Observability and unique continuation inequalities for the Schrödinger equation

Gengsheng Wang Ming Wang Yubiao Zhang

Abstract

In this paper, we present several observability and unique continuation inequalities for the free Schrödinger equation in the whole space. The observations in these inequalities are made either at two points in time or one point in time. These inequalities correspond to different kinds of controllability for the free Schrödinger equation. We also find that the observability inequality at two points in time is equivalent to the uncertainty principle built up in [21].

Keywords. Observability, unique continuation, controllability, free Schrödinger equation

1 Introduction

An interesting unique continuation property for Schrödinger equations was contained in [20] (see also [22]). It says that if u solves the following Schrödinger equation:

$$i\partial_t u + \Delta u + Vu = 0 \ \text{ in } \ \mathbb{R}^n \times (0,1), \tag{1.1}$$

(with a time-dependent potential V in some suitable conditions and with $n \in \mathbb{N}^+ \triangleq \{1, 2, \dots\}$), then

$$u = 0 \ \text{ in } \ B_R^c(0) \times \{0,1\} \Rightarrow u \equiv 0.$$

Here, $R > 0$, $B_R(0)$ is the closed ball in $\mathbb{R}^n$, centered at the origin and of radius R, and $B_R^c(0)$ denotes the complement of $B_R(0)$. In [12] (see also [15, Theorems 3-4]), it was presented that if u solves (1.1) (with V in some suitable conditions) and verifies that

$$\|e^{|x|^2/\alpha^2} u(x,0)\|_{L^2(\mathbb{R}^n;\mathbb{C})} + \|e^{|x|^2/\beta^2} u(x,1)\|_{L^2(\mathbb{R}^n;\mathbb{C})} < \infty$$

for some positive constants α, β with $\alpha\beta < 4$, then $u \equiv 0$. It was further proved that when $\alpha\beta = 4$, such property fails. The above mentioned two properties can be treated as the qualitative unique continuation at two points in time. It is natural to ask if one can have an observability inequality at two points in time?

G. Wang: Corresponding author, School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China; e-mail: wanggs62@yeah.net

M. Wang: School of Mathematics and Physics, China University of Geosciences, Wuhan, 430074, China; e-mail: mwangcug@outlook.com

Y. Zhang: Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China; e-mail: yubiao_zhang@yeah.net

Mathematics Subject Classification (2010): Primary 93B05; Secondary 35B60

In this paper, we will present several observability and unique continuation inequalities (at either two points in time or one point in time) for the following free Schrödinger equation (or the Schrödinger equation, for simplicity):

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) \in L^2(\mathbb{R}^n;\mathbb{C}). \end{cases} \tag{1.2}$$

(Here and throughout this paper, $n \in \mathbb{N}^+$ is arbitrarily fixed.) From perspective of applications, to these inequalities correspond different controllability properties for the Schrödinger equation.

The free Schrödinger equation (1.2) describes the evolution of the wave function for a particle without external field (see for instance [24]). Though the Schrödinger equation with a potential $V \neq 0$ is more attractive, the free Schrödinger equation is also important and there are many studies on it. For instance, [6, 27, 40] studied the local smoothing effect and Strichartz estimates of the free Schrödinger equation; [10] built up some unique continuation and convexity properties for the free Schrödinger equation; [26] obtained some observability inequality over time intervals for the free Schrödinger equation over a bounded domain (see also [25, 32, 41]).

Throughout this paper, we write either $u(x,t;u_0)$ (with $(x,t) \in \mathbb{R}^n \times (0,\infty)$) or $e^{i\Delta t}u_0$ (with $t \geq 0$) for the solution of (1.2) with the initial condition that $u(x,0) = u_0(x)$ over $\mathbb{R}^n$; The Fourier transform of $f \in L^1(\mathbb{R}^n;\mathbb{C}) \cap L^2(\mathbb{R}^n;\mathbb{C})$ is given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, \mathrm{d}x, \ \xi \in \mathbb{R}^n$$

and extended to all of $L^2(\mathbb{R}^n;\mathbb{C})$ in the usual way; Write respectively A^c and $|A|$ for the complement and the Lebesgue measure of a set A in $\mathbb{R}^n$; For each subset $A \subset \mathbb{R}^n$ and each $\lambda \in \mathbb{R}$, we let $\lambda A \triangleq \{\lambda x : x \in A\}$; For all $a, b \in \mathbb{R}$, we write $a \wedge b \triangleq \min\{a,b\}$; For each $x \in \mathbb{R}^n$, $|x|$ denotes to the $\mathbb{R}^n$-Euclidean norm of x; ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

There are three main theorems in this paper. The first one presents an observability inequality at two points in time for the equation (1.2).

Theorem 1.1. *Given x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$ and $T > S \geq 0$, there is a positive constant $C \triangleq C(n)$ so that*

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \, \mathrm{d}x \quad \leq \quad Ce^{Cr_1 r_2 \frac{1}{T-S}} \Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \, \mathrm{d}x + \int_{B^c_{r_2}(x'')} |u(x,T;u_0)|^2 \, \mathrm{d}x \Big) \tag{1.3}$$

for all $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$.

Several remarks on Theorem 1.1 are given in order:

(a1) Theorem 1.1 can be explained in the following manner: The integral on the left hand side of (1.3) can be treated as a recover term, while the integrals on the right hand side of (1.3) are regarded as observation terms. The inequality (1.3) is understood as follows: Through observing a solution at two different points in time, each time outside of a ball, one can estimate the recover term (which says, in plain language, that one can recover this solution). This inequality is equivalent to the exact controllability for the impulse controlled Schrödinger equation with controls acting at two points in time, each time outside of a ball (see Subsection 5.2).

(a2) The observability inequality (1.3) seems to be new for us. Most observability inequalities for Schrödinger equations, in published papers, have observations in time intervals. For instance, the paper [26] presents an observability inequality for the Schrödinger equation on a bounded domain Ω (in $\mathbb{R}^n$), with an analytic boundary $\partial\Omega$. In that inequality, the observation is made over $\hat{\omega} \times (0,T)$, where $T > 0$ and $\hat{\omega} \subset \partial\Omega$ is a subdomain satisfying the Geometric Control Condition. This condition was introduced in [4] and then was used in [8] to study the stabilization property and the exact controllability for the nonlinear Schrödinger equation on a two dimensional compact Riemannian manifold without boundary. The paper [32] builds up an observability estimate for the homogenous Schrödinger equation on a bounded domain Ω. In that inequality, the observation is made over $\omega \times (0,T)$, where $T > 0$ and $\omega \subset \Omega$ is a subdomain satisfying the Geometric Control Condition. The book [41] gives an observability inequality for the free Schrödinger equation over a rectangular domain in $\mathbb{R}^2$. There, the observation can be made over any $\omega \times (0,T)$, with ω an open (nonempty) subset (see [41, Theorem 8.5.1]). More recently, the paper [1] (see [1, Theorem 1.2]) presents an observability inequality for Schrödinger equations (with some potentials) on the disk of $\mathbb{R}^2$. The observation is made over $\omega \times (0,T)$, where ω is an open (nonempty) subset which may not satisfy the Geometric Control Condition.

(a3) The inequality (1.3) is "optimal" in the following sense: First, $\forall\, A \subset \mathbb{R}^n$, with $m(A^c) > 0$, $\forall\, T > 0$, the following conclusion is not true (see (b) of Remark 4.2): $\exists\, C > 0$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C \int_A |u(x,T;u_0)|^2 \,\mathrm{d}x, \ \forall\, u_0 \in L^2(\mathbb{R}^n;\mathbb{C}).$$

This means that we cannot recover a solution by observing it at one point in time and over a subset $A \subset \mathbb{R}^n$, with $|A^c| > 0$; Second, $\forall\, x',\ x'' \in \mathbb{R}^n$, $r_1,\ r_2 > 0$ and $T > S \geq 0$, the following conclusion is not true (see (a) of Remark 4.2): $\exists\, C > 0$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C\Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \,\mathrm{d}x + \int_{B_{r_2}(x'')} |u(x,T;u_0)|^2 \,\mathrm{d}x \Big), \ \forall\, u_0 \in L^2(\mathbb{R}^n;\mathbb{C}).$$

This means that we cannot recover a solution by observing it at two different points in time, one time in a ball, while another time outside of a ball; And last, $\forall\, x',\ x'' \in \mathbb{R}^n$, $r_1,\ r_2 > 0$ and $T > S \geq 0$, the following conclusion is not true (see (a) of Remark 4.2): $\exists\, C > 0$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C\Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \,\mathrm{d}x + \int_0^T \int_{B_{r_2}(x'')} |u(x,t;u_0)|^2 \,\mathrm{d}x\mathrm{d}t \Big), \ \forall\, u_0 \in L^2(\mathbb{R}^n;\mathbb{C}).$$

This can be comparable with the work in [29].

(a4) The proof of (1.3) is based on two properties as follows: First, the uncertainty principle built up in [21]; Second, the equivalence between the uncertainty principle and the observability estimate which grows like (1.3). The aforementioned equivalence is indeed a connection between the uncertainty principle and the observability (at two time points) for the Schrödinger equation. Such equivalence is obtained in this paper (see Lemma 2.3). Its proof relies on the identity [10, (1.2)] (see (2.6) in our paper).

(a5) The inequality (1.3) can be extended to the case where $B^c_{r_1}(x')$ and $B^c_{r_2}(x'')$ are replaced by two measurable sets A^c and B^c, with $|A| < \infty$ and $|B| < \infty$. This can be easily seen from the proof of (1.3), as well as Theorem 2.1 (which is the uncertainty principle built up in [21]) and Lemma 2.3.

(a6) From Theorem 1.1, one can directly derive the following observability inequality: Given $x_0 \in \mathbb{R}^n$, $r > 0$ and $T > 0$, there exists $C \triangleq C(n) > 0$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le Ce^{Cr^2/T} \int_0^T \left(\int_{B_r^c(x_0)} |u(x,t;u_0)|^2 \,\mathrm{d}x \right)^{1/2} \mathrm{d}t \quad \text{for all} \quad u_0 \in L^2(\mathbb{R}^n;\mathbb{C}).$$

This inequality is equivalent to the standard L^∞-exact controllability for the Schrödinger equation. The later is comparable to [36, Theorem 3.1].

The second main theorem gives a unique continuation inequality at one time point for a class of solutions to the equation (1.2). (This class of solutions consists of solutions whose initial data have exponential decay at infinity.)

Theorem 1.2. *The following conclusions are true for all $r > 0$, $a > 0$ and $T > 0$:*
(i) There is $C \triangleq C(n) > 0$ and $\theta \triangleq \theta(n) \in (0,1)$, depending only on n, so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le C\left(1 + \frac{r^n}{(aT)^n}\right) \left(\int_{B_r^c(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x \right)^{\theta^{1+\frac{r}{aT}}} \left(\int_{\mathbb{R}^n} e^{a|x|} |u_0(x)|^2 \,\mathrm{d}x \right)^{1-\theta^{1+\frac{r}{aT}}} \tag{1.4}$$

for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$.
(ii) There is $C \triangleq C(n) > 0$ so that for any $\beta > 1$ and $\gamma \in (0,1)$,

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le Ce^{\left(\frac{C^\beta r^\beta}{a(1-\gamma)T^\beta}\right)^{\frac{1}{\beta-1}}} \left(\int_{B_r^c(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x \right)^{\gamma} \left(\int_{\mathbb{R}^n} e^{a|x|^\beta} |u_0(x)|^2 \,\mathrm{d}x \right)^{1-\gamma}, \tag{1.5}$$

for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$.
(iii) Let $\alpha(s)$, $s \in \mathbb{R}^+$, be an increasing function with $\lim_{s\to\infty} \frac{\alpha(s)}{s} = 0$. Then for each $\gamma \in (0,1)$, there is no positive constant C so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \le C \left(\int_{B_r^c(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x \right)^{\gamma} \left(\int_{\mathbb{R}^n} e^{a\alpha(|x|)} |u_0(x)|^2 \,\mathrm{d}x \right)^{1-\gamma} \quad \textit{for all} \quad u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}).$$

The last main theorem gives another kind of unique continuation inequality at one time point for a class of solutions to the equation (1.2).

Theorem 1.3. *Given x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$, $a > 0$ and $T > 0$, the following estimate holds for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$:*

$$\begin{aligned} &\int_{B_{r_2}(x'')} |u(x,T;u_0)|^2 \,\mathrm{d}x \qquad (1.6) \\ \le\ & Cr_2^n\big((aT)\wedge r_1\big)^{-n} \left(\int_{B_{r_1}(x')} |u(x,T;u_0)|^2 \,\mathrm{d}x \right)^{\theta^p} \left(\int_{\mathbb{R}^n} e^{a|x|} |u_0(x)|^2 \,\mathrm{d}x \right)^{1-\theta^p}, \end{aligned}$$

where $C \triangleq C(n) > 0$, $\theta \triangleq \theta(n) \in (0,1)$ and

$$p \triangleq 1 + \frac{|x' - x''| + r_1 + r_2}{(aT)\wedge r_1}. \tag{1.7}$$

Several remarks on Theorem 1.2 and Theorem 1.3 are given in order:

(b1) The motivation to build up Theorem 1.2 is as follows: According to Remark (a3) after Theorem 1.1, one cannot recover a solution by observing it at one time point and outside of a ball. Hence, it should be interesting to ask what we can expect by observing solutions at one time point and outside of a ball. Theorem 1.2 gives an answer to the above question.

The motivation to present Theorem 1.3 is as follows: We will see that (1.4) does not hold when $B_r^c(0)$ is replaced by $B_r(0)$ (see (c) of Remark 4.2). Thus, it could be interesting to ask what we can expect by observing solutions at one time point and in a ball. Theorem 1.3 gives an answer to such question.

(b2) The inequalities (1.4) and (1.6) are two kinds of unique continuation inequalities at one point in time. From (1.4), one can easily see that

$$e^{\frac{a|x|}{2}}u_0(x) \in L^2(\mathbb{R}^n;\mathbb{C}) \text{ and } u(x,T;u_0)=0 \text{ over } B_r^c(0) \Rightarrow u(x,t;u_0)=0 \text{ over } \mathbb{R}^n\times[0,\infty).$$

From (1.6), one can easily check that

$$e^{\frac{a|x|}{2}}u_0(x) \in L^2(\mathbb{R}^n;\mathbb{C}) \text{ and } u(x,T;u_0)=0 \text{ over } B_{r_1}(x') \Rightarrow u(x,t;u_0)=0 \text{ over } \mathbb{R}^n\times[0,\infty).$$

(Indeed, the left hand side of the above, together with (1.6), indicates that for each $x''\in\mathbb{R}^n$ and each $r_2>0$, $u(\cdot,T;u_0)=0$ over $B_{r_2}(x'')$. Then by the arbitrariness of x'' and r_2, we see that $u(x,T;u_0)=0$ over $\mathbb{R}^n$. This leads to that $u(x,t;u_0)=0$ over $\mathbb{R}^n\times[0,\infty)$.)

From (1.6), we can also have that

$$u_0=0 \text{ over } B_{r_2}^c(x'') \text{ and } u(x,T;u_0)=0 \text{ over } B_{r_1}(x') \Rightarrow u(x,t;u_0)=0 \text{ over } \mathbb{R}^n\times[0,\infty).$$

(b3) The inequalities (1.4) and (1.6) can be explained from the following two perspectives:

Perspective One: The integral on the left hand side of (1.4) (or (1.6)) is treated as a recover term, while on the right hand side of (1.4) (or (1.6)), the integral over $B_r^c(0)$ (or $B_{r_1}(x')$) is regarded as an observation term and the integral over the whole space $\mathbb{R}^n$ is viewed as a prior term which provides some prior information on initial data ahead of observations. The inequality (1.4) (or (1.6)) can be explained in the following way: If one knows in advance that the initial datum of a solution has an exponential decay at infinity, then by observing this solution at one point in time and outside of a ball (or inside of a ball), one can estimate the recover term, which says, in plain language, that one can recover this solution (or this solution over $B_{r_2}(x'')$ at time T).

Perspective Two: The inequality (1.4) is equivalent to that $\exists\, C>0$ and $\theta\in(0,1)$ s.t. $\forall\, r,\, a,\, T>0$ and $\varepsilon>0$,

$$\int_{\mathbb{R}^n}|u_0(x)|^2\,\mathrm{d}x \le C\left(1+\frac{r^n}{(aT)^n}\right)\left(\varepsilon^{1-\theta^{-1-\frac{r}{aT}}}\int_{B_r^c(0)}|u(x,T;u_0)|^2\,\mathrm{d}x+\varepsilon\int_{\mathbb{R}^n}e^{a|x|}|u_0(x)|^2\,\mathrm{d}x\right)$$

for all $u_0\in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Thus, the inequality (1.4) can be understood as follows: Through observing a solution at one point in time and outside of a ball, we can approximately recover this solution, with the error:

$$C\left(1+\frac{r^n}{(aT)^n}\right)\varepsilon\int_{\mathbb{R}^n}e^{a|x|}|u_0(x)|^2\,\mathrm{d}x.$$

Notice that if $\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2 = \infty$, then the error is ∞.

The inequality (1.6) is equivalent to that $\exists\, C > 0$ and $\theta \in (0,1)$ s.t. $\forall\, x',\, x'' \in \mathbb{R}^n$, $r_1, r_2 > 0$, a, $T > 0$ and $\varepsilon > 0$,

$$\begin{aligned}&\int_{B_{r_2}(x'')} |u(x,T;u_0)|^2\,\mathrm{d}x\\ \leq\quad & Cr_2^n\big((aT)\wedge r_1\big)^{-n}\left(\varepsilon^{1-\theta^{-p}}\int_{B_{r_1}(x')}|u(x,T;u_0)|^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2\,\mathrm{d}x\right)\end{aligned}$$

for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Here, p is given by (1.7). Thus, the inequality (1.6) can be understood as follows: Through observing a solution at one point in time and in a ball, we can approximately recover this solution over $B_{r_2}(x'')$ at time T, with the error:

$$Cr_2^n\big((aT)\wedge r_1\big)^{-n}\varepsilon\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2\,\mathrm{d}x.$$

If $\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2 = \infty$, then the error is ∞.

Notice that the recover terms in (1.4) and (1.6) are different. By (1.4), we can recover approximately a solution over $\mathbb{R}^n \times \{0\}$, while by (1.6), we can recover approximately a solution over $B_{r_2}(x'') \times \{T\}$.

(b4) The inequality (1.4) is equivalent to a kind of approximate controllability for the impulse controlled Schrödinger equation with controls acting at one point in time, while the inequality (1.6) is equivalent to a kind of approximate null controllability for the initial controlled Schrödinger equation with controls acting at one point in time. Notice that the above two kinds of controllability are not standard (see Subsection 5.2).

(b5) Theorem 1.2 is "optimal" from two perspectives. Perspective One: If $\beta \geq 1$, then for any $r > 0$, $a > 0$ and $T > 0$, there is $C > 0$ and $\theta \in (0,1)$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2\,\mathrm{d}x \leq C\left(\int_{B_r^c(0)} |u(x,T;u_0)|^2\,\mathrm{d}x\right)^{\theta}\left(\int_{\mathbb{R}^n} e^{a|x|^\beta}|u_0(x)|^2\,\mathrm{d}x\right)^{1-\theta},\ \forall\, u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}),$$

while if $\beta \in (0,1)$, then for any $r > 0$, $a > 0$ and $T > 0$, there is no $C > 0$ or $\theta \in (0,1)$ so that the above inequality holds. Perspective Two: For each $r > 0$, $a > 0$ and $T > 0$, the following conclusion is not true (see (c) of Remark 4.2): $\exists\, C > 0$ and $\exists\, \theta \in (0,1)$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2\,\mathrm{d}x \leq C\left(\int_{B_r(0)} |u(x,T;u_0)|^2\,\mathrm{d}x\right)^{\theta}\left(\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2\,\mathrm{d}x\right)^{1-\theta},\ \forall\, u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}).$$

The above optimality implies in some sense that the choice of the weight $e^{a|x|}$, $x \in \mathbb{R}^n$ (with $a > 0$) is reasonable (to ensure the type of unique continuation estimates build up in (i) and (ii) of Theorem 1.2). In plain language, other types of weights are not expected.

(b6) The proofs of Theorem 1.2 and Theorem 1.3 are mainly based on [2, Theorem 1.3], which gives an analytic interpolation inequality (see also [42]), and an estimate for some kind of the Euler's integral in high dimension built up in Lemma 2.11 of the current paper and the identity [10, (1.2)] (see (2.6) in our paper).

We next present three consequences of the above main theorems.

Theorem 1.4. *Given $r > 0$, $T > 0$ and $N > 0$, the following estimate is true for all $u_0 \in L^2(\Omega;\mathbb{C})$ with supp $u_0 \subset B_N(0)$:*

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \quad\leq\quad e^{C\left(1+\frac{rN}{T}\right)} \int_{B_r^c(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x, \tag{1.8}$$

where $C \triangleq C(n) > 0$.

Theorem 1.5. *Given x_0, $x' \in \mathbb{R}^n$, $r > 0$, $a > 0$, $b > 0$ and $T > 0$, the following inequality holds for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and $\varepsilon \in (0,1)$:*

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{-b|x-x'|}|u(x,T;u_0)|^2 \,\mathrm{d}x \qquad (1.9)\\ \leq\quad & C(x_0,x',r,a,b,T)\left(\varepsilon\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2\,\mathrm{d}x + \varepsilon e^{\varepsilon^{-1-\frac{Cb^{-1}}{(aT)\wedge r}}}\int_{B_r(x_0)}|u(x,T;u_0)|^2\,\mathrm{d}x\right), \end{aligned}$$

where

$$C(x_0,x',r,a,b,T) \triangleq \exp\left\{C\left[1+\frac{|x_0-x'|+r+b^{-1}}{(aT)\wedge r}\right]\right\},$$

with $C \triangleq C(n) > 0$.

Theorem 1.6. *Given $x_0 \in \mathbb{R}^n$, $r > 0$, $a > 0$ and $T > 0$, the following estimate is true for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and $\varepsilon \in (0,1)$:*

$$\begin{aligned} &\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \qquad (1.10)\\ \leq\quad & C(x_0,r,a,T)\left(\varepsilon\Big(\int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2\mathrm{d}x + \|u_0\|^2_{H^{n+3}(\mathbb{R}^n;\mathbb{C})}\Big) + \varepsilon e^{e^{\varepsilon^{-2}}}\int_{B_r(x_0)}|u(x,T;u_0)|^2\,\mathrm{d}x\right), \end{aligned}$$

where

$$C(x_0,r,a,T) \triangleq (1+T)^{2n+6}\exp\left\{C^{1+\frac{|x_0|+r+1}{(aT)\wedge r}}\right\},$$

with $C \triangleq C(n) > 0$.

Two notes on Theorem 1.4-Theorem 1.6 are as follows:

(c1) The inequalities in Theorem 1.4-Theorem 1.6 are different kinds of unique continuation at one time point for the Schrödinger equation. They correspond to different kinds of controllability which are not standard controllability (see Subsection 5.3).

(c2) Theorem 1.4 is a direct consequence of the conclusion (i) in Theorem 1.2. Theorem 1.5 is a consequence of Theorem 1.3. Theorem 1.6 is based on Theorem 1.3, as well as a regularity propagation property for the Schrödinger equation (presented in Lemma 3.2 of this paper).

The main novelties of this paper are as follows: (a) We build up observability estimate at two points in time for the Schrödinger equation in $\mathbb{R}^n$. (b) We present several unique continuation (or observability) inequalities at one point in time for the Schrödinger equation in $\mathbb{R}^n$. These inequalities correspond to different kinds of controllability.

(c) We find an equivalence between the observability at two different points in time and the uncertainty principle built up in [21] (see Lemma 2.3).

It should be interesting to extend our results to the following equations: (a) Schrödinger equations with nonzero potentials. (b) Homogeneous Schrödinger equations on a bounded domain. Unfortunately, we are not able to extend our results to the above cases. Let us explain the reasons. Our methods rely heavily on an identity (see (2.6)). This identity holds for the free Schrödinger equation in $\mathbb{R}^n$. For the case that either $\mathbb{R}^n$ is replaced by a bounded domain or the Schrödinger equation has a nonzero potential, we are not able to find a suitable substitute of (2.6). The next question could also be interesting. Can we extend our results to the case that $\mathbb{R}^n$ is replaced by $\mathbb{R}^n_+$ (where $\mathbb{R}^n_+ \triangleq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$)? It might be done by symmetrizing initial data in $L^2(\mathbb{R}^n_+; \mathbb{C})$.

For observability and unique continuation inequalities of Schrödinger equations, we would like to mention [5, 7, 9, 10, 11, 12, 13, 14, 15, 19, 20, 25, 26, 28, 30, 38, 44, 45] and the references therein. For the uncertainty principle, we refer the readers to [18, 21, 23, 31, 39] and the references therein. We think of that the uncertainty principle built up in some of these papers may be used to get some observability estimates for Schrödinger equations. For interpolation inequalities of heat equations, we would like to mention [3, 16, 17, 33, 34, 35, 43] and the references therein.

The rest of the paper is organized as follows: Section 2 proves Theorem 1.1-Theorem 1.3. Section 3 proves Theorem 1.4-Theorem 1.6. Section 4 provides some further comments on the main results. Section 5 presents applications of Theorem 1.1-Theorem 1.6 to the controllability for the Schrödinger equation.

2 Proofs of the main results

This section is devoted to proving Theorem 1.1-Theorem 1.3.

2.1 Proof of Theorem 1.1

In this subsection, we will prove Theorem 1.1. We first introduce in Theorem 2.1 the uncertainty principle built up in [21], then show in Lemma 2.3 the equivalence between the uncertainty principle and the observability at two points in time, finally give the proof of Theorem 1.1.

Theorem 2.1. *Given subsets $S, \Sigma \subset \mathbb{R}^n$, with $|S| < \infty$ and $|\Sigma| < \infty$, there is a positive constant*

$$C(n, S, \Sigma) \triangleq Ce^{C \min\left\{|S||\Sigma|, |S|^{1/n} w(\Sigma), |\Sigma|^{1/n} w(S)\right\}}, \tag{2.1}$$

with $C \triangleq C(n)$, so that for each $f \in L^2(\mathbb{R}^n; \mathbb{C})$,

$$\int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x \le C(n, S, \Sigma) \left(\int_{\mathbb{R}^n_x \setminus S} |f(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n_\xi \setminus \Sigma} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \right).$$

Here, $w(S)$ (or $w(\Sigma)$) denotes the mean width of S (or Σ).

Remark 2.2. For the detailed definition of $w(S)$ (the mean width of S), we refer the readers to [21]. Here, we would like to mention what follows: First, when S is an open bounded subset of $\mathbb{R}^n$, $w(S) < \infty$; Second, when S is a ball in $\mathbb{R}^n$, $w(S)$ is the diameter of the ball.

Lemma 2.3. *Let A and B be two measurable subsets of $\mathbb{R}^n$. Then the following propositions are equivalent:*
(i) There exists a positive constant $C_1(n,A,B)$ so that for each $f \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$\int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x \quad \leq \quad C_1(n,A,B)\left(\int_A |f(x)|^2 \,\mathrm{d}x + \int_B |\hat{f}(\xi)|^2 \,\mathrm{d}\xi\right). \tag{2.2}$$

(ii) There exists a positive constant $C_2\big(n,A,B\big)$ so that for each $T>0$ and each $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \quad \leq \quad C_2(n,A,B)\Big(\int_A |u_0(x)|^2 \,\mathrm{d}x + \int_{2TB} |u(x,T;u_0)|^2 \,\mathrm{d}x\Big). \tag{2.3}$$

Furthermore, when one of the above two propositions holds, the constants $C_1(n,A,B)$ and $C_2(n,A,B)$ can be chosen as the same number.

