# Dynamic Bifurcation from Infinity of Nonlinear Evolution Equations*

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**Abstract.** This paper is concerned with dynamic bifurcation from infinity and multiplicity of stationary solutions for nonlinear evolution equations near resonance. First, we prove some new global continuation results and establish a general theorem on dynamic bifurcation from infinity in the framework of local semiflows on metric spaces. Then, by applying these abstract results, we derive more precise descriptions on the dynamic bifurcation from infinity of evolution equations in Banach spaces. Finally, we focus our attention on the parabolic equation $u_t - \Delta u = \lambda u + f(x, u)$ associated with the Dirichlet boundary condition, where $f$ satisfies appropriate Landesman-Laser type condition. A detailed discussion on the dynamical behavior and the multiplicity of stationary solutions of the equation near resonance will be presented.

**Key words.** Conley index, nonlinear evolution equation, bifurcation from infinity, parabolic equation, resonance

**AMS subject classifications.** 37B30, 35B32, 35K55, 34C23

## 1. Introduction.

This paper is concerned with the nonlinear evolution equation

$$\frac{du}{dt} + Au = \lambda u + f(u, \lambda) \tag{1}$$

on a Banach space $X$, where $A$ is a sectorial operator on $X$ with compact resolvent, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $f(u, \lambda)$ is a locally Lipschitz continuous mapping from $X^\alpha \times \mathbb{R}$ $(0 \le \alpha < 1)$ to $X$ which is sublinear as $||u||_\alpha \to \infty$ uniformly on bounded $\lambda$-intervals. We are basically interested in the dynamic bifurcation from infinity of the equation and its applications.

This topic can be traced back to the work of Rabinowitz [30], in which the author studied the bifurcation from infinity of stationary solutions of the equation in a general setting of operator equations of the following form:

$$u = \lambda Lu + K(u, \lambda), \tag{2}$$

where $L$ is a compact operator, $\lambda \in \mathbb{R}$, and $K(u, \lambda) = o(||u||)$ as $||u|| \to \infty$ uniformly on bounded $\lambda$-intervals. It was shown that if $\mu^{-1}$ is a real eigenvalue of $L$ of odd multiplicity, then $(\infty, \mu)$ is a bifurcation point. Furthermore, there is a continuum of solutions of (2) which goes to infinity as $\lambda \to \mu$. This result was partially extended by Toland [38], Dias and Hernandez [10] and Schmitt and Wang [36] to potential operator equations to cover the case of even multiplicity. The interested reader is referred to [1, 2, 5, 6, 13, 16, 22, 23, 27, 28, 31, 33],

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etc. for concrete examples on the bifurcation from infinity and multiplicity near resonance for differential equations under various boundary conditions.

For a nonlinear system as in (1), stationary solutions may be far from being adequate for understanding its dynamics. This is because that the dynamics of the system is usually determined not only by its stationary solutions, but also by all other bounded full ones. In fact, it is often the case that a system may have no stationary solutions. It is therefore of great importance to develop appropriate theories to analyze the bifurcation of bounded full solutions. A fundamental one in this line is the well-known Hopf's bifurcation theory, which was first developed in the very early work of Poincaré [29] around 1892. Actually it forms the central part of the classical dynamic bifurcation theory. The Hopf's bifurcation theory focuses on the case when there are exactly a pair of conjugate eigenvalues of the linearized equation crossing the imaginary axis, and was fully developed in the 20-th century. One can find a vast body of literature on how to determine Hopf bifurcations for nonlinear systems arising from applications. To deal with the general case, some other dynamic bifurcation theories need to be developed, and the Conley index theory, attractor theory and so on allow us to take a step; see e.g. [12, 18, 21, 34, 39, 40], etc.

Very recently, Li and Wang [18] established some new local and global bifurcation results in terms of invariant sets via the Conley index theory, completely extending the well-known Rabinowitz's global bifurcation theorem to the dynamic bifurcation of nonlinear evolution equations without requiring the "crossing odd-multiplicity" condition. Inspired by this work and some other ones mentioned above, in this paper we consider the dynamic bifurcation from infinity of (1). This problem was actually addressed in Ward [40]. The author first established a global continuation theorem (see Remark 3.2). Then he proved the following interesting result: For any real numbers $c < d$ such that the interval $[c, d]$ contains exactly one number $\mu \in \mathrm{Re}\,\sigma(A)$ with $c < \mu < d$, there exists a continuum $\mathscr{C} \subset X^\alpha \times [c, d]$ meeting $X^\alpha \times \{c, d\}$ such that

(1) for $\lambda \neq c, d$, $\mathscr{C}[\lambda]$ consists of bounded full solutions, where

$$\mathscr{C}[\lambda] = \{x :\ (x, \lambda) \in \mathscr{C}\};$$

(2) there is a sequence $\lambda_n \to \mu$ such that $\mathscr{C}[\lambda_n]$ is unbounded as $\lambda_n \to \mu$.

(See [40, Theorem 3.2].) Note that the continuum $\mathscr{C}$ in the above result may contain either all the connected branches of bounded full solutions of the equation meeting $X^\alpha \times \{c\}$, or all the connected branches of bounded full solutions meeting $X^\alpha \times \{d\}$, according to which side $\mathscr{C}$ will meet. Here, by using the techniques in [18] we will prove some new continuation results and establish an abstract theorem on bifurcation from infinity in terms of local semiflows on metric spaces. Then based on these theoretical results, we give some more precise descriptions on the dynamic bifurcation from infinity for (1).

As an example, we consider the parabolic equation

$$u_t - \Delta u = \lambda u + f(x, u), \qquad x \in \Omega \tag{3}$$

associated with the homogeneous Dirichlet boundary condition, where $\Omega$ is a bounded domain in $\mathbb{R}^n$, and $f$ is a bounded function satisfying the following Landesman-Laser type condition:

$$(4) \qquad \liminf_{s\to+\infty} f(x,s) \geq \overline{f} > 0, \qquad \limsup_{s\to-\infty} f(x,s) \leq -\underline{f} < 0$$

uniformly for $x \in \overline{\Omega}$ (where $\overline{f}$ and $\underline{f}$ are independent of $x$). First, we give a detailed discussion on the dynamic bifurcation from infinity of the equation near any eigenvalue $\mu_k$ of the operator $-\Delta$ (in $H_0^1(\Omega)$). Specifically, we prove that there exists $\delta > 0$ such that for each $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$, the maximal compact invariant set $S_\lambda$ of the equation has a Morse decomposition $\mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\}$ with $M_\lambda^1$ being uniformly bounded on $\Lambda_-$ while

$$\lim_{\lambda\to\mu_k^-} \min_{v\in M_\lambda^\infty} ||v|| = \infty.$$

Besides, there is at least one connecting trajectory $\gamma$ between $M_\lambda^\infty$ and $M_\lambda^1$. More interestingly, it will be shown that each of the following two sets

$$\mathscr{K}^1 = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^1 \times \{\lambda\})}, \quad \mathscr{K}^\infty = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^\infty \times \{\lambda\})}$$

contains a connected component $\Gamma$ with

$$\Gamma[\lambda] := \{u : (u,\lambda) \in \Gamma\} \neq \emptyset, \qquad \forall\, \lambda \in \Lambda_-.$$

The bifurcation and multiplicity of elliptic equations near resonance is always an interesting topic and has attracted much attention in the past decades. As a byproduct of our dynamical argument, we can naturally derive some bifurcation and multiplicity results on the stationary problem:

$$(5) \qquad \begin{cases} -\Delta u = \lambda u + f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

This problem was first studied by Mawhin and Schmitt [22], where the authors considered the case when $\lambda$ crosses an eigenvalue of odd multiplicity. Later Schmitt and Wang [36] developed a theory on bifurcation from infinity for potential operators, through which they extended the results in [22] to the case when $\lambda$ crosses any eigenvalue $\mu_k$. More specifically, under an abstract Landesman-Laser type condition on the Nemitski operator $\tilde{f} : H_0^1(\Omega) \to L^2(\Omega)$ corresponding to the function $f(x,s)$, the authors proved the following result and its "dual" version: there exists $\delta > 0$ such that for each $\lambda \in (\mu_k, \mu_k + \delta]$ the equation (5) has at least two solutions with one of which approaching $\infty$ as $\lambda \to \mu_k$, and for each $\lambda \in [\mu_k - \delta, \mu_k]$ it has at least one. (Some further development and extensions can be found in [4, 6, 9, 11, 27, 37], etc.) As an application of our dynamical bifurcation results, we show that the "dual" version of the above result holds true under the hypothesis (4). Moreover, there exists an open dense subset $\mathcal{D}$ of $\mathbb{R}$ such that for $\lambda \in \Lambda_- \cap \mathcal{D}$, where $\Lambda_- = [\mu_k - \delta, \mu_k)$, the problem has at least three distinct solutions.

Special attention will also be paid to the case where $f(x,s) = o(|s|)$ as $|s| \to 0$ uniformly for $x \in \overline{\Omega}$, in which we can say a little more on the multiplicity of nontrivial solutions of (5). Such a case was studied in Chiappinelli, Mawhin and Nugari [6], where the authors considered the multiplicity of solutions of the problem near the first eigenvalue $\mu_1$. Under appropriate Landesman-Laser type conditions, it was proved, among other things, that the problem has at least two distinct nontrivial solutions as $\lambda \to \mu_1^+$. (We mention that the nonlinearity in [6] was allowed to be unbounded.) Here we present some more precise information on the multiplicity of solutions for the problem near any eigenvalue $\mu_k$ under the condition (4). Roughly speaking, we show that (5) has at least two distinct nontrivial solutions for $\lambda \in \Lambda_-$, provided $\delta$ is sufficiently small. Furthermore, there is always a one-sided neighborhood $\Lambda_1$ of $\mu_k$ such that the problem has at least three distinct nontrivial stationary solutions for $\lambda \in \Lambda_1 \setminus \{\mu_k\}$.

It is worth mentioning that "dual" versions of all our results on (3) and (5) mentioned above hold true if, instead of (4), we assume

$$\limsup_{s\to+\infty} f(x,s) \le -\overline{f} < 0, \qquad \liminf_{s\to-\infty} f(x,s) \ge \underline{f} > 0 \tag{6}$$

uniformly for $x \in \overline{\Omega}$.

This work is organized as follows. In section 2 we make some preliminaries. In section 3 we first prove some new global continuation results by utilizing the theory of Conley index. Then we apply these results to establish a general dynamical bifurcation theorem from infinity for infinite dynamical systems. In section 4 we use the abstract results to prove some bifurcation theorems from infinity for nonlinear evolution equations. Finally in section 5, we discuss the dynamic bifurcation from infinity and multiplicity of stationary solutions for the parabolic equation mentioned above.

## 2. Preliminaries.

In this section we make some preliminaries.

### 2.1. Basic topological notions and results.

Let $X$ be a complete metric space with metric $d(\cdot,\cdot)$.

Let $A$ and $B$ be nonempty subsets of $X$. The *distance* $d(A,B)$ between $A$ and $B$ is defined as

$$d(A,B) = \inf\{d(x,y) : \ x \in A, \ y \in B\},$$

and the *Hausdorff semi-distance* and *Hausdorff distance* of $A$ and $B$ are defined, respectively, as

$$d_{\mathrm{H}}(A,B) = \sup_{x\in A} d(x,B), \quad \delta_{\mathrm{H}}(A,B) = \max\{d_{\mathrm{H}}(A,B), d_{\mathrm{H}}(B,A)\}.$$

The closure, interior and boundary of $A$ in $X$ are denoted, respectively, by $\overline{A}$, $\operatorname{int} A$ and $\partial A$. Sometimes we also write $\overline{A}$, $\operatorname{int} A$ and $\partial A$ as $\overline{A}^X$, $\operatorname{int}_X A$ and $\partial_X A$, respectively, to emphasize in which space these operations are taken.

The *$\varepsilon$-neighborhood* of $A$, denoted by $\mathrm{B}(A,\varepsilon)$ or $\mathrm{B}_X(A,\varepsilon)$, is defined to be the set $\{y \in X : d(y,A) < \varepsilon\}$.

A subset $U$ of $X$ is called a *neighborhood* of $A$, if $\overline{A} \subset \operatorname{int} U$.

Lemma 2.1. *[30] Let $X$ be a compact metric space, and let $A$ and $B$ be two disjoint closed subsets of $X$. Then either there exists a subcontinuum $C$ of $X$ such that*

$$A \cap C \ne \emptyset \ne B \cap C,$$

*or $X = X_A \cup X_B$, where $X_A$ and $X_B$ are disjoint compact subsets of $X$ containing $A$ and $B$, respectively.*

Lemma 2.2. *([3, pp. 41]), Let $X$ be a compact metric space. Denote $\mathcal{C}(X)$ the family of compact subsets of $X$ which is equipped with the Hausdorff metric $\delta_H(\cdot,\cdot)$. Then $\mathcal{C}(X)$ is a compact metric space.*

**2.2. Wedge/smash product of pointed spaces.** Let $(X, x_0)$ and $(Y, y_0)$ be two pointed spaces. The *wedge product* $(X, x_0) \vee (Y, y_0)$ and *smash product* $(X, x_0) \wedge (Y, y_0)$ are defined, respectively, as follows:

$$(X, x_0) \vee (Y, y_0) = (\mathcal{W},\, (x_0, y_0))\,, \quad (X, x_0) \wedge (Y, y_0) = ((X \times Y)/\mathcal{W},\, [\mathcal{W}])\,,$$

where $\mathcal{W} = X \times \{y_0\} \cup \{x_0\} \times Y$.

Denote $[(X, x_0)]$ the *homotopy type* of a pointed space $(X, x_0)$. Since the operations "$\vee$" and "$\wedge$" preserve homotopy equivalence relations, they can be naturally extended to the homotopy types of pointed spaces.

Let $\overline{0}$ be the homotopy type of the one-point space $(\{p\}, p)$. Denote $\Sigma^m$ $(m \geq 0)$ the homotopy type of a pointed $m$-dimensional sphere. Then

$$[(X, x_0)] \vee \overline{0} = [(X, x_0)], \quad \text{and } \Sigma^m \wedge \Sigma^n = \Sigma^{m+n} \ \ (\forall\, m, n \geq 0).$$

**2.3. Local semiflows on metric spaces.** For completeness and the reader's convenience, let us first collect some fundamental notions and facts on local semiflows.

**2.3.1. Local semiflows.** Let $X$ be a complete metric space.

A *local semiflow* $\Phi$ on $X$ is a continuous mapping from an open set $\mathcal{D}(\Phi) \subset \mathbb{R}_+ \times X$ to $X$ that enjoys the following properties:

(1) for each $x \in X$, there exists $0 < T_x \leq \infty$, called the *escape time* of $\Phi(t, x)$, such that

$$(t, x) \in \mathcal{D}(\Phi) \Longleftrightarrow t \in [0, T_x);$$

(2) $\Phi(0, \cdot) = id_X$, and

$$\Phi(t + s, x) = \Phi(t, \Phi(s, x))$$

for all $x \in X$ and $t, s \in \mathbb{R}_+$ with $t + s \leq T_x$.

Let $\Phi$ be a given local semiflow on $X$. For simplicity, we usually rewrite $\Phi(t, x)$ as $\Phi(t)x$.

Let $I \subset \mathbb{R}$ be an interval. A *trajectory* (or *solution*) of $\Phi$ on $I$ is a continuous mapping $\gamma : I \to X$ such that

$$\gamma(t) = \Phi(t - s)\gamma(s), \quad \forall t, s \in I,\ t \geq s.$$

A trajectory $\gamma$ on $\mathbb{R}$ is called a *full trajectory*.

The *orbit* of a trajectory $\gamma$ on $I$ is the set

$$\mathrm{orb}(\gamma) = \{\gamma(t) :\ t \in I\}.$$

The orbit of a full trajectory is simply called a *full orbit*.

