y_0)$, the existence of $u^*$ is guaranteed by (i) of Theorem 4.5, as well as (5.31).) Then we have that

$$\|u^*\|_{L^\infty(\mathbb{R}^+;U)} \leq M \text{ and } y(T(M,y_0); y_0, u^*) = 0. \tag{5.32}$$

Meanwhile, since $T^0(y_0) < T^1(y_0)$, by using (H1), we can apply (i) of Corollary 3 to see that

$$T(M, y_0) \in (0, \infty) \text{ and } M = N(T(M, y_0), y_0). \tag{5.33}$$

By (5.32) and (5.33), we see that the problem $(NP)^{T(M,y_0),y_0}$ makes sense, and find that

$$\|u^*|_{(0,T(M,y_0))}\|_{L^\infty(0,T(M,y_0);U)} \leq N(T(M, y_0), y_0) \tag{5.34}$$

and

$$\hat{y}(T(M, y_0); y_0, u^*|_{(0,T(M,y_0))}) = 0. \tag{5.35}$$

From (5.34), (5.35) and (1.16), it follows that $u^*|_{(0,T(M,y_0))}$ is a minimal norm control to the problem $(NP)^{T(M,y_0),y_0}$.

(ii) Arbitrarily fix a $T$ so that

$$T^0(y_0) < T < T^1(y_0). \tag{5.36}$$

Suppose that $v^*$ is a minimal norm control to $(NP)^{T,y_0}$. ( The existence of $v^*$ is guaranteed by (i) of Theorem 4.3, because of (5.36).) Write $\widetilde{v}^*$ for the zero extension of $v^*$ over $\mathbb{R}^+$. Then we have that

$$y(T; y_0, \widetilde{v}^*) = 0 \text{ and } \|\widetilde{v}^*\|_{L^\infty(\mathbb{R}^+;U)} \leq N(T, y_0). \tag{5.37}$$

Meanwhile, by (H1) and (5.36), we can apply (ii) of Corollary 3 to find that

$$N(T, y_0) \in (0, \infty) \text{ and } T = T(N(T, y_0), y_0). \tag{5.38}$$

From (5.37) and (5.38), it follows that the problem $(TP)^{N(T,y_0),y_0}$ makes sense and that

$$y\big(T(N(T, y_0), y_0); y_0, \widetilde{v}^*\big) = 0 \text{ and } \|\widetilde{v}^*\|_{L^\infty(\mathbb{R}^+;U)} \leq N(T, y_0).$$

These imply that $\widetilde{v}^*$ is a minimal time control to $(TP)^{N(T,y_0),y_0}$.

In summary, we finish the proof of this lemma. □

Now, we will present a maximum principle for $(TP)^{M,y_0}$, with $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$ in next Theorem 5.4. Two facts deserve to be mentioned: First, it is not obvious, at the first sight, that the region of pairs $(M, y_0)$ described in Theorem 5.4, is the same as $\mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$. However, from (ii) of Remark 10 and the definitions of $\mathcal{V}_{2,2}$ and $\mathcal{V}_{3,2}$ (see (1.32) and (1.34)), we can easily verify that they are the same. Second, the proof of Theorem 5.4 is based on Theorem 5.2 and the connections between $(NP)^{T,y_0}$ and $(TP)^{M,y_0}$ built up in Corollary 3 and Lemma 5.3.

**Theorem 5.4.** *Suppose that (H1) holds. Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then*

$$N(T^1(y_0), y_0) < N(T^0(y_0), y_0).$$

*Furthermore, for each $M \in \big(N(T^1(y_0), y_0), N(T^0(y_0), y_0)\big)$, the following conclusions are true:*
*(i) It holds that*

$$T^0(y_0) < T(M, y_0) < T^1(y_0). \tag{5.39}$$

*(ii) There is a vector $f^* \in Y_{T(M,y_0)} \setminus \{0\}$ so that each minimal time control $u^*$ to $(TP)^{M,y_0}$ satisfies that*

$$\langle u^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \le M} \langle w, f^*(t)\rangle_U \ \text{ for a.e. } t \in \big(0, T(M, y_0)\big). \tag{5.40}$$

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that

$$T^0(y_0) < T^1(y_0). \tag{5.41}$$

By (H1) and (5.41), we can see from (5.30) that

$$N(T^1(y_0), y_0) < N(T^0(y_0), y_0).$$

Arbitrarily fix a number $M$ so that

$$N(T^1(y_0), y_0) < M < N(T^0(y_0), y_0). \tag{5.42}$$

We now are going to show the conclusions (i)-(ii) in this theorem one by one.

(i) By (H1) and (5.41), we can apply (i) of Corollary 3 (more precisely, apply (3.56)) to get both (5.39) and the fact that

$$M = N(T(M, y_0), y_0). \tag{5.43}$$

(ii) By (H1), (5.41) and (5.39), we can apply Theorem 5.2 to get a vector $f^* \in Y_{T(M,y_0)} \setminus \{0\}$ so that every minimal norm control $v^*$ to $(NP)^{T(M,y_0),y_0}$ satisfies that

$$\langle v^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \le N(T(M,y_0),y_0)} \langle w, f^*(t)\rangle_U \ \text{ for a.e. } t \in \big(0, T(M, y_0)\big). \tag{5.44}$$

Next, we suppose that $u^*$ is a minimal time control to $(TP)^{M,y_0}$. (The existence of $u^*$ is guaranteed by (i) of Theorem 4.5, because of (5.41) and (5.42).) Then by (H1), (5.41) and (5.42), we can use (i) of Lemma 5.3 to see that $u^*|_{(0,T(M,y_0))}$ is a minimal norm control to $(NP)^{T(M,y_0),y_0}$. This, along with (5.44) and (5.43), leads to (5.40).

In summary, we finish the proof of this theorem. □

5.3. **Bang-bang properties of minimal time and norm controls.** In this section, we will present the bang-bang properties for $(NP)^{T,y_0}$, with $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, and $(TP)^{M,y_0}$, with $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$, under the assumptions (H1) and (H2). Their proof are based on Theorem 5.2 and Theorem 5.4.

**Theorem 5.5.** *Suppose that (H1) and (H2) hold. Let $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then for each $T \in \big(T^0(y_0), T^1(y_0)\big)$, $(NP)^{T,y_0}$ has the bang-bang property.*

*Proof.* Arbitrarily fix $y_0 \in X\backslash\{0\}$ so that $T^0(y_0) < T^1(y_0)$. Let $T \in \big(T^0(y_0), T^1(y_0)\big)$. Then according to (i) of Theorem 4.3, $(NP)^{T,y_0}$ has at least one minimal norm control. Arbitrarily fix a minimal norm control $v^*$ to this problem. By (H1), we can apply Theorem 5.2 to find a vector $f^* \in Y_T \setminus \{0\}$ so that

$$\langle v^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \leq N(T,y_0)} \langle w, f^*(t)\rangle_U \ \text{ for a.e. } \ t \in (0,T). \tag{5.45}$$

Meanwhile, since $f^* \neq 0$ in $Y_T$, we can derive from (H2) that $f^*(t) \neq 0$ for a.e. $t \in (0,T)$. This, along with (5.45), yields that $\|v^*(t)\|_U = N(T,y_0)$ for a.e. $t \in (0,T)$. Hence, $(NP)^{T,y_0}$ has the bang-bang property. We end the proof of this theorem. □

**Theorem 5.6.** *Suppose that (H1) holds. Let $y_0 \in X\backslash\{0\}$ satisfy that $T^0(y_0) < T^1(y_0)$. Then $N(T^1(y_0),y_0) < N(T^0(y_0),y_0)$. If further assume that (H2) holds, then for each $M \in \big(N(T^1(y_0),y_0), N(T^0(y_0),y_0)\big)$, $(TP)^{M,y_0}$ has the bang-bang property.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that $T^0(y_0) < T^1(y_0)$. By (H1), we can apply (i) of Proposition 6 to find that $N(T^1(y_0),y_0) < N(T^0(y_0),y_0)$. Arbitrarily fix an $M \in \big(N(T^1(y_0),y_0), N(T^0(y_0),y_0)\big)$. Then we can use (i) of Theorem 4.5 to find that $(TP)^{M,y_0}$ has at least one minimal time control. Next, we arbitrarily fix a minimal time control $u^*$ to $(TP)^{M,y_0}$. Then by (H1), we can apply (ii) of Theorem 5.4 to find a vector $f^*$ in $Y_{T(M,y_0)} \setminus \{0\}$ so that

$$\langle u^*(t), f^*(t)\rangle_U = \max_{\|w\|_U \leq M} \langle w, f^*(t)\rangle_U \ \text{ for a.e. } t \in \big(0, T(M,y_0)\big). \tag{5.46}$$

Meanwhile, since $f^* \neq 0$ in $Y_{T(M,y_0)}$, it follows from (H2) that

$$f^*(t) \neq 0 \ \text{ for a.e. } \ t \in \big(0, T(M,y_0)\big).$$

This, along with (5.46), yields that

$$\|u^*(t)\|_U = M \ \text{ for a.e. } \ t \in \big(0, T(M,y_0)\big). \tag{5.47}$$

Thus, $(TP)^{M,y_0}$ has at least one minimal time control and each minimal time control $u^*$ to this problem satisfies (5.47). Hence, $(TP)^{M,y_0}$ has the bang-bang property. this ends the proof of this theorem. □

## 6. **Proofs of main results.**

This section is devoted to prove the main theorems of this paper. They are Theorem 1.1, Theorem 1.2 and Theorem 1.3.

### 6.1. **Some preliminaries.**

Before proving the main theorems of this paper, we introduce the two theorems (Theorem 6.2 and Theorem 6.3), which concern with the conclusions (iii) and (iv) in Theorem 1.2. The proofs of these two theorems are based on the next Lemma 6.1.

**Lemma 6.1.** *Suppose that (H1) holds. Let*

$$\mathcal{O}_T := \big\{u \in L^\infty(0,T;U) \ : \ \hat{y}(T;0,u) = 0\big\}, \ \ \textit{with} \ \ T \in (0,\infty).$$

*Then $\mathcal{O}_T$ is a closed and infinitely dimensional subspace in $L^\infty(0,T;U)$.*

*Proof.* Let $0 < T < \infty$. It is clear that $\mathcal{O}_T$ is a closed subspace in $L^\infty(0,T;U)$. It remains to show that $\mathcal{O}_T$ is of infinite dimension. To this end, we define

$$\mathcal{O}_{t_1,t_2} := \big\{u \in \mathcal{O}_T \ : \ \mathrm{supp}(u) \subset (t_1,t_2)\big\},\ 0 < t_1 < t_2 < T. \tag{6.1}$$

The rest of the proof is organized by two steps.

*Step 1. To show that when $0 < t_1 < t_2 < T$, $\mathcal{O}_{t_1,t_2}$ is a closed subspace of $\mathcal{O}_T$ with $\dim O_{t_1,t_2} \geq 1$*

Define

$$Y_{T,t_1,t_2} := \{f \in L^1(0,T;U) \ : \ f|_{(t_1,t_2)} = g|_{(t_1,t_2)} \text{ for some } g \in Y_{t_2}\}. \tag{6.2}$$

We claim that

$$Y_{T,t_1,t_2} \text{ is a closed proper subspace in } L^1(0,T;U). \tag{6.3}$$

To this end, we first show that $Y_{T,t_1,t_2}$ is closed in $L^1(0,T;U)$. For this purpose, let $\{f_n\}_{n=1}^\infty \subset Y_{T,t_1,t_2}$ satisfy that

$$f_n \to \widehat{f} \text{ in } L^1(0,T;U), \ \text{ as } \ n \to \infty. \tag{6.4}$$

Since $\{f_n\}_{n=1}^\infty \subset Y_{T,t_1,t_2}$, from (6.2), there exists a sequence $\{g_n\}_{n=1}^\infty \subset Y_{t_2}$ so that for all $n \geq 1$, $f_n = g_n$ over $(t_1,t_2)$. This, as well as (6.4), yields that

$$g_n \to \widehat{f} \text{ in } L^1(t_1,t_2;U) \ \text{ as } \ n \to \infty. \tag{6.5}$$

Meanwhile, by (H1), we can use Lemma 2.3 to get the conclusion (iii) in Lemma 2.3. This, as well as (6.5), indicates that $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^1(0,t_2;U)$. Since $Y_{t_2}$ is closed in $L^1(0,t_2;U)$ (see (1.20)), we have that $g_n$ converges to a function $\hat{g}$ in $Y_{t_2}$. This, along with (6.5), shows that $\widehat{f} = \hat{g}$ over $(t_1,t_2)$, which, combined with (6.2), implies that $\widehat{f} \in Y_{T,t_1,t_2}$. Hence, the subspace $Y_{T,t_1,t_2}$ is closed in $L^1(0,T;U)$.

We next show that $Y_{T,t_1,t_2}$ is a proper subspace of $L^1(0,T;U)$. In fact, for each $f \in Y_{T,t_1,t_2}$, we obtain from (6.2) and (iii) of Lemma 2.3 that there is $p_2 > 1$ so that

$$f|_{(t_1,s)} \in L^{p_2}(t_1,s;U) \quad \text{for all} \quad s \in (t_1,t_2). \tag{6.6}$$

However, it is clear that not every function in $L^1(0,T;U)$ holds the property (6.6). Hence, $Y_{T,t_1,t_2}$ is strictly contained in $L^1(0,T;U)$. This finishes the proof of (6.3).

Now by (6.3), there is an $h \in L^1(0,T;U)\backslash Y_{T,t_1,t_2}$. Since $Y_{T,t_1,t_2}$ is closed subspace of $L^1(0,T;U)$, we can apply the Hahn-Banach separation theorem to find a function $u_h$ in $\big(L^1(0,T;U)\big)^*$ (which is $L^\infty(0,T;U)$) so that

$$0 = \int_0^T \langle u_h(t), f(t)\rangle_U \,\mathrm{d}t < \int_0^T \langle u_h(t), h(t)\rangle_U \,\mathrm{d}t \ \text{ for all } \ f \in Y_{T,t_1,t_2}. \tag{6.7}$$

For each $g \in L^1\big((0,t_1)\cup(t_2,T);U\big)$, let $\widetilde{g}(\cdot)$ be the zero extension of $g$ over $(0,T)$. Clearly, it follows from (6.2) that $\widetilde{g} \in Y_{T,t_1,t_2}$. Then by the first equality in (6.7), we find that

$$0 = \int_0^T \langle u_h(t), \widetilde{g}(t)\rangle_U \,\mathrm{d}t \ \text{ for all } \ g \in L^1\big((0,t_1)\cup(t_2,T);U\big).$$

This yields that

$$u_h = 0 \text{ over } (0,t_1)\cup(t_2,T). \tag{6.8}$$

Meanwhile, for each $z \in D(A^*)$, we define $\psi_z : (0,T) \to U$ by

$$\psi_z(t) = \begin{cases} B^*S^*(t_2-t)z, & t \in (t_1,t_2), \\ 0, & t \in (0,t_1] \cup [t_2,T). \end{cases}$$

It follows from (6.2) and (1.20) that for all $z \in D(A^*)$, $\psi_z \in Y_{T,t_1,t_2}$. Then we see from (1.13), (6.8) and the first equality in (6.7) that for each $z \in D(A^*)$,

$$\begin{aligned}\langle \hat{y}(t_2; 0, u_h), z\rangle_X &= \int_0^{t_2} \langle u_h(t), B^* S^*(t_2 - t) z\rangle_U \, \mathrm{d}t \\ &= \int_0^T \langle u_h(t), \psi_z(t)\rangle_U \, \mathrm{d}t = 0.\end{aligned}$$

Since $D(A^*)$ is dense in $X$, the above, as well as (6.8), yields that

$$\hat{y}(T; 0, u_h) = \hat{y}(t_2; 0, u_h) = 0,$$

which leads to that $u_h \in \mathcal{O}_T$. This, along with (6.1) and (6.8), implies that $u_h \in \mathcal{O}_{t_1,t_2}$.

Finally, we see from the second equality in (6.7) that $u_h \neq 0$ in $L^\infty(0, T; U)$. Hence, we have that $\dim O_{t_1,t_2} \geq 1$.

*Step 2. To show that* $\dim \mathcal{O}_T = +\infty$

By the conclusion in Step 1, we find that

$$\{0\} \neq \mathcal{O}_{T/2^{k+1}, T/2^k} \subset \mathcal{O}_T \ \text{ for all } \ k \in \mathbb{N}^+.$$

From (6.1), we see that

$$\mathcal{O}_{T/2^{i+1}, T/2^i} \cap \mathcal{O}_{T/2^{j+1}, T/2^j} = \{0\} \ \text{ for all } \ i, j \in \mathbb{N}^+, \ \text{ with } \ i \neq j.$$

Take a sequence $\{u_k\}$ so that for each $k \in \mathbb{N}^+$, $u_k \in O_{T/2^{k+1}, T/2^k}$. Arbitrarily take a finite subsequence $\{u_{k_n}\}_{n=1}^N$ from $\{u_k\}_{k=1}^\infty$. Let $\{\alpha_n\}_{n=1}^N \subset \mathbb{R}$ be so that $\sum_{n=1}^N \alpha_n u_{k_n} = 0$. Since for each $k$, the support of $u_k$ belongs to $(T/2^{k+1}, T/2^k)$, we can easily derive from the above equality that $\alpha_n = 0$ for all $n \in \{1, \cdots, N\}$. So $u_{k_1}, u_{k_2}, \cdots, u_{k_N}$ are linearly independent. Thus, we conclude that $\dim \mathcal{O}_T = \infty$.

In summary, we finish the proof of this lemma.

□

**Theorem 6.2.** *Let $y_0 \in X \setminus \{0\}$ satisfy that*

$$T^0(y_0) < T^1(y_0) \ \textit{ and } \ N(T^0(y_0), y_0) < \infty. \tag{6.9}$$

*Suppose that (H1) holds and that*

$$N(T^0(y_0), y_0) < M < \infty. \tag{6.10}$$

*Then $(TP)^{M,y_0}$ has infinitely many different minimal time controls so that among them, any finite number of controls are linearly independent in $L^\infty(\mathbb{R}^+; U)$.*

*Proof.* Arbitrarily fix a $y_0 \in X \setminus \{0\}$ so that (6.9) holds. Then fix an $M$ so that (6.10) holds. By (6.9) and (6.10), we can use (i) and (ii) of Corollary 6 to see that

$$T^0(y_0) = T(M, y_0) \in (0, \infty), \tag{6.11}$$

and to find a minimal time control $u^*$ so that $v^* := u^*|_{(0,T^0(y_0))}$ is a minimal norm control to $(NP)^{T^0(y_0),y_0}$. The latter, along with (6.11) and (6.10), yields that

$$y(T(M, y_0); y_0, u^*) = y(T^0(y_0); y_0, u^*) = \hat{y}(T^0(y_0); y_0, v^*) = 0 \tag{6.12}$$

and

$$\|u^*\|_{L^\infty(0,T(M,y_0);U)} = \|v^*\|_{L^\infty(0,T^0(y_0);U)} = N(T^0(y_0), y_0) < M. \tag{6.13}$$

Next, since $0 < T^0(y_0) < \infty$ (see (6.11)), by (H1), we can use Lemma 6.1 to find a sequence $\{u_k\}_{k=1}^{\infty} \subset L^\infty(0, T^0(y_0); U)$ so that

$$\hat{y}(T^0(y_0); 0, u_k) = 0 \text{ for all } k \in \mathbb{N}^+, \tag{6.14}$$

and so that any finite number of elements in $\{u_k\}_{k=1}^{\infty}$ are linearly independent in the space $L^\infty(0, T^0(y_0); U)$. Write $\hat{u}_k$, $k = 1, 2, \cdots$, for the zero extension of $u_k$ over $\mathbb{R}^+$. Then any finite number of elements in $\{\hat{u}_k\}_{k=1}^{\infty}$ are linearly independent in $L^\infty(\mathbb{R}^+; U)$. Arbitrarily fix a $k \in \mathbb{N}^+$. It follows from (6.11) and (6.14) that

$$y(T(M, y_0); 0, \hat{u}_k) = 0. \tag{6.15}$$

Because of (6.10), we can take $\varepsilon_k > 0$ so that

$$\varepsilon_k \|\hat{u}_k\|_{L^\infty(\mathbb{R}^+;U)} < M - N(T^0(y_0), y_0). \tag{6.16}$$

Define a control $u_k^*$ as follows:

$$u_k^* := \varepsilon_k \hat{u}_k + \chi_{(0,T(M,y_0))} u^* \text{ over } \mathbb{R}^+. \tag{6.17}$$

This, along with (6.12) and (6.15), yields that

$$y(T(M, y_0); y_0, u_k^*) = y(T(M, y_0); y_0, u^*) + \varepsilon_k y(T(M, y_0); 0, \hat{u}_k) = 0. \tag{6.18}$$

At same time, it follows from (6.17), (6.16) and (6.13) that

$$\|u_k^*\|_{L^\infty(\mathbb{R}^+;U)} < M. \tag{6.19}$$

Since $k$ was arbitrarily taken from $\mathbb{N}^+$, by (6.18) and (6.19), $\{u_k^*\}_{k=1}^{\infty}$ is a sequence of minimal time controls to $(TP)^{M,y_0}$. (Each $u_k^*$ is not a bang-bang control, see (6.19).)

Finally, we will show that any finite number of controls in $\{u_k^*\}_{k=1}^{\infty}$ are linearly independent in $L^\infty(\mathbb{R}^+; U)$. Suppose that there are a finite subsequence $\{u_{k_j}^*\}_{j=1}^N$ of $\{u_k^*\}_{k=1}^{\infty}$ and a sequence $\{\alpha_j\}_{j=1}^N \subset \mathbb{R}$ so that

$$\sum_{j=1}^N \alpha_j u_{k_j}^* = 0. \tag{6.20}$$

We aim to show that

$$\alpha_j = 0 \text{ for each } j \in \{1, 2, \cdots, N\}. \tag{6.21}$$

By (6.20) and (6.17), it follows that

$$\sum_{j=1}^N \alpha_j \varepsilon_{k_j} \hat{u}_{k_j} + \Big(\sum_{j=1}^N \alpha_j\Big) \chi_{(0,T(M,y_0))} u^* = 0. \tag{6.22}$$

Since $\hat{u}_{k_1}, \ldots, \hat{u}_{k_N}$ are linearly independent, we see from (6.22) that, to show (6.21), it suffices to prove that

$$\sum_{j=1}^N \alpha_j = 0. \tag{6.23}$$

By contradiction, suppose that (6.23) were not true. Then we would have

$$\sum_{j=1}^N \alpha_j \neq 0. \tag{6.24}$$

By (6.24) and (6.22), we know that $\chi_{(0,T(M,y_0))}u^*$ is a linear combination of $\hat{u}_{k_1},\cdots,\hat{u}_{k_N}$. This, along with (6.11) and (6.15), yields that

$$y(T^0(y_0);0,u^*)=y(T(M,y_0);0,u^*)=0,$$

which, together with (6.12), implies that

$$y(T^0(y_0);y_0,0)=y(T^0(y_0);y_0,u^*)-y(T^0(y_0);0,u^*)=0. \tag{6.25}$$

Notice that $T^0(y_0)\in(0,\infty)$ (see (6.11)). So the problem $(NP)^{T^0(y_0),y_0}$ is well defined. Then by (1.16) and (6.25), we see that $N(T^0(y_0),y_0)=0$. By this, we can use (iv) of Lemma 3.4 to find that $T^0(y_0)=T^1(y_0)$, which contradicts (6.9). So (6.23) is true and then any finite number of controls in $\{u_k^*\}_{k=1}^\infty$ are linearly independent in $L^\infty(\mathbb{R}^+;U)$. We end the proof of this theorem.

