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IMPULSE AND SAMPLED-DATA OPTIMAL CONTROL OF HEAT EQUATIONS, AND ERROR ESTIMATES*

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Abstract. We consider the optimal control problem of minimizing some quadratic functional over all possible solutions of an internally controlled multidimensional heat equation with a periodic terminal state constraint. This problem has a unique optimal solution, which can be characterized by an optimality system derived from the Pontryagin maximum principle. We define two approximations of this optimal control problem. The first one is an impulse approximation and consists of considering a system of linear heat equations with impulse control. The second one is obtained by the sample-and-hold procedure applied to the control, resulting in a sampled-data approximation of the controlled heat equation. We prove that both problems have a unique optimal solution, and we establish precise error estimates for the optimal controls and optimal states of the initial problem with respect to its impulse and sampled-data approximations.

Key words. heat equation, optimal control problem, impulse control, sampled-data control, error estimates

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1. Introduction and main results.

1.1. The context. There is a vast literature on numerical approximations of optimal control problems settled for parabolic differential equations. In the linear quadratic regulator (LQR) problem, many results exist concerning space semidiscretizations of the Riccati procedure. We refer to [1, 2, 14, 16, 27, 31] for general results showing convergence of the approximations of the Riccati operator, under assumptions of uniform exponential stabilizability, and of uniform boundedness of the sequence of approximate Riccati solutions. In [1, 22, 27], these sufficient conditions (and thus the convergence result) are proved to hold true in the general parabolic case and for unbounded control operators. Note that in such LQR problems, the final point is not fixed. When there is a terminal constraint the situation is more intricate, because things may go badly when discretizing optimal control problems in infinite dimension, due to interferences with the mesh that may cause the divergence of the optimization procedure when the mesh size is going to zero. These interferences are stronger when the terminal constraint has infinite codimension, in spite of strong dissipativity properties of parabolic equations. For the optimal control problem of minimizing the L^2 norm of the control (corresponding to the celebrated "Hilbert Uniqueness Method"), one can find results on uniform exact controllability and/or

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observability of discretized control systems in [4, 5, 7, 8, 18, 38] (see also references therein), for different discretization processes on different parabolic models. It can be noted that uniformity requires in general adding some appropriate viscosity terms in the numerical scheme. Besides, when the convergence is ensured, it is important to be able to derive error estimates which are as sharp as possible, and we refer the reader to [10, 12, 13, 15, 20, 21, 26, 32, 33, 34] for situations where Galerkin finite element approximations are used.

In many cases impulse control is an interesting alternative, not only to usual discretization schemes but also in order to deal with systems that cannot be acted on by means of continuous control inputs, as often occurs in applications. For example, relevant controls for acting on a population of bacteria should be impulsive, so that the density of the bactericide may change instantaneously; indeed continuous control would enhance drug resistance of bacteria. For more discussions and examples about impulse control or impulse control problems in infinite dimension, we refer the readers to [3, 36, 35] and references therein. It is also interesting to note that impulse control is as well an alternative to the well-known concept of digital control, or sampled-data control, which is much used in the engineering community.

To the best of our knowledge, error estimates for impulse approximations or for sampled-data approximations of an optimal control problem settled with partial differential equations and with continuous control inputs have not been investigated.

In this paper, we consider the problem of deriving precise error estimates for impulse approximations and for sampled-data approximations of a linear quadratic optimal control problem settled for an internally controlled linear homogeneous heat equation with periodic terminal state constraint. The latter periodicity requirement is motivated by the fact that steady solutions and periodic solutions are of particular interest when considering parabolic differential equations.

1.2. Definitions of the optimal control problems. Let $N \geqslant 1$ be an integer, let $\Omega \subset \mathbb{R}^N$ be a bounded open set having a C^2 boundary $\partial\Omega$, let $\omega \subset \Omega$ be an open nonempty subset, and let $T > 0$ and $y_d \in L^2(0,T;L^2(\Omega))$ be arbitrary. Throughout the paper, the norm in $L^2(\Omega)$ is denoted by $\|\ \|$.

The optimal control problem (OCP). We consider the optimal control problem (OCP) of minimizing the functional

$$J(y,u) = \frac{1}{2}\int_0^T \|y - y_d\|^2 \, \mathrm{d}t + \frac{1}{2}\int_0^T \|u\|^2 \, \mathrm{d}t \tag{1.1}$$

over all $(y,u) \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \times L^2(0,T;L^2(\Omega))$ such that

$$\begin{cases} \partial_t y - \triangle y = \chi_\omega u & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y(0) = y(T) & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

Here, χ_ω designates the characteristic function of ω, and y and u are functions of (x,t).

We will prove the following facts:

- Given any $u \in L^2(0,T;L^2(\Omega))$, there exists a unique solution $y \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ of (1.2).
- The problem (OCP) has a unique optimal solution (y^*,u^*).

- The Pontryagin maximum principle (PMP) of the problem (OCP) holds. More precisely, (y^*, u^*) is characterized by the existence of $p^* \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$ such that

$$\begin{cases} \partial_t y^* - \triangle y^* = \chi_\omega u^* & \text{in} \quad \Omega \times (0,T), \\ y^* = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y^*(0) = y^*(T) & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

$$\begin{cases} \partial_t p^* + \triangle p^* = y^* - y_d & \text{in} \quad \Omega \times (0,T), \\ p^* = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ p^*(0) = p^*(T) & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

and

$$u^* = \chi_\omega p^* \ \text{ in } \ \Omega \times (0,T). \tag{1.5}$$

Moreover, (1.3), (1.4), and (1.5) are necessary and sufficient conditions for optimality because the problem (OCP) is linear quadratic.

These three claims are easy to establish, but for completeness they are proved in section 2.1.

We are next going to design an approximating impulse optimal control problem $(\mathrm{IOCP})_\mathrm{n}$ and an approximating sampled-data optimal control problem $(\mathrm{SOCP})_\mathrm{n}$ for a linear heat equation with periodic terminal state constraint. Both problems have as well a unique solution, to which we will give the PMP. We will then establish error estimates between the optimal solutions of (OCP) and, respectively, $(\mathrm{IOCP})_\mathrm{n}$ and $(\mathrm{SOCP})_\mathrm{n}$.

The approximating impulse optimal control problem $(\mathrm{IOCP})_\mathrm{n}$. Let us define the approximating impulse optimal control problem $(\mathrm{IOCP})_\mathrm{n}$ for $n \geqslant 2$. We set

$$h_n = T/n, \qquad \tau_i = i\, h_n, \qquad i = 0, 1, \ldots, n,$$

$$X = \prod_{i=1}^{n} X_i, \qquad X_i = L^2(\tau_{i-1}, \tau_i; H_0^1(\Omega)) \cap H^1(\tau_{i-1}, \tau_i; H^{-1}(\Omega)), \qquad i = 1, 2, \ldots, n,$$

and we define the functional $J_n : X \times (L^2(\Omega))^{n-1} \to [0, +\infty)$ by

$$J_n(Y_n, U_n) = \frac{1}{2}\left(\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \|y_{i,n} - y_d\|^2 \,\mathrm{d}t + \frac{1}{h_n} \sum_{i=2}^{n} \|u_{i-1,n}\|^2 \right) \tag{1.6}$$

for $Y_n = (y_{1,n}, y_{2,n}, \ldots, y_{n,n}) \in X$ and $U_n = (u_{1,n}, u_{2,n}, \ldots, u_{n-1,n}) \in (L^2(\Omega))^{n-1}$. Here and throughout, $\|\cdot\|$ designates the norm in $L^2(\Omega)$. Accordingly, the inner product is denoted by $\langle \cdot, \cdot \rangle$.

We consider the impulse optimal control problem $(\mathrm{IOCP})_\mathrm{n}$, consisting of minimizing the functional J_n over all possible $(Y_n, U_n) \in X \times (L^2(\Omega))^{n-1}$ such that

$$\begin{cases} \partial_t y_{i,n} - \triangle y_{i,n} = 0 & \text{in} \quad \Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ y_{i,n} = 0 & \text{on} \quad \partial\Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ y_{i,n}(\tau_{i-1}) = y_{i-1,n}(\tau_{i-1}) + \chi_\omega u_{i-1,n} & \text{in} \quad \Omega, \quad 2 \leqslant i \leqslant n, \\ y_{1,n}(0) = y_{n,n}(T) & \text{in} \quad \Omega. \end{cases} \tag{1.7}$$

Proposition 1. *For every $n \geqslant 2$, the optimal control problem* $(\mathrm{IOCP})_{\mathrm{n}}$ *has a unique solution* (Y_n^*, U_n^*) *with* $Y_n^* = (y_{1,n}^*, y_{2,n}^*, \dots, y_{n,n}^*)$ *and* $U_n^* = (u_{1,n}^*, u_{2,n}^*, \dots, u_{n-1,n}^*)$. *The optimal solution* (Y_n^*, U_n^*) *of* $(\mathrm{IOCP})_{\mathrm{n}}$ *is characterized by the existence of* $p_n^* \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ *such that*

$$\begin{cases} \partial_t y_{i,n}^* - \triangle y_{i,n}^* = 0 & \text{in} \quad \Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ y_{i,n}^* = 0 & \text{on} \quad \partial\Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ y_{i,n}^*(\tau_{i-1}) = y_{i-1,n}^*(\tau_{i-1}) + \chi_\omega u_{i-1,n}^* & \text{in} \quad \Omega, \quad 2 \leqslant i \leqslant n, \\ y_{1,n}^*(0) = y_{n,n}^*(T) & \text{in} \quad \Omega, \end{cases} \tag{1.8}$$

$$\begin{cases} \partial_t p_n^* + \triangle p_n^* = y_n^* - y_d & \text{in} \quad \Omega \times (0,T), \\ p_n^* = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ p_n^*(0) = p_n^*(T) & \text{in} \quad \Omega, \end{cases} \tag{1.9}$$

and

$$u_{i-1,n}^* = h_n \chi_\omega p_n^*(\tau_{i-1}), \;\; 2 \leqslant i \leqslant n, \tag{1.10}$$

with

$$y_n^*(0) = y_n^*(T), \tag{1.11}$$

where $y_n^* \in L^\infty(0,T; L^2(\Omega))$ *is defined by*

$$y_n^*(t) = y_{i,n}^*(t), \;\; t \in (\tau_{i-1}, \tau_i], \;\; 1 \leqslant i \leqslant n, \tag{1.12}$$

and $y_{i,n}^* \in C([\tau_{i-1}, \tau_i]; L^2(\Omega))$, $1 \leqslant i \leqslant n$.

Proposition 1 is proved in section 2.2.

Remark 1. We could also consider the corresponding impulse version of the optimality system (1.3)–(1.4)–(1.5), but then its well-posedness would be hard to prove, and therefore obtaining error estimates in such a way seems difficult.

The approximating sampled-data optimal control problem $(\mathrm{SOCP})_{\mathrm{n}}$. Let us now define the approximating sampled-data optimal control problem $(\mathrm{SOCP})_{\mathrm{n}}$, for $n \geqslant 2$ by performing the usual sample-and-hold procedure on the control function. This consists of freezing the value of the control over a certain horizon of time, usually called sampling time. In other words, we replace the control function $u \in L^2(0,T; L^2(\Omega))$ with a control that is piecewise constant in time with values in $L^2(\Omega)$. We keep the same notation as in the definition of $(\mathrm{IOCP})_{\mathrm{n}}$, and we assume that the sampling time is equal to $h_n = T/n$. Recall that we have set $\tau_i = i\, h_n$ for $i = 0, \dots, n$. We consider the class of sampled-data controls $f_n \in L^2(0,T; L^2(\Omega))$ defined by

$$f_n(t) = v_{i,n} \quad \forall\, t \in (\tau_{i-1}, \tau_i], \quad 1 \leqslant i \leqslant n, \tag{1.13}$$

where $v_{i,n} \in L^2(\Omega)$ for every $i \in \{1, \dots, n\}$. This class of controls is therefore identified with $(L^2(\Omega))^n$.

Recall that the functional J is defined by (1.1). We consider the sampled-data optimal control problem $(\mathrm{SOCP})_{\mathrm{n}}$, consisting of minimizing the functional

$$J(y_n, f_n) = \frac{1}{2}\left(\int_0^T \|y_n - y_d\|^2 \,\mathrm{d}t + h_n \sum_{i=1}^n \|v_{i,n}\|^2\right)$$

over all $(y_n, V_n) \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))\times(L^2(\Omega))^n$, with $V_n = (v_{1,n},\dots,v_{n,n})$, such that

$$\begin{cases} \partial_t y_n - \triangle y_n = \chi_\omega f_n & \text{in} \quad \Omega\times(0,T),\\ y_n = 0 & \text{on} \quad \partial\Omega\times(0,T),\\ y_n(0) = y_n(T) & \text{in} \quad \Omega, \end{cases}$$

where $f_n \in L^2(0,T;L^2(\Omega))$ is the sampled-data control defined by (1.13).

PROPOSITION 2. *For every $n \geqslant 2$, the optimal control problem* $(\mathrm{SOCP})_\mathrm{n}$ *has a unique solution $(\bar{y}_n^*, V_n^*)$ with $V_n^* = (v_{1,n}^*,\dots,v_{n,n}^*)$. The optimal solution $(\bar{y}_n^*, V_n^*)$ of* $(\mathrm{SOCP})_\mathrm{n}$ *is characterized by the existence of $\bar{p}_n^* \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ such that*

$$\begin{cases} \partial_t \bar{y}_n^* - \triangle \bar{y}_n^* = \chi_\omega f_n^* & \textit{in} \quad \Omega\times(0,T),\\ \bar{y}_n^* = 0 & \textit{on} \quad \partial\Omega\times(0,T),\\ \bar{y}_n^*(0) = \bar{y}_n^*(T) & \textit{in} \quad \Omega, \end{cases} \tag{1.14}$$

$$\begin{cases} \partial_t \bar{p}_n^* + \triangle \bar{p}_n^* = \bar{y}_n^* - y_d & \textit{in} \quad \Omega\times(0,T),\\ \bar{p}_n^* = 0 & \textit{on} \quad \partial\Omega\times(0,T),\\ \bar{p}_n^*(0) = \bar{p}_n^*(T) & \textit{in} \quad \Omega, \end{cases} \tag{1.15}$$

and

$$v_{i,n}^* = \frac{1}{h_n}\chi_\omega \int_{\tau_{i-1}}^{\tau_i} \bar{p}_n^*(t)\,\mathrm{d}t, \quad 1 \leqslant i \leqslant n, \tag{1.16}$$

where $f_n^* \in L^2(0,T;L^2(\Omega))$ is the (optimal) sampled-data control given by

$$f_n^*(t) = v_{i,n}^* \ \forall\, t \in (\tau_{i-1},\tau_i],\ 1 \leqslant i \leqslant n. \tag{1.17}$$

Since the proof of Proposition 2 is similar to the one of Proposition 1, we do not provide any proof in the present paper. Note that the optimal sampled-data control f_n^*, defined by (1.17), is given by time-averages of the adjoint state $\bar{p}_n^*$ over the time-subdivision defined by the sampling time h_n (see (1.16)). This fact has been proved as well in a more general context in [6].