The *$\omega$-limit set* $\omega(\gamma)$ and $\omega^*$-limit set of a full trajectory $\gamma$ are defined as

$$\omega(\gamma) = \{y \in X :\ \ \text{there exists } t_n \to \infty \text{ such that } \gamma(t_n) \to y\},$$

$$\omega^*(\gamma) = \{y \in X : \text{ there exists } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Given $U \subset X$, denote $K_\infty(\Phi, U)$ the *union of all bounded full orbits* in $U$. In the case where $U = X$, we will simply write

$$K_\infty(\Phi, X) = K_\infty(\Phi).$$

Let $N \subset X$. We say that $\Phi$ *does not explode* in $N$, if

$$\Phi([0, T_x))x \subset N \Longrightarrow T_x = \infty.$$

Definition 2.1. *[32] $N \subset X$ is said to be admissible, if for any sequences $x_n \in N$ and $t_n \to \infty$ with $\Phi([0, t_n])x_n \subset N$ for all $n$, the sequence $\Phi(t_n)x_n$ has a convergent subsequence. $N$ is said to be strongly admissible, if in addition, $\Phi$ does not explode in $N$.*

Definition 2.2. *$\Phi$ is said to be asymptotically compact on $X$, if each bounded set $B \subset X$ is strongly admissible.*

Let $S \subset X$. $S$ is said to be *positively invariant* (resp. *invariant*), if $\Phi(t)S \subset S$ (resp. $\Phi(t)S = S$) for all $t \geq 0$.

A compact invariant set $\mathcal{A} \subset X$ is called an *attractor* of $\Phi$, if it attracts a neighborhood $U$ of itself, namely, $\lim_{t \to +\infty} d_H(\Phi(t)U, \mathcal{A}) = 0$.

Let $S$ be a compact invariant set of $\Phi$. An ordered collection $\mathcal{M} = \{M_1, \cdots, M_l\}$ of disjointed compact invariant subsets of $S$ is called a *Morse decomposition of $S$*, if for any full trajectory $\gamma$ contained in $S \setminus \left(\bigcup_{1 \leq k \leq l} M_k\right)$, there exist $i$ and $j$ with $i < j$ such that

$$\omega^*(\gamma) \subset M_j, \quad \omega(\gamma) \subset M_i. \tag{1}$$

Remark 2.1. *A full trajectory satisfying (1) will be referred to as a* connecting trajectory *between $M_i$ and $M_j$.*

Remark 2.2. *One may use equivalent definitions of Morse decompositions; see e.g. [32, Chap. III].*

**2.4. Conley index.** In this subsection we briefly recall the definition of Conley index. The interested reader is referred to [7, 25] and [32], etc. for details.

Let $\Phi$ be a local semiflow on $X$. Since $X$ may be an infinite dimensional space, we always assume $\Phi$ is asymptotically compact, hence each bounded subset of $X$ is *strongly admissible.*

A compact invariant set $S$ of $\Phi$ is said to be *isolated*, if there is a neighborhood $N$ of $S$ such that $S$ is the maximal compact invariant set in $\overline{N}$. Correspondingly, $N$ is called an *isolating neighborhood* of $S$.

Remark 2.3. *Note that we do not require an isolating neighborhood to be bounded, although the bounded ones are always of particular interest.*

An important example for isolating neighborhoods is the so called *isolating block*, which plays a crucial role in the computation of Conley index.

Let $B \subset X$ be a bounded closed set. $x \in \partial B$ is called a *strict egress* (resp. *strict ingress*, *bounce-off*) point of $B$, if for every trajectory $\gamma : [-\tau, s] \to X$ with $\gamma(0) = x$, where $\tau \geq 0$, $s > 0$, the following properties hold:

(1) there exists $0 < \varepsilon < s$ such that

$$\gamma(t) \notin B \ \ (\text{resp. } \ \gamma(t) \in \text{int}B, \ \ \text{resp. } \ \gamma(t) \notin B), \qquad \forall\, t \in (0, \varepsilon);$$

(2) if $\tau > 0$, then there exists $0 < \delta < \tau$ such that

$$\gamma(t) \in \text{int}B \ \ (\text{resp. } \ \gamma(t) \notin B, \ \ \text{resp. } \ \gamma(t) \notin B), \qquad \forall\, t \in (-\delta, 0).$$

Denote $B^e$ (resp. $B^i$, $B^b$) the set of all strict egress (resp. ingress, bounce-off) points of the closed set $B$, and set $B^- = B^e \cup B^b$.

$B$ is called an *isolating block* [32], if $B^-$ is closed and $\partial B = B^i \cup B^-$.

Let $N, E$ be two closed subsets of $X$. $E$ is called an *exit set* of $N$, if (1) $E$ is *N-positively invariant*, that is, for any $x \in E$ and $t \geq 0$,

$$\Phi([0,t])x \subset N \Longrightarrow \Phi([0,t])x \subset E;$$

and (2) for any $x \in N$, if $\Phi(t_1)x \notin N$ for some $t_1 > 0$, then there exists $t_0 \in [0, t_1]$ such that $\Phi(t_0) \in E$.

Let $S$ be a compact isolated invariant set. A pair of bounded closed subsets $(N, E)$ is called an *index pair* of $S$, if (1) $N \setminus E$ is an isolating neighborhood of $S$; and (2) $E$ is an exit set of $N$. We infer from [32] that if $B$ is a bounded isolating block, then $(B, B^-)$ is an index pair of the maximal compact invariant set $S = K_\infty(\Phi, B)$ in $B$.

Definition 2.3. *The homotopy Conley index of $S$, denoted by $h(\Phi, S)$, is defined to be the homotopy type $[(N/E, [E])]$ of the pointed space $(N/E, [E])$ for any index pair $(N, E)$ of $S$.*

Remark 2.4. *For convenience, if $U$ is an isolating neighborhood of a compact invariant set $S$ ($U$ need not to be bounded), we also write*

$$h(\Phi, U) = h(\Phi, S),$$

*hoping that this will not cause any confusion.*

*Example* 2.1. As an example (and also for later use), let us compute the Conley index of an asymptotically stable equilibrium $e$ ($e$ is an attractor of $\Phi$).

Let $L(x)$ be a Lyapunov function of $e$ defined on an open neighborhood $U$ of $e$ which is strictly decreasing along each trajectory of $\Phi$ in $U$ outside $e$ (see e.g. [15, pp. 226] for the construction of such a function). We may assume $L(e) = 0$ (hence $L(x) > 0$ for $x \in U \setminus \{e\}$). Take a $\delta > 0$ sufficiently small so that $B = \{x : \ L(x) \leq \delta\} \subset U$ and is a closed neighborhood of $e$. Then one easily sees that $B$ is an isolating block with $B^- = \emptyset$.

We claim that $B$ is contractible. Indeed, set

$$H(s,x) = \begin{cases} \Phi(s/(1-s))x, & x \in B, \ s \in [0,1); \\ x, & x \in B, \ s = 1. \end{cases}$$

Then $H$ is a strong deformation retraction.

Now by the definition of Conley index, we have

$$h(\Phi, \{e\}) = [(B/B^-, [B^-])] = \Sigma^0.$$

Let $S$ be a compact isolated invariant set of $\Phi$. Denote $H_*$ and $H^*$ the singular homology and cohomology theories with coefficient group $\mathbb{Z}$, respectively. Applying $H_*$ and $H^*$ to $h(\Phi, S)$ one immediately obtains the *homology* and *cohomology Conley indices* of $S$.

The *Poincaré polynomial* of $S$, denoted by $p(t, S)$, is the *formal polynomial*

$$p(t,S) = \sum_{q=0}^{\infty} \beta_q \, t^q$$

with $\beta_q = \operatorname{rank} H_q(h(\Phi, S))$. If $S$ has a Morse decomposition $\mathcal{M} = \{M_1, \cdots, M_l\}$, then the following *Morse equation*

$$p(t, M_1) + \cdots + p(t, M_l) = p(t, S) + (1 + t)Q(t)$$

holds for some formal polynomial $Q(t) = \sum_{q=0}^{\infty} d_q \, t^q$ with $d_q \in \mathbb{Z}_+$.

Let us also recall briefly the basic continuation property of the Conley index.

Let $\Phi_\lambda$ $(\lambda \in \Lambda)$ be a family of semiflows on $X$, where $\Lambda$ is a metric space. We say that $\Phi_\lambda$ *depends on* $\lambda$ *continuously*, if $\Phi_\lambda(t)x$ is defined at the point $(t, x, \lambda)$, then for any sequence $(t_n, x_n, \lambda_n)$ converging to $(t, x, \lambda)$, $\Phi_{\lambda_n}(t_n)x_n$ is defined as well for all $n$ sufficiently large, furthermore,

$$\Phi_{\lambda_n}(t_n)x_n \to \Phi_\lambda(t)x \quad \text{ as } n \to \infty.$$

Suppose $\Phi_\lambda$ depends on $\lambda$ continuously. Define

$$\Pi(t)(x, \lambda) = (\Phi_\lambda(t)x, \lambda), \qquad (x, \lambda) \in \mathscr{X} = X \times \Lambda.$$

Then $\Pi$ is a local semiflow on the product space $\mathscr{X}$, which will be called the *skew-product flow* of the family $\Phi_\lambda$ $(\lambda \in \Lambda)$.

We say that $\Phi_\lambda$ $(\lambda \in \Lambda)$ is $\lambda$-*locally uniformly asymptotically compact* ($\lambda$-l.u.a.c. in short), if its skew-product flow $\Pi$ is asymptotically compact.

Remark 2.5. *It is trivial to see that if $\Phi_\lambda$ $(\lambda \in \Lambda)$ is $\lambda$-l.u.a.c., then $\Lambda$ is necessarily locally compact.*

For convenience, given $K \subset \mathscr{X}$ and $\lambda \in \Lambda$, we will write

$$K[\lambda] = \{x : \ (x, \lambda) \in \mathscr{X}\}.$$

$K[\lambda]$ is called the $\lambda$-*section of* $K$. The following continuation result is actually a particular case of [32, Chap. I, Theorem 12.2].

Theorem 2.1. *Let $\Phi_\lambda$ $(\lambda \in \Lambda)$ be a family of semiflows on $X$, where $\Lambda$ is a connected compact metric space. Suppose $\Phi_\lambda$ depends on $\lambda$ continuously and is $\lambda$-l.u.a.c.*

*Let $K$ be a compact isolated invariant set of the skew-product flow $\Pi$ of $\Phi_\lambda$ $(\lambda \in \Lambda)$. Then*

$$h(\Phi_\lambda, K[\lambda]) \equiv const., \qquad \lambda \in \Lambda.$$

*Proof.* Take a bounded closed isolating neighborhood $\mathcal{U}$ of $K$ in $\mathscr{X}$. Then for each $\lambda \in \Lambda$, the $\lambda$-section $\mathcal{U}_\lambda$ of $\mathcal{U}$ is an isolating neighborhood of $K[\lambda]$. By the compactness of $K$ one can easily verify that $K[\lambda]$ is upper semicontinuous in $\lambda$. Consequently $\mathcal{U}_\lambda$ is also an isolating neighborhood of $K[\lambda']$ for $\lambda'$ near $\lambda$. The conclusion then directly follows from [32, Chap. I, Theorem 12.2.]. ■

Remark 2.6. *We emphasize that in the above theorem, we allow $K[\lambda'] = \emptyset$ for some $\lambda' \in \Lambda$. Note also that when such a case occurs, one necessarily has $h(\Phi_\lambda, K[\lambda]) = \overline{0}$ for all $\lambda \in \Lambda$.*

**2.5. Sectorial operators.** For the readers' convenience, we finally recall some basic notions concerning sectorial operators.

Let $X$ be a Banach space. A closed and densely defined linear operator $A : \mathrm{D}(A) \subset X \to X$ is called a *sectorial operator*, if there exist real numbers $\phi \in (0, \pi/2)$, $a \in \mathbb{R}$ and $M \geq 1$ such that the sector

$$S_{a,\phi} = \{\lambda : \phi \leq |\arg(\lambda - a)| \leq \pi, \quad \lambda \neq a\}$$

is contained in the resolvent set of $A$, moreover,

$$\|(\lambda I - A)^{-1}\| \leq M/|\lambda - a|$$

for all $\lambda \in S_{a,\phi}$, where $I$ denotes the identity on $X$.

Let $A$ be a sectorial operator in $X$. Denote $\sigma(A)$ the spectral of $A$. If $\min_{z\in\sigma(A)} \mathrm{Re}\, z > 0$, then $A$ generates an analytic semigroup $T(t) = e^{-At}$ with

$$||T(t)|| \leq Ce^{-\beta t}, \qquad t \geq 0$$

for some $C, \beta > 0$. This allows us to define the fractional powers of $A$ as follows: for each $\alpha > 0$,

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt,$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the Gamma function, and let $A^\alpha$ be the inverse of $A^{-\alpha}$ with $\mathrm{D}(A^\alpha) = \mathrm{R}(A^{-\alpha})$; see Henry [14, Chap. I] for details. We also assign $A^0 = I$.

Note that in general we may not have $\min_{z\in\sigma(A)} \mathrm{Re}\, z > 0$. However, one can always find a real number $a$ such that $\min_{z\in\sigma(A_1)} \mathrm{Re}\, z > 0$, where $A_1 = A + aI$. Hence we can define the fractional powers of $A_1$ as above. For each $\alpha \geq 0$, denote $X^\alpha = \mathrm{D}(A_1^\alpha)$. We equip $X^\alpha$ with the norm $\|\cdot\|_\alpha$ defined as

$$\|u\|_\alpha = \|A_1^\alpha u\|, \quad u \in X^\alpha.$$

Then $X^\alpha$ is a Banach space, which is called the *fractional power* of $X$. It is well known that the definition of $X^\alpha$ is independent of the choice of the number $a$, and different choices of $a$ give equivalent norms on $X^\alpha$ [14, Chap. I].

## 3. Continuation Theorems and Bifurcation from Infinity of Local Semiflows.

In this section, we establish some abstract continuation theorems on invariant sets and prove a general result on bifurcation from infinity in the framework of local semiflows by using Conley index.

Let $X$ be a complete metric space with metric $d(\cdot, \cdot)$, and set

$$\mathscr{X} = X \times \mathbb{R}, \quad \mathscr{X}_\pm = X \times \mathbb{R}_\pm.$$

$\mathscr{X}$ is equipped with the metric defined by

$$\varrho((u_1, \lambda_1), (u_2, \lambda_2)) = d(u_1, u_2) + |\lambda_1 - \lambda_2|, \qquad (u_1, \lambda_1), (u_2, \lambda_2) \in \mathscr{X}.$$

**3.1. Global continuation theorem.** Let $\Phi_\lambda$ ($\lambda \in \mathbb{R}$) be a family of local semiflows on $X$. Henceforth we always assume that $\Phi_\lambda$ depends on $\lambda$ continuously and is $\lambda$-l.u.a.c.

Given $\Lambda \subset \mathbb{R}$ and $U \subset X$, denote

$$\mathscr{K}(\Lambda, U) = \overline{\bigcup_{\lambda\in\Lambda} \left(K_\infty(\Phi_\lambda, U) \times \{\lambda\}\right)}. \tag{1}$$

For simplicity, we will write

$$\mathscr{K}(\Lambda, X) = \mathscr{K}(\Lambda).$$

Remark 3.1. *By the $\lambda$-l.u.a.c. property of $\Phi_\lambda$ ($\lambda \in \mathbb{R}$) and the invariance property of $\mathscr{K}(\Lambda, U)$, one can easily verify that if $\Lambda$ and $U$ are bounded then $\mathscr{K}(\Lambda, U)$ is compact.*

Theorem 3.1. *Let $S$ be a compact isolated invariant set of $\Phi_0$, and $U$ an isolating neighborhood of $S$. Denote $\mathscr{F}_\pm$ the family of components of $\mathscr{K}(\mathbb{R}_\pm)$ meeting $S \times \{0\}$.*

*Suppose $h(\Phi_0, S) \neq \overline{0}$. Then there is a $\Gamma \in \mathscr{F}_\pm$ such that either $\Gamma[0] \setminus U \neq \emptyset$, or $\Gamma$ is unbounded in the space $\mathscr{X}_\pm$.*

*Proof.* We only consider the case of $\mathscr{F}_+$. The argument for that of $\mathscr{F}_-$ is parallel.

We argue by contradiction and suppose the assertion in the theorem was false. Then each $\Gamma \in \mathscr{F}_+$ would be bounded in $\mathscr{X}_+$. Furthermore, $\Gamma[0] \subset U$ (hence $\Gamma[0] \subset S$).