Proof. Divide the proof into the following two steps:

Step 1. To show that (i)$\Rightarrow$(ii)
Suppose that (i) is true for $C_1(n,A,B)$. We first claim that for all $T>0$ and $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$\begin{aligned} &\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \\ \leq \quad & C_1\big(n,A,B\big)\left(\int_A |u_0(x)|^2 \,\mathrm{d}x + \frac{1}{(2T)^n}\int_{2TB} |\widehat{e^{i|\xi|^2/4T}u_0}(\xi)(x/2T)|^2 \,\mathrm{d}x\right). \end{aligned} \tag{2.4}$$

Indeed, for arbitrarily fixed $T>0$ and $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$, we define a function $\widetilde{u}_0(\cdot)$ over $\mathbb{R}^n$ in the following manner:

$$\widetilde{u}_0(x) \triangleq e^{i|x|^2/4T}u_0(x),\ x \in \mathbb{R}^n. \tag{2.5}$$

It is clear that $\widetilde{u}_0 \in L^2(\mathbb{R}^n;\mathbb{C})$. Then by (i), we have (2.2), with $f=\widetilde{u}_0$, i.e.,

$$\int_{\mathbb{R}^n} |\widetilde{u}_0(x)|^2 \,\mathrm{d}x \quad \leq \quad C_1\big(n,A,B\big)\left(\int_A |\widetilde{u}_0(x)|^2 \,\mathrm{d}x + \int_B |\widehat{\widetilde{u}_0}(x)|^2 \,\mathrm{d}x\right).$$

This, along with (2.5), leads to (2.4).

We next notice from [10, (1.2)] that for all $T>0$ and $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$(2iT)^{n/2}e^{-i|x|^2/4T}u(x,T;u_0) = \widehat{e^{i|\xi|^2/4T}u_0}(\xi)(x/2T),\ x \in \mathbb{R}^n. \tag{2.6}$$

Then from (2.4) and (2.6), it follows that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \quad \leq \quad C_1\big(n,A,B\big)\left(\int_A |u_0(x)|^2 \,\mathrm{d}x + \int_{2TB} |u(x,T;u_0)|^2 \,\mathrm{d}x\right).$$

Hence, the conclusion (ii) is true, and $C_2\big(n,A,B\big)$ can be taken as $C_1\big(n,A,B\big)$.

Step 2. To prove that (ii)$\Rightarrow$(i)
Suppose that (ii) is true for $C_2(n,A,B)$. Arbitrarily fix $f \in L^2(\mathbb{R}^n;\mathbb{C})$. Define a function u_f by

$$u_f(x) = e^{-i|x|^2/2}f(x),\ x \in \mathbb{R}^n. \tag{2.7}$$

From (2.7) and (2.6) (where $u_0 = u_f$ and $T=1/2$), it follows that

$$\hat{f}(\xi) = \widehat{e^{i|x|^2/2}u_f}(x)(\xi) = (i)^{n/2}e^{-i|\xi|^2/2}u(\xi,1/2;u_f),\ \xi \in \mathbb{R}^n.$$

This, along with (2.7) and (2.3) (where $u_0 = u_f$ and $T = 1/2$), yields that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |u_f(x)|^2 \,\mathrm{d}x &\leq C_2(n, A, B) \left(\int_A |u_f(x)|^2 \,\mathrm{d}x + \int_B |u(x, 1/2; u_f)|^2 \,\mathrm{d}x \right) \\ &\leq C_2(n, A, B) \left(\int_A |f(x)|^2 \,\mathrm{d}x + \int_B |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \right).\end{aligned}$$

Hence, the conclusion (i) is true and $C_1(n, A, B)$ can be taken as $C_2(n, A, B)$.

Finally, from Step 1 - Step 2, we find that when one of (i) and (ii) is true, the constants $C_1(n, A, B)$ and $C_2(n, A, B)$ can be chosen as the same positive number. This ends the proof of this lemma.

□

We now use Theorem 2.1 and Lemma 2.3 to prove Theorem 1.1.

Proof of Theorem 1.1. Let $x',\ x'' \in \mathbb{R}^n$, $r_1,\ r_2 > 0$ and $T > S \geq 0$. Define

$$A = B^c_{r_1}(x') \ \text{ and } \ B = B^c_{r_2}(x''). \tag{2.8}$$

By Theorem 2.1, we have (2.2), where

$$(A, B) \ \text{ is replaced by } \ \big(A, \frac{B}{2(T-S)}\big) \ \text{ and } \ C_1(n, A, B) \ \text{ is replaced by } \ C\big(n, A^c, \frac{B^c}{2(T-S)}\big),$$

with $C(n, \cdot, \cdot)$ given by (2.1). Thus we can apply Lemma 2.3 to get (2.3), where

$$(A, B) \ \text{ is replaced by } \ \big(A, \frac{B}{2(T-S)}\big) \ \text{ and } \ C_2(n, A, B) \ \text{ is replaced by } \ C\big(n, A^c, \frac{B^c}{2(T-S)}\big).$$

The latter, together with (2.1) and (2.8), indicates that there exists $C > 0$ (depending only on n) so that for each $u_0 \in L^2(\mathbb{R}^n; \mathbb{C})$,

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C\big(n, A^c, \frac{B^c}{2(T-S)}\big) \left(\int_{B^c_{r_1}(x')} |u_0(x)|^2 \,\mathrm{d}x + \int_{B^c_{r_2}(x'')} |u(x, T-S; u_0)|^2 \,\mathrm{d}x \right), \tag{2.9}$$

where

$$C\big(n, A^c, \frac{B^c}{2(T-S)}\big) = Ce^{C \min\left\{ \omega_n r_1^n \omega_n r_2^n \frac{1}{2^n (T-S)^n}, \omega_n^{\frac{1}{n}} r_1 r_2 \frac{1}{T-S}, \omega_n^{\frac{1}{n}} r_2 r_1 \frac{1}{T-S} \right\}} \leq Ce^{C\omega_n^{\frac{1}{n}} r_1 r_2 \frac{1}{T-S}}, \tag{2.10}$$

with ω_n the volume of the unit ball in $\mathbb{R}^n$.

Finally, by (2.9) and (2.10), we obtain that

$$\int_{\mathbb{R}^n} |u(x, S; u_0)|^2 \,\mathrm{d}x \leq Ce^{Cc_0^{\frac{1}{n}} r_1 r_2 \frac{1}{T-S}} \times \Big(\int_{B^c_{r_1}(x')} |u(x, S; u_0)|^2 \,\mathrm{d}x + \int_{B^c_{r_2}(x'')} |u(x, T; u_0)|^2 \,\mathrm{d}x \Big).$$

Because of the conservation law of the Schrödinger equation, the above leads to the inequality in Theorem 1.1. This ends the proof of this theorem.

□

2.2 Preliminaries on Theorem 1.2 and Theorem 1.3

In the proofs of Theorem 1.2 and Theorem 1.3, an interpolation inequality plays a key role. This inequality will be presented in Lemma 2.5. To prove Lemma 2.5, we need the following Lemma 2.4:

Lemma 2.4. *There exists an absolute constant C so that for each $a > 0$ and $\beta \in \mathbb{N}^n$,*

$$\left(\int_{\mathbb{R}^n} |\xi^{2\beta}| e^{-a|\xi|}\,\mathrm{d}\xi\right)^{1/2} \leq \left(\frac{2n}{a}\right)^{n/2} \beta! \left(\frac{Cn}{a}\right)^{|\beta|} . \tag{2.11}$$

Proof. First, we observe that for all $a > 0$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\xi^{2\beta}| e^{-a|\xi|}\,\mathrm{d}\xi &\leq \int_{\mathbb{R}^n} |\xi^{2\beta}| e^{-a(\Sigma_{i=1}^n |\xi_i|/n)}\,\mathrm{d}\xi \\
&= \Pi_{i=1}^n \int_{\mathbb{R}_{\xi_i}} |\xi_i|^{2\beta_i} e^{-a|\xi_i|/n}\,\mathrm{d}\xi_i \\
&= \Pi_{i=1}^n 2 \int_0^\infty r^{2\beta_i} e^{-ar/n}\,\mathrm{d}r \\
&= \Pi_{i=1}^n 2 \left(\frac{n}{a}\right)^{2\beta_i+1} \int_0^\infty t^{2\beta_i} e^{-t}\,\mathrm{d}t \\
&= 2^n \left(\frac{n}{a}\right)^{2|\beta|+n} \Pi_{i=1}^n \Gamma(2\beta_i + 1) \\
&= 2^n \left(\frac{n}{a}\right)^{2|\beta|+n} \Pi_{i=1}^n (2\beta_i)!,
\end{aligned} \tag{2.12}$$

where $\Gamma(\cdot)$ denotes the Euler's integral of the second kind or the Gamma function.

We next claim that there is an absolute constant $C > 0$ so that

$$\sqrt{(2\alpha)!} \leq \alpha! C^\alpha \quad \text{for all} \quad \alpha \in \mathbb{N}^+. \tag{2.13}$$

In fact, using the Stirling's approximation for factorials

$$\ln(\eta!) = \eta \ln \eta - \eta + O(\ln \eta), \ \forall\, \eta \in \mathbb{N}^+,$$

we see that for all $\alpha \in \mathbb{N}^+$,

$$\begin{aligned}
\ln \sqrt{(2\alpha)!} &= \frac{1}{2}\Big(2\alpha \ln(2\alpha) - 2\alpha + O\big(\ln(2\alpha)\big)\Big) \\
&= \ln \alpha! + \alpha \ln 2 + O(\ln \alpha).
\end{aligned}$$

Thus, there exists an absolute constant $C_1 > 1$ so that

$$\sqrt{(2\alpha)!} \leq \exp\left[\ln \alpha! + \alpha \ln C_1\right] = \alpha! C_1^\alpha \quad \text{for all} \quad \alpha \in \mathbb{N}^+,$$

which leads to (2.13).

Finally, (2.11) follows from (2.12) and (2.13) at once. This ends the proof of this lemma.

$\square$

We now present an interpolation estimate for L^2-functions whose Fourier transforms have compact supports.

Lemma 2.5. *Given x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$ and $a > 0$, there exist two constants $C \triangleq C(n) > 0$ and $\theta \triangleq \theta(n) \in (0,1)$ so that for each $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$,*

$$\int_{B_{r_2}(x'')} |f(x)|^2 \,\mathrm{d}x \le C r_2^n (a^{-n} + r_1^{-n}) \left(\int_{B_{r_1}(x')} |f(x)|^2 \,\mathrm{d}x \right)^{\theta^p} \left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|} \,\mathrm{d}\xi \right)^{1-\theta^p}, \tag{2.14}$$

where

$$p \triangleq 1 + \frac{|x' - x''| + r_1 + r_2}{a \wedge r_1}.$$

Proof. The proof is divided into two steps.

Step 1. To show that there is $C \triangleq C(n) > 0$ and $\theta \triangleq \theta(n) \in (0,1)$ so that (2.14), with $a = 1$, holds for all x', $x'' \in \mathbb{R}^n$, $r_1 > 0$, $r_2 > 0$ and $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$

Arbitrarily fix x', $x'' \in \mathbb{R}^n$, $r_1 > 0$, $r_2 > 0$ and $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. We first claim that there is an absolute constant $C > 1$ so that

$$\|\partial_x^\alpha f\|_{L^\infty(\mathbb{R}^n_x)} \le (2\pi)^{-\frac{n}{2}} (2n)^{n/2} (Cn)^{|\alpha|} \alpha! \sqrt{\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{|\xi|} \,\mathrm{d}\xi} \ \text{ for all } \ \alpha \in \mathbb{N}^n. \tag{2.15}$$

In fact, since $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$, we see that f is analytic and for each multi-index $\alpha \in \mathbb{N}^n$,

$$\partial_x^\alpha f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} (i\xi)^\alpha \hat{f}(\xi) \,\mathrm{d}\xi, \ x \in \mathbb{R}^n.$$

From the above equality and the Hölder inequality, we see that for each multi-index $\alpha \in \mathbb{N}^n$,

$$\|\partial_x^\alpha f\|_{L^\infty(\mathbb{R}^n_x)} \le (2\pi)^{-\frac{n}{2}} \sqrt{\int_{\mathbb{R}^n_\xi} |\xi^{2\alpha}| e^{-|\xi|} \,\mathrm{d}\xi} \sqrt{\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{|\xi|} \,\mathrm{d}\xi}.$$

This, along with Lemma 2.4, leads to (2.15).

We next claim that there is $C_1 \triangleq C_1(n) > 0$ and $\theta_1 \triangleq \theta_1(n) \in (0,1)$ (depending only on n) so that

$$\int_{B_{r_2}(x'')} |f(x)|^2 \,\mathrm{d}x \le \omega_n r_2^n (C_1 r_0^{-n/2} + 1)^2 \left(M^2\right)^{1-\frac{\theta_1}{2^K}} \left(\int_{B_{r_1}(x')} |f(x)|^2 \,\mathrm{d}x \right)^{\frac{\theta_1}{2^K}}, \tag{2.16}$$

where

$$M \triangleq \left(\frac{n}{\pi}\right)^{n/2} \sqrt{\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{|\xi|} \,\mathrm{d}\xi}, \quad r_0 \triangleq \frac{(Cn)^{-1} \wedge r_1}{5} < 1 \tag{2.17}$$

(with C given by (2.15)) and

$$K \triangleq \frac{|x' - x''| + r_1 + r_2}{r_0}. \tag{2.18}$$

Let M and r_0 be given by (2.17). From (2.15), we see that

$$|\partial_x^\alpha f(x)| \le M \frac{\alpha!}{(5r_0)^{|\alpha|}}, \ x \in B_{4r_0}(x').$$

Then we can apply [2, Theorem 1.3] where $R = 2r_0$ (see also [42]) to find that

$$\|f\|_{L^\infty(B_{2r_0}(x'))} \quad \le \quad C_1' M^{1-\theta_1'} \left(\omega_n^{1/2} |B_{r_0}(x')|^{-1} \|f\|_{L^1(B_{r_0}(x'))} \right)^{\theta_1'},$$

for some $C_1' \triangleq C_1'(n) > 0$ and $\theta_1' \triangleq \theta_1'(n) \in (0,1)$, depending only on n. Since $r_0 < r_1$ (see (2.17)), the above inequality, along with the Hölder inequality, yields that

$$\begin{aligned} \|f\|_{L^\infty(B_{2r_0}(x'))} &\leq C_1' M^{1-\theta_1'} \left(\omega_n^{1/2}|B_{r_0}(x')|^{-1/2}\|f\|_{L^2(B_{r_0}(x'))}\right)^{\theta_1'} \\ &\leq C_1' M^{1-\theta_1'} \left(r_0^{-n/2}\|f\|_{L^2(B_{r_1}(x'))}\right)^{\theta_1'}. \end{aligned} \tag{2.19}$$

Write $D_l(z)$ for the closed disk in the complex plane, centered at z and of radius l. It is clear that

$$D_{r_0}((k+1)r_0) \subset D_{2r_0}(kr_0),\ k = 1, 2, \ldots. \tag{2.20}$$

Arbitrarily fix $\vec{v} \in S^{n-1}$. Define a function g over the real line in the following manner:

$$g(s) = \frac{1}{M} f(x' + s\vec{v}),\ s \in \mathbb{R}. \tag{2.21}$$

From (2.21) and (2.15), one can easily check that g can be extended to be an analytic function over

$$\Omega_{r_0} \triangleq \{x + iy \in \mathbb{C} \ :\ x,\, y \in \mathbb{R},\ |y| < 5r_0\} \tag{2.22}$$

and that the extension, still denoted by g, has the property:

$$\|g\|_{L^\infty(\Omega_{r_0})} \leq 1. \tag{2.23}$$

By (2.21), (2.22) and (2.23), we see that the function $z \mapsto g(4r_0 z)$ is analytic over $D_1(0)$ and verifies that $\sup_{z\in D_1(0)}|g(4r_0z)| \leq 1$. Then we can apply [2, Lemma 3.2] (to the above function) to find that

$$\sup_{z\in D_{1/2}(0)} |g(4r_0z)| \leq C_2' \sup_{x\in\mathbb{R},\ |x|\leq 1/5} |g(4r_0x)|^{\theta_2'} \tag{2.24}$$

for some $C_2' \triangleq C_2'(n) > 0$ and $\theta_2' \triangleq \theta_2'(n) \in (0,1)$, depending only on n. Since $r_0 < r_1$ (see (2.17)), by (2.24) and (2.21), we obtain that

$$\|g\|_{L^\infty(D_{2r_0}(0))} \leq C_2' \left(\frac{1}{M}\|f\|_{L^\infty(B_{2r_0}(x'))}\right)^{\theta_2'}.$$

This, along with (2.19), yields that

$$\|g\|_{L^\infty(D_{2r_0}(0))} \leq C_2' C_1'^{\theta_2'} r_0^{-\theta_1'\theta_2' n/2} \left(\frac{1}{M}\|f\|_{L^2(B_{r_1}(x'))}\right)^{\theta_1'\theta_2'}. \tag{2.25}$$

Meanwhile, since g is analytic over Ω_{r_0}, we can apply the Hadamard three-circle theorem (see for instance [2, Theorem 3.1]) to get that for each $k = 1, 2, \ldots,$

$$\|g\|_{L^\infty(D_{2r_0}(kr_0))} \leq \|g\|^{1/2}_{L^\infty(D_{r_0}(kr_0))}\|g\|^{1/2}_{L^\infty(D_{4r_0}(kr_0))} \leq \|g\|^{1/2}_{L^\infty(D_{r_0}(kr_0))}. \tag{2.26}$$

(Here, we used (2.23).) By (2.26) and (2.20), we see that for each $k = 1, 2, \ldots,$

$$\|g\|_{L^\infty(D_{r_0}((k+1)r_0))} \leq \|g\|_{L^\infty(D_{2r_0}(kr_0))} \leq \|g\|^{1/2}_{L^\infty(D_{r_0}(kr_0))},$$

from which, it follows that for each $k = 1, 2, \ldots,$

$$\|g\|_{L^\infty(D_{r_0}((k+1)r_0))} \leq \|g\|^{\frac{1}{2}}_{L^\infty(D_{r_0}(kr_0))} \leq \cdots \leq \|g\|^{(\frac{1}{2})^k}_{L^\infty(D_{r_0}(r_0))}.$$

This, along with (2.18) and (2.23), yields that

$$\begin{aligned}\|g\|_{L^\infty(\cup_{1\le k\le q}D_{r_0}(kr_0))} &= \sup_{1\le k\le q}\|g\|_{L^\infty(D_{r_0}(kr_0))}\le \sup_{1\le k\le q}\|g\|^{(\frac12)^{k-1}}_{L^\infty(D_{r_0}(r_0))}\\ &\le \sup_{1\le k\le q}\|g\|^{(\frac12)^{q-1}}_{L^\infty(D_{r_0}(r_0))}\le \|g\|^{(\frac12)^{K}}_{L^\infty(D_{r_0}(r_0))},\end{aligned}\tag{2.27}$$

where q is the integer so that

$$qr_0\ge |x'-x''|+r_1+r_2>(q-1)r_0.\tag{2.28}$$

Because it follows by (2.28) that

$$\big[0,|x'-x''|+r_1+r_2\big]\subset \cup_{1\le k\le q}D_{r_0}(kr_0)\ \text{ and }\ D_{r_0}(r_0)\subset D_{2r_0}(0),$$

we see from (2.27) that for all $s\in\big[0,|x'-x''|+r_1+r_2\big]$,

$$|g(s)|\le \|g\|_{L^\infty(\cup_{1\le k\le q}D_{r_0}(kr_0))}\le \|g\|^{(\frac12)^{K}}_{L^\infty(D_{r_0}(r_0))}\le \|g\|^{(\frac12)^{K}}_{L^\infty(D_{2r_0}(0))}.\tag{2.29}$$

From (2.21), (2.29) and (2.25), we find that for all $s\in\big[0,|x'-x''|+r_1+r_2\big]$,

$$\begin{aligned}|f(x'+s\vec v)|&=M|g(s)|\le M\|g\|^{\frac{1}{2^K}}_{L^\infty(D_{2r_0}(0))}\\ &\le M\left[C_2'{C_1'}^{\theta_2'}r_0^{-\theta_1'\theta_2'n/2}\left(\frac{1}{M}\|f\|_{L^2(B_{r_1}(x'))}\right)^{\theta_1'\theta_2'}\right]^{\frac{1}{2^K}}\\ &=\left(C_2'{C_1'}^{\theta_2'}r_0^{-\theta_1'\theta_2'n/2}\right)^{2^{-K}}M^{1-\frac{\theta_1'\theta_2'}{2^K}}\|f\|^{\frac{\theta_1'\theta_2'}{2^K}}_{L^2(B_{r_1}(x'))}.\end{aligned}$$

Since the above inequality holds for all $\vec v\in S^{n-1}$ and $s\in\big[0,|x'-x''|+r_1+r_2\big]$, we see that

$$\sup_{|x-x'|\le|x'-x''|+r_1+r_2}|f(x)|\le \left(C_2'{C_1'}^{\theta_2'}r_0^{-\theta_1'\theta_2'n/2}\right)^{2^{-K}}M^{1-\frac{\theta_1'\theta_2'}{2^K}}\|f\|^{\frac{\theta_1'\theta_2'}{2^K}}_{L^2(B_{r_1}(x'))}.$$

Because $r_0<1$ (see (2.17)), it follows from the above that

$$\sup_{|x-x'|\le|x'-x''|+r_1+r_2}|f(x)|\ \le\ \left(C_2'{C_1'}^{\theta_2'}r_0^{-n/2}+1\right)M^{1-\frac{\theta_1'\theta_2'}{2^K}}\|f\|^{\frac{\theta_1'\theta_2'}{2^K}}_{L^2(B_{r_1}(x'))}.$$

Since $B_{r_2}(x'')\subset B_{|x'-x''|+r_1+r_2}(x')$, the above yields that

$$\begin{aligned}\int_{B_{r_2}(x'')}|f(x)|^2\,\mathrm{d}x &\le \omega_n r_2^n\sup_{|x-x'|\le|x'-x''|+r_1+r_2}|f(x)|^2\\ &\le \omega_n r_2^n\left(C_2'{C_1'}^{\theta_2'}r_0^{-n/2}+1\right)^2M^{2(1-\frac{\theta_1'\theta_2'}{2^K})}\|f\|^{\frac{2\theta_1'\theta_2'}{2^K}}_{L^2(B_{r_1}(x'))},\end{aligned}$$

from which, (2.16) follows at once.

Finally, by (2.17), we see that

$$M\ge\|f\|_{L^2(B_{r_1}(x'))}\ \text{ and }\ r_0\ge\frac{(Cn)^{-1}}{5}(1\wedge r_1).$$

These, combined with (2.16) and (2.18), yield that

$$
\begin{aligned}
&\int_{B_{r_2}(x'')} |f(x)|^2 \,\mathrm{d}x \\
\leq\ & \omega_n r_2^n (1+C_1)^2 (5Cn)^n [(1\wedge r_1)^{-n/2}+1]^2 M^2 \left(\frac{\|f\|^2_{L^2(B_{r_1}(x'))}}{M^2}\right)^{\alpha_1} \\
\leq\ & 4\omega_n (1+C_1)^2 (5Cn)^n r_2^n (r_1^{-n}+1) M^2 \left(\frac{\|f\|^2_{L^2(B_{r_1}(x'))}}{M^2}\right)^{\alpha_2},
\end{aligned}
\tag{2.30}
$$

where

$$
\alpha_1 \triangleq \theta_1 \left(\frac{1}{2}\right)^{\frac{|x'-x''|+r_1+r_2}{r_0}} \quad \text{and} \quad \alpha_2 \triangleq \min\left\{\theta_1, \left(\frac{1}{2}\right)^{5Cn}\right\}^{1+\frac{|x'-x''|+r_1+r_2}{1\wedge r_1}}.
$$

From (2.30) and (2.17), we see that f satisfies (2.14), with $a=1$. This proves the conclusion in Step 1.

Step 2. To show that there is $C \triangleq C(n) > 0$ and $\theta \triangleq \theta(n) \in (0,1)$ so that (2.14), with $a>0$, holds for all x', $x'' \in \mathbb{R}^n$, $r_1 > 0$, $r_2 > 0$ and $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$

Arbitrarily fix x', $x'' \in \mathbb{R}^n$, $r_1 > 0$, $r_2 > 0$, $a > 0$ and $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Define a function g by

$$
g(x) = a^{\frac{n}{2}} f(ax),\ x \in \mathbb{R}^n.
$$

It is clear that

$$
g \in L^2(\mathbb{R}^n;\mathbb{C}) \ \text{ and } \ \hat{g}(\xi) = a^{-\frac{n}{2}} \hat{f}(\xi/a),\ \xi \in \mathbb{R}^n.
$$

Since $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$, the above implies that $\hat{g} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Thus, we can use the conclusion in Step 1 to see that there is $C>0$ and $\theta \in (0,1)$, depending only on n, so that

$$
\begin{aligned}
&\int_{B_{\frac{r_2}{a}}(\frac{x''}{a})} |g(x)|^2 \,\mathrm{d}x \\
\leq\ & C\left(\frac{r_2}{a}\right)^n \left(1+\left(\frac{r_1}{a}\right)^{-n}\right) \left(\int_{B_{\frac{r_1}{a}}(\frac{x'}{a})} |g(x)|^2 \,\mathrm{d}x\right)^{\theta^{p'}} \left(\int_{\mathbb{R}^n_\xi} |\hat{g}(\xi)|^2 e^{|\xi|} \,\mathrm{d}\xi\right)^{1-\theta^{p'}},
\end{aligned}
\tag{2.31}
$$

where

$$
p' = 1 + \frac{|\frac{x'}{a} - \frac{x''}{a}| + \frac{r_1}{a} + \frac{r_2}{a}}{1 \wedge \frac{r_1}{a}} = 1 + \frac{|x'-x''|+r_1+r_2}{a \wedge r_1}.
$$

From (2.31), we find that

$$
\begin{aligned}
&\int_{B_{r_2}(x'')} |f(x)|^2 \,\mathrm{d}x = \int_{B_{\frac{r_2}{a}}(\frac{x''}{a})} |g(x)|^2 \,\mathrm{d}x \\
\leq\ & C r_2^n (a^{-n} + r_1^{-n}) \left(\int_{B_{r_1}(x')} |f(x)|^2 \,\mathrm{d}x\right)^{\theta^{p'}} \left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|} \,\mathrm{d}\xi\right)^{1-\theta^{p'}}.
\end{aligned}
$$

This proves the conclusion in Step 2 and completes the proof of this lemma.

□

Two consequences of Lemma 2.5 will be given in order. The first one (Corollary 2.6) is another interpolation estimate for L^2-functions whose Fourier transforms have compact supports, while the second one (Corollary 2.7) is a kind of spectral inequality. (The name of spectral inequality in $\mathbb{R}^n$ arose from [37], see [37, Theorem 3.1].)