Remark 2. It is clear that the sampled-data optimal control problem $(\mathrm{SOCP})_\mathrm{n}$ may be considered as an approximate version of (OCP), but it is less clear, at least intuitively, for the impulse optimal control problem $(\mathrm{IOCP})_\mathrm{n}$. Before establishing precise error estimates in the next section, let us provide a first intuitive explanation. First, any continuously distributed control may be discretized by the sample-and-hold procedure, leading to the sampled-data control $\sum_{i=1}^n \chi_{((i-1)h_n, i\,h_n)} v_i$ with $v_i \in L^2(\Omega)$. Second, this sampled-data control can be seen as an approximation of an impulsive control, in the sense that

$$\frac{1}{h_n}\chi_{(0,h_n)} \to \delta_{\{t=0\}}$$

in the distributional sense, where $\delta_{\{t=0\}}$ is the Dirac mass at $t=0$ (note that we have as well the convergence of the corresponding solutions; see Lemma 2) Denoting by

$y(u)$ the solution of (1.2), we have, noting that $u_{i-1,n} \simeq h_n u(i\,h_n)$, for $i = 2, \dots, n$,

$$y(u) \simeq y\left(\sum_{i=1}^{n} \chi_{((i-1)h_n, i\,h_n)}(\cdot) u(i\,h_n)\right) \simeq y\left(\sum_{i=2}^{n} \delta_{\{t=(i-1)\,h_n\}} h_n u(i\,h_n)\right)$$
$$\simeq y\left(\sum_{i=2}^{n} \delta_{\{t=(i-1)\,h_n\}} u_{i-1,n}\right)$$

and

$$\|u\|^2_{L^2(0,T;L^2(\Omega))} \simeq \sum_{i=1}^{n} h_n \|u(i\,h_n)\|^2 \simeq \frac{1}{h_n} \sum_{i=2}^{n} \|u_{i-1,n}\|^2.$$

Here, $y\left(\sum_{i=2}^{n} \delta_{\{t=(i-1)\,h_n\}} u_{i-1,n}\right)$ is the corresponding impulsive solution, and we have moreover $J(y,u) \simeq J_n(Y_n, U_n)$.

1.3. Error estimates. We keep all notation introduced in section 1.2. The main results of the paper are as follows.

Error estimates for the impulse approximation.

THEOREM 1. *We set*

$$u_n^*(t) = \frac{1}{h_n} u_{i-1,n}^*, \quad t \in (\tau_{i-1}, \tau_i], \ \ 1 \leqslant i \leqslant n, \qquad u_{0,n}^* = 0. \tag{1.18}$$

Then there exists $C(T) > 0$ such that

$$\|u^* - u_n^*\|_{L^2(0,T;L^2(\Omega))} \leqslant C(T) h_n^{1/2} \|y_d\|_{L^2(0,T;L^2(\Omega))} \tag{1.19}$$

and

$$|J_n(Y_n^*, U_n^*) - J(y^*, u^*)| \leqslant C(T) h_n^{1/2} \|y_d\|^2_{L^2(0,T;L^2(\Omega))}. \tag{1.20}$$

For every $r \in [2, +\infty)$, there exists $C(T,r)$ such that

$$\|y^* - y_n^*\|_{L^r(0,T;L^2(\Omega))} \leqslant C(T,r) h_n^{1/r} \|y_d\|_{L^2(0,T;L^2(\Omega))}, \quad r \in [2, +\infty). \tag{1.21}$$

Moreover, the constants $C(T)$ and $C(T,r)$ are independent of n and of y_d.

Theorem 1 is proved in section 2.3. Note that we have assumed that $r < +\infty$ in the statement. If $r = +\infty$, then the situation is more complicated, and we have the following result.

THEOREM 2. *We assume that the subset ω of Ω has a C^2 boundary. Let $q \in (1, +\infty)$ be arbitrary. If $\omega \neq \Omega$, then*

$$\|y^* - y_n^*\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \begin{cases} C(T) h_n^{1/2N} \|y_d\|_{L^2(0,T;L^2(\Omega))} & \text{for} \quad N \geqslant 3, \\ C(T,q) h_n^{1/4q} \|y_d\|_{L^2(0,T;L^2(\Omega))} & \text{for} \quad N = 2, \\ C(T) h_n^{1/4} \|y_d\|_{L^2(0,T;L^2(\Omega))} & \text{for} \quad N = 1. \end{cases} \tag{1.22}$$

If $\omega = \Omega$, then

$$\|y^* - y_n^*\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(T) h_n^{1/2} \|y_d\|_{L^2(0,T;L^2(\Omega))}.$$

The constants $C(T)$ and $C(T,q)$ are independent of n and of y_d.

Theorem 2 is proved in section 2.4.

Remark 3. The above error estimates are much easier to obtain when the control domain ω is equal to the whole domain Ω, that is, when $\omega = \Omega$. But in this case the optimal control problems (OCP) and $(\mathrm{IOCP})_\mathrm{n}$ have little interest. Actually, the main difficulty in obtaining our results is due to the fact that if $\omega \subsetneq \Omega$, then the function χ_ω is not smooth and the function $\chi_\omega p_n^*(\tau_{i-1})$ in (1.10) is not in $H_0^1(\Omega)$. In the proof of Theorem 2 (see section 2), in addition to more or less standard functional analysis arguments, to overcome the above-mentioned difficulty, we use smooth regularizations of the characteristic function χ_ω, the gradient of which we have to estimate in a refined way in some appropriate L^r norm. Of course, this gradient blows up as the regularization parameter tends to zero, but fortunately there is some room to design appropriate regularizations, with adequate blow-up exponents (which we compute in a sharp way) that can be compensated elsewhere in the estimates, using Sobolev embeddings and usual functional inequalities. Using this approach, and deriving nonstandard estimates for the linear heat equation, we ultimately establish the desired error estimates.

Remark 4. In Theorem 2, if $\omega = \Omega$ (trivial case, according to Remark 3), then the order of convergence of the state is $1/2$, and we conjecture that it is sharp.[1] If $\omega \subsetneq \Omega$, then we have obtained the error estimate (1.22) but we conjecture that it is not sharp and that the order of convergence $1/2$ should hold true as well.

Error estimates for the sampled-data approximation.

THEOREM 3. *There exists $C(T) > 0$ such that*

(1.23)
$$\|u^* - f_n^*\|_{L^2(0,T;L^2(\Omega))} = \left(\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \|u^* - v_{i,n}^*\|^2 \, \mathrm{d}t \right)^{1/2} \leqslant C(T) h_n \|y_d\|_{L^2(0,T;L^2(\Omega))},$$

$$\begin{aligned} &\|y^* - \bar{y}_n^*\|_{C([0,T];H_0^1(\Omega))} + \|y^* - \bar{y}_n^*\|_{L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \\ &\leqslant C(T) h_n \|y_d\|_{L^2(0,T;L^2(\Omega))}, \end{aligned} \tag{1.24}$$

and

$$|J(y^*, u^*) - J(\bar{y}_n^*, f_n^*)| \leqslant C(T) h_n \|y_d\|^2_{L^2(0,T;L^2(\Omega))}. \tag{1.25}$$

Moreover, the constant $C(T)$ is independent of n and of y_d.

Theorem 3 is proved in section 2.5.

Remark 5. It is interesting to note that the error estimates are better for the sampled-data approximation than for the impulse approximation. For instance, the control error estimate is of order 1 for the sampled-data approximation but is of order $1/2$ for the impulse approximation (in L^2 norm). This is not surprising because, as explained in Remark 2, the sampled-data optimal control problem $(\mathrm{SOCP})_\mathrm{n}$ can easily be recast as a classical approximation of (OCP), in the sense that the class of admissible sampled-data controls of $(\mathrm{SOCP})_\mathrm{n}$ is a subset of the class of admissible

[1] Actually, we are able to prove that the exponent $1/2$ is sharp in the estimate given in Lemma 2 (in section 2.3) and in Lemma 5 (in section 2.4.3), in the case where $\omega = \Omega$ and $r = 2$. We do not provide the proofs of these facts here.

controls of (OCP). In this sense, obtaining the error estimates of Theorem 3 could be expected. In contrast, the set of unknowns (Y_n, U_n) for the impulse optimal control problem $(\mathrm{IOCP})_\mathrm{n}$ is not a subset of the set of unknowns (y, u) for (OCP). This explains why the derivation of error estimates for $(\mathrm{IOCP})_\mathrm{n}$ is much more difficult.

1.4. Further comments. We have established error estimates for the optimal controls and states of impulse approximations and of sampled-data approximations of a linear quadratic optimal control problem for a linear heat equation, with internal control, and with periodic terminal constraints. To our knowledge, this is the first result providing such convergence results and estimates, in an infinite-dimensional context. Many questions are open, that are in order.

Terminal constraints. Here, we have considered periodic terminal constraints. This condition is instrumental in order to obtain existence and uniqueness results and to be able to derive a PMP (see, in particular, Lemma 1 in section 2.1). But it is of course of interest to consider other terminal conditions. For instance, one may want to consider the problem (OCP) with the fixed terminal conditions $y(0) = y^0 \in L^2(\Omega)$ and $y(T) = 0$. It is well known that this exact null controllability problem admits some solutions, without any specific requirement on the (open) domain of control ω. But it is well known too that the final adjoint state coming from the PMP lives in a very big space that is larger than any distribution space. This will lead to the lack of regularity of adjoint state and optimal control. Hence this raises an important difficulty from the functional analysis point of view, preventing us from extending our analysis to this setting.

Moreover, when considering more general equations (see the next item), if one considers an infinite-codimensional state constraint, then it is well known that the PMP may fail (see [24]), and then in this case even the basic fact of establishing an optimality system may raise some impassable obstacles.

More general evolution equations. We have considered the linear homogeneous heat equations. The question is open to extend our analysis to more general parabolic equations, of the kind $\partial_t y = Ay + \chi_\omega u$, with $A : D(A) \to L^2(\Omega)$ generating an analytic semigroup. For instance, one may want to replace the Dirichlet Laplacian with a general elliptic second-order differential operator, with various possible boundary conditions (Dirichlet, Neumann, Robin, mixed), or with the Stokes operator. It is likely that our results may be extended to this situation, but note anyway that, in our proofs, we repeatedly use the fact that we have Dirichlet conditions.

More generally, the question is open to investigate semilinear parabolic equations, of the kind $\partial_t y = Ay + f(y) + \chi_\omega u$. Even when A is the Dirichlet Laplacian, this extension seems to be challenging.

The case of hyperbolic equations is another completely open issue. Certainly, the first case to be investigated is the wave equation: in that case one replaces the first equation in (1.2) with the internally controlled linear homogeneous wave equation $\partial_{tt} y = \triangle y + \chi_\omega u$. In this case, it is well known that exact controllability holds true under the so-called Geometric Control Condition on (ω, T). What happens for the corresponding impulse model is open and is far from being clear (see [17] for results in that direction). Also, many estimates, that are quite standard for heat-like equations and that we use in this paper are no longer valid in the hyperbolic context.

More general control operators. In this paper, we have considered an internal control. Writing the control system in the abstract form $\partial_t y = Ay + Bu$, this corresponds to considering the control operator B defined by $Bu = \chi_\omega u$. In this case, the control operator is bounded, and we implicitly use this fact in many places in our proofs. We expect that our results can be extended to more general classes of bounded control operators, but the case of unbounded control operators seems much more challenging. For instance, what happens when considering a Dirichlet boundary control is open.

Time-varying control domains and optimal design. Another open question is how to derive our error estimates for time-varying control domains. Note that control issues for wave equations with time-varying domains have been investigated in [23]. In our context, this means that we consider a control domain $\omega(t)$ depending on t in (1.2). In this case, the definition of the approximating impulse control system (1.7) must be adapted as well, by considering $\omega(\tau_{i-1})$ at time τ_{i-1}. It is likely that our main results may be at least extended to the case where $\omega(t)$ depends continuously on t. The general case is open.

Related to this issue is the question of determining how to place and shape "optimally" the control domain. Of course, the optimization criterion has to be defined, and we refer to [28, 29, 30], where optimal design problems have been modeled and studied. In the context of the present paper, we could investigate the problem of designing the best possible control domain such that the constants appearing in our error estimates are minimal.

Let us be more precise and let us define the open problem. Given any $T > 0$, and any open subset ω of Ω, Theorem 1 asserts that there exists a constant $C(T) > 0$ such that the error estimates (1.19), (1.20), and (1.21) (with $r = 2$, for instance) are satisfied. Since this constant depends on ω, we rather denote it by $C_T(\omega)$. Given a real number $L \in (0, 1)$, we consider the optimal design problem

$$\inf_{\omega \in \mathcal{U}_L} C_T(\omega),$$

that is, the problem of finding, if it exists, the best possible control subset having a prescribed Lebesgue measure, such that the functional constant in the error estimates is as small as possible. This prescribed measure is $L|\Omega|$, that is, a fixed fraction of the total volume of the domain. We stress that the set of unknowns is the (very big) set of all possible measurable subsets of Ω of measure $L|\Omega|$. It does not share any good compactness properties that would be appropriate for deriving nice functional properties, and thus already the problem of the existence of an optimal set is far from obvious. However, following [30], where similar optimal design problems have been investigated in the parabolic setting, we conjecture that there exists a unique best control domain, in the sense given above. Proving this conjecture, and deriving characterizations of the optimal set, is an interesting open issue.

Note that, less ambitiously than the problem above, one could already consider simpler optimal design problems, where the problem consists, for instance, of optimizing the placement of a control domain having a prescribed shape, such as a ball: in this case the set of unknowns is finite-dimensional (centers of the balls).

Impulse Riccati theory. In the present paper, we have considered a problem within a finite horizon of time T. It would be interesting to consider the optimal control problems (OCP), $(\mathrm{IOCP})_\mathrm{n}$, and $(\mathrm{SOCP})_\mathrm{n}$ in infinite horizon, that is, when $T = +\infty$. In this case, the optimal control solution of (OCP) is obtained by the well-known Riccati theory (see, e.g., [37]), which gives, here,

$$u = \chi_\omega E(y - y_d),$$

where E, a linear and bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$, is the unique negative definite solution of the algebraic Riccati equation

$$\triangle E + E\triangle + E\chi_\omega E = \mathrm{id}.$$

For the approximating impulse problem $(\mathrm{IOCP})_\mathrm{n}$, to our knowledge, the Riccati procedure has not been investigated. In other words, until now there does not seem to exist a Riccati theory for impulse linear quadratic optimal control problems in infinite dimension. Developing such a theory is already a challenge in itself. Assuming that such a theory has been established, the next challenge would be to establish as well the corresponding error estimates on the control and on the state, as done in our paper.