□

**Theorem 6.3.** *Suppose that (H1) holds. Let $y_0\in X\setminus\{0\}$ satisfy that*

$$T^0(y_0)=T^1(y_0)<\infty. \tag{6.26}$$

*Then for each $M\in(0,\infty)$, $(TP)^{M,y_0}$ has infinitely many different minimal time controls so that among them, any finite number of controls are linearly independent in $L^\infty(\mathbb{R}^+;U)$.*

*Proof.* Arbitrarily fix a $y_0\in X\setminus\{0\}$ so that (6.26) holds. Let $M\in(0,\infty)$. Then by (6.26), we can use Corollary 9 to see that

$$0<T^1(y_0)=T(M,y_0)=T^0(y_0)<\infty \tag{6.27}$$

and to find that the null control is a minimal time control to $(TP)^{M,y_0}$, i.e.,

$$y(T(M,y_0);y_0,0)=0. \tag{6.28}$$

Meanwhile, since $0<T^0(y_0)<\infty$ (see (6.27)), by (H1), we can use Lemma 6.1 to find a sequence $\{u_k\}_{k=1}^\infty\subset L^\infty(0,T^0(y_0);U)$ so that

$$\hat{y}(T^0(y_0);0,u_k)=0 \ \text{ for all } \ k\in\mathbb{N}^+, \tag{6.29}$$

and so that any finite number of elements in $\{u_k\}_{k=1}^\infty$ are linearly independent in the space $L^\infty(0,T^0(y_0);U)$. Write $\hat{u}_k$, $k=1,2,\cdots$, for the zero extension of $u_k$ over $\mathbb{R}^+$. Then any finite number of elements in $\{\hat{u}_k\}_{k=1}^\infty$ are linearly independent in $L^\infty(\mathbb{R}^+;U)$. Arbitrarily fix a $k\in\mathbb{N}^+$. It follows from (6.27) and (6.29) that

$$y(T(M,y_0);0,\hat{u}_k)=0. \tag{6.30}$$

Since $M>0$, we can take $\varepsilon_k>0$ so that

$$\varepsilon_k\|\hat{u}_k\|_{L^\infty(\mathbb{R}^+;U)}<M. \tag{6.31}$$

Next, we define a control $u_k^*$ in the following manner:

$$u_k^*:=\varepsilon_k\hat{u}_k \ \text{ over } \ \mathbb{R}^+. \tag{6.32}$$

Then by (6.32), (6.28) and (6.30), we find that

$$y(T(M,y_0);y_0,u_k^*)=y(T(M,y_0);y_0,0)+\varepsilon_k y(T(M,y_0);0,\hat{u}_k)=0. \tag{6.33}$$

Meanwhile, by (6.32) and (6.31), we see that

$$\|u_k^*\|_{L^\infty(\mathbb{R}^+;U)}<M. \tag{6.34}$$

Since $k$ was arbitrarily taken from $\mathbb{N}^+$, it follows by (6.33) and (6.34) that for each $k\in\mathbb{N}^+$, $u_k^*$ is a minimal time control to $(TP)^{M,y_0}$ and has no the bang-bang property.

Finally, we will show that any finite number of controls in $\{u_k^*\}_{k=1}^\infty$ are linearly independent in $L^\infty(\mathbb{R}^+;U)$. Here is the argument: Suppose that there are a finite subsequence $\{u_{k_j}^*\}_{j=1}^N$ of $\{u_k^*\}_{k=1}^\infty$ and a sequence $\{\alpha_j\}_{j=1}^N \subset \mathbb{R}$ so that

$$\sum_{j=1}^N \alpha_j u_{k_j}^* = 0. \tag{6.35}$$

Then we will have that

$$\alpha_j = 0 \ \text{ for each } \ j \in \{1, 2, \cdots, N\}. \tag{6.36}$$

Indeed, by (6.35) and (6.32), it follows that

$$\sum_{j=1}^N \alpha_j \varepsilon_{k_j} \hat{u}_{k_j} = 0.$$

Since $\hat{u}_{k_1}, \ldots, \hat{u}_{k_N}$ are linearly independent, we find that for each $j \in \{1, 2, \cdots, N\}$, $\alpha_j \varepsilon_{k_j} = 0$. Because $\{\varepsilon_{k_j}\}_{j=1}^N \subset (0, \infty)$, we see that (6.36) holds. So any finite number of controls in $\{u_k^*\}_{k=1}^\infty$ are linearly independent in $L^\infty(\mathbb{R}^+;U)$. This ends the proof.

□

6.2. **Proofs of the main theorems.** We begin with the proof of Theorem 1.1, which gives the BBP decomposition for $(NP)^{T,y_0}$.

*Proof of Theorem 1.1.* (i) First of all, we observe from (1.23) and (1.25)-(1.30) that

$$\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3, \tag{6.37}$$

$$\mathcal{W}_1 = \mathcal{W}_{1,1} \cup \mathcal{W}_{1,2}, \tag{6.38}$$

$$\mathcal{W}_2 = \mathcal{W}_{2,1} \cup \mathcal{W}_{2,2} \cup \mathcal{W}_{2,3} \cup \mathcal{W}_{2,4}, \tag{6.39}$$

and

$$\mathcal{W}_3 = \mathcal{W}_{3,1} \cup \mathcal{W}_{3,2} \cup \mathcal{W}_{3,3} \cup \mathcal{W}_{3,4}. \tag{6.40}$$

To prove the conclusion (i), it suffices to show that

$$\mathcal{W} = \big( \cup_{j=1}^2 \mathcal{W}_{1,j} \big) \cup \big( \cup_{j=1}^4 \mathcal{W}_{2,j} \big) \cup \big( \cup_{j=1}^4 \mathcal{W}_{3,j} \big) \tag{6.41}$$

and

$$\mathcal{W}_{i,j} \cap \mathcal{W}_{i',j'} = \emptyset, \ \text{ when } \ (i,j) \neq (i',j'). \tag{6.42}$$

The equality (6.41) follows from (6.37), (6.38), (6.39) and (6.40) at once. To show (6.42), three observations are given in order: First, from (1.26), (1.28) and (1.30), we see that $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ are pairwise disjoint; Second, from (1.25), it follows that $\mathcal{W}_{1,1}$ and $\mathcal{W}_{1,2}$ are disjoint; Third, by (1.27) and (1.29), we see respectively that all $\mathcal{W}_{2,j}$, $j = 1, 2, 3, 4$ are pairwise disjoint, and that all $\mathcal{W}_{3,j}$, $j = 1, 2, 3, 4$ are pairwise disjoint. The above three observations, together with (6.38), (6.39) and (6.40), leads to (6.42). Thus, we end the proof of the conclusion (i).

(ii) First, we let $(T, y_0) \in \mathcal{W}_{1,2}$. Then by the definitions of $\mathcal{W}_{1,2}$ and $\mathcal{W}_1$ (see (1.25) and (1.26), respectively), we have that

$$T^0(y_0) \leq T < \infty \ \text{ and } \ N(T^0(y_0), y_0) = 0. \tag{6.43}$$

By the last equation in (6.43), we can use (iv) of Lemma 3.4 to obtain that $T^0(y_0) = T^1(y_0) < \infty$. From this and the first inequality in (6.43), we can apply (ii) of Corollary 5 to see that the null control is the unique minimal norm control to

$(NP)^{T,y_0}$. This, along with (1.16), yields that $N(T,y_0)=0$. Hence, $(NP)^{T,y_0}$ has the bang-bang property.

Next, we let $(T,y_0)\in\mathcal{W}_{2,4}$. Then by the definitions of $\mathcal{W}_{2,4}$ and $\mathcal{W}_2$ (see (1.27) and (1.28), respectively), we have that

$$T^1(y_0)\le T<\infty \ \text{ and } \ 0<N(T^0(y_0),y_0)<\infty. \tag{6.44}$$

By the last equation in (6.44), we can apply the conclusion (iii) in Lemma 3.4 to obtain that $T^0(y_0)<T^1(y_0)$. From this and the first inequality in (6.44), we can apply (ii) of Corollary 4 to see that the null control is the unique minimal norm control to $(NP)^{T,y_0}$. This, along with (1.16), yields that $N(T,y_0)=0$. Hence, $(NP)^{T,y_0}$ has the bang-bang property.

Finally, we let $(T,y_0)\in\mathcal{W}_{3,3}$. Then by the definitions of $\mathcal{W}_{3,3}$ and $\mathcal{W}_3$ (see (1.29) and (1.30), respectively), we have that

$$T^1(y_0)\le T<\infty \ \text{ and } \ N(T^0(y_0),y_0)=\infty. \tag{6.45}$$

By (6.45), we can use (i) of Lemma 3.4 to obtain that $T^0(y_0)<T^1(y_0)$. From this and the first inequality in (6.45), we can apply (ii) of Corollary 4 to see that the null control is the unique minimal norm control to $(NP)^{T,y_0}$. This, along with (1.16), yields that $N(T,y_0)=0$. Hence, $(NP)^{T,y_0}$ has the bang-bang property. This ends the proof of the conclusion (ii).

(iii) First, we let $(T,y_0)\in\mathcal{W}_{2,3}$. Then by the definition of $\mathcal{W}_{2,3}$ (see (1.27)), we have that $T^0(y_0)<T<T^1(y_0)$. From this and and the assumptions (H1)-(H2), we can apply Theorem 5.5 to find that $(NP)^{T,y_0}$ has the bang-bang property. The remainder is to show that the null control is not a minimal norm control to $(NP)^{T,y_0}$. In fact, since $T^0(y_0)<T<T^1(y_0)$, it follows from (iii) of Lemma 3.3 that $N(T,y_0)>0$, from which, we see that the null control is not a minimal norm control to $(NP)^{T,y_0}$.

Next, we let $(T,y_0)\in\mathcal{W}_{3,2}$. By the definition of $\mathcal{W}_{3,2}$ (see (1.29)), we find that $T\in(T^0(y_0),T^1(y_0))$. Then by the same way as that used for the above case that $(T,y_0)\in\mathcal{W}_{2,3}$, we see that $(NP)^{T,y_0}$ has the bang-bang property and the null control is not its minimal norm control. This ends the proof of the conclusion (iii).

(iv) First we let $(T,y_0)\in\mathcal{W}_{1,1}$. Then by the definitions of $\mathcal{W}_{1,1}$ and $\mathcal{W}_1$ (see (1.25) and (1.26), respectively), we have that

$$0<T<T^0(y_0) \ \text{ and } \ N(T^0(y_0),y_0)=0. \tag{6.46}$$

From the last equation in (6.46), we can apply (iv) of Lemma 3.4 to see that $T^0(y_0)=T^1(y_0)<\infty$. This, together with the first inequality in (6.46), yields that

$$T^0(y_0)=T^1(y_0) \ \text{ and } \ T\in\big(0,T^0(y_0)\big). \tag{6.47}$$

From (6.47), we can use (i) of Corollary 5 to find that $(NP)^{T,y_0}$ has no admissible control and so does not hold the bang-bang property.

Next we let $(T,y_0)\in\mathcal{W}_{2,1}$. Then by the definitions of $\mathcal{W}_{2,1}$ and $\mathcal{W}_2$ (see (1.27) and (1.28), respectively), we have that

$$0<T<T^0(y_0) \ \text{ and } \ 0<N(T^0(y_0),y_0)<\infty. \tag{6.48}$$

By the second inequality in (6.48), we can use (iii) of Lemma 3.4 to get that $T^0(y_0)<T^1(y_0)$. This, along with the first inequality in (6.48), yields that

$$0<T^0(y_0)<T^1(y_0) \ \text{ and } \ 0<T<T^0(y_0). \tag{6.49}$$

From (6.49), we can use (i) of Corollary 4 to get that $(NP)^{T,y_0}$ has no admissible control and so does not hold bang-bang property.

We now let $(T, y_0) \in \mathcal{W}_{3,1}$. Then by the definitions of $\mathcal{W}_{3,1}$ and $\mathcal{W}_3$ (see (1.29) and (1.30), respectively), we see that

$$T^0(y_0) < \infty, \ \ 0 < T \leq T^0(y_0) \ \text{ and } \ N(T^0(y_0), y_0) = \infty. \tag{6.50}$$

By the first inequality and the last equality in (6.50), we can use (i) of Lemma 3.4 to find that $T^0(y_0) < T^1(y_0)$. This, together with (6.50), indicates that

$$0 < T^0(y_0) < T^1(y_0), \ \ 0 < T \leq T^0(y_0) \ \text{ and } \ N(T^0(y_0), y_0) = \infty. \tag{6.51}$$

In the case that $T = T^0(y_0)$, from the first inequality in (6.51), we can use (i) of Corollary 4 to see that $(NP)^{T,y_0}$ has no admissible control and so does not have the bang-bang property. In the case when $T < T^0(y_0)$, from the last equality in (6.51), we can apply (v) of Theorem 4.3 to find that $(NP)^{T,y_0}$ has no admissible control and so does not have the bang-bang property.

Finally, we let $(T, y_0) \in \mathcal{W}_{3,4}$. Then by the definitions of $\mathcal{W}_{3,4}$ and $\mathcal{W}_3$ (see (1.29) and (1.30), respectively), we have that

$$0 < T < \infty, \ \ T^0(y_0) = \infty \ \text{ and } \ N(T^0(y_0), y_0) = \infty. \tag{6.52}$$

By the last two equalities in (6.52), we can use (i) of Lemma 3.4 to see that $T^0(y_0) = T^1(y_0) = \infty$, which, along with the first inequality in (6.52), yields that $T^0(y_0) = T^1(y_0)$ and $0 < T < T^0(y_0)$. From these, we can apply (i) of Corollary 5 to find that $(NP)^{T,y_0}$ has no admissible control and so does not hold the bang-bang property. This ends the proof of the conclusion (iv).

(v) Let $(T, y_0) \in \mathcal{W}_{2,2}$. Then by the definitions of $\mathcal{W}_{2,2}$ and $\mathcal{W}_2$ (see (1.27) and (1.28), respectively), we see that $0 < T = T^0(y_0) < \infty$ and $0 < N(T^0(y_0), y_0) < \infty$. From these, we can use (iii) of Theorem 4.3 to see that $(NP)^{T,y_0}$ has at least one minimal norm control. This ends the proof of the conclusion (v).

In summary, we finish the proof of Theorem 1.1.

□

Next, we prove Theorem 1.2, which gives the BBP decompositions for $(TP)^{M,y_0}$.

*Proof of Theorem 1.2.* (i) First of all, we observe from (1.24) and (1.31)-(1.35) that

$$\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3, \tag{6.53}$$

$$\mathcal{V}_2 = \mathcal{V}_{2,1} \cup \mathcal{V}_{2,2} \cup \mathcal{V}_{2,3} \cup \mathcal{V}_{2,4}, \tag{6.54}$$

and

$$\mathcal{V}_3 = \mathcal{V}_{3,1} \cup \mathcal{V}_{3,2} \cup \mathcal{V}_{3,3}. \tag{6.55}$$

To show the conclusion (i), it suffices to verify that

$$\mathcal{V} = \mathcal{V}_1 \cup (\cup_{j=1}^4 \mathcal{V}_{2,j}) \cup (\cup_{j=1}^3 \mathcal{V}_{3,j}) \tag{6.56}$$

and

$$\mathcal{V}_1 \cap \mathcal{V}_{i,j} = \emptyset, \ \ \mathcal{V}_{i',j'} \cap \mathcal{V}_{i'',j''} = \emptyset \ \text{ when } \ (i', j') \neq (i'', j''). \tag{6.57}$$

First of all, the equality (6.56) follows from (6.53), (6.54) and (6.55) at once. To prove (6.57), three observations are given in order: First, from (1.31), (1.33) and (1.35), we see that $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are pairwise disjoint. Second, from (1.32), we find that all $\mathcal{V}_{2,j}$, $j = 1, 2, 3, 4$, are pairwise disjoint. Third, from (1.34), we find that all $\mathcal{V}_{3,j}$, $j = 1, 2, 3$, are pairwise disjoint. The above three observations, along with (6.54) and (6.55), yield (6.57). Thus, we end the proof of the conclusion (i).

(ii) First we let $(M, y_0) \in \mathcal{V}_{2,2}$. By the definitions of $\mathcal{V}_{2,2}$ and $\mathcal{V}_2$ (see (1.32) and (1.33)), we have that

$$N(T^1(y_0), y_0) < M < N(T^0(y_0), y_0) \text{ and } 0 < N(T^0(y_0), y_0) < \infty. \tag{6.58}$$

By the second inequality in (6.58), we can use (iii) of Lemma 3.4 to see that $T^0(y_0) < T^1(y_0)$. By this, the first inequality in (6.58) and the assumptions (H1)-(H2), we can apply Theorem 5.6 to see that $(TP)^{M,y_0}$ has the bang-bang property.

Next, we let $(M, y_0) \in \mathcal{V}_{3,2}$. By the definitions of $\mathcal{V}_{3,2}$ and $\mathcal{V}_3$ (see (1.34) and (1.35)), we find that

$$T^0(y_0) < \infty \text{ and } N(T^1(y_0), y_0) < M < \infty = N(T^0(y_0), y_0). \tag{6.59}$$

From (6.59), we can use (i) of Lemma 3.4 to get that $T^0(y_0) < T^1(y_0)$. By this, the second conclusion in (6.59) and the assumptions (H1) and (H2), we can apply Theorem 5.6 to see that $(TP)^{M,y_0}$ has the bang-bang property. This ends the proof of the conclusion (ii).

(iii) Let $(M, y_0) \in \mathcal{V}_{2,4}$. By the definitions of $\mathcal{V}_{2,4}$ and $\mathcal{V}_2$ (see (1.32) and (1.33)), we find that

$$N(T^0(y_0), y_0) < M < \infty \text{ and } 0 < N(T^0(y_0), y_0) < \infty. \tag{6.60}$$

From the second inequality in (6.60), we can use (iii) of Lemma 3.4 to see that

$$T^0(y_0) < T^1(y_0). \tag{6.61}$$

By (6.61) and (6.60), we can use (i) and (ii) of Corollary 6 to find respectively that

$$T(M, y_0) = T^0(y_0) \in (0, \infty), \tag{6.62}$$

and that $(TP)^{M,y_0}$ has a minimal time control $u^*$ so that $u^*|_{(0,T^0(y_0))}$ is a minimal norm control to $(NP)^{T^0(y_0),y_0}$. The later, together with (6.62) and the first inequality in (6.60), indicates that

$$\|u^*\|_{L^\infty(0,T(M,y_0);U)} = \|u^*\|_{L^\infty(0,T^0(y_0);U)} = N(T^0(y_0), y_0) < M.$$

This implies that $(TP)^{M,y_0}$ does not hold the bang-bang property.

Meanwhile, according to (iii) of Corollary 6, the null control is not a minimal time control to $(TP)^{M,y_0}$.

The remainder is to show that $(TP)^{M,y_0}$ has infinitely many different minimal time controls. Fortunately, this follows from Theorem 6.2, since we already have (6.61), (6.60) and (H1). This ends the proof of the conclusion (iii).

(iv) Let $(M, y_0) \in \mathcal{V}_1$. By the definition of $\mathcal{V}_1$ (see (1.31)), we find that

$$N(T^0(y_0), y_0) = 0 < M < \infty. \tag{6.63}$$

Since $N(T^0(y_0), y_0) = 0$, it follows from (iv) of Lemma 3.4 that

$$T^0(y_0) = T^1(y_0) < \infty. \tag{6.64}$$

By (6.64) and (6.63), we can use (ii) of Corollary 9 to see that the null control is a minimal time control to $(TP)^{M,y_0}$. From this, we see that $(TP)^{M,y_0}$ does not hold the bang-bang property, since $M > 0$.

The remainder is to show that $(TP)^{M,y_0}$ has infinitely many different minimal time controls. Fortunately, this follows from Theorem 6.3, since we already have (6.64) and (H1). This ends the proof of the conclusion (iv).

(v) First, we let $(M, y_0) \in \mathcal{V}_{3,3}$. Then by the definitions of $\mathcal{V}_{3,3}$ and $\mathcal{V}_3$ (see (1.34) and (1.35)), we find that

$$T^0(y_0) = \infty \text{ and } N(T^0(y_0), y_0) = \infty. \tag{6.65}$$

From (6.65), we can use (i) of Lemma 3.4 to see that $T^0(y_0) = T^1(y_0) = \infty$. By this, we can apply Corollary 8 to find that $(TP)^{M,y_0}$ has no admissible control and so does not hold the bang-bang property.

Next, we let $(M, y_0) \in \mathcal{V}_{2,1}$. By the definitions of $\mathcal{V}_{2,1}$ and $\mathcal{V}_2$ (see (1.32) and (1.33)), we have that

$$0 < M \leq N(T^1(y_0), y_0) \text{ and } 0 < N(T^0(y_0), y_0) < \infty. \tag{6.66}$$

By the second inequality in (6.66), we can use (iii) of Lemma 3.4 to see that $T^0(y_0) < T^1(y_0)$. From this, the first inequality in (6.66) and the assumption (H1), we can apply (ii) of Corollary 7 to find that $(TP)^{M,y_0}$ has no admissible control and so does not hold the bang-bang property.

Finally, we let $(M, y_0) \in \mathcal{V}_{3,1}$. By the definitions of $\mathcal{V}_{3,1}$ and $\mathcal{V}_3$ (see (1.34) and (1.35)), we have that

$$T^0(y_0) < \infty, \ \ 0 < M \leq N(T^1(y_0), y_0) \text{ and } N(T^0(y_0), y_0) = \infty. \tag{6.67}$$

By the last equality and the first inequality in (6.67), we can use (i) of Lemma 3.4 to get that $T^0(y_0) < T^1(y_0)$. From this, the second inequality in (6.67) and the assumption (H1), we can use (ii) of Corollary 7 to find that $(TP)^{M,y_0}$ has no admissible control and so does not hold the bang-bang property. This ends the proof of the conclusion (v).

(vi) Let $(T, y_0) \in \mathcal{V}_{2,3}$. Then by the definitions of $\mathcal{V}_{2,3}$ and $\mathcal{V}_2$ (see (1.32) and (1.33)), we see that $0 < M = N(T^0(y_0), y_0) < \infty$. This, along with (iii) of Lemma 3.4, yields that $T^0(y_0) < T^1(y_0)$ and $N(T^0(y_0), y_0) = M < \infty$. From these, we can use (ii) of Corollary 6 to find that $(TP)^{M,y_0}$ has at least one minimal time control. This ends the proof of the conclusion (vi).

In summary, we finish the proof of Theorem 1.2.

□

We end this section with proving Theorem 1.3. To do it, we need three propositions. The first one is the following Proposition 8. It presents some equivalent conditions for the $L^\infty$-null controllability of $(A, B)$. Though there have been many literatures on such issue, we do not find the exactly same version of Proposition 8 in literatures. For the sake of the completeness of the paper, we provide the detailed proof in Appendix F.