Denote $\mathscr{C}(S)$ the family of all components of $S$. For each $Z \in \mathscr{C}(S)$, there is a (unique) $\Gamma_Z \in \mathscr{F}_+$ such that $Z \subset \Gamma_Z[0]$ (note that $\Gamma_Z[0]$ may not be connected). It can be easily seen that for any $Z_1, Z_2 \in \mathscr{C}(S)$, one has

$$\text{either } \Gamma_{Z_1} = \Gamma_{Z_2}, \text{ or } \Gamma_{Z_1} \cap \Gamma_{Z_2} = \emptyset. \tag{2}$$

Let $Z \in \mathscr{C}(S)$. Pick a number $\delta$ with $0 < \delta < d(Z, \partial U)$, and let $\mathcal{V}_\delta = \mathrm{B}_{\mathscr{X}_+}(\Gamma_Z, \delta)$ be the $\delta$-neighborhood of $\Gamma_Z$ in $\mathscr{X}_+$. Set

$$\mathcal{K} = \overline{\mathcal{V}}_\delta \cap \mathscr{K}(\mathbb{R}_+), \quad \mathcal{K}_\delta = \partial_+\mathcal{V}_\delta \cap \mathscr{K}(\mathbb{R}_+),$$

where $\partial_+\mathcal{V} = \partial_{\mathscr{X}_+}\mathcal{V}$ denotes the boundary of $\mathcal{V}$ in $\mathscr{X}_+$ for any $\mathcal{V} \subset \mathscr{X}_+$. Then by the boundedness of $\mathcal{V}_\delta$ and Remark 3.1 we easily deduce that both $\mathcal{K}$ and $\mathcal{K}_\delta$ are compact. Because $\Gamma$ is a component of $\mathscr{K}(\mathbb{R}_+)$ and $\Gamma_Z \cap \mathcal{K}_\delta = \emptyset$, by virtue of Lemma 2.1 there exist two disjoint closed subsets $\mathcal{K}_1, \mathcal{K}_2$ of $\mathcal{K}$ with $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ such that

$$\Gamma_Z \subset \mathcal{K}_1, \quad \mathcal{K}_\delta \subset \mathcal{K}_2\,.$$

Note that $\mathcal{K}_1$ is contained in the interior of $\mathcal{V}_\delta$ in $\mathscr{X}_+$.

Take a number $\delta_Z > 0$ with

$$\delta_Z < \frac{1}{4}\min\{\varrho(\mathcal{K}_1, \mathcal{K}_2), \varrho(\mathcal{K}_1, \partial_+\mathcal{V}_\delta)\}.$$

Let $\mathcal{V}_Z = \mathrm{B}_{\mathscr{X}_+}(\mathcal{K}_1, 2\delta_Z)$. Then by the choice of $\delta_Z$ we have

$$\mathrm{B}_{\mathscr{X}_+}(\partial_+\mathcal{V}_Z, \delta_Z) \cap \mathscr{K}(\mathbb{R}_+) = \emptyset. \tag{3}$$

Let $\lambda_1 = \max\{\lambda : \mathcal{V}_Z[\lambda] \neq \emptyset\}$. Thanks to Theorem 2.1, one deduces that

$$(4) \qquad h(\Phi_\lambda, \mathcal{V}_Z[\lambda]) \equiv const., \qquad \lambda \in [0, \lambda_1).$$

But $K_\infty(\Phi_\lambda, \overline{\mathcal{V}}_Z[\lambda]) = \emptyset$ if $\lambda$ is close to $\lambda_1$; see Fig. 3.1.

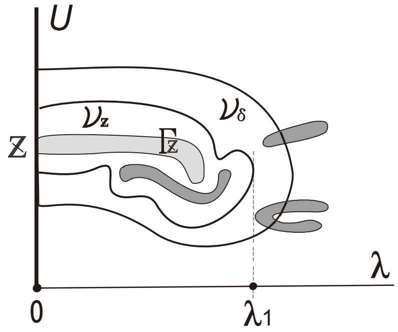


Fig. 3.1: $\mathcal{V}_Z$ is an isolating neighborhood.

By (4) it follows that

$$h(\Phi_\lambda, \mathcal{V}_Z[\lambda]) = \overline{0}, \qquad \lambda \in [0, \lambda_1).$$

In particular, we have

$$(5) \qquad h(\Phi_0, \Omega_Z) = \overline{0}, \quad \text{where } \Omega_Z = \mathcal{V}_Z[0].$$

Note that $\Omega_Z \subset U$. We also infer from (3) that

$$(6) \qquad \mathrm{B}_X(\partial\Omega_Z, \delta_Z) \cap S = \emptyset.$$

(Here $\partial\Omega_Z$ is the boundary of $\Omega_Z$ in $X$.) As $S$ is the maximal compact invariant set of $\Phi_0$ in $U$, (6) implies that $\Omega_Z$ is an isolating neighborhood of $\Phi_0$.

Since $S$ is compact, there exist a finite number of components $Z_1, \cdots, Z_l$ of $S$ such that $S \subset \bigcup_{i=1}^{l} \Omega_{Z_i}$. Let $W_1 = \Omega_{Z_1}$, and

$$W_k = \Omega_{Z_k} \setminus (\overline{\Omega}_{Z_1} \cup \cdots \cup \overline{\Omega}_{Z_{k-1}}), \qquad k = 2, \cdots, l.$$

Then $W_k{}'s$ are disjoint open sets in $X$, and

$$(7) \qquad \partial W_k \subset \bigcup_{i=1}^{k} \partial\Omega_{Z_i}.$$

As $S \bigcap \left( \bigcup_{i=1}^{l} \partial\Omega_{Z_i} \right) = \emptyset$ (see (6)), one finds that

$$S \subset \left( \bigcup_{i=1}^{l} \Omega_{Z_i} \right) \setminus \left( \bigcup_{i=1}^{l} \partial\Omega_{Z_i} \right) = \bigcup_{i=1}^{l} W_i.$$

Set $S_k = S \cap W_k$. We observe that if $w \in S_k$, then by (6),

$$d(w, \partial\Omega_{Z_i}) \geq \delta_{Z_i} \geq \min_{1 \leq i \leq l} \delta_{Z_i} > 0, \qquad 1 \leq i \leq l.$$

Thus by (7) it holds that

$$d(S_k, \partial W_k) > 0, \tag{8}$$

which implies that $S_k$ is compact. We also infer from (8) that $W_k$ is an isolating neighborhood of $S_k$ (with respect to $\Phi_0$). We claim that

$$h(\Phi_0, S_k) = \overline{0}. \tag{9}$$

Indeed, let $M_k = K_\infty(\Phi_0, \Omega_{Z_k}) \setminus S_k$. Then $M_k \subset \Omega_{Z_k} \setminus W_k$. Therefore by (8) we deduce that

$$d(S_k, M_k) > 0,$$

from which one can easily see that $M_k$ is compact. (5) then asserts that

$$\begin{aligned} \overline{0} = h(\Phi_0, \Omega_{Z_k}) &= h\left(\Phi_0, K_\infty(\Phi_0, \Omega_{Z_k})\right) \\ &= h(\Phi_0, S_k \cup M_k) = h(\Phi_0, S_k) \vee h(\Phi_0, M_k). \end{aligned}$$

By the basic knowledge in the theory of Conley index (see e.g. [32, pp. 52]) one immediately concludes the validity of (9).

Now since $S_k$ are disjoint isolated invariant sets of $\Phi_0$ and $S = \bigcup_{1\le k\le l} S_k$, we have

$$h(\Phi_0, S) = h(\Phi_0, S_1) \vee \cdots \vee h(\Phi_0, S_l) = \overline{0},$$

which leads to a contradiction. ∎

Remark 3.2. *In [40], Ward gave a continuation theorem asserting that* $\mathscr{S}_\pm = \bigcup_{\Gamma\in\mathscr{F}_\pm} \Gamma$ *either meets* $(X \setminus U) \times \{0\}$*, or is unbounded. Theorem 3.1 significantly improves this result.*

Theorem 3.2. *Let $S$ be an isolated invariant set of $\Phi_0$ with $h(\Phi_0, S) \neq \overline{0}$, and $U$ an isolating neighborhood of $S$. Let $0 < d \le \infty$, and denote $\Lambda$ either the interval $[0, d)$ or the one $(-d, 0]$. Denote $\mathscr{F}$ the family of components of $\mathscr{K}(\Lambda, U)$ meeting $S \times \{0\}$.*

*Then there exists $\Gamma \in \mathscr{F}$ such that one of the alternatives below holds:*

*(1) $\Gamma$ is unbounded; see Fig. 3.2.*
*(2) $\Gamma$ meeting $\partial U \times \Lambda$; see Fig. 3.3.*
*(3) $\Gamma[\lambda] \neq \emptyset$ for all $\lambda \in \Lambda$; see Fig. 3.4.*

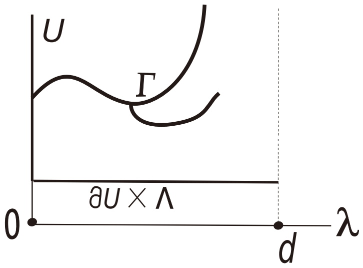

Fig. 3.2

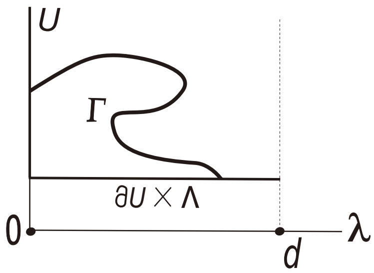

Fig. 3.3

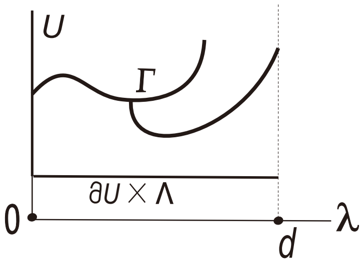


Fig. 3.4

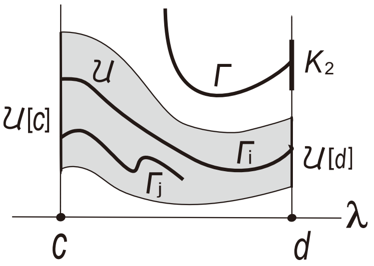


Fig. 3.5: $K_\infty(\Phi_c) \subset \mathcal{U}[c]$

*Proof.* The proof can be easily obtained by slightly modifying the one of Theorem 3.1. We omit the details. ∎

**3.2. An abstract theorem on bifurcation from infinity.** We now establish a new abstract theorem on dynamic bifurcation from infinity.

Let $\Phi_\lambda$ $(\lambda \in \mathbb{R})$ be as in subsection 3.1.

Theorem 3.3. *Let $\Lambda = [c, d]$ be a compact interval. Suppose both $K_\infty(\Phi_c)$ and $K_\infty(\Phi_d)$ are compact, furthermore,*

$$h\left(\Phi_c,\, K_\infty(\Phi_c)\right) \neq h\left(\Phi_d,\, K_\infty(\Phi_d)\right). \tag{10}$$

*Then the set $\mathscr{K}(\Lambda, X)$ has an unbounded component $\Gamma$ meeting $X \times \{c, d\}$.*

*Proof.* Denote $\mathscr{T}$ the family of connected components of $\mathscr{K}(\Lambda, X)$, and let

$$\mathscr{T}_c = \{\Gamma \in \mathscr{T} :\ \Gamma[c] \neq \emptyset\}, \quad \mathscr{T}_d = \{\Gamma \in \mathscr{T} :\ \Gamma[d] \neq \emptyset\}.$$

In the following we prove that if every $\Gamma \in \mathscr{T}_c$ is bounded, then there is a $\Gamma \in \mathscr{T}_d$ such that $\Gamma$ is unbounded.

Let $\mathcal{H} = X \times [c, d]$. Denote $\partial_{\mathcal{H}}\mathcal{V}$ the boundary of $\mathcal{V}$ in $\mathcal{H}$ for any $\mathcal{V} \subset \mathcal{H}$.

Let $\Gamma \in \mathscr{T}_c$. Since $\Gamma$ is bounded, as in Remark 3.1 one easily deduces by the $\lambda$-l.u.a.c. property of $\Phi_\lambda$ that $\Gamma$ is compact. Take a number $\varepsilon > 0$, and let

$$\mathcal{V}_\varepsilon = \mathrm{B}_{\mathcal{H}}(\Gamma, \varepsilon) := \{(x, \lambda) \in \mathcal{H} :\ \varrho((x, \lambda), \Gamma) < \varepsilon\}$$

be the $\varepsilon$-neighborhood of $\Gamma$ in $\mathcal{H}$. Set

$$\mathcal{C} = \overline{\mathcal{V}}_\varepsilon \cap \mathscr{K}(\Lambda, X), \quad \mathcal{C}_\varepsilon = \partial_{\mathcal{H}}\mathcal{V}_\varepsilon \cap \mathcal{C}.$$

By Remark 3.1 we see that both $\mathcal{C}$ and $\mathcal{C}_\varepsilon$ are compact. Since $\Gamma$ does not intersect any other component of $\mathcal{C}$, by Lemma 2.1 there exist two disjoint closed subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{C}$ with $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ such that

$$\Gamma \subset \mathcal{C}_1, \quad \mathcal{C}_\varepsilon \subset \mathcal{C}_2.$$

Clearly $\mathcal{C}_1$ is contained in the interior of $\mathcal{V}_\varepsilon$ in $\mathcal{H}$.

Pick a number $\varepsilon_\Gamma > 0$ with

$$\varepsilon_\Gamma < \frac{1}{4}\min\{\varrho(\mathcal{C}_1, \mathcal{C}_2),\ \varrho(\mathcal{C}_1, \partial_{\mathcal{H}}\mathcal{V}_\varepsilon)\}.$$

Let $\mathcal{U}_\Gamma = \mathrm{B}_{\mathcal{H}}(\mathcal{C}_1, 2\varepsilon_\Gamma)$ be the $2\varepsilon_\Gamma$-neighborhood of $\mathcal{C}_1$ in $\mathcal{H}$. Then by the choice of $\varepsilon_\Gamma$ we see that $\mathcal{U}_\Gamma \subset \mathcal{V}_\varepsilon$, and moreover,

$$\mathrm{B}_{\mathcal{H}}(\partial_{\mathcal{H}}\mathcal{U}_\Gamma, \varepsilon_\Gamma) \cap \mathscr{K}(\Lambda, X) = \emptyset. \tag{11}$$

Now we observe that $\mathscr{U} = \{\mathcal{U}_\Gamma[c]\}_{\Gamma\in\mathscr{T}_c}$ forms an open covering of $K_\infty(\Phi_c)$ in $X$. Thus there exist $\Gamma_1, \cdots, \Gamma_n \in \mathscr{T}_c$ such that

$$K_\infty(\Phi_c) \subset \bigcup_{1\le i\le n} \mathcal{U}_{\Gamma_i}[c].$$

Let $\mathcal{U} = \bigcup_{1\le i\le n} \mathcal{U}_{\Gamma_i}$. We infer from (11) that $\mathcal{U}$ is an isolating neighborhood of the skew-product flow $\Pi$ of $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ with $K_\infty(\Phi_c) \subset \mathcal{U}[c]$; see Fig. 3.5. Therefore by Theorem 2.1 one concludes that

$$h\left(\Phi_c,\, K_\infty(\Phi_c)\right) = h\left(\Phi_c,\, \mathcal{U}[c]\right) = h\left(\Phi_d,\, \mathcal{U}[d]\right) = h\left(\Phi_d,\, K_1\right), \tag{12}$$

where $K_1 = K_\infty\left(\Phi_d,\, \mathcal{U}[d]\right)$.

For any component $\Gamma$ of $\mathscr{K}(\Lambda, X)$, by (11) we have $\Gamma\cap\partial_{\mathcal{H}}\mathcal{U}_{\Gamma_i} = \emptyset$ for all $1 \le i \le n$. Hence one finds that

$$\text{either } \Gamma \subset \mathcal{U}, \quad \text{or } \Gamma \cap \overline{\mathcal{U}} = \emptyset.$$

Consequently, for any component $C$ of $K_\infty(\Phi_d)$, we have

$$\text{either } C \subset \mathcal{U}[d], \quad \text{or } C \cap \overline{\mathcal{U}}[d] = \emptyset.$$

Thus we deduce that $K_\infty(\Phi_d) = K_1 \cup K_2$, where

$$K_2 = \bigcup\{C :\ C \text{ is a component of } K_\infty(\Phi_d) \text{ with } C \cap \overline{\mathcal{U}}[d] = \emptyset\}.$$

As $K_1$ is isolated with $\mathcal{U}[d]$ being an isolating neighborhood, it is trivial to check that $K_2$ is isolated as well. Thereby

$$h\left(\Phi_d,\, K_\infty(\Phi_d)\right) = h\left(\Phi_d,\, K_1\right) \vee h\left(\Phi_d,\, K_2\right). \tag{13}$$

This, along with (10) and (12), yields that

$$h\left(\Phi_d,\, K_2\right) \neq \overline{0}.$$

Now by virtue of Theorem 3.1, one immediately concludes that there is a $\Gamma \in \mathscr{T}_d$ with $\Gamma[d] \subset K_2$ such that $\Gamma$ is unbounded; see Fig. 3.5. ■

**3.3. Two examples.** In this subsection we give two simple illustrating examples by considering ODE systems, which may help the reader have a better understanding to the abstract results given above.