For the approximating sampled-data problem $(\mathrm{SOCP})_\mathrm{n}$, few results exist in the literature. In [31] the authors have established a convergence result (which can certainly be improved, by combining it with the more recent results of [22, 27], for instance), but we are not aware of any result providing error estimates as in our paper.

For impulse systems in particular, such a theory would certainly be very useful for many practical issues, because, as already mentioned, impulse control may be an interesting alternative to discretization approaches, or to sample-and-hold procedures, which is sometimes better suited to the context of the study. Notice that although the theory of space semidiscretization of the Riccati procedure is complete in the parabolic case (but not in the hyperbolic case when the control operators are unbounded), to our knowledge the theory is far from complete for infinite-dimensional sampled-data control systems. Therefore, with respect to sample-and-hold procedures, this is one more motivation for developing an impulse Riccati theory and its approximations.

2. Proofs.

2.1. Preliminaries.

Existence and uniqueness. We start with an easy existence and uniqueness result, together with a useful estimate. Throughout the paper, we denote by $\{e^{t\triangle}\}_{t\geqslant 0}$ the semigroup generated by the Dirichlet Laplacian on $L^2(\Omega)$.

LEMMA 1. *Let $T>0$ be arbitrary. Let $f\in L^2(0,T;L^2(\Omega))$. Then the equation*

$$\begin{cases} \partial_t y - \triangle y = f & \text{in} \quad \Omega\times(0,T),\\ y=0 & \text{on} \quad \partial\Omega\times(0,T),\\ y(0)=y(T) & \text{in} \quad \Omega \end{cases} \tag{2.1}$$

has a unique solution $y\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$. Moreover, there exists $C(T)>0$, not depending on f and on y, such that

$$\|y\|_{C([0,T];H_0^1(\Omega))} + \|y\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))} \leqslant C(T)\|f\|_{L^2(0,T;L^2(\Omega))}. \tag{2.2}$$

This lemma is easy to prove. Such results are known; however, for the sake of completeness we give a proof in Appendix A, section A.1.

COROLLARY 1. *Let $T>0$ be arbitrary. Let $g\in L^2(0,T;L^2(\Omega))$. Then the equation*

$$\begin{cases} \partial_t \psi + \triangle \psi = g & \text{in} \quad \Omega\times(0,T),\\ \psi=0 & \text{on} \quad \partial\Omega\times(0,T),\\ \psi(0)=\psi(T) & \text{in} \quad \Omega \end{cases} \tag{2.3}$$

has a unique solution $\psi \in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$. *Moreover, there exists* $C(T)>0$, *not depending on* g *and on* ψ, *such that*

(2.4)
$$\|\psi\|_{C([0,T];H_0^1(\Omega))}+\|\psi\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))}\leqslant C(T)\|g\|_{L^2(0,T;L^2(\Omega))}.$$

This corollary is proved in Appendix A, section A.2.

Optimality system (PMP). The proof of existence and uniqueness of an optimal solution of (OCP) is easy. Since it is similar to, but simpler than, the proof of Proposition 1, we skip it.

Let (y^*,u^*) be the optimal solution of (OCP). For any $v\in L^2(0,T;L^2(\Omega))$ and $\lambda\in\mathbb{R}\setminus\{0\}$, let $y_{\lambda,v}$ be the solution of

$$\begin{cases} \partial_t y_{\lambda,v}-\triangle y_{\lambda,v}=\chi_\omega(u^*+\lambda v) & \text{in} \quad \Omega\times(0,T),\\ y_{\lambda,v}=0 & \text{on} \quad \partial\Omega\times(0,T),\\ y_{\lambda,v}(0)=y_{\lambda,v}(T) & \text{in} \quad \Omega.\end{cases}$$

Setting $z=\frac{y_{\lambda,v}-y^*}{\lambda}$, we have

$$\begin{cases} \partial_t z-\triangle z=\chi_\omega v & \text{in} \quad \Omega\times(0,T),\\ z=0 & \text{on} \quad \partial\Omega\times(0,T),\\ z(0)=z(T) & \text{in} \quad \Omega.\end{cases}$$

Moreover, by definition, $J(y_{\lambda,v},u^*+\lambda v)\geqslant J(y^*,u^*)$, for every $\lambda\neq 0$, and hence

$$\int_0^T\int_\Omega (y^*-y_d)z\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega u^*v\,\mathrm{d}x\,\mathrm{d}t=0. \tag{2.5}$$

Let p^* be the solution of

$$\begin{cases} \partial_t p^*+\triangle p^*=y^*-y_d & \text{in} \quad \Omega\times(0,T),\\ p^*=0 & \text{on} \quad \partial\Omega\times(0,T),\\ p^*(0)=p^*(T) & \text{in} \quad \Omega.\end{cases}$$

This, together with (2.5), yields that

$$0=\int_0^T\int_\Omega (y^*-y_d)z\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega u^*v\mathrm{d}x\mathrm{d}t=\int_0^T\int_\Omega(-\chi_\omega p^*+u^*)v\mathrm{d}x\mathrm{d}t.$$

Hence $u^*=\chi_\omega p^*$. This gives the PMP for the problem (OCP).

2.2. Proof of Proposition 1. Let us first prove existence and uniqueness of a solution of $(\text{IOCP})_\text{n}$. For $U_n=(u_{1,n},\dots,u_{n-1,n})\in(L^2(\Omega))^{n-1}$, we say that $Y_n=(y_{1,n},\dots,y_{n,n})\in X$ is a weak solution of (1.7) if

$$\langle\partial_t y_{i,n}(t),\varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}+\int_\Omega\nabla y_{i,n}(t)\cdot\nabla\varphi\,\mathrm{d}x=0 \ \text{ a.e. } t\in(\tau_{i-1},\tau_i),\ 1\leqslant i\leqslant n,$$

for each $\varphi\in H_0^1(\Omega)$ (this means that the differential equation is written in $H^{-1}(\Omega)$) and $y_{i,n}(\tau_{i-1})=y_{i-1,n}(\tau_{i-1})+\chi_\omega u_{i-1,n}$, for $2\leqslant i\leqslant n$, and $y_{1,n}(0)=y_{n,n}(T)$. By the same arguments as in the proof of Lemma 1, if we define

$$y_{1,n}(0)=(I-e^{T\Delta})^{-1}\sum_{j=2}^n e^{(T-\tau_{j-1})\Delta}\chi_\omega u_{j-1,n},$$

then $y_{1,n}(0)\in L^2(\Omega)$ and (1.7) has a unique weak solution.

Let $d^* = \inf J_n(Y_n, U_n) \geqslant 0$, where the infimum is taken over all pairs $(Y_n, U_n) \in X \times (L^2(\Omega))^{n-1}$ satisfying (1.7). By definition, there exists a sequence $(Y_{n,m}, U_{n,m})_{m\geqslant 1}$, with $Y_{n,m} = (y_{i,n,m})_{1\leqslant i\leqslant n}$ and $U_{n,m} = (u_{i,n,m})_{1\leqslant i\leqslant n-1}$ satisfying (1.7), such that

$$d^* \leqslant J_n(Y_{n,m}, U_{n,m}) \leqslant d^* + \frac{1}{m}. \tag{2.6}$$

It follows that

$$\sum_{i=2}^{n} \|u_{i-1,n,m}\|^2 \leqslant 2h_n(d^* + 1). \tag{2.7}$$

Integrating the equations given by (1.7), we get

(2.8)

$$\begin{cases} y_{1,n,m}(t) = e^{t\triangle} y_{1,n,m}(0), \quad t \in [0, \tau_1], \\ y_{i,n,m}(t) = e^{t\triangle} y_{1,n,m}(0) + \displaystyle\sum_{j=2}^{i} e^{(t-\tau_{j-1})\triangle} \chi_\omega u_{j-1,n,m}, \quad t \in [\tau_{i-1}, \tau_i], \quad 2 \leqslant i \leqslant n, \\ y_{1,n,m}(0) = e^{T\triangle} y_{1,n,m}(0) + \displaystyle\sum_{j=2}^{n} e^{(T-\tau_{j-1})\triangle} \chi_\omega u_{j-1,n,m}. \end{cases}$$

Using (2.7) and the third equality of (2.8), we infer that

$$\|y_{1,n,m}(0)\| \leqslant C \tag{2.9}$$

for every $m \geqslant 1$.[2] Here and throughout the proof, C designates a generic positive constant not depending on m. Multiplying the first equation of (1.7) (written for $y_{i,n,m}$) by $2y_{i,n,m}$ and integrating over $\Omega \times (\tau_{i-1}, t)$, we obtain that

$$\|y_{i,n,m}(t)\|^2 + 2\int_{\tau_{i-1}}^{t} \|\nabla y_{i,n,m}(s)\|^2 \,\mathrm{d}s \leqslant \|y_{i,n,m}(\tau_{i-1})\|^2 \quad \forall t \in [\tau_{i-1}, \tau_i], \quad 1 \leqslant i \leqslant n,$$

which implies that

$$\|y_{i,n,m}\|_{C([\tau_{i-1},\tau_i];L^2(\Omega))} + \|y_{i,n,m}\|_{X_i} \leqslant C\|y_{i,n,m}(\tau_{i-1})\|, \quad 1 \leqslant i \leqslant n.$$

This, together with (2.7), (2.9), and the third equation of (1.7), gives

$$\sum_{i=1}^{n} (\|y_{i,n,m}\|_{C([\tau_{i-1},\tau_i];L^2(\Omega))} + \|y_{i,n,m}\|_{X_i}) + \sum_{i=2}^{n} \|u_{i-1,n,m}\| \leqslant C.$$

Hence, up to some subsequence, we have

$$y_{i,n,m} \to y^*_{i,n} \text{ weakly in } X_i, \text{ strongly in } L^2(\tau_{i-1}, \tau_i; L^2(\Omega)), \ 1 \leqslant i \leqslant n,$$

and

$$u_{i-1,n,m} \to u^*_{i-1,n} \text{ weakly in } L^2(\Omega), \ 2 \leqslant i \leqslant n,$$

[2]Here, the L^2 norm is used. For $y_{1,n,m}(0)$, we may wish to consider the $H^1_0(\Omega)$ norm. But since $y_{i,n,m}(\tau_{i-1}) = y_{i-1,n,m}(\tau_{i-1}) + \chi_\omega u_{i-1,n}$ and $u_{i-1,n} \in L^2(\Omega)$ $(2 \leqslant i \leqslant n)$, it follows that $y_{i,n,m}(\tau_{i-1}) \in L^2(\Omega)$ for $2 \leqslant i \leqslant n$. Hence the $H^1_0(\Omega)$ norm does not seem to be useful.

for some $Y_n^* = (y_{1,n}^*, y_{2,n}^*, \ldots, y_{n,n}^*) \in X$ and $U_n^* = (u_{1,n}^*, u_{2,n}^*, \ldots, u_{n-1,n}^*) \in (L^2(\Omega))^{n-1}$. Passing to the limit in (2.6) and in (1.7), it is clear that (Y_n^*, U_n^*) is an optimal solution of $(\mathrm{IOCP})_\mathrm{n}$.

The uniqueness follows from the strict convexity of the functional $\widetilde{J}_n : (L^2(\Omega))^{n-1} \to [0,+\infty)$ defined by $\widetilde{J}_n(U_n) = J_n(Y_n, U_n)$, where Y_n is the unique solution of (1.7) corresponding to U_n.

Let us now prove the characterization of the optimal solution given in the proposition.

We assume that (Y_n^*, U_n^*) is the optimal solution of $(\mathrm{IOCP})_\mathrm{n}$. Let us prove the existence of the adjoint state. The argument goes by perturbation of the optimal solution. Given any $U_n = (u_{1,n}, u_{2,n}, \ldots, u_{n-1,n}) \in (L^2(\Omega))^{n-1}$ and any $\lambda \in (0,1)$, we set

$$U_{n,\lambda} = U_n^* + \lambda U_n. \tag{2.10}$$

Let $Y_{n,\lambda} = (y_{1,n,\lambda}, y_{2,n,\lambda}, \ldots, y_{n,n,\lambda})$ be the solution of

$$\begin{cases} \partial_t y_{i,n,\lambda} - \triangle y_{i,n,\lambda} = 0 & \text{in} \quad \Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ y_{i,n,\lambda} = 0 & \text{on} \quad \partial\Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ y_{i,n,\lambda}(\tau_{i-1}) = y_{i-1,n,\lambda}(\tau_{i-1}) + \chi_\omega(u_{i-1,n}^* + \lambda u_{i-1,n}) & \text{in} \quad \Omega, \quad 2 \leqslant i \leqslant n, \\ y_{1,n,\lambda}(0) = y_{n,n,\lambda}(T) & \text{in} \quad \Omega. \end{cases}$$

Setting $z_{i,n} = \frac{y_{i,n,\lambda} - y_{i,n}^*}{\lambda}$, $1 \leqslant i \leqslant n$, we have

$$\begin{cases} \partial_t z_{i,n} - \triangle z_{i,n} = 0 & \text{in} \quad \Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ z_{i,n} = 0 & \text{on} \quad \partial\Omega \times (\tau_{i-1}, \tau_i), \quad 1 \leqslant i \leqslant n, \\ z_{i,n}(\tau_{i-1}) = z_{i-1,n}(\tau_{i-1}) + \chi_\omega u_{i-1,n} & \text{in} \quad \Omega, \quad 2 \leqslant i \leqslant n, \\ z_{1,n}(0) = z_{n,n}(T) & \text{in} \quad \Omega. \end{cases} \tag{2.11}$$

Since (Y_n^*, U_n^*) is the optimal solution of $(\mathrm{IOCP})_\mathrm{n}$, we have $J_n(Y_{n,\lambda}, U_{n,\lambda}) - J_n(Y_n^*, U_n^*) \geqslant 0$. Dividing by λ and passing the limit $\lambda \to 0^+$, using (1.6), (2.10), and (2.11), we infer that

$$\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n}^* - y_d, z_{i,n} \rangle \, \mathrm{d}t + \frac{1}{h_n} \sum_{i=2}^n \langle u_{i-1,n}^*, u_{i-1,n} \rangle \geqslant 0 \qquad \forall U_n \in (L^2(\Omega))^{n-1}. \tag{2.12}$$

Let p_n^* be defined by (1.9) (same reasoning as in section 2.1). Multiplying the first equation of (2.11) by p_n^* and integrating over $\Omega \times (\tau_{i-1}, \tau_i)$, we get

$$\langle z_{i,n}(\tau_i), p_n^*(\tau_i) \rangle - \langle z_{i,n}(\tau_{i-1}), p_n^*(\tau_{i-1}) \rangle = \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n}^* - y_d, z_{i,n} \rangle \, \mathrm{d}t, \quad 1 \leqslant i \leqslant n,$$

and summing over $i = 1, 2, \ldots, n$, we obtain

$$\sum_{i=1}^n \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n}^* - y_d, z_{i,n} \rangle \, \mathrm{d}t = -\sum_{i=2}^n \langle \chi_\omega p_n^*(\tau_{i-1}), u_{i-1,n} \rangle, \tag{2.13}$$

which, combined with (2.12), yields (1.10).

