A Semi-linear Energy Critical Wave Equation with an Application*

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Abstract

In this paper we consider an energy critical wave equation ($3 \leq d \leq 5$, $\zeta = \pm 1$)

$$\partial_t^2 u - \Delta u = \zeta \phi(x) |u|^{4/(d-2)} u, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}$$

with initial data $(u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. Here $\phi \in C(\mathbb{R}^d; (0,1])$ converges as $|x| \rightarrow \infty$ and satisfies certain technical conditions. We generalize Kenig and Merle's results on the Cauchy problem of the equation $\partial_t^2 u - \Delta u = |u|^{4/(d-2)} u$. Following a similar compactness-rigidity argument we prove that any solution with a finite energy must scatter in the defocusing case $\zeta = -1$. While in the focusing case $\zeta = 1$ we give a criterion for global behaviour of the solutions, either scattering or finite-time blow-up when the energy is smaller than a certain threshold. As an application we give a similar a criterion on the global behaviour of radial solutions to the focusing, energy critical shifted wave equation $\partial_t^2 v - (\Delta_{\mathbb{H}^3} + 1) v = |v|^4 v$ on the hyperbolic space $\mathbb{H}^3$.

1 Introduction

In this work we consider a semi-linear energy critical wave equation in $\mathbb{R}^d$ with $3 \leq d \leq 5$:

$$\begin{cases} \partial_t^2 u - \Delta u = \zeta \phi(x) |u|^{p_c - 1} u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}; \\ u(\cdot, 0) = u_0 \in \dot{H}^1(\mathbb{R}^d); \\ \partial_t u(\cdot, 0) = u_1 \in L^2(\mathbb{R}^d); \end{cases} \qquad \text{(CP1)}$$

Here the coefficient function $\phi(x)$ satisfies

$$\phi \in C(\mathbb{R}^d; (0,1]), \qquad \phi(\infty) \doteq \lim_{|x| \rightarrow \infty} \phi(x) \text{ is well-defined.} \qquad (1)$$

The exponent $p_c = 1 + \frac{4}{d-2}$ is energy-critical and $\zeta = \pm 1$. If $\zeta = 1$, then the equation is called focusing, otherwise defocusing. Solutions to this equation satisfy an energy conservation law:

$$E_\phi(u, \partial_t u) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\partial_t u|^2 - \frac{\zeta}{2^*} \phi |u|^{2^*} \right) dx = E_\phi(u_0, u_1). \qquad (2)$$

Here the notation 2^* represents the constant $2^* = 2d/(d-2)$. The Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$ implies that the energy $E_\phi(u_0, u_1)$ is finite for any initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. The results of this work can be applied to deal with the radial solutions to the shifted wave equation $\partial_t^2 v - (\Delta_{\mathbb{H}^3} + 1) v = |v|^4 v$ on the hyperbolic space $\mathbb{H}^3$, as shown in the final section.

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1.1 Background

Pure Power-type Nonlinearity Wave equations with a similar nonlinearity have been extensively studied in many works over a few decades, in particular with a power-type nonlinearity $\zeta|u|^{p-1}u$. There is a large symmetric group acting on the set of solutions to an equation of this kind. For example, if $u(x,t)$ is a solution to

$$\partial_t^2 u - \Delta u = \zeta |u|^{p-1} u \tag{3}$$

with initial data (u_0, u_1), then $\tilde{u}(x,t) \doteq \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{x-x_0}{\lambda}, \frac{t-t_0}{\lambda}\right)$ is another solution to (3) with initial data

$$\left(\frac{1}{\lambda^{\frac{2}{p-1}}} u_0\left(\frac{x-x_0}{\lambda}\right), \frac{1}{\lambda^{\frac{2}{p-1}+1}} u_1\left(\frac{x-x_0}{\lambda}\right)\right)$$

at $t = t_0$, where $\lambda > 0$, $x_0 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$ are arbitrary constants. One can check that the energy defined by

$$E(u, \partial_t u) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\partial_t u|^2 - \frac{\zeta}{p+1}|u|^{p+1}\right) dx$$

is preserved under the transformations defined above, i.e. $E(u, \partial_t u) = E(\tilde{u}, \partial_t \tilde{u})$, if and only if $p = p_c \doteq 1 + \frac{4}{d-2}$. This is the reason why the exponent p_c is called the energy-critical exponent, and why the equation (3) with $p = p_c$ is called an energy-critical nonlinear wave equation.

Previous Results A large number of papers have been devoted to the study of wave equations with a power-type nonlinearity. For instance almost complete results about Strichartz estimates, which is the basis of a local theory, can be found in [8, 13]. Local and global well-posedness has been considered for example in [12, 22]. In particular, there are a lot of works regarding the global existence and well-posedness of solutions with small initial data such as [3, 6, 7, 18]. Questions on global behaviour of larger solutions, such as scattering and blow-up, are usually considered more subtle. Grillakis [9, 10] and Shatah-Struwe [26, 27] proved the global existence and scattering of solutions with any $\dot{H}^1 \times L^2$ initial data in the energy-critical, defocusing case in 1990's. The focusing, energy-critical case has been the subject of several more recent papers. This current work is motivated by one of them, F. Merle and C. Kenig's work [14]. Let us first briefly describe the main results and ideas of their work.

Merle and Kenig's work Let us consider the focusing, energy-critical wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{p_c-1}u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}; \\ u(\cdot, 0) = u_0 \in \dot{H}^1(\mathbb{R}^d); \\ \partial_t u(\cdot, 0) = u_1 \in L^2(\mathbb{R}^d); \end{cases} \tag{CP0}$$

Unlike the defocusing case, the solutions to this equation do not necessarily scatter. The ground states, defined as the solutions of (CP0) independent of the time t and thus solving the elliptic equation $-\Delta W = |W|^{p_c-1}W$, are among the most important counterexamples. One specific example of the ground states is given by the formula

$$W(x,t) = W(x) = \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}}.$$

Kenig and Merle's work classifies all solutions to (CP0) whose energy satisfies the inequality

$$E(u_0, u_1) \doteq \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|u_1|^2 - \frac{1}{2^*}|u_0|^{2^*}\right) dx < E(W, 0)$$

into two categories:

(I) If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then the solution u exists globally in time and scatters. The exact meaning of scattering is explained in Definition 2.13 blow.

(II) If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, then the solution blows up within finite time in both two time directions.

Please note that $\|\nabla u_0\|_{L^2} = \|\nabla W\|_{L^2}$ can never happen if $E(u_0, u_1) < E(W, 0)$. Thus the classification is complete under the assumption that $E(u_0, u_1) < E(W, 0)$. The scattering part of this result is proved via a compactness-rigidity argument, which consists of two major steps.

(I) If the scattering result were false, then there would exist a non-scattering solution to (CP0), called a "critical element", with a minimal energy among those non-scattering solutions, that has a compactness property up to dilations and space translations.

(II) A "critical element" as described above does not exist.

Solutions with a greater energy Before introducing the main results, the author would like to mention a few works that discuss the properties of the solutions to (CP0) with an energy $E \geq E(W, 0)$. These works include [4, 5, 19] (Radial case) and [20] (Non-radial Case).

1.2 Main Results of this work

In this work, we will prove that similar results as mentioned in the previous subsection still hold for the equation (CP1), at least for those ϕ's that satisfy some additional technical condition besides (1).

The Defocusing Case As in the case of the wave equation with a pure power-type nonlinearity, we expect that all solutions in the defocusing case scatter. In fact we have

Theorem 1.1. *Let $3 \leq d \leq 5$. Assume that the coefficient function $\phi \in C^1(\mathbb{R}^d)$ satisfies the condition* (1) *and*

$$\phi(x) - \frac{(d-2)x \cdot \nabla\phi(x)}{2(d-1)} > 0, \qquad \text{for any} \quad x \in \mathbb{R}^d. \tag{4}$$

Then the solution to the Cauchy Problem (CP1) in the defocusing case with any initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ exists globally in time and scatters.

Remark 1.2. *Any positive radial C^1 function satisfies the condition* (4) *as long as it decreases as the radius $r = |x|$ grows. For example, a bump-like function satisfies this condition.*

The Focusing Case As in the case of a pure power-type nonlinearity, we can classify all solutions with an energy smaller than a certain positive constant. Without loss of generality, we assume that[1]

$$\sup_{x\in\mathbb{R}^d} \phi(x) = \phi(x_0) = 1.$$

The threshold here is again the energy of the ground state W for the equation (CP0)

$$E_1(W, 0) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla W|^2 - \frac{1}{2^*}|W|^{2^*} \right) dx.$$

Please note that W is no longer a ground state of (CP1) and that the energy above is not the energy $E_\phi(W, 0)$ for the equation (CP1) as defined in (2). This can be explained by considering the function

$$W_{\lambda,x_0}(x) = \frac{1}{\lambda^{\frac{d-2}{2}}} W\left(\frac{x - x_0}{\lambda}\right).$$

[1]If $\sup \phi = 1$ but $\phi(x) < 1$ for all $x \in \mathbb{R}^d$, then the threshold remains the same value via a limiting process $x_0 \to \infty$; if $\sup \phi < 1$, then the threshold can be enlarged as in Corollary 2.22.

When $\lambda \to 0^+$, the function W_{λ,x_0} becomes "almost" a ground state for (CP1) with its energy $E_\phi(W_{\lambda,x_0},0) \to E_1(W,0)$, as shown by Lemma 3.15.

Theorem 1.3. *Let $3 \leq d \leq 5$. Assume the function $\phi \in C^1(\mathbb{R}^d)$ satisfies the condition* (1) *and*

$$2^*(1-\phi(x)) + (x \cdot \nabla\phi(x)) \geq 0, \qquad \text{for any} \quad x \in \mathbb{R}^d. \tag{5}$$

Given initial data $(u_0,u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ with an energy $E_\phi(u_0,u_1) < E_1(W,0)$, the global behaviour, and in particular, the maximal interval of existence $I = (-T_-(u_0,u_1), T_+(u_0,u_1))$ of the corresponding solution u to the Cauchy problem (CP1) in the focusing case can be determined by:

(i) If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then $I = \mathbb{R}$ and u scatters in both time directions.

(ii) If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, then u blows up within finite time in both two directions, namely

$$T_-(u_0,u_1) < +\infty; \qquad\qquad T_+(u_0,u_1) < +\infty.$$

Remark 1.4. *The function $\phi(x) = (\frac{|x|}{\sinh|x|})^\sigma$ satisfies the conditions in Theorem 1.3 as long as $2 \leq \sigma \leq 2^*$.*

Remark 1.5. *The compactness process and the blow-up part in Theorem 1.3 work for any ϕ that satisfies the basic assumption* (1)*. The main theorems would probably still work without the assumption* (4) *or* (5)*, if we could successfully develop a rigidity theory for more general ϕ's. For example, we show in Section 5.3 that the main theorems still hold without these assumptions in the three dimensional case, provided that both initial data and the coefficient function $\phi(x)$ are radial.*

Remark 1.6. *A mass critical Schrödinger equation with a similar nonlinearity has also been discussed. Please see Raphael-Szeftel [25] and citation therein, for instance.*

1.3 Idea of the proof

In this subsection we briefly describe the idea for the scattering part of our main theorems. We focus on the focusing case, but the defocusing case, that is relatively less difficult, can be handled in the same way. Let us first introduce $(M > 0)$

Statement 1.7 (SC(ϕ, M))**.** *There exists a function $\beta : [0,M) \to \mathbb{R}^+$, such that if the initial data $(u_0,u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ satisfy*

$$\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}, \qquad\qquad E_\phi(u_0,u_1) < M;$$

then the solution u to (CP1) in the focusing case with the initial data (u_0,u_1) exists globally in time, scatters in both two time directions with

$$\|u\|_{L^{\frac{d+2}{d-2}} L^{\frac{2(d+2)}{d-2}}(\mathbb{R}\times\mathbb{R}^d)} < \beta(E_\phi(u_0,u_1)).$$

Remark 1.8. *According to Remark 2.19, if $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then we have*

$$E_\phi(u_0,u_1) \simeq \|(u_0,u_1)\|^2_{\dot{H}^1 \times L^2} \geq 0.$$

Therefore we have

- *The expression $\beta(E_\phi(u_0,u_1))$ is always meaningful.*
- *Proposition 2.11 guarantees that the statement SC(ϕ, M) is always true if $M > 0$ is sufficiently small.*

Compactness Process It is clear that the statement SC(ϕ, $E_1(W,0)$) implies the scattering part of our main theorem 1.3. If the statement above broke down at $M_0 < E_1(W,0)$, i.e. SC(ϕ, M) holds for $M = M_0$ but fails for any $M > M_0$, then we would find a sequence of non-scattering solutions u_n's with initial data $(u_{0,n}, u_{1,n})$, such that $E_\phi(u_{0,n}, u_{1,n}) \to M_0$. In this case a critical element can be extracted as the limit of some subsequence of $\{u_n\}$ by applying the profile decomposition. This process is somewhat standard for the wave or Schrödinger equations. However, this is still some difference between our argument and that for a wave equation with a pure power-type nonlinearity. The point is that dilations and space translations are no longer contained in the symmetric group of this equation. The situation is similar when people are considering the compactness process for wave/Schrödinger equations on a space other than the Euclidean spaces, see [11, 21], for instance. We start by introducing the profile decomposition, before more details are discussed.

The profile decomposition One of the key components in the compactness process is the profile decomposition. Given a sequence $(u_{0,n}, u_{1,n}) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$, we can always find a subsequence of it, still denoted by $\{(u_{0,n}, u_{1,n})\}_{n\in\mathbb{Z}^+}$, a sequence of free waves (solutions to the linear wave equation), denoted by $\{V_j(x,t)\}_{j\in\mathbb{Z}^+}$, and a triple $(\lambda_{j,n}, x_{j,n}, t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$ for each pair (j,n), such that

- For each integer $J > 0$, we have the decomposition
$$(u_{0,n}, u_{1,n}) = \sum_{j=1}^{J} (V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0)) + (w_{0,n}^J, w_{0,n}^J).$$
Here $V_{j,n}$ is a modified version of V_j via the application of a dilation, a space translation and/or a time translation:
$$(V_{j,n}(x,t), \partial_t V_{j,n}(x,t)) = \left(\frac{1}{\lambda^{\frac{d-2}{2}}} V_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}}\right), \frac{1}{\lambda^{\frac{d}{2}}} \partial_t V_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}}\right)\right);$$
and $(w_{0,n}^J, w_{1,n}^J)$ represents a remainder that gradually becomes negligible as J and n grow.
- The sequences $\{(\lambda_{j,n}, x_{j,n}, t_{j,n})\}_{n\in\mathbb{Z}^+}$ and $\{(\lambda_{j',n}, x_{j',n}, t_{j',n})\}_{n\in\mathbb{Z}^+}$ are "almost orthogonal" for $j \neq j'$. More precisely we have
$$\lim_{n\to\infty} \left(\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}}\right) = +\infty.$$
- We can also assume $\lambda_{j,n} \to \lambda_j \in [0,\infty) \cup \{\infty\}$, $x_{j,n} \to x_j \in \mathbb{R}^d \cup \{\infty\}$ and $-t_{j,n}/\lambda_{j,n} \to t_j \in \mathbb{R} \cup \{\infty, -\infty\}$ as $n \to \infty$ for each fixed j.

The nonlinear profile Let us first consider the case with a pure power-type nonlinearity, namely the equation (CP0). Given $j \in \mathbb{Z}^+$ we can find a solution U_j to (CP0), called a nonlinear profile, so that the function
$$U_{j,n}(x,t) \doteq U_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}}\right) \tag{6}$$
serves as a more and more accurate approximation of the solution to (CP0) with initial data $(V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0))$ when $n \to \infty$. We then add these approximations up to obtain an approximation of u_n, thanks to the almost orthogonality. The fact that the equation (CP0) is invariant under dilations and space/time translations plays a crucial role in this argument. As a result, this can no longer be done for the equation (CP1). However, this problem can still be solved if we allow the use of nonlinear profiles that are not necessarily solutions to (CP1) but

possibly solutions to other related equations instead. In fact, the solution to (CP1) with initial data $(V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0))$ can be approximated by a nonlinear profile U_j as described below, up to a dilation, a space translation and/or a time translation.