Corollary 2.6. *There exist two constants $C \triangleq C(n) > 0$ and $\theta \triangleq \theta(n) \in (0,1)$ so that for each $r > 0$, $a > 0$ and each $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$,*

$$\int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x \le C\left(1+\frac{r^n}{a^n}\right)\left(\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x\right)^{\theta^{1+\frac{r}{a}}}\left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi\right)^{1-\theta^{1+\frac{r}{a}}}. \tag{2.32}$$

Proof. Arbitrarily fix $r > 0$, $a > 0$ and $f \in L^2(\mathbb{R}^n;\mathbb{C})$, with $\hat{f} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. First of all, we claim that there exist two constants $C_1 \triangleq C_1(n) > 0$ and $\theta_1 \triangleq \theta_1(n) \in (0,1)$ so that

$$\int_{B_r(0)} |f(x)|^2 \,\mathrm{d}x \le C_1\left(1+\frac{r^n}{a^n}\right)\left(\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x\right)^{\theta_1^{1+r/a}}\left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi\right)^{1-\theta_1^{1+r/a}}. \tag{2.33}$$

Indeed, for arbitrarily fixed $\vec{v} \in S^{n-1}$, we have that $B_r(2r\vec{v}) \subset B_r^c(0)$. Then according to Lemma 2.5, where $(x', x'', r_1, r_2) = (2r\vec{v}, 0, r, r)$, there is $C_{11} \triangleq C_{11}(n) > 0$ and $\theta_{11} \triangleq \theta_{11}(n) \in (0,1)$ so that

$$\begin{aligned}
&\int_{B_r(0)} |f(x)|^2\,\mathrm{d}x \\
\le\;& C_{11} r^n (a^{-n}+r^{-n})\left(\int_{B_r(2r\vec{v})} |f(x)|^2\,\mathrm{d}x\right)^{\theta_{11}^{1+\frac{4r}{a\wedge r}}}\left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi\right)^{1-\theta_{11}^{1+\frac{4r}{a\wedge r}}} \\
\le\;& C_{11} r^n (a^{-n}+r^{-n})\left(\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x\right)^{\theta_{11}^{1+\frac{4r}{a\wedge r}}}\left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi\right)^{1-\theta_{11}^{1+\frac{4r}{a\wedge r}}}.
\end{aligned} \tag{2.34}$$

Since

$$\frac{1}{a\wedge r} \le \frac{1}{a}+\frac{1}{r},\ \theta_{11}\in(0,1)\ \text{ and }\ \int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi,$$

we find from (2.34) that

$$\begin{aligned}
\int_{B_r(0)} |f(x)|^2\,\mathrm{d}x \;&\le\; C_{11} r^n (a^{-n}+r^{-n})\left(\frac{\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi}\right)^{\theta_{11}^{1+\frac{4r}{a\wedge r}}}\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi \\
&\le\; C_{11}(r^n a^{-n}+1)\left(\frac{\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi}\right)^{\theta_{11}^{5(1+\frac{r}{a})}}\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi,
\end{aligned}$$

which leads to (2.33).

Next, since

$$\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^n_x} |f(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2\,\mathrm{d}\xi \le \int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi,$$

we have that

$$\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x \le \left(\int_{B_r^c(0)} |f(x)|^2\,\mathrm{d}x\right)^{\theta_1^{1+r/a}}\left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|\xi|}\,\mathrm{d}\xi\right)^{1-\theta_1^{1+r/a}},$$

which, together with (2.33), leads to (2.32). this ends the proof of this corollary.

$\square$

Corollary 2.7. *There exists a positive constant $C \triangleq C(n)$ so that for each $r > 0$ and $N \geq 0$,*

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x \leq e^{C(1+rN)} \int_{B_r^c(0)} |f(x)|^2 \, \mathrm{d}x \tag{2.35}$$

for all $f \in L^2(\mathbb{R}^n; \mathbb{C})$ with supp $\hat{f} \subset B_N(0)$.

Proof. The proof is divided into the following two steps:

Step 1. To show that there is $C \triangleq C(n) > 0$ so that (2.35), with $r = 1$, holds for all $N \geq 0$ and $f \in L^2(\mathbb{R}^n; \mathbb{C})$, with supp $\hat{f} \subset B_N(0)$

Arbitrarily fix $N \geq 0$ and then fix $f \in L^2(\mathbb{R}^n; \mathbb{C})$, with supp $\hat{f} \subset B_N(0)$. By a standard density argument, we can apply Corollary 2.6 to verify that there is $C_1 \triangleq C_1(n) > 0$ and $\theta_1 \triangleq \theta_1(n) \in (0, 1)$ (only depending on n) so that

$$\int_{\mathbb{R}^n_x} |f(x)|^2 \, \mathrm{d}x \leq C_1 \left(\int_{B_1^c(0)} |f(x)|^2 \, \mathrm{d}x \right)^{\theta_1} \left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{|\xi|} \, \mathrm{d}\xi \right)^{1-\theta_1}. \tag{2.36}$$

Indeed, since $\hat{f}(\xi) e^{|\xi|/2} \in L^2(\mathbb{R}^n; \mathbb{C})$, we can choose $\{g_k\} \subset C_0^\infty(\mathbb{R}^n; \mathbb{C})$, with $\operatorname{supp} g_k \subset B_k(0)$, so that

$$\lim_{k \to \infty} \int_{\mathbb{R}^n_\xi} |g_k(\xi) - \hat{f}(\xi) e^{|\xi|/2}|^2 \, \mathrm{d}\xi = 0. \tag{2.37}$$

Meanwhile, since $\operatorname{supp} g_k \subset B_k(0)$ for all $k \in \mathbb{N}^+$, we can find $\{h_k\} \subset C_0^\infty(\mathbb{R}^n; \mathbb{C})$, with $\operatorname{supp} h_k \subset B_{k+1}(0)$, so that

$$\int_{\mathbb{R}^n_\xi} |h_k(\xi) - g_k(\xi) e^{-|\xi|/2}|^2 \, \mathrm{d}\xi \leq \frac{1}{k} e^{-k-1} \text{ for each } k \in \mathbb{N}^+.$$

This implies that for each $k \in \mathbb{N}^+$,

$$\int_{\mathbb{R}^n_\xi} |h_k(\xi) e^{|\xi|/2} - g_k(\xi)|^2 \, \mathrm{d}\xi = \int_{B_{k+1}(0)} |h_k(\xi) - g_k(\xi) e^{-|\xi|/2}|^2 e^{|\xi|} \, \mathrm{d}\xi \leq 1/k,$$

which, together with (2.37), yields that

$$\lim_{k \to \infty} \int_{\mathbb{R}^n_\xi} |h_k(\xi) - \hat{f}(\xi)|^2 e^{|\xi|} \, \mathrm{d}\xi = 0. \tag{2.38}$$

Let $\{f_k\} \subset L^2(\mathbb{R}^n; \mathbb{C})$ so that

$$\hat{f}_k(\xi) = h_k(\xi), \ \xi \in \mathbb{R}^n \text{ for each } k \in \mathbb{N}^+.$$

Then by (2.38), we find that

$$\{\hat{f}_k\} \subset C_0^\infty(\mathbb{R}^n; \mathbb{C}), \ \lim_{k \to \infty} \int_{\mathbb{R}^n_\xi} |\hat{f}_k(\xi) - \hat{f}(\xi)|^2 e^{|\xi|} \, \mathrm{d}\xi = 0 \text{ and } \lim_{k \to \infty} \|f_k - f\|_{L^2(\mathbb{R}^n; \mathbb{C})} = 0.$$

From these, we can apply Corollary 2.6 (where $a = 1$ and $r = 1$) to get (2.36).

Since supp $\hat{f} \subset B_N(0)$, it follows from (2.36) that

$$\int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x \leq C_1 \left(\int_{B_1^c(0)} |f(x)|^2 \,\mathrm{d}x \right)^{\theta_1} e^{(1-\theta_1)N} \left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \right)^{1-\theta_1}.$$

Since the Fourier transform is an isometry, we obtain from the above inequality that

$$\int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x \leq C_1^{1/\theta_1} e^{(1-\theta_1)N/\theta_1} \int_{B_1^c(0)} |f(x)|^2 \,\mathrm{d}x = e^{[\ln C_1 + (1-\theta_1)N]/\theta_1} \int_{B_1^c(0)} |f(x)|^2 \,\mathrm{d}x.$$

Hence, (2.35), with $r = 1$, is true.

Step 2. To show that there is $C \triangleq C(n) > 0$ so that (2.35), with $r > 0$, holds for all $N \geq 0$ and $f \in L^2(\mathbb{R}^n; \mathbb{C})$, with supp $\hat{f} \subset B_N(0)$

For this purpose, arbitrarily fix $N \geq 0$ and $r > 0$. Then fix $f \in L^2(\mathbb{R}^n; \mathbb{C})$ with supp $\hat{f} \subset B_N(0)$. Define a function g by

$$g(x) = r^{n/2} f(rx), \ x \in \mathbb{R}^n. \tag{2.39}$$

One can easily check that

$$\hat{g}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n_x} r^{n/2} f(rx) e^{-ix\cdot\xi} \,\mathrm{d}x = r^{-n/2} \hat{f}(\xi/r) \ \text{ for a.e. } \ \xi \in \mathbb{R}^n. \tag{2.40}$$

Since supp $\hat{f} \subset B_N(0)$, we see from (2.40) that supp $\hat{g} \subset B_{rN}(0)$. Thus, according to the conclusion in Step 1, there is $C \triangleq C(n)$ so that (2.35), with (f, r, N) replaced by $(g, 1, rN)$, is true. That is,

$$\int_{\mathbb{R}^n} |g(x)|^2 \,\mathrm{d}x \leq e^{C(1+rN)} \int_{B_1^c(0)} |g(x)|^2 \,\mathrm{d}x.$$

This, along with (2.39) and (2.40), yields that

$$\begin{aligned} & \int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |g(x)|^2 \,\mathrm{d}x \\ \leq \ & e^{C(1+rN)} \int_{B_1^c(0)} |g(x)|^2 \,\mathrm{d}x = e^{C(1+rN)} \int_{B_r^c(0)} |f(x)|^2 \,\mathrm{d}x. \end{aligned}$$

Hence, (2.35), with $r > 0$ is true. We end the proof of this corollary.

□

2.3 Proofs of Theorem 1.2 and Theorem 1.3

We first prove Theorem 1.2.

Proof of Theorem 1.2. Throughout this proof, we arbitrarily fix

$$r > 0, \ a > 0, \ T > 0 \ \text{ and } \ u_0 \in C_0^\infty(\mathbb{R}^n; \mathbb{C}).$$

Define a function f as follows:

$$f(x) \triangleq e^{-i|x|^2/4T} u(x, T; u_0), \ x \in \mathbb{R}^n. \tag{2.41}$$

From (2.41) and (2.6), we find that

$$(2iT)^{n/2}f(x) = \widehat{e^{i|\xi|^2/4T}u_0}(\xi)(x/2T),\ x\in\mathbb{R}^n.$$

This yields that for a.e. $\xi\in\mathbb{R}^n$,

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n_x} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x = \frac{(2iT)^{-n/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n_x}(2iT)^{n/2}f(x)e^{-ix\cdot\xi}\,\mathrm{d}x\\
&= \frac{(2iT)^{-n/2}}{(2\pi)^{n/2}}(2T)^n\int_{\mathbb{R}^n_x}(2iT)^{n/2}f(2Tx)e^{-ix\cdot(2T\xi)}\,\mathrm{d}x\\
&= \frac{(-2iT)^{n/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n_x}\widehat{e^{i|\eta|^2/4T}u_0}(\eta)(x)e^{ix\cdot(-2T\xi)}\,\mathrm{d}x\\
&= (-2iT)^{n/2}e^{i|\eta|^2/4T}u_0(\eta)|_{\eta=-2T\xi} = (-2iT)^{n/2}e^{iT|\xi|^2}u_0(-2T\xi).
\end{aligned}\tag{2.42}$$

We are going to prove the conclusions (i)-(iii) in the theorem one by one.

We first show the conclusion (i) of Theorem 1.2. By (2.41), we have that

$$\int_{\mathbb{R}^n_x}|u(x,T;u_0)|^2\,\mathrm{d}x = \int_{\mathbb{R}^n_x}|f(x)|^2\,\mathrm{d}x.$$

Then by Corollary 2.6, where a is replaced by $2Ta$, we find that

$$\begin{aligned}
\int_{\mathbb{R}^n_x}|u(x,T;u_0)|^2\,\mathrm{d}x &\le C\left(1+\frac{r^n}{(2Ta)^n}\right)\left(\frac{\int_{B^c_r(0)}|f(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^n_\xi}|\hat{f}(\xi)|^2e^{2Ta|\xi|}\,\mathrm{d}\xi}\right)^{\theta^{1+\frac{r}{2Ta}}}\int_{\mathbb{R}^n_\xi}|\hat{f}(\xi)|^2e^{2Ta|\xi|}\,\mathrm{d}\xi\\
&\le C\left(1+\frac{r^n}{(Ta)^n}\right)\left(\frac{\int_{B^c_r(0)}|f(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^n_\xi}|\hat{f}(\xi)|^2e^{2Ta|\xi|}\,\mathrm{d}\xi}\right)^{\theta^{1+\frac{r}{Ta}}}\int_{\mathbb{R}^n_\xi}|\hat{f}(\xi)|^2e^{2Ta|\xi|}\,\mathrm{d}\xi,
\end{aligned}$$

for some $C\triangleq C(n)>0$ and $\theta\triangleq\theta(n)\in(0,1)$ (depending only on n). From this, (2.41) and (2.42), after some computations, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n_x}|u(x,T;u_0)|^2\,\mathrm{d}x\\
\le\ & C\left(1+\frac{r^n}{(aT)^n}\right)\left(\int_{B^c_r(0)}|u(x,T;u_0)|^2\,\mathrm{d}x\right)^{\theta^{1+\frac{r}{aT}}}\left(\int_{\mathbb{R}^n_\xi}|u_0(\xi)|^2e^{a|\xi|}\,\mathrm{d}\xi\right)^{1-\theta^{1+\frac{r}{aT}}}.
\end{aligned}$$

The above inequality, together with the conservation law of the Schrödinger equation, leads to (1.4). Hence, the conclusion (i) of the theorem is true.

We next show the conclusion (ii) of Theorem 1.2. Arbitrarily fix $\beta>1$ and $\gamma\in(0,1)$. We divide the proof into the following two steps:

Step 1. To show that there exists $C\triangleq C(n)$ so that

$$\int_{\mathbb{R}^n_x}|f(x)|^2\,\mathrm{d}x \le Ce^{\left(\frac{C^\beta r^\beta}{aT^\beta(1-\gamma)}\right)^{\frac{1}{\beta-1}}}\left(\int_{B^c_r(0)}|f(x)|^2\,\mathrm{d}x\right)^\gamma\left(\int_{\mathbb{R}^n_\xi}e^{a|2T\xi|^\beta}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1-\gamma}\tag{2.43}$$

Indeed, for an arbitrarily fixed $N\ge 0$, we define two functions g_1 and g_2 in $L^2(\mathbb{R}^n;\mathbb{C})$ so that

$$\hat{g}_1\triangleq\chi_{B_N(0)}\hat{f}\ \text{ and }\ \hat{g}_2\triangleq\chi_{B^c_N(0)}\hat{f}.$$

It is clear that $f = g_1 + g_2$ in $L^2(\mathbb{R}^n;\mathbb{C})$. Then by applying Corollary 2.7 to g_1, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n_x} |f(x)|^2\,\mathrm{d}x &\leq 2\int_{\mathbb{R}^n_x} |g_1(x)|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^n_x} |g_2(x)|^2\,\mathrm{d}x \\ &\leq 2e^{C(1+rN)}\int_{B^c_r(0)} |g_1(x)|^2\,\mathrm{d}x + 2\int_{\mathbb{R}^n_x} |g_2(x)|^2\,\mathrm{d}x \\ &\leq 4e^{C(1+rN)}\int_{B^c_r(0)} \left(|f(x)|^2 + |g_2(x)|^2\right)\mathrm{d}x + 2\int_{\mathbb{R}^n_x} |g_2(x)|^2\,\mathrm{d}x \\ &\leq 4e^{C(1+rN)}\int_{B^c_r(0)} |f(x)|^2\,\mathrm{d}x + 6e^{C(1+rN)}\int_{\mathbb{R}^n_x} |g_2(x)|^2\,\mathrm{d}x, \end{aligned} \tag{2.44}$$

for some $C > 0$, depending only on n. Meanwhile, since the Fourier transform is an isometry, we have that

$$\begin{aligned}\int_{\mathbb{R}^n_x} |g_2(x)|^2\,\mathrm{d}x &= \int_{\mathbb{R}^n_\xi} |\hat{g_2}(\xi)|^2\,\mathrm{d}\xi = \int_{\mathbb{R}^n_\xi} |\chi_{B^c_N(0)}(\xi)\hat{f}(\xi)|^2\,\mathrm{d}\xi \\ &= e^{-a(2TN)^\beta}\int_{\mathbb{R}^n_\xi} |\chi_{B^c_N(0)}(\xi)\hat{f}(\xi)|^2 e^{a(2TN)^\beta}\,\mathrm{d}\xi.\end{aligned}$$

This, along with (2.44), yields that

$$\int_{\mathbb{R}^n_x} |f(x)|^2\,\mathrm{d}x \leq 4e^{C(1+rN)}\int_{B^c_r(0)} |f(x)|^2\,\mathrm{d}x + 6e^{C(1+rN)-a(2TN)^\beta}\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|2T\xi|^\beta}\,\mathrm{d}\xi. \tag{2.45}$$

Since it follows from the Young inequality that

$$\begin{aligned} CrN &= \left[Cr\big((1-\gamma)a(2T)^\beta\big)^{-\frac{1}{\beta}}\right]\left[\big((1-\gamma)a(2T)^\beta\big)^{\frac{1}{\beta}}N\right] \\ &\leq (1-\frac{1}{\beta})\left[Cr\big((1-\gamma)a(2T)^\beta\big)^{-\frac{1}{\beta}}\right]^{\frac{\beta}{\beta-1}} + \frac{1}{\beta}\left[\big((1-\gamma)a(2T)^\beta\big)^{\frac{1}{\beta}}N\right]^\beta \\ &\leq \left[(Cr)^\beta/\big(a(2T)^\beta(1-\gamma)\big)\right]^{\frac{1}{\beta-1}} + (1-\gamma)a(2TN)^\beta,\end{aligned}$$

we get from (2.45) that

$$\begin{aligned}&\int_{\mathbb{R}^n_x} |f(x)|^2\,\mathrm{d}x \\ \leq\ & 6e^{C+\left(\frac{C^\beta r^\beta}{a(2T)^\beta(1-\gamma)}\right)^{\frac{1}{\beta-1}}}\left(e^{(1-\gamma)a(2TN)^\beta}\int_{B^c_r(0)} |f(x)|^2\,\mathrm{d}x + e^{-\gamma a(2TN)^\beta}\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|2T\xi|^\beta}\,\mathrm{d}\xi\right).\end{aligned}$$

Since N was arbitrarily taken from $[0,\infty)$, the above indicates that for all $\varepsilon \in (0,1)$,

$$\int_{\mathbb{R}^n_x} |f(x)|^2\,\mathrm{d}x \leq 6e^{C+\left(\frac{C^\beta r^\beta}{a(2T)^\beta(1-\gamma)}\right)^{\frac{1}{\beta-1}}}\left(\varepsilon^{-(1-\gamma)}\int_{B^c_r(0)} |f(x)|^2\,\mathrm{d}x + \varepsilon^\gamma\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{a|2T\xi|^\beta}\,\mathrm{d}\xi\right).$$

One can directly check that the above inequality holds for all $\varepsilon > 0$. Minimizing it w.r.t. $\varepsilon > 0$ leads to (2.43). Here, we used the inequality:

$$\inf_{\varepsilon>0}\left(\varepsilon^{-(1-\gamma)}A + \varepsilon^\gamma B\right) \leq 2A^\gamma B^{1-\gamma} \ \text{ for all } \ A,\ B \geq 0.$$

This ends the proof of Step 1.

Step 2. To prove (1.5)

From (2.41), (2.43) and (2.42), after some computations, we see that

$$
\begin{aligned}
&\int_{\mathbb{R}^n_x} |u(x,T;u_0)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x \\
\leq\;& Ce^{\left(\frac{C^\beta r^\beta}{aT^\beta(1-\gamma)}\right)^{\frac{1}{\beta-1}}} \left(\int_{B^c_r(0)} |f(x)|^2 \,\mathrm{d}x\right)^{\gamma} \left(\int_{\mathbb{R}^n_\xi} e^{a|2T\xi|^\beta} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi\right)^{1-\gamma} \\
\leq\;& Ce^{\left(\frac{C^\beta r^\beta}{aT^\beta(1-\gamma)}\right)^{\frac{1}{\beta-1}}} \left(\int_{B^c_r(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x\right)^{\gamma} \left(\int_{\mathbb{R}^n_\xi} |u_0(\xi)|^2 e^{a|\xi|^\beta} \,\mathrm{d}\xi\right)^{1-\gamma},
\end{aligned}
$$

which, along with the conservation law of the Schrödinger equation, leads to (1.5). This ends the proof of the conclusion (ii).

(iii) By contradiction, suppose that the conclusion (iii) was not true. Then there would exist $\hat{r} > 0$, $\hat{a} > 0$, $\widehat{T} > 0$, $\hat{\gamma} \in (0,1)$, $\hat{C} > 0$ and an increasing function $\hat{\alpha}(s)$ defined over $[0,\infty)$, with $\lim_{s\to\infty} s^{-1}\hat{\alpha}(s) = 0$, so that for each $v_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$, the solution of (1.2) satisfies that

$$
\int_{\mathbb{R}^n} |v_0(x)|^2 \,\mathrm{d}x \leq \hat{C} \left(\int_{B^c_{\hat{r}}(0)} |u(x,\widehat{T};v_0)|^2 \,\mathrm{d}x\right)^{\hat{\gamma}} \left(\int_{\mathbb{R}^n} e^{\hat{a}\hat{\alpha}(|x|)} |v_0(x)|^2 \,\mathrm{d}x\right)^{1-\hat{\gamma}}. \tag{2.46}
$$

Arbitrarily fix $g \in L^2(\mathbb{R}^n;\mathbb{C})$ with $\hat{g} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Define $v_{0,g} \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ in the following manner:

$$
\hat{g}(\xi) = (-2i\widehat{T})^{n/2} e^{i\widehat{T}|\xi|^2} v_{0,g}(-2\widehat{T}\xi),\ \xi \in \mathbb{R}^n. \tag{2.47}
$$

One can easily check that

$$
g(x) = e^{-i|x|^2/4\widehat{T}} u(x,\widehat{T};v_{0,g}),\ x \in \mathbb{R}^n. \tag{2.48}
$$

Indeed, let f_g verify that

$$
f_g(x) = e^{-i|x|^2/4\widehat{T}} u(x,\widehat{T};v_{0,g}),\ x \in \mathbb{R}^n. \tag{2.49}
$$

Then by (2.41), (2.42) (where $(T,u_0) = (\widehat{T}, v_{0,g})$) and (2.47), we find that

$$
\hat{f}_g(\xi) = (-2i\widehat{T})^{n/2} e^{i\widehat{T}|\xi|^2} v_{0,g}(-2\widehat{T}\xi) = \hat{g}(\xi),\ \xi \in \mathbb{R}^n,
$$

which implies that $f_g = g$. This, along with (2.49), leads to (2.48).

By (2.48), the conservation law (for the Schrödinger equation), (2.46) and (2.47), we get that

$$
\begin{aligned}
\int_{\mathbb{R}^n_x} |g(x)|^2 \,\mathrm{d}x &= \int_{\mathbb{R}^n_x} |u(x,\widehat{T};v_{0,g})|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n_x} |v_{0,g}(x)|^2 \,\mathrm{d}x \\
&\leq \hat{C} \left(\int_{B^c_{\hat{r}}(0)} |u(x,\widehat{T};v_{0,g})|^2 \,\mathrm{d}x\right)^{\hat{\gamma}} \left(\int_{\mathbb{R}^n} e^{\hat{a}\hat{\alpha}(|x|)} |v_{0,g}(x)|^2 \,\mathrm{d}x\right)^{1-\hat{\gamma}} \\
&= \hat{C} \left(\int_{B^c_{\hat{r}}(0)} |g(x)|^2 \,\mathrm{d}x\right)^{\hat{\gamma}} \left(\int_{\mathbb{R}^n_\xi} e^{\hat{a}\hat{\alpha}(2\widehat{T}|\xi|)} |\hat{g}(\xi)|^2 \,\mathrm{d}\xi\right)^{1-\hat{\gamma}}.
\end{aligned}
$$

By this, using a standard density argument, we can show that for each $g \in L^2(\mathbb{R}^n;\mathbb{C})$ with $\operatorname{supp}\hat{g}$ compact,

$$
\int_{\mathbb{R}^n_x} |g(x)|^2 \,\mathrm{d}x \leq \hat{C} \left(\int_{B^c_{\hat{r}}(0)} |g(x)|^2 \,\mathrm{d}x\right)^{\hat{\gamma}} \left(\int_{\mathbb{R}^n_\xi} e^{\hat{a}\hat{\alpha}(2\widehat{T}|\xi|)} |\hat{g}(\xi)|^2 \,\mathrm{d}\xi\right)^{1-\hat{\gamma}}.
$$

Since $\hat{\alpha}(\cdot)$ is increasing and because the Fourier transform is an isometry, the above yields that that for each $N \geq 1$ and each $g \in L^2(\mathbb{R}^n;\mathbb{C})$ with $\operatorname{supp}\hat{g} \subset B_N(0)$,

$$\begin{aligned}\int_{\mathbb{R}^n_x} |g(x)|^2\,\mathrm{d}x &\leq \hat{C}\left(\int_{B^c_{\hat{r}}(0)} |g(x)|^2\,\mathrm{d}x\right)^{\hat{\gamma}}\left(\int_{\mathbb{R}^n_\xi} e^{\hat{a}\hat{\alpha}(2\widehat{T}N)}|\widehat{g}(\xi)|^2\,\mathrm{d}\xi\right)^{1-\hat{\gamma}} \\ &= \hat{C}e^{(1-\hat{\gamma})\hat{a}\hat{\alpha}(2\widehat{T}N)}\left(\int_{B^c_{\hat{r}}(0)} |g(x)|^2\,\mathrm{d}x\right)^{\hat{\gamma}}\left(\int_{\mathbb{R}^n_x} |g(x)|^2\,\mathrm{d}x\right)^{1-\hat{\gamma}}. \end{aligned} \tag{2.50}$$

Two observations are given in order: First, according to [37, Proposition 3.4], there is $C_0 > 0$ and $N_0 > 0$ so that for each $N \geq N_0$, there is $f_N \in L^2(\mathbb{R}^n;\mathbb{C}) \setminus \{0\}$ with $\operatorname{supp}\widehat{f_N} \subset B_N(0)$ such that

$$e^{C_0 N}\int_{B^c_{\hat{r}}(0)} |f_N(x)|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^n_x} |f_N(x)|^2\,\mathrm{d}x.$$

Second, (2.50) implies that $N \geq 1$ and each $g \in L^2(\mathbb{R}^n;\mathbb{C})$ with $\operatorname{supp}\hat{g} \subset B_N(0)$,

$$\int_{\mathbb{R}^n_x} |g(x)|^2\,\mathrm{d}x \leq \hat{C}^{\frac{1}{\hat{\gamma}}}e^{\frac{1-\hat{\gamma}}{\hat{\gamma}}\hat{a}\hat{\alpha}(2\widehat{T}N)}\int_{B^c_{\hat{r}}(0)} |g(x)|^2\,\mathrm{d}x.$$

These two observations show that for each $N \geq N_0$,

$$e^{C_0 N} \leq \hat{C}^{\frac{1}{\hat{\gamma}}}e^{\frac{1-\hat{\gamma}}{\hat{\gamma}}\hat{a}\hat{\alpha}(2\widehat{T}N)},$$

from which, it follows that

$$0 < \frac{\hat{\gamma}C_0}{2(1-\hat{\gamma})\hat{a}\widehat{T}} \leq \lim_{N\to\infty}\frac{\hat{\alpha}(2\widehat{T}N)}{2\widehat{T}N}.$$

This leads to a contradiction, since $\lim_{s\to\infty} s^{-1}\hat{\alpha}(s) = 0$. Hence, the conclusion (iii) is true.