Let us now prove the converse, that is, let us prove that if (Y_n^*, U_n^*) and p_n^* satisfy (1.8)–(1.9)–(1.10)–(1.11), then (Y_n^*, U_n^*) is the optimal solution of $(\mathrm{IOCP})_\mathrm{n}$.

Given any $U_n = (u_{1,n}, u_{2,n}, \ldots, u_{n-1,n}) \in (L^2(\Omega))^{n-1}$, we denote by $Y_n = (y_{1,n}, y_{2,n}, \ldots, y_{n,n})$ the corresponding solution of (1.7). By using arguments similar to those used to establish (2.13), we obtain that

$$\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n}^* - y_d, y_{i,n} - y_{i,n}^* \rangle \, \mathrm{d}t + \sum_{i=2}^{n} \langle \chi_\omega p_n^*(\tau_{i-1}), u_{i-1,n} - u_{i-1,n}^* \rangle = 0.$$

This, together with (1.10), implies that

$$\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n}^* - y_d, y_{i,n} - y_{i,n}^* \rangle \, \mathrm{d}t + \frac{1}{h_n} \sum_{i=2}^{n} \langle u_{i-1,n}^*, u_{i-1,n} - u_{i-1,n}^* \rangle = 0.$$

Hence

$$\begin{aligned} & J_n(Y_n, U_n) - J_n(Y_n^*, U_n^*) \\ & = \frac{1}{2} \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n} + y_{i,n}^* - 2y_d, y_{i,n} - y_{i,n}^* \rangle \, \mathrm{d}t + \frac{1}{2h_n} \sum_{i=2}^{n} \langle u_{i-1,n} + u_{i-1,n}^*, u_{i-1,n} - u_{i-1,n}^* \rangle \\ & \geqslant \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \langle y_{i,n}^* - y_d, y_{i,n} - y_{i,n}^* \rangle \, \mathrm{d}t + \frac{1}{h_n} \sum_{i=2}^{n} \langle u_{i-1,n}^*, u_{i-1,n} - u_{i-1,n}^* \rangle = 0. \end{aligned}$$

We conclude that (Y_n^*, U_n^*) is the optimal solution of $(\mathrm{IOCP})_\mathrm{n}$.

2.3. Proof of Theorem 1.

2.3.1. A first estimate. The following lemma compares two states generated by controls activated in different ways.

LEMMA 2. *Let $0 \leqslant T_1 < T_1 + \delta \leqslant T_2 < +\infty$ and let $u \in L^2(\Omega)$. Let z and w be the solutions of*

$$(2.14) \qquad \begin{cases} \partial_t z - \triangle z = \dfrac{1}{\delta} \chi_{(T_1, T_1+\delta)} \chi_\omega u & \text{in} \quad \Omega \times (T_1, T_2), \\ z = 0 & \text{on} \quad \partial\Omega \times (T_1, T_2), \\ z(T_1) = 0 & \text{in} \quad \Omega \end{cases}$$

and

$$(2.15) \qquad \begin{cases} \partial_t w - \triangle w = 0 & \text{in} \quad \Omega \times (T_1, T_2), \\ w = 0 & \text{on} \quad \partial\Omega \times (T_1, T_2), \\ w(T_1) = \chi_\omega u & \text{in} \quad \Omega. \end{cases}$$

For every $r \in [2, \infty)$, there exists $C(T_2, r) > 0$ such that

$$\|z - w\|_{L^r(T_1, T_2; L^2(\Omega))} \leqslant C(T_2, r) \delta^{\frac{1}{r}} \|\chi_\omega u\|.$$

Proof. By the definitions of z and w, we have

$$(2.16) \quad z(t) - w(t) = \int_{T_1}^{t} e^{\triangle(t-\tau)} \frac{1}{\delta} \chi_{(T_1, T_1+\delta)}(\tau) \chi_\omega u \, \mathrm{d}\tau - e^{\triangle(t-T_1)} \chi_\omega u \quad \forall t \in [T_1, T_2].$$

Let $q = \frac{r}{r-1}$ and let $f \in L^q(T_1, T_2; L^2(\Omega))$. Let φ be the solution of

$$(2.17) \qquad \begin{cases} \partial_t \varphi + \triangle \varphi = f & \text{in} \quad \Omega \times (T_1, T_2), \\ \varphi = 0 & \text{on} \quad \partial\Omega \times (T_1, T_2), \\ \varphi(T_2) = 0 & \text{in} \quad \Omega. \end{cases}$$

By [19, Theorem 1], there exists $C(T_2, r) > 0$ such that

$$\|\partial_t\varphi\|_{L^q(T_1,T_2;L^2(\Omega))} \leqslant C(T_2,r)\|f\|_{L^q(T_1,T_2;L^2(\Omega))}. \tag{2.18}$$

It follows from (2.16) and (2.17) that

$$\begin{aligned}\int_{T_1}^{T_2} \langle z(t) - w(t), f(t)\rangle \,\mathrm{d}t &= \left\langle \chi_\omega u, \varphi(T_1) - \frac{1}{\delta}\int_{T_1}^{T_1+\delta} \varphi(\tau)\,\mathrm{d}\tau \right\rangle \\ &= -\left\langle \chi_\omega u, \frac{1}{\delta}\int_{T_1}^{T_1+\delta}\int_{T_1}^{\tau} \partial_t\varphi \,\mathrm{d}t\,\mathrm{d}\tau \right\rangle,\end{aligned}$$

which, together with (2.18), yields

$$\begin{aligned}&\left|\int_{T_1}^{T_2} \langle z(t) - w(t), f(t)\rangle \,\mathrm{d}t\right| \\ &\leqslant \|\chi_\omega u\| \int_{T_1}^{T_1+\delta} \|\partial_t\varphi\|\,\mathrm{d}t \leqslant C(T_2,r)\delta^{\frac{1}{r}}\|\chi_\omega u\|\|f\|_{L^q(T_1,T_2;L^2(\Omega))}.\end{aligned}$$

This leads to the desired result and completes the proof. ∎

2.3.2. Proof of the control error estimate. In this section, our objective is to establish (1.19).

Recalling that u_n^* is defined by (1.18), we denote by $y(u_n^*)$ and by $p(u_n^*)$ the solutions of

$$\begin{cases} \partial_t y(u_n^*) - \triangle y(u_n^*) = \chi_\omega u_n^* & \text{in} \quad \Omega\times(0,T), \\ y(u_n^*) = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ y(u_n^*)(0) = y(u_n^*)(T) & \text{in} \quad \Omega \end{cases} \tag{2.19}$$

and

$$\begin{cases} \partial_t p(u_n^*) + \triangle p(u_n^*) = y(u_n^*) - y_d & \text{in} \quad \Omega\times(0,T), \\ p(u_n^*) = 0 & \text{on} \quad \partial\Omega\times(0,T), \\ p(u_n^*)(0) = p(u_n^*)(T) & \text{in} \quad \Omega. \end{cases} \tag{2.20}$$

The existence of these solutions follows from section 2.1. The proof goes in three steps.

Step 1. We claim that

$$\int_{\tau_1}^{T} \|u^* - u_n^*\|^2 \,\mathrm{d}t = I_1 + I_2 \tag{2.21}$$

with

$$\begin{aligned} I_1 &= \sum_{i=2}^{n}\int_{\tau_{i-1}}^{\tau_i} \langle \chi_\omega p^* - \chi_\omega p(u_n^*), u^* - u_n^*\rangle\,\mathrm{d}t, \\ I_2 &= \sum_{i=2}^{n}\int_{\tau_{i-1}}^{\tau_i} \langle \chi_\omega p(u_n^*) - \chi_\omega p_n^*(\tau_{i-1}), u^* - u_n^*\rangle\,\mathrm{d}t, \end{aligned}$$

where p^* and p_n^* are given by (1.4) and (1.9), respectively.

The claim follows from (1.5), (1.18), and (1.10) and from the fact that

$$\int_{\tau_1}^T \|u^* - u_n^*\|^2 \, \mathrm{d}t = \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \langle u^* - u_n^*, u^* - u_n^* \rangle \, \mathrm{d}t = \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \langle \chi_\omega p^* - \chi_\omega p_n^*(\tau_{i-1}), u^* - u_n^* \rangle \, \mathrm{d}t.$$

Step 2. We claim that

$$I_1 \leqslant C(T) h_n \|y_d\|^2_{L^2(0,T;L^2(\Omega))}. \tag{2.22}$$

We first infer from (1.18) that

$$\begin{aligned} I_1 &\equiv \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \langle \chi_\omega (p^* - p(u_n^*)), u^* - u_n^* \rangle \, \mathrm{d}t \\ &= \int_0^T \langle p^* - p(u_n^*), \chi_\omega(u^* - u_n^*) \rangle \, \mathrm{d}t - \int_0^{\tau_1} \langle p^* - p(u_n^*), \chi_\omega u^* \rangle \, \mathrm{d}t. \end{aligned} \tag{2.23}$$

Then, on one hand, by (1.3), (1.4), (2.19), and (2.20), we get that

$$\begin{cases} \partial_t (y^* - y(u_n^*)) - \triangle (y^* - y(u_n^*)) = \chi_\omega (u^* - u_n^*) & \text{in} \quad \Omega \times (0,T), \\ y^* - y(u_n^*) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ (y^* - y(u_n^*))(0) = (y^* - y(u_n^*))(T) & \text{in} \quad \Omega \end{cases} \tag{2.24}$$

and

$$\begin{cases} \partial_t (p^* - p(u_n^*)) + \triangle (p^* - p(u_n^*)) = y^* - y(u_n^*) & \text{in} \quad \Omega \times (0,T), \\ p^* - p(u_n^*) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ (p^* - p(u_n^*))(0) = (p^* - p(u_n^*))(T) & \text{in} \quad \Omega. \end{cases} \tag{2.25}$$

Multiplying the first equation of (2.24) by $p^* - p(u_n^*)$ and integrating over $\Omega \times (0,T)$, by (2.24) and (2.25), we obtain that

$$\int_0^T \langle p^* - p(u_n^*), \chi_\omega(u^* - u_n^*) \rangle \, \mathrm{d}t = -\int_0^T \|y^* - y(u_n^*)\|^2 \, \mathrm{d}t \leqslant 0. \tag{2.26}$$

On the other hand, since (Y_n^*, U_n^*) is the optimal pair for the problem $(\mathrm{IOCP})_\mathrm{n}$, we have $J_n(Y_n^*, U_n^*) \leqslant J_n(0,0)$. Then by (1.6), (1.12), and (1.18), it follows that

$$\int_0^T \|y_n^*\|^2 \, \mathrm{d}t + \int_0^T \|u_n^*\|^2 \, \mathrm{d}t \leqslant C \int_0^T \|y_d\|^2 \, \mathrm{d}t. \tag{2.27}$$

From (2.27), (2.19), (2.20), Lemma 1, and Corollary 1, we infer that

(2.28)
$$\|y(u_n^*)\|_{C([0,T];H_0^1(\Omega))} + \|p(u_n^*)\|_{C([0,T];H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))} \leqslant C(T) \|y_d\|_{L^2(0,T;L^2(\Omega))}.$$

Since (y^*, u^*) is the optimal pair for the problem (OCP), we have $J(y^*, u^*) \leqslant J(0,0)$. Then by (1.1), we get

$$\int_0^T \|y^*\|^2 \, \mathrm{d}t + \int_0^T \|u^*\|^2 \, \mathrm{d}t \leqslant C \int_0^T \|y_d\|^2 \, \mathrm{d}t, \tag{2.29}$$

which, combined with (1.4) and Corollary 1, implies that

$$\|p^*\|_{C([0,T];L^2(\Omega))} \leqslant C(T)\|y^* - y_d\|_{L^2(0,T;L^2(\Omega))} \leqslant C(T)\|y_d\|_{L^2(0,T;L^2(\Omega))}. \tag{2.30}$$

By (1.5), we have that

$$\int_0^{\tau_1} \langle p^* - p(u_n^*), \chi_\omega u^*\rangle \,\mathrm{d}t = \int_0^{\tau_1} \langle p^* - p(u_n^*), \chi_\omega p^*\rangle \,\mathrm{d}t. \tag{2.31}$$

Noticing that $\tau_1 = h_n$, by (2.23), (2.26), (2.31), (2.28), and (2.30), we get that

$$\begin{aligned} I_1 &\leqslant -\int_0^{\tau_1} \langle p^* - p(u_n^*), \chi_\omega p^*\rangle \,\mathrm{d}t \\ &\leqslant \tau_1 \|p^* - p(u_n^*)\|_{C([0,T];L^2(\Omega))}\|p^*\|_{C([0,T];L^2(\Omega))} \leqslant C(T)h_n\|y_d\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned}$$

and (2.22) follows.

Step 3. We claim that

$$I_2 \leqslant \frac{1}{2}\int_{\tau_1}^T \|u^* - u_n^*\|^2 \,\mathrm{d}t + C(T)h_n\|y_d\|^2_{L^2(0,T;L^2(\Omega))}. \tag{2.32}$$

We first note that

$$\begin{aligned} I_2 &\equiv \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \langle \chi_\omega p(u_n^*) - \chi_\omega p_n^*(\tau_{i-1}), u^* - u_n^*\rangle \,\mathrm{d}t \\ &\leqslant \frac{1}{2}\int_{\tau_1}^T \|u^* - u_n^*\|^2 \,\mathrm{d}t + \frac{1}{2}\sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \|p(u_n^*) - p_n^*(\tau_{i-1})\|^2 \,\mathrm{d}t. \end{aligned} \tag{2.33}$$

Then, we proceed with three substeps.

- *Substep* 3.1. Let us prove that

$$\begin{aligned} \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \|p(u_n^*) - p_n^*(\tau_{i-1})\|^2 \,\mathrm{d}t &\leqslant C(T)h_n^2\|y_d\|^2_{L^2(0,T;L^2(\Omega))} \\ &\quad + C(T)\int_0^T \|y(u_n^*) - y_n^*\|^2 \,\mathrm{d}t. \end{aligned} \tag{2.34}$$

By (2.28), we have

$$\begin{aligned} \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \|p(u_n^*) - p(u_n^*)(\tau_{i-1})\|^2 \,\mathrm{d}t &= \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \Big\|\int_{\tau_{i-1}}^t \partial_s p(u_n^*) \,\mathrm{d}s\Big\|^2 \,\mathrm{d}t \\ &\leqslant h_n^2\|\partial_t p(u_n^*)\|^2_{L^2(0,T;L^2(\Omega))} \leqslant C(T)h_n^2\|y_d\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{2.35}$$

Moreover, from (1.9), (2.20), and Corollary 1, we obtain that

$$\begin{aligned} \sum_{i=2}^n \int_{\tau_{i-1}}^{\tau_i} \|p(u_n^*)(\tau_{i-1}) - p_n^*(\tau_{i-1})\|^2 \,\mathrm{d}t &\leqslant T\|p(u_n^*) - p_n^*\|^2_{C([0,T];L^2(\Omega))} \\ &\leqslant C(T)\int_0^T \|y(u_n^*) - y_n^*\|^2 \,\mathrm{d}t. \end{aligned}$$

Combined with (2.35), this gives (2.34).