I (Expanding Profile) If $\lambda_j = \infty$, then the profile spreads out in the space as $n \to \infty$. Eventually a given compact set won't contain any significant part of the profile. The combination of this fact and our assumption $\lim_{|x|\to\infty} \phi(x) = \phi(\infty)$ implies that the nonlinear term $\phi(x)|u|^{p_c-1}u$ works as though $\phi \equiv \phi(\infty)$ is a constant if we make $n \to \infty$. As a result, the nonlinear profile U_j in this case is a solution to the non-linear wave equation $\partial_t^2 u - \Delta u = \phi(\infty)|u|^{p_c-1}u$.

II (Traveling Profile) If $\lambda_j < \infty$ but $x_j = \infty$, then the profile travels to the infinity as $n \to \infty$. Again this enables us to ignore the difference of ϕ and the constant $\phi(\infty)$ and choose a nonlinear profile from the solutions to the non-linear wave equation $\partial_t^2 u - \Delta u = \phi(\infty)|u|^{p_c-1}u$.

III (Stable Profile) If $\lambda_j \in \mathbb{R}^+$ and $x_j \in \mathbb{R}^d$, then the profile approaches a limiting scale and position as $n \to \infty$. Therefore the nonlinear profile U_j is still a solution to (CP1).

IV (Concentrating Profile) If $\lambda_j = 0$ and $x_j \in \mathbb{R}^d$, then the profile concentrates around a fixed point x_j as $n \to \infty$. The nonlinear term $\phi(x)|u|^{p_c-1}u$ performs almost the same as $\phi(x_j)|u|^{p_c-1}u$. As a result, the nonlinear profile U_j can be chosen as a solution to $\partial_t^2 u - \Delta u = \phi(x_j)|u|^{p_c-1}u$.

Extraction of a critical element After the nonlinear profiles $\{U_j\}$ are assigned, we can proceed step by step

(I) First of all, we show there is at least one non-scattering profile $U \in \{U_j | j \in \mathbb{Z}^+\}$, whose energy is at least M_0.

(II) By considering the estimates regarding the energy, we show that U is the only nonzero profile and its energy is exactly M_0. This also implies that this nonlinear profile is a solution to (CP1).

(III) Finally we prove that the solution U is "almost periodic", i.e. the set

$$\{(U(\cdot,t), \partial_t U(\cdot,t)) | t \in I\}$$

is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^d)$, where I is the maximal lifespan of U, by considering a new sequence of solutions derived from U via time translations and repeating the whole compactness process. A direct corollary is that the maximal lifespan I is actually $\mathbb{R}$.

Nonexistence of a critical element Finally we show that a critical element may never exist.

- In the defocusing case, we apply a Morawetz-type inequality, which gives a global integral estimate. This contradicts with the "almost periodicity".
- In the focusing case, we follow the same idea used in Kenig and Merle's work. We show that the derivative

$$\frac{d}{dt}\left[\int_{\mathbb{R}^d} (x \cdot \nabla u) u_t \varphi_R dx + \frac{d}{2}\int_{\mathbb{R}^d} \varphi_R u u_t dx\right]$$

has a negative upper bound but the integral itself is always bounded for all time t. This gives us a contradiction when we consider a long time interval. Here φ_R is a suitable cut-off function.

1.4 Structure of this Paper

This paper is organized as follows: In Section 2 We make a brief review on some preliminary results such as the Strichartz estimates, the local theory and some results regarding the wave equation with a pure power-type nonlinearity. We then consider the linear profile decomposition, define the nonlinear profiles and discuss their properties in Section 3. After finishing the preparation work, in Section 4 we perform the crucial compactness procedure and show that the failure of the scattering part of our main theorem would imply the existence of a critical element. Next we prove that the critical element can never exist, thus finish the proof of the scattering part of our main theorem in Section 5. Finally in Sections 6 we prove the blow-up part of our main theorem. Section 7 is an extra, showing an application of our main theorem, about the radial solutions to the focusing, energy-critical shifted wave equation on the 3-dimensional hyperbolic space.

2 Preliminary Results

2.1 Notations

Definition 2.1. *Throughout this paper the notation F represents the function $F(u) = \zeta|u|^{p_c-1}u$. The parameter $\zeta = \pm 1$ is determined by whether the equation in question is focusing ($\zeta = 1$) or defocusing ($\zeta = -1$).*

Definition 2.2 (Dilation-translation Operators)**.** *We define $\mathbf{T}_\lambda$ to be the dilation operator*

$$\mathbf{T}_\lambda \left(u_0(x), u_1(x)\right) = \left(\frac{1}{\lambda^{d/2-1}} u_0\left(\frac{x}{\lambda}\right), \frac{1}{\lambda^{d/2}} u_1\left(\frac{x}{\lambda}\right)\right);$$

and $\mathbf{T}_{\lambda,x_0}$ to be the dilation-translation operator

$$\mathbf{T}_{\lambda,x_0} \left(u_0(x), u_1(x)\right) = \left(\frac{1}{\lambda^{d/2-1}} u_0\left(\frac{x-x_0}{\lambda}\right), \frac{1}{\lambda^{d/2}} u_1\left(\frac{x-x_0}{\lambda}\right)\right);$$

Here x is the spatial variable of the functions. Similarly we can define these operators in the same manner when both the input and output are written as column vectors.

Definition 2.3. *Let $\mathbf{S}_L(t)$ be the linear propagation operator. More precisely, we define*

$$\mathbf{S}_L(t_0)(u_0, u_1) = (u(t_0), u_t(t_0)) \qquad\qquad \mathbf{S}_L(t_0)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t_0) \\ u_t(t_0) \end{pmatrix}$$

if u is the solution to the linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0; \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

In addition, we use the notation $\mathbf{S}_{L,0}(t_0)(u_0, u_1) \doteq u(t_0)$ if we want to ignore the velocity u_t.

Definition 2.4 (The energy)**.** *Let $\phi : \mathbb{R}^d \rightarrow [0, 1]$ be a function and $\zeta \in \{1, -1\}$. We define $E_{\zeta,\phi}(u_0, u_1)$ to be the energy of the solution to the nonlinear wave equation $\partial_t^2 u - \Delta u = \zeta\phi|u|^{p_c-1}u$ with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$:*

$$E_{\zeta,\phi}(u_0, u_1) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|u_1|^2 - \frac{\zeta}{2^*}\phi|u_0|^{2^*}\right) dx.$$

We may omit ζ and use $E_\phi(u_0, u_1)$ instead when the choice of ζ is obvious.

Definition 2.5 (Space-time Norms). *Assume $1 \leq q, r \leq \infty$ and let I be a time interval. The norm $\|u\|_{L^q L^r(I\times\mathbb{R}^d)}$ represents $\big\| \|u(x,t)\|_{L^r(\mathbb{R}^d;dx)} \big\|_{L^q(I;dt)}$. In particular, if $1 \leq q, r < \infty$, then we have*

$$\|u\|_{L^q L^r(I\times\mathbb{R}^d)} = \left(\int_I \left(\int_{\mathbb{R}^d} |u(x,t)|^r dx\right)^{q/r} dt\right)^{1/q}$$

Definition 2.6 (Function Spaces). *Let I be a time interval. We define the norms*

$$\|u\|_{Y(I)} = \|u\|_{L^{p_c} L^{2p_c}(I\times\mathbb{R}^d)}; \qquad \|(u_0,u_1)\|_H = \|(u_0,u_1)\|_{\dot{H}^1\times L^2(\mathbb{R}^d)}.$$

2.2 Local Theory

In this subsection we briefly discuss the local theory of the nonlinear equation (CP1). Our local theory is based on the Strichartz estimates.

Proposition 2.7 (Strichartz estimates, see [8]). *There is a constant C determined solely by the dimension $d \in \{3,4,5\}$, such that if u is a solution to the linear wave equation*

$$\begin{cases} \partial_t^2 u - \Delta u = F(x,t); & (x,t) \in \mathbb{R}^d \times I; \\ (u, \partial_t u)|_{t=0} = (u_0, u_1); \end{cases}$$

where I is a time interval containing 0; then we have the inequality

$$\sup_{t\in I} \|(u(\cdot,t), \partial_t u(\cdot,t))\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} + \|u\|_{Y(I)} \leq C \left[\|(u_0,u_1)\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} + \|F\|_{L^1 L^2(I\times\mathbb{R}^d)}\right].$$

Remark 2.8. *We can substitute Y norm in Proposition 2.7 by any $L^q L^r$ norm if*

$$2 \leq q \leq \infty, \qquad 2 \leq r < \infty, \qquad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - 1,$$

as shown in the paper [8]. These space-time norms are called (energy-critical) Strichartz norms.

Definition 2.9 (Solutions). *Let (u_0,u_1) be initial data in $\dot{H}^1 \times L^2(\mathbb{R}^d)$ and I be a time interval containing 0. We say $u(t)$ is a solution of (CP1) defined on the time interval I, if $(u(t), \partial_t u(t)) \in C(I; \dot{H}^1 \times L^2(\mathbb{R}^d))$ comes with a finite norm $\|u\|_{Y(J)}$ for any bounded closed interval $J \subseteq I$ and satisfies the integral equation*

$$u(\cdot,t) = \mathbf{S}_{L,0}(t)(u_0,u_1) + \int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} [\phi F(u(\cdot,\tau))]\, d\tau$$

holds for all time $t \in I$.

Combining the inequalities

$$\|\phi F(u)\|_{L^1L^2(I\times\mathbb{R}^d)} \leq \|u\|_{Y(I)}^{p_c};$$
$$\|\phi F(u_1) - \phi F(u_2)\|_{L^1L^2(I\times\mathbb{R}^d)} \leq C\|u_1 - u_2\|_{Y(I)} \left(\|u_1\|_{Y(I)}^{p_c-1} + \|u_2\|_{Y(I)}^{p_c-1}\right);$$

with the Strichartz estimate we can apply a fixed-point argument and obtain the following local theory. (Our argument is similar to those in a lot of earlier papers. Therefore we only give important statements but omit most of the proof here. Please see, for instance, [22] for more details.)

Proposition 2.10 (Local solution). *For any initial data $(u_0,u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$, there is a maximal interval $(-T_-(u_0,u_1), T_+(u_0,u_1))$ in which the Cauchy problem (CP1) has a solution.*

Proposition 2.11 (Scattering with small data)**.** *There exists $\delta > 0$ such that if the norm of the initial data $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} < \delta$, then the Cauchy problem (CP1) has a global-in-time solution u with $\|u\|_{Y(\mathbb{R})} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)}$.*

Lemma 2.12 (Standard finite blow-up criterion)**.** *Let u be a solution to (CP1) with a maximal lifespan $(-T_-, T_+)$. If $T_+ < \infty$, then $\|u\|_{Y([0,T_+))} = \infty$.*

Definition 2.13 (Scattering)**.** *We say a solution u to (CP1) with a maximal lifespan $I = (-T_-, T_+)$ scatters in the positive time direction, if $T_+ = \infty$ and there exists a pair $(v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$, such that*

$$\lim_{t \to +\infty} \left\| \begin{pmatrix} u(\cdot, t) \\ \partial_t u(\cdot, t) \end{pmatrix} - \mathbf{S}_L(t) \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} = 0.$$

In fact, the scattering can be characterized by a more convenient but still equivalent condition $\|u\|_{Y([T',T_+))} < \infty$. Here T' is an arbitrary time in I. The scattering at the negative time direction can be defined in the same manner.

Theorem 2.14 (Long-time perturbation theory, see also [2, 14, 15, 28])**.** *Let M be a positive constant. There exists a constant $\varepsilon_0 = \varepsilon_0(M) > 0$, such that if an approximation solution $\tilde{u}$ defined on $\mathbb{R}^d \times I$ $(0 \in I)$ and a pair of initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ satisfy*

$$\begin{aligned} &(\partial_t^2 - \Delta)(\tilde{u}) - \phi F(\tilde{u}) = e(x,t), \qquad (x,t) \in \mathbb{R}^d \times I; \\ &\|\tilde{u}\|_{Y(I)} < M; \qquad \|(\tilde{u}(0), \partial_t \tilde{u}(0))\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} < \infty; \\ &\varepsilon \doteq \|e(x,t)\|_{L^1 L^2(I \times \mathbb{R}^d)} + \|\mathbf{S}_{L,0}(t)(u_0 - \tilde{u}(0), u_1 - \partial_t \tilde{u}(0))\|_{Y(I)} \leq \varepsilon_0; \end{aligned}$$

then there exists a solution $u(x,t)$ of (CP1) defined in the interval I with the initial data (u_0, u_1) and satisfying

$$\|u(x,t) - \tilde{u}(x,t)\|_{Y(I)} \leq C(M)\varepsilon.$$

$$\sup_{t \in I} \left\| \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} - \begin{pmatrix} \tilde{u}(t) \\ \partial_t \tilde{u}(t) \end{pmatrix} - \mathbf{S}_L(t) \begin{pmatrix} u_0 - \tilde{u}(0) \\ u_1 - \partial_t \tilde{u}(0) \end{pmatrix} \right\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} \leq C(M)\varepsilon.$$

Proof. Let us first prove the perturbation theory when M is sufficiently small. Let I_1 be the maximal lifespan of the solution $u(x,t)$ to the equation (CP1) with the given initial data (u_0, u_1) and assume $[-T_1, T_2] \subseteq I \cap I_1$. By the Strichartz estimates, we have [2]

$$\begin{aligned} &\|\tilde{u} - u\|_{Y([-T_1,T_2])} \\ &\quad \leq \|\mathbf{S}_{L,0}(t)(u_0 - \tilde{u}(0), u_1 - \tilde{u}(0))\|_{Y([-T_1,T_2])} + C\|e + \phi F(\tilde{u}) - \phi F(u)\|_{L^1 L^2([-T_1,T_2] \times \mathbb{R}^d)} \\ &\quad \leq \varepsilon + C\|e\|_{L^1 L^2([-T_1,T_2] \times \mathbb{R}^d)} + C\|F(\tilde{u}) - F(u)\|_{L^1 L^2([-T_1,T_2] \times \mathbb{R}^d)} \\ &\quad \leq \varepsilon + C\varepsilon + C\|\tilde{u} - u\|_{Y([-T_1,T_2])} \left(\|\tilde{u}\|_{Y([-T_1,T_2])}^{p_c - 1} + \|(\tilde{u} - u)\|_{Y([-T_1,T_2])}^{p_c - 1} \right) \\ &\quad \leq C\varepsilon + C\|\tilde{u} - u\|_{Y([-T_1,T_2])} \left(M^{p_c - 1} + \|\tilde{u} - u\|_{Y([-T_1,T_2])}^{p_c - 1} \right). \end{aligned}$$

By a continuity argument in T_1 and T_2, there exist $M_0 = M_0(d) > 0$ and $\varepsilon_0 = \varepsilon_0(d) > 0$, such that if $M \leq M_0$ and $\varepsilon \leq \varepsilon_0$, we have

$$\|\tilde{u} - u\|_{Y([-T_1,T_2])} \leq C(d)\varepsilon.$$

Observing that this estimate does not depend on T_1 or T_2, we are actually able to conclude $I \subseteq I_1$ by the standard blow-up criterion and obtain

$$\|\tilde{u} - u\|_{Y(I)} \leq C(d)\varepsilon.$$

[2]The letter C in the argument represents a constant depending solely on the dimension d, it may represent different constants in different places.