In summary, we finish the proof of this theorem.

$\square$

We are on the position to prove Theorem 1.3.

Proof of Theorem 1.3. Arbitrarily fix x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$, $a > 0$, $T > 0$ and $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Define a function f as follows:

$$f(x) \triangleq e^{-i|x|^2/4T}u(x,T;u_0),\ x \in \mathbb{R}^n. \tag{2.51}$$

By the same way to get (2.42), we obtain that

$$\hat{f}(\xi) = (-2iT)^{n/2}e^{iT|\xi|^2}u_0(-2T\xi),\ \xi \in \mathbb{R}^n.$$

This, along with (2.51) and Lemma 2.5 (where a is replaced by $2aT$), yields that

$$\begin{aligned}&\int_{B_{r_2}(x'')} |u(x,T;u_0)|^2\,\mathrm{d}x = \int_{B_{r_2}(x'')} |f(x)|^2\,\mathrm{d}x \\ \leq\ & C_1 r_2^n\left((2aT)^{-n} + r_1^{-n}\right)\left(\int_{B_{r_1}(x')} |f(x)|^2\,\mathrm{d}x\right)^{\theta_1^{p_1}}\left(\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 e^{2aT|\xi|}\,\mathrm{d}\xi\right)^{1-\theta_1^{p_1}} \\ \leq\ & C_1 r_2^n\left((aT)^{-n} + r_1^{-n}\right)\left(\int_{B_{r_1}(x')} |u(x,T;u_0)|^2\,\mathrm{d}x\right)^{\theta_1^{p_1}}\left(\int_{\mathbb{R}^n_x} |u_0(x)|^2 e^{a|x|}\,\mathrm{d}x\right)^{1-\theta_1^{p_1}} \end{aligned} \tag{2.52}$$

for some $C_1 \triangleq C_1(n) > 0$ and $\theta_1 \triangleq \theta_1(n) \in (0,1)$, where

$$p_1 \triangleq 1 + \frac{|x' - x''| + r_1 + r_2}{(2aT) \wedge r_1}.$$

Since

$$(aT)^{-1} + r_1^{-1} \leq 2((aT) \wedge r_1)^{-1}, \ \ (aT) \wedge r_1 \leq (2aT) \wedge r_1 \ \text{ and } \ \theta_1 \in (0,1),$$

we get from (2.52) that

$$\begin{aligned}
&\int_{B_{r_2}(x'')} |u(x,T;u_0)|^2 \,\mathrm{d}x \\
\leq \ & C_1 r_2^n \big((aT)^{-1} + r_1^{-1}\big)^n \int_{\mathbb{R}^n} |u_0(x)|^2 e^{a|x|} \,\mathrm{d}x \left(\frac{\int_{B_{r_1}(x')} |u(x,T;u_0)|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^n} |u_0(x)|^2 e^{a|x|} \,\mathrm{d}x} \right)^{\theta_1^{\beta_1}} \\
\leq \ & C_1 r_2^n 2^n \big((aT) \wedge r_1\big)^{-n} \int_{\mathbb{R}^n} |u_0(x)|^2 e^{a|x|} \,\mathrm{d}x \left(\frac{\int_{B_{r_1}(x')} |u(x,T;u_0)|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^n} |u_0(x)|^2 e^{a|x|} \,\mathrm{d}x} \right)^{\theta_1^{\beta_2}},
\end{aligned}$$

with

$$\beta_1 \triangleq 1 + \frac{|x' - x''| + r_1 + r_2}{(2aT) \wedge r_1} \ \text{ and } \ \beta_2 \triangleq 1 + \frac{|x' - x''| + r_1 + r_2}{(aT) \wedge r_1}.$$

This implies that (1.6) is true. We end the proof of this theorem.

$\square$

3 Proofs of Theorem 1.4-Theorem 1.6

Theorem 1.4 is indeed a direct consequence of Theorem 1.2, while the proofs of both Theorem 1.5 and Theorem 1.6 rely on Theorem 1.3 and other properties. We begin with the proof of Theorem 1.4.

Proof of Theorem 1.4. Arbitrarily fix $r > 0$, $T > 0$, $N > 0$ and $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$ with $\operatorname{supp} u_0 \subset B_N(0)$. By a standard density argument, we can apply (i) of Theorem 1.2 (where $a = \frac{r}{T}$) to get that for some $C \triangleq C(n) > 0$ and $\theta \triangleq \theta(n) \in (0,1)$ (depending only on n),

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq 2C \left(\int_{B_r^c(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x \right)^{\theta^2} \left(\int_{\mathbb{R}^n} e^{\frac{r}{T}|x|} |u_0(x)|^2 \,\mathrm{d}x \right)^{1-\theta^2} \tag{3.1}$$

At the same time, since $\operatorname{supp} u_0 \subset B_N(0)$, we have that

$$\int_{\mathbb{R}^n} e^{\frac{r}{T}|x|} |u_0(x)|^2 \,\mathrm{d}x \leq e^{\frac{r}{T}N} \int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x.$$

This, along with (3.1), yields that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq (2C)^{\frac{1}{\theta^2}} e^{\frac{1-\theta^2}{\theta^2} \frac{rN}{T}} \int_{B_r^c(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x.$$

Hence, (1.8) stands. This ends the proof of Theorem 1.4.

$\square$

The following lemma will be used in the proofs of Theorem 1.5 and Theorem 1.6.

Lemma 3.1. *Let $x \in (0,1)$ and $\theta \in (0,1)$. Then the following conclusions are true:*
(i) For each $a > 0$,

$$\sum_{k=1}^{\infty} x^{\theta^k} e^{-ak} \le \frac{e^a}{|\ln\theta|}\Gamma\Big(\frac{a}{|\ln\theta|}\Big)|\ln x|^{-\frac{a}{|\ln\theta|}}, \tag{3.2}$$

where $\Gamma(\cdot)$ denotes the Euler's integral of the second kind.
(ii) For each $\varepsilon > 0$ and $\alpha > 0$,

$$\sum_{k=1}^{\infty} x^{\theta^k} k^{-1-\varepsilon} \le \frac{4}{\varepsilon}\alpha^{\varepsilon} e^{\varepsilon\ln\varepsilon+\varepsilon+e\alpha^{-1}\theta^{-1}}\big(\ln(\alpha|\ln x|+e)\big)^{-\varepsilon}. \tag{3.3}$$

Proof. (i) Since $\theta \in (0,1)$, it follows that

$$\sum_{k=1}^{\infty} x^{\theta^k} e^{-ak} \le \sum_{k=1}^{\infty}\int_k^{k+1} x^{\theta^\tau} e^{-a(\tau-1)}\,\mathrm{d}\tau = e^a\int_1^{\infty} x^{\theta^\tau} e^{-a\tau}\,\mathrm{d}\tau. \tag{3.4}$$

Next, because $x \in (0,1)$, we find that

$$x^{\theta^\tau} = \exp[-e^{\ln|\ln x|+\tau\ln\theta}].$$

Then, by changing variable $s = \ln|\ln x| + \tau\ln\theta$ and noticing that $\theta \in (0,1)$, we find that

$$\begin{aligned}\int_1^{\infty} x^{\theta^\tau} e^{-a\tau}\,\mathrm{d}\tau &= \int_{-\infty}^{\ln|\ln x|+\ln\theta} \frac{1}{|\ln\theta|} e^{-e^s} e^{\frac{a}{|\ln\theta|}(s-\ln|\ln x|)}\,\mathrm{d}s \\ &= \frac{|\ln x|^{-\frac{a}{|\ln\theta|}}}{|\ln\theta|}\int_{-\infty}^{\ln|\ln x|+\ln\theta} e^{\frac{a}{|\ln\theta|}s-e^s}\,\mathrm{d}s,\end{aligned}$$

from which, it follows that

$$\begin{aligned}&\int_1^{\infty} x^{\theta^\tau} e^{-a\tau}\,\mathrm{d}\tau = \frac{|\ln x|^{-\frac{a}{|\ln\theta|}}}{|\ln\theta|}\int_{-\infty}^{\ln|\ln x|+\ln\theta} (e^s)^{\frac{a}{|\ln\theta|}} e^{-e^s} e^{-s}\,\mathrm{d}e^s \\ =\;& \frac{|\ln x|^{-\frac{a}{|\ln\theta|}}}{|\ln\theta|}\int_0^{|\ln x|\theta} \eta^{\frac{a}{|\ln\theta|}-1} e^{-\eta}\,\mathrm{d}\eta \le \frac{|\ln x|^{-\frac{a}{\ln\theta}}}{|\ln\theta|}\int_0^{\infty} \eta^{\frac{a}{|\ln\theta|}-1} e^{-\eta}\,\mathrm{d}\eta.\end{aligned}$$

This, along with (3.4), leads to (3.2) and ends the proof of the conclusion (i).

(ii) Since $\theta \in (0,1)$, it follows that

$$\sum_{k=1}^{\infty} x^{\theta^k} k^{-1-\varepsilon} \le \sum_{k=1}^{\infty}\frac{(k+1)^{1+\varepsilon}}{k^{1+\varepsilon}}\int_k^{k+1} x^{\theta^\tau}\tau^{-1-\varepsilon}\,\mathrm{d}\tau \le 2^{1+\varepsilon}\int_1^{\infty} x^{\theta^\tau}\tau^{-1-\varepsilon}\,\mathrm{d}\tau. \tag{3.5}$$

Next, because $x \in (0,1)$, we see that

$$x^{\theta^\tau} = \exp[-|\ln x|e^{\tau\ln\theta}].$$

Since $\theta \in (0,1)$, the above yields that

$$\begin{aligned}&\int_1^{\infty} x^{\theta^\tau}\tau^{-1-\varepsilon}\,\mathrm{d}\tau = |\ln\theta|^{\varepsilon}\int_{-\infty}^{\ln\theta} e^{-|\ln x|e^s}|s|^{-1-\varepsilon}\,\mathrm{d}s \\ =\;& |\ln\theta|^{\varepsilon}\int_{-\infty}^{\ln\theta} e^{-\frac{|\ln x|}{e^{-s}}}|\ln e^{-s}|^{-1-\varepsilon}(-e^s)\,\mathrm{d}e^{-s} = |\ln\theta|^{\varepsilon}\int_{\frac{1}{\theta}}^{\infty} e^{-\frac{|\ln x|}{\eta}}|\ln\eta|^{-1-\varepsilon}\eta^{-1}\,\mathrm{d}\eta.\end{aligned}$$

From this, we find that for each $N \geq \frac{1}{\theta}$,

$$\begin{aligned}\int_1^\infty x^{\theta^\tau}\tau^{-1-\varepsilon}\,\mathrm{d}\tau &= |\ln\theta|^\varepsilon\Big[\int_{\frac{1}{\theta}}^N e^{-\frac{|\ln x|}{\eta}}|\ln\eta|^{-1-\varepsilon}\eta^{-1}\,\mathrm{d}\eta+\int_N^\infty e^{-\frac{|\ln x|}{\eta}}|\ln\eta|^{-1-\varepsilon}\eta^{-1}\,\mathrm{d}\eta\Big]\\ &\leq |\ln\theta|^\varepsilon\Big[e^{-\frac{|\ln x|}{N}}\int_{\frac{1}{\theta}}^N|\ln\eta|^{-1-\varepsilon}\eta^{-1}\,\mathrm{d}\eta+\int_N^\infty|\ln\eta|^{-1-\varepsilon}\eta^{-1}\,\mathrm{d}\eta\Big]\\ &= \frac{1}{\varepsilon}\Big[e^{-\frac{|\ln x|}{N}}\big(1-|\ln\theta|^\varepsilon(\ln N)^{-\varepsilon}\big)+|\ln\theta|^\varepsilon(\ln N)^{-\varepsilon}\Big]\\ &\leq \frac{1}{\varepsilon}\big[e^{-\frac{|\ln x|}{N}}+|\ln\theta|^\varepsilon(\ln N)^{-\varepsilon}\big].\end{aligned}$$

Let $\alpha>0$. Taking $N=\sqrt{\alpha|\ln x|+e\theta^{-2}}$ in the above inequality leads to that

$$\int_1^\infty x^{\theta^\tau}\tau^{-1-\varepsilon}\,\mathrm{d}\tau\leq\frac{1}{\varepsilon}\Big[e^{-\frac{|\ln x|}{\sqrt{\alpha|\ln x|+e\theta^{-2}}}}+|\ln\theta|^\varepsilon2^\varepsilon\big(\ln(\alpha|\ln x|+e\theta^{-2})\big)^{-\varepsilon}\Big] \tag{3.6}$$

Since

$$\begin{aligned}-\frac{|\ln x|}{\sqrt{\alpha|\ln x|+e\theta^{-2}}} &= -\frac{\alpha^{-1}(\alpha|\ln x|+e\theta^{-2})}{\sqrt{\alpha|\ln x|+e\theta^{-2}}}+\frac{\alpha^{-1}e\theta^{-2}}{\sqrt{\alpha|\ln x|+e\theta^{-2}}}\\ &\leq -\alpha^{-1}\sqrt{\alpha|\ln x|+e\theta^{-2}}+\alpha^{-1}e\theta^{-1},\end{aligned}$$

and because

$$0<\theta<1\ \text{ and }\ (\ln s)^\varepsilon\leq\alpha^\varepsilon e^{\varepsilon\ln\varepsilon-\varepsilon+\alpha^{-1}s}\ \text{ for all }\ s>1,$$

we find from (3.6) that

$$\begin{aligned}&\int_1^\infty x^{\theta^\tau}\tau^{-1-\varepsilon}\,\mathrm{d}\tau\leq\frac{1}{\varepsilon}\Big[e^{-\alpha^{-1}\sqrt{\alpha|\ln x|+e\theta^{-2}}+\alpha^{-1}e\theta^{-1}}+|\ln\theta^{-1}|^\varepsilon2^\varepsilon\big(\ln(\alpha|\ln x|+e\theta^{-2})\big)^{-\varepsilon}\Big]\\ \leq\;&\frac{1}{\varepsilon}\Big[\alpha^\varepsilon e^{\varepsilon\ln\varepsilon-\varepsilon+e\alpha^{-1}\theta^{-1}}2^\varepsilon\big(\ln(\alpha|\ln x|+e\theta^{-2})\big)^{-\varepsilon}+\alpha^\varepsilon e^{\varepsilon\ln\varepsilon-\varepsilon+\alpha^{-1}\theta^{-1}}2^\varepsilon\big(\ln(\alpha|\ln x|+e\theta^{-2})\big)^{-\varepsilon}\Big]\\ \leq\;&\frac{2}{\varepsilon}\alpha^\varepsilon e^{\varepsilon\ln\varepsilon-\varepsilon+e\alpha^{-1}\theta^{-1}}2^\varepsilon\big(\ln(\alpha|\ln x|+e)\big)^{-\varepsilon}.\end{aligned}$$

This, together with (3.5), leads to (3.3), and ends the proof of the conclusion (ii).

In summary, we finish the proof of this lemma.

□

We are now on the position to prove Theorem 1.5.

Proof of Theorem 1.5. Let $x_0,\ x'\in\mathbb{R}^n$, $r>0$, $a>0$, $b>0$ and $T>0$. It suffices to show the desired inequality (1.9) for any $u_0\in C_0^\infty(\mathbb{R}^n;\mathbb{C})\setminus\{0\}$ and $\varepsilon\in(0,1)$.

For this purpose, we arbitrarily fix $u_0\in C_0^\infty(\mathbb{R}^n;\mathbb{C})\setminus\{0\}$. Define the following three numbers

$$A_1\triangleq\int_{\mathbb{R}^n}|u_0(x)|^2e^{a|x|}\mathrm{d}x;\ B_1\triangleq\int_{B_r(x_0)}|u(x,T;u_0)|^2\,\mathrm{d}x;\ R_b\triangleq\int_{\mathbb{R}^n}e^{-b|x-x'|}|u(x,T;u_0)|^2\,\mathrm{d}x.$$

The proof is divided into the following several steps.

Step 1. To prove that there exist two positive constants $C_1\triangleq C_1(n)$ and $C_2\triangleq C_2(n)$ so that

$$R_b\leq C_3(x_0,x',r,a,b,T)g\Big(\frac{A_1}{B_1}\Big)A_1, \tag{3.7}$$

where

$$C_3(x_0,x',r,a,b,T) \triangleq 1 + C_1\Gamma\big(C_2 b((aT)\wedge r)\big)\exp\left[b^{-1}((aT)\wedge r)^{-1} + b(|x_0-x'|+r)\right], \tag{3.8}$$

and

$$g(\eta) \triangleq (\ln\eta)^{-C_2 b((aT)\wedge r)},\ \eta>1 \tag{3.9}$$

In fact, by Theorem 1.3 (with (x',x'',r_1,r_2) being replaced by $(x_0,x',r,2kb^{-1})$), with $k\in\mathbb{N}^+$, and the definitions of A_1 and B_1, we see that for each $k\in\mathbb{N}^+$,

$$\int_{B_{2kb^{-1}}(x')} |u(x,T;u_0)|^2\,\mathrm{d}x \le C2^n b^{-n}k^n\big((aT)\wedge r\big)^{-n}\Big(\frac{B_1}{A_1}\Big)^{\theta^{1+\frac{|x_0-x'|+2kb^{-1}+r}{(aT)\wedge r}}} A_1$$

for some $C>0$ and $\theta\in(0,1)$ depending only on n. This, along with the fact that $k\le ne^{\frac{1}{n}k}$ for all $k\in\mathbb{N}^+$, yields that

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-b|x-x'|}|u(x,T;u_0)|^2\,\mathrm{d}x \le \sum_{k=1}^{\infty}\int_{2(k-1)b^{-1}\le|x-x'|<2kb^{-1}} e^{-2(k-1)}|u(x,T;u_0)|^2\,\mathrm{d}x\\
\le\ & C(2n)^n b^{-n}\big((aT)\wedge r\big)^{-n}e^2\left(\sum_{k=1}^{\infty}e^{-k}\Big(\frac{B_1}{A_1}\Big)^{\theta^{1+\frac{|x_0-x'|+2kb^{-1}+r}{(aT)\wedge r}}}\right)A_1.
\end{aligned} \tag{3.10}$$

Meanwhile, since $B_1<A_1$ (which follows from the definitions of A_1 and B_1, the conservation law for the Schrödinger equation and the fact that $u_0\neq 0$), we can apply (i) of Lemma 3.1, where

$$(a,x,\theta) = \Big(1,\big(B_1/A_1\big)^{\theta^{1+\frac{|x_0-x'|+r}{(aT)\wedge r}}},\theta^{\frac{2}{b((aT)\wedge r)}}\Big),$$

to get that

$$\sum_{k=1}^{\infty}e^{-k}\Big(\frac{B_1}{A_1}\Big)^{\theta^{1+\frac{|x_0-x'|+2kb^{-1}+r}{(aT)\wedge r}}} \le \frac{eb((aT)\wedge r)}{2|\ln\theta|}\Gamma\left(\frac{b((aT)\wedge r)}{2|\ln\theta|}\right)\left[\theta^{1+\frac{|x_0-x'|+r}{(aT)\wedge r}}|\ln\frac{B_1}{A_1}|\right]^{-\frac{b((aT)\wedge r)}{2|\ln\theta|}}.$$

This, together with (3.10) and the facts that $x^{n-1}\le(n-1)!e^x$ for all $x>0$ and that $(aT)\wedge r\le r$, indicates that

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-b|x-x'|}|u(x,T;u_0)|^2\,\mathrm{d}x\\
\le\ & C(2n)^n b^{-n}\big((aT)\wedge r\big)^{-n}e^3\frac{b((aT)\wedge r)}{|\ln\theta|}\Gamma\left(\frac{b((aT)\wedge r)}{2|\ln\theta|}\right)\left[\theta^{1+\frac{|x_0-x'|+r}{(aT)\wedge r}}|\ln\frac{B_1}{A_1}|\right]^{-\frac{b((aT)\wedge r)}{2|\ln\theta|}}A_1\\
=\ & \frac{C(2n)^n e^3}{|\ln\theta|}\big(b((aT)\wedge r)\big)^{-n+1}e^{\frac{1}{2}b((aT)\wedge r+|x_0-x'|+r)}\Gamma\left(\frac{b((aT)\wedge r)}{2|\ln\theta|}\right)\left(\ln\frac{A_1}{B_1}\right)^{-\frac{b((aT)\wedge r)}{2|\ln\theta|}}A_1\\
\le\ & \frac{C(2n)^n e^3}{|\ln\theta|}(n-1)!e^{b^{-1}((aT)\wedge r)^{-1}+b(|x_0-x'|+r)}\Gamma\left(\frac{b((aT)\wedge r)}{2|\ln\theta|}\right)\left(\ln\frac{A_1}{B_1}\right)^{-\frac{b((aT)\wedge r)}{2|\ln\theta|}}A_1.
\end{aligned}$$

This, as well as (3.9), shows (3.7).

Step 2. To show (1.9) for the above-mentioned u_0 and any $\varepsilon\in(0,1)$

Let $C_1 \triangleq C_1(n)$ and $C_2 \triangleq C_2(n)$ be given by Step 1. Since

$$\varepsilon e^{\varepsilon^{-1-\frac{\alpha}{b((aT)\wedge r)}}} \leq \varepsilon e^{\varepsilon^{-1-\frac{\beta}{b((aT)\wedge r)}}}, \quad \text{when} \;\; 0<\alpha<\beta \;\; \text{and} \;\; \varepsilon \in (0,1),$$

it suffices to show that for each $\varepsilon \in (0,1)$,

$$R_b \leq C_4(x_0,x',r,a,b,T)\left(\varepsilon A_1 + \varepsilon e^{\varepsilon^{-1-\frac{1}{C_2 b((aT)\wedge r)}}} B_1\right), \tag{3.11}$$

where

$$C_4(x_0,x',r,a,b,T) \triangleq C_1 \exp\left\{2(C_1+C_2^{-1}+1)(C_2+1)\Big[1+\frac{b^{-1}+|x_0-x'|+r}{(aT)\wedge r}\Big]\right\}.$$

The proof of (3.11) is organized by two parts.

Part 2.1. To show (3.11) in the case that $b \leq \frac{1}{C_2((aT)\wedge r)}$

First, we claim that for each $\varepsilon \in (0,1)$,

$$R_b \leq C_3\left(\varepsilon A_1 + \varepsilon e^{\varepsilon^{-\frac{1}{C_2 b((aT)\wedge r)}}} B_1\right), \tag{3.12}$$

where $C_3 \triangleq C_3(x_0,x',r,a,b,T)$ is given by (3.8). In fact, for an arbitrarily fix $\varepsilon > 0$, there are only two possibilities: either $R_b \leq C_3\varepsilon A_1$ or $R_b > C_3\varepsilon A_1$. In the first case, (3.12) is obvious. In the second case, we first claim that

$$0 < \varepsilon < \frac{R_b}{C_3 A_1} < 1. \tag{3.13}$$

Indeed, the first and the second inequalities in (3.13) is clear. To prove the last inequality in (3.13), two facts are given in order: First, we observe from (3.8) that $C_3 > 1$. Second, by the definitions of A_1 and R_b, using the conservation law of the Schrödinger equation, we find that

$$R_b = \int_{\mathbb{R}^n} e^{-b|x-x'|}|u(x,T;u_0)|^2dx \leq \int_{\mathbb{R}^n} |u(x,T;u_0)|^2dx = \int_{\mathbb{R}^n} |u_0(x)|^2dx \leq \int_{\mathbb{R}^n} e^{a|x|}|u_0(x)|^2dx = A_1.$$

These two facts lead to the last inequality in (3.13) at once.

Since $b \leq \frac{1}{C_2((aT)\wedge r)}$, we see that the function $x \mapsto xe^{x^{-\frac{1}{C_2 b((aT)\wedge r)}}}$, with its domain $(0,1)$, is decreasing. This, along with (3.13), indicates that

$$\frac{R_b}{C_3A_1} e^{(\frac{R_b}{C_3A_1})^{-\frac{1}{C_2 b((aT)\wedge r)}}} \leq \varepsilon e^{\varepsilon^{-\frac{1}{C_2 b((aT)\wedge r)}}}. \tag{3.14}$$

Meanwhile, since the function: $f(x) = e^{x^{-\frac{1}{C_2 b((aT)\wedge r)}}}$, with its domain $(0,\infty)$, is decreasing and its inverse is the function g (given by (3.9)), we get from (3.7) that

$$\frac{A_1}{B_1} = f(g(\frac{A_1}{B_1})) \leq f(\frac{R_b}{C_3A_1}) = e^{(\frac{R_b}{C_3A_1})^{-\frac{1}{C_2 b((aT)\wedge r)}}}. \tag{3.15}$$

From (3.15) and (3.14), it follows that

$$\begin{aligned} R_b &= C_3\frac{R_b}{C_3A_1}\frac{A_1}{B_1}B_1 \leq C_3\Big[\frac{R_b}{C_3A_1}e^{(\frac{R_b}{C_3A_1})^{-\frac{1}{C_2 b((aT)\wedge r)}}}\Big]B_1 \\ &\leq C_3\varepsilon e^{\varepsilon^{-\frac{1}{C_2 b((aT)\wedge r)}}} B_1. \end{aligned}$$

Since ε was arbitrarily taken from $(0,1)$, the above leads to (3.12) for the case that $R_b > C_3\varepsilon A_1$. Hence, (3.12) is true.

Next, we claim that

$$C_3(x_0,x',r,a,b,T) \le \exp\left\{2(C_1+C_2^{-1}+1)\Big[1+\frac{b^{-1}+|x_0-x'|+r}{(aT)\wedge r}\Big]\right\}. \tag{3.16}$$

To this end, we first observe that for each $s\in(0,1]$,

$$\begin{aligned}\Gamma(s) &= \int_0^\infty e^{-x}x^{s-1}\,\mathrm{d}x = \int_0^1 e^{-x}x^{s-1}\,\mathrm{d}x + \int_1^\infty e^{-x}x^{s-1}\,\mathrm{d}x\\ &\le \sum_{k=0}^\infty \int_{e^{-k-1}}^{e^{-k}} e^{(1-s)(k+1)}\,\mathrm{d}x + \int_1^\infty e^{-x}\,\mathrm{d}x\\ &= (e-1)\frac{1}{e^s-1}+e^{-1} \le (e-1)s^{-1}+1 \le 2s^{-1}+1 \le e^{2s^{-1}}.\end{aligned} \tag{3.17}$$

Since we are in the case that $b\le \frac{1}{C_2((aT)\wedge r)}$, it follows from (3.8) and (3.17), with $s=C_2b((aT)\wedge r)$, that

$$\begin{aligned}C_3(x_0,x',r,a,b,T) &\le 1+e^{C_1}e^{2C_2^{-1}b^{-1}((aT)\wedge r)^{-1}}\exp\left[b^{-1}((aT)\wedge r)^{-1}+b(|x_0-x'|+r)\right]\\ &\le e\cdot\exp\left[C_1+(2C_2^{-1}+1)b^{-1}((aT)\wedge r)^{-1}+C_2^{-1}\frac{|x_0-x'|+r}{(aT)\wedge r}\right].\end{aligned}$$

This leads to (3.16).