- *Substep* 3.2. We claim that

$$\|e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))\|_{L^r(0,T;L^2(\Omega))}\leqslant C(T,r)h_n^{1/r}\|y_d\|_{L^2(0,T;L^2(\Omega))} \tag{2.36}$$

for every $r\in[2,\infty)$.
Indeed, by (1.8), (1.11), (2.19), and (1.18), we have

$$\begin{cases} y_n^*(0)=e^{T\triangle}y_n^*(0)+\displaystyle\sum_{j=2}^{n}e^{(T-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*, \\ y(u_n^*)(0)=e^{T\triangle}y(u_n^*)(0)+\displaystyle\sum_{j=2}^{n}\frac{1}{h_n}\int_{\tau_{j-1}}^{\tau_j}e^{(T-t)\triangle}\chi_\omega u_{j-1,n}^*\,\mathrm{d}t. \end{cases}$$

Then

$$\begin{aligned} y(u_n^*)(0)-y_n^*(0)&=e^{T\triangle}(y(u_n^*)(0)-y_n^*(0)) \\ &\quad+\sum_{j=2}^{n}\left(\frac{1}{h_n}\int_{\tau_{j-1}}^{\tau_j}e^{(T-\tau)\triangle}\chi_\omega u_{j-1,n}^*\mathrm{d}\tau-e^{(T-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*\right). \end{aligned} \tag{2.37}$$

It follows that

$$\begin{aligned} &\|e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))\|_{L^r(0,T;L^2(\Omega))} \\ &\leqslant(1-e^{-\lambda_1T})^{-1} \\ &\quad\sum_{j=2}^{n}\left\|\frac{1}{h_n}\int_{\tau_{j-1}}^{\tau_j}e^{(T-\tau+t)\triangle}\chi_\omega u_{j-1,n}^*\mathrm{d}\tau-e^{(T-\tau_{j-1}+t)\triangle}\chi_\omega u_{j-1,n}^*\right\|_{L^r(0,T;L^2(\Omega))} \\ &\leqslant C(T) \\ &\quad\sum_{j=2}^{n}\left\|\frac{1}{h_n}\int_{\tau_{j-1}}^{\tau_j}e^{(t-\tau)\triangle}\chi_\omega u_{j-1,n}^*\mathrm{d}\tau-e^{(t-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*\right\|_{L^r(\tau_j,2T;L^2(\Omega))}, \end{aligned}$$

where $-\lambda_1<0$ is the first eigenvalue of the Dirichlet Laplacian. Moreover, by Lemma 2 with $T_1=\tau_{j-1},\delta=h_n,T_2=2T$, and $u=u_{j-1,n}^*$, we get that

$$\begin{aligned} &\left\|\frac{1}{h_n}\int_{\tau_{j-1}}^{t}e^{(t-\tau)\triangle}\chi_{(\tau_{j-1},\tau_j)}(\tau)\chi_\omega u_{j-1,n}^*\mathrm{d}\tau\right. \\ &\qquad\left.-e^{(t-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*\right\|_{L^r(\tau_{j-1},2T;L^2(\Omega))} \\ &\leqslant C(T,r)h_n^{1/r}\|\chi_\omega u_{j-1,n}^*\| \end{aligned} \tag{2.38}$$

for every $j\in\{2,\ldots,n\}$. Since

$$\begin{aligned} &\int_{\tau_{j-1}}^{t}e^{(t-\tau)\triangle}\chi_{(\tau_{j-1},\tau_j)}(\tau)\chi_\omega u_{j-1,n}^*\mathrm{d}\tau \\ &=\int_{\tau_{j-1}}^{\tau_j}e^{(t-\tau)\triangle}\chi_\omega u_{j-1,n}^*\mathrm{d}\tau\quad\forall t\in[\tau_j,2T], \end{aligned}$$

by (2.38) and (1.18), we obtain that

$$
\begin{aligned}
&\|e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))\|_{L^r(0,T;L^2(\Omega))}\\
&\quad\leqslant C(T,r)\sum_{j=2}^{n}h_n^{1/r}\|u_{j-1,n}^*\|\\
&\quad\leqslant C(T,r)h_n^{1/r}\left(\sum_{j=2}^{n}h_n\right)^{\frac{1}{2}}\left(\sum_{j=2}^{n}\frac{1}{h_n}\|u_{j-1,n}^*\|^2\right)^{\frac{1}{2}} \qquad (2.39)\\
&\quad\leqslant C(T,r)h_n^{1/r}\|u_n^*\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

which, combined with (2.27), implies (2.36).

- *Substep* 3.3. Let us prove that

$$
\|y(u_n^*)-y_n^*\|_{L^r(0,T;L^2(\Omega))}\leqslant C(T,r)h_n^{1/r}\|y_d\|_{L^2(0,T;L^2(\Omega))} \qquad (2.40)
$$

for every $r\in[2,+\infty)$.

Let $(z_{j,n})_{1\leqslant j\leqslant n}$ and $(w_{j,n})_{1\leqslant j\leqslant n}$ be solutions of (2.14) and (2.15), respectively, with $T_1=\tau_{j-1},\delta=h_n,T_2=T$, and $u=u_{j-1,n}^*$. We set

$$
\widetilde{z}_{j,n}(t)=\begin{cases}0, & t\in(0,\tau_{j-1}],\\ z_{j,n}(t), & t\in(\tau_{j-1},T),\end{cases}\quad\text{and}\quad \widetilde{w}_{j,n}(t)=\begin{cases}0, & t\in(0,\tau_{j-1}],\\ w_{j,n}(t), & t\in(\tau_{j-1},T).\end{cases}
$$

By (1.8), (1.11), (1.18), and (2.19), we have

$$
\begin{aligned}
y(u_n^*)(t)-y_n^*(t)&=e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))\\
&\quad+\sum_{j=1}^{i}\left(\int_{\tau_{j-1}}^{\tau_j}\chi_{(\tau_{j-1},t)}(s)e^{(t-s)\triangle}\chi_\omega u_n^*(s)\,\mathrm{d}s\right.\\
&\qquad\left.-e^{(t-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*\right) \qquad (2.41)\\
&=e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))+\sum_{j=1}^{i}(z_{j,n}(t)-w_{j,n}(t))
\end{aligned}
$$

for every $t\in(\tau_{i-1},\tau_i]$ and every $i\in\{1,\dots,n\}$. Then

(2.42)

$$
y(u_n^*)(t)-y_n^*(t)=e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))+\sum_{j=1}^{n}\chi_{(\tau_{j-1},T)}(t)\left(\widetilde{z}_{j,n}(t)-\widetilde{w}_{j,n}(t)\right)
$$

for every $t\in(0,T)$. Indeed, given any $t\in(0,T)$, let $i_0\in\{1,\dots,n\}$ be such that $t\in(\tau_{i_0-1},\tau_{i_0}]$. It follows from (2.41) that

$$
\begin{aligned}
y(u_n^*)(t)-y_n^*(t)&=e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))+\sum_{j=1}^{i_0}(z_{j,n}(t)-w_{j,n}(t))\\
&=e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))+\sum_{j=1}^{i_0}\left(\widetilde{z}_{j,n}(t)-\widetilde{w}_{j,n}(t)\right)\\
&=e^{t\triangle}(y(u_n^*)(0)-y_n^*(0))+\sum_{j=1}^{i_0}\chi_{(\tau_{j-1},T)}(t)\left(\widetilde{z}_{j,n}(t)-\widetilde{w}_{j,n}(t)\right),
\end{aligned}
$$

which yields (2.42). By (2.42), (2.36), and Lemma 2, we obtain that

$$\begin{aligned}&\|y(u_n^*) - y_n^*\|_{L^r(0,T;L^2(\Omega))}\\&\leqslant \|e^{t\triangle}(y(u_n^*)(0) - y_n^*(0))\|_{L^r(0,T;L^2(\Omega))} + \sum_{j=1}^{n}\|z_{j,n} - w_{j,n}\|_{L^r(\tau_{j-1},T;L^2(\Omega))}\\&\leqslant C(T,r)\left(h_n^{1/r}\|y_d\|_{L^2(0,T;L^2(\Omega))} + \sum_{j=1}^{n} h_n^{1/r}\|u_{j-1,n}^*\|\right).\end{aligned}$$

Using (2.27) and the same arguments as in (2.39), we obtain (2.40).
Step 3 follows immediately from (2.33), (2.34), and (2.40).

Finally, by (2.21), (2.22), and (2.32), we get that

$$\int_{\tau_1}^{T}\|u^* - u_n^*\|^2\mathrm{d}t \leqslant C(T)h_n\|y_d\|^2_{L^2(0,T;L^2(\Omega))}. \tag{2.43}$$

Moreover, it follows from (1.5), (1.18), and (2.30) that

$$\begin{aligned}\int_0^{\tau_1}\|u^* - u_n^*\|^2\mathrm{d}t &= \int_0^{\tau_1}\|\chi_\omega p^*\|^2\mathrm{d}t\\&\leqslant \tau_1\|p^*\|^2_{C([0,\tau_1];L^2(\Omega))} \leqslant C(T)h_n\|y_d\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

This, combined with (2.43), yields the control error estimate (1.19).

Remark 6. Although more general estimates for $r \in [2, +\infty)$ in Substeps 3.2 and 3.3 are provided, only the case $r = 2$ is required in the proof of (1.19).

2.3.3. Proof of the state and cost functional error estimates. In this section, our objective is to establish (1.20) and (1.21).

We start with the case $2 \leqslant r < +\infty$. By the triangular inequality, we have

$$\|y^* - y_n^*\|_{L^r(0,T;L^2(\Omega))} \leqslant \|y^* - y(u_n^*)\|_{L^r(0,T;L^2(\Omega))} + \|y(u_n^*) - y_n^*\|_{L^r(0,T;L^2(\Omega))}. \tag{2.44}$$

We infer from (1.3), (2.19) and from Lemma 1 that

$$\begin{aligned}\|y^* - y(u_n^*)\|_{L^r(0,T;L^2(\Omega))} &\leqslant T^{\frac{1}{r}}\|y^* - y(u_n^*)\|_{C([0,T];L^2(\Omega))}\\&\leqslant C(T,r)\|u^* - u_n^*\|_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{2.45}$$

Since $r \geqslant 2$, it follows from (2.45) and (1.19) that

$$\begin{aligned}\|y^* - y(u_n^*)\|_{L^r(0,T;L^2(\Omega))} &\leqslant C(T,r)\sqrt{h_n}\|y_d\|_{L^2(0,T;L^2(\Omega))}\\&\leqslant C(T,r)h_n^{1/r}\|y_d\|_{L^2(0,T;L^2(\Omega))},\end{aligned}$$

which, combined with (2.44) and (2.40), gives (1.21).

Finally, (1.20) follows from (1.1), (1.6), (1.12), (1.18), (1.19), (1.21), (2.27), and (2.29).

2.4. Proof of Theorem 2. In this part, Lemma 5 plays an important role in the proof of Theorem 2. In Lemma 5, as mentioned in Remark 3, the difficulty for proving (2.64) comes from the following fact: if $\omega \subsetneq \Omega$, then the characteristic function χ_ω is not smooth and the function $\chi_\omega z_0 (z_0 \in H_0^1(\Omega))$ of (2.62) may not belong to $H_0^1(\Omega)$. This causes a lack of regularity of the solution z to (2.62), more precisely, $\partial_t z \in L^2(T_1, T_2; L^2(\Omega))$ does not hold for any $\chi_\omega z_0$ $(z_0 \in H_0^1(\Omega))$. To overcome this difficulty, we design smooth regularizations χ_ω^ε of χ_ω (see (2.53)). Then we need to estimate $\chi_\omega^\varepsilon - \chi_\omega$ and $\nabla\chi_\omega^\varepsilon$ in a refined way (see Lemma 4). In order to prove Lemma 4, a geometric result (Lemma 3) is required.

2.4.1. A general result in measure theory.

LEMMA 3. *Let ω be a measurable subset of Ω having a C^2 boundary $\partial\omega$. For $\varepsilon > 0$, we define*

$$\omega_\varepsilon = \{x \in \mathbb{R}^N \ \mid \ d(x, \partial\omega) \leqslant \varepsilon\}, \tag{2.46}$$

where $d(x, \partial\omega) = \inf\{|x-y| \ \mid \ y \in \partial\omega\}$. There exists $\mu > 0$ such that

$$|\omega_\varepsilon| = \int_0^\varepsilon |\partial\omega_\eta| \, \mathrm{d}\eta \leqslant 2(1+\varepsilon/\mu)^{N-1} |\partial\omega| \varepsilon \tag{2.47}$$

for every $\varepsilon \in (0, \mu)$.

In (2.47), without ambiguity, $|\omega_\varepsilon|$ designates the Lebesgue measure of ω_ε, and $|\partial\omega_\eta| = \mathcal{H}^{N-1}(\partial\omega_\eta)$ designates the $(N-1)$-Hausdorff measure of $\partial\omega_\eta$.

We give a proof of this result for completeness, borrowing arguments from [11, pp. 354–355].

Remark 7. In the proof below, the assumption $\partial\omega \in C^2$ is required. For the general case, whether (2.47) holds seems to be open.