In addition, by the Strichartz estimate we have

$$\begin{aligned}
&\sup_{t\in I}\left\|\begin{pmatrix} u(t)\\ \partial_t u(t)\end{pmatrix}-\begin{pmatrix}\tilde{u}(t)\\ \partial_t\tilde{u}(t)\end{pmatrix}-\mathbf{S}_L(t)\begin{pmatrix}u_0-\tilde{u}(0)\\ u_1-\partial_t\tilde{u}(0)\end{pmatrix}\right\|_{\dot{H}^1\times L^2(\mathbb{R}^d)}\\
&\quad\le C\|\phi F(u)-\phi F(\tilde{u})-e\|_{L^1L^2(I\times\mathbb{R}^d)}\\
&\quad\le C\left(\|e\|_{L^1L^2(I\times\mathbb{R}^d)}+\|F(u)-F(\tilde{u})\|_{L^1L^2(I\times\mathbb{R}^d)}\right)\\
&\quad\le C\left[\varepsilon+\|u-\tilde{u}\|_{Y(I)}\left(\|\tilde{u}\|_{Y(I)}^{p_c-1}+\|u-\tilde{u}\|_{Y(I)}^{p_c-1}\right)\right]\\
&\quad\le C\varepsilon.
\end{aligned}$$

This finishes the proof as $M\le M_0$. To deal with the general case, we can separate the time interval I into finite number of subintervals $\{I_j\}_{1\le j\le n}$, so that $\|\tilde{u}\|_{Y(I_j)}<M_0$, and then iterate our argument above. □

Remark 2.15. *If K is a compact subset of the space $\dot{H}^1\times L^2(\mathbb{R}^d)$, then there exists $T=T(K)>0$ such that for any $(u_0,u_1)\in K$, we have $T_+(u_0,u_1)>T(K)$ and $T_-(u_0,u_1)>T(K)$. This is a direct corollary of the perturbation theory.*

2.3 Ground States and the Energy Trapping

In this subsection we make a brief review on the properties of ground states for the equation (CP0) and understand the "energy trapping" phenomenon. Let us first recall a particular ground state

$$W(x)=\frac{1}{\left(1+\frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}}.$$

The ground state is not unique, because given any constants $\lambda\in\mathbb{R}^+$ and $x_0\in\mathbb{R}^d$, the function

$$W_{\lambda,x_0}(x)=\frac{1}{\lambda^{\frac{d-2}{2}}}W\left(\frac{x-x_0}{\lambda}\right)$$

is also a ground state. Any ground state constructed in this way share the same $\dot{H}^1$ and L^{2^*} norms as W. The ground state W, or any other ground state we constructed above, can be characterized by the following lemma.

Lemma 2.16 (Please see [32])**.** *The function W gives the best constant C_d in the Sobolev embedding $\dot{H}^1(\mathbb{R}^d)\hookrightarrow L^{2^*}(\mathbb{R}^d)$. Namely, the inequality*

$$\|u\|_{L^{2^*}}\le C_d\|\nabla u\|_{L^2}$$

holds for any function $u\in\dot{H}^1(\mathbb{R}^d)$ and becomes an equality for $u=W$.

Remark 2.17. *Because the function W is a solution to $-\Delta W=|W|^{\frac{4}{d-2}}W$, we also have*

$$\int_{\mathbb{R}^d}|\nabla W|^2dx=\int_{\mathbb{R}^d}|W|^{2^*}dx\Longrightarrow\|\nabla W\|_{L^2}^2=\|W\|_{L^{2^*}}^{2^*}=C_d^{2^*}\|\nabla W\|_{L^2}^{2^*}$$

As a result, we have $C_d^{2^*}\|\nabla W\|_{L^2}^{2^*-2}=1$ and $E_1(W,0)=(1/d)\|\nabla W\|_{L^2}^2>0$.

Proposition 2.18 (Energy Trapping, see also [14, 15])**.** *Let $\delta>0$. If u is a solution to (CP1) in the focusing case with initial data (u_0,u_1) so that*

$$\|\nabla u_0\|_{L^2}<\|\nabla W\|_{L^2},\qquad E_\phi(u_0,u_1)<(1-\delta)E_1(W,0);$$

then for any time t in the maximal lifespan I of u we have

$$\|(u(\cdot,t),\partial_t u(\cdot,t))\|_{\dot H^1\times L^2(\mathbb{R}^d)} < (1-2\delta/d)^{1/2}\|\nabla W\|_{L^2} \tag{7}$$

$$\int_{\mathbb{R}^d}\left(|\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\right)dx \simeq_\delta \int_{\mathbb{R}^d}|\nabla u(x,t)|^2 dx \tag{8}$$

$$\int_{\mathbb{R}^d}|u_t(x,t)|^2 dx + \int_{\mathbb{R}^d}\left(|\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\right)dx \simeq_\delta E_\phi(u_0,u_1). \tag{9}$$

Proof. If we have $\|\nabla u(\cdot,t_0)\|_{L^2} < \|\nabla W\|_L^2$ for some time $t_0\in I$, then we have

$$\|u(\cdot,t_0)\|_{L^{2^*}} \le C_d\|\nabla u(\cdot,t_0)\|_{L^2} < C_d\|\nabla W\|_{L^2} = \|W\|_{L^{2^*}}.$$

Therefore

$$\begin{aligned}\frac{1}{2}\|(u(\cdot,t_0),\partial_t u(\cdot,t_0))\|^2_{\dot H^1\times L^2(\mathbb{R}^d)} &= E_\phi(u_0,u_1) + \frac{1}{2^*}\int_{\mathbb{R}^d}\phi|u(x,t_0)|^{2^*}dx\\ &< (1-\delta)E_1(W,0) + \frac{1}{2^*}\int_{\mathbb{R}^d}|W|^{2^*}dx\\ &= \frac{1-\delta}{d}\|\nabla W\|^2 + \frac{1}{2^*}\|\nabla W\|^2\\ &= \left(\frac{1}{2}-\frac{\delta}{d}\right)\|\nabla W\|^2.\end{aligned}$$

Because we know $(u,\partial_t u)\in C(I;\dot H^1\times L^2(\mathbb{R}^d))$, this implies that the inequality (7) holds for each t in a small neighbourhood of t_0. Using a continuity argument and the fact $\|\nabla u_0\|_{L^2} < \|\nabla W\|_L^2$, we know the inequality (7) holds for all $t\in I$. Applying this inequality and the Sobolev embedding we have

$$\begin{aligned}\int_{\mathbb{R}^d}|u(x,t)|^{2^*}dx = \|u(\cdot,t)\|^{2^*}_{L^{2^*}} &\le C_d^{2^*}\|\nabla u(\cdot,t)\|^{2^*}_{L^2} = C_d^{2^*}\|\nabla u(\cdot,t)\|^{2^*-2}_{L^2}\int_{\mathbb{R}^d}|\nabla u(x,t)|^2dx\\ &\le C_d^{2^*}(1-2\delta/d)^{(2^*-2)/2}\|\nabla W\|^{2^*-2}_{L^2}\int_{\mathbb{R}^d}|\nabla u(x,t)|^2dx\\ &\le (1-2\delta/d)^{(2^*-2)/2}\int_{\mathbb{R}^d}|\nabla u(x,t)|^2dx\end{aligned} \tag{10}$$

Here we use the identity $C_d^{2^*}\|\nabla W\|^{2^*-2}_{L^2} = 1$. Since the constant $(1-2\delta/d)^{(2^*-2)/2} < 1$, we obtain the estimate (8). The estimate (9) immediately follows as a direct corollary. ☐

Remark 2.19. *If $(u_0,u_1)\in\dot H^1\times L^2(\mathbb{R}^d)$ satisfies $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then as we did in (10), we can show (This inequality becomes an identity only if $u_0\equiv 0$)*

$$\int_{\mathbb{R}^d}|u_0|^{2^*}dx \le \int_{\mathbb{R}^d}|\nabla u_0|^2dx.$$

Therefore we have $E_\phi(u_0,u_1)\simeq E_1(u_0,u_1)\simeq E_c(u_0,u_1)\simeq\|(u_0,u_1)\|^2_{\dot H^1\times L^2(\mathbb{R}^d)}$. Here c is any constant in $(0,1)$. Furthermore, if any of the energy above is zero, then we have $(u_0,u_1)=0$.

2.4 Known Results with a Pure Power-type Nonlinearity

The defocusing case The problem about the global behaviour of solutions in the defocusing, energy-critical case with a pure power-type nonlinearity was completely solved by Grillakis [9, 10] and Shatah-Struwe [26, 27] in 1990's.

Theorem 2.20. *Let* $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ *with* $3 \leq d \leq 5$*. Then the solution to the Cauchy problem*

$$\begin{cases} \partial_t^2 u - \Delta u = -|u|^{4/(d-2)} u, & (x,t) \in \mathbb{R}^d \times \mathbb{R}; \\ u(\cdot, 0) = u_0; \\ \partial_t u(\cdot, 0) = u_1; \end{cases}$$

exists globally in time and scatters.

The focusing case The global behaviour of solutions in the focusing case is much more complicated and subtle. It has not been completely understood. The following result is the main theorem in [14], on the global well-posedness, scattering and blow-up of solutions to the focusing, energy-critical non-linear wave equation, as we mentioned in the introduction.

Theorem 2.21. *Let* $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ *with* $3 \leq d \leq 5$*. Assume that* $E_1(u_0, u_1) < E_1(W, 0)$*. Let* u *be the corresponding solution to the Cauchy problem (CP0) with a maximal interval of existence* $I = (-T_-(u_0, u_1), T_+(u_0, u_1))$*. Then*

(i) If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$*, then* $I = \mathbb{R}$ *and* u *scatters in both time directions.*

(ii) If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$*, then* u *blows up within finite time in both two directions, namely*

$$T_-(u_0, u_1) < +\infty; \qquad T_+(u_0, u_1) < +\infty.$$

Nonlinearity with a coefficient Assume that c is a positive constant. If u is a solution to the equation

$$\begin{cases} \partial_t^2 u - \Delta u = c|u|^{4/(d-2)} u, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1); \end{cases} \tag{11}$$

then $c^{\frac{d-2}{4}} u$ is a solution to the equation (CP0) with initial data $(c^{\frac{d-2}{4}} u_0, c^{\frac{d-2}{4}} u_1)$. This transformation immediately gives us

Corollary 2.22. *Let* $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ *with* $3 \leq d \leq 5$*. Assume that* $E_c(u_0, u_1) < c^{-\frac{d-2}{2}} E_1(W, 0)$*. Let* u *be the corresponding solution to the Cauchy problem* (11) *with a maximal interval of existence* $I = (-T_-(u_0, u_1), T_+(u_0, u_1))$*. Then*

(i) If $\|\nabla u_0\|_{L^2} < c^{-\frac{d-2}{4}} \|\nabla W\|_{L^2}$*, then* $I = \mathbb{R}$ *and* u *scatters in both time direction.*

(ii) If $\|\nabla u_0\|_{L^2} > c^{-\frac{d-2}{4}} \|\nabla W\|_{L^2}$*, then* u *blows up within finite time in both two directions, namely*

$$T_-(u_0, u_1) < +\infty; \qquad T_+(u_0, u_1) < +\infty.$$

2.5 Technical Lemma

Lemma 2.23. *Let* $\{(w_{0,n}, w_{1,n})\}_{n \in \mathbb{Z}^+}$ *be a bounded sequence in the space* $\dot{H}^1 \times L^2(\mathbb{R}^d)$*. If we have* $\|\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n})\|_{Y(\mathbb{R})} \to 0$ *as* $n \to \infty$*, then the pairs* $(w_{0,n}, w_{1,n})$ *weakly converge to* 0 *in* $\dot{H}^1 \times L^2(\mathbb{R}^d)$*.*

Proof. If the weak limit $(w_{0,n}, w_{1,n}) \rightharpoonup 0$ were not true, we could assume $(w_{0,n}, w_{1,n}) \rightharpoonup (w_0, w_1) \neq 0$ in $\dot{H}^1 \times L^2(\mathbb{R}^d)$ by possibly passing to a subsequence. As a result, we have $\mathbf{S}_{L,0}(t)(w_{0,n}, w_{1,n}) \rightharpoonup \mathbf{S}_{L,0}(t)(w_0, w_1)$ in the space $Y(\mathbb{R})$. Thus we have $\mathbf{S}_{L,0}(t)(w_0, w_1) = 0 \Longrightarrow (w_0, w_1) = 0$. This is a contradiction. □

3 Profile Decomposition

In this section we review the linear profile decomposition derived in [1], introduce nonlinear profiles and give their properties as a preparation for our compactness procedure. In order to save space we use the notation H for the space $\dot{H}^1 \times L^2(\mathbb{R}^d)$ when necessary.

3.1 Linear Profile Decomposition

Theorem 3.1 (Linear Profile Decomposition)**.** *Given a sequence $(u_{0,n}, u_{1,n}) \in H$ so that $\|(u_{0,n}, u_{1,n})\|_H \leq A$, these exist a subsequence of $(u_{0,n}, u_{1,n})$ (We still use the notation $(u_{0,n}, u_{1,n})$ for the subsequence), a sequence of free waves $V_j(x,t) = \mathbf{S}_{L,0}(t)(v_{0,j}, v_{1,j})$, a family of triples $(\lambda_{j,n}, x_{j,n}, t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}$, which are "almost orthogonal", i.e. we have*

$$\lim_{n\to+\infty}\left(\frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{|x_{j,n}-x_{j',n}|}{\lambda_{j,n}} + \frac{|t_{j,n}-t_{j',n}|}{\lambda_{j,n}}\right) = +\infty$$

for $j \neq j'$; such that

(I) We have the following decomposition for each fixed $J \geq 1$,

$$(u_{0,n}, u_{1,n}) = \sum_{j=1}^{J} (V_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0)) + (w_{0,n}^J, w_{1,n}^J).$$

Here $V_{j,n}$ is another free wave derived from V_j:

$$\begin{aligned}(V_{j,n}(\cdot,t), \partial_t V_{j,n}(\cdot,t)) &= \left(\frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right), \frac{1}{\lambda_{j,n}^{d/2}} \partial_t V_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right)\\ &= \mathbf{S}_L(t-t_{j,n})\mathbf{T}_{\lambda_{j,n},x_{j,n}}(v_{0,j}, v_{1,j});\end{aligned}$$

(II) We have the following limits regarding the remainder $(w_{0,n}^J, w_{1,n}^J)$ as $J \to \infty$.

$$\limsup_{n\to\infty} \left\|\mathbf{S}_{L,0}(t)(w_{0,n}^J, w_{1,n}^J)\right\|_{Y(\mathbb{R})} \to 0; \quad \limsup_{n\to\infty} \left\|\mathbf{S}_{L,0}(t)(w_{0,n}^J, w_{1,n}^J)\right\|_{L^\infty L^{2^*}(\mathbb{R}\times\mathbb{R}^d)} \to 0;$$

(III) For each given $J \geq 1$, we have (the error $o_J(n) \to 0$ as $n \to \infty$)

$$\|(u_{0,n}, u_{1,n})\|_H^2 = \sum_{j=1}^{J} \|V_j\|_H^2 + \left\|(w_{0,n}^J, w_{1,n}^J)\right\|_H^2 + o_J(n).$$

Here the notation $\|V_j\|_H$ represents $\|(V_j(\cdot,t), \partial_t V_j(\cdot,t))\|_H$ which is a constant independent of $t \in \mathbb{R}$.

Please see [1] for a proof. There are a few remarks. First of all, the original paper is only for the three-dimensional case but the same argument also works for higher dimensions. Second, only the limit with Y norm in part (II) of the conclusion is mentioned in the original theorem. However, we can substitute Y norm by any Strichartz norm $L^q L^r(\mathbb{R} \times \mathbb{R}^d)$ with $q > 2$, as mentioned in Page 136 of the paper. Here we use the $L^\infty L^{2^*}$ norm. Finally, the original paper proves the theorem under an additional assumption labelled (1.6) there. But this condition can be eliminated according to Remark 5 on Page 159 of the paper.

Remark 3.2. *Passing to a subsequence if necessary, we can always assume the following limits hold as $n \to \infty$ in addition.*

$$\lambda_{j,n} \to \lambda_j \in [0,\infty], \qquad \frac{-t_{j,n}}{\lambda_{j,n}} \to t_j \in [-\infty,\infty], \qquad x_{j,n} \to x_j \in \mathbb{R}^3 \cup \{\infty\}$$

Here $x_{j,n} \to \infty$ means $|x_{j,n}| \to \infty$. Furthermore, by adjusting the free waves V_j's, we can always assume each of the triples (λ_j, x_j, t_j) satisfies one of the following conditions

(I) $\lambda_j = 0$;

(II) $\lambda_j = 1$, x_j *is either* 0 *or* ∞*;*

(III) $\lambda_j = +\infty$.