*Example* 3.1. Consider the planar system

$$\begin{cases} \dot{x} = x - \lambda x(x^2 + y^2), & x = x(t) \in \mathbb{R}, \\ \dot{y} = y - \lambda y(x^2 + y^2), & y = y(t) \in \mathbb{R}, \end{cases} \tag{14}$$

where $\lambda$ is the bifurcation parameter.

Denote $\Phi_\lambda$ the semiflow on $X = \mathbb{R}^2$ generated by the system. Multiplying the first equation in (14) by $x$ and the second one by $y$, summing the results we obtain that

$$\frac{d}{dt}r^2 = 2r^2(1 - \lambda r^2), \tag{15}$$

where $r^2 = x^2 + y^2$. Let $\lambda \le 0$. Then by (15) we have

$$\frac{d}{dt}r^2 = 2r^2(1 - \lambda r^2) \ge 2r^2, \tag{16}$$

by which we deduce that $K_\infty(\Phi_\lambda) = \{(0,0)\}$ and is a repeller of the system. Let $B = \overline{\mathrm{B}}(2)$, where $\mathrm{B}(r)$ denotes the ball in $X$ centered at $(0,0)$ with radius $r$. By (16) it is clear that $B$ is an isolating block of $K_\infty(\Phi_\lambda)$ with $B^- = \partial\mathrm{B}(2)$. Hence

$$h(\Phi_\lambda, K_\infty(\Phi_\lambda)) = [(B/B^-, [B^-])] = \Sigma^2.$$

Now assume $\lambda > 0$. By (15) we find that

$$\frac{d}{dt}r^2 \le -r^2 \tag{17}$$

as long as $r(t) \ge \sqrt{2/\lambda}$, from which it can be easily seen that the system is dissipative with $K_\infty(\Phi_\lambda)$ being the global attractor. Let $\lambda = 1$. Then we infer from (17) that $B = \overline{\mathrm{B}}(2)$ is an isolating block of $K_\infty(\Phi_1)$ with $B^- = \emptyset$. Since $B$ is contractible, one has

$$h(\Phi_1, K_\infty(\Phi_1)) = [(B/B^-, [B^-])] = \Sigma^0.$$

Let $\Lambda = [-1, 1]$. Then $h(\Phi_{-1}, K_\infty(\Phi_{-1})) \ne h(\Phi_1, K_\infty(\Phi_1))$. By Theorem 3.3 one immediately concludes that the set $\mathscr{K}(\Lambda, X)$ has an unbounded component $\Gamma$ meeting $X \times \{\pm 1\}$.

One can also discuss the bifurcation phenomena of the system by choosing appropriate isolating neighborhoods of the system and applying Theorem 3.2. For instance, take $U = X \setminus \mathrm{B}\left(\frac{1}{2}\right)$. Then for $\lambda \in [0, 1]$, we have by (15) that

$$\frac{d}{dt}r^2 = 2r^2(1 - \lambda r^2) > 0, \quad \text{if } (x(t), y(t)) \in \partial U, \tag{18}$$

from which one easily deduces that

$$K_\infty(\Phi_\lambda, U) \cap \partial U = \emptyset, \qquad \forall\, \lambda \in [0, 1]. \tag{19}$$

Since $K_\infty(\Phi_\lambda, U) \subset K_\infty(\Phi_\lambda)$ and hence is compact for all $\lambda$, by (19) we find that $U$ is an isolating neighborhood of $\Phi_\lambda$ for each $\lambda \in [0, 1]$.

Set $S = K_\infty(\Phi_1, U)$. We infer from the above argument that $S \subset B := \overline{\mathrm{B}}(2) \setminus \mathrm{B}\left(\frac{1}{2}\right)$; furthermore, $B$ is an isolating block of $S$ with $\mathrm{B}^- = \emptyset$. We have

$$h(\Phi_1, S) = [(B/\emptyset, [\emptyset])] = [(B \cup \{q\}, q)] \ne \overline{0},$$

where $q$ is an element with $q \notin B$. By virtue of Theorem 3.2 one concludes that $\mathscr{K}((0,1],U)$ has a component $\Gamma_U$ meeting $S \times \{1\}$ such that one of the alternatives (1)-(3) in the theorem holds true. We claim that $\Gamma_U$ is unbounded. To see this, we first observe that $K_\infty(\Phi_0, U) = \emptyset$. Now we argue by contradiction and suppose the contrary. Then one can easily verify that $\Gamma_U[\lambda] \subset K_\infty(\Phi_\lambda, U)$ for all $\lambda \in (0,1]$. It follows by (19) that the second alternative (2) in Theorem 3.2 does not occur. Thus we necessarily have $\Gamma_U[\lambda] \neq \emptyset$ for all $\lambda \in (0,1]$. But this and the boundedness of $\Gamma_U$ then imply that $\Gamma_U[0] \neq \emptyset$, which leads to a contradiction and proves our claim.

Now let us give a simple observation that justifies our theoretical results obtained above. By (15) we see that the circle

$$C_\lambda : \; r = r_\lambda := 1/\sqrt{\lambda}$$

is a closed orbit of the system for each $\lambda > 0$, which depends on $\lambda$ continuously. Clearly $r_\lambda \to \infty$ as $\lambda \to 0$.

It is also worth mentioning that the bifurcating branches $\Gamma$ and $\Gamma_U$ given above may be different. In fact, it is easy to check that for $\lambda \in (0,1]$, $\Gamma_U[\lambda]$ consists of exactly the closed orbit $C_\lambda$, whereas $\Gamma[\lambda]$ may contain $C_\lambda$ and the equilibrium $(0,0)$ and also the connecting orbits between them.

*Example* 3.2. Consider the following non-autonomous scalar equation

$$\dot{x} = -(\lambda + h(t))x + e^{-x^2} \tag{20}$$

on $\mathbb{R}$, where $h \in C(\mathbb{R})$ is a $T$-periodic function ($T > 0$). To have a better understanding of the dynamics of the equation, as usual we embed the equation into a cocycle system below:

$$\dot{x} = -(\lambda + p(t))x + e^{-x^2}, \qquad p \in \mathcal{H}, \tag{21}$$

where $\mathcal{H} = \{h(\tau + \cdot) : \; \tau \in \mathbb{R}\}$, which is equipped with the topology of uniform convergence on $[0,T]$ (and hence on $\mathbb{R}$). It is a basic knowledge that due to the periodicity of $h$, $\mathcal{H}$ is homeomorphic to the unit circle (or, one-dimensional sphere) $S^1$.

Let $X = \mathbb{R} \times \mathcal{H}$, and denote $\phi_\lambda(t,p)x_0$ the unique solution of (21) with $x(0) = x_0$. Set

$$\Phi_\lambda(t)(x,p) = (\phi_\lambda(t,p)x, \theta_t p), \qquad (x,p) \in X,$$

where $\theta_t$ is the translation group on $\mathcal{H}$,

$$(\theta_t p)(\cdot) = p(t + \cdot), \qquad \forall\, p \in \mathcal{H}, \; t \in \mathbb{R}.$$

Then $\Phi_\lambda$ is a flow on $X$, called the *skew-product flow* of (21).

For the sake of simplicity, we may assume $\max_{\mathbb{R}} |h(t)| \leq 1$. Let $\Lambda = [-2,2]$. For $\lambda = -2$, multiplying the equation (21) by $x$ we find that

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}x^2 &= (2 - p(t))x^2 + xe^{-x^2} \\ &\geq x^2 - |x| = |x|(|x| - 1), \end{aligned} \tag{22}$$

from which it is clear that for any solution $x(t)$ of (21), if $|x(t_0)| > 1$ for some $t_0 \in \mathbb{R}$ then $|x(t)| > 1$ for all $t \geq t_0$; moreover, $|x(t)| \to \infty$ as $t \to +\infty$. It follows that

$$K_\infty(\Phi_{-2}) \subset [-1, 1] \times \mathcal{H}.$$

Let $B_1 = [-2, 2] \times \mathcal{H}$. Making use of (22) it is trivial to check that $B_1$ is an isolating block of $K_\infty(\Phi_{-2})$ with $B_1^- = \{\pm 2\} \times \mathcal{H}$. Thus

$$h\left(\Phi_{-2}, K_\infty(\Phi_{-2})\right) = [(B_1/B_1^-, [B_1^-])].$$

Since $B_1$ is pass-connected, one can easily verify that the quotient space $B_1/B_1^-$ is pass-connected as well. Hence

$$H_0\left(h\left(\Phi_{-2}, K_\infty(\Phi_{-2})\right)\right) = H_0\left((B_1/B_1^-, [B_1^-])\right) = 0. \tag{23}$$

Now we consider the case where $\lambda = 2$. A fully analogous argument as above applies to show that $K_\infty(\Phi_2) \subset [-1, 1] \times \mathcal{H}$ with $B_2 = [-2, 2] \times \mathcal{H}$ being an isolating block with $B_2^- = \emptyset$. ($K_\infty(\Phi_2)$ is actually the global attractor of $\Phi_2$.) Thus we have

$$\begin{aligned} h\left(\Phi_2, K_\infty(\Phi_2)\right) &= [(B_2/\emptyset, [\emptyset])] = [(([-2, 2] \times \mathcal{H})/\emptyset, [\emptyset])] \\ &= [(S^1/\emptyset, [\emptyset])] = [(S^1 \cup \{q\}, q)], \end{aligned}$$

where $q$ is an element with $q \notin S^1$. Therefore

$$H_0\left(h\left(\Phi_2, K_\infty(\Phi_2)\right)\right) = H_0\left((S^1 \cup \{q\}, q)\right) = \mathbb{Z}. \tag{24}$$

(23) and (24) indicate that $h\left(\Phi_{-2}, K_\infty(\Phi_{-2})\right) \neq h\left(\Phi_2, K_\infty(\Phi_2)\right)$. Applying Theorem 3.3 one immediately concludes that the system $\Phi_\lambda$ undergoes a dynamic bifurcation from infinity as $\lambda$ varies in the interval $\Lambda$, although we know little about where and how this bifurcation occurs.

**4. Bifurcation from Infinity of Nonlinear Evolution Equations.** In this section we use our general results in section 3 to discuss the bifurcation phenomena from infinity of the nonlinear evolution equation

$$\frac{du}{dt} + Au - \lambda u - f(u, \lambda) = 0 \tag{1}$$

on a Banach space $X$, where $A$ is a sectorial operator on $X$ with compact resolvent, $\lambda \in \mathbb{R}$, and $f(u, \lambda)$ is a locally Lipschitz continuous mapping from $X^\alpha \times \mathbb{R}$ to $X$ for some $0 \leq \alpha < 1$. Our main goal is to present some more precise descriptions on the dynamic bifurcation from infinity.

Denote $\|\cdot\|$ and $\|\cdot\|_\alpha$ the norms of $X$ and $X^\alpha$, respectively.

**4.1. Existence of unbounded bifurcating branch.** It is well known (see e.g. [14, Theorem 3.3.3]) that the Cauchy problem of (1) is well-posed in $X^\alpha$, that is, for any $u_0 \in X^\alpha$, there exist $T > 0$ and a (unique) continuous function $u : [0,T) \to X^\alpha$ with $u(0) = u_0$, called the *strong solution* of the problem, such that $u(t) \in \mathrm{D}(A)$ and $\frac{d}{dt}u(t)$ exists for $t \in (0,T)$, moreover, the differential equation (1) is satisfied on $(0,T)$.

Denote $\Phi_\lambda$ the local semiflow generated by the equation. By the continuity property of $f$ in $\lambda$, one can easily verify that $\Phi_\lambda$ depends on $\lambda$ continuously. Also, by very standard argument (see e.g. [32, Chap. I, Theorem 4.4]), it can be shown that the family $\Phi_\lambda$ ($\lambda \in \mathbb{R}$) is $\lambda$-l.u.a.c.

We always assume $f$ satisfies the following *sublinear condition*:

**(A)** $\lim_{\|u\|_\alpha \to \infty} \|f(u,\lambda)\|/\|u\|_\alpha = 0$ uniformly on compact $\lambda$-intervals.

Hence $\Phi_\lambda$ is actually a global semiflow on $X^\alpha$ for each $\lambda$.

Definition 4.1. *We say that (1) bifurcates from infinity at $\lambda = \mu$ (or, $(\infty, \mu)$ is a bifurcation point), if for any $\varepsilon > 0$, there exist $\lambda \in \mathbb{R}$ with $|\lambda - \mu| < \varepsilon$ and a bounded full solution $u_\lambda = u_\lambda(t)$ of (1) such that*

$$\|u_\lambda\|_\infty > 1/\varepsilon,$$

*where $\|u_\lambda\|_\infty = \sup_{t\in\mathbb{R}} ||u_\lambda(t)||_\alpha$.*

Denote $\sigma(A)$ the spectral of $A$, and write

$$\mathrm{Re}\,\sigma(A) = \{\mathrm{Re}\, z : \ z \in \sigma(A)\}.$$

Theorem 4.1. *Let $\mu \in \mathrm{Re}\,\sigma(A)$. Then $(\infty, \mu)$ is a bifurcation point of (1). Specifically, for any $c, d \in \mathbb{R}$ with $c < \mu < d$ and $\mathrm{Re}\,\sigma(A) \cap [c,d] = \{\mu\}$, the set $\mathscr{K}([c,d])$ (see (1) in section 3 for the definition) has a component $\Gamma$ meeting $X^\alpha \times \{c,d\}$ such that for some sequence $\lambda_n \to \mu$,*

$$\sup\{\|x\|_\alpha : \ x \in \Gamma[\lambda_n]\} \to \infty \quad \text{as } n \to \infty. \tag{2}$$

*Proof.* Let us begin with the following linear equation

$$\frac{du}{dt} + Au - \lambda u = 0. \tag{3}$$

Let $c, d$ be the numbers given in the theorem. Then if $\lambda = c, d$, the set $\{0\}$ is an isolated invariant set for the semiflow $\phi_\lambda$ in $X^\alpha$ generated by (3). By [32] (see Chap. I, Corollary 11.2) there exist two nonnegative integers $p$ and $q$ with $q - p > 0$ such that

$$h(\phi_c, \{0\}) = \Sigma^p, \quad h(\phi_d, \{0\}) = \Sigma^q. \tag{4}$$

($q-p$ is actually the total algebraic multiplicity of all the eigenvalues $z$ of the operator $A$ with $\mathrm{Re}\, z = \mu$.)

Now consider the nonlinear equation

$$\frac{du}{dt} + Au - \lambda u - \nu f(u,\lambda) = 0, \tag{5}$$

where $\nu \in [0,1]$ is the homotopy parameter. By appropriately modifying the argument in the proof of [32, Chap. II, Theorem 5.1] (see also the proof of [40, Theorem 3.2]), it can be shown that for any $\varepsilon > 0$ with

$$c < \mu - \varepsilon < \mu + \varepsilon < d,$$

there exists $R_\varepsilon > 0$ such that for any bounded full solution $u = u(t)$ of (5) with $\lambda \in [c, \mu - \varepsilon] \cup [\mu + \varepsilon, d]$ and $\nu \in [0,1]$, we have

$$\|u\|_\infty < R_\varepsilon. \tag{6}$$

Denote $\phi_\lambda^\nu$ the semiflow generated by (5). By virtue of the continuation property of Conley index, we conclude that

$$\begin{aligned} h\left(\Phi_\lambda, K_\infty(\Phi_\lambda)\right) &= h\left(\phi_\lambda^1, K_\infty(\phi_\lambda^1)\right) \\ &= h\left(\phi_\lambda^0, K_\infty(\phi_\lambda^0)\right) = h(\phi_\lambda, \{0\}) = \Sigma^p \end{aligned} \tag{7}$$

for $\lambda \in [c, \mu - \varepsilon]$, and

$$\begin{aligned} h\left(\Phi_\lambda, K_\infty(\Phi_\lambda)\right) &= h\left(\phi_\lambda^1, K_\infty(\phi_\lambda^1)\right) \\ &= h\left(\phi_\lambda^0, K_\infty(\phi_\lambda^0)\right) = h(\phi_\lambda, \{0\}) = \Sigma^q \end{aligned} \tag{8}$$

for $\lambda \in [\mu + \varepsilon, d]$. Thanks to Theorem 3.3, one immediately concludes that $\mathscr{K}([c,d])$ has an unbounded connected component $\Gamma$ meeting $X^\alpha \times \{c, d\}$. On the other hand, (6) implies that for any $\varepsilon > 0$,

$$\Gamma[\lambda] \subset \mathrm{B}_{X^\alpha}(R_\varepsilon), \qquad \forall\, \lambda \in [c, \mu - \varepsilon] \cup [\mu + \varepsilon, d],$$

where $\mathrm{B}_{X^\alpha}(R_\varepsilon)$ denotes the ball in $X^\alpha$ centered at 0 with radius $R_\varepsilon$. Thus there exists a sequence $\lambda_n \to \mu$ such that (2) holds true. ∎

Remark 4.1. *In Theorem 4.1 one should distinguish two cases of the bifurcation. One is that $K_\infty(\Phi_\mu)$ is unbounded. When this occurs we say that (1) undergoes a* vertical bifurcation from infinity *at $\lambda = \mu$. The other is that $K_\infty(\Phi_\mu)$ is bounded, in which case we deduce that there is a sequence $\lambda_n \to \mu$ ($\lambda_n \neq \mu$ for all $n$) such that $\Gamma[\lambda_n]$ is unbounded, where $\Gamma$ is the connected bifurcating branch given in the theorem. Note that both cases may occur. This can be seen from the following two simple examples.*

*Example* 4.1. Consider the linear equation

$$\dot{u} + u = \lambda u, \qquad u = u(t) \in \mathbb{R}, \tag{9}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. Then we can see that $\mu = 1$ is a bifurcation value, at which each constant function $u(t) = c$ ($c \in \mathbb{R}$) is a bounded full solution of the equation. Hence the equation undergoes a vertical bifurcation from infinity at $\lambda = 1$.