Proof. For every $y \in \partial\omega$, let $\nu(y)$ and $\Gamma(y)$ respectively denote the unit inner normal to $\partial\omega$ at y and the tangent hyperplane to $\partial\omega$ at y. The curvatures of $\partial\omega$ at a fixed point $y_0 \in \partial\omega$ are determined as follows. By a rotation of coordinates, we assume that the x_N coordinate axis lies in the direction $\nu(y_0)$. In some neighborhood $\mathcal{N}(y_0)$ of y_0, we have $\mathcal{N}(y_0) \cap \partial\omega = \{x_N = \varphi(x')\}$, where $x' = (x_1, \dots, x_{N-1})$, $\varphi \in C^2(\Gamma(y_0) \cap \mathcal{N}(y_0))$, and $D\varphi(y_0') = 0$. The eigenvalues $\kappa_1, \dots, \kappa_{N-1}$ of the Hessian matrix $D^2\varphi(y_0')$ are the principal curvatures of $\partial\omega$ at y_0 and the corresponding eigenvectors are the principal directions of $\partial\omega$ at y_0. By an additional rotation of coordinates, we assume that the $x_1, \dots, x_{N-1}$ axes lie along principal directions corresponding to $\kappa_1, \dots, \kappa_{N-1}$ at y_0. Such a coordinate system is said to be a principal coordinate system at y_0. The Hessian matrix $D^2\varphi(y_0')$ with respect to the principal coordinate system at y_0 described above is given by $D^2\varphi(y_0') = \operatorname{diag}(\kappa_1, \dots, \kappa_{N-1})$. The unit inner normal vector $\nu(y)$ at the point $y = (y', \varphi(y')) \in \mathcal{N}(y_0) \cap \partial\omega$ is given by

$$\nu_i(y) = -\frac{D_i\varphi(y')}{\sqrt{1+|D\varphi(y')|^2}}, \quad 1 \leqslant i \leqslant N-1, \qquad \nu_N(y) = \frac{1}{\sqrt{1+|D\varphi(y')|^2}}.$$

Hence, with respect to the principal coordinate system at y_0, we have

$$D_j\nu_i(y_0) = -\kappa_i\delta_{ij}, \quad i, j = 1, \dots, N-1. \tag{2.48}$$

Since $\partial\omega$ is C^2, $\partial\omega$ satisfies a uniform interior and exterior sphere condition, i.e., at each point $y_0 \in \partial\omega$, there exist two balls B_1 and B_2 depending on y_0 such that $\overline{B}_1 \cap (\mathbb{R}^N - \omega) = \{y_0\}$ and $\overline{B}_2 \cap \overline{\omega} = \{y_0\}$, and the radii of the balls B_1 and B_2 are bounded below by a positive constant denoted by μ. It is easy to show that μ^{-1} bounds the principal curvatures of $\partial\omega$.

The rest of the proof goes in two steps.

Step 1. Let us prove that ω_ε $(0<\varepsilon<\mu)$ has a C^1-smooth manifold structure.

Given any point x such that $d(x,\partial\omega)<\mu$, there exists a unique point $y=y(x)\in\partial\omega$ satisfying $|x-y|=d(x,\partial\omega)$. We have $x=y+\nu(y)d(x,\partial\omega)$ if $x\in\omega$ and $x=y-\nu(y)d(x,\partial\omega)$ if $x\notin\omega$. Now we give a construction of a C^1-smooth manifold structure on ω_ε. For this purpose, we fix a $y_0\in\partial\omega$ and we define the C^1 map Φ_0 from $\mathcal{U}=(\Gamma(y_0)\cap\mathcal{N}(y_0))\times(-\mu,\mu)$ to $\mathbb{R}^N$ by

$$\Phi_0(y',d)=y+\nu(y)d\quad\forall(y',d)\in(\Gamma(y_0)\cap\mathcal{N}(y_0))\times(-\mu,\mu),\tag{2.49}$$

where $y=(y',\varphi(y'))$. By (2.48), the Jacobian matrix of Φ_0 at (y_0',d) is $D\Phi_0(y_0',d)=\mathrm{diag}(1-\kappa_1 d,\dots,1-\kappa_{N-1}d,1)$, and hence $\det D\Phi_0(y_0',d)=(1-\kappa_1 d)\cdots(1-\kappa_{N-1}d)\neq 0$, for every $d\in(-\mu,\mu)$. It follows from the inverse function theorem that Φ_0 is a local C^1-diffeomorphism in a neighborhood of any point of the line $\{y_0'\}\times(-\mu,\mu)$. Then by compactness of $[-\varepsilon,\varepsilon]$, we can choose $\mathcal{U}_0=B_0\times[-\varepsilon,\varepsilon]$, with B_0 an open ball in $\Gamma(y_0)\cap\mathcal{N}(y_0)$, so that Φ_0 is a C^1-diffeomorphism from $\mathcal{U}_0$ to $\Phi_0(\mathcal{U}_0)$. This shows that $(\Phi_0(\mathcal{U}_0),\Phi_0^{-1})$ is a coordinate chart centered at y_0 in the topological space ω_ε.

We carry on the above process for each $y\in\partial\omega$ and we define an atlas $\{(V_\alpha,\Phi_\alpha^{-1})\}$ for ω_ε, where V_α is an open neighborhood of $y_\alpha\in\partial\omega$, $\Phi_\alpha^{-1}(V_\alpha)=\mathcal{U}_\alpha=B_\alpha\times[-\varepsilon,\varepsilon]$ and B_α is an open ball in $\Gamma(y_\alpha)\cap\mathcal{N}(y_\alpha)$. By the definition of Φ_α (similar to (2.49)), one can check that any two charts in $\{(V_\alpha,\Phi_\alpha^{-1})\}$ are C^1-smoothly compatible one with each other. Hence $\{(V_\alpha,\Phi_\alpha^{-1})\}$ is a C^1 atlas for ω_ε. This atlas induces a C^1 structure on ω_ε.

Step 2. Let us establish (2.47).

By [11, Lemma 14.16, p. 355], we have $d(\cdot,\partial\omega)\in C^2(\omega_\varepsilon)$ and $|\nabla d(\cdot,\partial\omega)|=1$ in ω_ε, which, combined with the coarea formula (see, e.g., [9]) applied to $f=d(\cdot,\partial\omega)$, gives

$$\begin{aligned}|\omega_\varepsilon|&=\int_{\omega_\varepsilon}|\nabla d(x,\partial\omega)|\,\mathrm{d}x=\int_0^\varepsilon\mathcal{H}^{N-1}(\{d(\cdot,\partial\omega)=\eta\})\,\mathrm{d}\eta\\&=\int_0^\varepsilon|\partial\omega_\eta|\,\mathrm{d}\eta=\int_0^\varepsilon(|\partial\omega_\eta^+|+|\partial\omega_\eta^-|)\,\mathrm{d}\eta,\end{aligned}\tag{2.50}$$

where $\partial\omega_\delta^+$ and $\partial\omega_\delta^-$ are the inner and outer parts (with respect to ω) of $\partial\omega_\delta$ for each $\delta\in(0,\varepsilon)$, respectively. Now, given any $\eta\in(0,\varepsilon)$, to compute $|\partial\omega_\eta^+|$ and $|\partial\omega_\eta^-|$, we define $\psi_\eta:\partial\omega\to\partial\omega_\eta^+$ by $\psi_\eta(y_\alpha)=\Phi_\alpha\circ\tau_\eta\circ\Phi_\alpha^{-1}(y_\alpha)$ for every $y_\alpha\in\partial\omega$, where τ_η is the mapping given by $\tau_\eta(z,0)=(z,\eta)$ for every $z\in\mathbb{R}^{N-1}$. From Step 1, we take two arbitrary coordinate charts $\{(V_\beta,\Phi_\beta^{-1})\}$ and $\{(V_\gamma,\Phi_\gamma^{-1})\}$, where V_β and V_γ are an open neighborhood of y_β and y_γ ($y_\beta,y_\gamma\in\partial\omega$), respectively. Then by the definitions of Φ_β and Φ_γ (similar to (2.49)), one can check that

$$\Phi_\beta\circ\tau_\eta\circ\Phi_\beta^{-1}(y)=\Phi_\gamma\circ\tau_\eta\circ\Phi_\gamma^{-1}(y)\quad\forall\,y\in V_\beta\cap V_\gamma\cap\partial\omega.\tag{2.51}$$

We recall from Step 1 that each Φ_α^{-1} is C^1 diffeomorphic from V_α to $\mathcal{U}_\alpha=\Phi_\alpha(V_\alpha)$. Therefore, by (2.51), ψ_η is C^1 diffeomorphic from $\partial\omega$ onto $\partial\omega_\eta^+$ and

$$\begin{aligned}&\det(\Phi_\alpha\circ\tau_\eta\circ\Phi_\alpha^{-1})(y_\alpha)\\&=(1-\kappa_1(y_\alpha)\eta)\cdots(1-\kappa_{N-1}(y_\alpha)\eta)\in((1-\varepsilon\mu^{-1})^{N-1},(1+\varepsilon\mu^{-1})^{N-1})\end{aligned}$$

for every $y_\alpha\in\partial\omega$. This, together with the definition of ψ_η and (2.51), implies that

$$|\partial\omega_\eta^+|=\int_{\partial\omega}\det[\psi_\eta](x)\,\mathrm{d}\sigma\leqslant(1+\eta\mu^{-1})^{N-1}|\partial\omega|\quad\forall\eta\in(0,\varepsilon).\tag{2.52}$$

Similarly, we have $|\partial\omega_\eta^-| \leqslant (1+\eta\mu^{-1})^{N-1}|\partial\omega|$ for every $\eta \in (0,\varepsilon)$. Then, (2.47) follows from the latter inequality, from (2.52) and (2.50).

This completes the proof. ∎

2.4.2. Smooth regularizations of characteristic functions. We define the C^∞ function $\chi_\omega^\varepsilon : \mathbb{R}^N \to \mathbb{R}$ by

$$\chi_\omega^\varepsilon(x) = \int_\omega \eta_\varepsilon(x-y)\chi_\omega(y)\,\mathrm{d}y \tag{2.53}$$

for every $x \in \mathbb{R}^N$, where

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right), \qquad \eta(x) = \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1, \\ 0 & \text{if } |x| \geqslant 1, \end{cases} \tag{2.54}$$

with $c>0$ such that $\int_{\mathbb{R}^N} \eta(x)\,\mathrm{d}x = 1$.

Lemma 4. *Let μ be as in Lemma 3 and let $\varepsilon \in (0,\mu)$. For every $r \in [1,+\infty]$, we have*

$$\|\nabla\chi_\omega^\varepsilon\|_{L^r(\Omega)} \leqslant C\varepsilon^{-1+\frac{1}{r}} \quad \textit{and} \quad \|\chi_\omega^\varepsilon - \chi_\omega\|_{L^r(\Omega)} \leqslant C\varepsilon^{\frac{1}{r}}. \tag{2.55}$$

Here and throughout the proof, C is a generic positive constant independent of r and ε.

Proof. Note that the case $r=+\infty$ follows by passing to the limit. Therefore it suffices to prove (2.55) for $1 \leqslant r < +\infty$. We set $\omega_\varepsilon^1 = \{x \in \omega : d(x,\partial\omega) > \varepsilon\}$ and $\omega_\varepsilon^2 = \{x \notin \omega : d(x,\partial\omega) > \varepsilon\}$. Then ω_ε^1 and ω_ε^2 are an open subsets of $\mathbb{R}^N$ such that

$$\omega_\varepsilon^1 \cup \omega_\varepsilon^2 \cup \omega_\varepsilon = \mathbb{R}^N, \tag{2.56}$$

where ω_ε is defined by (2.46).

On the one hand, by (2.53) and (2.54), we get that

$$\chi_\omega^\varepsilon(x) = \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N}\eta\left(\frac{x-y}{\varepsilon}\right)\chi_\omega(y)\,\mathrm{d}y, \tag{2.57}$$

which, combined with (2.54), yields

$$\begin{aligned} \nabla\chi_\omega^\varepsilon(x) &= \int_{\mathbb{R}^N} \frac{1}{\varepsilon^N}\frac{1}{\varepsilon}\chi_\omega(y)\nabla\eta\left(\frac{x-y}{\varepsilon}\right)\,\mathrm{d}y = \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\chi_\omega(x-\varepsilon y)\nabla\eta(y)\,\mathrm{d}y \\ &= \frac{1}{\varepsilon}\int_{\{y\in\mathbb{R}^N:|y|\leqslant 1\}} \chi_\omega(x-\varepsilon y)\nabla\eta(y)\,\mathrm{d}y. \end{aligned} \tag{2.58}$$

On the other hand, by (2.57), we have

$$\chi_\omega^\varepsilon(x) = \int_{\mathbb{R}^N}\eta(y)\chi_\omega(x-\varepsilon y)\,\mathrm{d}y = \int_{\{y\in\mathbb{R}^N:|y|<1\}}\eta(y)\chi_\omega(x-\varepsilon y)\,\mathrm{d}y. \tag{2.59}$$

This implies that

$$\chi_\omega^\varepsilon(x) = \begin{cases} 1 & \text{if} \quad x \in \omega_\varepsilon^1, \\ 0 & \text{if} \quad x \in \omega_\varepsilon^2. \end{cases} \tag{2.60}$$

It follows from (2.56), (2.58), and (2.60) that

$$\|\nabla\chi_\omega^\varepsilon\|_{L^r(\Omega)}^r \leqslant \|\nabla\chi_\omega^\varepsilon\|_{L^r(\mathbb{R}^N)}^r = \|\nabla\chi_\omega^\varepsilon\|_{L^r(\omega_\varepsilon)}^r \leqslant (C\varepsilon^{-1})^r|\omega_\varepsilon|,$$

which, combined with Lemma 3, yields $\|\nabla\chi_\omega^\varepsilon\|_{L^r(\Omega)} \leqslant C\varepsilon^{-1+\frac{1}{r}}$.

Besides, by (2.59), we have

$$0 \leqslant \chi_\omega^\varepsilon(x) \leqslant 1 \quad \forall x \in \mathbb{R}^N,$$

which indicates

$$-1 \leqslant \chi_\omega^\varepsilon(x) - \chi_\omega(x) \leqslant 1 \quad \forall x \in \mathbb{R}^N. \tag{2.61}$$

Moreover, by (2.60), we get

$$\chi_\omega^\varepsilon(x) - \chi_\omega(x) = 0 \quad \forall x \in \omega_\varepsilon^1 \cup \omega_\varepsilon^2.$$

This, together with (2.61), (2.56), and Lemma 3, implies that $\|\chi_\omega^\varepsilon - \chi_\omega\|_{L^r(\Omega)} \leqslant |\omega_\varepsilon|^{\frac{1}{r}} \leqslant C\varepsilon^{\frac{1}{r}}$. This completes the proof. ∎

2.4.3. A useful estimate. The following estimate for a linear heat equation is not standard.

LEMMA 5. *Let $\omega \subset \Omega$ be a subset having a C^2 boundary. Let $r \in (1, +\infty)$, let T_1 and T_2 be two nonnegative real numbers such that $T_1 < T_2$, and let $z_0 \in H_0^1(\Omega)$. Let z be the solution of*

$$\begin{cases} \partial_t z - \triangle z = 0 & \text{in} \quad \Omega \times (T_1, T_2), \\ z = 0 & \text{on} \quad \partial\Omega \times (T_1, T_2), \\ z(T_1) = \chi_\omega z_0 & \text{in} \quad \Omega. \end{cases} \tag{2.62}$$

If $\omega = \Omega$, then

$$\|z(s) - z(T_1)\| \leqslant (s - T_1)^{\frac{1}{2}}\|z_0\|_{H_0^1(\Omega)} \tag{2.63}$$

for every $s \in [T_1, T_2]$.

If $\omega \neq \Omega$, then

$$\|z(s) - z(T_1)\| \leqslant \begin{cases} C(T_2)(s - T_1)^{\frac{1}{2N}}\|z_0\|_{H_0^1(\Omega)} & \text{if} \quad N \geqslant 3, \\ C(T_2, r)(s - T_1)^{\frac{1}{4r}}\|z_0\|_{H_0^1(\Omega)} & \text{if} \quad N = 2, \\ C(T_2)(s - T_1)^{\frac{1}{4}}\|z_0\|_{H_0^1(\Omega)} & \text{if} \quad N = 1 \end{cases} \tag{2.64}$$

for every $s \in [T_1, T_2]$, for some constants $C(T_2) > 0$ and $C(T_2, r)$ not depending on z_0.