Definition 3.3. *For each linear profile* V_j *as above, we choose a coefficient function* $\phi_j(x)$ *by:*

- *If* $\lambda_j = 0$ *and* $x_j \in \mathbb{R}^d$*, then* $\phi_j(x) \equiv \phi(x_j)$*;*
- *If* $(\lambda_j, x_j) = (1, \mathbf{0})$*, then* $\phi_j(x) = \phi(x)$*;*
- *If* $\lambda_j = +\infty$ *or* $x_j = \infty$*, then* $\phi_j(x) \equiv \phi(\infty)$*;*

In fact, this function $\phi_j(x)$ is the limit of $\phi(\lambda_{j,n}x + x_{j,n})$ as $n \to \infty$ in the sense of measure. More precisely, we have

$$\lim_{n\to\infty} \left|\left\{x \in \mathbb{R}^d : |\phi(\lambda_{j,n}x + x_{j,n}) - \phi_j(x)| > \varepsilon_0\right\}\right| = 0, \qquad \text{for any } \varepsilon_0 > 0.$$

Combining this limit with the inequalities $|\phi(x)| \leq 1$ and $|\phi_j(x)| \leq 1$, we can apply the basic real analysis theory and obtain

Proposition 3.4. *Assume* $1 \leq p_1, q_1, r_1 < \infty$. *If* $u \in L^{p_1}(\mathbb{R}^d)$ *and* $v \in L^{q_1}L^{r_1}(\mathbb{R}\times\mathbb{R}^d)$*, then we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(\lambda_{j,n}x + x_{j,n})|u(x)|^{p_1}dx = \int_{\mathbb{R}^d} \phi_j(x)|u(x)|^{p_1}dx.$$
$$\lim_{n\to\infty} \|[\phi(\lambda_{j,n}x + x_{j,n}) - \phi_j(x)]\, v(x,t)\|_{L^{q_1}L^{r_1}(\mathbb{R}\times\mathbb{R}^d)} = 0.$$

In the rest of this section, we prove a few properties about the linear profiles.

Remark 3.5 (Almost Orthogonality in the Energy Space)**.** *A basic computation shows* $(j \neq j')$

$$\begin{aligned}
&\left\langle \begin{pmatrix} V_{j,n}(\cdot,t_0) \\ \partial_t V_{j,n}(\cdot,t_0)\end{pmatrix}, \begin{pmatrix} V_{j',n}(\cdot,t_0) \\ \partial_t V_{j',n}(\cdot,t_0)\end{pmatrix} \right\rangle_H \\
&= \left\langle \mathbf{S}_L(t_0 - t_{j,n})\mathbf{T}_{\lambda_{j,n},x_{j,n}} \begin{pmatrix} v_{0,j} \\ v_{1,j}\end{pmatrix}, \mathbf{S}_L(t_0 - t_{j',n})\mathbf{T}_{\lambda_{j',n},x_{j',n}} \begin{pmatrix} v_{0,j'} \\ v_{1,j'}\end{pmatrix} \right\rangle_H \\
&= \left\langle \mathbf{T}_{\lambda_{j,n}/\lambda_{j',n}} \begin{pmatrix} v_{0,j} \\ v_{1,j}\end{pmatrix}, \mathbf{T}_{1,\frac{x_{j',n}-x_{j,n}}{\lambda_{j',n}}}\mathbf{S}_L\left(\frac{t_{j,n}-t_{j',n}}{\lambda_{j',n}}\right) \begin{pmatrix} v_{0,j'} \\ v_{1,j'}\end{pmatrix} \right\rangle_H.
\end{aligned}$$

Since the triples $(\lambda_{j,n}, x_{j,n}, t_{j,n})$ *and* $(\lambda_{j',n}, x_{j',n}, t_{j',n})$ *are almost orthogonal, we have*

$$\lim_{n\to\infty} \left\langle \begin{pmatrix} V_{j,n}(\cdot,t_0) \\ \partial_t V_{j,n}(\cdot,t_0)\end{pmatrix}, \begin{pmatrix} V_{j',n}(\cdot,t_0) \\ \partial_t V_{j',n}(\cdot,t_0)\end{pmatrix} \right\rangle_H = 0$$

for each $t_0 \in \mathbb{R}$ *and* $j \neq j'$.

Lemma 3.6. *Let* $V(x,t) = \mathbf{S}_{L,0}(t)(v_0, v_1)$ *be a solution to the linear wave equation with a finite energy. We have the limit*

$$\lim_{t\to\pm\infty} \|V(\cdot,t)\|_{L^{2^*}(\mathbb{R}^d)} = 0.$$

Proof. By the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$ and the fact that the linear propagation preserves the $\dot{H}^1 \times L^2$ norm, it is sufficiently to prove this lemma for each (v_0, v_1) in a dense subset of $\dot{H}^1 \times L^2$. Thus we only need to consider smooth and compactly supported initial data. In this case the kernel of the wave propagation gives a well-known estimate

$$\|V(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d-1}{2}}.$$

On the other hand the volume of the support of $V(\cdot,t)$ satisfies $|\mathrm{Supp}(V(\cdot,t))| \lesssim |t|^d$ when $|t|$ is large. This immediately gives the estimate $\|V(\cdot,t)\|_{L^{2^*}} \lesssim |t|^{-1/2}$ and finishes the proof. $\square$

Lemma 3.7. *Given $j \in \mathbb{Z}^+$, we have:*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(x)|V_{j,n}(x,0)|^{2^*} dx = \begin{cases} 0, & \text{if } t_j = \pm\infty; \\ \int_{\mathbb{R}^d} \phi_j(x)|V_j(x,t_j)|^{2^*} dx, & \text{if } t_j \in \mathbb{R}. \end{cases}$$

Proof. By Lemma 3.6, the case $t_j = \pm\infty$ is trivial. Thus we only need to consider the case that t_j is finite. By our assumption $-t_{j,n}/\lambda_{j,n} \to t_j$, the fact $V_j(\cdot,t) \in C(\mathbb{R}; \dot{H}^1(\mathbb{R}^d))$ and the Sobolev embedding $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{2^*}(\mathbb{R}^d)$, we have

$$\left\| V_{j,n}(\cdot,0) - \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j\left(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}, t_j\right) \right\|_{L^{2^*}(\mathbb{R}^d)} = \left\| V_j\left(\cdot, \frac{-t_{j,n}}{\lambda_{j,n}}\right) - V_j(\cdot, t_j) \right\|_{L^{2^*}(\mathbb{R}^d)} \to 0.$$

This implies $\lim_{n\to\infty} \|V_{j,n}(\cdot,0)\|_{L^{2^*}(\mathbb{R}^d;\phi dx)} = \lim_{n\to\infty} \left\| \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j\left(\frac{\cdot - x_{j,n}}{\lambda_{j,n}}, t_j\right) \right\|_{L^{2^*}(\mathbb{R}^d;\phi dx)}$. Thus we have

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(x)|V_{j,n}(x,0)|^{2^*} dx &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(x) \left| \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}, t_j\right) \right|^{2^*} dx \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi(\lambda_{j,n}x + x_{j,n})|V_j(x,t_j)|^{2^*} dx \\ &= \int_{\mathbb{R}^d} \phi_j(x)|V_j(x,t_j)|^{2^*} dx. \end{aligned}$$

In the last step we apply Proposition 3.4. □

Lemma 3.8. *Assume $\tilde{V}_1, \tilde{V}_2 \in L^{2^*}(\mathbb{R}^d)$. If the pairs $\{(\lambda_{j,n}, x_{j,n})\}_{j=1,2;n\in\mathbb{Z}^+}$ satisfy*

$$\lim_{n\to+\infty} \left(\frac{\lambda_{2,n}}{\lambda_{1,n}} + \frac{\lambda_{1,n}}{\lambda_{2,n}} + \frac{|x_{1,n} - x_{2,n}|}{\lambda_{1,n}} \right) = \infty,$$

then we have the following limit as $n \to \infty$

$$N(n) = \left\| \frac{1}{\lambda_{1,n}^{(d-2)/2}} \tilde{V}_1\left(\frac{x - x_{1,n}}{\lambda_{1,n}}\right) \cdot \frac{1}{\lambda_{2,n}^{(d-2)/2}} \tilde{V}_2\left(\frac{x - x_{2,n}}{\lambda_{2,n}}\right) \right\|_{L^{2^*/2}(\mathbb{R}^d)} \to 0.$$

Proof. Let us first define

$$\tilde{V}_{j,n}(x) = \frac{1}{\lambda_{j,n}^{(d-2)/2}} \tilde{V}_j\left(\frac{x - x_{j,n}}{\lambda_{j,n}}\right).$$

Observing the continuity of the map

$$\Phi : L^{2^*}(\mathbb{R}^d) \times L^{2^*}(\mathbb{R}^d) \to l^\infty, \qquad \Phi(\tilde{V}_1, \tilde{V}_2) = \left\{ \left\| \tilde{V}_{1,n} \tilde{V}_{2,n} \right\|_{L^{2^*/2}} \right\}_{n\in\mathbb{Z}^+},$$

we can also assume, without loss of generality, that

$$|\tilde{V}_j(x)| \leq M_j, \text{ for any } x \in \mathbb{R}^d; \qquad \operatorname{Supp}(\tilde{V}_j) \subseteq \{x : |x| < R_j\}$$

for $j = 1, 2$ and some constants M_j, R_j, because the functions satisfying these conditions are dense in the space $L^{2^*}(\mathbb{R}^d)$. If the conclusion were false, we would find a sequence $n_1 < n_2 < n_3 < \cdots$ and a positive constant ε_0 such that $N(n_k) \geq \varepsilon_0$. There are three cases

(I) $\limsup_{k\to\infty} \lambda_{1,n_k}/\lambda_{2,n_k} = \infty$. First of all, we observe that the product $\tilde{V}_{1,n_k}\tilde{V}_{2,n_k}$ is supported in a d-dimensional ball centred at x_{2,n_k} with a radius $\lambda_{2,n_k}R_2$ since $\tilde{V}_{2,n_k}$ is supported in this ball. On the other hand, we have

$$\left|\tilde{V}_{1,n_k}\tilde{V}_{2,n_k}\right| \le \lambda_{1,n_k}^{-\frac{d-2}{2}}\lambda_{2,n_k}^{-\frac{d-2}{2}} M_1M_2.$$

A basic computation shows

$$N(n_k) = \left\|\tilde{V}_{1,n_k}\tilde{V}_{2,n_k}\right\|_{L^{2^*/2}(\mathbb{R}^d)} \le C(d)M_1M_2R_2^{d-2}\left(\frac{\lambda_{2,n_k}}{\lambda_{1,n_k}}\right)^{\frac{d-2}{2}}.$$

The upper bound tends to zero as $\lambda_{1,n_k}/\lambda_{2,n_k} \to \infty$. Thus we have a contradiction.

(II) $\limsup_{k\to\infty} \lambda_{2,n_k}/\lambda_{1,n_k} = \infty$. This can be handled in the same way as case (I).

(III) $\lambda_{1,n_k} \simeq \lambda_{2,n_k}$. Thus we also have $\dfrac{|x_{1,n_k} - x_{2,n_k}|}{\lambda_{1,n_k}} \to \infty$. This implies $\mathrm{Supp}(\tilde{V}_{1,n_k}) \cap \mathrm{Supp}(\tilde{V}_{2,n_k}) = \emptyset$ when k is sufficiently large thus gives a contradiction.

□

Lemma 3.9. *For any $j \neq j'$, we have the limit*

$$\lim_{n\to\infty} \|V_{j,n}(\cdot,0)V_{j',n}(\cdot,0)\|_{L^{2^*/2}(\mathbb{R}^d)} = 0.$$

Proof. If $t_j = \pm\infty$, then Lemma 3.6 guarantees

$$\|V_{j,n}(\cdot,0)\|_{L^{2^*}(\mathbb{R}^d)} = \left\|V_j\left(\cdot, \frac{-t_{j,n}}{\lambda_{j,n}}\right)\right\|_{L^{2^*}(\mathbb{R}^d)} \to 0$$

On the other hand, we know

$$\|V_{j',n}(\cdot,0)\|_{L^{2^*}(\mathbb{R}^d)} \lesssim \|\nabla V_{j',n}(\cdot,0)\|_{L^2} \le \|V_j\|_H \le A. \tag{12}$$

This immediately finishes our proof. Thus we only need to consider the case that $t_j, t_{j'}$ are both finite. In this case the almost orthogonality of the triples gives

$$\lim_{n\to+\infty}\left(\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}}\right) = \infty.$$

By the fact $V_j(\cdot,t), V_{j'}(\cdot,t) \in C(\mathbb{R}; L^{2^*}(\mathbb{R}^d))$, we have

$$\left\|V_{j,n}(\cdot,0) - \frac{1}{\lambda_{j,n}^{(d-2)/2}}V_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, t_j\right)\right\|_{L^{2^*}} = \left\|V_j\left(\cdot, \frac{-t_{j,n}}{\lambda_{j,n}}\right) - V_j(\cdot,t_j)\right\|_{L^{2^*}} \to 0;$$

$$\left\|V_{j',n}(\cdot,0) - \frac{1}{\lambda_{j',n}^{(d-2)/2}}V_{j'}\left(\frac{x-x_{j',n}}{\lambda_{j',n}}, t_j'\right)\right\|_{L^{2^*}} = \left\|V_{j'}\left(\cdot, \frac{-t_{j',n}}{\lambda_{j',n}}\right) - V_{j'}(\cdot,t_{j'})\right\|_{L^{2^*}} \to 0.$$

Therefore we only need to show

$$\lim_{n\to\infty}\left\|\frac{1}{\lambda_{j,n}^{(d-2)/2}}V_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, t_j\right)\cdot\frac{1}{\lambda_{j',n}^{(d-2)/2}}V_{j'}\left(\frac{x-x_{j',n}}{\lambda_{j',n}}, t_{j'}\right)\right\|_{L^{2^*/2}} = 0.$$

This immediately follows Lemma 3.8. □

Lemma 3.10. *The profile decomposition we obtain above satisfies*

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi(x)|u_{0,n}(x)|^{2^*}dx=\sum_{j=1}^{\infty}\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi(x)|V_{j,n}(x,0)|^{2^*}dx. \tag{13}$$

Proof. First of all, we know each limit on the right hand of (13) exists by Lemma 3.7. Let us first show

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi\left|\sum_{j=1}^{J}V_{j,n}(x,0)\right|^{2^*}dx=\sum_{j=1}^{J}\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi\left|V_{j,n}(x,0)\right|^{2^*}dx. \tag{14}$$

This can proved via an induction by observing (take the value of $V_{j,n}$ at $t=0$ and let $G(u)=|u|^{2^*}$)

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^d}\left|G\left(\sum_{j=1}^{J}V_{j,n}(\cdot,0)\right)-G\left(\sum_{j=1}^{J-1}V_{j,n}(\cdot,0)\right)-G(V_{J,n}(\cdot,0))\right|\phi dx\\
\le&\limsup_{n\to\infty}\int_{\mathbb{R}^d}\left|\left[V_{J,n}\int_0^1G'\left(\tau V_{J,n}+\sum_{j=1}^{J-1}V_{j,n}\right)d\tau\right]-\left[V_{J,n}\int_0^1G'(\tau V_{J,n})d\tau\right]\right|dx\\
=&\limsup_{n\to\infty}\int_{\mathbb{R}^d}\left|\left[V_{J,n}\sum_{j=1}^{J-1}V_{j,n}\right]\left[\int_0^1\int_0^1G''\left(\tau V_{J,n}+\tilde{\tau}\sum_{j=1}^{J-1}V_{j,n}\right)d\tilde{\tau}d\tau\right]\right|dx\\
\le&\limsup_{n\to\infty}C\left(\sum_{j=1}^{J-1}\|V_{J,n}V_{j,n}\|_{L^{2^*/2}(\mathbb{R}^d)}\right)\left(\sum_{j=1}^{J}\|V_{j,n}\|_{L^{2^*}(\mathbb{R}^d)}\right)^{2^*-2}\\
=&0.
\end{aligned}$$

Here we use Lemma 3.9 and the estimate (12). Next we can rewrite (14) into

$$\left(\sum_{j=1}^{J}\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi\left|V_{j,n}(x,0)\right|^{2^*}dx\right)^{1/2^*}=\lim_{n\to\infty}\left\|\sum_{j=1}^{J}V_{j,n}(x,0)\right\|_{L^{2^*}(\mathbb{R}^d;\phi dx)}$$

The conclusion (II) of profile decomposition gives

$$\limsup_{n\to\infty}\left\|w_{0,n}^J\right\|_{L^{2^*}(\mathbb{R}^d;\phi dx)}=o(J)\to 0,\ \text{as } J\to 0.$$

By the identity $u_{0,n}=\sum_{j=1}^{J}V_{j,n}(\cdot,0)+w_{0,n}^J$ and the limits above, we have

$$\begin{aligned}
\limsup_{n\to\infty}\|u_{0,n}\|_{L^{2^*}(\mathbb{R}^d;\phi dx)}-o(J)\le&\left(\sum_{j=1}^{J}\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi\left|V_{j,n}(x,0)\right|^{2^*}dx\right)^{1/2^*}\\
&\le\liminf_{n\to\infty}\|u_{0,n}\|_{L^{2^*}(\mathbb{R}^d;\phi dx)}+o(J)
\end{aligned}$$

Letting $J\to\infty$, we finish the proof. □

Now we can summarize all properties of our profile decomposition for future use.