**Proposition 8.** *The following conclusions are equivalent:*
*(i) The pair $(A^*, B^*)$ is $L^1$-observable, i.e., the condition (H3) holds, i.e., for each $T \in (0, \infty)$, there exists a positive constant $C_1(T)$ so that*

$$\|S^*(T)z\|_X \leq C_1(T) \int_0^T \|B^* S^*(T-t)z\|_U \,\mathrm{d}t \ \text{ for all } z \in D(A^*). \tag{6.68}$$

*(ii) The pair $(A, B)$ has the $L^\infty$-null controllability with a cost, i.e., for each $T \in (0, \infty)$, there is a positive constant $C_2(T)$ so that for each $y_0 \in X$, there exists a control $v \in L^\infty(0, T; U)$ satisfying that*

$$\hat{y}(T; y_0, v) = 0 \text{ and } \|v\|_{L^\infty(0,T;U)} \leq C_2(T)\|y_0\|_X. \tag{6.69}$$

*(iii) The pair $(A, B)$ is $L^\infty$-null controllable, i.e., for each $T \in (0, \infty)$ and each $y_0 \in X$, there exists a control $v \in L^\infty(0, T; U)$ so that $\hat{y}(T; y_0, v) = 0$.*

*Furthermore, when one of the above three conclusions is valid, the constants $C_1(T)$ in (6.68) and $C_2(T)$ in (6.69) can be taken as the same number.*

The next two propositions concern some connections among assumptions (H1)-(H4).

**Proposition 9.** *Suppose that (H3) holds. Then (H1) is true.*

*Proof.* Suppose that (H3) holds. Arbitrarily fix $T$ and $t$ so that $0 < t < T < \infty$. Then by (H3), there exists a positive number $C_1(T-t)$ (depending on $(T-t)$) so that

$$\|S^*(T-t)z\| \leq C_1(T-t)\int_0^{T-t} \|B^*S^*(T-t-s)z\|_U \, \mathrm{d}s \ \text{ for all } \ z \in D(A^*),$$

which implies that

$$\|S^*(T-t)z\| \leq C_1(T-t)\int_t^{T} \|B^*S^*(T-s)z\|_U \, \mathrm{d}s \ \text{ for all } \ z \in D(A^*).$$

This, together with (2.2), yields that for each $z \in D(A^*)$,

$$\begin{aligned} \|B^*S^*(T-\cdot)z\|_{L^2(0,t;U)} &= \big\|B^*S^*(t-\cdot)\big(S^*(T-t)z\big)\big\|_{L^2(0,t;U)} \\ &\leq \sqrt{C(t)}\|S^*(T-t)z\|_X \\ &\leq \sqrt{C(t)}C_1(T-t)\|B^*S^*(T-\cdot)z\|_{L^1(t,T;U)}, \end{aligned}$$

where $C(t)$ is given by (2.2). Then by the definition of $Y_T$ (see (1.20)), the above yields that

$$\|g\|_{L^2(0,t;U)} \leq \sqrt{C_1(t)}C(T-t)\|g\|_{L^1(t,T;U)} \ \text{ for all } \ g \in Y_T. \tag{6.70}$$

Notice that (6.70) is exactly the statement (iii) in Lemma 2.3, where $p_2 = 2$. Thus we can apply Lemma 2.3 to get the conclusion (i) of Lemma 2.3 which is exactly the condition (H1). Hence, (H1) follows from (H3). This ends the proof of this proposition.

□

**Proposition 10.** *Suppose that (H3) and (H4) are true. Then (H2) holds.*

*Proof.* Let $T \in (0,\infty)$. Suppose that $f \in Y_T$ satisfies that

$$f = 0 \ \text{ over } \ E, \tag{6.71}$$

where the subset $E \subset (0,T)$ is of positive measure. We are going to use (H3) and (H4) to show that

$$f = 0 \ \text{ over } \ (0,T). \tag{6.72}$$

When this is done, we obtain (H2) from (H3) and (H4).

The rest is to show (6.72). By (1.20), there exists a sequence $\{z_n\} \subset D(A^*)$ so that

$$B^*S^*(T-\cdot)z_n \to f(\cdot) \text{ in } L^1(0,T;U), \ \text{ as } \ n \to \infty. \tag{6.73}$$

In particular, $\{B^*S^*(T-\cdot)z_n\}$ is a Cauchy sequence in $L^1(0,T;U)$. Take a sequence $\{T_k\} \subset (0,T)$ so that $T_k \nearrow T$. Then by (H3), we find that for each $k$, $\{S^*(T-T_k)z_n\}$ is a Cauchy sequence in $X$. Hence, for each $k$, there is a $\hat{z}_k \in X$ so that

$$S^*(T-T_k)z_n \to \hat{z}_k \ \text{ strongly in } \ X, \ \text{ as } \ n \to \infty. \tag{6.74}$$

By (6.74) and (2.2), we see that for each $k$, $\{B^*S^*(T-\cdot)z_n\}$ is a Cauchy sequence in $L^2(0,T_k;U)$. This, along with (6.74) and (1.22), indicates that for each $k$,

$$B^*S^*(T-\cdot)z_n \to \widetilde{B^*S^*}(T_k-\cdot)\hat{z}_k \ \text{ in } \ L^2(0,T_k;U), \ \text{ as } \ n \to \infty. \tag{6.75}$$

By (6.73) and (6.75), we find that for each $k$,

$$f(\cdot) = \widetilde{B^*S^*}(T_k - \cdot)\hat{z}_k \ \text{ over } \ (0, T_k). \tag{6.76}$$

Since $T_k \nearrow T$, we see that for each $k$ large enough, $E_k := E \cap (0, T_k)$ has a positive measure. Then from (6.76) and (6.71), we observe that for each $k$ large enough,

$$\widetilde{B^*S^*}(T_k - \cdot)\hat{z}_k = 0 \ \text{ over } \ E_k.$$

This, along with (H4), yields that for all $k$ large enough,

$$\widetilde{B^*S^*}(T_k - \cdot)\hat{z}_k = 0 \ \text{ over } \ (0, T_k). \tag{6.77}$$

Now, (6.72) follows from (6.76) and (6.77). This ends the proof. □

**Remark 12.** Since $Y_T$ is the completion of the space $X_T$ in the norm $\|\cdot\|_{L^1(0,T;U)}$ (see (1.20)), it is hard to characterize elements of $Y_T$ in general. However, when the assumption (H3) holds, we have that $Y_T = \mathcal{Y}_T$, where

$$\mathcal{Y}_T := \Big\{ f \in L^1(0,T;U) \ : \ \forall\, t \in (0,T),\, \exists\, z^t \in X \text{ s.t. } f(\cdot)|_{(0,t)} = \widetilde{B^*S^*}(t - \cdot)z^t \Big\}.$$

Indeed, on one hand, by (H3), we get (6.76), from which, it follows that $Y_T \subset \mathcal{Y}_T$. On the other hand, from (H1) and (ii) of Lemma 2.4, we find that $\mathcal{Y}_T \subset Y_T$. (Notice that (H1) is ensured by (H3), see Proposition 9.) For time varying systems, we do not know if these two spaces are the same in general. (In the proof of Lemma 2.4, we used the time-invariance of the system.)

We now are on the position to show Theorem 1.3.

*Proof of Theorem 1.3.* (i) We first claim that

$$T^0(y_0) = 0 \ \text{ and } \ N(T^0(y_0), y_0) = \infty \ \text{ for all } \ y_0 \in X \setminus \{0\}. \tag{6.78}$$

Indeed, by (H3), we can use Proposition 8 to get the $L^\infty$-null controllability for $(A, B)$, which, along with the definition of $T^0(\cdot)$ (see (1.17)), yields the first equality in (6.78). This, together with (iv) of Lemma 3.3, leads to the second equality in (6.78).

We next claim that

$$\mathcal{W} = \mathcal{W}_{3,2} \cup \mathcal{W}_{3,3}. \tag{6.79}$$

In fact, by the second equality in (6.78) and the definition of $\mathcal{W}_1$ and $\mathcal{W}_2$ (see (1.26) and (1.28)), we find that $\mathcal{W}_1 \cup \mathcal{W}_2 = \emptyset$. Meanwhile, by the first equality in (6.78) and the definitions of $\mathcal{W}_{3,1}$ and $\mathcal{W}_{3,4}$ (see (1.29)), we find that $\mathcal{W}_{3,1} \cup \mathcal{W}_{3,4} = \emptyset$. These, along with (i) of Theorem 1.1, lead to (6.79).

We then claim that

$$\mathcal{V} = \mathcal{V}_{3,1} \cup \mathcal{V}_{3,2}. \tag{6.80}$$

Indeed, by the second equality in (6.78) and the definitions of $\mathcal{V}_1$ and $\mathcal{V}_2$ (see (1.31) and (1.33)), we see that $\mathcal{V}_1 \cup \mathcal{V}_2 = \emptyset$. Meanwhile, the first equality in (6.78) and the definition of $\mathcal{V}_{3,3}$ (see (1.34)), we find that $\mathcal{V}_{3,3} = \emptyset$. These, along with (i) of Theorem 1.2, lead to (6.80).

Now, (1.38) follows from (6.79) and (6.80) at once.

Finally, we verify (1.39). On one hand, by the definitions of $\gamma_1$ and $\mathcal{W}_{2,2}$ (see (1.36) and (1.27)), we see that $\gamma_1 = \mathcal{W}_{2,2}$. On the other hand, from (i) of Lemma 3.3 and (ii) of Lemma 3.2, it follows that

$$N(T^0(y_0), y_0) \geq N(T^1(y_0), y_0) \ \text{ for all } \ y_0 \in X \setminus \{0\}.$$

Then by the definitions of $\gamma_2$ and $\mathcal{V}_{2,3}$ (see (1.37) and (1.32)), one can directly check that $\gamma_2 = \mathcal{V}_{2,3}$. Since we already knew that $\mathcal{W}_2 = \emptyset$, $\mathcal{V}_2 = \emptyset$, $\mathcal{W}_{2,2} \subset \mathcal{W}_2$ and $\mathcal{V}_{2,3} \subset \mathcal{V}_2$, (1.39) follows at once. Thus we end the proof of the conclusion (i) of Theorem 1.3.

(ii) Since (H3) and (H4) hold, we find from Proposition 9 and Proposition 10 that both (H1) and (H2) hold. Then by the conclusions (ii) and (v) of Theorem 1.2, as well as the second equality in (1.38), we get the conclusion (ii) of Theorem 1.3.

(iii) By (H3) and (H4), we can use Proposition 9 and Proposition 10 to get (H1) and (H2). Then by (ii) and (iii) of Theorem 1.1, as well as the first equality in (1.38), we are led to the conclusion (iii) of Theorem 1.3.

In summary, we finish the proof of Theorem 1.3.

□

7. **Applications.** Two applications of the main theorems of this paper will be given in this section. The first one is an application of Theorem 1.3, while the second one is an application of Theorem 1.1, Theorem 1.2.

7.1. **Application to boundary controlled heat equations.** In this subsection, we will use Theorem 1.3 to study the BBP decompositions for minimal time and minimal norm control problems for boundary controlled heat equations. We begin with introducing the controlled equations. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with a smooth boundary $\partial\Omega$. Let $\Gamma$ be a nonempty open subset of $\partial\Omega$. Consider the following two controlled equations:

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in} \quad \Omega \times (0, \infty), \\ y = u & \text{on} \quad \Gamma \times (0, \infty), \\ y = 0 & \text{on} \quad (\partial\Omega \setminus \Gamma) \times (0, \infty), \\ y(0) = y_0 & \text{in} \quad \Omega \end{cases} \tag{7.1}$$

and

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in} \quad \Omega \times (0, T), \\ y = v & \text{on} \quad \Gamma \times (0, T), \\ y = 0 & \text{on} \quad (\partial\Omega \setminus \Gamma) \times (0, T), \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{7.2}$$

Here, $y_0 \in H^{-1}(\Omega)$, $0 < T < \infty$, $u \in L^\infty(\mathbb{R}^+; L^2(\Gamma))$ and $v \in L^\infty(0, T; L^2(\Gamma))$. Write $y_1(\cdot; y_0, u)$ and $\hat{y}_1(\cdot; y_0, v)$ for the solutions of (7.1) and (7.2), respectively.

We will put the above systems in our framework where $X := H^{-1}(\Omega)$, $U := L^2(\Gamma)$, $A := A_1$ and $B := B_1$. Here, $A_1 = \Delta$, with $D(A_1) = H_0^1(\Omega)$, and $B_1$ is defined in the following manner: Let $D : L^2(\partial\Omega) \to L^2(\Omega)$ be defined by $Dv := f_v$, for all $v \in L^2(\partial\Omega)$, where $f_v$ solves the equation

$$\begin{cases} -\Delta f = 0 & \text{in} \ \ \Omega, \\ f = v & \text{on} \ \ \partial\Omega. \end{cases} \tag{7.3}$$

Then let $B_1 := -\Delta D$. We regard $L^2(\Gamma)$ as a subspace of $L^2(\partial\Omega)$. Let $X_{-1} := (D(A_1^*))'$ be the dual of $D(A_1^*)$ with respect to the pivot space $X$.

To prove that the above $X$, $U$ and $(A_1, B_1)$ are in our framework, we will use some results in [39] where both state and control spaces are assumed to be complex Hilbert spaces. Thus, we will consider the complexifications of our spaces. Write

$\mathcal{H}^{-1}(\Omega)$ and $\mathcal{H}_0^1(\Omega)$ for the complexifications of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, respectively. Write $\mathcal{X} := \mathcal{H}^{-1}(\Omega)$ and $\mathcal{U} := L^2(\Gamma;\mathbb{C})$. Let $\mathcal{A}_1 := \Delta$, with $D(\mathcal{A}_1) = \mathcal{H}_0^1(\Omega)$. Define $\mathcal{D} : L^2(\partial\Omega;\mathbb{C}) \to L^2(\Omega;\mathbb{C})$ given by $\mathcal{D}w = g_w$, for all $w \in L^2(\partial\Omega;\mathbb{C})$, where $g_w$ solves (7.3) with $v = w$. Then let $\mathcal{B}_1 := -\Delta\mathcal{D}$. The space $L^2(\Gamma;\mathbb{C})$ is regarded as a subspace of $L^2(\partial\Omega;\mathbb{C})$. Let $\mathcal{X}_{-1} := (D(\mathcal{A}_1^*))'$ be the dual of $D(\mathcal{A}_1^*)$ with respect to the pivot space $\mathcal{X}$. Then, from [39, Proposition 10.7.1], it follows that $\mathcal{A}_1$ generates a $C_0$-semigroup $\{\mathcal{S}_1(t)\}_{t\in\mathbb{R}^+}$ over $\mathcal{H}^{-1}(\Omega)$; $\mathcal{B}_1 \in \mathcal{L}(\mathcal{U};\mathcal{X}_{-1}) \setminus \{0\}$ is an admissible control operator for the semigroup $\{\mathcal{S}_1(t)\}_{t\in\mathbb{R}^+}$.

Several observations are given in order: First, $\mathcal{A}_1|_{D(A_1)} = A_1$ and $\mathcal{B}_1|_{L^2(\Gamma)} = B_1$; Second, $\{\mathcal{S}_1(t)|_X\}_{t\in\mathbb{R}^+}$ is a $C_0$-semigroup over $H^{-1}(\Omega)$, with its generator $A_1$; Third, $B_1 \in \mathcal{L}(U, X_{-1}) \setminus \{0\}$ is an admissible control operator for the semigroup $\{\mathcal{S}_1(t)|_X\}_{t\in\mathbb{R}^+}$. From these observations, we see that if $S_1(t) := \mathcal{S}_1(t)|_X$, $t \in \mathbb{R}^+$, then the systems (7.1) and (7.2) can be rewritten respectively as

$$y'(t) = A_1 y(t) + B_1 u(t),\ t > 0;\ \ y(0) = y_0;$$

$$y'(t) = A_1 y(t) + B_1 v(t),\ 0 < t \le T;\ \ y(0) = y_0.$$

The corresponding two optimal control problems are as follows: The first one is the minimal time control problem $(TP)_1^{M,y_0}$, with $y_0 \in H^{-1}(\Omega) \setminus \{0\}$ and $M \in (0,\infty)$:

$$T_1(M, y_0) := \{\hat{t} > 0\ :\ \exists\, u \in \mathcal{U}_1^M\ \text{ s.t. }\ y_1(\hat{t}; y_0, u) = 0\},$$

where

$$\mathcal{U}_1^M := \{u \in L^\infty(\mathbb{R}^+, L^2(\Gamma))\ :\ \|u(t)\|_{L^2(\Gamma)} \le M\ \text{ a.e. }\ t \in \mathbb{R}^+\}.$$

The second one is the minimal norm control problem $(NP)_1^{T,y_0}$, (with $y_0 \in H^{-1}(\Omega)\setminus \{0\}$ and $T \in (0,\infty)$) as follows:

$$N_1(T, y_0) := \inf\{\|v\|_{L^\infty(0,T;L^2(\Gamma))}\ :\ \hat{y}_1(T; y_0, v) = 0\}.$$

**Lemma 7.1.** *The conditions (H3) and (H4) hold for the pair $(A_1, B_1)$. Furthermore, $N_1(T^1(y_0), y_0) = 0$ for each $y_0 \in H^{-1}(\Omega) \setminus \{0\}$, where $T^1(y_0)$ is given by (1.18) where $\{S(t)\}_{t\in\mathbb{R}^+}$ is replaced by $\{S_1(t)\}_{t\in\mathbb{R}^+}$.*

*Proof.* First, the condition (H3) follows from Proposition 8 and the $L^\infty$-null boundary controllability of the heat equation (see, for instance, subsection 3.2.1 in [44]).

Next, we prove that (H4) holds for $(A_1, B_1)$. For this purpose, let $0 < T < \infty$ and $E \subset (0, T)$ be a measurable subset of positive measure. Then fix a $\hat{z} \in X$ so that

$$\widetilde{B_1^* S_1^*}(T - \cdot)\hat{z} = 0\ \text{ over }\ E, \tag{7.4}$$

where $\widetilde{B_1^* S_1^*}(T - \cdot)\hat{z}$ is given by (1.22). We will use the real analyticity of $\{S_1(t)\}_{t\in\mathbb{R}}$ to show that

$$\widetilde{B_1^* S_1^*}(T - \cdot)\hat{z} = 0\ \text{ over }\ (0, T). \tag{7.5}$$

Indeed, from subsection 3.2.1 in [44], it follows that the semigroup $\{\mathcal{S}_1(t)\}_{t\in\mathbb{R}^+}$ can be extended to an analytic semigroup. Thus, the semigroup $\{\mathcal{S}_1^*(t)\}_{t\in\mathbb{R}^+}$ is also analytic. Then by [29, Theorem 5.2 in Chapter 2], we find that

$$\mathcal{S}_1^*(\cdot)\ \text{ is real analytic over }\ (0,\infty);\ \text{ and }\ \|\mathcal{S}_1^*(t)\|_{\mathcal{L}(\mathcal{X},D(\mathcal{A}_1^*))} \le \widehat{C}/t,\ t > 0, \tag{7.6}$$

where the constant $\widehat{C}$ is independent of $t > 0$. Since $\mathcal{S}_1(\cdot)|_X = S_1(\cdot)$ over $\mathbb{R}^+$, we have that $\mathcal{S}_1^*(\cdot)|_X = S_1^*(\cdot)$ over $\mathbb{R}^+$, which, along with (7.6), implies that

$$S_1^*(\cdot)\ \text{ is real analytic over }\ (0,\infty);\ \text{ and }\ \|S_1^*(t)\|_{\mathcal{L}(X,D(A_1^*))} \le \widehat{C}/t,\ t > 0. \tag{7.7}$$

Arbitrarily fix an $\varepsilon \in (0, T)$ so that

$$|E \cap (0, T-\varepsilon)| > 0. \tag{7.8}$$

Let $\{z_n\} \subset D(A_1^*)$ so that $\lim_{n\to\infty} z_n = \hat{z}$ in $X$. Because $B_1 \in \mathcal{L}(D(A_1^*), U)$, we find from the second conclusion in (7.7) that when $n$ goes to $\infty$,

$$\|B_1^* S_1^*(\cdot) z_n - B_1^* S_1^*(\cdot)\hat{z}\|_{L^2(\varepsilon,T;U)} \quad = \quad \|B_1^* S_1^*(\cdot - \varepsilon) S_1^*(\varepsilon)(z_n - \hat{z})\|_{L^2(\varepsilon,T;U)} \to 0.$$

This, along with (1.22), yields that

$$\widetilde{B_1^* S_1^*}(T - \cdot)\hat{z} = B_1^* S_1^*(T - \cdot)\hat{z} \ \text{ over } \ (0, T-\varepsilon),$$

which, together with the first conclusion in (7.7), shows that $\widetilde{B_1^* S_1^*}(T - \cdot)\hat{z}$ is real analytic over $(0, T-\varepsilon)$. Then, by (7.8) and (7.4), we see that

$$\widetilde{B_1^* S_1^*}(T - \cdot)\hat{z} = 0 \ \text{ over } \ (0, T-\varepsilon).$$

Sending $\varepsilon \to 0$ in the above leads to (7.5). Hence, (H4) holds for $(A_1, B_1)$.

Finally, we will prove that

$$N_1(T^1(y_0), y_0) = 0 \ \text{ for all } \ y_0 \in H^{-1}(\Omega) \setminus \{0\}. \tag{7.9}$$

According to (vi) of Lemma 3.3, (7.9) is equivalent to that

$$N_1(\infty, y_0) = 0 \ \text{ for all } \ y_0 \in H^{-1}(\Omega) \setminus \{0\}. \tag{7.10}$$

To prove (7.10), we arbitrarily fix a $y_0 \in H^{-1}(\Omega) \setminus \{0\}$ and then fix a $\hat{t} \in (0, \infty)$. Notice that the semigroup $\{S_1(t)\}_{t\geq 0}$ has the following property: there exist $C > 0$ and $\delta > 0$, independent of $\hat{t}$, so that

$$\|S_1(\hat{t})y_0\|_{H^{-1}(\Omega)} \leq Ce^{-\delta\hat{t}}\|y_0\|_{H^{-1}(\Omega)}. \tag{7.11}$$

Meanwhile, according to the $L^\infty$-null controllability of the boundary controlled heat equation, there exist a positive constant $C'$ (independent of $\hat{t}$) and a control $u_{\hat{t}} \in L^\infty(0, 1; L^2(\Gamma))$ so that

$$\hat{y}_1(1; S_1(\hat{t})y_0, u_{\hat{t}}) = 0 \ \text{ and } \ \|u_{\hat{t}}\|_{L^\infty(0,1;L^2(\Gamma))} \leq C'\|S_1(\hat{t})y_0\|_{H^{-1}(\Omega)}. \tag{7.12}$$

Define another control

$$v_{\hat{t}}(\tau) = \begin{cases} 0, & \tau \in (0, \hat{t}], \\ u_{\hat{t}}(\tau - \hat{t}), & \tau \in (\hat{t}, \hat{t}+1). \end{cases}$$

From this and (7.12), we find that

$$\hat{y}_1(\hat{t}+1; y_0, v_{\hat{t}}) = \hat{y}_1(1; S_1(\hat{t})y_0, u_{\hat{t}}) = 0;$$

$$\|v_{\hat{t}}\|_{L^\infty(0,\hat{t}+1;L^2(\Gamma))} = \|u_{\hat{t}}\|_{L^\infty(0,1;L^2(\Gamma))} \leq C'\|S_1(\hat{t})y_0\|_{H^{-1}(\Omega)}.$$

These, along with the optimality of $N_1(\hat{t}+1, y_0)$ and (7.11), yield that

$$N_1(\hat{t}+1, y_0) \leq \|v_{\hat{t}}\|_{L^\infty(0,\hat{t}+1;L^2(\Gamma))} \leq C'\|S_1(\hat{t})y_0\|_{H^{-1}(\Omega)} \leq C'Ce^{-\delta\hat{t}}\|y_0\|_{H^{-1}(\Omega)}.$$

By this and the first equality in (1.19), we obtain (7.10). Hence, (7.9) has been proved. This ends the proof of this lemma.