Proof. Since the proof of (2.63) is similar to obtain but simpler than the one of (2.64), we assume that we are in the (more difficult) case where $\omega \neq \Omega$. Let μ be as in Lemma 3 and let $s \in (T_1, T_2]$. We set

$$c_0 = \frac{\min\{\mu, 1\}}{2\max\{\sqrt{T_2}, 1\}} \quad \text{and} \quad \varepsilon = c_0\sqrt{s - T_1}. \tag{2.65}$$

Note that $\varepsilon < \min\{\mu, 1\}$. Recalling the definition of χ_ω^ε in (2.53), by (2.62), we have

$$z(s) = e^{(s-T_1)\triangle}(\chi_\omega - \chi_\omega^\varepsilon)z_0 + e^{(s-T_1)\triangle}\chi_\omega^\varepsilon z_0 = z_1(s) + z_2(s) \tag{2.66}$$

for every $s \in [T_1, T_2]$. We have

$$\|z_2(s) - z_2(T_1)\|^2 = \int_\Omega \Big| \int_{T_1}^s \partial_t z_2 \,\mathrm{d}t \Big|^2 \mathrm{d}x \leqslant (s - T_1) \int_{T_1}^{T_2} \|\partial_t z_2\|^2 \,\mathrm{d}t$$

for every $s \in [T_1, T_2]$. By definition, z_2 is the unique solution of the Dirichlet heat equation with initial condition $z_2(T_1) = \chi_\omega^\varepsilon z_0$. By integration by parts, we have

$$\begin{aligned}\int_{T_1}^{T_2} \|\partial_t z_2\|^2 \,\mathrm{d}t &= \int_{T_1}^{T_2} \int_\Omega (\partial_t z_2(x,t))^2 \,\mathrm{d}x \,\mathrm{d}t = \int_{T_1}^{T_2} \int_\Omega \partial_t z_2(x,t) \cdot \triangle z_2(x,t) \,\mathrm{d}x \,\mathrm{d}t \\ &= \frac{1}{2} \int_\Omega |\nabla z_2(x, T_1)|^2 \mathrm{d}x - \frac{1}{2} \int_\Omega |\nabla z_2(x, T_2)|^2 \mathrm{d}x \leqslant \|\nabla z_2(T_1)\|^2,\end{aligned}$$

and therefore we get that

$$\|z_2(s) - z_2(T_1)\|^2 \leqslant (s - T_1) \|\nabla z_2(T_1)\|^2$$

for every $s \in [T_1, T_2]$. It follows from (2.66) that

$$\begin{aligned}\|z(s) - z(T_1)\| &\leqslant \|z_1(s) - z_1(T_1)\| + \|z_2(s) - z_2(T_1)\| \\ &\leqslant 2\|(\chi_\omega - \chi_\omega^\varepsilon) z_0\| + \sqrt{s - T_1} \|\nabla(\chi_\omega^\varepsilon z_0)\|.\end{aligned} \tag{2.67}$$

If $N \geqslant 3$, then, using the Hölder inequality, the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, Lemma 4, and (2.65), we obtain that

$$\|(\chi_\omega - \chi_\omega^\varepsilon) z_0\| \leqslant \|z_0\|_{L^{\frac{2N}{N-2}}(\Omega)} \|\chi_\omega - \chi_\omega^\varepsilon\|_{L^N(\Omega)} \leqslant C \|z_0\|_{H_0^1(\Omega)} \varepsilon^{\frac{1}{N}}$$

and

$$\|\nabla(\chi_\omega^\varepsilon z_0)\| \leqslant \|z_0\|_{H_0^1(\Omega)} + \|z_0\|_{L^{\frac{2N}{N-2}}(\Omega)} \|\nabla \chi_\omega^\varepsilon\|_{L^N(\Omega)} \leqslant C \|z_0\|_{H_0^1(\Omega)} (1 + \varepsilon^{\frac{1}{N} - 1}).$$

These estimates, together with (2.67), imply that

$$\|z(s) - z(T_1)\| \leqslant C \|z_0\|_{H_0^1(\Omega)} \Big(\varepsilon^{\frac{1}{N}} + \sqrt{s - T_1} \varepsilon^{\frac{1}{N} - 1} \Big). \tag{2.68}$$

From (2.68) and (2.65) it follows that (2.64) holds.

If $N = 2$, then, similarly, using the Hölder inequality, the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2r}{r-1}}(\Omega)$ $(r > 1)$, Lemma 4, and (2.65), we get that

$$\|(\chi_\omega - \chi_\omega^\varepsilon) z_0\| \leqslant \|z_0\|_{L^{\frac{2r}{r-1}}(\Omega)} \|\chi_\omega - \chi_\omega^\varepsilon\|_{L^{2r}(\Omega)} \leqslant C(r) \|z_0\|_{H_0^1(\Omega)} \varepsilon^{\frac{1}{2r}},$$

$$\|\nabla(\chi_\omega^\varepsilon z_0)\| \leqslant \|z_0\|_{H_0^1(\Omega)} + \|z_0\|_{L^{\frac{2r}{r-1}}(\Omega)} \|\nabla \chi_\omega^\varepsilon\|_{L^{2r}(\Omega)} \leqslant C(r) \|z_0\|_{H_0^1(\Omega)} (1 + \varepsilon^{\frac{1}{2r} - 1}),$$

and using (2.67) we infer that

$$\|z(s) - z(T_1)\| \leqslant C(r) \|z_0\|_{H_0^1(\Omega)} \Big(\varepsilon^{\frac{1}{2r}} + \sqrt{s - T_1} \varepsilon^{\frac{1}{2r} - 1} \Big). \tag{2.69}$$

It follows from (2.69) and (2.65) that (2.64) holds.

If $N = 1$, then, by the Sobolev embedding $H_0^1(\Omega) \hookrightarrow C(\overline{\Omega})$, Lemma 4, and (2.65), we have that

$$\|(\chi_\omega - \chi_\omega^\varepsilon)z_0\| \leqslant \|z_0\|_{C(\overline{\Omega})}\|\chi_\omega - \chi_\omega^\varepsilon\| \leqslant C\|z_0\|_{H_0^1(\Omega)}\varepsilon^{\frac{1}{2}},$$

$$\|\nabla(\chi_\omega^\varepsilon z_0)\| \leqslant \|z_0\|_{H_0^1(\Omega)} + \|z_0\|_{C(\overline{\Omega})}\|\nabla\chi_\omega^\varepsilon\| \leqslant C\|z_0\|_{H_0^1(\Omega)}(1+\varepsilon^{-\frac{1}{2}}),$$

which, combined with (2.67), imply that

$$\|z(s) - z(T_1)\| \leqslant C\|z_0\|_{H_0^1(\Omega)}\Big(\varepsilon^{\frac{1}{2}} + \sqrt{s-T_1}\,\varepsilon^{-\frac{1}{2}}\Big). \tag{2.70}$$

By (2.70) and (2.65), we obtain (2.64). The proof is complete. ∎

2.4.4. Proof of the state error estimates. We prove (1.22) only when $N \geqslant 3$, the other cases being similar. Let u^* and U_n^* be the optimal controls solutions of (OCP) and $(\text{IOCP})_\text{n}$, respectively, where $U_n^* = (u_{1,n}^*, u_{2,n}^*, \ldots, u_{n-1,n}^*) \in (L^2(\Omega))^{n-1}$. Let u_n^* be given by (1.18). We have

$$\|y^* - y_n^*\|_{L^\infty(0,T;L^2(\Omega))} \leqslant \|y^* - y(u_n^*)\|_{L^\infty(0,T;L^2(\Omega))} + \|y(u_n^*) - y_n^*\|_{L^\infty(0,T;L^2(\Omega))}. \tag{2.71}$$

By (1.3), (2.19), and Lemma 1, we infer that

$$\|y^* - y(u_n^*)\|_{C([0,T];L^2(\Omega))} \leqslant C(T)\|u^* - u_n^*\|_{L^2(0,T;L^2(\Omega))}. \tag{2.72}$$

Besides, we claim that

$$\|y(u_n^*) - y_n^*\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C(T)h_n^{1/2N}\|y_d\|_{L^2(0,T;L^2(\Omega))}. \tag{2.73}$$

Then (1.22) follows from (2.71), (2.72), (2.73), and Theorem 1.

Let us prove (2.73). On the one hand, by (2.37) and (1.10), we have

(2.74)

$$\begin{aligned}
&\|y_n^*(0) - y(u_n^*)(0)\| \\
&\quad\leqslant \left\|(I - e^{T\triangle})^{-1}\sum_{j=2}^n\left(\frac{1}{h_n}\int_{\tau_{j-1}}^{\tau_j} e^{(T-s)\triangle}\chi_\omega u_{j-1,n}^*\,\mathrm{d}s - e^{(T-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*\right)\right\| \\
&\quad\leqslant C(T)\left\|\sum_{j=2}^n\int_{\tau_{j-1}}^{\tau_j} e^{(T-s)\triangle}\left(I - e^{(s-\tau_{j-1})\triangle}\right)\chi_\omega p_n^*(\tau_{j-1})\,\mathrm{d}s\right\| \\
&\quad\leqslant C(T)\sum_{j=2}^n\int_{\tau_{j-1}}^{\tau_j}\left\|\left(I - e^{(s-\tau_{j-1})\triangle}\right)\chi_\omega p_n^*(\tau_{j-1})\right\|\mathrm{d}s.
\end{aligned}$$

On the other hand, by (2.41), (1.18), and (1.10), we infer that

- for every $t \in [0, \tau_1]$,

$$y(u_n^*)(t) - y_n^*(t) = e^{t\triangle}(y(u_n^*)(0) - y_n^*(0)); \tag{2.75}$$

- for every $t \in (\tau_1, \tau_2]$,

$$\begin{aligned}
y(u_n^*)(t) - y_n^*(t) &= e^{t\triangle}(y(u_n^*)(0) - y_n^*(0)) \\
&\quad + \int_{\tau_1}^t e^{(t-s)\triangle}\chi_\omega h_n^{-1}u_{1,n}^*\,\mathrm{d}s - e^{(t-\tau_1)\triangle}\chi_\omega u_{1,n}^* \\
&= e^{t\triangle}(y(u_n^*)(0) - y_n^*(0)) \\
&\quad + \int_{\tau_1}^t e^{(t-s)\triangle}\chi_\omega p_n^*(\tau_1)\,\mathrm{d}s - \int_{\tau_1}^{\tau_2} e^{(t-\tau_1)\triangle}\chi_\omega p_n^*(\tau_1)\,\mathrm{d}s;
\end{aligned} \tag{2.76}$$

- for every $t \in (\tau_{i-1}, \tau_i]$, with $i \geqslant 3$,

$$
\begin{aligned}
(2.77)\quad y(u_n^*)(t) - y_n^*(t) &= e^{t\triangle}(y(u_n^*)(0) - y_n^*(0)) \\
&\quad + \sum_{j=2}^{i-1}\Big(\int_{\tau_{j-1}}^{\tau_j} e^{(t-s)\triangle}\chi_\omega u_n^*(s)\,\mathrm{d}s - e^{(t-\tau_{j-1})\triangle}\chi_\omega u_{j-1,n}^*\Big) \\
&\quad + \int_{\tau_{i-1}}^{t} e^{(t-s)\triangle}\chi_\omega u_n^*(s)\,\mathrm{d}s - e^{(t-\tau_{i-1})\triangle}\chi_\omega u_{i-1,n}^* \\
&= e^{t\triangle}(y(u_n^*)(0) - y_n^*(0)) \\
&\quad + \sum_{j=2}^{i-1}\int_{\tau_{j-1}}^{\tau_j} e^{(t-s)\triangle}[I - e^{(s-\tau_{j-1})\triangle}]\chi_\omega p_n^*(\tau_{j-1})\,\mathrm{d}s \\
&\quad + \int_{\tau_{i-1}}^{t} e^{(t-s)\triangle}\chi_\omega p_n^*(\tau_{i-1})\,\mathrm{d}s - \int_{\tau_{i-1}}^{\tau_i} e^{(t-\tau_{i-1})\triangle}\chi_\omega p_n^*(\tau_{i-1})\,\mathrm{d}s.
\end{aligned}
$$

It follows from (1.9), Corollary 1, and (2.27) that

$$
(2.78)\qquad \|p_n^*\|_{C([0,T];H_0^1(\Omega))} \leqslant C(T)\|y_n^* - y_d\|_{L^2(0,T;L^2(\Omega))} \leqslant \|y_d\|_{L^2(0,T;L^2(\Omega))},
$$

which, combined with (2.75), (2.76), (2.77), and (2.74), implies that

(2.79)

$$
\begin{aligned}
&\|y(u_n^*) - y_n^*\|_{L^\infty(0,T;L^2(\Omega))} \\
\leqslant\ & C(T)\sum_{j=2}^{n}\int_{\tau_{j-1}}^{\tau_j} \|[I - e^{(s-\tau_{j-1})\triangle}]\chi_\omega p_n^*(\tau_{j-1})\|\,\mathrm{d}s + C(T)h_n\|y_d\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned}
$$

For each $j, 2 \leqslant j \leqslant n$, we set

$$
z_j(t) = e^{(t-\tau_{j-1})\Delta}\chi_\omega p_n^*(\tau_{j-1}), \quad t \in [\tau_{j-1}, T].
$$

It is obvious that

$$
\begin{cases}
\partial_t z_j - \triangle z_j = 0 & \text{in} \quad \Omega \times (\tau_{j-1}, T), \\
z_j = 0 & \text{on} \quad \partial\Omega \times (\tau_{j-1}, T), \\
z_j(\tau_{j-1}) = \chi_\omega p_n^*(\tau_{j-1}) & \text{in} \quad \Omega.
\end{cases}
$$

By Lemma 5 (with $T_1 = \tau_{j-1}, T_2 = T$, and $z_0 = p_n^*(\tau_{j-1})$), we have that

$$
\|z_j(t) - z_j(\tau_{j-1})\| \leqslant C(T)(t - \tau_{j-1})^{\frac{1}{2N}}\|p_n^*(\tau_{j-1})\|_{H_0^1(\Omega)} \quad \forall t \in [\tau_{j-1}, T].
$$

This, together with (2.79) and (2.78), yields

$$
\begin{aligned}
&\|y(u_n^*) - y_n^*\|_{L^\infty(0,T;L^2(\Omega))} \\
&\leqslant C(T)\sum_{j=2}^{n}\int_{\tau_{j-1}}^{\tau_j} (s - \tau_{j-1})^{\frac{1}{2N}}\|p_n^*(\tau_{j-1})\|_{H_0^1(\Omega)}\,\mathrm{d}s + C(T)h_n\|y_d\|_{L^2(0,T;L^2(\Omega))} \\
&\leqslant C(T)h_n^{1/2N}\|y_d\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}
$$

and (2.73) follows. This ends the proof.