Proposition 3.11 (Profile Decomposition). *Given a sequence $(u_{0,n},u_{1,n})\in H, n\in\mathbb{Z}^+$ so that $\|(u_{0,n},u_{1,n})\|_H\le A$, there exist a subsequence of $(u_{0,n},u_{1,n})$ (We still use the notation*

$(u_{0,n},u_{1,n})$ *for the subsequence), a sequence of free waves* $V_j(x,t) = \mathbf{S}_{L,0}(t)(v_{0,j},v_{1,j})$ *and a family of triples* $(\lambda_{j,n},x_{j,n},t_{j,n}) \in \mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R}$*, which are "almost orthogonal", i.e. we have*

$$\lim_{n\to+\infty}\left(\frac{\lambda_{j',n}}{\lambda_{j,n}}+\frac{\lambda_{j,n}}{\lambda_{j',n}}+\frac{|x_{j,n}-x_{j',n}|}{\lambda_{j,n}}+\frac{|t_{j,n}-t_{j',n}|}{\lambda_{j,n}}\right)=+\infty$$

for $j\neq j'$*; such that*

(I) We have the decomposition

$$(u_{0,n},u_{1,n})=\sum_{j=1}^{J}(V_{j,n}(\cdot,0),\partial_t V_{j,n}(\cdot,0))+(w_{0,n}^J,w_{1,n}^J)$$

Here $V_{j,n}$ *is another free wave derived from* V_j*:*

$$\begin{aligned}(V_{j,n}(\cdot,t),\partial_t V_{j,n}(\cdot,t)) &= \left(\frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}}V_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}},\frac{t-t_{j,n}}{\lambda_{j,n}}\right),\frac{1}{\lambda_{j,n}^{d/2}}\partial_t V_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}},\frac{t-t_{j,n}}{\lambda_{j,n}}\right)\right)\\ &= \mathbf{S}_L(t-t_{j,n})\mathbf{T}_{\lambda_{j,n},x_{j,n}}(v_{0,j},v_{1,j});\end{aligned}$$

(II) We have the following limits as $n\to\infty$*:*

$$\lambda_{j,n}\to\lambda_j\in\{0,1,+\infty\};\qquad x_{j,n}\to x_j\in\mathbb{R}^d\cup\{\infty\};\qquad \frac{-t_{j,n}}{\lambda_{j,n}}\to t_j\in[-\infty,\infty].$$

In addition, if $\lambda_j=1$*, then we must have* $x_j=0$ *or* $x_j=\infty$*.*

(III) The remainders $(w_{0,n}^J,w_{1,n}^J)$ *satisfy:*

$$\begin{aligned}&\limsup_{n\to\infty}\left\|\mathbf{S}_{L,0}(t)(w_{0,n}^J,w_{1,n}^J)\right\|_{Y(\mathbb{R})}\to 0,\ \text{as } J\to\infty.\\ &\limsup_{n\to\infty}\left\|(w_{0,n}^J,w_{1,n}^J)\right\|_H\le 2A.\end{aligned}$$

(IV) We have the limits

$$\sum_{j=1}^{\infty}\|V_j\|_H^2=\sum_{j=1}^{\infty}\|(v_{0,j},v_{1,j})\|_H^2\le\liminf_{n\to\infty}\|(u_{0,n},u_{1,n})\|_H^2$$

$$\sum_{j=1}^{\infty}\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi(x)|V_{j,n}(x,0)|^{2^*}dx=\lim_{n\to\infty}\int_{\mathbb{R}^d}\phi(x)|u_{0,n}(x)|^{2^*}dx$$

3.2 Nonlinear Profiles

Given any linear profile decomposition as in Proposition 3.11, we can assign a nonlinear profile to each linear profile V_j. Let us start by introducing the definition of a nonlinear profile.

Definition 3.12 (A nonlinear profile). *Fix* $\tilde{\phi}\in C(\mathbb{R}^d;[-1,1])$*. Let* $V(x,t)=\mathbf{S}_{L,0}(t)(v_0,v_1)$ *be a free wave and* $\tilde{t}\in[-\infty,\infty]$ *be a time. We say that* $U(x,t)$ *is a nonlinear profile associated to* $(V,\tilde{\phi},\tilde{t})$ *if* $U(x,t)$ *is a solution to the nonlinear wave equation*

$$\partial_t^2 u-\Delta u=\tilde{\phi}F(u)\tag{15}$$

with a maximal timespan I *so that* I *contains a neighbourhood*[3] *of* $\tilde{t}$ *and*

$$\lim_{t\to\tilde{t}}\|(U(\cdot,t),\partial_t U(\cdot,t))-(V(\cdot,t),\partial_t V(\cdot,t))\|_H=0.$$

[3] A neighbourhood of $+\infty$ is $(M,+\infty)$, where M is any real number. Similarly a neighbourhood of $-\infty$ is $(-\infty,M)$.

Remark 3.13. *Given a triple $(V, \tilde{\phi}, \tilde{t})$ as above, one can show there is always a unique nonlinear profile. Please see Remark 2.13 in [15] for the idea of proof. In particular, if $\tilde{t}$ is finite, then the nonlinear profile is simply the solution to the equation* (15) *with initial data $(V(\cdot,\tilde{t}), \partial_t V(\cdot,\tilde{t}))$ at the time $t=\tilde{t}$. We will also use the fact that the nonlinear profile automatically scatters in the positive time direction if $\tilde{t}=+\infty$.*

Definition 3.14 (Nonlinear Profiles)**.** *Given $j \in \mathbb{Z}^+$, we define U_j to be the nonlinear profile associated to (V_j, ϕ_j, t_j). Here the coefficient function ϕ_j is given in Definition 3.3. For convenience we will call U_j the nonlinear profile associated to V_j and use the notation*

$$U_{j,n}(x,t) \doteq \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right).$$

Lemma 3.15 ($U_{j,n}$ is an approximation solution to (CP1))**.** *If I_j' is a time interval so that $\|U_j\|_{Y(I_j')} < \infty$, then the error term*

$$e_{j,n} = (\partial_t^2 - \Delta)U_{j,n} - \phi F(U_{j,n})$$

satisfies the estimate

$$\lim_{n\to\infty} \|e_{j,n}\|_{L^1L^2((t_{j,n}+\lambda_{j,n}I_j')\times\mathbb{R}^d)} = 0.$$

Proof. Because U_j satisfies

$$\partial_t^2 U_j - \Delta U_j = \tilde{\phi}(x)F(U_j) \implies (\partial_t^2 - \Delta)U_{j,n} = \phi_j(x)\left(\frac{x-x_{j,n}}{\lambda_{j,n}}\right) F(U_{j,n})$$

we can finish the proof by calculating

$$\begin{aligned}\|e_{j,n}\|_{L^1L^2((t_{j,n}+\lambda_{j,n}I_j')\times\mathbb{R}^d)} &= \left\|\left(\phi_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}\right) - \phi(x)\right)F(U_{n,j})\right\|_{L^1L^2((t_{j,n}+\lambda_{j,n}I_j')\times\mathbb{R}^d)} \\ &= \|(\phi_j(x) - \phi(\lambda_{j,n}x + x_{j,n}))F(U_j)\|_{L^1L^2(I_j'\times\mathbb{R}^d)} \to 0. \qquad (16)\end{aligned}$$

In the last step we apply Proposition 3.4. ☐

We also need the following lemmata in order to find an upper bound for the Y norm of $\sum_j U_{j,n}$.

Lemma 3.16 (Almost Orthogonality of $U_{j,n}$)**.** *Assume $\|\tilde{U}_j\|_{L^qL^r(I_j'\times\mathbb{R}^d)} < \infty$ for $j=1,2$. Here L^qL^r is a Strichartz norm with $q<\infty$, i.e. we have*

$$2 \le q, r < \infty; \qquad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - 1.$$

Let $\{(\lambda_{1,n}, x_{1,n}, t_{1,n})\}_{n\in\mathbb{Z}^+}$ and $\{(\lambda_{2,n}, x_{2,n}, t_{2,n})\}_{n\in\mathbb{Z}^+}$ be two "almost orthogonal" sequences of triples, i.e.

$$\lim_{n\to+\infty}\left(\frac{\lambda_{2,n}}{\lambda_{1,n}} + \frac{\lambda_{1,n}}{\lambda_{2,n}} + \frac{|x_{1,n}-x_{2,n}|}{\lambda_{1,n}} + \frac{|t_{1,n}-t_{2,n}|}{\lambda_{1,n}}\right) = +\infty.$$

If I_n is a sequence of time intervals, such that $I_n \subseteq (t_{1,n}+\lambda_{1,n}I_1') \cap (t_{2,n}+\lambda_{2,n}I_2')$, then we have

$$N(n) = \left\|\tilde{U}_{1,n}\tilde{U}_{2,n}\right\|_{L_t^{q/2}L_x^{r/2}(I_n\times\mathbb{R}^d)} \to 0, \qquad as\ n\to\infty.$$

Here $\tilde{U}_{j,n}$ is defined as usual

$$\tilde{U}_{j,n} = \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}}\tilde{U}_j\left(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}}\right)$$

Proof. (See also Lemma 2.7 in [17]) First of all, we can always assume $I'_j = \mathbb{R}$ and $I_n = \mathbb{R}$. Otherwise we can define $\tilde{U}_j(x,t) = 0$ for all $t \notin I'_j$ and thus expand the domain of each $\tilde{U}_j$ to $\mathbb{R} \times \mathbb{R}^d$. Observing the continuity of the map

$$\Phi : L^q L^r(\mathbb{R} \times \mathbb{R}^d) \times L^q L^r(\mathbb{R} \times \mathbb{R}^d) \to l^\infty, \quad \Phi(\tilde{U}_1, \tilde{U}_2) = \left\{ \left\| \tilde{U}_{1,n} \tilde{U}_{2,n} \right\|_{L^{q/2} L^{r/2}(\mathbb{R}\times\mathbb{R}^d)} \right\}_{n\in\mathbb{Z}^+},$$

we can also assume, without loss of generality, that

$$|\tilde{U}_j(x,t)| \le M_j, \text{ for any } (x,t) \in \mathbb{R}^d \times \mathbb{R}; \qquad \mathrm{Supp}(\tilde{U}_j) \subseteq \{(x,t) : |x|, |t| < R_j\}$$

for each $j = 1, 2$ and some constants M_j, R_j, since the functions satisfying these conditions are dense in the space $L^q L^r(\mathbb{R} \times \mathbb{R}^d)$. If the conclusion were false, we would find a sequence $n_1 < n_2 < n_3 < \cdots$ and a positive constant ε_0 such that $N(n_k) \ge \varepsilon_0$. There are three cases

(I) $\limsup_{k\to\infty} \lambda_{1,n_k}/\lambda_{2,n_k} = \infty$. On one hand the product $\tilde{U}_{1,n_k} \tilde{U}_{2,n_k}$ is supported in the $(d+1)$-dimensional circular cylinder centred at (x_{2,n_k}, t_{2,n_k}) with radius $\lambda_{2,n_k} R_2$ and height $2\lambda_{2,n_k} R_2$ since $\tilde{U}_{2,n_k}$ is supported in this cylinder. On the other hand, we have

$$\left| \tilde{U}_{1,n_k} \tilde{U}_{2,n_k} \right| \le \lambda_{1,n_k}^{-\frac{d-2}{2}} \lambda_{2,n_k}^{-\frac{d-2}{2}} M_1 M_2.$$

A basic computation shows

$$N(n_k) = \left\| \tilde{U}_{1,n_k} \tilde{U}_{2,n_k} \right\|_{L^{q/2} L^{r/2}(\mathbb{R}\times\mathbb{R}^d)} \le C(d) M_1 M_2 R_2^{d-2} \left(\frac{\lambda_{2,n_k}}{\lambda_{1,n_k}} \right)^{\frac{d-2}{2}}.$$

The upper bound tends to zero as $\lambda_{1,n_k}/\lambda_{2,n_k} \to \infty$. Thus we have a contradiction.

(II) $\limsup_{k\to\infty} \lambda_{2,n_k}/\lambda_{1,n_k} = \infty$. This can be handled in the same way as case (I).

(III) $\lambda_{1,n_k} \simeq \lambda_{2,n_k}$. By the "almost orthogonality" of the sequences of triples, we also have

$$\frac{|x_{1,n_k} - x_{2,n_k}|}{\lambda_{1,n_k}} + \frac{|t_{1,n_k} - t_{2,n_k}|}{\lambda_{1,n_k}} \to \infty.$$

This implies $\mathrm{Supp}(\tilde{U}_{1,n_k}) \cap \mathrm{Supp}(\tilde{U}_{2,n_k}) = \emptyset$ when k is sufficiently large thus gives a contradiction.

□

Lemma 3.17. *Let $\{I'_j\}_{j=1,2,\cdots,J}$ be time intervals such that $\|U_j(x,t)\|_{Y(I'_j)} < \infty$ holds for each $1 \le j \le J$. Suppose $\{J_n\}_{n\in\mathbb{Z}^+}$ is a sequence of time intervals, so that $J_n \subseteq \cap_{j=1}^J (t_{j,n} + \lambda_{j,n} I'_j)$ holds for sufficiently large n. Then we have*

$$\lim_{n\to\infty} \left\| F\left(\sum_{j=1}^J U_{j,n} \right) - \sum_{j=1}^J F(U_{j,n}) \right\|_{L^1 L^2(J_n\times\mathbb{R}^d)} = 0.$$

$$\limsup_{n\to\infty} \left\| \sum_{j=1}^J U_{j,n} \right\|_{Y(J_n)} \le \left(\sum_{j=1}^J \|U_j\|_{Y(I'_j)}^{p_c} \right)^{1/p_c}.$$

Proof First of all, for any $J \geq 2$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \left\| F\left(\sum_{j=1}^{J} U_{j,n}\right) - F\left(\sum_{j=1}^{J-1} U_{j,n}\right) - F(U_{J,n}) \right\|_{L^1L^2(J_n\times\mathbb{R}^d)} \\
=&\lim_{n\to\infty} \left\| \left[U_{J,n}\int_0^1 F'\left(\tau U_{J,n} + \sum_{j=1}^{J-1} U_{j,n}\right) d\tau \right] - \left[U_{J,n}\int_0^1 F'(\tau U_{J,n}) d\tau \right] \right\|_{L^1L^2(J_n\times\mathbb{R}^d)} \\
=&\lim_{n\to\infty} \left\| \left(U_{J,n}\sum_{j=1}^{J-1} U_{j,n}\right) \left(\int_0^1\int_0^1 F''\left(\tau U_{J,n} + \tilde{\tau}\sum_{j=1}^{J-1} U_{j,n}\right) d\tilde{\tau} d\tau \right) \right\|_{L^1L^2(J_n\times\mathbb{R}^d)} \\
\leq&\lim_{n\to\infty} \left(\sum_{j=1}^{J-1} \|U_{J,n}U_{j,n}\|_{L^{p_c/2}L^{p_c}(J_n\times\mathbb{R}^d)} \right) \left(\sum_{j=1}^{J} \|U_{j,n}\|_{Y(J_n)} \right)^{p_c-2} \\
\leq&\lim_{n\to\infty} \left(\sum_{j=1}^{J-1} \|U_{J,n}U_{j,n}\|_{L^{p_c/2}L^{p_c}(J_n\times\mathbb{R}^d)} \right) \left(\sum_{j=1}^{J} \|U_{j}\|_{Y(I_j')} \right)^{p_c-2} \\
=&0.
\end{aligned}$$

In the final step we apply Lemma 3.16. As a result we can prove the first limit by an induction. The second limit is a corollary.

$$\begin{aligned}
\limsup_{n\to\infty} \left\| \sum_{j=1}^{J} U_{j,n} \right\|_{Y(J_n)}^{p_c} &= \limsup_{n\to\infty} \left\| F\left(\sum_{j=1}^{J} U_{j,n}\right) \right\|_{L^1L^2(J_n\times\mathbb{R}^d)} \\
&= \limsup_{n\to\infty} \left\| \sum_{j=1}^{J} F(U_{j,n}) \right\|_{L^1L^2(J_n\times\mathbb{R}^d)} \\
&\leq \limsup_{n\to\infty} \left(\sum_{j=1}^{J} \|F(U_{j,n})\|_{L^1L^2(J_n\times\mathbb{R}^d)} \right) \\
&= \limsup_{n\to\infty} \left(\sum_{j=1}^{J} \|U_{j,n}\|_{Y(J_n)}^{p_c} \right) \\
&\leq \sum_{j=1}^{J} \|U_j\|_{Y(I_j')}^{p_c}.
\end{aligned}$$

Lemma 3.18 (Energy of a Nonlinear Profile)**.** *Let U_j be the nonlinear profile as above. Then we have the energy defined by*

$$E(U_j) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla U_j(x,t)|^2 + \frac{1}{2}|\partial_t U_j(x,t)|^2 - \frac{\zeta\phi_j(x)}{2^*}|U_j(x,t)|^{2^*} \right) dx$$

satisfies

$$E(U_j) = \frac{1}{2}\|V_j\|_H^2 - \frac{\zeta}{2^*}\lim_{n\to\infty}\int_{\mathbb{R}^d} \phi(x)|V_{j,n}(x,0)|^{2^*} dx.$$

Proof. First of all, our definition of nonlinear profiles guarantees that the energy defined above does not depend on the time t. We prove the lemma by considering two different cases.