□

The BBP decompositions for $(A_1, B_1)$ are presented in the following Theorem 7.2:

**Theorem 7.2.** *Let $\mathcal{W}$, $\mathcal{W}_{3,2}$, $\mathcal{V}$ and $\mathcal{V}_{3,2}$ be respectively given by (1.23), (1.29), (1.24) and (1.34), where $(A,B)=(A_1,B_1)$. Then the following conclusions are true:*
*(i) $\mathcal{W}=\mathcal{W}_{3,2}$ and $\mathcal{V}=\mathcal{V}_{3,2}$.*
*(ii) For each $(M,y_0)\in(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$, the problem $(TP)_1^{M,y_0}$ has the bang-bang property.*
*(iii) For each $(T,y_0)\in(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$, the problem $(NP)_1^{T,y_0}$ has the bang-bang property and the null control is not a minimal norm control to this problem.*

*Proof.* (i) By Lemma 7.1, we see that (H3) and (H4) holds for $(A_1,B_1)$. Then we can use Theorem 1.3 to find that

$$\mathcal{W}=\mathcal{W}_{3,2}\cup\mathcal{W}_{3,3}\ \text{ and }\ \mathcal{V}=\mathcal{V}_{3,1}\cup\mathcal{V}_{3,2}. \tag{7.13}$$

On one hand, by the backward uniqueness property for $\{S_1(t)\}_{t\in\mathbb{R}^+}$, we have that $T^1(y_0)=\infty$ for all $y_0\in X\setminus\{0\}$. On the other hand, by Lemma 7.1, we also have that $N_1(T^1(y_0),y_0)=0$ for all $y_0\in H^{-1}(\Omega)\setminus\{0\}$. These, along with the definitions of $\mathcal{W}_{3,3}$ and $\mathcal{V}_{3,1}$ (see (1.29) (1.34)), yield that $\mathcal{W}_{3,3}=\emptyset$ and $\mathcal{V}_{3,1}=\emptyset$ in this case. From this and (7.13), we get the conclusion (i) of this theorem.

(ii) Notice that $\mathcal{V}=(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$ in this case. (For the definition of $\mathcal{V}$, see (1.24).) Then by the second equality in the conclusion (i) of this theorem and the assumptions (H3) and (H4), we can apply (ii) of Theorem 1.3 to get the conclusion (ii) of this theorem.

(iii) Notice that $\mathcal{W}=(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$ in this case. (For the definition of $\mathcal{W}$, see (1.23)) Then by the first equality in the conclusion (i) of this theorem and the assumptions (H3) and (H4), we can apply (iii) of Theorem 1.3 to get the conclusion (iii) of this theorem.

In summary, we finish the proof of this theorem.

□

**Remark 13.** (i) From Theorem 7.2, we see that the BBP decomposition for $(NP)_1^{T,y_0}$ has only one part which is $\mathcal{W}=(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$ and that for each $(T,y_0)$ in $\mathcal{W}$, the corresponding $(NP)_1^{T,y_0}$ has the bang-bang property. The reason to cause such decomposition is that $(A_1,B_1)$ is $L^\infty$-null controllable. The same can be said about the BBP decomposition for $(NP)^{T,y_0}$ built up in Theorem 1.3.

(ii) From Theorem 7.2, we see that the BBP decomposition for $(TP)_1^{M,y_0}$ has only one part which is $\mathcal{V}=(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$ and that for each $(M,y_0)$ in $\mathcal{V}$, the corresponding $(TP)_1^{M,y_0}$ has the bang-bang property. The reasons to cause such decomposition are that $(A_1,B_1)$ is $L^\infty$-null controllable and $N_1(T^1(y_0),y_0)=0$ for all $y_0\in(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$. (Compare this BBP decomposition with the BBP decomposition (P1) given by (1.6).) The above-mentioned second property (i.e., $N_1(T^1(y_0),y_0)=0$ for all $y_0\in(0,\infty)\times(H^{-1}(\Omega)\setminus\{0\})$) holds, because solutions of the controlled system (governed by $(A_1,B_1)$), with the null control, tend to zero as time goes to infinity.

7.2. **Application to some special controlled evolution systems.** In this subsection, we will use Theorem 1.1 and Theorem 1.2 to study the BBP decompositions for minimal time and minimal norm control problems in a special setting. The controlled system in this setting is taken from [15].

Let $X$ and $U$ be two real separable Hilbert spaces. Let $A := A_2$ and $B := B_2$, where $A_2$ and $B_2$ are defined in the following manner: Arbitrarily fix a Riesz basis $\{\phi_j\}_{j\geq 1}$ in $X$ and a biorthogonal sequence $\{\psi_j\}_{j\geq 1}$ of the aforementioned Riesz basis. Take a sequence $\Lambda := \{\lambda_j\}_{j\geq 1} \subset \mathbb{R}^+$ so that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots; \text{ and } \Sigma_{j\geq 1} 1/\lambda_j < \infty. \tag{7.14}$$

Write $X_1 := \{y \in X \ : \ \|y\|_1 < \infty\}$ with the norm $\|y\|_{X_1} := \sqrt{\sum_{j\geq 1} \lambda_j^2 \langle y, \psi_j\rangle_X^2}$. Define $A_2 : \ D(A_2) := X_1 \subset X \to X$ by setting

$$A_2 x := -\sum_{j\geq 1} \lambda_j \langle x, \psi_j\rangle_X \phi_j \ \text{ for each } \ x \in D(A_2). \tag{7.15}$$

Write $X_{-1} := (D(A_2^*))'$ (the dual of $D(A_2^*)$ with respect to the pivot space $X$). Then let $B_2 \in \mathcal{L}(U, X_{-1}) \setminus \{0\}$.

One can directly check the following facts: First, the operator $A_2$ generates a $C_0$-semigroup $\{S_2(t)\}_{t\in\mathbb{R}^+}$ over $X$; Second, the semigroup $\{S_2(t)\}_{t\in\mathbb{R}^+}$ has the expression:

$$S_2(t)x = \sum_{j=1}^{\infty} x_j e^{-\lambda_j t}\phi_j, \ t \geq 0, \ \text{ for each } \ x = \sum_{j=1}^{\infty} x_j \phi_j \in X. \tag{7.16}$$

Third, the dual semigroup $\{S_2^*(t)\}_{t\geq 0}$ has the expression:

$$S_2^*(t)x = \sum_{j=1}^{\infty} \hat{x}_j e^{-\lambda_j t}\psi_j, \ t \geq 0, \ \text{ for each } \ x = \sum_{j=1}^{\infty} \hat{x}_j \psi_j \in X. \tag{7.17}$$

In this setting, the systems (1.11) and (1.12) read respectively as follows:

$$y'(t) = A_2 y(t) + B_2 u(t), \ t > 0; \ \ y(0) = y_0; \tag{7.18}$$

$$y'(t) = A_2 y(t) + B_2 v(t), \ 0 < t \leq T; \ \ y(0) = y_0. \tag{7.19}$$

Here, $y_0 \in X$, $0 < T < \infty$, $u \in L^\infty(\mathbb{R}^+; U)$ and $v \in L^\infty(0, T; U)$. Write $y_2(\cdot; y_0, u)$ and $\hat{y}_2(\cdot; y_0, v)$ for the solutions of (7.18) and (7.19), respectively. There are many controlled PDEs governed by $(A_2, B_2)$, we refer the readers to [15], [16] and [17].

For each $y_0 \in X\setminus\{0\}$ and each $M \in (0,\infty)$, we consider the minimal time control problem:

$$(TP)_2^{M,y_0} \qquad T_2(M, y_0) := \inf\{\hat{t} > 0 \ : \ \exists\, u \in \mathcal{U}_3^M \ \text{ s.t. } \ y(\hat{t}; y_0, u) = 0\},$$

where

$$\mathcal{U}_3^M := \{u \in L^\infty(\mathbb{R}^+; U) \ : \ \|u(t)\|_U \leq M \ \text{ a.e. } \ t \in \mathbb{R}^+\}.$$

For each $y_0 \in X \setminus \{0\}$ and each $T \in (0,\infty)$, we consider the minimal norm control problem:

$$(NP)_2^{T,y_0} \qquad N_2(T, y_0) := \inf\{\|v\|_{L^\infty(0,T;U)} \ : \ \hat{y}_2(T; y_0, v) = 0\}.$$

We will prove that $(A_2, B_2)$ satisfies (H1) and (H2). To do this, we need three lemmas. The first one is very similar to [17, Lemma 4.6]. We will give its proof in Appendix G of this paper. To state it, we define

$$\mathcal{P} := \Big\{z \to \sum_{j=1}^{N} c_j e^{-\lambda_j z}, \ z \in \mathbb{C}^+ \ : \ \{c_j\}_{j=1}^N \subset \mathbb{C}, \ N \in \mathbb{N}^+\Big\}, \tag{7.20}$$

where $\mathbb{C}^+ := \{x+iy \in \mathbb{C} \ : \ x \geq 0\}$. And then for each $\theta_0 \in (0, \frac{\pi}{2})$ and $\varepsilon > 0$, define

$$S_{\varepsilon,\theta_0} := \Big\{ z = x + iy \in \mathbb{C} \ : \ x \geq \varepsilon, \ \frac{|y|}{x} \leq \frac{1}{2} \cot \theta_0 \Big\}. \tag{7.21}$$

**Lemma 7.3.** *For each $\theta_0 \in (0, \frac{\pi}{2})$, $\varepsilon > 0$, and each $T > 0$, there exist two positive constants $C_1 := C_1(\theta_0, \varepsilon, T)$ and $C_2 := C_2(\theta_0)$ so that*

$$|p(z)| \leq C_1 e^{-C_2 Re\, z} \|p|_{(0,T)}\|_{L^1(0,T;\mathbb{C})} \ \textit{for all} \ p \in \mathcal{P} \ \textit{and} \ z \in S_{\varepsilon,\theta_0}. \tag{7.22}$$

*Here, $p|_{(0,T)}$ denotes the restriction of $p$ on $(0,T)$.*

To state the second lemma, we write $\widetilde{U}$ for the complexification of $U$ and then define

$$\mathcal{P}_{\widetilde{U}} := \Big\{ z \to \sum_{j=1}^{N} c_j e^{-\lambda_j z} B_2^* \psi_j, \ z \in \mathbb{C}^+ \ : \ \{c_j\}_{j=1}^N \subset \mathbb{C}, \ N \in \mathbb{N}^+ \Big\}. \tag{7.23}$$

Notice that each element in $\mathcal{P}_{\widetilde{U}}$ is a vector-valued function, with its domain $\mathbb{C}$ and its range $\widetilde{U}$. With the aid of Lemma 7.3, we build up an estimate in the second lemma as follows:

**Lemma 7.4.** *For each $\theta_0 \in (0, \frac{\pi}{2})$, $\varepsilon > 0$ and each $T > 0$, there exist two positive constants $C_1 := C_1(\theta_0, \varepsilon, T)$ and $C_2 := C_2(\theta_0)$ so that*

$$\|f(z)\|_{\widetilde{U}} \leq C_1 e^{-C_2 Re\, z} \|f|_{(0,T)}\|_{L^1(0,T;\widetilde{U})} \ \textit{for all} \ f \in \mathcal{P}_{\widetilde{U}} \ \textit{and} \ z \in S_{\varepsilon,\theta_0}, \tag{7.24}$$

*where, $S_{\varepsilon,\theta_0}$ and $\mathcal{P}_{\widetilde{U}}$ are defined by (7.21) and (7.23), respectively, and $f|_{(0,T)}$ denotes the restriction of $f$ on $(0,T)$.*

*Proof.* Arbitrarily fix $f \in \mathcal{P}_{\widetilde{U}}$. Then by (7.23), there is $N \in \mathbb{N}^+$ and $\{c_j\}_{j=1}^N \subset \mathbb{C}$ so that

$$f(z) = \sum_{j=1}^{N} c_j e^{-\lambda_j z} B_2^* \psi_j \ \text{ for all } \ z \in \mathbb{C}^+.$$

Arbitrarily fix a $v \in \widetilde{U}$. Since

$$f_v(z) := \langle f(z), v \rangle_{\widetilde{U}} = \sum_{j=1}^{N} c_j \langle B_2^* \psi_j, v \rangle_{\widetilde{U}} e^{-\lambda_j z}, \ z \in \mathbb{C}^+,$$

it follows from (7.20) that $f_v \in \mathcal{P}$. Then according to Lemma 7.3, for each $\theta_0 \in (0, \frac{\pi}{2})$, each $\varepsilon > 0$ and each $T > 0$, there are two positive constants $C_1(\theta_0, \varepsilon, T)$ and $C_2(\theta_0)$ (independent of $f$ and $v$) so that

$$|\langle f(z), v \rangle_{\widetilde{U}}| \leq C_1(\theta_0, \varepsilon, T) e^{-C_2(\theta_0) Re\, z} \int_0^T |\langle f|_{(0,T)}(t), v \rangle_{\widetilde{U}}| \, \mathrm{d}t \ \text{ for each } \ z \in S_{\varepsilon,\theta_0}.$$

Since for each $z \in S_{\varepsilon,\theta_0}$, the above inequality holds for all $v \in \widetilde{U}$, we find that for each $z \in S_{\varepsilon,\theta_0}$,

$$\|f(z)\|_{\widetilde{U}} = \sup_{\|v\|_{\widetilde{U}} \leq 1} |\langle f(z), v \rangle_{\widetilde{U}}| \leq C_1(\theta_0, \varepsilon, T) e^{-C_2(\theta_0) Re\, z} \int_0^T \|f|_{(0,T)}(t)\|_{\widetilde{U}} \, \mathrm{d}t.$$

Since $f$ was arbitrarily taken from $\mathcal{P}_{\widetilde{U}}$, the above inequality leads to (7.24). This ends the proof of this lemma.

□

With the aid of Lemma 7.4, we obtain the third lemma which will play a key role in the proof of the conclusion that (H1) and (H2) hold for $(A_2, B_2)$.

**Lemma 7.5.** *Let $\theta_0 \in (0, \frac{\pi}{2})$. Then for each $T \in (0,\infty)$, each $\varepsilon \in (0,T)$ and each $f \in Y_T$ (which is defined by (1.20) with $(A^*, B^*)$ being replaced by $(A_2^*, B_2^*)$), there is a continuous and weakly analytic function $\widetilde{g}_{\varepsilon,f} : S_{\varepsilon,\theta_0} \to \widetilde{U}$ so that*

$$\widetilde{g}_{\varepsilon,f}|_{(\varepsilon,T)}(T-t) = f(t) \ \text{ for each } \ t \in (0, T-\varepsilon), \tag{7.25}$$

*and so that*

$$\|\widetilde{g}_{\varepsilon,f}\|_{L^\infty(S_{\varepsilon,\theta_0};\widetilde{U})} \le C_1(\theta_0,\varepsilon,\varepsilon)\|f\|_{L^1(T-\varepsilon,T;U)}, \tag{7.26}$$

*where $C_1(\theta_0,\varepsilon,\varepsilon)$ is given by (7.24).*

*Proof.* Let $\theta_0 \in (0, \frac{\pi}{2})$ be given. Arbitrarily fix $T \in (0,\infty)$, $\varepsilon \in (0,T)$ and $f \in Y_T$. First of all, since $\{\psi_j\}_{j\ge1}$ is a biorthogonal sequence of the Riesz basis $\{\phi_j\}_{j\ge1}$ in $X$, it follows by (7.15) that each element $w \in D(A_2^*)$ can be expressed by $w = \sum_{j=1}^\infty \alpha_j\psi_j$, with $\{\alpha_j\}_{j=1}^\infty \subset \mathbb{R}$, and satisfies that

$$\Big\|\sum_{j=1}^N \alpha_j\psi_j - w\Big\|_{D(A_2^*)} = \sqrt{\sum_{j\ge N}\lambda_j^2\alpha_j^2} \to 0, \ \text{ as } \ N \to \infty, \tag{7.27}$$

Since $B_2^* \in \mathcal{L}(D(A_2^*), U)$, it follows from (7.27) that

$$B_2^*S_2^*(T-\cdot)\sum_{j=1}^N \alpha_j\psi_j \to B_2^*S_2^*(T-\cdot)w \ \text{ in } \ L^1(0,T;U), \ \text{ as } \ N\to\infty. \tag{7.28}$$

Since $f \in Y_T$, according to (1.20) and (7.28), there is a sequence $\{w_N\}_{N=1}^\infty$ in $D(A_2^*)$ so that for each $N \in \mathbb{N}^+$,

$$w_N = \sum_{j=1}^{K_N} \alpha_j(w_N)\psi_j, \ \text{ with } \ K_N \in \mathbb{N}^+ \ \text{ and } \ \{\alpha_j(w_N)\}_{j=1}^{K_N} \subset \mathbb{R}, \tag{7.29}$$

and so that

$$B_2^*S_2^*(T-\cdot)w_N \to f(\cdot) \ \text{ in } \ L^1(0,T;U), \ \text{ as } \ N\to\infty. \tag{7.30}$$

Next, for each $N \in \mathbb{N}^+$, define $g_N : \mathbb{C}^+ \to \widetilde{U}$ by

$$g_N(z) := \sum_{j=1}^{K_N} \alpha_j(w_N)e^{-\lambda_j z}B_2^*\psi_j, \ z \in \mathbb{C}^+. \tag{7.31}$$

By (7.31), (7.29) and (7.17), we see that

$$g_N|_{(0,T)}(t) = B_2^*S_2^*(t)w_N \ \text{ for each } \ t \in (0,T). \tag{7.32}$$

Meanwhile, from (7.31) and (7.23), we see that $g_N \in \mathcal{P}_{\widetilde{U}}$ for all $N \in \mathbb{N}^+$. This, along with Lemma 7.4, yields that for each $N \in \mathbb{N}^+$,

$$\|g_N|_{S_{\varepsilon,\theta_0}}\|_{L^\infty(S_{\varepsilon,\theta_0};\widetilde{U})} \le C_1(\theta_0,\varepsilon,\varepsilon)\|g_N|_{(0,\varepsilon)}\|_{L^1(0,\varepsilon;\widetilde{U})},$$

where $C_1(\theta_0,\varepsilon,\varepsilon)$ is given by (7.24). Since for each $t \in \mathbb{R}^+$, we have that $g_N(t) \in U$ (see (7.31) and (7.29)), the above inequality can be rewritten as:

$$\|g_N|_{S_{\varepsilon,\theta_0}}\|_{L^\infty(S_{\varepsilon,\theta_0};\widetilde{U})} \le C_1(\theta_0,\varepsilon,\varepsilon)\|g_N|_{(0,\varepsilon)}\|_{L^1(0,\varepsilon;U)}, \tag{7.33}$$

By (7.32) and (7.30), we see that

$$g_N|_{(0,\varepsilon)}(\cdot) \to f(T-\cdot) \ \text{ in } \ L^1(0,\varepsilon;U). \tag{7.34}$$

Hence, $\{g_N|_{(0,\varepsilon)}\}_{N=1}^{\infty}$ is a Cauchy sequence in $L^1(0,\varepsilon;U)$. From this and (7.33), we can easily see that there exists a function $\widetilde{g}_{\varepsilon,f} \in L^\infty(S_{\varepsilon,\theta_0};\widetilde{U})$ so that

$$g_N|_{S_{\varepsilon,\theta_0}} \to \widetilde{g}_{\varepsilon,f} \text{ in } L^\infty(S_{\varepsilon,\theta_0};\widetilde{U}), \text{ as } N\to\infty. \tag{7.35}$$

We claim that

$$\widetilde{g}_{\varepsilon,f} : S_{\varepsilon,\theta_0} \to \widetilde{U} \text{ is continuous and weakly analytic over } S_{\varepsilon,\theta_0}. \tag{7.36}$$

First, by (7.31), we see that for each $N\in\mathbb{N}^+$, the function $g_N|_{S_{\varepsilon,\theta_0}}$ is continuous. This, along with (7.35), yields that the function $\widetilde{g}_{\varepsilon,f}$ is continuous over $S_{\varepsilon,\theta_0}$, and that

$$g_N|_{S_{\varepsilon,\theta_0}} \to \widetilde{g}_{\varepsilon,f} \text{ in } C(S_{\varepsilon,\theta_0};\widetilde{U}), \text{ as } N\to\infty. \tag{7.37}$$

Next, we prove the weak analyticity of the function $\widetilde{g}_{\varepsilon,f}$. Arbitrarily fix a $v\in\widetilde{U}$. By (7.37), we find that

$$\langle g_N|_{S_{\varepsilon,\theta_0}}, v\rangle_{\widetilde{U}} \to \langle \widetilde{g}_{\varepsilon,f}, v\rangle_{\widetilde{U}} \text{ in } C(S_{\varepsilon,\theta_0};\mathbb{C}), \text{ as } N\to\infty. \tag{7.38}$$

Meanwhile, by (7.31), we see that for each $N\in\mathbb{N}^+$, the function $z\to\langle g_N|_{S_{\varepsilon,\theta_0}}(z), v\rangle_{\widetilde{U}}$ is analytic over $S_{\varepsilon,\theta_0}$. By this and (7.38), we can use [35, Theorem 10.28] to see that the function $z\to\langle\widetilde{g}_{\varepsilon,f}(z), v\rangle_{\widetilde{U}}$ is analytic over $S_{\varepsilon,\theta_0}$. Since $v$ was arbitrarily taken from $\widetilde{U}$, $\widetilde{g}_{\varepsilon,f}$ is weakly analytic over $S_{\varepsilon,\theta_0}$. Hence, conclusions in (7.36) are true.

We now show that the above function $\widetilde{g}_{\varepsilon,f}$ satisfies (7.25). Indeed, by (7.21), we see that $(\varepsilon,T)\subset S_{\varepsilon,\theta_0}$. This, together with (7.37), yields that

$$g_N|_{(\varepsilon,T)} \to \widetilde{g}_{\varepsilon,f}|_{(\varepsilon,T)} \text{ in } C((\varepsilon,T);\widetilde{U}), \text{ as } N\to\infty. \tag{7.39}$$

From (7.39) and (7.32), it follows that

$$B_2^*S_2^*(T-\cdot)w_N \to \widetilde{g}_{\varepsilon,f}|_{(\varepsilon,T)}(T-\cdot) \text{ in } C((0,T-\varepsilon);\widetilde{U}), \text{ as } N\to\infty. \tag{7.40}$$

From (7.30) and (7.40), the desired equality (7.25) follows at once.