It is also interesting to note that for each $\lambda \neq 1$, the equation has no bounded full solutions other than the trivial one.

*Example* 4.2. Consider the non-homogenous equation

$$\dot{u} + u = \lambda u + 1, \qquad u \in \mathbb{R}, \tag{10}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. Again $\mu = 1$ is a bifurcation value, at which each solution of (10) is given by $u = t + c$ $(c \in \mathbb{R})$. Clearly $K_\infty(\Phi_\mu) = \emptyset$.

On the other hand, if we let $[c, d] = [0, 2]$, then by Theorem 4.1 we see that $\mathscr{K}([0, 2])$ has an unbounded connected component $\Gamma$ in the space $\mathbb{R} \times [0, 2]$ with $\Gamma \cap (\mathbb{R} \times \{0, 2\}) \neq \emptyset$. Actually, for $\lambda \neq 1$, the unique bounded full solution of the equation is the stationary one $u_\lambda(t) = (1 - \lambda)^{-1}$. Hence

$$\Gamma = \{(u_\lambda, \lambda) : \ 0 \le \lambda < 1\}$$

is a component of $\mathscr{K}([0, 2])$ fulfilling all the requirements in the theorem.

**4.2. Further results on dynamic bifurcation from infinity.** We infer from Theorem 4.1 that there is a sequence $\lambda_n \to \mu$ such that for each $\lambda = \lambda_n$, (1) has a bounded full solution $u_n = u_n(t)$ with $\|u_n\|_\infty \to \infty$. In what follows we give another result on the bifurcation of the equation from infinity, which seems to be more precise in some aspects.

Theorem 4.2. *Assume $f$ satisfies the sublinear condition* (**A**) *in Theorem 4.1. Let $\mu \in \operatorname{Re}\sigma(A)$. Then one of the following alternatives holds.*

*(1) There is a sequence $u_n$ of bounded full solutions of* (1) *at $\lambda = \mu$ such that $\lim_{n\to\infty} \|u_n\|_\infty = \infty$.*

*(2) There is a one-sided neighborhood $\Lambda_1$ of $\mu$ such that for each $\lambda \in \Lambda_1 \setminus \{\mu\}$,* (1) *has two distinct bounded full solutions $u_\lambda$ and $v_\lambda$ such that*

$$\lim_{\lambda\to\mu} \|u_\lambda\|_\infty = \infty, \tag{11}$$

*whereas $\|v_\lambda\|_\infty$ remains bounded on the $\lambda$-interval $\Lambda_1$.*

*(3) There is a two-sided neighborhood $\Lambda$ of $\mu$ such that for each $\lambda \in \Lambda \setminus \{\mu\}$, the equation* (1) *has a bounded full solution $u_\lambda$ satisfying* (11)*.*

*Proof.* If (1) holds true then we are done. Thus we assume the contrary, and hence $S_\mu$ is a bounded set, where (and below) $S_\lambda = K_\infty(\Phi_\lambda)$.

Take two numbers $c, d \in \mathbb{R}$ as in Theorem 4.1. Since the number $\varepsilon$ in (7) and (8) is arbitrary, we infer from (7) and (8) that

$$h(\Phi_\lambda, S_\lambda) = \Sigma^p \ (\lambda \in [c, \mu)\,), \quad h(\Phi_\lambda, S_\lambda) = \Sigma^q \ (\lambda \in (\mu, d]) \tag{12}$$

for some nonnegative integers $p$ and $q$ with $p < q$.

Pick a bounded closed isolating neighborhood $U$ of $S_\mu$. Choose a $\delta > 0$ sufficiently small so that $U$ is an isolating neighborhood of $\Phi_\lambda$ for all $\lambda \in \Lambda = [\mu - \delta, \mu + \delta]$. Then

$$h\,(\Phi_\lambda, U) \equiv const. \tag{13}$$

Two possibilities may occur.

Case 1) $h(\Phi_\mu, S_\mu) \neq \bar{0}$. In such a case we show that the second assertion (2) holds true. It is obvious that

$$\text{either } h(\Phi_\mu, S_\mu) \neq \Sigma^p, \quad \text{or} \quad h(\Phi_\mu, S_\mu) \neq \Sigma^q.$$

Let us first consider the case where $h(\Phi_\mu, S_\mu) \neq \Sigma^p$. By (12) we have

$$(14) \qquad h(\Phi_\lambda, S_\lambda) \neq h(\Phi_\mu, S_\mu), \qquad \lambda \in [c, \mu).$$

We claim that

$$(15) \qquad S_\lambda \setminus U \neq \emptyset, \qquad \forall\, \lambda \in \Lambda_- := [\mu - \delta, \mu).$$

Indeed, if $S_\lambda \subset U$ for some $\lambda \in \Lambda_-$, then by (12) and (13) one finds that

$$h\left(\Phi_\mu, S_\mu\right) = h\left(\Phi_\mu, U\right) = h\left(\Phi_\lambda, U\right) = h\left(\Phi_\lambda, S_\lambda\right) = \Sigma^p,$$

which leads to a contradiction.

For each $\lambda \in \Lambda_-$, pick an $x_\lambda \in S_\lambda \setminus U$. Let $u_\lambda$ be a full trajectory of $\Phi_\lambda$ contained in $S_\lambda$ with $u_\lambda(0) = x_\lambda$. We show that $u_\lambda$ fulfills (11).

Suppose the contrary. Then there would exist a sequence $\lambda_n \to \mu$ ($\lambda_n \neq \mu$) such that the sequence $u_n = u_{\lambda_n}$ is uniformly bounded on $\mathbb{R}$. By very standard argument it can be shown that $u_n$ has a subsequence converging to a bounded full trajectory $u_0$ of $\Phi_\mu$ uniformly on any compact interval of $\mathbb{R}$. $u_0$ is necessarily contained in $S_\mu$. On the other hand, since $u_n(0) = x_{\lambda_n} \notin U$, we deduce that $u_0(0) \notin \operatorname{int} U$. This leads to a contradiction.

Now assume that $h(\Phi_\mu, S_\mu) \neq \Sigma^q$. Then by a fully analogous argument as above, one concludes that for each $\lambda \in \Lambda_+ = (\mu, \mu + \delta]$, the equation has a bounded full solution $u_\lambda$ satisfying (11).

Since $h(\Phi_\mu, S_\mu) \neq \overline{0}$, by (13) we have

$$h\left(\Phi_\lambda, U\right) = h\left(\Phi_\mu, U\right) = h(\Phi_\mu, S_\mu) \neq \overline{0}, \qquad \lambda \in \Lambda.$$

It follows that $K_\infty(\Phi_\lambda, U) \neq \emptyset$. For each $\lambda \in \Lambda$, pick a full solution $v_\lambda$ in $K_\infty(\Phi_\lambda, U)$. Then $||v_\lambda||_\infty$ remains bounded on $\Lambda$.

Case 2) $\quad h(\Phi_\mu, S_\mu) = \overline{0}$. In this case, we have

$$\Sigma^p \neq h(\Phi_\mu, S_\mu) \neq \Sigma^q.$$

The same argument as in Case 1) applies to show that for each $\lambda \in \Lambda_- \cup \Lambda_+$, the equation has a bounded full solution $u_\lambda$ satisfying (11). Hence the assertion (3) holds. ■

## 5. Dynamic Bifurcation and Multiplicity for Parabolic Equations.

In this section we consider the following boundary value problem:

$$(1) \qquad \begin{cases} u_t - \Delta u = \lambda u + f(x, u), & x \in \Omega; \\ u(x, t) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$, and $f \in C^1(\overline{\Omega} \times \mathbb{R})$.

Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. By $(\cdot, \cdot)$ and $|\cdot|$ we denote the usual inner product and norm on $H$, respectively. The norm $\|\cdot\|$ on $V$ is defined by

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}, \qquad u \in V.$$

Denote $A$ the operator $-\Delta$ associated with the homogenous Dirichlet boundary condition. $A$ is a sectorial operator and has a compact resolvent. Denote

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$$

the eigenvalues of $A$.

We may convert (1) into an abstract equation on $V$:

$$u_t + Au = \lambda u + \tilde{f}(u), \quad u = u(t) \in V, \tag{2}$$

where $\tilde{f}(u)$ is the Nemitski operator from $V$ to $H$ given by

$$\tilde{f}(u)(x) = f(x, u(x)), \qquad u \in V.$$

If we assume that

$$f(x, s) = o(|s|) \quad \text{as } |s| \to \infty \tag{3}$$

uniformly with respect to $x \in \overline{\Omega}$, then one can trivially verify that the Nemitski operator $\tilde{f}$ in (2) satisfies the sublinear condition (**A**) in section 4. Thus applying the abstract results in section 4, one can immediately obtain some interesting information on the bifurcation of the equation. For instance, we have

Theorem 5.1. *Let $\mu_k$ be an eigenvalue of $A$. Then one of the following alternatives holds.*

*(1) There is a sequence $u_n$ of bounded full solutions of (2) at $\lambda = \mu_k$ such that*

$$\lim_{n\to\infty} \|u_n\|_\infty = \infty.$$

*(2) There is a one-sided neighborhood $\Lambda_1$ of $\mu_k$ such that for $\lambda \in \Lambda_1 \setminus \{\mu_k\}$, the equation (2) has at least two distinct bounded full solutions $u_\lambda$ and $v_\lambda$ such that*

$$\lim_{\lambda\to\mu_k} \|u_\lambda\|_\infty = \infty, \tag{4}$$

*whereas $\|v_\lambda\|$ remains bounded on $\Lambda_1$.*

*(3) There is a two-sided neighborhood $\Lambda$ of $\mu_k$ such that for each $\lambda \in \Lambda \setminus \{\mu_k\}$, (2) has at least one bounded full solution $u_\lambda$ satisfying (4).*

In this present work, we are basically interested in a particular but very important case, namely, the case where $f$ satisfies the Landesman-Laser type condition (4) in section 1. We will give some precise descriptions on the bifurcation of the equation and discuss the multiplicity of stationary solutions of the equation.

Henceforth we always assume

**(H)** $f$ satisfies the Landesman-Laser type condition (4) in section 1.

Denote $\Phi_\lambda$ the semiflow associated with (2), namely, for each $u_0 \in X^\alpha$,

$$u(t) = \Phi_\lambda(t)u_0$$

is the solution of the equation on $\mathbb{R}_+$ with initial value $u(0) = u_0$.

**5.1. Preliminaries.** Let us begin with a fundamental result on $f$.

Given a function $w$ on $\Omega$, we use $w_\pm$ to denote the positive and negative parts of $w$, respectively,

$$w_\pm = \max\{\pm w(x), 0\}, \qquad x \in \Omega.$$

Then $w = w_+ - w_-$. We have

Lemma 5.1. *For any $R, \varepsilon > 0$, there exists $s_0 > 0$ such that*

$$\int_\Omega f(x, v + sw) w dx \geq \int_\Omega (\overline{f} w_+ + \underline{f} w_-) dx - \varepsilon$$

*for all $s \geq s_0$, $v \in \overline{\mathrm{B}}_H(R)$ and $w \in \overline{\mathrm{B}}_H(1)$, where $B_H(r)$ denotes the ball in $H$ centered at 0 with radius $r$.*

*Proof.* This is a slightly modified version of [17, Lemma 6.7]. Here we give the details of the proof for completeness and the reader's convenience.

Let

$$I = \int_\Omega f(x, v + sw) w dx - \int_\Omega (\overline{f} w_+ + \underline{f} w_-) dx.$$

Since $w = w_+ - w_-$, we can rewrite $I$ as $I_+ - I_-$, where

$$I_+ = \int_\Omega (f(x, v + sw) - \overline{f}) w_+ dx, \quad I_- = \int_\Omega (f(x, v + sw) + \underline{f}) w_- dx.$$

In what follows, let us estimate $I_+$ for $v \in \overline{\mathrm{B}}_H(R)$ and $w \in \overline{\mathrm{B}}_H(1)$.

We observe that

$$R^2 \geq \int_\Omega |v|^2 dx \geq \int_{\{|v| \geq \sigma\}} |v|^2 dx \geq \sigma^2 |\{|v| \geq \sigma\}|,$$

from which it can be easily seen that $|\{|v| \geq \sigma\}| \to 0$ as $\sigma \to +\infty$ uniformly with respect to $v \in \overline{\mathrm{B}}_H(R)$. (Here and below $|E|$ denotes the Lebesgue measure for any measurable subset $E$ of $\mathbb{R}^n$.) Therefore there exists $\sigma > 0$ such that

$$|\{|v| \geq \sigma\}|^{1/2} < \delta := \varepsilon / 8\|f\|(|\Omega| + 1), \qquad v \in \overline{\mathrm{B}}_H(R), \tag{5}$$

where $\|f\| = \sup_{x \in \overline{\Omega},\, s \in \mathbb{R}} |f(x, s)|$.

For each $v \in \overline{\mathrm{B}}_H(R)$ and $w \in \overline{\mathrm{B}}_H(1)$, let

$$D = D_{v,w} := \{|v| < \sigma\} \cap \{w_+ > \delta\}.$$

Then $\Omega = D \cup \{|v| \geq \sigma\} \cup \{w_+ \leq \delta\}$. Hence

$$\begin{aligned} I_+ &\geq \textstyle\int_D (f(x, v + sw) - \overline{f}) w_+ dx - \int_{\{|v| \geq \sigma\}} |f(x, v + sw) - \overline{f}| w_+ dx \\ &\quad - \textstyle\int_{\{w_+ \leq \delta\}} |f(x, v + sw) - \overline{f}| w_+ dx \\ &\geq \textstyle\int_D (f(x, v + sw) - \overline{f}) w_+ dx - 2\|f\| \left( \int_{\{|v| \geq \sigma\}} w_+ dx + \int_{\{w_+ \leq \delta\}} w_+ dx \right). \end{aligned}$$

Note that

$$\begin{aligned}\textstyle\int_{\{|v|\geq\sigma\}} w_+dx &\leq \left(\textstyle\int_{\{|v|\geq\sigma\}} w_+^2dx\right)^{1/2} |\{|v|\geq\sigma\}|^{1/2}\\ &\leq (\text{by } (5)) \leq |w|\delta \leq \delta.\end{aligned}$$

It is obvious that

$$\int_{\{w_+\leq\delta\}} w_+dx \leq |\Omega|\delta.$$

Thereby

$$\begin{aligned}I_+ &\geq \textstyle\int_D(f(x,v+sw)-\overline{f})w_+dx - 2\|f\|(|\Omega|+1)\delta\\ &= \textstyle\int_D(f(x,v+sw)-\overline{f})w_+dx - \frac{\varepsilon}{4}.\end{aligned} \tag{6}$$

Since $z+s\eta\to+\infty$ (as $s\to+\infty$) uniformly for $z\in[-\sigma,\sigma]$ and $\eta\geq\delta$, there exists $s_1>0$ (independent of $v$ and $w$) such that if $s>s_1$ then

$$f(x,z+s\eta)-\overline{f}\geq -\frac{\varepsilon}{4|\Omega|^{1/2}},\qquad \forall z\in[-\sigma,\sigma],\ \eta\geq\delta.$$

Now suppose that $s>s_1$. Then by the definition of $D$, we have (note that $w=w_+$ on $D$)

$$\begin{aligned}\textstyle\int_D(f(x,v+sw)-\overline{f})w_+dx &\geq -\tfrac{\varepsilon}{4|\Omega|^{1/2}}\textstyle\int_D w_+dx\\ &\geq -\tfrac{\varepsilon}{4|\Omega|^{1/2}}|D|^{1/2}(\textstyle\int_D |w|^2dx)^{1/2}\geq -\tfrac{\varepsilon}{4}.\end{aligned}$$

Thus by (6) we see that

$$I_+\geq \int_D(f(x,v+sw)-\overline{f})w_+dx-\frac{\varepsilon}{4} > -\frac{\varepsilon}{2}.$$

Similarly it can be shown that there exists $s_2>0$ (independent of $v$ and $w$) such that $I_- < \frac{\varepsilon}{2}$, provided $s>s_2$.