Case 1 If $t_j = +\infty$, then the definition of the nonlinear profile gives

$$\lim_{t\to+\infty} \left\| \begin{pmatrix} U_j(\cdot,t) \\ \partial_t U_j(\cdot,t) \end{pmatrix} - \begin{pmatrix} V_j(\cdot,t) \\ \partial_t V_j(\cdot,t) \end{pmatrix} \right\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} = 0. \tag{17}$$

By the Sobolev embedding, we also have $\|U_j(\cdot,t) - V_j(\cdot,t)\|_{L^{2^*}} \to 0$ as $t \to +\infty$. Applying Lemma 3.6, we obtain

$$\lim_{t\to+\infty} \int_{\mathbb{R}^d} |U_j(x,t)|^{2^*} dx = 0.$$

Combining this with (17), we can evaluate the energy at larger and larger times t and finally obtain $E(U_j) = \frac{1}{2}\|V_j\|_H^2$. This finishes the proof, since we also know

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |V_{j,n}(x,0)|^{2^*} dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} |V_j(x, -t_{j,n}/\lambda_{j,n})|^{2^*} dx = 0.$$

The same argument works if $t_j = -\infty$.

Case 2 If t_j is finite, we can immediately conclude the proof if we evaluate the energy at $t = t_j$ by using Lemma 3.7 and the fact $(U_j(\cdot,t_j), \partial_t U_j(\cdot,t_j)) = (V_j(\cdot,t_j), \partial_t V_j(\cdot,t_j))$. □

Combining Lemma 3.18 and part (IV) of our profile decomposition (see Proposition 3.11), we obtain

Corollary 3.19. *Let U_j's be the nonlinear profiles with energy $E(U_j)$ as defined above. Then we have the inequality*

$$\sum_{j=1}^{\infty} E(U_j) \le \liminf_{n\to\infty} E_\phi(u_{0,n}, u_{1,n}).$$

4 Compactness Procedure

In this section we prove the following proposition

Proposition 4.1. *Assume that ϕ satisfies the condition* (1). *If the statement SC(ϕ, M) breaks down at $M_0 < E_1(W,0)$, i.e. the statement holds for all $M \le M_0$ but fails for any $M > M_0$, then there exists a critical element u, which is a solution to (CP1) in the focusing case with initial data (u_0, u_1) such that*

(I) The energy $E_\phi(u_0, u_1) = M_0$;

(II) The solution exists globally in time with $\|u\|_{Y([0,\infty))} = \|u\|_{Y((-\infty,0])} = +\infty$.

(III) The norm $\|(u(\cdot,t), u_t(\cdot,t))\|_{\dot{H}^1\times L^2(\mathbb{R}^d)} < \|\nabla W\|_{L^2}$ for each $t \in \mathbb{R}$.

(IV) The set $\{(u(t), \partial_t u(t)) | t \in \mathbb{R}\}$ is pre-compact in the space $\dot{H}^1 \times L^2(\mathbb{R}^d)$.

Remark 4.2. *The compactness procedure in the defocusing case is similar. We can substitute the statement SC(ϕ, M) and Proposition 4.1 by the statement SC'(ϕ, M) and Proposition 4.4 as below.*

Statement 4.3 (SC'(ϕ, M))**.** *There exists a function $\beta : [0, M) \to \mathbb{R}^+$, such that if the initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ satisfy $E_\phi(u_0, u_1) < M$, then the corresponding solution u to (CP1) in the defocusing case exists globally in time, scatters in both two time directions with*

$$\|u\|_{Y(\mathbb{R})} < \beta(E_\phi(u_0, u_1)).$$

Proposition 4.4. *Assume that ϕ satisfies the condition (1). If the statement SC'(ϕ, M) breaks down at M_0, i.e. the statement holds for all $M \leq M_0$ but fails for any $M > M_0$, then there exists a critical element u, which is a solution to (CP1) in the defocusing case with initial data (u_0, u_1) such that*

- *The energy $E_\phi(u_0, u_1) = M_0$;*
- *The solution exists globally in time with $\|u\|_{Y([0,\infty))} = \|u\|_{Y((-\infty,0])} = +\infty$.*
- *The set $\{(u(t), \partial_t u(t)) | t \in \mathbb{R}\}$ is pre-compact in the space $\dot{H}^1 \times L^2(\mathbb{R}^d)$.*

4.1 Set-up of the Proof

If the statement SC(ϕ, M) broke down at $M_0 < E_1(W, 0)$, then the statement $SC(\phi, M_0 + 2^{-n})$ would not be true for each positive integer n. Fix a positive integer N_0 so that $M_0 + 2^{-n} < E_1(W, 0)$ for each $n \geq N_0$. Under these assumptions we have

Lemma 4.5. *We can find a sequence of solutions $\{u_n\}_{n \geq N_0}$ with initial data $(u_{0,n}, u_{1,n}) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$, such that*

(I) $\|\nabla u_{0,n}\|_{L^2} < \|\nabla W\|_{L^2}$ *and* $E_\phi(u_{0,n}, u_{1,n}) < M_0 + 2^{-n}$;

(II) Let $(-T_-(u_{0,n}, u_{1,n}), T_+(u_{0,n}, u_{1,n}))$ be the maximal lifespan of u_n. We have

$$\|u_n\|_{Y((-T_-(u_{0,n},u_{1,n}),0])} > 2^n; \qquad \|u_n\|_{Y([0,T_+(u_{0,n},u_{1,n})))} > 2^n.$$

Proof. Given $n \geq N_0$, we claim that there exists a solution v_n with initial data $(v_{0,n}, v_{1,n})$ and maximal lifespan $(-T_-, T_+)$, so that part (I) of the conclusion above holds and

$$\|v_n\|_{Y((-T_-,T_+))} > 2^{n+\frac{1}{p_c}}.$$

Otherwise the statement $SC(\phi, M_0 + 2^{-n})$ holds since we can choose $\beta \equiv 2^{n+\frac{1}{p_c}} + 1$ to be a constant function. Next we can pick a time $t_0 \in (-T_-, T_+)$, so that $\|v_n\|_{Y^\star([t_0,T_+))} > 2^n$ and $\|v_n\|_{Y^\star((-T_-,t_0])} > 2^n$. Finally we finish the proof by choosing

$$(u_{0,n}, u_{1,n}) = (v_n(\cdot, t_0), \partial_t v_n(\cdot, t_0)); \qquad u_n(\cdot, t) = v_n(\cdot, t + t_0).$$

Note that the conservation law of energy and Lemma 2.18 guarantee the new initial data $(u_{0,n}, u_{1,n})$ still satisfy (I). □

Application of the profile decomposition Let us consider the solutions u_n and initial data $(u_{0,n}, u_{1,n})$ given above. According to Lemma 2.18, there exists a constant $0 < \bar{\delta} < 1$, such that

$$\|(u_{0,n}, u_{1,n})\|_{\dot{H}^1 \times L^2} < (1 - \bar{\delta})\|\nabla W\|_{L^2}$$

holds for large n. Thus we are able to apply the linear profile decomposition (Proposition 3.11) on the sequence $\{(u_{0,n}, u_{1,n})\}_{n \in \mathbb{Z}^+}$, then assign a nonlinear profile U_j to each linear profile V_j as we did in the previous section. Finally we obtain the decomposition ($J \in \mathbb{Z}^+$)

$$\begin{aligned}(u_{0,n}, u_{1,n}) =& \sum_{j=1}^{J} (V_{j,n}(\cdot, 0), \partial_t V_{j,n}(\cdot, 0)) + (w_{0,n}^J, w_{1,n}^J) \\ =& (S_{J,n}(\cdot, 0), \partial_t S_{J,n}(\cdot, 0)) + (\tilde{w}_{0,n}^J, \tilde{w}_{1,n}^J) \qquad (18)\end{aligned}$$

Here $\{(u_{0,n}, u_{1,n})\}_{n \in \mathbb{Z}^+}$ is actually a subsequence of the original sequence of initial data, although we still use the same notation. One can check that the conclusion of Lemma 4.5 still holds for this subsequence (along with the corresponding solutions u_n). The notations $S_{J,n}$ and $(\tilde{w}_{0,n}^J, \tilde{w}_{1,n}^J)$ represent

$$S_{J,n} = \sum_{j=1}^{J} U_{j,n};$$

$$(\tilde{w}_{0,n}^J, \tilde{w}_{1,n}^J) = (w_{0,n}^J, w_{1,n}^J) + \sum_{j=1}^{J} (V_{j,n}(\cdot,0) - U_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0) - \partial_t U_{j,n}(\cdot,0)).$$

By our definition of nonlinear profiles, we have ($j \in \mathbb{Z}^+$)

$$\lim_{n\to\infty} \|(V_{j,n}(\cdot,0) - U_{j,n}(\cdot,0), \partial_t V_{j,n}(\cdot,0) - \partial_t U_{j,n}(\cdot,0))\|_{\dot{H}^1 \times L^2} = 0.$$

Thus we still have

$$\limsup_{n\to\infty} \left\|\mathbf{S}_{L,0}(t)(\tilde{w}_{0,n}^J, \tilde{w}_{1,n}^J)\right\|_{Y(\mathbb{R})} \to 0, \qquad \text{as } J \to 0; \tag{19}$$

$$\limsup_{n\to\infty} \left\|(\tilde{w}_{0,n}^J, \tilde{w}_{1,n}^J)\right\|_H < 2\|\nabla W\|_{L^2}. \tag{20}$$

In addition, part (IV) of the conclusion in the profile decomposition gives

$$\sum_{j=1}^{\infty} \|V_j\|_H^2 \le \liminf_{n\to\infty} \|(u_{0,n}, u_{1,n})\|_H^2 \le (1-\bar{\delta})^2 \|\nabla W\|_{L^2}^2. \tag{21}$$

By the definition of nonlinear profiles, we also have

$$\lim_{t\to t_j} \|(U_j(\cdot,t), \partial_t U_j(\cdot,t))\|_H = \|V_j\|_H. \tag{22}$$

Combining (21) and (22), we obtain

- For each given j, we have

  $$\lim_{t\to t_j} \|(U_j(\cdot,t), \partial_t U_j(\cdot,t))\|_H \le (1-\bar{\delta})\|\nabla W\|_{L^2} \tag{23}$$

  According to Remark 2.19, we have $E(U_j) > 0$ unless U_j is identically zero. By Corollary 3.19 we also have $\sum_{j=1}^{\infty} E(U_j) \le M_0$.

- As $j \to \infty$, we have

  $$\lim_{t\to t_j} \|(U_j(\cdot,t), \partial_t U_j(\cdot,t))\|_H = \|V_j\|_H \to 0.$$

  Thus we know U_j is globally defined in time and scatters with $\|U_j\|_{Y(\mathbb{R})} \lesssim \|V_j\|_H$ for each sufficiently large $j > J_0$. Combining this upper bound on $\|U_j\|_{Y(\mathbb{R})}$ and the inequality (21), we have

  $$\sum_{j=J_0+1}^{\infty} \|U_j\|_{Y(\mathbb{R})}^{p_c} < \infty. \tag{24}$$

4.2 The Extraction of a Critical Element

In this subsection, we show there is exactly one nonzero profile in the profile decomposition, whose corresponding nonlinear profile is exactly the critical element we are looking for. We start by proving

Lemma 4.6. *If each nonlinear profile U_j we obtained in the previous subsection scatters in the positive time direction, then u_n scatters in the positive time direction for sufficiently large $n > N_0$ and*

$$\sup_{n>N_0} \|u_n\|_{Y([0,\infty))} < \infty.$$

Proof. Let us consider the approximate solution $S_{J,n} = \sum_{j=1}^{J} U_{j,n}$ which satisfies the equation

$$(\partial_t^2 - \Delta)S_{J,n} - \phi F(S_{J,n}) = e'_{J,n}, \qquad t \in [0,\infty)$$

with the error term

$$e'_{J,n} = \phi \left[\sum_{j=1}^{J} F(U_{j,n}) - F(S_{J,n}) \right] + \sum_{j=1}^{J} \left[(\partial_t^2 - \Delta)U_{j,n} - \phi F(U_{j,n}) \right]$$

and the initial data $(u_{0,n}, u_{1,n}) = (S_{J,n}(\cdot,0), \partial_t S_{J,n}(\cdot,0)) + (\tilde{w}_{0,n}^J, \tilde{w}_{1,n}^J)$. We use the notation I_j for the maximal lifespan of U_j. By our assumption on scattering we can choose an interval $I'_j \subseteq I_j$ for each j in the following way so that $\|U_j\|_{Y(I'_j)} < \infty$,

$$I'_j = \begin{cases} (-\infty,\infty) & \text{if } j > J_0 \text{ or } t_j = -\infty; \\ [t_j^-,\infty) & \text{if } j \leq J_0 \text{ and } t_j > -\infty, \text{ here } t_j^- \in I_j \text{ is a time smaller than } t_j. \end{cases}$$

One can check that $[0,\infty) \subseteq \lambda_{j,n} I'_j + t_{j,n}$ holds for all $j \in \mathbb{Z}^+$ as long as n is sufficiently large. Thus we can apply Lemma 3.15 as well as Lemma 3.17 and obtain for each J

$$\lim_{n\to\infty} \|e'_{J,n}\|_{L^1L^2([0,\infty)\times\mathbb{R}^d)} = 0; \tag{25}$$

$$\limsup_{n\to\infty} \|S_{J,n}\|_{Y([0,\infty))} \leq \left(\sum_{j=1}^{J} \|U_j\|_{Y(I'_j)}^{p_c} \right)^{1/p_c} \leq \left(\sum_{j=1}^{\infty} \|U_j\|_{Y(I'_j)}^{p_c} \right)^{1/p_c} < \infty. \tag{26}$$

Here we use our estimate (24). Let

$$M_1 = \left(\sum_{j=1}^{\infty} \|U_j\|_{Y(I'_j)}^{p_c} \right)^{1/p_c} + 1$$

and $\varepsilon_0 = \varepsilon_0(M_1)$ be the constant given in the long-time perturbation theory (Theorem 2.14). Let us first fix a J_1 so that

$$\limsup_{n\to\infty} \left\| \mathbf{S}_{L,0}(t)(\tilde{w}_{0,n}^{J_1}, \tilde{w}_{1,n}^{J_1}) \right\|_{Y(\mathbb{R})} < \varepsilon_0/2.$$