Finally, since

$$\int_0^\varepsilon \|f(T-t)\|_U\,\mathrm{d}t = \int_{T-\varepsilon}^T \|f(t)\|_U\,\mathrm{d}t,$$

by (7.35) and (7.34), we can pass to the limit for $N\to\infty$ in (7.33) to see that the above function $\widetilde{g}_{\varepsilon,f}$ satisfies (7.26). This ends the proof.

□

**Proposition 11.** *The condition (H1), with $p_0 = 2$, and the condition (H2) hold for $(A_2,B_2)$.*

*Proof.* From Lemma 2.3, we see that in order to show the condition (H1) (with $p_0=2$) for $(A_2,B_2)$, it suffices to prove the property (iii) in Lemma 2.3 (with $p_2=2$) for $(A_2,B_2)$. To prove the later, we arbitrarily fix $\hat{t}$ and $T$ so that $0<\hat{t}<T<\infty$. Let $f\in Y_T$, which is defined by (1.20) with $(A^*,B^*)$ being replaced by $(A_2^*,B_2^*)$. Then by Lemma 7.5 (where $\varepsilon=T-\hat{t}$), we see that $f$ satisfies (7.25) and (7.26) (with $\varepsilon=T-\hat{t}$) for some continuous and weakly analytic function $\widetilde{g}_{\varepsilon,f}: S_{\varepsilon,\theta_0}\to\widetilde{U}$ with some $\theta_0\in(0,\frac{\pi}{2})$. By (7.25), one can easily check that

$$\begin{aligned}\|\widetilde{g}_{\varepsilon,f}(\cdot)\|_{L^\infty(S_{\varepsilon,\theta_0};\widetilde{U})} &\geq \|\widetilde{g}_{\varepsilon,f}|_{(\varepsilon,T)}(\cdot)\|_{L^\infty(\varepsilon,T;\widetilde{U})} \geq \|\widetilde{g}_{\varepsilon,f}|_{(\varepsilon,T)}(T-\cdot)\|_{L^\infty(0,\hat{t};\widetilde{U})}\\ &= \|f(\cdot)\|_{L^\infty(0,\hat{t};U)} \geq \hat{t}^{-1/2}\|f(\cdot)\|_{L^2(0,\hat{t};U)}.\end{aligned}$$

This, along with (7.26) (where $\varepsilon = T - \hat{t}$), yields that

$$\|f\|_{L^2(0,\hat{t};U)} \le \hat{t}^{1/2} C_1(\theta_0, \varepsilon, \varepsilon)\|f\|_{L^1(\hat{t},T;U)} =: C(T, \hat{t}, \theta_0)\|f\|_{L^1(\hat{t},T;U)},$$

which leads to the property (iii) in Lemma 2.3 (with $p_2 = 2$) for $(A_2, B_2)$. Hence, (H1) with $p_0 = 2$ holds for $(A_2, B_2)$.

We next show that (H2) holds for $(A_2, B_2)$. Arbitrarily fix $T \in (0, \infty)$. Assume that there is $f \in Y_T$ and a subset $E \subset (0, T)$ with a positive measure so that

$$f = 0 \text{ over } E. \tag{7.41}$$

We will show that

$$f = 0 \text{ over } (0, T). \tag{7.42}$$

In fact, since $|E| > 0$, we can arbitrarily take $\varepsilon \in (0, |E|)$. It is clear that

$$|E \cap (0, T - \varepsilon)| \ge |E| - \varepsilon > 0. \tag{7.43}$$

Since $f \in Y_T$, by Lemma 7.5, we see that $f$ satisfies (7.25) and (7.26) for some continuous and weakly analytic function $\widetilde{g}_{\varepsilon,f} : S_{\varepsilon,\theta_0} \to \widetilde{U}$ with some $\theta_0 \in (0, \frac{\pi}{2})$. Then by (7.25) and the weak analyticity of $\widetilde{g}_{\varepsilon,f}$, we find that for each $v \in U$, the function $t \to \langle f(t), v\rangle_U$ is real analytic on $(0, T - \varepsilon)$. This, along with (7.41) and (7.43), yields that for each $v \in U$,

$$\langle f(t), v\rangle_U = 0 \text{ for each } t \in (0, T - \varepsilon).$$

Sending $\varepsilon \to 0$ in the above leads to (7.42). Hence, (H2) holds for $(A_2, B_2)$. This ends the proof. □

To get the BBP decompositions for $(TP)_2^{M,y_0}$ and $(NP)_2^{T,y_0}$, we also need the following lemma:

**Lemma 7.6.** *Let functions $T^0(\cdot)$ and $T^1(\cdot)$ be given respectively by (1.17) and (1.18) where $(A, B) = (A_2, B_2)$. Then the following conclusions are true:*
*(i) For each $y_0 \in X \setminus \{0\}$, $T^1(y_0) = \infty$.*
*(ii) If $y_0 \in X \setminus \{0\}$ satisfies that $T^0(y_0) < \infty$, then $N_2(T^1(y_0), y_0) = 0$.*

*Proof.* (i) By contradiction, suppose that $T^1(y_0) < \infty$ for some $y_0 \in X \setminus \{0\}$. Then from (1.18), we see that

$$S_2(T)y_0 = 0 \text{ for each } T \in \big(T^1(y_0), \infty\big). \tag{7.44}$$

Arbitrarily fix a $w_0 \in X$. Then we see from (7.44) that

$$\langle S_2(T)y_0, w_0\rangle_X = 0 \text{ for each } T \in \big(T^1(y_0), \infty\big). \tag{7.45}$$

Since $\{\psi_j\}_{j\ge1}$ is a biorthogonal sequence of the Riesz basis $\{\phi_j\}_{j\ge1}$ in $X$, we can write $y_0$ and $w_0$ in the following manner:

$$y_0 = \sum_{i=1}^{\infty} y_{0,i}\phi_i \text{ and } w = \sum_{j=1}^{\infty} w_{0,j}\psi_j. \tag{7.46}$$

It is clear that $\sum_{i=1}^{\infty} y_{0,i}^2 < \infty$ and $\sum_{j=1}^{\infty} w_{0,j}^2 < \infty$. These, along with the Cauchy-Schwarz inequality, yield that

$$\sum_{k=1}^{\infty} |y_{0,k}||w_{0,k}| \le \Big(\sum_{k=1}^{\infty} y_{0,k}^2\Big)^{1/2}\Big(\sum_{k=1}^{\infty} w_{0,k}^2\Big)^{1/2} < \infty. \tag{7.47}$$

Meanwhile, from (7.46) and (7.16), it follows that

$$\langle S_2(t)y_0, w_0\rangle_X = \sum_{k=1}^{\infty} e^{-\lambda_k t} y_{0,k} w_{0,k} \text{ for all } t \in \mathbb{R}^+. \tag{7.48}$$

Since $\lambda_k > 0$ for all $k \geq 1$, and because the function $z \to \sum_{k=1}^{N} e^{-\lambda_k z} y_{0,k} w_{0,k}$ ($N \in \mathbb{N}^+$) is analytic over $\mathbb{C}^+$, it follows from (7.48) and (7.47) that the function $t \to \langle S_2(t)y_0, w\rangle_X$ is real analytic over $(0,\infty)$. This, along with (7.45), yields that $\langle S_2(T)y_0, w_0\rangle_X = 0$ for each $T \in (0,\infty)$. Because $w_0$ was arbitrarily taken from $X$, we conclude from the above that $S_2(T)y_0 = 0$ for each $T \in (0,\infty)$. This implies that $y_0 = \lim_{T\to 0^+} S_2(T)y_0 = 0$, which contradicts the assumption that $y_0 \in X \setminus \{0\}$. Hence, $T^1(y_0) = \infty$.

(ii) Suppose that $y_0 \in X \setminus \{0\}$ satisfy that $T^0(y_0) < \infty$. Arbitrarily fix a $\hat{t} \in \big(T^0(y_0), \infty\big)$. Then it follows from Corollary 2 that

$$N_2(\hat{t}, y_0) = \sup_{w\in D(A_2^*), B_2^* S_2^*(\hat{t}-\cdot)w\neq 0} \frac{\langle S_2(\hat{t})y_0, w\rangle_X}{\|B_2^* S_2^*(\hat{t}-\cdot)w\|_{L^1(0,\hat{t};U)}} < \infty. \tag{7.49}$$

Write

$$y_0 = \sum_{j\geq 1} y_{0,j}\phi_k \text{ for some } \{y_{0,j}\} \subset \mathbb{R}. \tag{7.50}$$

Arbitrarily fix such a $w \in D(A_2^*)$ that

$$w = \sum_{j=1}^{N} w_j \psi_k \text{ for some } \{w_j\} \subset \mathbb{R} \text{ and } N \in \mathbb{N}^+. \tag{7.51}$$

The rest of the proof is organized by three steps.

*Step 1. To show that there are positive constants $C_1$ and $C_2$ so that for each $s \in (2\hat{t}, \infty)$,*

$$|\langle S_2(s)y_0, w\rangle_X| \leq C_1 e^{-C_2 s} \int_{\hat{t}}^{2\hat{t}} |\langle S_2(t)y_0, w\rangle_X| \,\mathrm{d}t \tag{7.52}$$

Observe from (7.50), (7.51) and (7.16) that

$$\langle S_2(t)y_0, w\rangle_X = \sum_{j=1}^{N} y_{0,j} w_j e^{-\lambda_j t} \text{ for each } t \in \mathbb{R}^+. \tag{7.53}$$

Define a function $g_1$ over $\mathbb{C}_+$ in the following manner: $g_1(z) := \sum_{j=1}^{N} y_{0,j} w_j e^{-\lambda_j z}$ for each $z \in \mathbb{C}_+$. Then by (7.20) and (7.53), we find that

$$g_1(\cdot + \hat{t})|_{\mathbb{C}_+} \in \mathcal{P}; \text{ and } g_1(\cdot) = \langle S_2(\cdot)y_0, w\rangle_X \text{ over } \mathbb{R}^+. \tag{7.54}$$

These, together with (7.22), yield that there exist two positive constants $C_1$ and $C_2$, independent of $w$, so that for each $s \in (2\hat{t}, \infty)$,

$$\begin{aligned} |\langle S_2(s)y_0, w\rangle_X| &= \big|g_1\big((s-\hat{t}) + \hat{t}\big)\big| \leq C_1 e^{-C_2(s-\hat{t})} \int_0^{\hat{t}} |g_1(t+\hat{t})| \,\mathrm{d}t \\ &= C_1 e^{-C_2(s-\hat{t})} \int_{\hat{t}}^{2\hat{t}} |g_1(t)| \,\mathrm{d}t = C_1 e^{-C_2(s-\hat{t})} \int_{\hat{t}}^{2\hat{t}} |\langle S_2(t)y_0, w\rangle_X| \,\mathrm{d}t, \end{aligned}$$

which implies (7.52).

*Step 2. To show that There are positive constants $C_1'$ and $C_2'$ so that for each $s \in (\hat{t}, \infty)$,*

$$\int_0^s \|B_2^* S_2^*(t) w\|_U \,\mathrm{d}t \geq (1 - C_1' e^{-C_2' s}) \int_0^\infty \|B_2^* S_2^*(t) w\|_U \,\mathrm{d}t \tag{7.55}$$

From (7.51) and (7.17), we find that

$$B_2^* S_2^*(t) w = \sum_{j=1}^N w_j e^{-\lambda_j t} B_2^* \psi_j \quad \text{for each} \quad t \in \mathbb{R}^+. \tag{7.56}$$

Write $\widetilde{U}$ for the complexification of $U$. Define a function $g_2 : \mathbb{C}^+ \to \widetilde{U}$ in the following manner:

$$g_2(z) := \sum_{j=1}^N w_j e^{-\lambda_j z} B_2^* \psi_j, \;\; z \in \mathbb{C}_+,$$

This, along with (7.23) and (7.56), yields that

$$g_2(\cdot) \in \mathcal{P}_{\widetilde{U}} \;\; \text{and} \;\; g_2(t) = B_2^* S_2^*(t) w \in U \;\; \text{for each} \;\; t \in \mathbb{R}^+.$$

These, together with Lemma 7.4, yield that there exist two positive constants $C_1'$ and $C_2'$, independent of $w$, so that for each $t \in (\hat{t}, \infty)$,

$$\begin{aligned} \|B_2^* S_2^*(t) w\|_U &= \|g_2(t)\|_U \leq C_1' e^{-C_2' t} \|g_2(\cdot)\|_{L^1(0,\hat{t};U)} = C_1' e^{-C_2' t} \|B_2^* S_2^*(\cdot) w\|_{L^1(0,\hat{t};U)} \\ &\leq C_1' e^{-C_2' t} \|B_2^* S_2^*(\cdot) w\|_{L^1(\mathbb{R}^+;U)}. \end{aligned}$$

Thus, we find that for each $s \in (\hat{t}, \infty)$,

$$\begin{aligned} \int_0^s \|B_2^* S_2^*(t) w\|_U \,\mathrm{d}t &= \int_0^\infty \|B_2^* S_2^*(t) w\|_U \,\mathrm{d}t - \int_s^\infty \|B_2^* S_2^*(t) w\|_U \,\mathrm{d}t \\ &\geq \|B_2^* S_2^*(\cdot) w\|_{L^1(\mathbb{R}^+;U)} - \int_s^\infty \big( C_1' e^{-C_2' t} \|B_2^* S_2^*(\cdot) w\|_{L^1(\mathbb{R}^+;U)} \big) \,\mathrm{d}t \\ &\geq \big(1 - C_1' e^{-C_2' s} / C_2'\big) \|B_2^* S_2^*(\cdot) w\|_{L^1(\mathbb{R}^+;U)}, \end{aligned}$$

which implies (7.55).

*Step 3. To show that $N_2(T^1(y_0), y_0) = 0$*

We first claim that for each $t \in (\hat{t}, 2\hat{t})$,

$$|\langle S_2(t) y_0, w \rangle_X| \leq N_2(t, y_0) \|B_2^* S_2^*(t - \cdot) w\|_{L^1(0,t;U)}. \tag{7.57}$$

To this end, fix a $t \in (\hat{t}, 2\hat{t})$. There are only two possibilities on $B_2^* S_2^*(t - \cdot) w$: either $B_2^* S_2^*(t - \cdot) w \neq 0$ in $L^1(0, t; U)$ or $B_2^* S_2^*(t - \cdot) w = 0$ in $L^1(0, t; U)$.

In first case, since $\hat{t} > T^0(y_0)$, we see from Corollary 2 that (7.57) holds. In the second case, it follows from (ii) of Lemma 3.2 and (7.49) that

$$N_2(t, y_0) \leq N_2(\hat{t}, y_0) < \infty.$$

So $(NP)_2^{t,y_0}$ has at least one admissible control. Then there exists a control $u \in L^\infty(0, t; U)$ so that $\hat{y}(t; y_0, u) = 0$. Thus, from (1.13), we obtain that

$$\langle S_2(t) y_0, w \rangle_X = -\int_0^t \langle u(\tau), B_2^* S_2^*(t - \tau) w \rangle_U \,\mathrm{d}\tau = 0,$$

which implies (7.57) in the case that $B_2^* S_2^*(t - \cdot) w = 0$. So (7.57) is proved.

Next, by (7.52), (7.57) and (7.55), we find that for each large enough $s \in (2\hat{t}, \infty)$,

$$\begin{aligned}\langle S_2(s)y_0, w\rangle_X &\leq C_1 e^{-C_2 s}\int_{\hat{t}}^{2\hat{t}} |\langle S_2(t)y_0, w\rangle_X| \,\mathrm{d}t \\ &\leq C_1 e^{-C_2 s}\int_{\hat{t}}^{2\hat{t}} N_2(t,y_0)\|B_2^* S_2^*(t-\cdot)w\|_{L^1(0,t;U)}\,\mathrm{d}t \\ &\leq C_1 e^{-C_2 s}\|B_2^* S_2^*(\cdot)w\|_{L^1(\mathbb{R}^+;U)}\int_{\hat{t}}^{2\hat{t}} N_2(t,y_0)\,\mathrm{d}t \\ &\leq \Big(\frac{C_1 e^{-C_2 s}}{1-C_1' e^{-C_2' s}}\int_{\hat{t}}^{2\hat{t}} N_2(t,y_0)\,\mathrm{d}t\Big)\|B_2^* S_2^*(s-\cdot)w\|_{L^1(0,s;U)}.\end{aligned}$$

Since $w$ was arbitrarily taken as in (7.51), the above, along with Corollary 2 and (ii) of Lemma 3.2, yields that for each large enough $s \in (2\hat{t}, \infty)$,

$$N_2(s,y_0) \leq \frac{C_1 e^{-C_2 s}}{1-C_1' e^{-C_2' s}}\int_{\hat{t}}^{2\hat{t}} N_2(t,y_0)\,\mathrm{d}t \leq \frac{C_1 e^{-C_2 s}}{1-C_1' e^{-C_2' s}}\big(\hat{t}N_2(\hat{t},y_0)\big),$$

which, together with (7.49) and the first equality in (1.19), implies that $N_2(\infty, y_0) = 0$. This, along with (vi) of Lemma 3.3, yields that the conclusion (ii) is true.

In summary, we end the proof of Lemma 7.6.

□

The BBP decompositions for $(A_2, B_2)$ are presented in the following Theorem 7.7.

**Theorem 7.7.** *Let $\mathcal{W}$, $\mathcal{W}_{2,j}$ $(j = 1, 2, 3)$, $\mathcal{W}_{3,j}$ $(j = 1, 2, 4)$, $\mathcal{V}$, $\mathcal{V}_{2,j}$ $(j = 2, 3, 4)$ and $\mathcal{V}_{3,j}$ $(j = 2, 3)$ be respectively given by (1.23), (1.27), (1.29), (1.24), (1.32) and (1.34) where $(A, B) = (A_2, B_2)$. Then the following conclusions are valid:*
*(i) The set $\mathcal{W}$ is the disjoint union of the above mentioned subsets $\mathcal{W}_{i,j}$, and $\mathcal{V}$ is the disjoint union of the above mentioned subsets $\mathcal{V}_{i,j}$.*
*(ii) For each $(T, y_0) \in \mathcal{W}_{2,1} \cup \mathcal{W}_{3,1} \cup \mathcal{W}_{3,4}$, $(NP)_2^{T,y_0}$ has no admissible control and does not hold the bang-bang property; For each $(T, y_0) \in \mathcal{W}_{2,3} \cup \mathcal{W}_{3,2}$, $(NP)_2^{T,y_0}$ has the bang-bang property and the null control is not a minimal norm control to this problem; For each $(T, y_0) \in \mathcal{W}_{2,2}$, $(NP)_2^{T,y_0}$ has at least one minimal norm control.*
*(iii) For each $(M, y_0) \in \mathcal{V}_{3,3}$, $(TP)_2^{M,y_0}$ has no admissible control and does not hold the bang-bang property; For each $(M, y_0) \in \mathcal{V}_{2,2} \cup \mathcal{V}_{3,2}$, $(TP)_2^{M,y_0}$ has the bang-bang property; For each $(M, y_0) \in \mathcal{V}_{2,4}$, $(TP)_2^{M,y_0}$ has infinitely many different minimal time controls (not including the null control), and does not hold the bang-bang property; For each $(M, y_0) \in \mathcal{V}_{2,3}$, $(TP)_2^{M,y_0}$ has at least one minimal time control.*

*Proof.* By Proposition 11, we see that (H1) and (H2) hold for $(A_2, B_2)$. Thus, all conclusions in Theorem 1.1 and Theorem 1.2 are true. From these conclusions, we see that to prove this theorem, it suffices to show that

$$\mathcal{W}_{1,1} \cup \mathcal{W}_{1,2} \cup \mathcal{W}_{2,4} \cup \mathcal{W}_{3,3} = \emptyset;\ \mathcal{V}_1 \cup \mathcal{V}_{2,1} \cup \mathcal{V}_{3,1} = \emptyset. \tag{7.58}$$

Here, $\mathcal{W}_{1,j}$ $(j = 1, 2)$, $\mathcal{W}_{2,4}$, $\mathcal{W}_{3,3}$, $\mathcal{V}_1$, $\mathcal{V}_{2,1}$ and $\mathcal{V}_{3,1}$ are respectively given by (1.25), (1.27), (1.29), (1.31), (1.32) and (1.34), where $(A, B) = (A_2, B_2)$.

To show (7.58), we use Lemma 7.6 to get that

$$T^1(y_0) = \infty \ \text{ and } \ N_2(T^1(y_0), y_0) = 0 \ \text{ for all } \ y_0 \in X \setminus \{0\}. \tag{7.59}$$

By the first equality in (7.59) and (iv) of Lemma 3.4, we deduce that

$$N_2(T^0(y_0), y_0) > 0 \quad \text{for all} \quad y_0 \in X \setminus \{0\}. \tag{7.60}$$

We now show the first equality in (7.58). On one hand, by the definitions of $\mathcal{W}_{1,j}$ ($j = 1, 2$) (see (1.25)), we find from (7.60) that $\mathcal{W}_{1,1} \cup \mathcal{W}_{1,2}$ is empty. On the other hand, by contradiction, suppose that $\mathcal{W}_{2,4} \cup \mathcal{W}_{3,3}$ were not empty. Then there would be a pair $(\hat{T}, \hat{y}_0) \in \mathcal{W}_{2,4} \cup \mathcal{W}_{3,3}$. Hence, by the definitions of $\mathcal{W}_{2,4} \cup \mathcal{W}_{3,3}$ (see (1.27) and (1.29)), it follows that

$$T^1(\hat{y}_0) \leq \hat{T} < \infty,$$

which contradicts the first equality in (7.59). So $\mathcal{W}_{2,4} \cup \mathcal{W}_{3,3}$ is empty. Thus, we have proved the first equality in (7.58).

Finally, we prove the second equality in (7.58). On one hand, by the definitions of $\mathcal{V}_1$ (see (1.31)), we find from (7.60) that $\mathcal{V}_1$ is empty. On the other hand, by contradiction, suppose that $\mathcal{V}_{2,1} \cup \mathcal{V}_{3,1}$ were not empty. Then there would be a pair $(\hat{M}, \hat{y}_0) \in \mathcal{V}_{2,1} \cup \mathcal{V}_{3,1}$. So by the definitions of $\mathcal{V}_{2,1} \cup \mathcal{V}_{3,1}$ (see (1.32) and (1.34)), it follows that

$$0 < \hat{M} \leq N_2(T^1(\hat{y}_0), \hat{y}_0),$$

which contradicts the second equality in (7.59). Therefore, $\mathcal{V}_{2,1} \cup \mathcal{V}_{3,1}$ is empty. Thus, we have proved the second equality in (7.58).