Set $s_0=\max\{s_1,s_2\}$. Then if $s>s_0$, we have

$$I\geq I_+-I_- > -\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=-\varepsilon$$

for all $v\in\overline{\mathrm{B}}_H(R)$ and $w\in\overline{\mathrm{B}}_H(1)$. This completes the proof of the lemma. ■

Now we prove some basic facts concerning the dynamical behavior of the equation (2).

Let $L=A-\mu_k$, where $\mu_k$ is an eigenvalue of $A$. The space $H$ can be decomposed into the orthogonal direct sum of its subspaces $H^-$, $H^0$ and $H^+$ corresponding to the negative, zero and positive eigenvalues of $L$, respectively. Note that both $H^-$ and $H^0$ are of finite-dimensional. Denote $P^\sigma$ ($\sigma\in\{0,\pm\}$) the projection from $H$ to $H^\sigma$. Set

$$V^\sigma = V\cap H^\sigma,\qquad \sigma\in\{0,\pm\}.$$

By the finite dimensionality of $H^-$ and $H^0$, one finds that $V^-$ and $V^0$ coincide with $H^-$ and $H^0$, respectively. We also have

$$V=V^-\oplus V^0\oplus V^+.$$

Lemma 5.2. *Assume $\lambda \le \mu_k + \eta$, where $\eta = (\mu_{k+1} - \mu_k)/2$. Then there exists $\rho_0 > 0$ (independent of $\lambda$) such that for any solution $u = u(t)$ of (2) on $\mathbb{R}_+$,*

$$\|u^+(t)\|^2 \le \|u_0^+\|^2\, e^{-\eta t} + \rho_0^2(1 - e^{-\eta t}), \qquad \forall\, t \ge 0.$$

*Here $u^+ = P^+u$.*

*Proof.* Taking the inner product of the equation with $Au^+$ in $H$, it yields

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|u^+\|^2 + |Au^+|^2 \quad &= \lambda\|u^+\|^2 + (Au^+, \tilde{f}(u))\\ &\le \lambda\|u^+\|^2 + \varepsilon|Au^+|^2 + C_\varepsilon\end{aligned}$$

for any $\varepsilon > 0$, where $C_\varepsilon$ is a positive constant depending only upon $\varepsilon$ and the upper bound of $|\tilde{f}(s)|$. Hence

$$\frac{1}{2}\frac{d}{dt}\|u^+\|^2 + (1-\varepsilon)|Au^+|^2 \le \lambda\|u^+\|^2 + C_\varepsilon. \tag{7}$$

Note that $|Au^+|^2 \ge \mu_{k+1}\|u^+\|^2$. Therefore by (7) we have

$$\frac{1}{2}\frac{d}{dt}\|u^+\|^2 \le -\left((1-\varepsilon)\mu_{k+1} - \lambda\right)\|u^+\|^2 + C_\varepsilon. \tag{8}$$

Fix an $\varepsilon > 0$ sufficiently small so that $(1-\varepsilon)\mu_{k+1} > \mu_k + \frac{3}{2}\eta$. Then for $\lambda \le \mu_k + \eta$, one has

$$(1-\varepsilon)\mu_{k+1} - \lambda > \left(\mu_k + \frac{3}{2}\eta\right) - (\mu_k + \eta) = \eta/2.$$

Now the conclusion follows from (8) and the classical Gronwall Lemma. ■

Denote

$$\Xi_\rho = \{v \in V:\ \|P^+v\| \le \rho\}, \qquad \rho > 0.$$

As a direct consequence of Lemma 5.2, we have

Corollary 5.1. *Assume $\lambda \le \mu_k + \eta$. Then*

$$K_\infty(\Phi_\lambda) \subset \Xi_{\rho_0}.$$

*Furthermore, $\Xi_\rho$ is positively invariant under $\Phi_\lambda$ for any $\rho > \rho_0$.*

Set $W = V^- \oplus V^0$, and let

$$P_W = P^- + P^0$$

be the projection from $V$ to $W$. Given $0 \le a < b \le \infty$ and $\rho > 0$, denote

$$\Xi_\rho[a,b] = \{u \in \Xi_\rho:\ a \le |P_W u| \le b\}. \tag{9}$$

Lemma 5.3. *Let $\eta$ and $\rho_0$ be as in Lemma 5.2, and $\rho > \rho_0$. Then there exist $R_0, c_0 > 0$ such that the following assertions hold.*

*(1) If $\lambda \in [\mu_k, \mu_k + \eta]$, then for any solution $u(t)$ of the equation (2) in $\Xi_\rho[R_0, \infty]$, we have*

$$\frac{d}{dt}|w(t)|^2 \geq c_0|w(t)|, \tag{10}$$

*where $w(t) = P_W u(t)$.*

*(2) For any $R > R_0$, there exists $0 < \theta \leq \eta$ such that if $\lambda \in [\mu_k - \theta, \mu_k)$, then (10) holds true for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, R]$.*

*Proof.* Let $\lambda \in [\mu_k - \eta, \mu_k + \eta]$, and $u = u(t)$ a solution of (2) in $\Xi_\rho$. Taking the inner product of (2) with $w = P_W u$ in $H$, it yields

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \|w\|^2 = \lambda|w|^2 + (\tilde{f}(u), w). \tag{11}$$

Because $\|w\|^2 \leq \mu_k|w|^2$, by (11) we have

$$\frac{1}{2}\frac{d}{dt}|w|^2 \geq (\lambda - \mu_k)|w|^2 + (\tilde{f}(u), w). \tag{12}$$

Let us first estimate the last term in (12).

As the norm $\|\cdot\|_{L^1(\Omega)}$ of $L^1(\Omega)$ and that of $H = L^2(\Omega)$ are equivalent on $W$, one easily sees that

$$\min\{\|v\|_{L^1(\Omega)} : \ v \in W, \ |v| = 1\} := m > 0. \tag{13}$$

Pick a number $\delta > 0$ with $\delta \leq \min\{\overline{f}, \underline{f}\}$. By virtue of Lemma 5.1 there exists $s_0 > 0$ (depending only upon $\rho$) such that if $s \geq s_0$, then

$$(\widetilde{f}(h + sv), v) = \int_\Omega f(x, h + sv)v\,dx \geq \int_\Omega \left(\overline{f}v_+ + \underline{f}v_-\right)dx - \frac{1}{2}m\delta \tag{14}$$

for all $h \in \overline{\mathrm{B}}_H(\rho)$ and $v \in \overline{\mathrm{B}}_H(1)$.

Now we rewrite

$$w = sv, \ \text{ where } s = |w|.$$

Then $|v| = 1$. Suppose $s \geq s_0$. Noticing that $||u^+|| \leq \rho$, by (14) one finds that

$$(\tilde{f}(u), w) = s(f(x, u^+ + sv), v) \geq s\left(\int_\Omega \left(\overline{f}v_+ + \underline{f}v_-\right)dx - \frac{1}{2}m\delta\right).$$

Observing that

$$\begin{aligned} & \textstyle\int_\Omega \left(\overline{f}v_+ + \underline{f}v_-\right)dx - \frac{1}{2}m\delta \\ \geq \ & \delta \textstyle\int_\Omega |v|dx - \frac{1}{2}m\delta \geq (\text{by } (13)) \geq \frac{1}{2}m\delta, \end{aligned}$$

we conclude that

$$(\tilde{f}(u), w) \geq \frac{1}{2}m\delta s = \frac{1}{2}m\delta|w|. \tag{15}$$

Now we combine (15) and (12) together to obtain that

$$\frac{d}{dt}|w(t)|^2 \geq 2\,(\lambda-\mu_k)\,|w|^2 + m\delta|w(t)| \tag{16}$$

as long as $|w(t)| \geq s_0$.

Set $R_0 = s_0$, $c_0 = m\delta/2$. Assume $\lambda \in [\mu_k, \mu_k+\eta]$. Then $\lambda-\mu_k \geq 0$, and we infer from (16) that

$$\frac{d}{dt}|w(t)|^2 \geq m\delta|w(t)| > c_0|w(t)|$$

at any point $t$ where $|w(t)| \geq R_0$. Hence the assertion (1) holds.

Now assume $\lambda < \mu_k$. Let $R > R_0$. Choose a $\theta > 0$ with $\theta R^2 < m\delta s_0/4$. Then if $\lambda \in [\mu_k-\theta, \mu_k)$, for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, R]$, by (16) we conclude that

$$\begin{aligned}\tfrac{d}{dt}|w(t)|^2 &\geq -2|\lambda-\mu_k|\,R^2 + m\delta|w(t)|\\ &\geq c_0|w(t)| + \left(c_0|w(t)| - 2\theta R^2\right)\\ &\geq c_0|w(t)| + \left(c_0 s_0 - 2\theta R^2\right) \geq c_0|w(t)|,\end{aligned}$$

which justifies the second assertion (2). ■

### 5.2. Dynamic bifurcation from infinity.

We are now ready to discuss the bifurcation of the equation (2) near $\lambda = \mu_k$.

Let $\Phi_\lambda$ be the semiflow generated by (2). First, as a consequence of Lemma 5.3 we have the following basic fact.

Proposition 5.1. *Assume the hypothesis (**H**). Then $K_\infty(\Phi_\lambda)$ is uniformly bounded in $V$ for $\lambda \in [\mu_k, \mu_k+\eta]$, and*

$$h\left(\Phi_{\mu_k}, K_\infty(\Phi_{\mu_k})\right) = h\left(\Phi_\lambda, K_\infty(\Phi_\lambda)\right) = \Sigma^{p+r}, \tag{17}$$

*where $p$ is the sum of the multiplicities of the eigenvalues $\mu_i$ $(0 \leq i \leq k-1)$ of $A$, and $r$ the multiplicity of $\mu_k$.*

*Proof.* Let $\eta$ and $\rho_0$ be the numbers given in Lemma 5.2. Fix a number $\rho > \rho_0$. Then there exist $R_0, c_0 > 0$ such that the first assertion (1) in Lemma 5.3 holds true, by which one easily deduces that

$$K_\infty(\Phi_\lambda) \subset \Xi_\rho[0, R_0], \qquad \forall\,\lambda \in [\mu_k, \mu_k+\eta]. \tag{18}$$

On the other hand, as in (12) in section 4 it can be shown that

$$h(\Phi_\lambda, K_\infty(\Phi_\lambda)) = \begin{cases} \Sigma^{p+r}, & \lambda \in (\mu_k, \mu_{k+1});\\ \Sigma^{p}, & \lambda \in (\mu_{k-1}, \mu_k).\end{cases} \tag{19}$$

By (18) and the continuation property of Conley index we immediately conclude that $h\left(\Phi_{\mu_k}, K_\infty(\Phi_{\mu_k})\right) = \Sigma^{p+r}$. ■

Now we state and prove the main result in this subsection on the dynamic bifurcation from infinity of the equation near each eigenvalue $\mu_k$.

Theorem 5.2. *Assume the hypothesis (*$\mathbf{H}$*). Then $S_\lambda := K_\infty(\Phi_\lambda)$ is nonvoid for all $\lambda \in \mathbb{R}$, and there exists $\delta > 0$ such that the following assertions hold.*

*(1) For each $\lambda \in \Lambda_- := [\mu_k - \delta, \mu_k)$, $S_\lambda$ has a Morse decomposition $\mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\}$. Furthermore, there is at least one connecting trajectory $\gamma$ between $M_\lambda^1$ and $M_\lambda^\infty$.*

*(2) $M_\lambda^1$ remains uniformly bounded on $\Lambda_-$, whereas*

$$\lim_{\lambda\to\mu_k^-} \min_{v\in M_\lambda^\infty} ||v|| = \infty. \tag{20}$$

*(3) Each of the sets $\mathscr{K}^1$ and $\mathscr{K}^\infty$ has a component $\Gamma$ with $\Gamma[\lambda] \neq \emptyset$ for all $\lambda \in \Lambda_-$, where*

$$\mathscr{K}^1 = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^1 \times \{\lambda\})}, \quad \mathscr{K}^\infty = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^\infty \times \{\lambda\})}.$$

*Proof.* (i) We infer from the proof of Theorem 4.1 (see (6) in section 4) and Proposition 5.1 that $S_\lambda$ is a compact subset of $V$ for all $\lambda \in \mathbb{R}$. Since the Conley index of $S_\lambda$ is nontrivial (see (17) and (19)), one concludes that $S_\lambda \neq \emptyset$.

(ii) Corollary 5.1 asserts that

$$S_\lambda \subset \Xi_{\rho_0}, \qquad \forall\, \lambda \leq \mu_k + \eta,$$

where $\rho_0$ is the number in Lemma 5.2. Fix a $\rho > \rho_0$, and let $R_0$ and $c_0$ be the numbers given by Lemma 5.3. Pick a bounded isolating neighborhood $N_1$ of $S_{\mu_k}$ with

$$\Xi_\rho[0, R_0] \subset N_1. \tag{21}$$

Then one can restrict $\delta > 0$ sufficiently small so that $N_1$ is also an isolating neighborhood of $\Phi_\lambda$ for all $\lambda \in \Lambda := [\mu_k - \delta, \mu_k + \delta]$. Hence

$$h(\Phi_\lambda, M_\lambda^1) \equiv const., \qquad \lambda \in \Lambda, \tag{22}$$

where $M_\lambda^1 = K_\infty(\Phi_\lambda, N_1)$. Further by Proposition 5.1 we deduce that

$$h(\Phi_\lambda, M_\lambda^1) = h(\Phi_{\mu_k}, M_{\mu_k}^1) = h(\Phi_{\mu_k}, S_{\mu_k}) = \Sigma^{p+r}, \qquad \lambda \in \Lambda. \tag{23}$$

It is clear that $M_\lambda^1 \subset S_\lambda \subset \Xi_\rho$. Therefore

$$M_\lambda^1 \subset N_1 \cap \Xi_\rho := \widetilde{N}_1, \qquad \lambda \in \Lambda. \tag{24}$$

Because $\widetilde{N}_1$ is bounded, one can find a number $R_1 > 0$ such that

$$\widetilde{N}_1 \subset \Xi_\rho[0, R_1/2]. \tag{25}$$

By Lemma 5.3 (2), there exists $\theta_1 > 0$ such that if $\lambda \in [\mu_k - \theta_1, \mu_k)$, then for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, R_1]$, one has

$$\frac{d}{dt}|w(t)|^2 \geq c_0|w(t)| \geq c_0 R_0 > 0, \tag{26}$$

where $w(t) = P_W u(t)$. We may assume $\delta \le \theta_1$. Let $\lambda \in \Lambda_- := [\mu_k - \delta, \mu_k)$. Then for any bounded full solution $u(t)$ of (2) in $\Xi_\rho$ with $u(t_0) \in \Xi_\rho[R_0, R_1]$ for some $t_0$, by (26) one easily deduces that there exists $T > 0$ such that

$$u(t) \in \Xi_\rho[0, R_0] \ (t < -T), \quad \text{and } \ u(t) \in \Xi_\rho[R_1, \infty] \ (t > T). \tag{27}$$

Combining (24), (25) and (27) together, it yields that

$$M_\lambda^1 \subset \Xi_\rho[0, R_0], \qquad \forall\, \lambda \in \Lambda_-. \tag{28}$$

As $M_\lambda^1$ is the maximal compact invariant set of $\Phi_\lambda$ in $N_1$, (21) and (28) imply that $M_\lambda^1$ is the maximal compact invariant set in $\Xi_\rho[0, R_0]$.