Now, by (3.12) and (3.16), we reach the aim of Part 2.1.

Part 2.2. To show (3.11) in the case that $b>\frac{1}{C_2((aT)\wedge r)}$
In this case, it follows from the definition of R_b that $R_b\le R_{\frac{1}{C_2((aT)\wedge r)}}$. Then by (3.12) and (3.16) (where b is replaced by $\frac{1}{C_2((aT)\wedge r)}$), we find that for each $\varepsilon\in(0,1)$,

$$\begin{aligned}R_b &\le \exp\left\{2(C_1+C_2^{-1}+1)\Big[1+\frac{C_2((aT)\wedge r)+|x_0-x'|+r}{(aT)\wedge r}\Big]\right\}\left(\varepsilon A_1+\varepsilon e^{\varepsilon^{-1}}B_1\right)\\ &\le \exp\left\{2(C_1+C_2^{-1}+1)\Big[1+C_2+\frac{b^{-1}+|x_0-x'|+r}{(aT)\wedge r}\Big]\right\}\left(\varepsilon A_1+\varepsilon e^{\varepsilon^{-1-\frac{1}{C_2b((aT)\wedge r)}}}B_1\right),\end{aligned}$$

from which, we reach the aim of Part 2.2.

In summary, we finish the proof of (3.11), which completes the proof of the theorem.

$\square$

Next, we are going to prove Theorem 1.6. Before it, one lemma will be introduced.

Lemma 3.2. *Given $k\in\mathbb{N}^+$, there exists a constant $C(k,n)$ so that for any $T>0$ and $u_0\in C_0^\infty(\mathbb{R}^n;\mathbb{C})$,*

$$\int_{\mathbb{R}^n}|x|^{2k}|u(x,T;u_0)|^2\,\mathrm{d}x \le C(k,n)(1+T)^{2k}\left(\|u_0\|^2_{H^{2k}(\mathbb{R}^n;\mathbb{C})}+\int_{\mathbb{R}^n}|x|^{4k}|u_0(x)|^2\,\mathrm{d}x\right). \tag{3.18}$$

Proof. Arbitrarily fix $k\in\mathbb{N}^+$, $T>0$ and $u_0\in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. For each $x\in\mathbb{R}^n$, write $x=(x_1,\cdots,x_n)$. One can directly check that for each $j\in\{1,\ldots,n\}$, the operators $\left(x_j+2i(t-T)\partial_{x_j}\right)^k$ and $i\partial_t+\Delta$ are commutative. This yields that for each $j\in\{1,\ldots,n\}$,

$$\begin{aligned}(i\partial_t+\Delta)\left(x_j+2i(t-T)\partial_{x_j}\right)^k u(x,t;u_0) &= \left(x_j+2i(t-T)\partial_{x_j}\right)^k(i\partial_t+\Delta)u(x,t;u_0)\\ &= 0,\ (x,t)\in\mathbb{R}^n\times\mathbb{R}^+,\end{aligned}$$

from which, it follows that for each $j \in \{1, \ldots, n\}$,

$$u(x,t;u_j) = \big(x_j + 2i(t-T)\partial_{x_j}\big)^k u(x,t;u_0),\ (x,t) \in \mathbb{R}^n \times \mathbb{R}^+,$$

where $u_j(x) \triangleq (x_j - 2iT\partial_{x_j})^k u_0(x),\ x \in \mathbb{R}^n$. In particular, we have that for each $j \in \{1, \ldots, n\}$,

$$u(x,T;u_j) = x_j^k u(x,T;u_0),\ x \in \mathbb{R}^n.$$

These, along with the conservation law for the Schrödinger equation, yields that for each $j \in \{1, \ldots, n\}$,

$$\begin{aligned} & \int_{\mathbb{R}^n} |x_j^k u(x,T;u_0)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |u(x,T;u_j)|^2 \,\mathrm{d}x \\ = & \int_{\mathbb{R}^n} |u_j(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |(x_j - 2iT\partial_{x_j})^k u_0(x)|^2 \,\mathrm{d}x. \end{aligned} \tag{3.19}$$

Next, we claim that there exists $C_1(k,n) > 0$ so that for each $j \in \{1, \ldots, n\}$,

$$\int_{\mathbb{R}^n} |(x_j - 2iT\partial_{x_j})^k u_0(x)|^2 \,\mathrm{d}x \leq C_1(k,n)(1+T)^{2k} \left(\|u_0\|^2_{H^{2k}(\mathbb{R}^n;\mathbb{C})} + \int_{\mathbb{R}^n} |x|^{4k} |u_0(x)|^2 \,\mathrm{d}x \right). \tag{3.20}$$

For this purpose, we arbitrarily fix j from $\{1, \ldots, n\}$. Since the operator $(x_j - i\partial_{x_j})^{2k}$ is a polynomial of x_j and ∂_{x_j}, with degree $2k$, and because

$$[\partial_{x_j}, x_j] \triangleq \partial_{x_j} x_j - x_j \partial_{x_j} = 1,$$

the polynomial $(x_j - i\partial_{x_j})^{2k}$ is a linear combination of the following monomials

$$\left\{ x_j^r \partial_{x_j}^s \ :\ 0 \leq r+s \leq 2k,\ r,\ s \in \mathbb{N}^+ \cup \{0\} \right\}.$$

By this, we see that

$$\begin{aligned} & \int_{\mathbb{R}^n} |(x_j - i\partial_{x_j})^k v(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} \left\langle (x_j - i\partial_{x_j})^{2k} v(x), v(x) \right\rangle_{\mathbb{C}} \,\mathrm{d}x \\ \leq & \ C_2(k,n) \sum_{0 \leq r+s \leq 2k} \int_{\mathbb{R}^n} |\langle \partial_{x_j}^s v(x), x_j^r v(x) \rangle_{\mathbb{C}}| \,\mathrm{d}x, \end{aligned} \tag{3.21}$$

where v is the function defined by

$$v(x) \triangleq u_0(\sqrt{2T}x),\ x \in \mathbb{R}^n, \tag{3.22}$$

and where and through the proof, $C_2(k,n)$ stands for a positive constant (depending only on k, n), which may vary in different contexts.

From (3.22) and (3.21), we find that

$$\begin{aligned} & \int_{\mathbb{R}^n} |(x_j - 2iT\partial_{x_j})^k u_0(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |(x_j - 2iT\partial_{x_j})^k v(\frac{x}{\sqrt{2T}})|^2 \,\mathrm{d}x \\ = & \ (2T)^{k+\frac{n}{2}} \int_{\mathbb{R}^n} |(x_j - i\partial_{x_j})^k v(x)|^2 \,\mathrm{d}x \\ \leq & \ C_2(k,n)(2T)^{k+\frac{n}{2}} \sum_{0 \leq r+s \leq 2k} \int_{\mathbb{R}^n} |\langle \partial_{x_j}^s v(x), x_j^r v(x) \rangle_{\mathbb{C}}| \,\mathrm{d}x \\ = & \ C_2(k,n) \sum_{0 \leq r+s \leq 2k} (2T)^{\frac{2k+s-r}{2}} \int_{\mathbb{R}^n} |\langle \partial_{x_j}^s u_0(x), x_j^r u_0(x) \rangle_{\mathbb{C}}| \,\mathrm{d}x \\ \leq & \ C_2(k,n)(1+T)^{2k} \sum_{0 \leq r+s \leq 2k} \left(\int_{\mathbb{R}^n} |\partial_{x_j}^s u_0(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} |x_j^r u_0(x)|^2 \,\mathrm{d}x \right) \\ \leq & \ C_2(k,n)(1+T)^{2k} \left(\|u_0\|^2_{H^{2k}(\mathbb{R}^n;\mathbb{C})} + \int_{\mathbb{R}^n} |x|^{4k} |u_0(x)|^2 \,\mathrm{d}x \right). \end{aligned}$$

This leads to (3.20).

Finally, since

$$|x|^{2k} = n^k\Big(\frac{x_1^2+\cdots+x_n^2}{n}\Big)^k \le n^{k-1}\big(x_1^{2k}+\cdots+x_n^{2k}\big),\ x\in\mathbb{R}^n,$$

it follows from (3.19) that

$$\int_{\mathbb{R}^n}|x|^{2k}|u(x,T;u_0)|^2\,\mathrm{d}x \le n^{k-1}\sum_{j=1}^n\int_{\mathbb{R}^n}|x_j^k u(x,T;u_0)|^2\,\mathrm{d}x$$
$$\le\ n^{k-1}\sum_{j=1}^n\int_{\mathbb{R}^n}|(x_j-2iT\partial_{x_j})^k u_0(x)|^2\,\mathrm{d}x.$$

This, along with (3.20), leads to (3.18). We end the proof of this lemma.

□

Remark 3.3. Lemma 3.2 gives a quantitative property for solutions of (1.2). This quantitative property is comparable with the following qualitative property for solutions of (1.2): If $u_0\in L^2(|x|^{4k}dx)\cap H^{2k}$ for some $k\in\mathbb{N}^+\cup\{0\}$, then

$$e^{iT\triangle}u_0\in L^2(|x|^{4k}dx)\cap H^{2k}\ \text{ for all }\ T\in\mathbb{R}^+.$$

The above-mention qualitative property was given in [30].

We now give the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $x_0\in\mathbb{R}^n$, $r>0$, $a>0$ and $T>0$. When $u_0=0$, (1.10) holds clearly for all $\varepsilon\in(0,1)$. We now arbitrarily fix $u_0\in C_0(\mathbb{R}^n;\mathbb{C})\setminus\{0\}$. Define the following three numbers:

$$A_2 \triangleq \int_{\mathbb{R}^n}|u_0(x)|^2e^{a|x|}\mathrm{d}x+\|u_0\|^2_{H^{n+3}(\mathbb{R}^n;\mathbb{C})},\ B_2\triangleq\int_{B_r(x_0)}|u(x,T;u_0)|^2\,\mathrm{d}x,$$
$$A_3 \triangleq \int_{\mathbb{R}^n_\xi}|u_0(x)|^2e^{a|x|}\mathrm{d}x.$$

Step 1. To prove that there exists a constant $C_1\triangleq C_1(n)>1$ so that

$$\sup_{1\le\eta\le2}\int_{\mathbb{R}^n}(1+|x|)^{-n-1-\eta}|u(x,T;u_0)|^2\,\mathrm{d}x \quad\le\quad C(x_0,r,a,T)\tilde g\left(\frac{A_2}{B_2}\right)A_2, \tag{3.23}$$

where the constant $C(x_0,r,a,T)$ is given by

$$C(x_0,r,a,T)\triangleq e^{C_1^{1+\frac{|x_0|+r+1}{(aT)\wedge r}}}, \tag{3.24}$$

and the function $\tilde g$ is defined by

$$\tilde g(\eta)\triangleq\frac{1}{\ln(\ln\eta+e)},\ \eta\ge1 \tag{3.25}$$

By the definitions of A_2 and A_3, we see that $A_3\le A_2$. Then by Theorem 1.3 (where (x',x'',r_1,r_2) is replaced by $(x_0,0,r,k)$) and the definitions of A_2 and B_2, we find that when $k\in\mathbb{N}^+$,

$$\int_{B_k}|u(x,T;u_0)|^2\,\mathrm{d}x \quad\le\quad Ck^n\big((aT)\wedge r\big)^{-n}B_2^{\theta^{1+\frac{|x_0|+k+r}{(aT)\wedge r}}}A_3^{1-\theta^{1+\frac{|x_0|+k+r}{(aT)\wedge r}}}$$
$$\le\quad Ck^n\big((aT)\wedge r\big)^{-n}B_2^{\theta^{1+\frac{|x_0|+k+r}{(aT)\wedge r}}}A_2^{1-\theta^{1+\frac{|x_0|+k+r}{(aT)\wedge r}}}$$

for some $C>0$ and $\theta\in(0,1)$ depending only on n. The above inequality yields that for each $\eta\in[1,2]$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}(1+|x|)^{-n-1-\eta}|u(x,T;u_0)|^2\,\mathrm{d}x \le \sum_{k=1}^{\infty}\int_{k-1\le|x|<k} k^{-n-1-\eta}|u(x,T;u_0)|^2\,\mathrm{d}x\\
\le\quad & C\big((aT)\wedge r\big)^{-n}\left(\sum_{k=1}^{\infty}k^{-1-\eta}\Big(\frac{B_2}{A_2}\Big)^{\theta^{1+\frac{|x_0|+k+r}{(aT)\wedge r}}}\right)A_2. \qquad (3.26)
\end{aligned}
$$

Since $u_0\neq 0$, by the definitions of A_2 and B_2, and by the conservation law for the Schrödinger equation, we obtain that $B_2<A_2$. Then by (ii) of Lemma 3.1, where

$$
(x,\theta,\varepsilon,\alpha)=\left(\Big(B_2/A_2\Big)^{\theta^{1+\frac{|x_0|+r}{(aT)\wedge r}}},\theta^{\frac{1}{(aT)\wedge r}},\eta,\theta^{-1-\frac{|x_0|+r}{(aT)\wedge r}}\right),
$$

we see that for each $\eta\in[1,2]$,

$$
\sum_{k=1}^{\infty}k^{-1-\eta}\Big(\frac{B_2}{A_2}\Big)^{\theta^{1+\frac{|x_0|+k+r}{(aT)\wedge r}}}\le\frac{4}{\eta}\theta^{-\eta-\eta\frac{|x_0|+r}{(aT)\wedge r}}e^{\eta\ln\eta+\eta+e\theta^{1+\frac{|x_0|+r-1}{(aT)\wedge r}}}\frac{1}{\big(\ln(|\ln\frac{B_2}{A_2}|+e)\big)^{\eta}}. \qquad (3.27)
$$

Therefore, we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}(1+|x|)^{-n-1-\eta}|u(x,T;u_0)|^2\,\mathrm{d}x\\
\le\quad & \frac{4C}{\eta}\big((aT)\wedge r\big)^{-n}\theta^{-\eta-\eta\frac{|x_0|+r}{(aT)\wedge r}}e^{\eta\ln\eta+\eta+e\theta^{1+\frac{|x_0|+r-1}{(aT)\wedge r}}}\frac{A_2}{\big(\ln(|\ln\frac{B_2}{A_2}|+e)\big)^{\eta}}\\
\le\quad & 4C((aT)\wedge r)^{-n}\theta^{-2-2\frac{|x_0|+r}{(aT)\wedge r}}e^{2\ln 2+2+e\theta^{-\frac{1}{(aT)\wedge r}}}\frac{A_2}{\ln(|\ln\frac{B_2}{A_2}|+e)}\\
\le\quad & 4Cn!e^{\frac{1}{(aT)\wedge r}}e^{\theta^{-2-2\frac{|x_0|+r}{(aT)\wedge r}}}e^{2\ln 2+2+e\theta^{-\frac{1}{(aT)\wedge r}}}\frac{A_2}{\ln(|\ln\frac{B_2}{A_2}|+e)}\\
\le\quad & 4Cn!e^{2\ln 2+2}e^{(\theta^{-2}+e+1)\theta^{-2\frac{|x_0|+r+1}{(aT)\wedge r}}}\frac{A_2}{\ln(|\ln\frac{B_2}{A_2}|+e)}. \qquad (3.28)
\end{aligned}
$$

(In the first inequality of (3.28), we used (3.26) and (3.27); In the last three inequalities of (3.28), we used the facts that

$$
\theta\in(0,1)\ \text{ and }\ ((aT)\wedge r)^{-n}\le n!e^{\frac{1}{(aT)\wedge r}}\le n!e^{\theta^{-2\frac{1}{(aT)\wedge r}}}.)
$$

Since $\theta\in(0,1)$, (3.23) follows from (3.28), as well as (3.24) and (3.25). This ends the proof of Step 1.

Step 2. To show that there exists $C_2\triangleq C_2(n)>1$ so that

$$
\int_{\mathbb{R}^n}|u_0(x)|^2\,\mathrm{d}x\le C_3(x_0,r,a,T)\frac{A_2}{\sqrt{\ln(\ln\frac{A_2}{B_2}+e)}}, \qquad (3.29)
$$

where

$$
C_3(x_0,r,a,T)\triangleq(1+T)^{2n+6}e^{C_2^{1+\frac{|x_0|+r+1}{(aT)\wedge r}}} \qquad (3.30)
$$

Choose $\eta_0\in\{1,2\}$ so that

$$
n+1+\eta_0=0\pmod 2.
$$

By Lemma 3.2 (where $k=\frac{n+1+\eta_0}{2}$), it follows that

$$\int_{\mathbb{R}^n}|x|^{n+1+\eta_0}|u(x,T;u_0)|^2\,\mathrm{d}x\le C_{31}(1+T)^{n+1+\eta_0}\Big(\|u_0\|^2_{H^{n+1+\eta_0}(\mathbb{R}^n;\mathbb{C})}+\int_{\mathbb{R}^n}|x|^{2(n+1+\eta_0)}|u_0(x)|^2\,\mathrm{d}x\Big)$$

for some $C_{31}>0$ depending only on n. The above inequality yields that

$$\begin{aligned}&\int_{\mathbb{R}^n}(1+|x|)^{n+1+\eta_0}|u(x,T;u_0)|^2\,\mathrm{d}x\le\int_{\mathbb{R}^n}2^{n+1+\eta_0}(1+|x|^{n+1+\eta_0})|u(x,T;u_0)|^2\,\mathrm{d}x\\ \le\;& C_{32}(1+T)^{n+1+\eta_0}\Big(\int_{\mathbb{R}^n}|u(x,T;u_0)|^2\,\mathrm{d}x+\|u_0\|^2_{H^{n+1+\eta_0}(\mathbb{R}^n;\mathbb{C})}+\int_{\mathbb{R}^n}|x|^{2(n+1+\eta_0)}|u_0(x)|^2\,\mathrm{d}x\Big)\end{aligned}\tag{3.31}$$

for some $C_{32}>0$ depending only on n. Since

$$(a|x|)^{2(n+1+\eta_0)}\le[2(n+1+\eta_0)]!e^{a|x|},\ x\in\mathbb{R}^n,$$

and because

$$\begin{aligned}\max\{1,a^{-2(n+1+\eta_0)}\}&=\max\{1,(aT)^{-2(n+1+\eta_0)}T^{2(n+1+\eta_0)}\}\\&\le(1+T)^{2(n+1+\eta_0)}\max\{1,(aT)\wedge r)^{-2(n+1+\eta_0)}\}\\&\le(1+T)^{3(n+3)}\big(1+((aT)\wedge r)^{-1}\big)^{2(n+3)},\end{aligned}$$

we obtain from (3.31) and the definition of A_2 that

$$\begin{aligned}&\int_{\mathbb{R}^n}(1+|x|)^{n+1+\eta_0}|u(x,T;u_0)|^2\,\mathrm{d}x\\ \le\;& C_{33}(1+T)^{n+1+\eta_0}\Big(\|u_0\|^2_{H^{n+3}(\mathbb{R}^n;\mathbb{C})}+\int_{\mathbb{R}^n}a^{-2(n+1+\eta_0)}e^{a|x|}|u_0(x)|^2\,\mathrm{d}x\Big)\\ \le\;& C_{33}(1+T)^{n+1+\eta_0}\max\{1,a^{-2(n+1+\eta_0)}\}A_2\\ \le\;& C_{33}(1+T)^{4(n+3)}\big(1+((aT)\wedge r)^{-1}\big)^{2(n+3)}A_2\end{aligned}\tag{3.32}$$

for some $C_{33}>0$ depending only on n.

Now, by the conservation law for the Schrödinger equation, (3.32) and (3.23), we find that

$$\begin{aligned}&\int_{\mathbb{R}^n}|u_0(x)|^2\,\mathrm{d}x=\int_{\mathbb{R}^n}|u(x,T;u_0)|^2\,\mathrm{d}x\\ \le\;&\Big(\int_{\mathbb{R}^n}(1+|x|)^{n+1+\eta_0}|u(x,T;u_0)|^2\,\mathrm{d}x\Big)^{\frac12}\Big(\int_{\mathbb{R}^n}(1+|x|)^{-n-1-\eta_0}|u(x,T;u_0)|^2\,\mathrm{d}x\Big)^{\frac12}\\ \le\;&\sqrt{C_{33}}(1+T)^{2n+6}(1+((aT)\wedge r)^{-1})^{n+3}\sqrt{C(x_0,r,a,T)}\frac{A_2}{\sqrt{\ln(\ln\frac{A_2}{B_2}+e)}}\\ \le\;&\sqrt{C_{33}}(1+T)^{2n+6}(n+3)!e^{1+((aT)\wedge r)^{-1}}\sqrt{C(x_0,r,a,T)}\frac{A_2}{\sqrt{\ln(\ln\frac{A_2}{B_2}+e)}}.\end{aligned}\tag{3.33}$$

(Notice that in the last inequality in (3.33), we used that $x^{n+3}\le(n+3)!e^x$ for all $x>0$.) Now, (3.29) follows from (3.33) and (3.24) at once. This ends the proof of Step 2.

Step 3. To show (1.10) for the above-mentioned u_0 and each $\varepsilon\in(0,1)$

It suffices to show that for each $\varepsilon\in(0,1)$,

$$S\triangleq\int_{\mathbb{R}^n}|u_0(x)|^2\,\mathrm{d}x\le C_3\Big(\varepsilon A_2+\varepsilon e^{e^{\varepsilon^{-2}}}B_2\Big),\tag{3.34}$$

where $C_3 \triangleq C_3(x_0, r, a, T)$ is given by (3.30). In fact, for an arbitrarily fixed $\varepsilon > 0$, there are only two possibilities: either $S \leq C_3\varepsilon A_2$ or $S > C_3\varepsilon A_2$. In the first case, (3.34) is obvious. In the second case, since $C_3 > 1$ (see (3.30)), it follows from the definitions of S and A_2 that

$$0 < \varepsilon < \frac{S}{C_3 A_2} < 1. \tag{3.35}$$

Since the function: $x \mapsto x e^{e^{x^{-2}}}$, with its domain $(0,1)$, is decreasing, we see from (3.35) that

$$\frac{S}{C_3 A_2} e^{e^{(\frac{S}{C_3 A_2})^{-2}}} \leq \varepsilon e^{e^{\varepsilon^{-2}}}. \tag{3.36}$$

Meanwhile, since the function $x \mapsto e^{-e} e^{e^{x^{-2}}}$, with its domain $(0,1)$, is decreasing and because the inverse of the aforementioned function is the function: $x \mapsto \frac{1}{\sqrt{\ln(\ln x + e)}}$, with its domain $(1, \infty)$, we get from (3.29) that

$$\frac{A_2}{B_2} \leq e^{-e} e^{e^{(\frac{S}{C_3 A_2})^{-2}}}. \tag{3.37}$$

Now, it follows from (3.37) and (3.36) that

$$\begin{aligned} S &= C_3 \frac{S}{C_3 A_2} \frac{A_2}{B_2} B_2 \leq C_3 \Big[\frac{S}{C_3 A_2} e^{-e} e^{e^{(\frac{S}{C_3 A_2})^{-2}}} \Big] B_2 \\ &\leq C_3 \varepsilon e^{-e} e^{e^{\varepsilon^{-2}}} B_2 \leq C_3 \varepsilon e^{e^{\varepsilon^{-2}}} B_2. \end{aligned}$$

Because ε was arbitrarily taken from $(0,1)$, the above leads to (3.34). This ends the proof of (1.10).

In summary, we complete the proof of this theorem.

□

4 Further comments on the main results

The purpose of this section is to present the next Theorem 4.1. From it, we can see that the inequalities in Theorem 1.1 and Theorem 1.2 cannot be improved greatly (see Remark 4.2). For instance, in the inequality (1.3) in Theorem 1.1, $(B^c_{r_1}(x'), B^c_{r_2}(x''))$ cannot be replaced by $(B^c_{r_1}(x'), B_{r_2}(x''))$.

Theorem 4.1. *The following conclusions are true:*
(i) Let x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$ and $T > 0$. Then there exists a sequence $\{u_k\} \subset L^2(\mathbb{R}^n; \mathbb{C})$, with

$$\int_{\mathbb{R}^n} |u_k(x)|^2 \, \mathrm{d}x = 1 \ \textit{for all} \ \ k \in \mathbb{N}^+, \tag{4.1}$$

so that

$$\lim_{k\to\infty} \int_{B^c_{r_1}(x')} |u_k(x)|^2 \, \mathrm{d}x = \lim_{k\to\infty} \int_{B_{r_2}(x'')} |u(x, T; u_k)|^2 \, \mathrm{d}x = 0. \tag{4.2}$$

(ii) Let x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$, $S_1 > 0$ and $S_2 > 0$. Then there exists a sequence $\{u_k\} \subset L^2(\mathbb{R}^n; \mathbb{C})$, with

$$\int_{\mathbb{R}^n} |u_k(x)|^2 \, \mathrm{d}x = 1 \ \textit{for all} \ \ k \in \mathbb{N}^+, \tag{4.3}$$

so that

$$\lim_{k\to\infty}\int_{B^c_{r_1}(x')}|u(x,S_1;u_k)|^2\,\mathrm{d}x=\lim_{k\to\infty}\int_0^{S_2}\int_{B_{r_2}(x'')}|u(x,t;u_k)|^2\,\mathrm{d}x\mathrm{d}t=0. \tag{4.4}$$

(iii) For each subset $A\subset\mathbb{R}^n$, with $m(A^c)>0$, and each $T>0$, there does not exist a positive constant $C>0$ so that

$$\int_{\mathbb{R}^n}|u_0(x)|^2\,\mathrm{d}x\le C\int_A|u(x,T;u_0)|^2\,\mathrm{d}x \tag{4.5}$$

for all $u_0\in L^2(\mathbb{R}^n;\mathbb{C})$.
(iv) For each $x_0\in\mathbb{R}^n$, $r>0$, $a>0$ and $T>0$, there exists a sequence of $\{u_k\}\subset C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and $M>0$ so that

$$\int_{\mathbb{R}^n}e^{a|x|}|u_k(x)|^2\,\mathrm{d}x\le M\ \ \text{and}\ \ \int_{\mathbb{R}^n}|u_k(x)|^2\,\mathrm{d}x=1\ \ \text{for all}\ \ k\in\mathbb{N}^+ \tag{4.6}$$

and so that

$$\lim_{k\to\infty}\int_{B_r(x_0)}|u(x,T;u_k)|^2\,\mathrm{d}x=0. \tag{4.7}$$

Proof. For each $\tau\in\mathbb{R}\setminus\{0\}$ and $f\in L^2(\mathbb{R}^n;\mathbb{C})$, we define a function $u_{\tau,f}$ by

$$u_{\tau,f}(x)\triangleq e^{-i|x|^2/4\tau}f(x),\ x\in\mathbb{R}^n. \tag{4.8}$$

By [10, (1.2)] and (4.8), we see that for all $\tau\in\mathbb{R}\setminus\{0\}$ and $f\in L^2(\mathbb{R}^n;\mathbb{C})$,

$$(2i\tau)^{n/2}e^{-i|x|^2/4\tau}u(x,\tau;u_{\tau,f})=\widehat{e^{i|\xi|^2/4\tau}u_{\tau,f}}(\xi)(x/2\tau)=\hat{f}(x/2\tau),\ x\in\mathbb{R}^n.$$

(Here and in what follows, $u(x,\tau;u_{\tau,f})=(e^{i\Delta\tau}u_{\tau,f})(x)$ when $\tau<0$.) Thus, one has that for all $\tau\in\mathbb{R}\setminus\{0\}$ and $f\in L^2(\mathbb{R}^n;\mathbb{C})$,

$$u(x,\tau;u_{\tau,f})=(2i\tau)^{-n/2}e^{i|x|^2/4\tau}\hat{f}(x/2\tau),\ x\in\mathbb{R}^n. \tag{4.9}$$

Now, we prove the conclusions (i)-(iv) one by one.