In summary, we end the proof of this theorem. □

We end this subsection with presenting such phenomenon that *for some pairs* $(A_2, B_2)$*, the corresponding function* $T^0(\cdot)$ *(given by (1.17) with* $(A, B)$ *being replaced by* $(A_2, B_2)$*) has the following property:* $T^0(y_0) \in (0, \infty)$ *for some* $y_0 \in X$. To see it, some preliminaries are needed. First we notice that the operator $A_2$ depends on the choices of $\{\phi_j\}_{j\geq 1}$, $\{\psi_j\}_{j\geq 1}$ and $\Lambda$; the operator $B_2$ can be arbitrarily taken from $\mathcal{L}(U, X_{-1}) \setminus \{0\}$. For each pair $(A_2, B_2)$, we define

$$T_2(A_2, B_2) := \inf\{T \in (0, \infty) \,:\, (A_2, B_2) \text{ has } L^2\text{-null controllability at } T\}. \tag{7.61}$$

(By the $L^2$-null controllability at $T$ for $(A_2, B_2)$, we mean that for each $y_0 \in X$, there is a control $v \in L^2(0, T; U)$ so that $\hat{y}_2(T; y_0, v) = 0$.) Sometimes, we will use $T_2$ to denote $T_2(A_2, B_2)$, if there is no risk causing any confusion. It is proved in [15] and [17] that $T_2 \in (0, \infty)$ for some pairs $(A_2, B_2)$. One such example (taken from [17]) is as follows:

**Example 7.8.** Consider the following controlled system

$$\begin{cases} \partial_t y - \partial_{xx} y = \delta_{x_0} v & \text{in} \quad (0, \pi) \times (0, \infty), \\ y(0, \cdot) = y(\pi, \cdot) = 0 & \text{in} \quad (0, \infty), \\ y(\cdot, 0) \in L^2(0, \pi). \end{cases}$$

One can directly check that this example can be put into the framework $(A_2, B_2)$. According to Corollary 6.4 and Theorem 6.5 in [17], there are many $x_0 \in (0, \pi)$ so that the corresponding $T_2 \in (0, \infty)$.

In the current paper, controls are taken from $L^\infty$ spaces. Thus, we define for each pair $(A_2, B_2)$,

$$T_\infty(A_2, B_2) := \inf\{T \in (0, \infty) \,:\, (A_2, B_2) \text{ has } L^\infty\text{-null controllability at } T\} \tag{7.62}$$

Also, we simply use $T_\infty$ to denote $T_\infty(A_2, B_2)$, if there is no risk to cause any confusion.

**Lemma 7.9.** *For each pair $(A_2, B_2)$, the corresponding $T_2$ and $T_\infty$ (defined by (7.61) and (7.62), respectively) are the same.*

*Proof.* It suffices to show that

$$T_\infty \le T_2. \tag{7.63}$$

By contradiction, suppose that it was not true. Then there would be two numbers $\hat{t}$ and $\hat{t}'$ so that

$$T_2 < \hat{t} < \hat{t}' < T_\infty. \tag{7.64}$$

Arbitrarily fix a $y_0 \in X$. According to the definition of $T_2$, there exists a control $u \in L^2(0, \hat{t}; U)$ so that

$$\hat{y}_2(\hat{t}; y_0, u) = 0. \tag{7.65}$$

Write $\widetilde{u}$ for the zero extension of $u$ over $(0, \hat{t}')$. According to Proposition 11, the pair $(A_2, B_2)$ satisfies the condition (H1) with $p_0 = 2$. Thus, we apply (H1), where $p_0 = 2$ and $T = \hat{t}'$ and $t = \hat{t}$, to find a control $v_u \in L^\infty(0, \hat{t}'; U)$ so that $\hat{y}_2(\hat{t}'; 0, \widetilde{u}) = \hat{y}_2(\hat{t}'; 0, v_u)$, which implies that

$$\hat{y}_2(\hat{t}'; y_0, \widetilde{u}) = \hat{y}_2(\hat{t}'; y_0, 0) + \hat{y}_2(\hat{t}'; 0, \widetilde{u}) = \hat{y}_2(\hat{t}'; y_0, v_u).$$

This, along with (7.65), yields that

$$\hat{y}_2(\hat{t}'; y_0, v_u) = S_2(\hat{t}' - \hat{t})\hat{y}_2(\hat{t}; y_0, u) = 0.$$

Since $y_0$ was arbitrarily taken from $X$, the above implies that the pair $(A_2, B_2)$ has $L^\infty$-null controllability at time $\hat{t}'$. By this and the definition of $T_\infty$, we deduce that $T_\infty \le \hat{t}'$, which contradicts (7.64). So (7.63) holds. We end the proof of this lemma. $\square$

**Remark 14.** There are systems (under the framework $(A_2, B_2)$) so that $0 < T_\infty < \infty$ (see Example 7.8 and Lemma 7.9). With the aid of this, we can prove that for some pair $(A_2, B_2)$, the corresponding function $T^0(\cdot)$, defined by (1.17), satisfies that $T^0(y_0) \in (0, \infty)$ for some $y_0 \in X$.

Here is the argument: Suppose that for some $(A_2, B_2)$,

$$0 < T_\infty(A_2, B_2) = T_\infty < \infty. \tag{7.66}$$

On one hand, by the first inequality in (7.66) and the definition of $T_\infty$, we can find $T \in (0, T_\infty)$ so that the pair $(A_2, B_2)$ is not $L^\infty$-null controllable. Thus there is $\hat{y}_0 \in X$ so that for any $v \in L^\infty(0, T; U)$, $\hat{y}_2(T; \hat{y}_0, v) \neq 0$. Then by the definition of $T^0(\hat{y}_0)$ (see (1.17)), we see that $T \le T^0(\hat{y}_0)$, which leads to that $T^0(\hat{y}_0) > 0$.

On the other hand, by the last inequality in (7.66) and the definition of $T_\infty$, we can find $\hat{T} \in (T_\infty, \infty)$ so that the pair $(A_2, B_2)$ is the $L^\infty$-null controllable at $\hat{T}$. Thus, for each $y_0 \in X$ there is a control $v \in L^\infty(0, \hat{T}; U)$ so that $\hat{y}(\hat{T}; y_0, v) = 0$. This, along with the definition of $T^0(y_0)$ (see (1.17)), yields that $T_0(y_0) \le \hat{T} < \infty$ for all $y_0 \in X$.

In summary, we conclude that $T^0(\hat{y}_0) \in (0, \infty)$.

## 8. **Appendix.**

8.1. **Appendix A.** In Appendix A, we will use the Kalman controllability decomposition to prove the following Proposition:

**Proposition 12.** *For each pair of matrices $(A, B)$ in $\mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$ (with $n, m \geq 1$), the corresponding decompositions (P1) and (P2) (given by (1.6) and (1.9), respectively) hold.*

*Proof.* Arbitrarily fix $(A, B) \in \mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$. Let $\mathcal{R}$ be given by (1.8). Since $B \neq 0$, we have that

$$p := \dim \mathcal{R} > 0 \ \text{ and } \ \mathcal{R} \setminus \{0\} \neq \emptyset. \tag{8.1}$$

We now recall the Kalman controllability decomposition of $(A, B)$ (see, for instance, Lemma 3.3.3 and Lemma 3.3.4 in [38]): There exist $K \in GL(n)$, $A_1 \in \mathbb{R}^{p\times p}$, $A_2 \in \mathbb{R}^{p\times(n-p)}$, $A_3 \in \mathbb{R}^{(n-p)\times(n-p)}$ and $B_1 \in \mathbb{R}^{p\times m}$ so that

$$K^{-1}AK = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \ \text{ and } \ K^{-1}B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \tag{8.2}$$

where the pair $(A_1, B_1)$ is controllable, which is equivalent to

$$\text{rank}\ (B_1, A_1B_1, \cdots, A_1^pB_1) = p. \tag{8.3}$$

Notice that when $p = n$, the decomposition is trivial. In this case, $A_1 = A$, $B_1 = B$ and $A_2$ and $A_3$ are not there.

We organize the proof by two steps as follows:

*Step 1. The proof of (P2)*

For each $z_0 \in \mathbb{R}^n \setminus \{0\}$ and $T \in (0, \infty)$, we define an affiliated minimal norm control problem:

$$(\mathcal{NP})_K^{T,z_0} \qquad \mathcal{N}_K(T, z_0) := \inf\{\|v\|_{L^\infty(0,T;\mathbb{R}^m)} \ : \ \hat{z}(T; z_0, v) = 0\}, \tag{8.4}$$

where $\hat{z}(\cdot; z_0, v)$ is the solution to the equation:

$$\begin{cases} z'(t) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} z(t) + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} v(t), \ 0 < t \leq T, \\ z(0) = z_0. \end{cases} \tag{8.5}$$

By the invertibility of $K$, one can easily show that when $z_0 = K^{-1}y_0$, the problems $(\mathcal{NP})^{T,y_0}$ and $(\mathcal{NP})_K^{T,z_0}$ (given by (1.3) and (8.4), respectively) are equivalent, i.e., either they have the same minimal norm controls or both of them have no admissible control. From (1.8), (8.2) and (8.3), it follows that

$$\mathcal{R} = \text{span}\,(B, AB, \cdots, A^nB) = \text{span}\, K \begin{pmatrix} B_1, A_1B_1, \cdots, A_1^nB_1 \\ 0 \end{pmatrix} = K(\widetilde{\mathbb{R}}^p), \tag{8.6}$$

where the span of a matrix denotes the subspace generated by all columns of the matrix, and $\widetilde{\mathbb{R}}^p$ is the following subspace:

$$\widetilde{\mathbb{R}}^p := \big\{(z_1, z_2, \cdots, z_n) \in \mathbb{R}^n \ : \ z_{p+1} = \cdots = z_n = 0\big\}. \tag{8.7}$$

By (8.1), we see that $\widetilde{\mathbb{R}}^p \setminus \{0\} \neq \emptyset$. From the equivalence of $(\mathcal{NP})^{T,y_0}$ and $(\mathcal{NP})_K^{T,z_0}$ (with $z_0 = K^{-1}y_0$), (8.6) and (1.9), we see that to prove (P2), it suffices to show the following BBP decomposition for $(\mathcal{NP})_K^{T,z_0}$:

**(Q2)**
- When $(T, z_0) \in (0, \infty) \times (\widetilde{\mathbb{R}}^p \setminus \{0\})$, $(\mathcal{NP})_K^{T,z_0}$ has the bang-bang property.
- When $(T, z_0) \in (0, \infty) \times (\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^p)$, $(\mathcal{NP})_K^{T,z_0}$ has no admissible control.

To show the first conclusion in (Q2), we let

$$(T, z_0) \in (0, \infty) \times (\widetilde{\mathbb{R}}^p \setminus \{0\}).$$

Write $z_{0,1}$ for the first $p$ components of $z_0$. Since $z_0 \in \widetilde{\mathbb{R}}^p$, it follows that $z_0 = (z_{0,1}, 0)$, if $p < n$; and $z_0 = z_{0,1}$, if $p = n$. Thus, for each $v \in L^\infty(0, T; \mathbb{R}^m)$, the solution $\hat{z}(\cdot; z_0, v)$ of the equation (8.5) satisfies that

$$\hat{z}(t; z_0, v) = \begin{cases} \big(\hat{z}_1(t; z_{0,1}, v), 0\big) \text{ for all } t \in [0, T], \text{ when } p < n, \\ \hat{z}_1(t; z_{0,1}, v) \text{ for all } t \in [0, T], \text{ when } p = n, \end{cases}$$

where $\hat{z}_1(\cdot; z_{0,1}, v)$ solves the following equation:

$$z_1'(t) = A_1 z_1(t) + B_1 v(t),\ 0 < t \le T;\ \ z_1(0) = z_{0,1}.$$

This, along with the controllability of $(A_1, B_1)$ (which follows from (8.3), see, for instance, Theorem 3 on Page 89 in [38]), indicates that $(\mathcal{NP})_K^{T,z_0}$ has an admissible control. Then by a standard way (see for instance [7, Lemma 1.1]), we can deduce that $(\mathcal{NP})_K^{T,z_0}$ has a minimal norm control.

Meanwhile, according to the Pontryagin maximum principle for $(\mathcal{NP})_K^{T,z_0}$ (see, for instance, [6, Theorem 1.1.1]), there is $\eta_1$ in $\mathbb{R}^p \setminus \{0\}$ so that each minimal norm control $v^*$ to $(\mathcal{NP})_K^{T,z_0}$ verifies that for a.e. $t \in \big(0, T\big)$,

$$\big\langle v^*(t), B_1^* e^{A_1^*(T-t)} \eta_1 \big\rangle_{\mathbb{R}^m} = \max_{\|w\|_{\mathbb{R}^m} \le \mathcal{N}_K(T, z_0)} \big\langle w, B_1^* e^{A_1^*(T-t)} \eta_1 \big\rangle_{\mathbb{R}^m}, \tag{8.8}$$

where $\mathcal{N}_K(T, z_0)$ is given by (8.4). Besides, since $\eta_1 \neq 0$ and the function $t \to B_1^* e^{A_1^*(T-t)}$ is real analytic over $\mathbb{R}$, it follows from (8.3) that the following set

$$\big\{t \in (0, T)\ :\ B_1^* e^{A_1^*(T-t)} \eta_1 = 0\big\}$$

has measure zero. From this and (8.8), we see that $(\mathcal{NP})_K^{T,z_0}$ has the bang-bang property. So the first conclusion in (Q2) is true.

To verify the second conclusion in (Q2), we first notice that when $p = n$, $\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^p$ is empty. Thus, we can assume, without loss of generality, that $p < n$. Arbitrarily fix $(T, z_0) \in (0, \infty) \times (\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^p)$. Then from the equation (8.5), we see that any control $v$ has no influence to the last $(n-p)$ components of the solution $\hat{z}(\cdot; z_0, v)$. Thus, for each control $v$ in $L^\infty(0, T; \mathbb{R}^m)$, the solution $\hat{z}(\cdot; z_0, v)$ of the equation (8.5) satisfies that $\hat{z}(T; z_0, v) \neq 0$. Hence, $(\mathcal{NP})_K^{T,z_0}$ has no admissible control. This proves the second conclusion in (Q2). Hence, the decomposition (Q2) holds. Consequently, (P2) is true.

*Step 2. The proof of (P1)*

For each $z_0 \in \mathbb{R}^n \setminus \{0\}$ and $M \in (0, \infty)$, we define an affiliated minimal time control problem:

$$(\mathcal{TP})_K^{M,z_0} \qquad \mathcal{T}_K(M, z_0) := \{\hat{t} > 0\ :\ \exists\, u \in \mathbb{U}^M \text{ s.t. } z(\hat{t}; z_0, u) = 0\}, \tag{8.9}$$

where $\mathbb{U}^M$ is given by (1.2), and $z(\cdot; z_0, u)$ is the solution to the equation:

$$\begin{cases} z'(t) = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} z(t) + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u(t),\ t > 0, \\ z(0) = z_0. \end{cases} \tag{8.10}$$

Two observations are given in order: First, by the invertibility of $K$, one can easily see that the problems $(\mathcal{TP})^{M,y_0}$ and $(\mathcal{TP})_K^{M,z_0}$ (given by (1.1) and (8.9), respectively) are equivalent, i.e., either they have the same minimal time controls or both of them have no admissible control. Second, from (1.3), one can easily check that when $y_0 \in \mathcal{R} \setminus \{0\}$, the function $\mathcal{N}(\cdot, y_0)$ has the properties: it is decreasing over $(0, \infty)$; for each $T \in (0, \infty)$, $\mathcal{N}(T, y_0) \in (0, \infty)$. Hence, for each $y_0 \in \mathcal{R} \setminus \{0\}$, $\lim_{T\to\infty} \mathcal{N}(T, y_0)$ exists and is a finite and non-negative number. Meanwhile, by the equivalence between $(\mathcal{NP})^{T,y_0}$ and $(\mathcal{NP})^{T,z_0}_K$ (with $z_0 = K^{-1}y_0$), it follows that for each $T > 0$, $\mathcal{N}(T, y_0) = \mathcal{N}_K(T, z_0)$. These imply that

$$\lim_{T\to\infty} \mathcal{N}(T, y_0) = \lim_{T\to\infty} \mathcal{N}_K(T, z_0) < \infty \text{ when } z_0 = K^{-1}y_0 \text{ and } y_0 \in \mathcal{R} \setminus \{0\} \tag{8.11}$$

From the above-mentioned two observations, as well as (8.6) and (1.6), we find that to prove (P1), it suffices to show the following BBP decomposition for $(\mathcal{TP})^{M,z_0}_K$:

**(Q1)**
- For each $(M, z_0) \in \mathcal{D}^K_{bbp}$, $(\mathcal{TP})^{M,z_0}_K$ has the bang-bang property.
- For each $(M, z_0) \in \big((0, \infty) \times (\mathbb{R}^n \setminus \{0\})\big) \setminus \mathcal{D}^K_{bbp}$, $(\mathcal{TP})^{M,z_0}_K$ has no any admissible control .

Here,

$$\mathcal{D}^K_{bbp} := \big\{(M, z_0) \in (0, \infty) \times (\widetilde{\mathbb{R}}^p \setminus \{0\}) \ : \ M > \lim_{T\to\infty} \mathcal{N}_K(T, z_0)\big\}, \tag{8.12}$$

where $\widetilde{\mathbb{R}}^p$ and $\mathcal{N}_K(T, z_0)$ are given by (8.7) and (8.4), respectively. From (8.12), (8.1), (1.7) and (8.11), one can easily check that

$$\mathcal{D}^K_{bbp} \neq \emptyset \quad \text{and} \quad \mathcal{D}_{bbp} \neq \emptyset. \tag{8.13}$$

Before proving the decomposition (Q1), we observe that by the first conclusion in (Q2), we can use the same way used in the proof of [43, Proposition 4.4] to get the following conclusion: When $z_0 \in \widetilde{\mathbb{R}}^p \setminus \{0\}$,

$$(\mathcal{TP})^{M,z_0}_K \text{ has a minimal time control } \iff \infty > M > \lim_{T\to\infty} \mathcal{N}_K(T, z_0). \tag{8.14}$$

To show the first conclusion in (Q1), we let $(M, z_0) \in \mathcal{D}^K_{bbp}$. Then, it follows from (8.14) and (8.12) that $(\mathcal{TP})^{M,z_0}_K$ has at least one minimal time control.

Write $z_{0,1}$ for the first $p$ components of $z_0$. Since $z_0 \in \widetilde{\mathbb{R}}^p$, it follows that $z_0 = (z_{0,1}, 0)$ when $p < n$; while $z_0 = z_{0,1}$ when $p = n$. Then by (8.10), we can easily check that

$$z(t; z_0, v) = \begin{cases} \big(z_1(t; z_{0,1}, u), 0\big) & \text{for all } t \geq 0, \text{ when } p < n, \\ z_1(t; z_{0,1}, u) & \text{for all } t \geq 0, \text{ when } p = n, \end{cases}$$

where $z_1(\cdot; z_{0,1}, u)$ solves the following equation:

$$z_1'(t) = A_1 z_1(t) + B_1 u(t), \ 0 < t < \infty, \ \ z_1(0) = z_{0,1}.$$

From this, we can use the Pontryagin maximum principle for $(\mathcal{TP})^{M,z_0}_K$ (see, for instance, [6, Theorem 1.1.1]) to find $\eta_2 \in \mathbb{R}^p \setminus \{0\}$ so that each minimal time control $u^*$ to $(\mathcal{TP})^{M,z_0}_K$ verifies that for a.e. $t \in \big(0, \mathcal{T}_K(M, z_0)\big)$,

$$\big\langle u^*(t), B_1^* e^{A_1^*(\mathcal{T}_K(M,z_0)-t)}\eta_2\big\rangle_{\mathbb{R}^m} = \max_{\|w\|_{\mathbb{R}^m}\leq M} \big\langle w, B_1^* e^{A_1^*(\mathcal{T}_K(M,z_0)-t)}\eta_2\big\rangle_{\mathbb{R}^m}. \tag{8.15}$$

Meanwhile, since $\eta^* \neq 0$ and the function $t \to B_1^* e^{A_1^*(T-t)}$ is real analytic over $\mathbb{R}$, the set $\big\{t \in \big(0, \mathcal{T}_K(M, z_0)\big) \ : \ B_1^* e^{A_1^*(\mathcal{T}_K(M,z_0)-t)}\eta_2 = 0\big\}$ has measure zero. This, along with (8.15), yields that $(\mathcal{TP})^{M,z_0}_K$ has the bang-bang property. Hence, the first conclusion in (Q1) is true.

To show the second conclusion in (Q1), we let

$$(M, z_0) \in \big((0, \infty) \times (\mathbb{R}^n \setminus \{0\})\big) \setminus \mathcal{D}^K_{bbp}.$$

Then, there are only two possibilities on the pair $(M, z_0)$ as follows: First, $(M, z_0)$ verifies that $z_0 \in \widetilde{\mathbb{R}}^p \setminus \{0\}$ and $0 < M \leq \lim_{T\to\infty} \mathcal{N}_K(T, z_0)$; Second, $(M, z_0) \in (0, \infty) \times (\mathbb{R}^n \setminus \widetilde{\mathbb{R}}^p)$. In the first case, it follows from (8.14) that $(\mathcal{TP})_K^{M,z_0}$ has no admissible control. In the second case, we have that $p < n$ and the last $(n-p)$ components of $z_0$ are not all zero. Then by (8.10), we find that $z(T; z_0, u) \neq 0$ for all $u \in L^\infty(\mathbb{R}^+; \mathbb{R}^m)$ and $T \in (0, \infty)$. This implies that $(\mathcal{TP})_K^{M,z_0}$ has no admissible control. Hence, the second conclusion in (Q1) is also true. So the BBP decomposition (Q1) holds. Consequently, (P1) stands.

In summary, we end the proof of (P1) and (P2), through using the Kalman controllability decomposition.

□

8.2. **Appendix B.** In Appendix B, we will show that each pair of matrices $(A, B)$ in $\mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$ (with $n, m \geq 1$) holds the properties (H1) and (H2).

**Proposition 13.** *Any pair of matrices $(A, B) \in \mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$ (with $n, m \geq 1$) satisfies (H1) (with $p_0 = 2$) and (H2).*

*Proof.* Arbitrarily fix $(A, B) \in \mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$. We organize the proof by two steps.

In Step 1, we show that (H1) (with $p_0 = 2$) holds for the pair $(A, B)$. For this purpose, we will show that $(A, B)$ satisfies the conclusion (iii) of Lemma 2.3 (with $p_2 = 2$). When the later is done, it follows from Lemma 2.3 that (H1) (with $p_0 = 2$) holds for the pair $(A, B)$.