Set

$$M_\lambda^\infty = K_\infty\left(\Phi_\lambda,\, \Xi_\rho[R_1, \infty]\right).$$

Then $M_\lambda^\infty \subset K_\infty(\Phi_\lambda) = S_\lambda$. We prove that $\mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\}$ forms a Morse decomposition of $S_\lambda$. For this purpose, let us first show that if $u = u(t)$ is a full solution in $S_\lambda \setminus (M_\lambda^1 \cup M_\lambda^\infty)$, then

$$\omega^*(u) \subset M_\lambda^1, \quad \omega(u) \subset M_\lambda^\infty. \tag{29}$$

Indeed, let $u$ be such a solution. Then since $S_\lambda \subset \Xi_\rho$ and $M_\lambda^1$ and $M_\lambda^\infty$ are the maximal compact invariant sets in $\Xi_\rho[0, R_0]$ and $\Xi_\rho[R_1, \infty]$, respectively, there exists $t_0 \in \mathbb{R}$ such that $u(t_0) \in \Xi_\rho[R_0, R_1]$. Hence (29) directly follows from (27).

Now we check that $M_\lambda^\infty \ne \emptyset$. Thus $\mathcal{M}$ is a Morse decomposition of $S_\lambda$. Suppose the contrary. Then by (29) we find that $S_\lambda = M_\lambda^1$. Hence

$$h(\Phi_\lambda, M_\lambda^1) = h(\Phi_\lambda, S_\lambda) = (\text{by } (19)) = \Sigma^p,$$

which contradicts (23).

To complete the proof of (1), there remains to check the existence of a connecting trajectory between $M_\lambda^1$ and $M_\lambda^\infty$. To this end, we consider the Morse equation of $\mathcal{M}$:

$$p(t, M_\lambda^1) + p(t, M_\lambda^\infty) = p(t, S_\lambda) + (1+t)Q(t). \tag{30}$$

Recalling that $h(\Phi_\lambda, M_\lambda^1) = \Sigma^{p+r}$ and $h(\Phi_\lambda, S_\lambda) = \Sigma^p$, we have

$$p(t, M_\lambda^1) = t^{p+r}, \quad p(t, S_\lambda) = t^p.$$

Thus (30) reads

$$t^{p+r} + p(t, M_\lambda^\infty) = t^p + (1+t)Q(t), \tag{31}$$

which implies that $Q(t) \ne 0$. By the basic knowledge in the Morse theory of invariant sets (see [32, Chap. III, Theorem 3.5]), one immediately concludes that there is at least one connecting trajectory between $M_\lambda^1$ and $M_\lambda^\infty$.

We also infer from (31) that $p(t, M_\lambda^\infty) \neq 0$. Consequently

$$h(\Phi_\lambda, M_\lambda^\infty) \neq \bar{0}, \qquad \forall\, \lambda \in \Lambda_-. \tag{32}$$

(iii) Clearly $M_\lambda^1$ remains uniformly bounded on $\Lambda$.

For any $R > R_1$, by Lemma 5.3 there exists $0 < \theta < \theta_1$ such that when $\lambda \in [\mu_k - \theta, \mu_k)$, the differential inequality (26) holds true for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, 2R]$. Using this basic fact, it can be easily seen that if $\lambda \in [\mu_k - \theta, \mu_k)$, any bounded full solution in $\Xi_\rho[R_1, \infty]$ is necessarily contained in $\Xi_\rho[R, \infty]$. Hence

$$M_\lambda^\infty \subset \Xi_\rho[R, \infty], \tag{33}$$

which implies what we desired in (20) and completes the proof of (2).

(iv) Finally, let us verify the validity of (3).

Let $U = \Xi_\rho[R_1, \infty]$. Then $\partial U = C_1 \cup C_2$, where

$$C_1 = \{v : \|v^+\| = \rho,\ |w| \geq R_1\}, \quad C_2 = \{v : \|v^+\| \leq \rho,\ |w| = R_1\}.$$

Here $v^+ = P^+ v$, and $w = P_W v$. Let $\lambda \in \Lambda_-$. By the choice of $\rho$ and Lemma 5.2, we see that $M_\lambda^\infty \cap C_1 = \emptyset$. Fix an $R > R_1$. Then we infer from the above argument in (iii) that one can restrict $\delta > 0$ to be sufficiently small so that (33) holds. Consequently $M_\lambda^\infty \cap C_2 = \emptyset$. Thus

$$M_\lambda^\infty \cap \partial U = \emptyset,$$

namely, $U$ is an isolating neighborhood of $M_\lambda^\infty$.

Because $h\left(\Phi_{\mu_k - \delta}, U\right) \neq \bar{0}$ (by (32)), $\mathscr{K}^\infty$ has a connected component $\Gamma$ with $\Gamma[\mu_k - \delta] \neq \emptyset$ such that one of the alternatives (1)-(3) in Theorem 3.2 holds true. As $\Gamma[\lambda] \subset \operatorname{int} U$ for all $\lambda \in \Lambda_-$, we conclude that either $\Gamma$ is unbounded, or $\Gamma[\mu_k] \neq \emptyset$. Because $\Gamma[\lambda]$ is uniformly bounded on $[\mu_k - \delta, \mu_k - \varepsilon]$ for any $\varepsilon \in (0, \delta)$, in any case we deduce that $\Gamma[\lambda] \neq \emptyset$ for all $\lambda \in \Lambda_-$.

The argument for $\mathscr{K}^1$ is similar. We omit the details. ■

**5.3. Bifurcation and multiplicity of stationary solutions.** We now turn to the static bifurcation and multiplicity of stationary solutions of (2). Since the equation has a natural Lyapunov function $J(u)$ defined by

$$J(u) = \frac{1}{2}(\|u\|^2 - \lambda |u|^2) - \int_\Omega F(x, u)dx, \qquad u \in V,$$

where $F(x, s) = \int_0^s f(x, t)dt$, this problem can be treated in the framework of dynamical systems.

Theorem 5.3. *Assume the hypothesis* (**H**). *Let $\delta > 0$ be the same as in Theorem 5.2. Then*

*(1) $\Phi_\lambda$ has at least one equilibrium $e_\lambda$ for all $\lambda \in \mathbb{R}$;*

*(2) there exists $\delta > 0$ such that $\Phi_\lambda$ has at least two distinct equilibria $e_\lambda^\infty$ and $e_\lambda^c$ for each $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$, and*

$$\lim_{\lambda \to \mu_k^-} \|e_\lambda^\infty\| = \infty, \tag{34}$$

*whereas $e^c_\lambda$ remains bounded on $\Lambda_-$; and*

*(3) there is an open dense subset $\mathcal{D}$ of $\mathbb{R}$ such that for each $\lambda \in \Lambda_- \cap \mathcal{D}$, $\Phi_\lambda$ has at least three distinct equilibria.*

*Proof.* (1) Since each nonempty compact invariant set contains at least one stationary solution, the conclusion (1) directly follows from Theorem 5.2.

(2) Let $N_1$ be the isolating neighborhood of $S_{\mu_k}$ given in the proof of Theorem 5.2, and let $\mathcal{M} = \{M^\infty_\lambda, M^1_\lambda\}$ be the Morse decomposition of $S_\lambda$ for $\lambda \in \Lambda_-$. Then $M^1_\lambda \subset N_1$. By (25) we have

$$N_1 \cap \Xi_\rho = \widetilde{N}_1 \subset \Xi_\rho[0, R_1/2], \qquad \lambda \in \Lambda_-. \tag{35}$$

As $M^\infty_\lambda \subset \Xi_\rho[R_1, \infty]$, by (35) we find that $M^\infty_\lambda \cap N_1 = \emptyset$. Pick two stationary solutions $e_\lambda$ and $e^\infty_\lambda$ from $M^1_\lambda$ and $M^\infty_\lambda$, respectively. Then $e_\lambda$ and $e^\infty_\lambda$ fulfill the requirements in (2).

(3) By slightly modifying the proof of [35, Theorem 2.1], it can be shown that there is an open dense subset $\mathcal{D}$ of $\mathbb{R}$ such that all the equilibria of $\Phi_\lambda$ are hyperbolic if $\lambda \in \mathcal{D}$. Now assume $\lambda \in \Lambda_- \cap \mathcal{D}$. We show that there is another equilibrium $z^\infty_\lambda \in M^\infty_\lambda$ with $z^\infty_\lambda \neq e^\infty_\lambda$. Consequently $\Phi_\lambda$ has at least three distinct equilibria.

We argue by contradiction and suppose $M^\infty_\lambda$ consists of exactly one hyperbolic stationary solution $e^\infty_\lambda$. Then $p(t, M^\infty_\lambda) = t^m$ for some $m \geq 0$. Accordingly the Morse equation (30) reads

$$t^{p+r} + t^m = t^p + (1+t)Q(t).$$

But this is impossible for any formal polynomial $Q(t)$ with coefficients in $\mathbb{Z}_+$, as the sum of the coefficients of the left-hand side does not equal that of the right-hand side.

The proof of the theorem is finished. ■

Remark 5.1. *We infer from the above argument that for each $\lambda \in \Lambda_- \cap \mathcal{D}$, $\Phi_\lambda$ has at least two distinct equilibria outside the domain $N_1$.*

Finally, we pay some special attention to the particular case where

**(F)** $f(x, s) = o(|s|)$ as $|s| \to 0$ uniformly for $x \in \overline{\Omega}$.

We prove some new multiplicity results on stationary solutions for the equation (2) near each eigenvalue $\mu_k$. The main results are summarized in the following theorem.

Theorem 5.4. *Assume $f$ satisfies the hypotheses (**H**) and (**F**). Denote $W^c_{loc}(0)$ the local center manifold of $\Phi_{\mu_k}$ at the trivial equilibrium $0$, and let $\phi$ be the restriction of $\Phi_{\mu_k}$ on $W^c_{loc}(0)$. Suppose $0$ is an isolated equilibrium of $\Phi_{\mu_k}$. Then there exists $\delta > 0$ such that one of the following assertions holds:*

*(1) $0$ is an attractor of $\phi$. In this case, the system $\Phi_\lambda$ has at least two distinct nontrivial equilibria $e^c_\lambda$ and $e^\infty_\lambda$ for $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$, whereas it has at least three distinct ones $e^1_\lambda$, $e^2_\lambda$ and $e^c_\lambda$ for $\lambda \in \Lambda_+ = (\mu_k, \mu_k + \delta]$.*

*(2) $0$ is a repeller of $\phi$ (i.e., an attractor of the inverse flow $\phi^{-1}$). In this case, $\Phi_\lambda$ has at least three distinct nontrivial equilibria $e^1_\lambda$, $e^2_\lambda$ and $e^\infty_\lambda$ for each $\lambda \in \Lambda_-$.*

*(3) 0 is neither an attractor nor a repeller of $\phi$. When this occurs, $\Phi_\lambda$ has at least three nontrivial equilibria $e^1_\lambda$, $e^c_\lambda$ and $e^\infty_\lambda$ for $\lambda \in \Lambda_-$, whereas it has at least two distinct ones $e^1_\lambda$ and $e^c_\lambda$ for $\lambda \in \Lambda_+$.*

*Furthermore, we have*

$$\lim_{\lambda\to\mu_k} \|e^\infty_\lambda\| = \infty, \qquad \lim_{\lambda\to\mu_k} \|e^i_\lambda\| = 0 \;\; (i = 1, 2), \tag{36}$$

*and*

$$0 < \liminf_{\lambda\to\mu_k} \|e^c_\lambda\| \le \limsup_{\lambda\to\mu_k} \|e^c_\lambda\| < \infty. \tag{37}$$

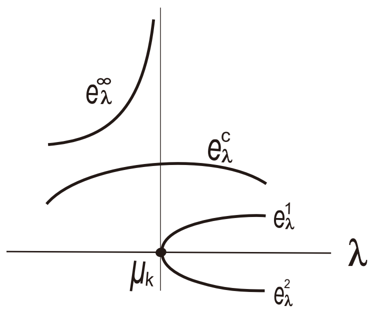


Fig. 5.1

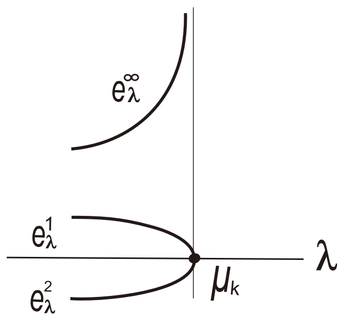


Fig. 5.2

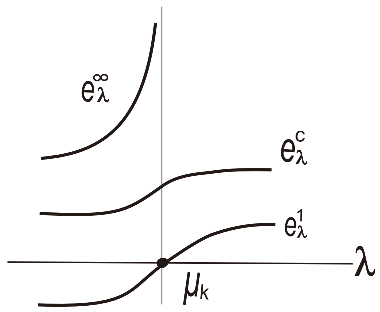


Fig. 5.3

*Proof.* In the following argument, we always assume that $\delta > 0$ is sufficiently small so that the conclusions in Theorem 5.2 and Theorem 5.3 are valid.

(1) The case where 0 is an attractor of $\phi$.

Let $N_1$ be the isolating neighborhood of $S_{\mu_k}$ in the proof of Theorem 5.3. Then by Theorem 5.3, for each $\lambda \in \Lambda_-$ the system $\Phi_\lambda$ always has an equilibrium $e^\infty_\lambda$ outside $N_1$ satisfying the first equation in (36).

Pick a number $\beta > 0$ and an isolating neighborhood $N_0$ of 0 with

$$N_0 \subset \mathrm{B}_V(\beta) \subset N_1.$$

We may restrict $\delta$ so that both $N_0$ and $N_1$ are isolating neighborhoods of $\Phi_\lambda$ for all $\lambda \in \Lambda := [\mu_k - \delta, \mu_k + \delta]$. Let

$$K^i_\lambda = K_\infty(\Phi_\lambda, N_i), \qquad i = 0, 1.$$

Then for each $i$,

$$h(\Phi_\lambda, K^i_\lambda) \equiv const., \qquad \lambda \in \Lambda. \tag{38}$$

It is trivial to check that

$$d_H\left(K^0_\lambda,\, \{0\}\right) \to 0 \;\text{ as }\; \lambda \to \mu_k. \tag{39}$$

It can be assumed that $N_0$ is sufficiently small so that the product formula of Conley index given in [32, Chap. II, Theorem 3.1] holds true, hence

$$h(\Phi_{\mu_k}, \{0\}) = \Sigma^p \wedge h(\phi, \{0\}).$$

We infer from Example 2.1 that $h(\phi, \{0\}) = \Sigma^0$. Therefore

$$h(\Phi_{\mu_k}, \{0\}) = \Sigma^p \wedge \Sigma^0 = \Sigma^p.$$

It then follows by (38) that

$$h(\Phi_\lambda, K_\lambda^0) = h(\Phi_{\mu_k}, K_{\mu_k}^0) = h(\Phi_{\mu_k}, \{0\}) = \Sigma^p, \qquad \lambda \in \Lambda. \tag{40}$$

By (38) and Proposition 5.1 we also deduce that

$$h(\Phi_\lambda, K_\lambda^1) = h(\Phi_{\mu_k}, K_{\mu_k}^1) = h(\Phi_{\mu_k}, S_{\mu_k}) = \Sigma^{p+r}, \qquad \lambda \in \Lambda. \tag{41}$$

Thus we see that $K_\lambda^1 \neq K_\lambda^0$. As $K_\lambda^0$ is the maximal invariant set in $N_0$, one concludes that

$$K_\lambda^1 \setminus N_0 \neq \emptyset, \qquad \lambda \in \Lambda. \tag{42}$$

For each $\lambda \in \Lambda$, pick a $v_\lambda \in K_\lambda^1 \setminus N_0$. Let $u_\lambda(t)$ be a bounded full trajectory of $\Phi_\lambda$ in $K_\lambda^1$ with $u_\lambda(0) = v_\lambda$. We claim that if $\delta$ is small enough then

$$\text{either } \omega(u_\lambda) \setminus N_0 \neq \emptyset, \quad \text{or } \omega^*(u_\lambda) \setminus N_0 \neq \emptyset. \tag{43}$$

Indeed, if this was false, there would exist a sequence $\lambda_n \to \mu_k$ (as $n \to \infty$) such that both $\omega(u_n)$ and $\omega^*(u_n)$ are contained in $N_0$ and hence in $K_{\lambda_n}^0$, where $u_n = u_{\lambda_n}$. Thus by (39) we deduce that

$$\lim_{n\to\infty} \max_{v\in\omega(u_n)} |J(v)| = 0 = \lim_{n\to\infty} \max_{v\in\omega^*(u_n)} |J(v)|. \tag{44}$$

Set

$$\Gamma_n = \overline{\mathrm{orb}(u_n)} = \mathrm{orb}(u_n) \cup \omega(u_n) \cup \omega^*(u_n).$$

Then

$$\min_{v\in\Gamma_n} J(v) = \min_{v\in\omega(u_n)} J(v), \quad \max_{v\in\Gamma_n} J(v) = \max_{v\in\omega^*(u_n)} J(v).$$

It follows by (44) that

$$\max_{v\in\Gamma_n} |J(v)| \to 0 \quad \text{as } n \to \infty. \tag{45}$$

On the other hand, since $\Gamma_n \subset K_{\lambda_n}^1 \subset N_1$ and $\Phi_\lambda$ is $\lambda$-l.u.a.c., it is easy to verify that $\bigcup_{\lambda_n\in\Lambda} \Gamma_n$ is precompact. Hence by Lemma 2.2 it can be assumed that $\Gamma_n$ converges to a nonempty compact invariant set $K$ of $\Phi_{\mu_k}$ (in the sense of Hausdorff distance). Noticing that $\Gamma_n \cap K_{\lambda_n}^0 \neq \emptyset$, by (39) we find that $0 \in K$. Because each $\Gamma_n$ is connected, $K$ is connected as well. (45) implies that $J(v) \equiv 0$ on $K$. Thereby each point in $K$ is an equilibrium of $\Phi_{\mu_k}$. As

$$u_n(0) \in \Gamma_n \setminus N_0 \tag{46}$$

for all $n$, we deduce that $K \setminus \mathrm{int}\, N_0 \neq \emptyset$. Further by the connectedness of $K$ one concludes that $K \cap \partial V \neq \emptyset$ for any small neighborhood $V$ of 0, which leads to a contradiction and completes the proof of our claim.