Using this upper limit as well as (25) and (26), we can find a number $N_0 \in \mathbb{Z}^+$, such that if $n > N_0$, then

$$\left\| e'_{J_1,n} \right\|_{L^1L^2([0,\infty)\times\mathbb{R}^d)} < \varepsilon_0/2; \qquad \left\| \mathbf{S}_{L,0}(t)(\tilde{w}_{0,n}^{J_1}, \tilde{w}_{1,n}^{J_1}) \right\|_{Y(\mathbb{R})} < \varepsilon_0/2;$$

$$\|S_{J_1,n}\|_{Y([0,\infty))} < M_1 = \left(\sum_{j=1}^{\infty} \|U_j\|_{Y(I'_j)}^{p_c} \right)^{1/p_c} + 1.$$

These estimates enable us to apply the long-time perturbation theory on the approximation solution $S_{J_1,n}$, initial data $(u_{0,n}, u_{1,n})$ and the time interval $[0,\infty)$, and then to conclude u_n scatters in the positive time direction with

$$\|u_n\|_{Y([0,\infty))} \leq \|S_{J_1,n}\|_{Y([0,\infty))} + \|u_n - S_{J_1,n}\|_{Y([0,\infty))} \leq M_1 + C(M_1)\varepsilon_0 < \infty$$

for each $n > N_0$. □

Critical Element Because we have assumed that $\|u_n\|_{Y[0,T_+(u_{0,n},u_{1,n}))} > 2^n$, Lemma 4.6 implies that there is at least one nonlinear profile, say U_{j_0}, that fails to scatter in the positive time direction. In addition, the limit (23) implies $\|\nabla U_{j_0}(\cdot,t)\|_{L^2} < \|\nabla W\|_{L^2}$ when t is close to t_{j_0}. According to our assumption that $SC(\phi, M)$ is true for any $M \leq M_0$ and Corollary 2.22, we obtain

$$E(U_{j_0}) = \begin{cases} \geq M_0, & \text{if } (\lambda_{j_0}, x_{j_0}) = (1, \mathbf{0}); \\ \geq E_1(W,0) > M_0, & \text{otherwise.} \end{cases}$$

Combining this with the already known facts that $\sum_{j=1}^{\infty} E(U_j) \leq M_0$ and $E(U_j) \geq 0$ (please see the bottom part of Subsection 4.1), we obtain that

- U_{j_0} is a solution to (CP1) with an energy $E(U_{j_0}) = E_\phi(U_{j_0}, \partial_t U_{j_0}) = M_0$.
- U_{j_0} is the only nonlinear profile with a positive energy;
- Any other profile U_j is identically zero;
- By Lemma 2.18, the inequality $\|(U_{j_0}(\cdot,t), \partial_t U_{j_0}(\cdot,t))\|_H < \|\nabla W\|_{L^2}$ holds for all time t in the maximal lifespan I_{j_0} of the nonlinear profile U_{j_0}.

Remark 4.7. *A similar result as Lemma 4.6 holds for the negative time direction, because the wave equation is time-invertible. This implies that U_{j_0} fails to scatter in the negative time direction as well. A direct corollary follows that t_{j_0} is finite.*

4.3 Almost Periodicity

Let u be the critical element (U_{j_0}) we obtained in the previous subsection and I be its maximal lifespan. In this subsection we prove that

Proposition 4.8. *The set $\{(u(\cdot,t), \partial_t u(\cdot,t)) | t \in I\}$ is pre-compact in $\dot{H}^1 \times L^2(\mathbb{R}^d)$.*

Proof. Given an arbitrary sequence of time $\{t_n\}_{n\in\mathbb{Z}^+}$ so that $t_n \in I$, we know $u_n = u(\cdot, t+t_n)$ is still a solution to (CP1) with initial data $(u_{0,n}, u_{1,n}) = (u(\cdot,t_n), \partial_t u(\cdot,t_n))$. This sequence of solutions still satisfies the conclusion of Lemma 4.5. Thus we can repeat the process we followed in the previous subsections. Finally we can find a subsequence $\{u_{n_k}\}_{k\in\mathbb{Z}^+}$ with a single linear profile V_1, a single nonlinear profile U_1 and a sequence of triples $(\lambda_{1,k}, x_{1,k}, t_{1,k})$ such that

(a) $(n_{0,n_k}, u_{1,n_k}) = (U_{1,k}(\cdot,0), \partial_t U_{1,k}(\cdot,0)) + (\tilde{w}_{0,k}, \tilde{w}_{1,k})$;

(b) $\limsup_{k\to\infty} \|\mathbf{S}_{L,0}(t)(\tilde{w}_{0,k}, \tilde{w}_{1,k})\|_{Y(\mathbb{R})} = 0$;

(c) $\lambda_{1,k} \to 1$, $x_{1,k} \to \mathbf{0}$ and $-t_{1,k}/\lambda_{1,k} \to t_1 \in \mathbb{R}$;

(d) $E_\phi(U_1, \partial_t U_1) = M_0$ and $\limsup_{k\to\infty} \|(\tilde{w}_{0,k}, \tilde{w}_{1,k})\|_H < 2\|\nabla W\|_{L^2}$;

(e) By the fact $U_1(\cdot,t_1) = V_1(\cdot,t_1)$, Lemma 3.7 and part (IV) of Proposition 3.11, we have

$$\int_{\mathbb{R}^d} \phi(x) |U_1(x,t_1)|^{2^*} dx = \lim_{k\to\infty} \int_{\mathbb{R}^d} \phi(x) |u_{0,n_k}(x)|^{2^*} dx.$$

Now let us prove that (u_{0,n_k}, u_{1,n_k}) converges to $(U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1))$ strongly in H, as $k \to \infty$. First of all, the condition (c) above implies that

$$(U_{1,k}(\cdot,0), \partial_t U_{1,k}(\cdot,0)) \to (U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1)) \text{ strongly in } H \tag{27}$$

Thus we can substitute Condition (a) above by

$$(n_{0,n_k}, u_{1,n_k}) = (U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1)) + (\tilde{w}_{0,k}, \tilde{w}_{1,k}) \tag{28}$$

Here the remainders $(\tilde{w}_{0,k}, \tilde{w}_{1,k})$ may be different from the original ones, but they still satisfy the same estimates in (b) and (d). According to Lemma 2.23, we know $(\tilde{w}_{0,k}, \tilde{w}_{1,k})$ converges to zero weakly in H. Thus

$$\lim_{k\to\infty} \langle (U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1)), (\tilde{w}_{0,k}, \tilde{w}_{1,k})\rangle_H = 0. \tag{29}$$

In addition, using Condition (e) and the fact $E_\phi(U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1)) = M_0 = E_\phi(n_{0,n_k}, u_{1,n_k})$, we obtain

$$\lim_{k\to\infty} \|(n_{0,n_k}, u_{1,n_k})\|_H^2 = \|(U_1(\cdot,t_1), \partial_t U_1(\cdot,t_1))\|_H^2. \tag{30}$$

Finally we can combine (28), (29) and (30) to conclude that $\lim\limits_{k\to\infty} \|(\tilde{w}_{0,k}, \tilde{w}_{1,k})\|_H \to 0$, which is equivalent to the strong convergence of (n_{0,n_k}, u_{1,n_k}), namely the strong convergence of $(u(\cdot,t_n), \partial_t u(\cdot,t_n))$, in light of (28). □

4.4 Global existence in time

According to Remark 2.15, the almost periodic property guarantees the existence of a constant $T > 0$, so that if t is contained in the maximal lifespan I of u, then $(t-T, t+T) \subseteq I$. This implies $I = \mathbb{R}$. Collecting all the properties of the critical element u we obtained earlier, we can conclude the proof of Proposition 4.1.

5 Rigidity

In this section we show that the critical element obtained in the previous section can never exist, thus finish the proof of the scattering part of our main theorem. There are two cases.

- In the defocusing case, A Morawetz-type estimate is sufficient to finish the job.
- In the focusing case, we follow the same idea as Kenig and Merle used to eliminate the critical element for the equation (CP0) in the paper [14].

5.1 The Defocusing Case: A Morawetz-type Estimate

In this subsection we introduce the following Morawetz-type estimate and use it to "kill" the critical element.

Proposition 5.1 (A Morawetz-type Inequality)**.** *Assume $\phi \in C^1(\mathbb{R}^d)$ satisfies the condition* (1) *and*

$$\eta(x) \doteq \phi(x) - \frac{(d-2)x\cdot\nabla\phi(x)}{2(d-1)} > 0, \qquad x \in \mathbb{R}^d$$

Let u be a solution to the Cauchy problem (CP1) in the defocusing case with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ and a maximal lifespan $(-T_-, T_+)$. Then we have

$$\int_{-T_-}^{T_+} \int_{\mathbb{R}^d} \eta(x) \frac{|u|^{2^*}}{|x|} dx dt \leq \frac{2d}{d-1} E_\phi(u_0, u_1).$$

The idea of proof The main idea is to choose a suitable function $a(x)$ and then apply the informal computation

$$\begin{aligned} & -\frac{d}{dt}\int_{\mathbb{R}^d} u_t \left(\nabla a\cdot\nabla u + u\cdot\frac{\Delta a}{2}\right) dx \\ =& \int_{\mathbb{R}^d} \left[\nabla u\cdot \mathbf{D}^2 a\cdot(\nabla u)^T\right] dx - \frac{1}{4}\int_{\mathbb{R}^d} \left(|u|^2 \Delta\Delta a\right) dx + \int_{\mathbb{R}^d} \left(\frac{1}{d}\phi\Delta a - \frac{1}{2^*}\nabla\phi\cdot\nabla a\right)|u|^{2^*} dx, \end{aligned}$$

for a solution u to (CP1) in the defocusing case. In order to obtain a Morawetz inequality, the same idea has been used in [23] for the defocusing wave equation with a pure power-type nonlinearity and in [28] for the defocusing shifted wave equation on the hyperbolic spaces. Here we choose the function $a(x) = |x| = r$, which satisfies

$$\nabla a = \frac{x}{r}; \qquad \Delta a = \frac{d-1}{r}; \qquad \mathbf{D}^2 a \geq 0; \qquad \nabla \Delta a = -\frac{(d-1)x}{r^3}; \qquad \Delta\Delta a \leq 0.$$

Since the original solution does not necessarily possess sufficient smoothness, we need to apply some smoothing and cut-off techniques. Please see [23, 28] for more details on this argument.

Nonexistence of a critical element Applying Proposition 5.1, we obtain a global integral estimate

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^d} \eta(x) \frac{|u|^{2^*}}{|x|} dxdt \leq \frac{2d}{d-1} E_\phi(u_0, u_1) < \infty$$

for the critical element u. However, the almost periodicity implies that the integral above should have been infinite, as shown in the following lemma. This gives us a contradiction and finishes the proof in the defocusing case.

Lemma 5.2. *Assume that $\beta(x)$ is a positive measurable function defined on $\mathbb{R}^d$. Let u be a nontrivial solution to the equation (CP1) defined for all $t \in \mathbb{R}$, so that the set $\{(u(\cdot,t), \partial_t u(\cdot,t)) : t \in \mathbb{R}\}$ is precompact in the space $\dot{H}^1 \times L^2(\mathbb{R}^d)$. Then for any given $\tau > 0$, there exists a positive constant δ_1, such that the following inequalities hold for any $t_0 \in \mathbb{R}$.*

$$\int_{t_0}^{t_0+\tau} \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dxdt \geq \delta_1, \qquad \int_{t_0}^{t_0+\tau} \int_{\mathbb{R}^d} \beta(x)|u|^{2^*} dxdt \geq \delta_1.$$

Proof. If the lemma were false for some $\tau > 0$, then there would exist a sequence of time $\{t_n\}_{n\in\mathbb{Z}^+}$ such that

$$\int_{t_n}^{t_n+\tau} \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dxdt < 2^{-n} \qquad \text{or} \qquad \int_{t_n}^{t_n+\tau} \int_{\mathbb{R}^d} \beta(x)|u(x,t)|^{2^*} dxdt < 2^{-n}. \tag{31}$$

By the almost periodicity we know the sequence $\{(u(\cdot,t_n), u_t(\cdot,t_n))\}_{n\in\mathbb{Z}^+}$ converges to some pair (v_0, v_1) strongly in the space $\dot{H}^1 \times L^2(\mathbb{R}^d)$. Let v be the corresponding solution to (CP1) with initial data (v_0, v_1) and $\tau_1 \in (0, \tau]$ be a small time contained in the lifespan of v. Thus we have $\|v\|_{Y([0,\tau_1])} < \infty$. Applying the long-time perturbation theory on the solution v, the time interval $[0, \tau_1]$ and the initial data $(u(\cdot,t_n), u_t(\cdot,t_n))$, we obtain that

$$\lim_{n\to\infty} \sup_{t\in[0,\tau_1]} \|u(\cdot, t_n + t) - v(\cdot,t)\|_{\dot{H}^1(\mathbb{R}^d)} = 0, \qquad \lim_{n\to\infty} \|u(\cdot, \cdot + t_n) - v\|_{Y([0,\tau_1])} = 0.$$

Combining this with our assumption (31), we obtain

$$\int_0^{\tau_1} \int_{\mathbb{R}^d} |\nabla v(x,t)|^2 dxdt = 0 \qquad \text{or} \qquad \int_0^{\tau_1} \int_{\mathbb{R}^d} \beta(x)|v(x,t)|^{2^*} dxdt = 0.$$

In either case we have $v(x,t) = 0$ for $t \in [0, \tau_1]$. This means that $(v_0, v_1) = (0,0)$ and $\|(u(\cdot,t_n), u_t(\cdot,t_n))\|_{\dot{H}^1\times L^2} \to 0$. The conservation law of energy immediately gives the energy $E_\phi(u, \partial_t u) = 0$. Finally we apply Remark 2.19 and conclude $u \equiv 0$. This gives a contradiction. □

5.2 The Focusing Case

The idea is to show the derivative

$$\frac{d}{dt}\left[\int_{\mathbb{R}^d}(x\cdot\nabla u)u_t\varphi_R dx+\frac{d}{2}\int_{\mathbb{R}^d}uu_t\varphi_R dx\right]$$

has a negative upper bound but the integral itself is bounded, which gives a contradiction when we consider a long time interval. Here φ_R is a cut-off function defined below and the parameter R is to be determined. It is necessary to apply the cut-off techniques here since the functions $(x\cdot\nabla u)u_t$ and uu_t may not be integrable in the whole space.

Definition 5.3 (Cut-off function)**.** *Let us fix a radial, smooth, nonnegative cut-off function $\varphi:\mathbb{R}^d\to[0,1]$ satisfying*

$$\varphi(x)=\begin{cases}1, & \text{if } |x|<1;\\ 0, & \text{if } |x|>2;\end{cases}$$

and define its rescaled version

$$\varphi_R(x)=\varphi(x/R).$$

Definition 5.4. *If $R>0$, then we define*

$$\kappa(R)=\sup_{t\in\mathbb{R}}\int_{|x|>R}\left(|u_t(x,t)|^2+|\nabla u(x,t)|^2+\frac{|u(x,t)|^2}{|x|^2}+|u(x,t)|^{2^*}\right)dx$$

Lemma 5.5. *Let u be a critical element as in Proposition 4.1. Then $\kappa(R)$ is bounded and converges to zero as $R\to\infty$.*

Proof. This is a direct corollary of the pre-compactness of $\{(u(\cdot,t),\partial_t u(\cdot,t))|t\in\mathbb{R}\}$ and the Hardy Inequality. □

Lemma 5.6 (Hardy Inequality)**.** *Assume $f\in\dot{H}^1(\mathbb{R}^d)$. We have*

$$\left(\int_{\mathbb{R}^d}\frac{|f(x)|^2}{|x|^2}dx\right)^{1/2}\lesssim\|f\|_{\dot{H}^1(\mathbb{R}^d)}.$$

Lemma 5.7 (Calculation of Derivatives)**.** *Fix $R>0$ and let φ_R be the cut-off function as in Definition 5.3. We have the following derivatives in t*

$$\begin{aligned}\frac{d}{dt}\left[\int_{\mathbb{R}^d}(x\cdot\nabla u)u_t\varphi_R dx\right]&=-\frac{d}{2}\int_{\mathbb{R}^d}|u_t|^2dx+\frac{d-2}{2}\int_{\mathbb{R}^d}\left(|\nabla u|^2-\phi|u|^{2^*}\right)dx\\&\quad-\frac{1}{2^*}\int(\nabla\phi\cdot x)|u|^{2^*}\varphi_R\,dx+O(\kappa(R));\\ \frac{d}{dt}\left[\int_{\mathbb{R}^d}\varphi_R uu_t dx\right]&=\int_{\mathbb{R}^d}|u_t|^2dx-\int_{\mathbb{R}^d}\left(|\nabla u|^2-\phi|u|^{2^*}\right)dx+O(\kappa(R)).\end{aligned}$$

Here $O(\kappa(R))$ represents an error term that can be dominated by a constant multiple of $\kappa(R)$.