The remainder of this step is to show that (iii) of Lemma 2.3 (with $p_2 = 2$) holds for the pair $(A, B)$. Arbitrarily fix $0 < t < T < \infty$. Define the following two spaces:

$$\mathcal{O}_1 := \{B^* e^{A^*(T-\cdot)} z|_{(0,t)} \in L^2(0, t; \mathbb{R}^m) \ : \ z \in \mathbb{R}^n\}, \ \text{ with the norm } \ \|\cdot\|_{L^2(0,t;\mathbb{R}^m)},$$

and

$$\mathcal{O}_2 := \{B^* e^{A^*(T-\cdot)} z|_{(t,T)} \in L^1(t, T; \mathbb{R}^m) \ : \ z \in \mathbb{R}^n\}, \ \text{ with the norm } \ \|\cdot\|_{L^1(t,T;\mathbb{R}^m)}.$$

It is clear that they are finitely dimensional spaces. Then define a map $\mathcal{F} : \mathcal{O}_2 \to \mathcal{O}_1$ by setting

$$\mathcal{F}\big(B^* e^{A^*(T-\cdot)} z|_{(t,T)}\big) := B^* e^{A^*(T-\cdot)} z|_{(0,t)} \ \text{ for each } \ z \in \mathbb{R}^n. \tag{8.16}$$

By the analyticity of the function $t \mapsto B^* e^{A^* t}$, $t \in \mathbb{R}$, one can easily check that the map $\mathcal{F}$ is well defined. It is clear that $\mathcal{F}$ is linear (from the finitely dimensional space $\mathcal{O}_2$ to the finitely dimensional space $\mathcal{O}_1$). Thus, $\mathcal{F}$ is bounded. Then it follows by (8.16) that there is a positive constant $C(T, t)$ so that

$$\|B^* e^{A^*(T-\cdot)} z\|_{L^2(0,t;\mathbb{R}^m)} \leq C(T, t) \|B^* e^{A^*(T-\cdot)} z\|_{L^1(t,T;\mathbb{R}^m)} \ \text{ for each } \ z \in \mathbb{R}^n.$$

This, along with the definition of $Y_T$ (see (1.20)), yields that

$$\|g\|_{L^2(0,t;\mathbb{R}^m)} \leq C(T, t) \|g\|_{L^1(t,T;\mathbb{R}^m)} \ \text{ for each } \ g \in Y_T,$$

which leads to the conclusion (iii) of Lemma 2.3 (with $p_2 = 2$).

In Step 2, we will prove that (H2) holds for the pair $(A, B)$. To this end, we first show that (H4) holds for the pair $(A, B)$. In the finitely dimensional setting, we have that for each $z \in \mathbb{R}^n$ and each $T > 0$, the function $\widetilde{B^* S^*}(T - \cdot)z$ (defined by (1.22)) is the same as $B^* e^{A^*(T-\cdot)}$ over $[0, T]$. From this and the analyticity of the function $t \mapsto B^* e^{A^* t}$, $t \in \mathbb{R}$, one can easily check that (H4) holds for the pair $(A, B)$. Next, we claim that for each $T \in (0, \infty)$, the space $X_T$ (defined by (1.21)) is the same as $Y_T$. In fact, it follows from (1.21) that for each $T > 0$, $X_T$ is a finitely dimensional subspace in $L^1(0,T;\mathbb{R}^m)$. Thus, for each $T > 0$, $X_T$ is closed in $L^1(0,T;\mathbb{R}^m)$. Then we find from (1.20) that $X_T = Y_T$ for all $T \in (0,\infty)$. From this, it follows that the conditions (H4) and (H2) are the same. Therefore, (H2) holds for the pair $(A,B)$. This ends the proof of this proposition.

□

8.3. **Appendix C.** In Appendix C, we will explain that the BBP decompositions (P1) and (P2) (given by (1.6) and (1.9), respectively) are consequences of Theorem 1.1 and Theorem 1.2. To see these, we need one lemma. In the proof of this lemma, the following well known result (see, for instance, [38, Section 3.3, Chapetr 3]) is used.

**Lemma 8.1.** *Let $(A,B) \in \mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$ (with $n, m \geq 1$). Let $\mathcal{R}_T$ and $\mathcal{R}_T^0$ (with $T > 0$) be given respectively by (1.41) and (1.42). Let $\mathcal{R}$ be given by (1.8). Define the following subspace*

$$\mathcal{C}_T := \{y_0 \in \mathbb{R}^n \ : \ \exists\, v \in L^\infty(0,T;\mathbb{R}^m) \ \text{ s.t. } \ \hat{y}_1(T;y_0,v) = 0\}, \ T > 0, \tag{8.17}$$

*where $\hat{y}_1(\cdot;y_0,v)$ denotes the solution of (1.4). Then it holds that*

$$\mathcal{C}_T = \widehat{\mathcal{R}} = \mathcal{R}_T = \mathcal{R}_T^0 \ \text{ for all } \ T > 0.$$

The following lemma concern some special properties on the functions $T^0(\cdot)$ and $T^1(\cdot)$ (defined respectively by (1.17) and (1.18)).

**Lemma 8.2.** *Let $(A,B) \in \mathbb{R}^{n\times n} \times (\mathbb{R}^{n\times m} \setminus \{0\})$. Let $\mathcal{R}$ be given by (1.8). Then the functions $T^0(\cdot)$ and $T^1(\cdot)$ (defined respectively by (1.17) and (1.18)) have the following properties:*
*(i) For each $y_0 \in \mathcal{R}$, $T^0(y_0) = 0$, while for each $y_0 \in \mathbb{R}^n \setminus \mathcal{R}$, $T^0(y_0) = \infty$.*
*(ii) For each $y_0 \in \mathbb{R}^n \setminus \{0\}$, $N(T^0(y_0), y_0) = \infty$.*
*(iii) For each $y_0 \in \mathbb{R}^n \setminus \{0\}$, $T^1(y_0) = \infty$.*
*(iv) For each $y_0 \in \mathcal{R} \setminus \{0\}$, $N(T^1(y_0), y_0) < \infty$.*

*Proof.* (i) We first prove that $T^0(y_0) = 0$ for each $y_0 \in \mathcal{R}$. Arbitrarily fix $y_0 \in \mathcal{R}$ and $\hat{t} \in (0,\infty)$. According to Lemma 8.1, there is $v \in L^\infty(0,\hat{t};\mathbb{R}^m)$ so that $\hat{y}_1(\hat{t};y_0,v) = 0$ From this and the definition of $T^0(y_0)$ (see (1.17)), we deduce that $T^0(y_0) \leq \hat{t}$. Since $\hat{t}$ was arbitrarily taken from $(0,\infty)$, it follows that $T^0(y_0) = 0$.

Next, we verify that $T^0(y_0) = \infty$ for each $y_0 \in \mathbb{R}^n \setminus \mathcal{R}$. By contradiction, suppose that $T^0(\hat{y}_0) < \infty$ for some $\hat{y}_0 \in \mathbb{R}^n \setminus \mathcal{R}$. Then from the definition of $T^0(\hat{y}_0)$ (see (1.17)), there would be $\hat{t}' \in \big(T^0(\hat{y}_0), \infty\big)$ and $\hat{v} \in L^\infty(0,\hat{t}';\mathbb{R}^m)$ so that $\hat{y}_1(\hat{t}';\hat{y}_0,\hat{v}) = 0$. This, along with the definition of $\mathcal{C}_{\hat{t}'}$ (given by (8.17) with $T = \hat{t}'$), implies that $\hat{y}_0 \in \mathcal{C}_{\hat{t}'}$. Then by Lemma 8.1, we find that $\hat{y}_0 \in \mathcal{R}$, which contradicts the assumption that $\hat{y}_0 \in \mathbb{R}^n \setminus \mathcal{R}$. This ends the proof of the conclusion (i).

(ii) Let $y_0 \in \mathbb{R}^n \setminus \{0\}$. There are only two possibilities on $y_0$: either $y_0 \in \mathcal{R} \setminus \{0\}$ or $y_0 \in \mathbb{R}^n \setminus \mathcal{R}$. In the case that $y_0 \in \mathcal{R} \setminus \{0\}$, we see from (i) of this lemma that $T^0(y_0) = 0$. Then by (iv) of Lemma 3.3, we have that $N(T^0(y_0), y_0) = N(0, y_0) = \infty$. In the case that $y_0 \in \mathbb{R}^n \setminus \mathcal{R}$, we find from (i) of this lemma that $T^0(y_0) = \infty$. Then by (ii) of Lemma 3.4, it follows that $N(T^0(y_0), y_0) = \infty$.

(iii) Let $y_0 \in \mathbb{R}^n \setminus \{0\}$. Since $\{e^{At}\}_{t\in\mathbb{R}^+}$ has the backward uniqueness property, we find from the definition of $T^1(y_0)$ (see (1.18)) that the conclusion (iii) holds.

(iv) Let $y_0 \in \mathcal{R} \setminus \{0\}$. Then it follows by the conclusion (i) of this lemma that $T^0(y_0) = 0$. This, along with (v) of Lemma 3.4, yields that $N(T^1(y_0), y_0) < \infty$.

In summary, we finish the proof of this lemma.

□

**Proposition 14.** *For each pair $(A,B)$ in $\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$ (with $n,m\geq 1$), the BBP decompositions (P1) and (P2) (given respectively by (1.6) and (1.9)), are the consequences of Theorem 1.1 and Theorem 1.2 respectively.*

*Proof.* Arbitrarily fix a pair $(A,B)$ in $\mathbb{R}^{n\times n}\times(\mathbb{R}^{n\times m}\setminus\{0\})$. By Proposition 13, $(A,B)$ satisfies (H1) and (H2). Then all conclusions in Theorem 1.1 and Theorem 1.2 hold. By (i)-(iii) of Lemma 8.2, (vi) of Lemma 3.3, the first conclusion in Theorem 1.1 and the first conclusion in Theorem 1.2, we can easily check that

$$\mathcal{W}=\mathcal{W}_{3,2}\cup\mathcal{W}_{3,4},\ \mathcal{V}=\mathcal{V}_{3,1}\cup\mathcal{V}_{3,2}\cup\mathcal{V}_{3,3};$$

$$\mathcal{W}_{3,2}=\mathcal{R}\setminus\{0\},\ \mathcal{W}_{3,4}=\mathbb{R}^n\setminus\mathcal{R},\ \mathcal{V}_{3,2}=\mathcal{D}_{bbp},\ \mathcal{V}_{3,1}\cup\mathcal{V}_{3,3}=\mathcal{X}_1\setminus\mathcal{D}_{bbp}.$$

(Here, $\mathcal{R}$, $\mathcal{D}_{bbp}$ and $\mathcal{X}_1$ are respectively given by (1.8), (1.7) and (1.5)). These, along with the conclusions (iii) and (iv) in Theorem 1.1 and the conclusions (ii) and (v) in Theorem 1.2, yields that the BBP decompositions (P1) and (P2) holds for the pair $(A,B)$. This ends the proof.

□

8.4. **Appendix D.** In Appendix D, we provide the proofs of Proposition 1 and Lemma 2.1, respectively.

*Proof of Proposition 1.* Arbitrarily fix $T\in(0,\infty)$, $v\in L^\infty(0,T;U)$ and $z\in D(A^*)$. Since $X_{-1}$ is the dual of $D(A^*)$ with respect to the pivot space $X$, we have that

$$\Big\langle\int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t,z\Big\rangle_X=\Big\langle\int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t,z\Big\rangle_{X_{-1},D(A^*)}. \tag{8.18}$$

Because $S_{-1}(T-\cdot)Bv(\cdot)\in L^1(0,T;X_{-1})$, we have that

$$\Big\langle\int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t,z\Big\rangle_{X_{-1},D(A^*)}=\int_0^T\langle S_{-1}(T-t)Bv(t),z\rangle_{X_{-1},D(A^*)}\,\mathrm{d}t. \tag{8.19}$$

We next claim that

$$(S_{-1})^*(T-t)z=S^*(T-t)z\ \text{ in }\ D(A^*),\ \text{ for all }\ t\in[0,T]. \tag{8.20}$$

Indeed, since $\{S_{-1}(t)\}_{t\in\mathbb{R}^+}$ is the extension of $\{S(t)\}_{t\in\mathbb{R}^+}$ on $X_{-1}$, and because $X_{-1}$ is the dual of $D(A^*)$ with respect to the pivot space $X$ , we find that for each $s\geq 0$ and $w\in X$,

$$\begin{aligned}\langle S^*_{-1}(s)z,w\rangle_{D(A^*),X_{-1}} &= \langle z,S_{-1}(s)w\rangle_{D(A^*),X_{-1}}=\langle z,S(s)w\rangle_{D(A^*),X_{-1}}\\ &= \langle z,S(s)w\rangle_X=\langle S^*(s)z,w\rangle_X=\langle S^*(s)z,w\rangle_{D(A^*),X_{-1}}.\end{aligned}$$

Since $X$ is dense in $X_{-1}$, the above implies that for all $s\geq 0$ and $\hat{w}\in X_{-1}$,

$$\langle S^*_{-1}(s)z,\hat{w}\rangle_{D(A^*),X_{-1}}=\langle S^*(s)z,\hat{w}\rangle_{D(A^*),X_{-1}}.$$

This leads to (8.20). From (8.20), we find that

$$\int_0^T\langle S_{-1}(T-t)Bv(t),z\rangle_{X_{-1},D(A^*)}\,\mathrm{d}t=\int_0^T\langle v(t),B^*S^*(T-t)z\rangle_{X_{-1},D(A^*)}\,\mathrm{d}t. \tag{8.21}$$

Now, (2.1) follows from (8.18), (8.19) and (8.21) immediately. This ends the proof of Proposition 1.

□

*Proof of Lemma 2.1.* Arbitrarily fix $0 < T < \infty$ and $z \in D(A^*)$. Then it follows from (2.1) that

$$\begin{aligned} \|B^* S^*(T-\cdot)z\|_{L^2(0,T;U)} &= \sup_{\|u\|_{L^2(0,T;U)}\le 1} \Big\langle z, \int_0^T S_{-1}(T-\cdot)Bu(t)\,\mathrm{d}t \Big\rangle_X \\ &\le \sup_{\|u\|_{L^2(0,T;U)}\le 1} \|z\|_X \Big\| \int_0^T S_{-1}(T-\cdot)Bu(t)\,\mathrm{d}t \Big\|_X, \end{aligned}$$

which, along with (1.10), leads to (2.2). This ends the proof of Lemma 2.1. □

8.5. **Appendix E.** In Appendix E, we give the proof of Lemma 3.1.

*Proof of Lemma 3.1.* Suppose that (3.1) holds for some $\{T_n\}_{n=1}^\infty$, $\widehat{T}$ in $[0,\infty)$, some $\{u_n\}_{n=1}^\infty$ and $\hat{u}$ in $L^2(\mathbb{R}^+;U)$. Arbitrarily fix a $y_0 \in X$. We will prove (3.2) by two steps as follows.
*Step 1. To show that there is a positive constant $C$ so that*

$$\|y(T_n; y_0, u_n)\|_X \le C \ \textit{for all}\ n \tag{8.22}$$

We first claim that there is a positive constant $C_1$ so that for each $s \in (0, \widehat{T}+1)$ and each $u_s \in L^2(0,s;U)$,

$$\Big\| \int_0^s S_{-1}(s-\tau)Bu_s(\tau)\,\mathrm{d}\tau \Big\|_X \le C_1 \|u_s\|_{L^2(0,s;U)}. \tag{8.23}$$

To this end, we arbitrarily fix $s \in (0, \widehat{T}+1)$ and $u_s \in L^2(0,s;U)$. Let

$$v_{u_s,s}(t) = \begin{cases} 0, & t \in (0, \widehat{T}+1-s], \\ u_s(t+s-\widehat{T}-1), & t \in (\widehat{T}+1-s, \widehat{T}+1). \end{cases}$$

Then, we have that $\|v_{u_s,s}\|_{L^2(0,\widehat{T}+1;U)} = \|u_s\|_{L^2(0,s;U)}$ and

$$\int_0^{\widehat{T}+1} S_{-1}(\widehat{T}+1-\tau)Bv_{u_s,s}(\tau)\,\mathrm{d}\tau = \int_0^s S_{-1}(s-\tau)Bu_s(\tau)\,\mathrm{d}\tau.$$

These, along with (1.10), yield that

$$\begin{aligned} \Big\| \int_0^s S_{-1}(s-\tau)Bu_s(\tau)\,\mathrm{d}\tau \Big\|_X &= \Big\| \int_0^{\widehat{T}+1} S_{-1}(\widehat{T}+1-\tau)Bv_{u_s,s}(\tau)\,\mathrm{d}\tau \Big\|_X \\ &\le C_1 \|v_{u_s,s}\|_{L^2(0,\widehat{T}+1;U)} = C_1 \|u_s\|_{L^2(0,s;U)}, \end{aligned}$$

where $C_1 := C_1(\widehat{T}+1)$ is given by (1.10). Hence, (8.23) is true.

Next, it follows from (1.14) that

$$y(T_n; y_0, u_n) = S(T_n)y_0 + \int_0^{T_n} S_{-1}(T_n - t)Bu_n(t)\,\mathrm{d}t \ \text{ for all }\ n \in \mathbb{N}^+. \tag{8.24}$$

Because of the first convergence in (3.1), we can assume, without loss of generality, that $T_n \le \widehat{T}+1$ for all $n$. This, along with (8.23) and (8.24), yields that

$$\|y(T_n; y_0, u_n)\|_X \le \sup_{0\le t\le \widehat{T}+1} \|S(t)\|_{\mathcal{L}(X,X)} \|y_0\|_X + C_1 \|u_n\|_{L^2(0,T_n;U)} \ \text{ for all }\ n \tag{8.25}$$

Meanwhile, it follows from the second convergence in (3.1) that there is a $\widehat{C} > 0$ so that $\|u_n\|_{L^2(\mathbb{R}^+;U)} \le \widehat{C}$ for all $n$, which, along with (8.25), implies (8.22).
*Step 2. To show (3.2)*

Arbitrarily fix a $z \in D(A^*)$. Define two functions $\psi_n^z(\cdot)$ and $\widehat{\psi}^z(\cdot)$ over $(-1, \widehat{T}+1)$ in the following manners:

$$\psi_n^z(t) := 0 \text{ for all } t \in (T_n, \widehat{T}+1) \text{ and } \psi_n^z(t) := B^* S^*(T_n - t)z \text{ for all } t \in (-1, T_n];$$

$$\widehat{\psi}^z(t) := 0 \text{ for all } t \in (\widehat{T}, \widehat{T}+1) \text{ and } \widehat{\psi}^z(t) := B^* S^*(\widehat{T} - t)z \text{ for all } t \in (-1, \widehat{T}].$$

We claim that for a.e. $t \in (-1, \widehat{T}+1)$,

$$\lim_{n\to\infty} \psi_n^z(t) = \widehat{\psi}^z(t) \text{ in } U. \tag{8.26}$$

In fact, by the first convergence in (3.1), we see that for each $t \in (\widehat{T}, \widehat{T}+1)$, there is $N_1(t) \geq 1$ so that $t \in (T_n, \widehat{T}+1)$ for all $n \geq N_1(t)$. Thus, we see that for each $t \in (\widehat{T}, \widehat{T}+1)$,

$$\psi_n^z(t) - \widehat{\psi}^z(t) = 0 \text{ for all } n \geq N_1(t). \tag{8.27}$$

Meanwhile, given $t \in (-1, \widehat{T})$, there is $N_2(t) \geq 1$ so that $t \in (-1, T_n)$ for all $n \geq N_2(t)$. This yields that for each $n \geq N_2(t)$,

$$\begin{aligned}\|\psi_n^z(t) - \widehat{\psi}^z(t)\|_U \quad &\leq \quad \|B^*\|_{\mathcal{L}(D(A^*),U)}\Big(\|S^*(T_n - t)z - S^*(\widehat{T} - t)z\|_X \\ &\qquad + \|S^*(T_n - t)A^*z - S^*(\widehat{T} - t)A^*z\|_X\Big).\end{aligned} \tag{8.28}$$

(Here, we used that $B^* \in \mathcal{L}(D(A^*), U)$.) Since $\{S^*(t)\}_{t\in\mathbb{R}^+}$ is a $C_0$-semigroup in $X$, it follows from (8.28) that for each $t \in (-1, \widehat{T})$, $\psi_n^z(t) \to \widehat{\psi}^z(t)$ in $U$, as $n \to \infty$. This, along with (8.27), leads to (8.26).

Next, since $B^* \in \mathcal{L}(D(A^*), U)$ and $0 \leq T_n \leq \widehat{T}+1$, $n \in \mathbb{N}^+$, one can easily check that for all $n \in \mathbb{N}^+$ and $t \in (-1, \widehat{T}+1)$,

$$\|\psi_n^z(t)\|_U \leq \|B^*\|_{\mathcal{L}(D(A^*),U)} \max_{0\leq s\leq \widehat{T}+2} \|S^*(s)\|_{\mathcal{L}(X,X)} \|z\|_{D(A^*)}. \tag{8.29}$$

By (8.26) and (8.29), we can use the Lebesgue dominated convergence theorem to get that $\psi_n^z \to \widehat{\psi}^z$ in $L^2(-1, \widehat{T}+1; U)$, as $n \to \infty$. This, along with (1.13), yields that for each $z \in D(A^*)$,

$$\langle y(T_n; y_0, u_n), z\rangle_X \to \langle y(\widehat{T}; y_0, \hat{u}), z\rangle_X, \text{ as } n \to \infty. \tag{8.30}$$

Since $D(A^*)$ is dense in $X$, (3.2) follows from (8.30) at once. This ends the proof of Lemma 3.1.

□

8.6. **Appendix F.** In Appendix F, we provide the proof of Proposition 8.

*Proof of Proposition 8* . We divide the proof into the following several steps.

*Step 1. To show that (i)⇒(ii)*

Suppose that (i) holds. Let $T \in (0, \infty)$ and let $C_1(T)$ be given by (6.68). Arbitrarily fix $y_0 \in X$. Define a map $\mathcal{F}_{T,y_0} : X_T \to \mathbb{R}$ (where $X_T$ is given by (1.21)) in the following manner:

$$\mathcal{F}_{T,y_0}\big(B^*S^*(T - \cdot)z|_{(0,T)}\big) = \langle y_0, S^*(T)z\rangle_X \text{ for each } z \in D(A^*). \tag{8.31}$$

We first claim that $\mathcal{F}_{T,y_0}$ is well defined. In fact, if

$$z_1, z_2 \in D(A^*) \text{ s.t. } B^*S^*(T - \cdot)z_1 = B^*S^*(T - \cdot)z_2 \text{ over } (0, T),$$

then by (6.68), it follows that $S^*(T)z_1 = S^*(T)z_2$ in $X$. Hence, $\mathcal{F}_{T,y_0}$ is well defined. Besides, one can easily check that $\mathcal{F}_{T,y_0}$ is linear. By (6.68), we can also find that

$$\left|\mathcal{F}_{T,y_0}\big(B^*S^*(T-\cdot)z|_{(0,T)}\big)\right| \leq C_1(T)\|y_0\|_X\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)},\ \forall\, z \in D(A^*).$$

From this, we see that

$$\|\mathcal{F}_{T,y_0}\|_{\mathcal{L}(X_T,\mathbb{R})} \leq C_1(T)\|y_0\|_X. \tag{8.32}$$

Since $X_T$ is a subspace of $L^1(0,T;U)$ (see (1.20)), we can apply the Hahn-Banach theorem to find a functional $\widetilde{\mathcal{F}}_{T,y_0} \in (L^1(0,T;U))^*$ so that

$$\|\mathcal{F}_{T,y_0}\|_{\mathcal{L}(X_T,\mathbb{R})} = \|\widetilde{\mathcal{F}}_{T,y_0}\|_{(L^1(0,T;U))^*} \ \text{ and } \ \mathcal{F}_{T,y_0}(g) = \widetilde{\mathcal{F}}_{T,y_0}(g) \ \text{ for all } \ g \in X_T.$$

From these, we can apply the Riesz representation theorem to find a function $v \in L^\infty(0,T;U)$ so that

$$\|\mathcal{F}_{T,y_0}\|_{\mathcal{L}(X_T,\mathbb{R})} = \|v\|_{L^\infty(0,T;U)} \tag{8.33}$$

and so that

$$\mathcal{F}_{T,y_0}(g) = \int_0^T \langle g(t), v(t)\rangle_U \,\mathrm{d}t \ \text{ for all } \ g \in X_T. \tag{8.34}$$

From (8.31), (8.34), (1.21) and (2.1) in Proposition 1, we see that for each $z \in D(A^*)$,

$$\begin{aligned}\langle S(T)y_0, z\rangle_X &= \mathcal{F}_{T,y_0}\big(B^*S^*(T-\cdot)z|_{(0,T)}\big) = \int_0^T \big\langle v(t), B^*S^*(T-t)z\big\rangle_U \,\mathrm{d}t \\ &= \Big\langle \int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t, z\Big\rangle_X.\end{aligned}$$

This, along with (1.14), indicates that $\langle \hat{y}(T;y_0,-v), z\rangle_X = 0$ for all $z \in D(A^*)$. Since $D(A^*)$ is dense in $X$, the above leads to that $\hat{y}(T;y_0,-v) = 0$. Meanwhile, it follows from (8.33) and (8.32) that $\|v\|_{L^\infty(0,T;U)} \leq C_1(T)\|y_0\|_X$. From these, (6.69) (with $C_2(T) = C_1(T)$) follows at once.