In view of (43), for each $\lambda \in \Lambda$ we can pick an equilibrium $e^c_\lambda$ of $\Phi_\lambda$ with

$$e^c_\lambda \in (\omega(u_\lambda) \cup \omega^*(u_\lambda)) \setminus N_0 \subset N_1 \setminus N_0. \tag{47}$$

Hence if $\lambda \in \Lambda_-$, the system $\Phi_\lambda$ has at least two distinct nontrivial equilibria $e^c_\lambda$ and $e^\infty_\lambda$.

We infer from the attractor bifurcation theory (see e.g. Ma and Wang [20, Theorem 4.3], [19, Theorem 6.1] or Li and Wang [18, Theorem 4.2]) that $K^0_\lambda$ contains at least two distinct equilibrium points $e^1_\lambda$ and $e^2_\lambda$ for $\lambda \in \Lambda_+$, provided $\delta$ is sufficiently small. By (47) one concludes that $\Phi_\lambda$ has at least three distinct nontrivial equilibria for $\lambda \in \Lambda_+$.

(2) 0 is a repeller of $\phi$. In this case, as in (1), by applying the attractor bifurcation theory we deduce that $K^0_\lambda$ contains at least two distinct equilibria $e^1_\lambda$ and $e^2_\lambda$ for $\lambda \in \Lambda_-$. Since $\Phi_\lambda$ has a nontrivial equilibrium $e^\infty_\lambda$ outside $N_1$ for each $\lambda \in \Lambda_-$, it has at least three distinct ones for $\lambda \in \Lambda_-$.

(3) Finally, let us consider the case where 0 is neither an attractor nor a repeller of $\phi$. By Li and Wang [18, Theorem 4.4] we deduce that the system bifurcates at each side of $\mu_k$ a nonempty compact invariant set $M_\lambda \subset N_0$ with $0 \notin M_\lambda$ and

$$d_H\left(M_\lambda, \{0\}\right) \to 0 \quad \text{as} \quad \lambda \to \mu_k. \tag{48}$$

$M_\lambda$ contains at least one nontrivial equilibrium $e^1_\lambda$.

We show that

$$h(\Phi_{\mu_k}, \{0\}) \neq \Sigma^{p+r}, \tag{49}$$

which fact will yield another equilibrium $e^c_\lambda \in N_1 \setminus N_0$ at both sides of $\mu_k$.

Consider the local center-unstable manifold $W^{cu}_{loc}(0)$ of $\Phi_{\mu_k}$ at 0. Denote $\psi$ the restriction of $\Phi_{\mu_k}$ on $W^{cu}_{loc}(0)$. Then

$$h(\Phi_{\mu_k}, \{0\}) = h(\psi, \{0\}). \tag{50}$$

Thus to prove (49), it suffices to check that

$$H_*(h(\psi, \{0\})) \neq H_*(\Sigma^{p+r}). \tag{51}$$

We argue by contradiction and suppose the contrary. Then

$$H_{p+r}(h(\psi, \{0\})) = H_{p+r}(\Sigma^{p+r}) = \mathbb{Z}.$$

Therefore by the Poincaré-Lefschetz duality theory of the Conley index (see McCord [24, Theorem 2.1] and Mrozek and Srzednicki [26, pp. 164]),

$$H^0(h(\psi^{-1}, \{0\})) = H_{p+r}(h(\psi, \{0\})) = \mathbb{Z}.$$

On the other hand, pick a pass-connected isolating block $B \subset W^{cu}_{loc}(0)$ of $S_0 = \{0\}$ with respect to the inverse flow $\psi^{-1}$. (Such an isolating block is always available due to [8, Theorem 1.5].) Since $S_0$ is not an attractor of $\psi^{-1}$ (note that $S_0$ is not an attractor of $\phi^{-1}$ on $W^c_{loc}(0)$), we necessarily have

$$B^- \neq \emptyset.$$

Thus by the basic knowledge in the theory of algebraic topology, one easily deduces that $H^0(B, B^-) = 0$. Consequently

$$H^0(h(\psi^{-1}, \{0\})) = H^0(B, B^-) = 0,$$

which leads to a contradiction and justifies the validity of (49).

Recall that (see (41))

$$h(\Phi_\lambda, K^1_\lambda) = \Sigma^{p+r}, \qquad \lambda \in \Lambda = [\mu_k - \delta, \mu_k + \delta].$$

Noticing that

$$h(\Phi_\lambda, K^0_\lambda) = h(\Phi_{\mu_k}, K^0_{\mu_k}) = h(\Phi_{\mu_k}, \{0\}) \neq \Sigma^{p+r}, \qquad \forall \lambda \in \Lambda,$$

we conclude that $K^1_\lambda \neq K^0_\lambda$. As $K^0_\lambda$ is the maximal invariant set in $N_0$, one finds that

$$K^1_\lambda \setminus N_0 \neq \emptyset, \qquad \lambda \in \Lambda.$$

We are now in a quite similar situation as in (42). Repeating the same argument below (42), it can be easily shown that the system has an equilibrium $e^c_\lambda$ in $N_1 \setminus N_0$.

In conclusion, there are at least two distinct nontrivial equilibria in $N_1$ for $\lambda \in \Lambda \setminus \{\mu_k\}$. Because $\Phi_\lambda$ has an equilibrium $e^\infty_\lambda$ outside $N_1$ for $\lambda \in \Lambda_-$, the system has at least three distinct nontrivial equilibria as $\lambda \in \Lambda_-$. This completes the proof of (3).

The second equation in (36) follows from (39) and (48). (37) is a direct consequence of the choice that $e^c_\lambda \in N_1 \setminus N_0$. ■

Remark 5.2. *It is interesting to note that there is always a one-sided neighborhood $\Lambda_1$ of $\mu_k$ such that the equation has at least three distinct nontrivial stationary solutions for $\lambda \in \Lambda_1$.*

Remark 5.3. *Dual versions of all the results in this section hold true if, instead of (**H**), we assume that (6) in section 1 is fulfilled.*

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## REFERENCES

[1] J. M. Arrieta, R. Pardo, and A. Rodríguez-Bernal, *Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity*, Proc. R. Soc. Edin. A, 137 (2007), pp. 225–252, https://www.mysciencework.com/publication/show/da4370bc8f4ae47d1209c26d3c04bb47.

[2] J. M. Arrieta, R. Pardo, and A. Rodríguez-Bernal, *Equilibria and global dynamics of a problem with bifurcation from infinity*, J. Differential Equations, 246 (2009), pp. 2055–2080, https://doi.org/10.1016/j.jde.2008.09.002.

[3] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, vol. 580, Springer-Verlag, Berlin, 1977.
[4] X. Chang and Y. Li, *Existence and multiplicity of nontrivial solutions for semilinear elliptic Dirichlet problems across resonance*, Topological Methods in Nonlinear Analysis, 36 (2010), pp. 285–310, http://projecteuclid.org/euclid.tmna/1461251091.
[5] R. Chiappinelli and D. G. de Figueiredo, *Bifurcation from infinity and multiple solutions for an elliptic system*, Differential and Integral Equations, 6 (1993), pp. 757–771.
[6] R. Chiappinelli, J. Mawhin, and R. Nugari, *Bifurcation from infinity and multiple solutions for some Dirichlet problems with unbounded nonlinearities*, Nonlinear Anal. TMA, 18 (1992), pp. 1099–1112, https://doi.org/10.1016/0362-546X(92)90155-8.
[7] C. Conley, *Isolated Invariant Sets and the Morse Index*, Regional Conference Series in Mathematics 38, Amer. Math. Soc., Providence RI, 1978.
[8] C. Conley and R. Easton, *Isolated invariant sets and isolating blocks*, Trans. Amer. Math. Soc., 158 (1971), pp. 35–61, https://doi.org/10.2307/1995770.
[9] F. O. de Paiva and E. Massa, *Semilinear elliptic problems near resonance with a nonprincipal eigenvalue*, J. Math. Anal. Appl., 342 (2008), pp. 638–650, https://doi.org/10.1016/j.jmaa.2007.12.053.
[10] J. P. Dias and J. Hernández, *A remark on a paper by J. F. Toland and some applications to unilateral problems*, Proc. R. Soc. Edin. A, 75 (1976), pp. 179–182, https://doi.org/10.1017/S0308210500017911.
[11] M. Filippakis, L. Gasiński, and N. S. Papageorgiou, *A multiplicity result for semilinear resonant elliptic problems with nonsmooth potential*, Nonlinear Anal. TMA, 61 (2005), pp. 61–75, https://doi.org/10.1016/j.na.2004.11.012.
[12] X. C. Fu and K. H. Xu, *The Conley index and bifurcation points*, Nonlinear Anal. TMA, 19 (1992), pp. 1137–1142, https://doi.org/10.1016/0362-546X(92)90187-J.
[13] J. L. Gámez and J. F. Ruiz-Hidalgo, *A detailed analysis on local bifurcation from infinity for nonlinear elliptic problems*, J. Math. Anal. Appl., 338 (2008), pp. 1458–1468, https://doi.org/10.1016/j.jmaa.2007.06.019.
[14] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes in Math. 840, Springer Verlag, Berlin New York, 1981.
[15] L. Kapitanski and I. Rodnianski, *Shape and Morse theory of attractors*, Comm. Pure Appl. Math., (2000), pp. 218–242, https://doi.org/10.1002/(SICI)1097-0312(200002)53:23.0.CO.
[16] W. Kryszewski and A. Szulkin, *Bifurcation from infinity for an asymptotically linear Schr ödinger equation*, J. Fixed Point Theory Appl., 16 (2014), pp. 411–435, https://doi.org/10.1007/s11784-015-0221-8.
[17] D. S. Li, G. L. Shi, and X. F. Song, *Linking theorems of local semiflows on complete metric spaces*, May 2015, https://arxiv.org/abs/1312.1868.
[18] D. S. Li and Z. Q. Wang, *Local and global dynamic bifurcations of nonlinear evolution equations*, Indiana Univ. Math. J., in press. https://arxiv.org/abs/1612.08128
[19] T. Ma and S. H. Wang, *Bifurcation Theory and Applications*, vol. 53, World Scientific, 2005.
[20] T. Ma and S. H. Wang, *Dynamic bifurcation of nonlinear evolution equations*, Chinese Ann. Math., 26 (2005), pp. 185–206, https://doi.org/10.1142/S0252959905000166.
[21] T. Ma and S. H. Wang, *Stability and Bifurcation of Nonlinear Evolution Equations*, Science Press, Beijing, 2007.
[22] J. Mawhin and K. Schmitt, *Landesman-Lazer type problems at an eigenvalue of odd multiplicity*, Results Math., 14 (1988), pp. 138–146, https://doi.org/10.1007/BF03323221.
[23] J. Mawhin and K. Schmitt, *Nonlinear eigenvalue problems with the parameter near resonance*, Ann. Polon. Math., 51 (1990), pp. 241–248.
[24] C. McCord, *Poincaré-Lefschetz duality for the homology Conley index*, Trans. Amer. Math. Soc., 329 (1992), pp. 233–252, http://www.ams.org/journals/tran/1992-329-01/S0002-9947-1992-1036005-X/.
[25] K. Mischaikow and M. Mrozek, *Conley index*, Handbook of dynamical systems, 2 (2002), pp. 393–460, https://doi.org/10.1016/S1874-575X(02)80030-3.
[26] M. Mrozek and R. Srzednicki, *On time-duality of the Conley index*, Results Math., 24 (1993), pp. 161–167, https://doi.org/10.1007/BF03322325.
[27] N. S. Papageorgiou and F. Papalini, *Multiple solutions for nearly resonant nonlinear Dirichlet problems*, Potential Anal., 37 (2012), pp. 247–279, https://doi.org/10.1007/s11118-011-9255-8.

[28] R. Pardo, *Bifurcation for an elliptic problem with nonlinear boundary conditions*, Revista Integración, 30 (2012), pp. 151–226, http://www.scielo.org.co/scielo.php?pid=S0120-419X2012000200005&script=sci_arttext&tlng=pt.

[29] H. Poincaré, *"Les Méthodes Nouvelles de la Mécanique Céleste"*, vol. I, Paris, 1892.

[30] P. H. Rabinowitz, *On bifurcation from infinity*, J. Differential Equations, 14 (1973), pp. 462–475, https://doi.org/10.1016/0022-0396(73)90061-2.

[31] M. Ramos and L. Sanchez, *A variational approach to multiplicity in elliptic problems near resonance*, Proc. R. Soc. Edin. A, 127 (1997), pp. 385–394, https://doi.org/10.1017/S0308210500023696.

[32] K. P. Rybakowski, *The Homotopy Index and Partial Differential Equations*, Springer-Verlag, Berlin. Heidelberg, 1987.

[33] K. Rybicki, *On bifurcation from infinity for $S^1$-equivariant potential operators*, Nonlinear Anal.TMA, 31 (1998), pp. 343–361, https://doi.org/10.1016/S0362-546X(96)00314-8.

[34] J. M. Sanjurjo, *Global topological properties of the Hopf bifurcation*, J. Differential Equations, 243 (2007), pp. 238–255, https://doi.org/10.1016/j.jde.2007.05.001.

[35] J. C. Saut and R. Temam, *Generic properties of nonlinear boundary value problems*, Comm. Partial Differential Equations, 4 (1979), pp. 293–319, https://doi.org/10.1080/03605307908820096.

[36] K. Schmitt and Z. Q. Wang, *On bifurcation from infinity for potential operators*, Differential and Integral Equations, 4 (1991), pp. 933–943.

[37] J. Su and C. Tang, *Multiplicity results for semilinear elliptic equations with resonance at higher eigenvalues*, Nonlinear Anal. TMA, 44 (2001), pp. 311–321, https://doi.org/10.1016/S0362-546X(99)00265-5.

[38] J. F. Toland, *Bifurcation and asymptotic bifurcation for non-compact non- symmetric gradient operators*, Proc. R. Soc. Edin. A, 73 (1975), pp. 137–147, https://doi.org/10.1017/S0308210500016334.

[39] J. R. Ward Jr, *Bifurcating continua in infinite dimensional dynamical systems and applications to differential equations*, J. Differential Equations, 125 (1996), pp. 117–132, https://doi.org/10.1006/jdeq.1996.0026.

[40] J. R. Ward Jr, *A global continuation theorem and bifurcation from infinity for infinite-dimensional dynamical systems*, Proc. R. Soc. Edin. A, 126 (1996), pp. 725–738, https://doi.org/10.1017/S0308210500023039.