2.5. Proof of Theorem 3.

2.5.1. Proof of the control error estimate. Let us establish (1.23). As in section 2.3.2, the proof goes in three steps.

Step 1. We claim that

$$\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\|u^*-v^*_{i,n}\|^2\,\mathrm{d}t=I_1+I_2 \tag{2.80}$$

with

$$I_1=\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\langle\chi_\omega p^*-\chi_\omega\bar{p}^*_n,u^*-v^*_{i,n}\rangle\,\mathrm{d}t$$

and

$$I_2=\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\left\langle\chi_\omega\bar{p}^*_n-\frac{1}{h_n}\chi_\omega\int_{\tau_{i-1}}^{\tau_i}\bar{p}^*_n(s)\,\mathrm{d}s,u^*-v^*_{i,n}\right\rangle\mathrm{d}t,$$

where p^* is defined by (1.4) and $\bar{p}^*_n$ is defined by (1.15).

The claim follows from (1.5), (1.16) and from the fact that

$$\begin{aligned}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\|u^*-v^*_{i,n}\|^2\,\mathrm{d}t&=\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\langle u^*-v^*_{i,n},u^*-v^*_{i,n}\rangle\,\mathrm{d}t\\&=\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\left\langle\chi_\omega p^*-\frac{1}{h_n}\chi_\omega\int_{\tau_{i-1}}^{\tau_i}\bar{p}^*_n(s)\,\mathrm{d}s,u^*-v^*_{i,n}\right\rangle\mathrm{d}t.\end{aligned}$$

Step 2. We claim that

$$I_1\leqslant 0. \tag{2.81}$$

Indeed, using (1.3), (1.4), (1.14), and (1.15), we get that

$$\begin{cases}\partial_t(y^*-\bar{y}^*_n)-\triangle(y^*-\bar{y}^*_n)=\chi_\omega(u^*-f^*_n) & \text{in}\quad\Omega\times(0,T),\\ y^*-\bar{y}^*_n=0 & \text{on}\quad\partial\Omega\times(0,T),\\ (y^*-\bar{y}^*_n)(0)=(y^*-\bar{y}^*_n)(T) & \text{in}\quad\Omega\end{cases} \tag{2.82}$$

and

$$\begin{cases}\partial_t(p^*-\bar{p}^*_n)+\triangle(p^*-\bar{p}^*_n)=y^*-\bar{y}^*_n & \text{in}\quad\Omega\times(0,T),\\ p^*-\bar{p}^*_n=0 & \text{on}\quad\partial\Omega\times(0,T),\\ (p^*-\bar{p}^*_n)(0)=(p^*-\bar{p}^*_n)(T) & \text{in}\quad\Omega.\end{cases} \tag{2.83}$$

Multiplying the first equation of (2.82) by $p^*-\bar{p}^*_n$ and integrating over $\Omega\times(0,T)$, by (2.82) and (2.83), we obtain that

$$\int_0^T\langle p^*-\bar{p}^*_n,\chi_\omega(u^*-f^*_n)\rangle\,\mathrm{d}t=-\int_0^T\|y^*-\bar{y}^*_n\|^2\,\mathrm{d}t\leqslant 0,$$

which, combined with (1.17), gives (2.81).

Step 3. We claim that

$$|I_2| \leqslant C(T)h_n\|y_d\|_{L^2(0,T;L^2(\Omega))}\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}\|u^*-v_{i,n}^*\|^2\,\mathrm{d}t\right)^{\frac12}. \tag{2.84}$$

Indeed, on one hand, we first note that

$$|I_2| \leqslant \sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}\left\|\bar{p}_n^* - \frac{1}{h_n}\int_{\tau_{i-1}}^{\tau_i}\bar{p}_n^*(s)\,\mathrm{d}s\right\|\|u^*-v_{i,n}^*\|\,\mathrm{d}t. \tag{2.85}$$

It is easy to check that, for every $t\in[\tau_{i-1},\tau_i]$,

$$\begin{aligned}\left\|\bar{p}_n^*(t) - \frac{1}{h_n}\int_{\tau_{i-1}}^{\tau_i}\bar{p}_n^*(s)\,\mathrm{d}s\right\| &= \left\|\frac{1}{h_n}\int_{\tau_{i-1}}^{\tau_i}(\bar{p}_n^*(t)-\bar{p}_n^*(s))\,\mathrm{d}s\right\|\\ &= \frac{1}{h_n}\left\|\int_{\tau_{i-1}}^{\tau_i}\int_s^t\partial_\tau\bar{p}_n^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\right\|\\ &\leqslant \frac{1}{h_n}\int_{\tau_{i-1}}^{\tau_i}\int_{\tau_{i-1}}^{\tau_i}\|\partial_\tau\bar{p}_n^*(\tau)\|\,\mathrm{d}\tau\,\mathrm{d}s\\ &= \int_{\tau_{i-1}}^{\tau_i}\|\partial_\tau\bar{p}_n^*(\tau)\|\,\mathrm{d}\tau\\ &\leqslant h_n^{1/2}\left(\int_{\tau_{i-1}}^{\tau_i}\|\partial_\tau\bar{p}_n^*(\tau)\|^2\,\mathrm{d}\tau\right)^{1/2}.\end{aligned} \tag{2.86}$$

It follows from (2.85), (2.86) and from the Hölder inequality that

$$\begin{aligned}|I_2| &\leqslant \sum_{i=1}^n h_n^{1/2}\left(\int_{\tau_{i-1}}^{\tau_i}\|\partial_t\bar{p}_n^*\|^2\,\mathrm{d}t\right)^{\frac12}\int_{\tau_{i-1}}^{\tau_i}\|u^*-v_{i,n}^*\|\,\mathrm{d}t\\ &\leqslant h_n\sum_{i=1}^n\left(\int_{\tau_{i-1}}^{\tau_i}\|\partial_t\bar{p}_n^*\|^2\,\mathrm{d}t\right)^{\frac12}\left(\int_{\tau_{i-1}}^{\tau_i}\|u^*-v_{i,n}^*\|^2\,\mathrm{d}t\right)^{\frac12}\\ &\leqslant h_n\left(\int_0^T\|\partial_t\bar{p}_n^*\|^2\,\mathrm{d}t\right)^{\frac12}\left(\sum_{i=1}^n\int_{\tau_{i-1}}^{\tau_i}\|u^*-v_{i,n}^*\|^2\,\mathrm{d}t\right)^{\frac12}.\end{aligned} \tag{2.87}$$

On the other hand, since $(\bar{y}_n^*, V_n^*)$ is optimal (with $V_n^* = (v_{1,n}^*,\ldots,v_{n,n}^*)$), we have $J(\bar{y}_n^*, f_n^*) \leqslant J(0,0)$, from which it follows that

$$\int_0^T\|\bar{y}_n^*-y_d\|^2\,\mathrm{d}t \leqslant \int_0^T\|y_d\|^2\,\mathrm{d}t \tag{2.88}$$

and

$$\int_0^T\|f_n^*\|^2\,\mathrm{d}t = h_n\sum_{i=1}^n\|v_{i,n}^*\|^2 \leqslant \int_0^T\|y_d\|^2\,\mathrm{d}t. \tag{2.89}$$

By (1.15), (2.88), and Corollary 1, we get that

$$\int_0^T\|\partial_t\bar{p}_n^*\|^2\,\mathrm{d}t \leqslant C(T)\int_0^T\|\bar{y}_n^*-y_d\|^2\,\mathrm{d}t \leqslant C(T)\int_0^T\|y_d\|^2\,\mathrm{d}t.$$

This, combined with (2.87), implies (2.84).

Finally, (1.23) follows from (2.80), (2.81), and (2.84).

2.5.2. Proof of the state and cost functional error estimates. We start with establishing (1.24). Using (1.3) and (1.14), we have

$$\begin{cases} \partial_t(y^* - \bar{y}_n^*) - \triangle(y^* - \bar{y}_n^*) = \chi_\omega(u^* - f_n^*) & \text{in} \quad \Omega \times (0,T), \\ y^* - \bar{y}_n^* = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ (y^* - \bar{y}_n^*)(0) = (y^* - \bar{y}_n^*)(T) & \text{in} \quad \Omega. \end{cases} \tag{2.90}$$

Equation (1.24) follows from (2.90), Lemma 1, and (1.23).

Finally, by (1.1), (1.23), (1.24), (2.88), and (2.89), we obtain (1.25).

Appendix A.

A.1. Proof of Lemma 1. As a preliminary remark, we recall that, given $y_0 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$ arbitrary, there exists a unique weak solution $y(\cdot;y_0,f) \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \subset C([0,T];L^2(\Omega))$ of $\partial_t y - \triangle y = f$ in $\Omega \times (0,T)$, with $y = 0$ along $\partial\Omega \times (0,T)$, such that $y(0) = y_0$ (see [25], for instance). Here, "weak" means that the differential equation is written in $H^{-1}(\Omega)$. Moreover, if $y_0 \in H_0^1(\Omega)$, then $y(\cdot;y_0,f) \in L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \subset C([0,T];H_0^1(\Omega))$.

Given any $f \in L^2(0,T;L^2(\Omega))$, let us prove the existence and uniqueness of a weak solution of (2.1). Since $\|e^{T\triangle}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leqslant e^{-\lambda_1 T} < 1$, it follows that $(I - e^{T\triangle})^{-1}$ exists and $\|(I - e^{T\triangle})^{-1}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leqslant (1 - e^{-\lambda_1 T})^{-1}$, where $-\lambda_1 < 0$ is the first eigenvalue of the Dirichlet Laplacian. Now we define

$$y_0^f = (I - e^{T\triangle})^{-1} \int_0^T e^{(T-t)\triangle} f(t)\,\mathrm{d}t \tag{A.1}$$

and

$$y(t;y_0^f,f) = e^{t\triangle} y_0^f + \int_0^t e^{(t-s)\triangle} f(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{A.2}$$

Then $y_0^f \in L^2(\Omega)$ and $y(\cdot;y_0^f,f) \in L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ is the weak solution of $\partial_t y - \triangle y = f$ in $\Omega \times (0,T)$, with $y = 0$ along $\partial\Omega \times (0,T)$, such that $y(0) = y_0^f$. Using (A.1) and (A.2), we have

$$\begin{aligned} y(T;y_0^f,f) &= e^{T\triangle} y_0^f + \int_0^T e^{(T-t)\triangle} f(t)\,\mathrm{d}t \\ &= e^{T\triangle}(I - e^{T\triangle})^{-1} \int_0^T e^{(T-t)\triangle} f(t)\,\mathrm{d}t + \int_0^T e^{(T-t)\triangle} f(t)\,\mathrm{d}t \\ &= (I - e^{T\triangle})^{-1} \int_0^T e^{(T-t)\triangle} f(t)\,\mathrm{d}t = y(0;y_0^f,f), \end{aligned}$$

which gives the periodicity requirement. Hence $y(\cdot;y_0^f,f)$ is a weak solution of (2.1).

Now, if y_1 and y_2 are two weak solutions of (2.1) associated with f, then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_1(t) - y_2(t)\|_{L^2(\Omega)}^2 + \int_\Omega |\nabla y_1(t) - \nabla y_2(t)|^2 \mathrm{d}x = 0 \ \text{ a.e. } \ t \in (0,T).$$

Integrating the latter equality over $(0,T)$, we deduce from the periodicity condition that $y_1 = y_2$. Therefore the weak solution is unique.

It remains to prove that the weak solution y of (2.1) actually belongs to $L^2(0,T; H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ and to prove the estimate (2.2). Using the preliminary remark, we have $y(T)\in H_0^1(\Omega)$, and since $y(0)=y(T)$, it follows that $y(0)\in H_0^1(\Omega)$. Therefore $y\in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$. Now, multiplying the differential equation by $2y$ and integrating over Ω, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2+2\|\nabla y\|^2=2\langle f,y\rangle \text{ a.e. } t\in(0,T). \tag{A.3}$$

Using the Poincaré inequality $\|\varphi\|\leqslant C\|\nabla\varphi\|$, valid for any $\varphi\in H_0^1(\Omega)$, combined with (A.3) and the Young inequality, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2+2\|\nabla y\|^2\leqslant\|\nabla y\|^2+C\|f\|^2 \text{ a.e. } t\in(0,T).$$

Here and throughout, C designates a generic positive constant only depending on Ω. Integrating over $(0,T)$, we obtain that

$$\int_0^T\|\nabla y\|^2\,\mathrm{d}t\leqslant C\int_0^T\|f\|^2\,\mathrm{d}t. \tag{A.4}$$

Besides, multiplying the first equation of (2.1) by $-2t\triangle y$ and integrating over Ω, we have

$$t\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y\|^2+2t\|\triangle y\|^2=-2t\langle f,\triangle y\rangle\leqslant t\|\triangle y\|^2+t\|f\|^2.$$

Integrating again over $(0,T)$, we obtain that

$$T\|\nabla y(T)\|^2\leqslant\int_0^T\|\nabla y\|^2\,\mathrm{d}t+T\int_0^T\|f\|^2\,\mathrm{d}t,$$

which, combined with (A.4) and the third equation of (2.1), gives $\|\nabla y(0)\|^2\leqslant\frac{C+T}{T}\int_0^T\|f\|^2\,\mathrm{d}t$. This, together with the first and second equations of (2.1), implies that

$$\begin{aligned}&\|y\|_{C([0,T];H_0^1(\Omega))}+\|y\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))}\\&\qquad\leqslant C(\|\nabla y(0)\|+\|f\|_{L^2(0,T;L^2(\Omega))})\leqslant\frac{C(T+1)}{T}\|f\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

This completes the proof of the lemma.

A.2. Proof of Corollary 1. Set

$$f(t)=-g(T-t)\quad\forall t\in(0,T).$$

Then $f\in L^2(0,T;L^2(\Omega))$. By Lemma 1, the equation

$$\begin{cases}\partial_t y-\triangle y=f & \text{in} \quad \Omega\times(0,T),\\ y=0 & \text{on} \quad \partial\Omega\times(0,T),\\ y(0)=y(T) & \text{in} \quad \Omega\end{cases} \tag{A.5}$$

has a unique solution $y\in H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$. Moreover, there exists $C(T)>0$, not depending on f and on y, such that

$$\|y\|_{C([0,T];H_0^1(\Omega))}+\|y\|_{H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))}\leqslant C(T)\|f\|_{L^2(0,T;L^2(\Omega))}. \tag{A.6}$$

Define

$$\psi(t) = y(T-t) \quad \forall t \in [0,T].$$

Then it follows from (A.5) and (A.6) that ψ satisfies (2.3) and (2.4). The uniqueness follows from a very similar argument as Lemma 1. This completes the proof.

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