*Step 2. To prove that (ii)⇒(i)*

Suppose that (ii) holds. Let $T \in (0,\infty)$ and let $C_2(T)$ be given by (ii). Arbitrarily fix $y_0 \in X$. By (ii), there is $v \in L^\infty(0,T;U)$ so that

$$\hat{y}(T;y_0,v) = 0 \ \text{ and } \ \|v\|_{L^\infty(0,T;U)} \leq C_2(T)\|y_0\|_X. \tag{8.35}$$

By the first equality in (8.35) and (1.13), we find that

$$\langle y_0, S^*(T)z\rangle_X = -\int_0^T \langle v(t), B^*S^*(T-t)z\rangle_U \,\mathrm{d}t \ \text{ for all } \ z \in D(A^*).$$

This, along with the second inequality in (8.35), yields that

$$\langle y_0, S^*(T)z\rangle_X \leq C_2(T)\|y_0\|_X\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)} \ \text{ for all } \ z \in D(A^*).$$

Since $y_0$ was arbitrarily taken from $X$, the above implies that for all $z \in D(A^*)$,

$$\|S^*(T)z\|_X = \sup_{y_0 \in X\setminus\{0\}} \frac{\langle y_0, S^*(T)z\rangle_X}{\|y_0\|_X} \leq C_2(T)\|B^*S^*(T-\cdot)z\|_{L^1(0,T;U)},$$

which leads to (6.68) with $C_1(T) = C_2(T)$.

*Step 3. To show that (ii)⇔(iii)*

It is clear that (ii)⇒(iii). We now show the reverse. Suppose that (iii) holds. Let $T \in (0,\infty)$. Define a linear operator $\mathcal{G}_T : L^\infty(0,T;U) \to X$ by setting

$$\mathcal{G}_T(v) = \int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t \;\text{ for each }\; v \in L^\infty(0,T;U). \tag{8.36}$$

Then it follows from (1.10) that $\mathcal{G}_T$ is bounded. By (iii), we know that for each $y_0 \in X$, there is $v \in L^\infty(0,T;U)$ so that $\hat{y}(T;y_0,v) = 0$. This, along with (1.14), yields that

$$0 = S(T)y_0 + \int_0^T S_{-1}(T-t)Bv(t)\,\mathrm{d}t. \tag{8.37}$$

From (8.36) and (8.37), we see that

$$\operatorname{Range} S(T) \subset \operatorname{Range} \mathcal{G}_T. \tag{8.38}$$

Write $Q_T$ for the quotient space of $L^\infty(0,T;U)$ with respect to $\operatorname{Ker}\mathcal{G}_T$, i.e.,

$$Q_T := L^\infty(0,T;U)/\operatorname{Ker}\mathcal{G}_T.$$

Let $\pi_T : L^\infty(0,T;U) \to Q_T$ be the quotient map. Then $\pi_T$ is surjective and it holds that

$$\|\pi_T(v)\|_{Q_T} = \inf\big\{\|w\|_{L^\infty(0,T;U)} \;:\; w \in v + \operatorname{Ker}\mathcal{G}_T\big\} \;\text{ for each }\; v \in L^\infty(0,T;U). \tag{8.39}$$

Define a map $\hat{\mathcal{G}}_T : Q_T \to X$ in the following manner:

$$\hat{\mathcal{G}}_T(\pi_T(v)) = \mathcal{G}_T(v) \;\text{ for each }\; \pi_T(v) \in Q_T. \tag{8.40}$$

One can easily check that $\hat{\mathcal{G}}_T$ is linear and bounded. By (8.40) and (8.38), we see that $\hat{\mathcal{G}}_T$ is injective and that

$$\operatorname{Range} S(T) \subset \operatorname{Range} \hat{\mathcal{G}}_T.$$

From these, we find that given $y_0 \in X$, there is a unique $\pi_T(v_{y_0}) \in Q_T$ so that

$$S(T)y_0 = \hat{\mathcal{G}}_T\big(\pi_T(v_{y_0})\big). \tag{8.41}$$

We next define another map $\mathcal{T}_T : X \to Q_T$ by

$$\mathcal{T}_T(y_0) = \pi_T(v_{y_0}) \;\text{ for each }\; y_0 \in X. \tag{8.42}$$

One can easily check that $\mathcal{T}_T$ is well defined and linear. We will use the closed graph theorem to show that $\mathcal{T}_T$ is bounded. For this purpose, we let $\{y_n\} \subset X$ satisfy that

$$y_n \to \hat{y} \;\text{ in }\; X \;\text{ and }\; \mathcal{T}_T(y_n) \to \hat{h} \;\text{ in }\; Q_T, \;\text{ as }\; n \to \infty. \tag{8.43}$$

Because $\hat{\mathcal{G}}_T$ and $S(T)$ are linear and bounded, it follows from (8.43), (8.42) and (8.41) that

$$\hat{\mathcal{G}}_T(\hat{h}) = \lim_{n\to\infty} \hat{\mathcal{G}}_T\big(\mathcal{T}_T(y_n)\big) = \lim_{n\to\infty} \hat{\mathcal{G}}_T\big(\pi_T(v_{y_n})\big) = \lim_{n\to\infty} S(T)y_n = S(T)\hat{y}. \tag{8.44}$$

Meanwhile, by (8.41) and (8.42), we find that $S(T)\hat{y} = \hat{\mathcal{G}}_T\big(\pi_T(v_{\hat{y}})\big) = \hat{\mathcal{G}}_T\big(\mathcal{T}_T(\hat{y})\big)$. This, together with (8.44), yields that $\hat{\mathcal{G}}_T(\hat{h}) = \hat{\mathcal{G}}_T\big(\mathcal{T}_T(\hat{y})\big)$, which, together with the injectivity of $\hat{\mathcal{G}}_T$, indicates that $\hat{h} = \mathcal{T}_T(\hat{y})$. So the graph of $\mathcal{T}_T$ is closed. Now we can apply the closed graph theorem to see that $\mathcal{T}_T$ is bounded. Hence, there is a constant $C(T) > 0$ so that $\|\mathcal{T}_T(y_0)\|_{Q_T} \le C(T)\|y_0\|_X$ for all $y_0 \in X$. This, along with (8.42), indicates that

$$\|\pi_T(v_{y_0})\|_{Q_T} \le C(T)\|y_0\|_X \;\text{ for each }\; y_0 \in X. \tag{8.45}$$

Meanwhile, by (8.39), we see that for each $y_0 \in X$, there is $v'_{y_0}$ so that

$$v'_{y_0} \in v_{y_0} + \operatorname{Ker} \mathcal{G}_T \ \text{ and } \ \|v'_{y_0}\|_{L^\infty(0,T;U)} \le 2\|\pi_T(v_{y_0})\|_{Q_T}. \tag{8.46}$$

From (8.41), (8.40), (8.46) and (8.45) , we find that for each $y_0 \in X$, there is a control $v'_{y_0} \in L^\infty(0,T;U)$ so that

$$S(T)y_0 = \mathcal{G}_T(v'_{y_0}) \ \text{ and } \ \|v'_{y_0}\|_{L^\infty(0,T;U)} \le 2C(T)\|y_0\|_X. \tag{8.47}$$

Then by (1.14), (8.36) and (8.47), we see that for each $y_0 \in X$, there is a control $v'_{y_0} \in L^\infty(0,T;U)$ so that

$$\hat{y}(T; y_0, -v'_{y_0}) = 0 \ \text{ and } \ \|v'_{y_0}\|_{L^\infty(0,T;U)} \le 2C(T)\|y_0\|_X.$$

These lead to (6.69) with $C_2(T) = 2C(T)$.

*Step 4. About the constants $C_1(T)$ and $C_2(T)$*

From the proofs in Step 1-Step 3, we find that the constants $C_1(T)$ in (6.68) and $C_2(T)$ in (6.69) can be taken as the same number, provided that one of the conclusions (i)-(iii) holds.

In summary, we end the proof of this proposition.

□

8.7. **Appendix G.** In Appendix G, we provide the proof of Lemma 7.3.

*Proof of Lemma 7.3.* Recall $\mathcal{P}$ is given by (7.20), where $\Lambda := \{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}^+$ satisfies (7.14). Arbitrarily fix $\theta_0 \in (0, \frac{\pi}{2})$. By [17, Proposition 4.5], there is a sequence $\{r_n\}_{n=1}^\infty \subset (0,\infty)$ so that

$$r_n \nearrow \infty \ \text{ and } \ \lim_{n\to\infty} r_n^{-1} \log|W(r_n e^{i\theta})| = 0 \text{ uniformly in } |\theta| \le \theta_0, \tag{8.48}$$

where $W(\lambda)$ is given by

$$\begin{cases} W(\lambda) = \prod_{k\ge1} \delta_k \frac{1-\lambda/\lambda_k}{1+\lambda/\overline{\lambda}_k}, \ \lambda \in \mathbb{C}^+, \\ \text{with } \ \delta_k = \frac{\lambda_k}{\overline{\lambda}_k} \frac{|\lambda_k - 1|}{|\lambda_k+1|} \frac{\overline{\lambda}_k+1}{\overline{\lambda}_k - 1} \ \text{ if } \ \lambda_k \neq 1; \ \ \delta_k = 1 \ \text{ if } \ \lambda_k = 1. \end{cases} \tag{8.49}$$

(Notice that in [17], $\lambda_j$ was a complex number, while in the current case, we take it as a real number. So $\lambda_j = \overline{\lambda}_j$ in the current case. To avoid the inconformity, we still use the notation $\overline{\lambda}_j$.) Since $W(\lambda_k) = 0$ for each $k \ge 1$, and because of (7.14) and (8.48), we can select a subsequence from $\{r_n\}_{n=1}^\infty$ (denoted in the same manner,) having two properties as follows: First, $\{\lambda_j\}_{j=1}^\infty \bigcap \{r_n\}_{n=1}^\infty = \emptyset$. Second, for each $n \in \mathbb{N}^+$, the set

$$G_n := \{z = re^{i\theta} \ : \ r_n < |z| < r_{n+1}, \ |\theta| < \theta_0\}$$

contains at least an element of $\Lambda := \{\lambda_j\}_{j=1}^\infty$. The sequence $\{G_n\}_{n\ge1}$ and the function $W(\cdot)$, as well as their properties, will be used later.

Let $J$ be a function defined by

$$J(\lambda) = \frac{W(\lambda)}{(1+\lambda)^2}, \ \lambda \in \mathbb{C}^+. \tag{8.50}$$

For each $j \ge 1$, define a function $J_j$ by

$$J_j(\lambda) = \frac{J(\lambda)}{J'(\lambda_j)(\lambda - \lambda_j)}, \ \lambda \in \mathbb{C}^+. \tag{8.51}$$

According to [17, Theorem 4.1] (see also the proof of [17, Theorem 4.1]), there exists a biorthogonal family $\{q_j\}_{j\geq 1}$ to $\{e^{-\lambda_j t}\}$ in $L^2(\mathbb{R}^+;\mathbb{C})$ so that the Laplace transform of $\bar{q}_j$ is $J_j$ for each $j \in \mathbb{N}^+$.

To prove the desired inequality (7.22), we will build up two inequalities for $p \in \mathcal{P}$. The first one reads: For each $\varepsilon > 0$, there is $C(\theta_0, \varepsilon) > 0$ so that for each $p \in \mathcal{P}$,

$$|p(z)| \leq C(\theta_0, \varepsilon) e^{-\frac{1}{8}|\lambda_1| \cos\theta_0 Re\, z} \|p\|_{L^1(\mathbb{R}^+;\mathbb{C})} \text{ for all } z \in S_{\varepsilon,\theta_0}, \tag{8.52}$$

where $S_{\varepsilon,\theta_0}$ is given by (7.21). The second one reads: For each $T \in (0, \infty)$, there exists $C := C(T) > 0$ so that

$$\|p\|_{L^1(\mathbb{R}^+;\mathbb{C})} \leq C \|p\|_{L^1(0,T;\mathbb{C})} \text{ for all } p \in \mathcal{P}. \tag{8.53}$$

We now show (8.52). Let $p \in \mathcal{P}$. By (7.20), we can express $p$ in the following manner:

$$p(z) = \sum_{j=1}^{N} c_j e^{-\lambda_j z},\ z \in \mathbb{C}^+, \text{ with } N \in \mathbb{N}^+ \text{ and } \{c_j\}_{j=1}^N \subset \mathbb{C}. \tag{8.54}$$

Since each $\{G_n\}_{n\geq 1}$ contains at least an element of $\Lambda := \{\lambda_j\}_{j=1}^\infty$ and $\lambda_j \nearrow \infty$, there is an $m := m(N) \in \mathbb{N}^+$ so that $\{\lambda_j\}_{j=1}^N \subset \bigcup_{k=1}^m G_k$. This, along with (8.54), yields that

$$p(z) = \sum_{k=1}^{m} \sum_{\lambda_j \in G_k} c_j e^{-\lambda_j z} := \sum_{k=1}^{m} g_k(z),\ z \in \mathbb{C}^+. \tag{8.55}$$

Meanwhile, since $\{q_j\}_{j=1}^\infty$ is a biorthogonal family to $\{e^{-\lambda_j t}\}$ in $L^2(\mathbb{R}^+;\mathbb{C})$, it follows from (8.54) that

$$c_j = \int_0^\infty p(t)\overline{q}_j(t)\,\mathrm{d}t, \text{ with } 1 \leq j \leq N.$$

From this and (8.55), we have that for each $k \in \{1, \cdots, m\}$,

$$g_k(z) = \int_0^\infty p(t)\Big(\sum_{\lambda_j \in G_k} \overline{q}_j(t) e^{-\lambda_j z}\Big)\,\mathrm{d}t,\ z \in \mathbb{C}^+.$$

This yields that for each $k \in \{1, \cdots, m\}$ and each $z \in \mathbb{C}^+$,

$$|g_k(z)| \leq \|p(\cdot)\|_{L^1(\mathbb{R}^+;\mathbb{C})} \|\mathcal{G}_k(\cdot, z)\|_{L^\infty(\mathbb{R}^+;\mathbb{C})}. \tag{8.56}$$

where

$$\mathcal{G}_k(t, z) := \sum_{\lambda_j \in G_k} \overline{q}_j(t) e^{-\lambda_j z},\ \ t \in \mathbb{R}^+. \tag{8.57}$$

Arbitrarily fix a $k \in \{1, \cdots, m\}$. Since for each $j \in \mathbb{N}^+$, the Laplace transform of $\bar{q}_j$ is $J_j$, we see that for each $z \in \mathbb{C}^+$, the Laplace transform of $\mathcal{G}_k(t, z)$ is given by

$$\int_0^\infty \mathcal{G}_k(t, z) e^{-\lambda t}\,\mathrm{d}t = \sum_{\lambda_j \in G_k} J_j(\lambda) e^{-\lambda_j z},\ \lambda \in \mathbb{C}^+, \tag{8.58}$$

Since $q_j(t) = 0$ for all $t < 0$ and $j \in \mathbb{N}^+$, we see from (8.57) that for each $z \in \mathbb{C}^+$, $\mathcal{G}_k(t, z) = 0$ for all $t < 0$. This, along with (8.58), yields that for each $z \in \mathbb{C}^+$, the function $\tau \to \sum_{\lambda_j\in G_k} J_j(i\tau)e^{-\lambda_j z}$, $\tau\in\mathbb{R}$, is the Fourier transform of $\mathcal{G}_k(\cdot,z)$. Then by the inverse Fourier transform, we see that for each $z\in\mathbb{C}^+$,

$$\begin{aligned}\|\mathcal{G}_k(\cdot,z)\|_{L^\infty(\mathbb{R}^+;\mathbb{C})} &= \sup_{t\in\mathbb{R}^+}\Big|\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\sum_{\lambda_j\in G_k} J_j(i\tau)e^{-\lambda_j z}\Big)e^{i\tau t}\,\mathrm{d}\tau\Big|\\ &\le \frac{1}{2\pi}\int_{\mathbb{R}}\Big|\sum_{\lambda_j\in G_k} J_j(i\tau)e^{-\lambda_j z}\Big|\,\mathrm{d}\tau. \qquad (8.59)\end{aligned}$$

Meanwhile, by (8.50), (8.49) and (7.20), we find that each $\lambda_j$ is a simple root of $J$. Thus, by (8.51), we can use the residue theorem to see that

$$\sum_{\lambda_j\in G_k} J_j(i\tau)e^{-\lambda_j z} = \frac{J(i\tau)}{2\pi i}\int_{\Gamma_k}\frac{e^{-\xi z}}{J(\xi)(i\tau-\xi)}\,\mathrm{d}\xi, \qquad (8.60)$$

where $\Gamma_k$ denotes the boundary of $G_k$. From (8.60) and (8.59), it follows that for each $k\in\{1\dots,m\}$ and each $z\in\mathbb{C}^+$,

$$\begin{aligned}\|\mathcal{G}_k(\cdot,z)\|_{L^\infty(\mathbb{R}^+;\mathbb{C})} &\le \frac{1}{4\pi^2}\int_{\mathbb{R}}\Big|\int_{\Gamma_k} J(i\tau)\frac{e^{-\xi z}}{J(\xi)(i\tau-\xi)}\,\mathrm{d}\xi\Big|\,\mathrm{d}\tau\\ &\le \frac{\|J\|_{L^1(i\mathbb{R};\mathbb{C})}}{4\pi^2\rho}\int_{\Gamma_k}\Big|\frac{e^{-\xi z}}{J(\xi)}\Big|\,|\mathrm{d}\xi|, \qquad (8.61)\end{aligned}$$

where $\rho=\min_{k\ge1} d(i\mathbb{R},\Gamma_k)>0$. From (8.55), (8.56) and (8.61), we get that

$$|p(z)|\le \frac{\|J\|_{L^1(i\mathbb{R};\mathbb{C})}}{4\pi^2\rho}\|p\|_{L^1(\mathbb{R}^+;\mathbb{C})}\Big(\sum_{k=1}^m\int_{\Gamma_k}\Big|\frac{e^{-\xi z}}{J(\xi)}\Big|\,|\mathrm{d}\xi|\Big),\ \forall\, z\in\mathbb{C}^+. \qquad (8.62)$$

Starting from (8.62), using the same way as that used in the proof of estimating (4.12) in [17, Lemma 4.6] (see [17, Pages 2113-2115]), we can get the inequality (8.52).

Now we prove the second inequality (8.53). By contradiction, suppose that it were not true. Then there would be a $T>0$ and a sequence $\{p_n\}_{n=1}^\infty\subset\mathcal{P}$ so that

$$\|p_n\|_{L^1(\mathbb{R}^+;\mathbb{C})}=1 \ \text{ and } \ \|p_n\|_{L^1(0,T;\mathbb{C})}<1/n \ \text{ for each } \ n\ge1. \qquad (8.63)$$

Arbitrarily fix $\varepsilon_0\in(0,T/2)$. Then choose a $s_0\in(T,\infty)$ so that

$$\int_{s_0}^\infty C(\theta_0,\varepsilon_0)e^{-\frac18|\lambda_1|\cos\theta_0 t}\,\mathrm{d}t<1/2, \qquad (8.64)$$

where $C(\theta_0,\varepsilon_0)$ is given by (8.52). From (8.52), we find that for all $m,n\in\mathbb{N}^+$,

$$\begin{aligned}\int_0^\infty |(p_n-p_m)(t)|\,\mathrm{d}t &\le \int_0^{s_0}|(p_n-p_m)(t)|\,\mathrm{d}t+\\ &\quad\int_{s_0}^\infty\Big(C(\theta_0,\varepsilon_0)e^{-\frac18|\lambda_1|\cos\theta_0 t}\int_0^\infty|(p_n-p_m)(s)|\,\mathrm{d}s\Big)\,\mathrm{d}t.\end{aligned}$$

This, along with (8.64), implies that for all $m,n\in\mathbb{N}^+$,

$$\int_0^\infty|(p_n-p_m)(t)|\,\mathrm{d}t\le 2\int_0^{s_0}|(p_n-p_m)(t)|\,\mathrm{d}t. \qquad (8.65)$$

Two observations are given in order: First, by (8.52) and the first equality in (8.63), we find that $\{\|p_n\|_{C(S_{\varepsilon_0,\theta_0},\mathbb{C})}\}_{n=1}^\infty$ is bounded. Second, each $p_n$ (with $n\in\mathbb{N}^+$) is analytic over $S_{\varepsilon_0,\theta_0}$. From these observations, we can use the Montel theorem to find a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ of $\{p_n\}_{n=1}^{\infty}$ and an analytic function $\hat{p}$ over $S_{\varepsilon_0,\theta_0}$ so that

$$p_{n_k} \to \hat{p} \ \text{ uniformly on each compact set of } \ S_{\varepsilon_0,\theta_0}, \ \text{ as } \ k \to \infty. \tag{8.66}$$

Since $0 < 2\varepsilon_0 < T < s_0$, it follows from (8.66) and the second inequality in (8.63) that

$$p_{n_k} \to 0 \ \text{ in } \ L^1(0,T;\mathbb{C}) \ \text{ and } \ p_{n_k} \to \hat{p} \ \text{ in } \ L^1(T,s_0;\mathbb{C}), \ \text{ as } \ k \to \infty.$$

These, along with (8.65), (8.66) and the first equality in (8.63), indicates that

$$\|\hat{p}\|_{L^1(T,\infty;\mathbb{C})} = 1 \ \text{ and } \ \|\hat{p}\|_{L^1(2\varepsilon_0,T;\mathbb{C})} = 0. \tag{8.67}$$

Since $\hat{p}$ is analytic over $S_{\varepsilon_0,\theta_0}$, from the second assertion in (8.67), we get that $\hat{p} \equiv 0$ over $S_{\varepsilon_0,\theta_0}$. This contradicts the first assertion in (8.67). So (8.53) is true.

Finally, the desired inequality (7.22) follows from (8.52) and (8.53) at once. This ends the proof of Lemma 7.3.

□

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