(i) Let x', $x''\in\mathbb{R}^n$, r_1, $r_2>0$ and $T>0$. Let g be a function so that

$$g\in C_0^\infty(\mathbb{R}^n;\mathbb{C})\ \ \text{and}\ \ \|g\|_{L^2(\mathbb{R}^n;\mathbb{C})}=1. \tag{4.10}$$

For each $k\in\mathbb{N}^+$, let

$$g_k(x)\triangleq k^{n/2}g(k(x-x')),\ x\in\mathbb{R}^n. \tag{4.11}$$

We define a sequence of $\{u_k\}\subset L^2(\mathbb{R}^n;\mathbb{C})$ as follows:

$$u_k(x)\triangleq e^{-i|x|^2/4T}g_k(x),\ x\in\mathbb{R}^n,\ k\in\mathbb{N}^+. \tag{4.12}$$

By (4.8) and (4.12), we have that

$$u_{T,g_k}=u_k\ \ \text{for all}\ \ k\in\mathbb{N}^+.$$

From this, (4.9) and (4.11), after some computations, we see that for each $k \in \mathbb{N}^+$,

$$u(x,T;u_k) = (2iT)^{-n/2} e^{i|x|^2/4T} k^{-n/2} \hat{g}(\frac{x}{2Tk}) e^{-ix\cdot x'/2T},\ x \in \mathbb{R}^n. \tag{4.13}$$

Three observations are given in order: First, by (4.12) and (4.11), we find that

$$\lim_{k\to\infty} \int_{B^c_{r_1}(x')} |u_k(x)|^2 \,\mathrm{d}x \quad = \quad \lim_{k\to\infty} \int_{B^c_{kr_1}(0)} |g(x)|^2 \,\mathrm{d}x = 0;$$

Second, from (4.12), (4.11) and (4.10), we see that

$$\int_{\mathbb{R}^n_x} |u_k(x)|^2 \,\mathrm{d}x \quad = \quad \int_{\mathbb{R}^n_x} |g_k(x)|^2 \,\mathrm{d}x = 1 \ \text{ for all } \ k \in \mathbb{N}^+;$$

Third, from (4.13) and (4.10), we obtain that

$$\lim_{k\to\infty} \int_{B_{r_2}(x'')} |u(x,T;u_k)|^2 \,\mathrm{d}x = \lim_{k\to\infty} \int_{B_{\frac{r_2}{2Tk}}(\frac{x''}{2Tk})} |\hat{g}(x)|^2 \,\mathrm{d}x = 0.$$

Now, from the above three observations, we get (4.1) and (4.2). This ends the proof the conclusion (i).

(ii) Let $x',\ x'' \in \mathbb{R}^n$, $r_1,\ r_2 > 0$, $S_1 > 0$ and $S_2 > 0$. Let g and g_k, with $k \in \mathbb{N}^+$, satisfy (4.10) and (4.11), respectively. Since the Schrödinger equation is time-reversible, we can find a sequence $\{u_k\} \subset L^2(\mathbb{R}^n;\mathbb{C})$ so that

$$v_k(x) \triangleq u(x,S_1;u_k) = g_k(x),\ x \in \mathbb{R}^n,\ k \in \mathbb{N}^+. \tag{4.14}$$

By (4.14), (4.10) and (4.11), we find that

$$\lim_{k\to\infty} \int_{B^c_{r_1}(x')} |v_k(x)|^2 \,\mathrm{d}x \quad = \quad \lim_{k\to\infty} \int_{B^c_{kr_1}(0)} |g(x)|^2 \,\mathrm{d}x = 0 \tag{4.15}$$

and

$$\int_{\mathbb{R}^n_x} |v_k(x)|^2 \,\mathrm{d}x \quad = \quad \int_{\mathbb{R}^n_x} |g_k(x)|^2 \,\mathrm{d}x = 1 \ \text{ for all } \ k \in \mathbb{N}^+. \tag{4.16}$$

Next, by (4.14) and (4.8), we have that

$$v_k = u_{\tau,f} \ \text{ with } \ (\tau,f) = (t, e^{i|\cdot|^2/4t} g_k(\cdot)).$$

Then by (4.9), we get that for each $k \in \mathbb{N}^+$,

$$u(x,t;v_k) = (2it)^{-n/2} e^{i|x|^2/4t} \widehat{e^{i|\xi|^2/4t} g_k}(\xi)(x/2t),\ (x,t) \in \mathbb{R}^n \times (\mathbb{R}\setminus\{0\}). \tag{4.17}$$

Meanwhile, from (4.11), it follows that for all $t \in \mathbb{R}\setminus\{0\}$ and a.e. $x \in \mathbb{R}^n$,

$$\begin{aligned}
\widehat{e^{i|\xi|^2/4t} g_k}(\xi)(x) \quad &= \quad (2\pi)^{-n/2} \int_{\mathbb{R}^n_\xi} e^{-ix\cdot\xi} e^{i|\xi|^2/4t} g_k(\xi) \,\mathrm{d}\xi \\
&= \quad (2\pi)^{-n/2} \int_{\mathbb{R}^n_\xi} e^{-ix\cdot\xi} e^{i|\xi|^2/4t} k^{n/2} g(k(\xi - x')) \,\mathrm{d}\xi \\
&= \quad (2\pi)^{-n/2} k^{-n/2} e^{-ix\cdot x'} \int_{\mathbb{R}^n_\xi} e^{-ix\cdot\xi/k} e^{i|\xi/k + x'|^2/4t} g(\xi) \,\mathrm{d}\xi.
\end{aligned}$$

This, along with (4.17) and (4.10), yields that for each $t \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}\int_{B_{r_2}(x'')} |u(x,t;v_k)|^2 \,\mathrm{d}x &\leq |B_{r_2}(x'')| \sup_{x\in B_{r_2}(x'')} |u(x,t;v_k)|^2 \\ &\leq |B_{r_2}(x'')| \Big((4\pi|t|k)^{-n/2} \int_{\mathbb{R}^n_\xi} |g(\xi)| \,\mathrm{d}\xi \Big)^2,\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \int_{B_{r_2}(x'')} |u(x,t;v_k)|^2 \,\mathrm{d}x = 0 \ \text{ for each } \ t \in \mathbb{R} \setminus \{0\}. \tag{4.18}$$

At the same time, by the conservation law for the Schrödinger equation and (4.16), we find that for all k and $t \in \mathbb{R} \setminus \{0\}$,

$$\int_{B_{r_2}(x'')} |u(x,t;v_k)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^n} |u(x,t;v_k)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |v_k(x)|^2 \,\mathrm{d}x = 1.$$

By this and (4.18), we can apply the Lebesgue dominated convergence theorem to get that

$$\lim_{k\to\infty} \int_{-S_1}^{S_2-S_1} \int_{B_{r_2}(x'')} |u(x,t;v_k)|^2 \,\mathrm{d}x\mathrm{d}t = 0. \tag{4.19}$$

Since $v_k(x) = u(x, S_1; u_k)$, $x \in \mathbb{R}^n$ (see (4.14)), by (4.15), (4.16) and (4.19), one can directly check that the above-mentioned sequence $\{u_k\}$ satisfies (4.3) and (4.4). This ends the proof of the conclusion (ii).

(iii) By contradiction, suppose that the conclusion (iii) in this theorem was not true. Then there would exist $A_0 \subset \mathbb{R}^n$, with $m(A_0^c) > 0$, $C_1 > 0$ and $T > 0$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C_1 \int_{A_0} |u(x,T;u_0)|^2 \,\mathrm{d}x \ \text{ for all } \ u_0 \in L^2(\mathbb{R}^n;\mathbb{C}). \tag{4.20}$$

From (4.8), (4.20) and (4.9), we find that for each $f \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$\int_{\mathbb{R}^n_\xi} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^n_x} |f(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^n} |u_{T,f}(x)|^2 \,\mathrm{d}x \leq C_1 \int_{A_0} |u(x,T;u_{T,f})|^2 \,\mathrm{d}x = C_1 \int_{A_0/2T} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi.$$

Since $|A_0^c| > 0$, by taking $f \in L^2(\mathbb{R}^n;\mathbb{C}) \setminus \{0\}$ with $\operatorname{supp} \hat{f} \subset A_0^c/2T$ in the above inequality, we are led to a contradiction. Hence, the conclusion (iii) in this theorem is true.

(iv) Arbitrarily fix $x_0 \in \mathbb{R}^n$, $r > 0$, $a > 0$ and $T > 0$. Let $g \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ be a function so that

$$\int_{\mathbb{R}^n_\xi} |\hat{g}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^n_x} |g(x)|^2 \,\mathrm{d}x = 1. \tag{4.21}$$

Let $\vec{v} \in S^{n-1}$. We define a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^n;\mathbb{C})$ by

$$u_k(x) \triangleq e^{-i|x|^2/4T} e^{-kix\cdot\vec{v}} g(x), \ x \in \mathbb{R}^n. \tag{4.22}$$

By (4.22) and (4.8), we have that

$$u_k = u_{\tau,f}, \ \text{ with } \ \tau = T \ \text{ and } \ f(x) = e^{-kix\cdot\vec{v}} g(x), \ \ x \in \mathbb{R}^n,$$

from which and (4.9), it follows that for each $k \in \mathbb{N}^+$,

$$u(x,T;u_k) = (2iT)^{-n/2} e^{i|x|^2/4T} \hat{g}\big(\frac{x + k\vec{v}}{2T}\big), \ x \in \mathbb{R}^n, \ k \in \mathbb{N}^+.$$

This yields that for each $k \in \mathbb{N}^+$,

$$\int_{B_r(x_0)} |u(x,T;u_k)|^2 \,\mathrm{d}x = \int_{B_{\frac{r}{2T}}(\frac{x_0+k\vec{v}}{2T})} |\hat{g}(x)|^2 \,\mathrm{d}x.$$

Since $\int_{\mathbb{R}^n} |\hat{g}(x)|^2 \,\mathrm{d}x < \infty$ (see (4.21)), the above implies that

$$\int_{B_r(x_0)} |u(x,T;u_k)|^2 \,\mathrm{d}x \to 0 \ \text{ as } \ k \to \infty. \tag{4.23}$$

Meanwhile, from (4.22) and (4.21), we find that for each $k \in \mathbb{N}^+$,

$$\int_{\mathbb{R}^n_x} e^{a|x|} |u_k(x)|^2 \,\mathrm{d}x \quad = \quad \int_{\mathbb{R}^n_x} e^{a|x|} |g(x)|^2 \,\mathrm{d}x < \infty$$

and

$$\int_{\mathbb{R}^n_x} |u_k(x)|^2 \,\mathrm{d}x \quad = \quad \int_{\mathbb{R}^n_x} |g(x)|^2 \,\mathrm{d}x = 1.$$

From these and (4.23), we obtain (4.6) and (4.7). This ends the proof the conclusion (iv).

In summary, we finish the proof of this theorem.

$\square$

Remark 4.2. (a) From (i) and (ii) of Theorem 4.1, one can easily check that for any x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$ and $T > S \geq 0$, there is no constant $C > 0$ so that any of the following inequalities holds:

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C\Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \,\mathrm{d}x + \int_{B_{r_2}(x'')} |u(x,T;u_0)|^2 \,\mathrm{d}x \Big), \ \forall\, u_0 \in L^2(\mathbb{R}^n;\mathbb{C});$$

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C\Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \,\mathrm{d}x + \int_0^T \int_{B_{r_2}(x'')} |u(x,t;u_0)|^2 \,\mathrm{d}x\mathrm{d}t \Big), \ \forall\, u_0 \in L^2(\mathbb{R}^n;\mathbb{C}).$$

Hence, the terms on the right hand side of (1.3) in Theorem 1.1 cannot be replaced by either

$$C\Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \,\mathrm{d}x + \int_{B_{r_2}(x'')} |u(x,T;u_0)|^2 \,\mathrm{d}x \Big)$$

or

$$C\Big(\int_{B^c_{r_1}(x')} |u(x,S;u_0)|^2 \,\mathrm{d}x + \int_0^T \int_{B_{r_2}(x'')} |u(x,t;u_0)|^2 \,\mathrm{d}x\mathrm{d}t \Big).$$

(b) From (iii) of Theorem 4.1, we see that in order to have (4.5) (the observability at one point in time), it is necessary that $|A^c| = 0$. That is, in order to recover a solution by observing it at one point in time, we must observe it at one time point and over the whole $\mathbb{R}^n$. From this, conclusions in (a) of this remark and Theorem 1.1, we see that the observability at two points in time is "optimal".

(c) From (iv) of Theorem 4.1, we find that for any $r > 0$, $a > 0$ and $T > 0$, there is no $C > 0$ or $\theta \in (0,1)$ so that

$$\int_{\mathbb{R}^n} |u_0(x)|^2 \,\mathrm{d}x \leq C \left(\int_{B_r(0)} |u(x,T;u_0)|^2 \,\mathrm{d}x \right)^{\theta} \left(\int_{\mathbb{R}^n} e^{a|x|} |u_0(x)|^2 \,\mathrm{d}x \right)^{1-\theta}$$

for all $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Hence, the inequality in (i) of Theorem 1.2 will not be true if $B^c_r(0)$ is replaced by $B_r(0)$.

5 Applications

In this section, we consider the applications of Theorems 1.1-1.6 to different controllability properties (for impulse controlled Schrödinger equations), which are Theorems 5.3, 5.5, 5.7, 5.9, 5.11 and 5.13, respectively. The main differences among these controllability properties are explained as follows:

- From the perspective of control location and control instant, Theorem 5.3 holds controls active at two different time points and each time outside of a ball; Theorem 5.5 and Theorem 5.9 hold controls active at one time point and outside of a ball; Theorem 5.7, Theorem 5.11 and Theorem 5.13 hold controls active at one time point and inside of a ball.

- From the perspective of controllability type, Theorem 5.3 studies the exact controllability (see Remark 5.4 for the detailed explanations); Theorem 5.9 studies a nonstandard exact controllability (see Remark 5.10 for the detailed explanations); Theorem 5.5 and Theorem 5.13 give two kinds of nonstandard approximate controllability (see Remarks 5.6 and 5.14 for the detailed explanations, respectively); Theorem 5.7 and Theorem 5.11 build up two kinds of nonstandard approximate null controllability (see Remarks 5.8 and 5.12 for the detailed explanations, respectively).

5.1 A functional analysis framework

This subsection presents an equivalence lemma (Lemma 5.1) between some observability and some controllability in an abstract framework. With the aid of it, we can use inequalities in Theorems 1.1-1.6 to study some controllability for the Schrödinger equation.

Lemma 5.1. *Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let X, Y and Z be three Banach spaces over $\mathbb{K}$, with their dual spaces X^*, Y^* and Z^*. Let $R \in \mathcal{L}(Z, X)$ and $O \in \mathcal{L}(Z, Y)$. Then the following two propositions are equivalent:*
(i) There exists $\widehat{C}_0 > 0$ and $\hat{\varepsilon}_0 > 0$ so that for each $z \in Z$,

$$\|Rz\|_X^2 \leq \widehat{C}_0 \|Oz\|_Y^2 + \hat{\varepsilon}_0 \|z\|_Z^2. \tag{5.1}$$

(ii) There exists $C_0 > 0$ and $\varepsilon_0 > 0$ so that for each $x^* \in X^*$, there is $y^* \in Y^*$ satisfying that

$$\frac{1}{C_0}\|y^*\|_{Y^*}^2 + \frac{1}{\varepsilon_0}\|R^* x^* - O^* y^*\|_{Z^*}^2 \leq \|x^*\|_{X^*}^2. \tag{5.2}$$

Furthermore, when one of the above two propositions holds, the constant pairs (C_0, ε_0) and $(\widehat{C}_0, \hat{\varepsilon}_0)$ can be chosen to be the same.

Remark 5.2. The part (i) of Lemma 5.1 presents a non-standard observability. In this part, Z is a state space, Y is an observation space, we call X as a state transformation space of Z. Further, O is an observation operator, while we call R as a state transformation operator. The inequality (5.1) means that we can approximately recover the transferred state Rz by observing Oz, the error is governed by $\sqrt{\hat{\varepsilon}_0}\|z\|_Z$.

The part (ii) of Lemma 5.1 presents a non-standard controllability. In this part, Y^* is a control space, X^* is a state space, and we call Z^* as a state transformation space of X^*. Furthermore, O^* is a control operator, while we call R^* as a state transformation operator. The inequality (5.2) can be understood as follows: For each state x^*, there is a control y^* so that O^*y^* is close to the target R^*x^*, with the distance less that $\sqrt{\varepsilon_0}\|x^*\|_{X^*}$. Moreover, the norm of this control is governed by $\sqrt{C_0}\|x^*\|_{X^*}$.

Proof of Lemma 5.1. The proof is divided into the following several steps.

Step 1. To show that (ii)⇒(i)

Suppose that (ii) is true. Then, for each $x^* \in X^*$, there exists $y^*_{x^*} \in Y^*$ so that (5.2), with $y^* = y^*_{x^*}$, is true. From this, it follows that for any $x^* \in X^*$ and $z \in Z$,

$$\begin{aligned}\langle Rz, x^*\rangle_{X,X^*} &= \langle z, R^*x^*\rangle_{Z,Z^*} = \langle z, R^*x^* - O^*y^*_{x^*}\rangle_{Z,Z^*} + \langle z, O^*y^*_{x^*}\rangle_{Z,Z^*}\\ &= \langle z, R^*x^* - O^*y^*_{x^*}\rangle_{Z,Z^*} + \langle Oz, y^*_{x^*}\rangle_{Y,Y^*}.\end{aligned}$$

By this and the Cauchy-Schwarz inequality, we deduce that for each $x^* \in X^*$ and $z \in Z$,

$$\begin{aligned}|\langle Rz, x^*\rangle_{X,X^*}| &\le \left(\sqrt{C_0}\|z\|_Z\right)\left(\frac{1}{\sqrt{C_0}}\|R^*x^* - O^*y^*_{x^*}\|_{Z^*}\right) + \left(\sqrt{\varepsilon_0}\|Oz\|_Y\right)\left(\frac{1}{\sqrt{\varepsilon_0}}\|y^*_{x^*}\|_{Y^*}\right)\\ &\le \left(C_0\|z\|_Z^2 + \varepsilon_0\|Oz\|_Y^2\right)^{1/2}\left(\frac{1}{C_0}\|R^*x^* - O^*y^*_{x^*}\|^2_{Z^*} + \frac{1}{\varepsilon_0}\|y^*_{x^*}\|^2_{Y^*}\right)^{1/2}\\ &\le \left(C_0\|z\|_Z^2 + \varepsilon_0\|Oz\|_Y^2\right)^{1/2}\|x^*\|_{X^*}.\end{aligned}$$

Hence, (5.1), with $(\widehat{C}_0, \hat{\varepsilon}_0) = (C_0, \varepsilon_0)$, is true.

Step 2. To show that (i)⇒(ii)

Suppose that (i) is true. Define a subspace E of $Y \times Z$ in the following manner:

$$E \triangleq \left\{\left(\sqrt{\widehat{C}_0}Oz, \sqrt{\hat{\varepsilon}_0}z\right) \ :\ z \in Z\right\}.$$

The norm of E is inherited form the following usual norm of $Y \times Z$:

$$\|(f,g)\|_{Y\times Z} \triangleq \left(\|f\|_Y^2 + \|g\|_Z^2\right)^{1/2},\ (f,g) \in Y \times Z. \tag{5.3}$$

Arbitrarily fix $x^* \in X^*$. Define an operator $\mathcal{T}_{x^*}$ by

$$\begin{aligned}\mathcal{T}_{x^*}\ :\ E &\to \mathbb{K}\\ \left(\sqrt{\widehat{C}_0}Oz, \sqrt{\hat{\varepsilon}_0}z\right) &\mapsto \langle x^*, Rz\rangle_{X^*,X}.\end{aligned} \tag{5.4}$$

By (5.1) and (5.4), we can easily check that $\mathcal{T}_{x^*}$ is well defined and linear. We now claim that

$$\|\mathcal{T}_{x^*}\|_{\mathcal{L}(E,\mathbb{K})} \le \|x^*\|_{X^*}. \tag{5.5}$$

Indeed, by the definition of E, we see that given $(f,g) \in E$, there is $z \in Z$ so that

$$(f,g) = \left(\sqrt{\widehat{C}_0}Oz, \sqrt{\hat{\varepsilon}_0}z\right).$$

Then by (5.4), we find that

$$|\mathcal{T}_{x^*}(f,g)| = |\langle x^*, Rz\rangle_{X^*,X}| \le \|x^*\|_{X^*}\|Rz\|_X.$$

This, along with (5.1), shows (5.5).

Since $\mathcal{T}_{x^*}$ is a linear and bounded functional, we can apply the Hahn-Banach extension theorem to find $\widetilde{\mathcal{T}}_{x^*}$ in $(Y\times Z)^*$ so that

$$\widetilde{\mathcal{T}}_{x^*}(f,g)=\mathcal{T}_{x^*}(f,g) \text{ for all } (f,g)\in E \tag{5.6}$$

and so that

$$\|\widetilde{\mathcal{T}}_{x^*}\|_{\mathcal{L}(Y\times Z,\mathbb{K})}=\|\mathcal{T}_{x^*}\|_{\mathcal{L}(E,\mathbb{K})}. \tag{5.7}$$

These, together with (5.3) and (5.5), yield that

$$\begin{aligned}|\widetilde{\mathcal{T}}_{x^*}(f,0)| &\leq \|x^*\|_{X^*}\|f\|_Y \text{ for all } f\in Y,\\ |\widetilde{\mathcal{T}}_{x^*}(0,g)| &\leq \|x^*\|_{X^*}\|g\|_Z \text{ for all } g\in Z.\end{aligned}$$

Thus, there exists $(y^*_{x^*},z^*_{x^*})\in Y^*\times Z^*$ so that

$$\begin{aligned}\widetilde{\mathcal{T}}_{x^*}(f,0) &= \langle y^*_{x^*},f\rangle_{Y^*,Y} \text{ for all } f\in Y,\\ \widetilde{\mathcal{T}}_{x^*}(0,g) &= \langle z^*_{x^*},g\rangle_{Z^*,Z} \text{ for all } g\in Z,\end{aligned}$$

from which, it follows that

$$\widetilde{\mathcal{T}}_{x^*}(f,g)=\langle y^*_{x^*},f\rangle_{Y^*,Y}+\langle z^*_{x^*},g\rangle_{Z^*,Z} \text{ for any } (f,g)\in Y\times Z. \tag{5.8}$$

Two observations are given in order: The first one reads

$$\|y^*_{x^*}\|^2_{Y^*}+\|z^*_{x^*}\|^2_{Z^*}\leq\|x^*\|^2_{X^*}, \tag{5.9}$$

while the second one is as

$$R^*x^*-O^*(\sqrt{\widehat{C}_0}y^*_{x^*})=\sqrt{\hat{\varepsilon}_0}z^*_{x^*} \text{ in } Z^*. \tag{5.10}$$

When (5.9) and (5.10) are proved, the conclusion (ii) (with (C_0,ε_0)= $(\widehat{C}_0,\hat{\varepsilon}_0)$) follows at once.

To prove (5.9), we see from (5.8), (5.7) and (5.3) that for each $(f,g)\in Y\times Z$,

$$|\langle y^*_{x^*},f\rangle_{Y^*,Y}+\langle z^*_{x^*},g\rangle_{Z^*,Z}|\leq\|x^*\|_{X^*}\left(\|f\|^2_Y+\|g\|^2_Z\right)^{1/2}.$$

Meanwhile, for each $\delta\in(0,1)$, we can choose $(f_\delta,g_\delta)\in Y\times Z$ so that

$$\begin{aligned}\langle y^*_{x^*},f_\delta\rangle_{Y^*,Y} &= \|y^*_{x^*}\|^2_{Y^*}+o_1(1),\ \|f_\delta\|_Y=\|y^*_{x^*}\|_{Y^*},\\ \langle z^*_{x^*},g_\delta\rangle_{Z^*,Z} &= \|z^*_{x^*}\|^2_{Z^*}+o_2(1),\ \|g_\delta\|_Z=\|z^*_{x^*}\|_{Z^*},\end{aligned}$$

where $o_1(1)$ and $o_2(1)$ are so that

$$\lim_{\delta\to0^+}o_1(1)=\lim_{\delta\to0^+}o_2(1)=0.$$

From these, it follows that

$$\|y^*_{x^*}\|^2_{Y^*}+\|z^*_{x^*}\|^2_{Z^*}-|o_1(1)|-|o_2(1)|\leq\|x^*\|_{X^*}\left(\|y^*_{x^*}\|^2_{Y^*}+\|z^*_{x^*}\|^2_{Z^*}\right)^{1/2}.$$

Sending $\delta \to 0^+$ in the above inequality leads to (5.9).

To prove (5.10), we find from (5.4), (5.6) and (5.8) that for all $z \in Z$,

$$\langle x^*, Rz \rangle_{X^*,X} = \langle y^*_{x^*}, \sqrt{\widehat{C}_0} Oz \rangle_{Y^*,Y} + \langle z^*_{x^*}, \sqrt{\hat{\varepsilon}_0} z \rangle_{Z^*,Z},$$

which yields that for all $z \in Z$,

$$\langle R^* x^*, z \rangle_{Z^*,Z} = \langle O^*(\sqrt{\widehat{C}_0} y^*_{x^*}), z \rangle_{Z^*,Z} + \langle \sqrt{\hat{\varepsilon}_0} z^*_{x^*}, z \rangle_{Z^*,Z}.$$

This leads to (5.10).

Step 3. About the constant pairs (C_0, ε_0) and $(\widehat{C}_0, \hat{\varepsilon}_0)$

From the proofs in Step 1 and Step 2, we see that when one of the propositions (i) and (ii) holds, (C_0, ε_0) and $(\widehat{C}_0, \hat{\varepsilon}_0)$ can be chosen to be the same pair. This ends the proof of this lemma.

□

We end this subsection with introducing the following dual equation:

$$\begin{cases} i\partial_t \varphi(x,t) + \Delta \varphi(x,t) = 0, & (x,t) \in \mathbb{R}^n \times (0,T), \\ \varphi(x,T) = z(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.11}$$

where $T > 0$ and $z \in L^2(\mathbb{R}^n)$. Write $\varphi(\cdot,\cdot;T,z)$ for the solution to (5.11). The equation (5.11) will play an important role in the studies of different controllability for the Schrödinger equation.

5.2 Applications of Theorem 1.1-Theorem 1.3 to controllability

First, we will use Theorem 1.1, as well as Lemma 5.1, to prove the exact controllability for the following impulse controlled Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = \delta_{\{t=\tau_1\}} \chi_{B^c_{r_1}(x')}(x) h_1(x) + \delta_{\{t=\tau_2\}} \chi_{B^c_{r_2}(x'')}(x) h_2(x), & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.12}$$

where x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$, T, τ_1 and τ_2 are three numbers with $0 \le \tau_1 < \tau_2 \le T$, $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$, controls h_1 and h_2 are taken from the space $L^2(\mathbb{R}^n;\mathbb{C})$. Write $u_1(\cdot,\cdot;u_0,h_1,h_2)$ for the solution to the equation (5.12).

Theorem 5.3. *Let x', $x'' \in \mathbb{R}^n$ and r_1, $r_2 > 0$. Let T, τ_1 and τ_2 be three numbers with $0 \le \tau_1 < \tau_2 \le T$. Then for each $u_0 \in L^2(\mathbb{R}^n;\mathbb{C})$ and $u_T \in L^2(\mathbb{R}^n;\mathbb{C})$, there is a pair of controls (h_1,h_2) in $L^2(\mathbb{R}^n;\mathbb{C}) \times L^2(\mathbb{R}^n;\mathbb{C})$ so that*

$$u_1(x,T;u_0,h_1,h_2) = u_T(x),\ x \in \mathbb{R}^n \tag{5.13}$$

and so that

$$\|h_1\|^2_{L^2(\mathbb{R}^n;\mathbb{C})} + \|h_2\|^2_{L^2(\mathbb{R}^n;\mathbb{C})} \le Ce^{Cr_1 r_2 \frac{1}{\tau_2-\tau_1}} \left\| u_T - e^{i\Delta T} u_0 \right\|^2_{L^2(\mathbb{R}^n;\mathbb{C})}, \tag{5.14}$$

where the constant $C \triangleq C(n)$ is given by Theorem 1.1.

Proof. We organize the proof by the following two steps:

In Step 1, we aim to prove that for each $z \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$\int_{\mathbb{R}^n} |z(x)|^2 \,\mathrm{d}x \le Ce^{Cr_1r_2\frac{1}{\tau_2-\tau_1}}\Big(\int_{B^c_{r_1}(x')} |\varphi(x,\tau_1;T,z)|^2 \,\mathrm{d}x + \int_{B^c_{r_2}(x'')} |\varphi(x,\tau_2;T,z)|^2 \,\mathrm{d}x\Big), \quad (5.15)$$

where $C \triangleq C(n)$ is given by Theorem 1.1. To this end, we set

$$u_1(x) \triangleq \varphi(x,\tau_1;T,z), \quad x \in \mathbb{R}^n.$$

Then it follows from (1.2) and (5.11) that for each $t \in [0, \tau_2-\tau_1]$,

$$u(x,t;u_1) = (e^{i\Delta t}u_1)(x) = (e^{i\Delta t}e^{i\Delta(\tau_1-T)}z)(x) = \varphi(x,t+\tau_1;T,z), \; x \in \mathbb{R}^n. \quad (5.16)$$

By Theorem 1.1 (where $u_0 = u_1$ and $T = \tau_2-\tau_1$), we find that

$$\int_{\mathbb{R}^n} |u_1(x)|^2 \,\mathrm{d}x \quad \le \quad Ce^{Cr_1r_2\frac{1}{\tau_2-\tau_1}}\Big(\int_{B^c_{r_1}(x')} |u_1(x)|^2 \,\mathrm{d}x + \int_{B^c_{r_2}(x'')} |u(x,\tau_2-\tau_1;u_1)|^2 \,\mathrm{d}x\Big).$$

This, along with (5.16), implies that

$$\int_{\mathbb{R}^n} |\varphi(x,\tau_1;T,z)|^2 \,\mathrm{d}x \le Ce^{Cr_1r_2\frac{1}{\tau_2-\tau_1}}\Big(\int_{B^c_{r_1}(x')} |\varphi(x,\tau_1;T,z)|^2 \,\mathrm{d}x + \int_{B^c_{r_2}(x'')} |\varphi(x,\tau_2;T,z)|^2 \,\mathrm{d}x\Big).$$

Because of the conservation law of the Schrödinger equation, the above inequality leads to (5.15).

In Step 2, we aim to use Lemma 5.1 and (5.15) to prove (5.13) and (5.14). For this purpose, we let

$$X \triangleq L^2(\mathbb{R}^n;\mathbb{C}) = X^*, \; Y \triangleq L^2(\mathbb{R}^n;\mathbb{C}) \times L^2(\mathbb{R}^n;\mathbb{C}) = Y^* \;\text{ and }\; Z \triangleq L^2(\mathbb{R}^n;\mathbb{C}) = Z^* \quad (5.17)$$

and define two operators $\mathcal{R} : Z \to X$ and $\mathcal{O} : Z \to Y$ as follows:

$$\mathcal{R}z \triangleq z; \quad \mathcal{O}z \triangleq \Big(\chi_{B^c_{r_1}(x')}(\cdot)\varphi(\cdot,\tau_1;T,z), \chi_{B^c_{r_2}(x'')}(\cdot)\varphi(\cdot,\tau_2;T,z)\Big) \;\text{ for each }\; z \in Z. \quad (5.18)$$

By (5.18) and (5.17), one can directly check, that

$$\mathcal{R}^*f = f, \; \forall\, f \in L^2(\mathbb{R}^n;\mathbb{C}); \quad \mathcal{O}^*(h_1,h_2) = u_1(\cdot,T;0,h_1,h_2), \; \forall\,(h_1,h_2) \in L^2(\mathbb{R}^n;\mathbb{C}) \times L^2(\mathbb{R}^n;\mathbb{C}). \quad (5.19)$$

Arbitrarily fix $k \in \mathbb{N}^+$. From (5.15) and (5.18), we find that for each $z \in L^2(\mathbb{R}^n;\mathbb{C})$,

$$\|\mathcal{R}z\|^2_X \le Ce^{Cr_1r_2\frac{1}{\tau_2-\tau_1}}\|\mathcal{O}z\|^2_Y + \frac{1}{k}\|z\|^2_Z. \quad (5.20)$$

where $C > 0$ is given by (5.15) and $\|\cdot\|_Y$ denotes the usual norm of $L^2(\mathbb{R}^n;\mathbb{C}) \times L^2(\mathbb{R}^n;\mathbb{C})$.

Arbitrarily fix $u_0,\, u_T \in L^2(\mathbb{R}^n;\mathbb{C})$. Define a function over $\mathbb{R}^n$ in the following manner:

$$f(x) \triangleq u_T(x) - e^{i\Delta T}u_0(x), \quad x \in \mathbb{R}^n. \quad (5.21)$$

By Lemma 5.1 and (5.20), it follows that there exists $(h^f_{1,k}, h^f_{2,k}) \in Y^*$ so that

$$C^{-1}e^{-Cr_1r_2\frac{1}{\tau_2-\tau_1}}\|(h^f_{1,k}, h^f_{2,k})\|^2_{Y^*} + k\|\mathcal{R}^*f - \mathcal{O}^*(h^f_{1,k}, h^f_{2,k})\|^2_{Z^*} \le \|f\|^2_{X^*}. \quad (5.22)$$

By (5.17) and (5.22), one can easily find that there exits a subsequence $\{k_j\}_{j=1}^{\infty}$ of $\mathbb{N}^+$ and $(h_1^f, h_2^f) \in L^2(\mathbb{R}^n;\mathbb{C})\times L^2(\mathbb{R}^n;\mathbb{C})$ so that

$$(h_{1,k_j}^f, h_{2,k_j}^f) \to (h_1^f, h_2^f) \ \text{ weakly in } \ L^2(\mathbb{R}^n;\mathbb{C}) \times L^2(\mathbb{R}^n;\mathbb{C}), \ \text{ as } \ j \to \infty$$

and so that

$$\mathcal{R}^* f - \mathcal{O}^*(h_{1,k_j}^f, h_{2,k_j}^f) \to \mathcal{R}^* f - \mathcal{O}^*(h_1^f, h_2^f) \ \text{ weakly in } \ L^2(\mathbb{R}^n;\mathbb{C}), \ \text{ as } \ j \to \infty.$$

(Here, we used the fact that the operator O is linear and bounded. This fact follows from (5.18).) These yield that

$$\|(h_1^f, h_2^f)\|_{L^2(\mathbb{R}^n;\mathbb{C})\times L^2(\mathbb{R}^n;\mathbb{C})} \leq \liminf_{j\to\infty} \|(h_{1,k_j}^f, h_{2,k_j}^f)\|^2_{L^2(\mathbb{R}^n;\mathbb{C})\times L^2(\mathbb{R}^n;\mathbb{C})}, \ \text{ as } \ j \to \infty$$

and that

$$\|\mathcal{R}^* f - \mathcal{O}^*(h_1^f, h_2^f)\|_{L^2(\mathbb{R}^n;\mathbb{C})} \leq \liminf_{j\to\infty} \|\mathcal{R}^* f - \mathcal{O}^*(h_{1,k_j}^f, h_{2,k_j}^f)\|_{L^2(\mathbb{R}^n;\mathbb{C})}, \ \text{ as } \ j \to \infty.$$

From these and (5.22), it follows that

$$\mathcal{R}^* f = \mathcal{O}^*(h_1^f, h_2^f) \ \text{ and } \ \|(h_1^f, h_2^f)\|^2_{L^2(\mathbb{R}^n;\mathbb{C})\times L^2(\mathbb{R}^n;\mathbb{C})} \leq C e^{C r_1 r_2 \frac{1}{\tau_2 - \tau_1}} \|f\|^2_{L^2(\mathbb{R}^n;\mathbb{C})}. \tag{5.23}$$

Now, (5.13) and (5.14) follow from (5.23), (5.19) and (5.21) at once. This ends the proof of this theorem.

□

Remark 5.4. The above theorem can be understood as follows: For each $u_0, u_T \in L^2(\mathbb{R}^n;\mathbb{C})$, there exists a pair of controls (in $L^2(\mathbb{R}^n;\mathbb{C}) \times L^2(\mathbb{R}^n;\mathbb{C})$) steering the solution of (5.12) from u_0 at time 0 to u_T at time T. Moreover, a bound of the norm of this pair of controls is explicitly given.

Next, we will use the inequality (1.4) in (i) of Theorem 1.2, as well as Lemma 5.1, to get some kind of approximate controllability for the following impulse controlled Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = \delta_{\{t=\tau\}} \chi_{B_r^c(0)}(x) h(x,t), & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{5.24}$$

where $T > \tau \geq 0$ and $r > 0$, both the initial data u_0 and the control h are taken from the space $L^2(\mathbb{R}^n;\mathbb{C})$. Write $u_2(\cdot,\cdot;u_0,h)$ for the solution to the equation (5.24). Define, for each $a > 0$, a Banach space:

$$X_a \triangleq \left\{ f \in L^2(\mathbb{R}^n;\mathbb{C}) \ : \ \int_{\mathbb{R}^n} e^{a|x|} |f(x)|^2 \, \mathrm{d}x < \infty \right\}, \tag{5.25}$$

endowed with the norm:

$$\|f\|_{X_a} \triangleq \left(\int_{\mathbb{R}^n} e^{a|x|} |f(x)|^2 \, \mathrm{d}x \right)^{1/2}, \ \ f \in X_a.$$

One can directly check that for each $a > 0$, the dual space of X_a reads

$$X_a^* = \overline{C_0^\infty(\mathbb{R}^n;\mathbb{C})}^{\|\cdot\|_{X_a^*}}, \tag{5.26}$$

with the norm $\|\cdot\|_{X_a^*}$ given by

$$\|g\|_{X_a^*} \triangleq \left(\int_{\mathbb{R}^n} e^{-a|x|} |g(x)|^2 \, \mathrm{d}x \right)^{1/2}, \ \ g \in X_a^*.$$

Theorem 5.5. *Let $r>0$, $a>0$ and $T>\tau\geq 0$. Let $C>0$ and $\theta\in(0,1)$ be given by (i) of Theorem 1.2. Write*

$$p \triangleq \theta^{1+\frac{r}{a(T-\tau)}}\in(0,1).$$

Then for any $\varepsilon>0$, u_0, $u_T\in L^2(\mathbb{R}^n;\mathbb{C})$, there is a control $h\in L^2(\mathbb{R}^n;\mathbb{C})$ so that

$$\begin{aligned}
&\varepsilon^{\frac{1-p}{p}}\int_{\mathbb{R}^n}|h(x)|^2\,\mathrm{d}x+\varepsilon^{-1}\|u_2(\cdot,T;u_0,h)-u_T(\cdot)\|^2_{X_a^*}\\
\leq\ & C\left(1+\frac{r^n}{(a(T-\tau))^n}\right)\int_{\mathbb{R}^n}|u_T(x)-e^{i\Delta T}u_0(x)|^2\,\mathrm{d}x,
\end{aligned}\tag{5.27}$$

Proof. First of all, we claim that for each $z\in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and each $\varepsilon>0$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}|z(x)|^2\,\mathrm{d}x\\
\leq\ & C\left(1+\frac{r^n}{(a(T-\tau))^n}\right)\left(\varepsilon\int_{\mathbb{R}^n}e^{a|x|}|z(x)|^2\,\mathrm{d}x+\varepsilon^{-\frac{1-p}{p}}\int_{B_r^c(0)}|\varphi(x,\tau;T,z)|^2\,\mathrm{d}x\right).
\end{aligned}\tag{5.28}$$

To this end, arbitrarily fix $z\in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. It follows from (1.2) and (5.11) that

$$\overline{u(x,t;\bar z)}=\varphi(x,T-t;T,z),\ (x,t)\in\mathbb{R}^n\times[0,T].\tag{5.29}$$

Then by (i) of Theorem 1.2 (where u_0 and T are replaced by $\bar z$ and $T-\tau$, respectively), we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\overline{z(x)}|^2\,\mathrm{d}x\\
\leq\ & C\left(1+\frac{r^n}{(a(T-\tau))^n}\right)\left(\int_{B_r^c(0)}|\overline{u(x,T-\tau;\bar z)}|^2\,\mathrm{d}x\right)^p\left(\int_{\mathbb{R}^n}e^{a|x|}|\overline{z(x)}|^2\,\mathrm{d}x\right)^{1-p},
\end{aligned}$$

from which and (5.29), we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|z(x)|^2\,\mathrm{d}x\\
\leq\ & C\left(1+\frac{r^n}{(a(T-\tau))^n}\right)\left(\int_{B_r^c(0)}|\varphi(x,\tau;T,z)|^2\,\mathrm{d}x\right)^p\left(\int_{\mathbb{R}^n}e^{a|x|}|z(x)|^2\,\mathrm{d}x\right)^{1-p}.
\end{aligned}$$

This, along with the Young inequality, yields (5.28).

Next, we will use Lemma 5.1 and (5.28) to prove (5.27). For this purpose, we let

$$X\triangleq L^2(\mathbb{R}^n;\mathbb{C})=X^*,\ Y\triangleq L^2(\mathbb{R}^n;\mathbb{C})=Y^*\ \text{ and }\ Z\triangleq X_a,\tag{5.30}$$

where the space X_a is given by (5.25). Define two operators $\mathcal{R}:Z\to X$ and $\mathcal{O}:Z\to Y$ by

$$\mathcal{R}z\triangleq z\ \text{ for each }\ z\in X_a;\quad \mathcal{O}z\triangleq\chi_{B_r^c(0)}(\cdot)\varphi(\cdot,\tau;T,z)\ \text{ for each }\ z\in X_a.\tag{5.31}$$

One can directly check that

$$\mathcal{R}^*f=f,\ \forall\, f\in L^2(\mathbb{R}^n;\mathbb{C});\quad \mathcal{O}^*h=u_2(\cdot,T;0,h),\ \forall\, h\in L^2(\mathbb{R}^n;\mathbb{C}).\tag{5.32}$$

Arbitrarily fix $\varepsilon>0$. By (5.28), (5.31) and (5.25), we can use a standard density argument to verify that

$$\|\mathcal{R}z\|^2_{L^2(\mathbb{R}^n;\mathbb{C})}\leq C_2\|\mathcal{O}z\|^2_{L^2(\mathbb{R}^n;\mathbb{C})}+\varepsilon_2\|z\|^2_{X_a}\ \text{ for each }\ z\in X_a\tag{5.33}$$

where

$$C_2 \triangleq C\left(1+\frac{r^n}{(a(T-\tau))^n}\right)\varepsilon^{-\frac{1-p}{p}} \text{ and } \varepsilon_2 \triangleq C\left(1+\frac{r^n}{(a(T-\tau))^n}\right)\varepsilon. \tag{5.34}$$

Arbitrarily fix u_0 and u_T in $L^2(\mathbb{R}^n;\mathbb{C})$. Define a function f by

$$f \triangleq u_T - e^{i\Delta T}u_0 \text{ over } \mathbb{R}^n. \tag{5.35}$$

According to Lemma 5.1 and (5.33), there exists $h^f \in L^2(\mathbb{R}^n;\mathbb{C})$ (depending on ε, u_0 and u_T) so that

$$\frac{1}{C_2}\|h^f\|_{Y^*}^2 + \frac{1}{\varepsilon_2}\|\mathcal{R}^*f - \mathcal{O}^*h^f\|_{Z^*}^2 \le \|f\|_{X^*}^2.$$

From this, (5.30), (5.32), (5.34), (5.35) and (5.26), we obtain (5.27). This ends the proof of this theorem.

□

Remark 5.6. The above theorem can be understood follows: For each $u_0, u_T \in L^2(\mathbb{R}^n;\mathbb{C})$ and $\varepsilon > 0$, there exists a control (in $L^2(\mathbb{R}^n;\mathbb{C})$) steering the solution of (5.24) from u_0 at time 0 to the target $B_\varepsilon^{X_a^*}(u_T)$ at time T. (Here, $B_\varepsilon^{X_a^*}(u_T)$ denotes the closed ball in X_a^*, centered at u_T and of radius ε.) Moreover, a bound of the norm of this control is explicitly given.

Finally, we will use the inequality (1.6) in Theorem 1.3, as well as Lemma 5.1, to get some kind of approximate null controllability for the following impulse controlled Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = \delta_{\{t=0\}}\chi_{B_{r_1}(x')}(x)h(x,t), & (x,t)\in\mathbb{R}^n\times(0,T), \\ u(x,0) = u_0, & x\in\mathbb{R}^n, \end{cases} \tag{5.36}$$

where $T > 0$, $x' \in \mathbb{R}^n$ and $r_1 > 0$, both the initial data u_0 and the control h are taken from the space $L^2(\mathbb{R}^n;\mathbb{C})$. Write $u_3(\cdot,\cdot;u_0,h)$ for the solution to the equation (5.36). Define, for each $r_2 > 0$ and $x'' \in \mathbb{R}^n$, the following subspace:

$$\widetilde{L}^2(B_{r_2}(x'');\mathbb{C}) \triangleq \{f \in L^2(\mathbb{R}^n;\mathbb{C}) \ : \ f = 0 \text{ over } B_{r_2}^c(x'')\}. \tag{5.37}$$

Theorem 5.7. *Let x', $x'' \in \mathbb{R}^n$, r_1, $r_2 > 0$, $a > 0$ and $T > 0$. Let $C > 0$ and $p > 0$ be given by Theorem 1.3. Then for each $\varepsilon > 0$ and $u_0 \in \widetilde{L}^2(B_{r_2}(x'');\mathbb{C})$, there is a control $h \in L^2(\mathbb{R}^n;\mathbb{C})$ so that*

$$\varepsilon^{\frac{1-\theta^p}{\theta^p}}\int_{\mathbb{R}^n}|h(x)|^2\,\mathrm{d}x + \varepsilon^{-1}\|u_3(\cdot,T;u_0,h)\|_{X_a^*}^2 \le Cr_2^n\big((aT)\wedge r_1\big)^{-n}\int_{B_{r_2}(x'')}|u_0(x)|^2\,\mathrm{d}x. \tag{5.38}$$

Proof. First of all, we claim that for each $z \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and each $\varepsilon > 0$,

$$\begin{aligned} &\int_{B_{r_2}(x'')}|\varphi(x,0;T,z)|^2\,\mathrm{d}x \\ \le\ & Cr_2^n\big((aT)\wedge r_1\big)^{-n}\Big(\varepsilon^{-\frac{1-\theta^p}{\theta^p}}\int_{B_{r_1}(x')}|\varphi(x,0;T,z)|^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^n}e^{a|x|}|z(x)|^2\,\mathrm{d}x\Big). \end{aligned} \tag{5.39}$$

To this end, we arbitrarily fix $z \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. It follows from (1.2) and (5.11) that

$$\overline{u(x,t;\bar{z})} = \varphi(x,T-t;T,z),\ (x,t)\in\mathbb{R}^n\times[0,T]. \tag{5.40}$$

Then by Theorem 1.3 (where $u_0 = \bar{z}$), we find that

$$\begin{aligned}&\int_{B_{r_2}(x'')} |\overline{u(x,T;\bar{z})}|^2 \,\mathrm{d}x \\ \leq\;& Cr_2^n\big((aT)\wedge r_1\big)^{-n}\left(\int_{B_{r_1}(x')} |\overline{u(x,T;\bar{z})}|^2 \,\mathrm{d}x\right)^{\theta^p}\left(\int_{\mathbb{R}^n} e^{a|x|}|\overline{z(x)}|^2 \,\mathrm{d}x\right)^{1-\theta^p}.\end{aligned}$$

This, along with (5.40), leads to that

$$\begin{aligned}&\int_{B_{r_2}(x'')} |\varphi(x,0;T,z)|^2 \,\mathrm{d}x \\ \leq\;& Cr_2^n\big((aT)\wedge r_1\big)^{-n}\left(\int_{B_{r_1}(x')} |\varphi(x,0;T,z)|^2 \,\mathrm{d}x\right)^{\theta^p}\left(\int_{\mathbb{R}^n} e^{a|x|}|z(x)|^2 \,\mathrm{d}x\right)^{1-\theta^p}.\end{aligned}$$

Now (5.39) follows from the above inequality and the Young inequality at once.

Next, we will use Lemma 5.1 and (5.39) to prove (5.38). For this purpose, we let

$$X \triangleq \widetilde{L}^2(B_{r_2}(x'');\mathbb{C}) = X^*,\; Y \triangleq L^2(\mathbb{R}^n;\mathbb{C}) = Y^* \;\text{ and }\; Z \triangleq X_a, \tag{5.41}$$

where the space X_a is given by (5.25). Define two operators $\mathcal{R}: Z \to X$ and $\mathcal{O}: Z \to Y$ by

$$\begin{aligned}\mathcal{R}z &\triangleq \chi_{B_{r_2}(x'')}(\cdot)\varphi(\cdot,0;T,z) \;\text{ for each }\; z \in X_a;\\ \mathcal{O}z &\triangleq \chi_{B^c_{r_1}(x')}(\cdot)\varphi(\cdot,0;T,z) \;\text{ for each }\; z \in X_a,\end{aligned} \tag{5.42}$$

One can directly check that

$$\mathcal{R}^*f = u_3(\cdot,T;f,0),\;\forall\, f \in \widetilde{L}^2(B_{r_2}(x'');\mathbb{C});\quad \mathcal{O}^*h = u_3(\cdot,T;0,h),\;\forall\, h \in L^2(\mathbb{R}^n;\mathbb{C}). \tag{5.43}$$

Arbitrarily fix $\varepsilon > 0$. By (5.39), (5.42) and (5.25), we can use a standard density argument to verify that

$$\|\mathcal{R}z\|_X^2 \leq C_3\|\mathcal{O}z\|_Y^2 + \varepsilon_3\|z\|_Z^2 \;\text{ for all }\; z \in Z, \tag{5.44}$$

where

$$C_3 \triangleq Cr_2^n\big((aT)\wedge r_1\big)^{-n}\varepsilon^{-\frac{1-\theta^p}{\theta^p}} \;\text{ and }\; \varepsilon_3 \triangleq Cr_2^n\big((aT)\wedge r_1\big)^{-n}\varepsilon. \tag{5.45}$$

Arbitrarily fix $u_0 \in \widetilde{L}^2(B_{r_2}(x'');\mathbb{C})$ (given by (5.37)). From Lemma 5.1 and (5.44), we find that there exists h^{u_0} (depending on ε and u_0) so that

$$\frac{1}{C_3}\|h^{u_0}\|_{Y^*}^2 + \frac{1}{\varepsilon_3}\|\mathcal{R}^*u_0 - \mathcal{O}^*h^{u_0}\|_{Z^*}^2 \leq \|u_0\|_{X^*}^2.$$

This, along with (5.41), (5.43), (5.45) and (5.26), yields (5.38). This ends the proof of this theorem.

□

Remark 5.8. The above theorem can be understood as follows: For each $u_0 \in \widetilde{L}^2(B_{r_2}(x'');\mathbb{C})$ and $\varepsilon > 0$, there exists a control (in $L^2(\mathbb{R}^n;\mathbb{C})$) steering the solution of (5.36) from u_0 at time 0 to the target $B_\varepsilon^{X_a^*}(0)$ at time T. Moreover, a bound of the norm of this control is explicitly given.

5.3 The applications of Theorem 1.4-Theorem 1.6 to controllability

First, we will use the inequality (1.8) in Theorem 1.4, as well as Lemma 5.1, to get some kind of exact controllability for the following impulse controlled Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = \delta_{\{t=\tau\}}\chi_{B_r^c(0)}(x)h(x,t), & (x,t)\in\mathbb{R}^n\times(0,T), \\ u(x,0)=u_0, & x\in\mathbb{R}^n, \end{cases} \tag{5.46}$$

where T and τ be two numbers with $0\le\tau<T$, $r>0$, both the initial data u_0 and the control h are taken from the space $L^2(\mathbb{R}^n;\mathbb{C})$. Write $u_4(\cdot,\cdot;u_0,h)$ for the solution to the equation (5.46).

Theorem 5.9. *Let $0\le\tau<T$, $r>0$ and $N>0$. Let $C\triangleq C(n)>0$ be given by Theorem 1.4. Then for each $u_0,\ u_T\in L^2(\mathbb{R}^n;\mathbb{C})$, there is a control $h\in L^2(\mathbb{R}^n;\mathbb{C})$ so that*

$$u_4(x,T;u_0,h)=u_T,\ x\in B_N(0) \tag{5.47}$$

and so that

$$\|h\|_{L^2(\mathbb{R}^n;\mathbb{C})}\le e^{\frac{C}{2}\left(1+\frac{N}{T-\tau}\right)}\|u_T-e^{i\Delta T}u_0\|_{L^2(\mathbb{R}^n;\mathbb{C})}. \tag{5.48}$$

Proof. First of all, we claim that for each $z\in\widetilde{L}^2(B_N(0);\mathbb{C})$,

$$\int_{\mathbb{R}^n}|z(x)|^2\,\mathrm{d}x\le e^{C\left(1+\frac{rN}{T-\tau}\right)}\int_{B_r^c(0)}|\varphi(x,\tau;T,z)|^2\,\mathrm{d}x. \tag{5.49}$$

(Here, $\widetilde{L}^2(B_N(0);\mathbb{C})$ is given by (5.37), with $B_{r_2}(x'')$ being replaced by $B_N(0)$.) To this end, arbitrarily fix $z\in\widetilde{L}^2(B_N(0);\mathbb{C})$. It follows from (1.2) and (5.11) that

$$\overline{u(x,t;\bar{z})}=\varphi(x,T-t;T,z),\ (x,t)\in\mathbb{R}^n\times[0,T]. \tag{5.50}$$

Then by Theorem 1.4 (where u_0 and T are replaced by $\bar{z}$ and $T-\tau$, respectively), we find that

$$\int_{\mathbb{R}^n}|\overline{z(x)}|^2\,\mathrm{d}x\le e^{C\left(1+\frac{rN}{T-\tau}\right)}\int_{B_r^c(0)}|\overline{u(x,T-\tau;\bar{z})}|^2\,\mathrm{d}x,$$

where $C>0$ is given by Theorem 1.4. This, along with (5.50), leads to (5.49).

Next, we will use Lemma 5.1 and (5.49) to prove (5.47) and (5.48). Let

$$X\triangleq L^2(\mathbb{R}^n;\mathbb{C})=X^*,\ Y\triangleq L^2(\mathbb{R}^n;\mathbb{C})=Y^*\ \text{ and }\ Z\triangleq\widetilde{L}^2(B_N(0);\mathbb{C})=Z^*. \tag{5.51}$$

Define two operators $\mathcal{R}:Z\to X$ and $\mathcal{O}:Z\to Y$ by

$$\begin{aligned} \mathcal{R}z &\triangleq z\ \text{ for each }\ z\in\widetilde{L}^2(B_N(0);\mathbb{C}); \\ \mathcal{O}z &\triangleq \chi_{B_r^c(0)}(\cdot)\varphi(\cdot,\tau;T,z)\ \text{ for each }\ z\in\widetilde{L}^2(B_N(0);\mathbb{C}). \end{aligned} \tag{5.52}$$

One can directly check that

$$\mathcal{R}^*f=\chi_{B_N(0)}f,\ \forall\, f\in L^2(\mathbb{R}^n;\mathbb{C});\quad \mathcal{O}^*h=\chi_{B_N(0)}u_4(\cdot,T;0,h),\ \forall\, h\in L^2(\mathbb{R}^n;\mathbb{C}). \tag{5.53}$$

From (5.49) and (5.52), we find that

$$\|\mathcal{R}z\|_X^2\le e^{C\left(1+\frac{rN}{T-\tau}\right)}\|\mathcal{O}\tilde{z}\|_Y^2+\frac{1}{k}\|z\|_Z^2\ \text{ for all }\ k\in\mathbb{N}^+,\ z\in Z. \tag{5.54}$$

Arbitrarily fix $u_0,\, u_T \in L^2(\mathbb{R}^n;\mathbb{C})$. Define a function f by

$$f \triangleq u_T - e^{i\Delta T}u_0 \ \text{ over } \ \mathbb{R}^n. \tag{5.55}$$

By Lemma 5.1 and (5.54), it follows that there exists $h_k^f \in L^2(\mathbb{R}^n;\mathbb{C})$ so that

$$e^{-C\left(1+\frac{rN}{T-\tau}\right)}\|h_k^f\|_{Y^*}^2 + k\|\mathcal{R}^*f - \mathcal{O}^*h_k^f\|_{Z^*}^2 \le \|f\|_{X^*}^2 \ \text{ for all } \ k\in\mathbb{N}^+. \tag{5.56}$$

Since $\{h_k^f\}_{k=1}^\infty$ is bounded in $L^2(\mathbb{R}^n;\mathbb{C})$ (see (5.56) and (5.51)), there exits a subsequence $\{k_j\}_{j=1}^\infty$ of $\mathbb{N}^+$ and $h^f \in L^2(\mathbb{R}^n;\mathbb{C})$ so that

$$h_{k_j}^f \to h^f \ \text{ weakly in } \ L^2(\mathbb{R}^n;\mathbb{C}), \ \text{ as } \ j\to\infty$$

and so that

$$\mathcal{R}^*f - \mathcal{O}^*h_{k_j}^f \to \mathcal{R}^*f - \mathcal{O}^*h^f \ \text{ weakly in } \ L^2(B_N(0);\mathbb{C}), \ \text{ as } \ j\to\infty.$$

These yield that

$$\|h^f\|_{L^2(\mathbb{R}^n;\mathbb{C})} \le \liminf_{j\to\infty}\|h_{k_j}^f\|_{L^2(\mathbb{R}^n;\mathbb{C})}^2; \ \ \|\mathcal{R}^*f - \mathcal{O}^*h^f\|_{L^2(B_N(0);\mathbb{C})} \le \liminf_{j\to\infty}\|\mathcal{R}^*f - \mathcal{O}^*h_{k_j}^f\|_{L^2(B_N(0);\mathbb{C})}.$$

From these and (5.56), it follows that

$$\mathcal{R}^*f = \mathcal{O}^*h^f \ \text{ over } \ B_N(0) \ \text{ and } \ \|h^f\|_{L^2(\mathbb{R}^n;\mathbb{C})}^2 \le e^{C\left(1+\frac{rN}{T-\tau}\right)}\|f\|_{L^2(\mathbb{R}^n;\mathbb{C})}^2. \tag{5.57}$$

Now, (5.13) and (5.14) follow from (5.51), (5.57), (5.53) and (5.55) at once. This ends the proof of this theorem.

□

Remark 5.10. The above theorem can be understood as follows: For each $u_0,\, u_T \in L^2(\mathbb{R}^n;\mathbb{C})$ and $N > 0$, there exists a control in $L^2(\mathbb{R}^n;\mathbb{C})$ steering the solution of (5.46) from u_0 at time 0 to u_T at time T over $B_N(0)$. Moreover, a bound of the norm of this control is explicitly given.

Next, we will use the inequality (1.9) in Theorem 1.5, as well as Lemma 5.1, to get some kind of approximate null controllability for the following impulse controlled Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = \delta_{\{t=0\}}\chi_{B_r(x_0)}(x)h(x,t), & (x,t)\in\mathbb{R}^n\times(0,T),\\ u(x,0) = u_0, & x\in\mathbb{R}^n, \end{cases} \tag{5.58}$$

where $T>0$, $x_0\in\mathbb{R}^n$ and $r>0$, both the initial data u_0 and the control h are taken from the space $L^2(\mathbb{R}^n;\mathbb{C})$. Write $u_5(\cdot,\cdot;u_0,h)$ for the solution to the equation (5.58). Before state the main result, we define, for each $b>0$ and $x'\in\mathbb{R}^n$, the following space:

$$X_{b,x'} \triangleq \left\{ f\in L^2(\mathbb{R}^n;\mathbb{C}) \ : \ \int_{\mathbb{R}^n} e^{b|x-x'|}|f(x)|^2\,\mathrm{d}x < \infty \right\},$$

with the norm $\|\cdot\|_{X_{b,x'}}$ given by

$$\|f\|_{X_{b,x'}} \triangleq \left(\int_{\mathbb{R}^n} e^{b|x-x'|}|f(x)|^2\,\mathrm{d}x\right)^{1/2}, \ \ f\in X_{b,x'}.$$

One can directly check that the dual space of $X_{b,x'}$ is as

$$X^*_{b,x'} = \overline{C_0^\infty(\mathbb{R}^n;\mathbb{C})}^{\|\cdot\|_{X^*_{b,x'}}},$$

with the norm $\|\cdot\|_{X^*_{b,x'}}$ given by

$$\|g\|_{X^*_{b,x'}} \triangleq \left(\int_{\mathbb{R}^n} e^{-b|x-x'|}|g(x)|^2\,\mathrm{d}x\right)^{1/2}, \quad g \in C_0^\infty(\mathbb{R}^n;\mathbb{C}).$$

Theorem 5.11. *Let $x_0, x' \in \mathbb{R}^n$, $r > 0$, $a > 0$, $b > 0$ and $T > 0$. Let $C(x_0, x', r, a, b, T)$ and C be given by Theorem 1.5. Then for each $\varepsilon \in (0,1)$ and $u_0 \in X_{b,x'}$, there is a control $h \in L^2(\mathbb{R}^n;\mathbb{C})$ so that*

$$\frac{1}{\varepsilon}e^{-(\frac{1}{\varepsilon})^{1+\frac{1}{Cb((aT)\wedge r)}}}\int_{\mathbb{R}^n}|h(x)|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\|u_5(\cdot,T;u_0,h)\|^2_{X_a^*} \leq C(x_0,x',r,a,b,T)\|u_0\|^2_{X_{b,x'}}. \tag{5.59}$$

Proof. First of all, we claim that for each $z \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and each $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-b|x-x'|}|\varphi(x,0;T,z)|^2\,\mathrm{d}x \qquad (5.60)\\
\leq\ & C(x_0,x',r,a,b,T)\Big(\varepsilon e^{\varepsilon^{-1-\frac{1}{Cb((aT)\wedge r)}}}\int_{B_r(x_0)}|\varphi(x,0;T,z)|^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^n}e^{a|x|}|z(x)|^2\,\mathrm{d}x\Big).
\end{aligned}$$

To this end, we arbitrarily fix $z \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. It follows from (1.2) and (5.11) that

$$\overline{u(x,t;\bar{z})} = \varphi(x, T-t; T, z),\ (x,t) \in \mathbb{R}^n \times [0,T]. \tag{5.61}$$

Then by Theorem 1.5 (where $u_0 = \bar{z}$), we find that for each $\varepsilon \in (0,1)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-b|x-x'|}|\overline{u(x,T;\bar{z})}|^2\,\mathrm{d}x\\
\leq\ & C(x_0,x',r,a,b,T)\Big(\varepsilon e^{\varepsilon^{-1-\frac{1}{Cb((aT)\wedge r)}}}\int_{B_r(x_0)}|\overline{u(x,T;\bar{z})}|^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^n}e^{a|x|}|\overline{z(x)}|^2\,\mathrm{d}x\Big).
\end{aligned}$$

This, along with (5.61), leads to (5.60).

Next, we will use Lemma 5.1 and (5.60) to prove (5.59). For this purpose, we let

$$X \triangleq X^*_{b,x'},\ Y \triangleq L^2(\mathbb{R}^n;\mathbb{C}) = Y^* \ \text{ and } \ Z \triangleq X_a,$$

where the space X_a is given by (5.25). Define two operators $\mathcal{R} : Z \to X$ and $\mathcal{O} : Z \to Y$ by

$$\mathcal{R}z \triangleq \varphi(\cdot,0;T,z);\quad \mathcal{O}z \triangleq \chi_{B_r^c(x_0)}(\cdot)\varphi(\cdot,0;T,z) \ \text{ for all } \ z \in X_a. \tag{5.62}$$

One can directly check that

$$\mathcal{R}^*f = u_5(\cdot,T;f,0),\ \forall\, f \in X_{b,x'};\quad \mathcal{O}^*h = u_5(\cdot,T;0,h),\ \forall\, h \in L^2(\mathbb{R}^n;\mathbb{C}). \tag{5.63}$$

Arbitrarily fix $\varepsilon \in (0,1)$. From (5.60), (5.62) and (5.25), we can use a standard density argument to get that

$$\|\mathcal{R}z\|^2_X \leq C_5\|\mathcal{O}z\|^2_Y + \varepsilon_5\|z\|^2_Z \ \text{ for each } \ z \in Z, \tag{5.64}$$

where

$$C_5 \triangleq C(x_0,x',r,a,b,T)\varepsilon e^{\varepsilon^{-1-\frac{1}{Cb((aT)\wedge r)}}} \ \text{ and } \ \varepsilon_5 \triangleq C(x_0,x',r,a,b,T)\varepsilon. \tag{5.65}$$

Arbitrarily fix $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. Define a function f by

$$f(x) \triangleq u_0(x),\ x \in \mathbb{R}^n. \tag{5.66}$$

Then by Lemma 5.1 and (5.64), there exists h^f (depending on ε and u_0) so that

$$\frac{1}{C_5}\|h^f\|_{Y^*}^2 + \frac{1}{\varepsilon_5}\|\mathcal{R}^* f - \mathcal{O}^* h^f\|_{Z^*}^2 \le \|f\|_{X^*}^2.$$

This, along with (5.63), (5.65), (5.66) and (5.26), yields that (5.59) holds. This ends the proof of this theorem.

□

Remark 5.12. The above theorem can be understood as follows: For each $u_0 \in X_{b,x'}$ and $\varepsilon > 0$, there exists a control (in $L^2(\mathbb{R}^n;\mathbb{C})$) steering the solution of (5.58) from u_0 at time 0 to the target $B_\varepsilon^{X_a^*}(0)$ at time T. Moreover, a bound of the norm of this control is explicitly given.

Finally, we will use the inequality (1.10) in Theorem 1.6, as well as Lemma 5.1, to get some kind of approximate controllability for the following impulse controlled Schrödinger equation:

$$\begin{cases} i\partial_t u(x,t) + \Delta u(x,t) = \delta_{\{t=\tau\}}\chi_{B_r(x_0)}(x)h(x,t), & (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0, & x \in \mathbb{R}^n, \end{cases} \tag{5.67}$$

where $T > \tau \ge 0$, $x_0 \in \mathbb{R}^n$ and $r > 0$, both the initial data u_0 and the control h are taken from the space $L^2(\mathbb{R}^n;\mathbb{C})$. Write $u_6(\cdot,\cdot;u_0,h)$ for the solution to the equation (5.67). For each $a > 0$, we write Q_a for the completion of $C_0^\infty(\mathbb{R}^n;\mathbb{C})$ in the following norm:

$$\|f\|_{Q_a} \triangleq \left(\int_{\mathbb{R}^n} e^{a|x|}|f(x)|^2\,\mathrm{d}x + \|f\|_{H^{n+3}(\mathbb{R}^n;\mathbb{C})}^2\right)^{\frac{1}{2}},\quad f \in C_0^\infty(\mathbb{R}^n;\mathbb{C}). \tag{5.68}$$

One can easily check that the space Q_a is continuously imbedded to $L^2(\mathbb{R}^n;\mathbb{C})$. Denote by Q_a^* the dual space of Q_a with respect to the pivot space $L^2(\mathbb{R}^n;\mathbb{C})$.

Theorem 5.13. *Let $x_0 \in \mathbb{R}^n$, $r > 0$, $a > 0$ and $T > \tau \ge 0$. Let $C(x_0,r,a,T-\tau)$ be given by Theorem 1.6, with T being replaced by $T-\tau$. Then for each $\varepsilon \in (0,1)$ and u_0, $u_T \in L^2(\mathbb{R}^n;\mathbb{C})$, there is a control $h \in L^2(\mathbb{R}^n;\mathbb{C})$ so that*

$$\begin{aligned} &\varepsilon^{-1}e^{-e^{\varepsilon^{-2}}}\int_{\mathbb{R}^n}|h(x)|^2\,\mathrm{d}x + \varepsilon^{-1}\|u_6(\cdot,T;u_0,h) - u_T(\cdot)\|_{Q_a^*}^2 \\ \le\ & C(x_0,r,a,T-\tau)\|u_T - e^{i\Delta T}u_0\|_{L^2(\mathbb{R}^n;\mathbb{C})}^2. \end{aligned} \tag{5.69}$$

Proof. First of all, we claim that for each $z \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$ and each $\varepsilon \in (0,1)$,

$$\begin{aligned} \int_{\mathbb{R}^n}|z(x)|^2\,\mathrm{d}x \ \le\ & \overline{C}(x_0,r,a,T-\tau)\Big(\varepsilon e^{e^{\varepsilon^{-2}}}\int_{B_r(x_0)}|\varphi(x,\tau;T,z)|^2\,\mathrm{d}x \\ & +\varepsilon\Big(\int_{\mathbb{R}^n}e^{a|x|}|z(x)|^2\,\mathrm{d}x + \|z\|_{H^{n+3}(\mathbb{R}^n;\mathbb{C})}^2\Big)\Big). \end{aligned} \tag{5.70}$$

To this end, we arbitrarily fix $z \in C_0^\infty(\mathbb{R}^n;\mathbb{C})$. It follows from (1.2) and (5.11) that

$$\overline{u(x,t;\bar{z})} = \varphi(x,T-t;T,z),\ (x,t) \in \mathbb{R}^n \times [0,T]. \tag{5.71}$$

Then by Theorem 1.6 (where (u_0, T) is replaced by $(\bar{z}, T-\tau)$), we find that for each $\varepsilon \in (0,1)$,

$$\begin{aligned}&\int_{\mathbb{R}^n} |\overline{z(x)}|^2 \,\mathrm{d}x \\ \leq\ & \overline{C}(x_0, r, a, T-\tau)\left(\varepsilon\Big(\int_{\mathbb{R}^n} |\overline{z(x)}|^2 e^{a|x|}\mathrm{d}x + \|\bar{z}\|^2_{H^{n+3}(\mathbb{R}^n;\mathbb{C})}\Big) + \varepsilon e^{e^{\varepsilon^{-2}}}\int_{B_r(x_0)} |\overline{u(x, T-\tau; \bar{z})}|^2 \,\mathrm{d}x\right),\end{aligned}$$

This, along with (5.71), leads to (5.70).

Next, we will use Lemma 5.1 and (5.70) to prove (5.69). Let

$$X \triangleq L^2(\mathbb{R}^n;\mathbb{C}) = X^*,\ Y \triangleq L^2(\mathbb{R}^n;\mathbb{C}) = Y^* \ \text{ and } \ Z \triangleq Q_a,$$

where Q_a is given by (5.68). Define two operators $\mathcal{R} : Z \to X$ and $\mathcal{O} : Z \to Y$ by

$$\mathcal{R}z \triangleq z; \quad \mathcal{O}z \triangleq \chi_{B_r^c(x_0)}(\cdot)\varphi(\cdot, \tau; T, z) \ \text{ for each } \ z \in Z. \tag{5.72}$$

One can directly check that

$$\mathcal{R}^* f = f,\ \forall\, f \in L^2(\mathbb{R}^n;\mathbb{C}); \quad \mathcal{O}^* h = u_6(\cdot, T; 0, h),\ \forall\, h \in L^2(\mathbb{R}^n;\mathbb{C}). \tag{5.73}$$

Arbitrarily fix $\varepsilon \in (0,1)$. From (5.70), (5.72) and (5.68), we can use a standard density argument to get that

$$\|\mathcal{R}z\|^2_X \leq C_6\|\mathcal{O}z\|^2_Y + \varepsilon_6\|z\|^2_Z \ \text{ for each } \ z \in Q_a \tag{5.74}$$

where

$$C_6 \triangleq C(x_0, r, a, T-\tau)\varepsilon e^{e^{\varepsilon^{-2}}} \ \text{ and } \ \varepsilon_6 \triangleq C(x_0, r, a, T-\tau)\varepsilon. \tag{5.75}$$

Arbitrarily fix u_0 and u_T in $L^2(\mathbb{R}^n;\mathbb{C})$. Define a function f by

$$f \triangleq u_T - e^{i\Delta T}u_0 \ \text{ over } \ \mathbb{R}^n. \tag{5.76}$$

Then by Lemma 5.1 and (5.74), there exists h^f (depending on ε, u_0 and u_T) so that

$$\frac{1}{C_6}\|h^f\|^2_{Y^*} + \frac{1}{\varepsilon_6}\|\mathcal{R}^* f - \mathcal{O}^* h^f\|^2_{Z^*} \leq \|f\|^2_{X^*},$$

which, along with (5.73), (5.75) and (5.76), leads to (5.69). This ends the proof of the theorem.

□

Remark 5.14. The above theorem can be understood as follows: For each u_0, $u_T \in L^2(\mathbb{R}^n;\mathbb{C})$ and $\varepsilon > 0$, there exists a control (in $L^2(\mathbb{R}^n;\mathbb{C})$) steering the solution of (5.67) from u_0 at time 0 to the target $B_\varepsilon^{Q_a^*}(u_T)$ at time T. Here, $B_\varepsilon^{Q_a^*}(u_T)$ denotes the closed ball in Q_a^*, centered at u_T and of radius ε. Moreover, a bound of the norm of this control is explicitly given.

Acknowledgments. The first and third authors were supported by the National Natural Science Foundation of China under grant 11571264. The second author was supported by the National Natural Science Foundation of China under grant 11426209, and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) under grant G1323521638.

References

[1] Anantharaman N., Léautaud M., Macià F.: Wigner measures and observability for the Schrödinger equation on the disk. Invent. Math. 206, 485–599 (2016)

[2] Apraiz J., Escauriaza L.: Null control and measurable sets. ESAIM: COCV 19, 239–254 (2013)

[3] Apraiz J., Escauriaza L., Wang G., Zhang C.: Observability inequalities and measurable sets. J. Eur. Math. Soc. 16, 2433–2475 (2014)

[4] Bardos C., Lebeau G., Rauch J.: Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. SIAM J. Control Optim. 305, 1024–1065 (1992)

[5] Bourgain J.: On the compactness of the support of solutions of dispersive equations. Internat. Math. Res. Notices 9, 437–447 (1997)

[6] Cazenave T.: Semilinear Schrödinger Equations, American Mathematical Society, Providence, RI (2003)

[7] Cruz-Sampedro J.: Unique continuation at infinity of solutions to Schrödinger equations with complex-valued potentials. Proc. Edinburgh Math. Soc. 42, 143–153 (1999)

[8] Dehman B., Gérard P., Lebeau G.: Stabilization and control for the nonlinear Schrödinger equation on a compact surface. Mathematische Zeitschrift 254, 729–749 (2006)

[9] Escauriaza L., Kenig C., Ponce G., Vega L.: On uniqueness properties of solutions of Schrödinger equations. Communications in Partial Differential Equations 31, 1811–1823 (2006)

[10] Escauriaza L., Kenig C., Ponce G., Vega L.: Convexity properties of solutions to the free Schrödinger equation with Gaussian decay. Math. Res. Lett. 15, 957–971 (2008)

[11] Escauriaza L., Kenig C., Ponce G., Vega L.: Hardy's uncertainty principle, convexity and Schrödinger evolutions. J. Eur. Math. Soc. 10, 883–907 (2008)

[12] Escauriaza L., Kenig C., Ponce G., Vega L.: The sharp Hardy uncertainty principle for Schrödinger evolutions. Duke Math. J. 155, 163–187 (2010)

[13] Escauriaza L., Kenig C., Ponce G., Vega L.: Uncertainty principle of Morgan type and Schrödinger evolutions. J. London Math. Soc. 83, 187–207 (2011)

[14] Escauriaza L., Kenig C., Ponce G., Vega L.: Unique continuation for Schrödinger evolutions, with applications to profiles of concentration and traveling waves. Comm. Math. Phys. 305, 487–512 (2011)

[15] Escauriaza L., Kenig C., Ponce G., Vega L.: Uniqueness properties of solutions to Schrödinger equations. Bull. Amer. Math. Soc. (N.S.) 49, 415–442 (2012)

[16] Escauriaza L., Montaner S. and Zhang C. : Observation from measurable sets for parabolic analytic evolutions and applications. J. Math. Pures Appl. 104, 837–867 (2015)

[17] Escauriaza L., Montaner S. and Zhang C. : Analyticity of solutions to parabolic evolutions and applications. SIAM J. Math. Anal., to appear

[18] Havin V., Jöricke B.: The Uncertainty Principle in Harmonic Analysis, Springer, Berlin (1994)

[19] Ionescu A., Kenig C.: L^p Carleman inequalities and uniqueness of solutions of nonlinear Schrödinger equations. Acta Math. 193, 193–239 (2004)

[20] Ionescu A., Kenig C.: Uniqueness properties of solutions of Schrödinger equations. J. Funct. Anal. 232, 90–136 (2006)

[21] Jaming P.: Nazarov uncertainty principles in higher dimension. Journal of Approximation Theory 149, 30–41 (2007)

[22] Kenig C., Ponce G., Vega L.: On unique continuation for nonlinear Schrödinger equations. Comm. Pure Appl. Math. 56, 1247–1262 (2002)

[23] Kovrijkine O.: Some results related to the Logvinenko-Sereda theorem. Proc. Amer. Math. Soc. 129, 3037–3047 (2001)

[24] Landau L. D., Lifshitz E. M.: Quantum Mechanics: Non-relativistic Theory, Vol. 3, Pergamon Press Ltd. (1977)

[25] Laurent C.: Internal control of the Schrödinger equation. Math. Control Relat. Fields 4, 161–186 (2014)

[26] Lebeau G.: Contrôle de l'équation de Schrödinger. J. Math. Pures Appl. 71, 267–291 (1992)

[27] Linares F., Ponce G.: Introduction to Nonlinear Dispersive Equations, Springer Science & Bussiness Media, LLC (2009)

[28] Masuda K.: A unique continuation theorem for solutions of the Schrödinger equations. Proc. Japan Acad. 43, 361–364 (1967)

[29] Micu S., Zuazua E.: On the lack of null-controllability of the heat equation of the half-line. Trans. Amer. Math. Soc. 353, 1635–1659 (2000)

[30] Nahas J., Ponce G.: On the persistent properties of solutions to semi-linear Schrödinger equation. Communications in Partial Differential Equations 34, 1208–1227 (2009)

[31] Nazarov F. L.: Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. Algebra i Analiz 5, 3–66 (1993) (in Russian); Translation in St. Petersburg Math. J. 5, 663–717 (1994)

[32] Phung K. D.: Observability and control of Schrödinger equations. SIAM J. Control Optim. 40, 211–230 (2001)

[33] Phung K. D., Wang G. : Quantitative unique continuation for the semilinear heat equation in a convex domain. J. Funct. Anal. 259, 1230–1247 (2010)

[34] Phung K. D., Wang G.: An observability estimate for parabolic equations from a measurable set in time and its applications. J. Eur. Math. Soc. 15, 681–703 (2013)

[35] Phung K. D., Wang L., Zhang C.: Bang-bang property for time optimal control of semilinear heat equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 31, 477–499 (2014)

[36] Rosier L., Zhang B.: Exact boundary controllability of the nonlinear Schrödinger equation. J. Differential Equations 246, 4129–4153 (2009)

[37] Rousseau J., Moyano I.: Null-controllability of the Kolmogorov equation in the whole phase space. J. Differential Equations 260, 3193–3233 (2016)

[38] Seo I.: Global unique continuation from a half space for the Schrödinger equation. J. Funct. Anal. 266, 85–98 (2014)

[39] Sitaram A., Sundari M., Thangavelu S.: Uncertainty principles on certain Lie groups. Proc. Indian Acad. Sci. Math. Sci. 105, 135–151 (1995)

[40] Tao T.: Nonlinear Dispersive Equations: Local and Global Analysis, American Mathematical Society (2006)

[41] Tucsnak M., Weiss G.: Observation and Control for Operator Semigroups, Birkhäuser Verlag AG (2009)

[42] Vessella S.: A continuous dependence result in the analytic continuation problem. Forum Math. 11, 695–703 (1999)

[43] Wang L., Kunisch K.: The bang-bang property of time optimal controls for the Burgers equation. Discrete and Continous Dynamical Systems 34, 3611–3637 (2014)

[44] Zhang B.: Unique continuation properties for the nonlinear Schrödinger equation. Proc. Roy. Soc. Edinburgh Sect. A 127, 191–205 (1997)

[45] Zuazua E.: Remarks on the controllability of the Schrödinger equation. Quantum control: mathematical and numerical challenges. CRM Proc. Lecture Notes, Amer. Math. Soc. 33, 193–211 (2003)