Proof. Let us assume that u is a smooth solution to (CP1). Otherwise we can apply basic smoothing techniques. The idea is to apply integration by parts

$$\begin{aligned}
\frac{d}{dt}\left[\int_{\mathbb{R}^d}(x\cdot\nabla u)u_t\varphi_R dx\right] =&\int_{\mathbb{R}^d}(x\cdot\nabla u_t)u_t\varphi_R dx+\int_{\mathbb{R}^d}(x\cdot\nabla u)u_{tt}\varphi_R dx\\
=&\frac{1}{2}\int_{\mathbb{R}^d}\varphi_R x\cdot\nabla(|u_t|^2)dx+\int_{\mathbb{R}^d}(x\cdot\nabla u)(\Delta u+\phi|u|^{4/(d-2)}u)\varphi_R dx\\
=&-\frac{d}{2}\int_{\mathbb{R}^d}\varphi_R|u_t|^2dx-\frac{1}{2}\int_{\mathbb{R}^d}(\nabla\varphi_R\cdot x)|u_t|^2dx\\
&-\int_{\mathbb{R}^d}\nabla(\varphi_R x\cdot\nabla u)\cdot\nabla u dx+\frac{1}{2^*}\int_{\mathbb{R}^d}\varphi_R\phi x\cdot\nabla(|u|^{2^*})dx\\
=&-\frac{d}{2}\int_{\mathbb{R}^d}|u_t|^2dx+O(\kappa(R))+I_1+I_2 \qquad (32)
\end{aligned}$$

In the calculation below, we let u_i, u_{ij} represent the derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_j\partial x_i}$, respectively. We have

$$\begin{aligned}
I_1=&-\int_{\mathbb{R}^d}\nabla(\varphi_R x\cdot\nabla u)\cdot\nabla u dx\\
=&-\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\left(\frac{\partial\varphi_R}{\partial x_j}x_iu_iu_j+\varphi_R\delta_{ij}u_iu_j+\varphi_R x_iu_{ij}u_j\right)dx\\
=&-\int_{\mathbb{R}^d}\varphi_R|\nabla u|^2dx-\frac{1}{2}\int_{\mathbb{R}^d}\varphi_R x\cdot\nabla(|\nabla u|^2)dx+O(\kappa(R))\\
=&-\int_{\mathbb{R}^d}|\nabla u|^2dx+\frac{d}{2}\int_{\mathbb{R}^d}\varphi_R|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^d}(x\cdot\nabla\varphi_R)|\nabla u|^2dx+O(\kappa(R))\\
=&\frac{d-2}{2}\int_{\mathbb{R}^d}|\nabla u|^2dx+O(\kappa(R))
\end{aligned}$$

$$\begin{aligned}
I_2=&\frac{1}{2^*}\int_{\mathbb{R}^d}\varphi_R\phi x\cdot\nabla(|u|^{2^*})dx\\
=&-\frac{d}{2^*}\int_{\mathbb{R}^d}\varphi_R\phi|u|^{2^*}dx-\frac{1}{2^*}\int_{\mathbb{R}^d}(\nabla\phi\cdot x)\varphi_R|u|^{2^*}dx-\frac{1}{2^*}\int_{\mathbb{R}^d}(\nabla\varphi_R\cdot x)\phi|u|^{2^*}dx\\
=&-\frac{d-2}{2}\int_{\mathbb{R}^d}\phi|u|^{2^*}dx-\frac{1}{2^*}\int_{\mathbb{R}^d}(\nabla\phi\cdot x)\varphi_R|u|^{2^*}dx+O(\kappa(R)).
\end{aligned}$$

Plugging I_1, I_2 into (32), we finish the calculation of the first derivative. The second derivative can be dealt with in the same manner:

$$\begin{aligned}
\frac{d}{dt}\left[\int_{\mathbb{R}^d}\varphi_R uu_t dx\right]=&\int_{\mathbb{R}^d}\varphi_R|u_t|^2dx+\int_{\mathbb{R}^d}\varphi_R uu_{tt}dx\\
=&\int_{\mathbb{R}^d}|u_t|^2dx+\int_{\mathbb{R}^d}\varphi_R u(\Delta u+\phi|u|^{4/(d-2)}u)dx+O(\kappa(R))\\
=&\int_{\mathbb{R}^d}|u_t|^2dx-\int_{\mathbb{R}^d}\varphi_R|\nabla u|^2dx+\int_{\mathbb{R}^d}\varphi_R\phi|u|^{2^*}dx+O(\kappa(R))\\
=&\int_{\mathbb{R}^d}|u_t|^2dx-\int_{\mathbb{R}^d}\left(|\nabla u|^2-\phi|u|^{2^*}\right)dx+O(\kappa(R)).
\end{aligned}$$

□

Nonexistence of a critical element Now we can show that a critical element does not exist in the focusing case. Consider the function ($R > 0$)

$$G_R(t) = \int_{\mathbb{R}^d} (x \cdot \nabla u(x,t)) u_t(x,t) \varphi_R dx + \frac{d}{2} \int_{\mathbb{R}^d} \varphi_R u(x,t) u_t(x,t) dx.$$

Applying Lemma 5.7, we obtain

$$\begin{aligned} G_R'(t) &= - \int_{\mathbb{R}^d} \left(|\nabla u|^2 - \phi |u|^{2^*} \right) dx - \frac{1}{2^*} \int (\nabla \phi \cdot x) |u|^{2^*} \varphi_R \, dx + O(\kappa(R)) \\ &= - \int_{\mathbb{R}^d} \left(|\nabla u|^2 - |u|^{2^*} \right) dx - \int_{\mathbb{R}^d} (1-\phi) |u|^{2^*} \varphi_R dx - \frac{1}{2^*} \int (\nabla \phi \cdot x) |u|^{2^*} \varphi_R \, dx + O(\kappa(R)) \\ &\leq - C(E) \int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^d} [2^*(1-\phi) + (\nabla \phi \cdot x)] |u|^{2^*} \varphi_R \, dx + O(\kappa(R)) \qquad (33) \end{aligned}$$

In the last step above, we apply Proposition 2.18. The positive constant $C(E)$ only depends on the energy $E = E_\phi(u, u_t)$, but not on t or R. According to our assumption that $2^*(1-\phi(x)) + (x \cdot \nabla \phi(x)) \geq 0$ holds for all $x \in \mathbb{R}^d$, the integrand of the second integral in (33) is always nonnegative. Thus we have

$$G_R'(t) \leq -C(E) \int_{\mathbb{R}^d} |\nabla u|^2 dx + O(\kappa(R))$$

for all $R > 0$ and $t \in \mathbb{R}$. Fix $\tau > 0$ and let δ_1 be the constant in Lemma 5.2. Since $\lim_{R \to \infty} \kappa(R) = 0$, we can fix a large R so that

$$G_R'(t) \leq -C(E) \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{C(E)\delta_1}{2\tau}, \qquad \text{for any } t \in \mathbb{R}.$$

Integrating both sides from $t = 0$ to $t = n\tau$ for an positive integer n and applying Lemma 5.2, we obtain

$$G_R(n\tau) - G_R(0) \leq -C(E) \int_0^{n\tau} \int_{\mathbb{R}^d} |\nabla u|^2 dx dt + \frac{C(E)\delta_1}{2\tau} \cdot n\tau \leq -\frac{C(E)\delta_1 \cdot n}{2}.$$

This is impossible as $n \to \infty$, because $|G_R(t)|$ has a uniform upper bound for all t if we fix R.

$$|G_R(t)| \lesssim R \int_{\mathbb{R}^d} \left(|\nabla u(x,t)|^2 + |u_t(x,t)|^2 + \frac{|u(x,t)|^2}{|x|^2} \right) dx \lesssim RE.$$

Here we need to use Hardy's inequality.

5.3 Three dimensional Case with Radial Data

If $d = 3$, then we can substitute the assumptions (4) and (5) in the main theorems with the radial assumptions on ϕ and initial data (u_0, u_1). The complete proof will be pretty long but follow the same "channel of energy" method introduced in [4], thus we only give important statements here but omit the details.

Step 1: Existence and regularity of "ground states" Section 6 of the author's recent paper [31] gives the following results about the radial solution to the elliptic equation $-\Delta W = \zeta \phi(x) |W|^{p-1} W$.

Proposition 5.8. *Given any constant A, there exist a radius $R = R(\zeta\phi, A) \geq 0$ and a radial solution $W_A \in C^2(\mathbb{R}^3 \setminus \bar{B}(0,R))$ to the elliptic equation $-\Delta W = \zeta \phi |W|^4 W$, such that*

(a) *The behaviour of* $W_A(x)$ *as* $|x|\to\infty$ *is characterized by*

$$\left|W_A(x)-\frac{A}{|x|}\right|\lesssim\frac{1}{|x|^3}, \qquad\qquad |\nabla W_A(x)|\lesssim\frac{1}{|x|^2}.$$

(b) *If* $R(\zeta\phi,A)>0$*, then* $\limsup_{|x|\to R(\zeta\phi,A)^+}|W_A(x)|=+\infty.$

Proposition 5.9. *Let* $W\in C^2(\mathbb{R}^3\setminus\{0\})$ *be a radial solution to the elliptic equation* $-\Delta W=\zeta\phi(x)|W|^4W$ *so that*

$$\lim_{|x|\to 0^+}|x|^{1/2}|W(x)|=0.$$

Then we can extend the domain of W *to the whole space* $\mathbb{R}^3$ *by continuity so that* $W\in C^2(\mathbb{R}^3)$ *gives a classic solution to the elliptic equation.*

Step 2: The coincidence of a critical element and a ground state Following the same "channel of energy" argument as given in [4], we obtain

(1) There exists a constant $A\neq 0$, so that the behaviour of the critical element u near infinity is characterized by

$$\left|u(x,t)-\frac{A}{|x|}\right|\lesssim\frac{1}{|x|^3}.$$

(2) The critical element u coincides with the ground state W_A given in Proposition 5.8 when $|x|>R(\zeta\phi,A)$, i.e. we have

$$u(x,t)=W_A(x),\quad \text{for all } t\in\mathbb{R} \text{ and } |x|>R(\zeta\phi,A).$$

Step 3: C^2 regularity of the ground state $W_A(x)$ Fix a time t. Since $u(\cdot,t)\in\dot H^1(\mathbb{R}^d)$ is a radial function, we have (see Lemma 3.2 of [16] and appendix of [28])

$$|u(x,t)|\lesssim\frac{\|u(\cdot,t)\|_{\dot H^1}}{|x|^{1/2}}; \qquad\qquad \lim_{|x|\to 0}|x|^{1/2}\,|u(x,t)|=0.$$

Combining the first inequality and conclusion (b) of Proposition 5.8, we conclude that $R(\zeta\phi,A)=0$. Therefore we can substitute $u(x,t)$ with $W_A(x)$ in the second inequality above and apply Proposition 5.9 to conclude $W_A(x)\in C^2(\mathbb{R}^d)$.

Step 4: Contradiction Multiplying both sides of the equation $-\Delta W=\zeta\phi|W|^4W$ by W and applying integration by parts, we obtain an identity

$$\int_{\mathbb{R}^d}|\nabla W_A(x)|^2dx=\zeta\int_{\mathbb{R}^d}\phi(x)|W_A(x)|^6dx. \tag{34}$$

This immediately gives a contradiction in the defocusing case $\zeta=-1$. In the focusing case we have $\|\nabla W_A\|_{L^2}=\|\nabla u(\cdot,0)\|_{L^2}<\|\nabla W\|_{L^2}$ by the third property of the critical element given in Proposition 4.4. Therefore the energy trapping (Remark 2.19) applies

$$\int_{\mathbb{R}^d}|\nabla W_A(x)|^2dx>\int_{\mathbb{R}^d}\phi(x)|W_A(x)|^6dx. \tag{35}$$

A comparison of the identity (34) and the inequality (35) gives a contradiction.

6 Finite Time Blow-up

In this section, we prove the second part of my main theorem. Namely, if $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and $E_\phi(u_0, u_1) < E_1(W, 0)$, then the corresponding solution to (CP1) in the focusing case blows up within finite time in both two time directions. Since our argument here is similar to the one used in section 7 of [14], we will omit some details.

The idea If the initial data $u_0 \in L^2(\mathbb{R}^d)$, then the blow-up of the solution u can be proved by considering the function $y(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx$ and showing that this function has to blow up in finite time. In the general case, we have to use a cut-off technique. Let us assume $T_+(u_0, u_1) = +\infty$ and show a contradiction. Applying integration by parts and smoothing approximation techniques, we have

Lemma 6.1. *Let $\varphi_R(x)$ be the cut-off function as given in Definition 5.3. If we define $y_R(t) = \int_{\mathbb{R}^d} |u(x,t)|^2 \varphi_R(x) dx$, then*

$$\begin{aligned} y_R'(t) &= 2\int_{\mathbb{R}^d} u_t(x,t) u(x,t) \varphi_R(x) dx \\ y_R''(t) &= -\frac{4d}{d-2} \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2 - \frac{1}{2^*}\phi |u|^{2^*} \right) \varphi_R dx + \frac{4(d-1)}{d-2} \int_{\mathbb{R}^d} |u_t|^2 \varphi_R dx \\ &\quad + \frac{4}{d-2} \int_{\mathbb{R}^d} |\nabla u|^2 \varphi_R dx - 2 \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \varphi_R) u dx. \end{aligned}$$

Tail Estimate For any given initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$ we always have

$$\lim_{R\to\infty} \|((1-\varphi_R(x))u_0, (1-\varphi_R(x))u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} = 0.$$

Thus for any $\varepsilon > 0$, there exists a number $R_0 = R_0(\varepsilon)$, such that

$$\|((1-\varphi_{R_0}(x))u_0, (1-\varphi_{R_0}(x))u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} < \varepsilon.$$

When ε is sufficiently small, our local theory guarantees that the solution u_{R_0} to (CP1) with initial data $((1-\varphi_{R_0}(x))u_0, (1-\varphi_{R_0}(x))u_1)$ exists globally in time and scatters with

$$\sup_{t\in\mathbb{R}} \|(u_{R_0}(\cdot,t), \partial_t u_{R_0}(\cdot,t))\|_{\dot{H}^1 \times L^2(\mathbb{R}^d)} < 2\varepsilon.$$

As a result, we have the following estimate for each $t \in R$:

$$\int_{\mathbb{R}^d} \left(|\nabla u_{R_0}(x,t)|^2 + |\partial_t u_{R_0}(x,t)|^2 + |u_{R_0}(x,t)|^{2^*} + \frac{|u_{R_0}(x,t)|^2}{|x|^2} \right) dx \leq C\varepsilon^2.$$

Here $C > 1$ is a constant depending only on the dimension d. Since our center-cutoff version of initial data $((1-\varphi_{R_0}(x))u_0, (1-\varphi_{R_0}(x))u_1)$ remain the same as the original initial data (u_0, u_1) in the region $\{x : |x| \geq 2R_0\}$, finite speed of propagation immediately gives

$$\int_{|x|>2R_0(\varepsilon)+|t|} \left(|\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 + |u(x,t)|^{2^*} + \frac{|u(x,t)|^2}{|x|^2} \right) dx \leq C\varepsilon^2 \qquad (36)$$

Proof of the blow-up part If $R > 2R_0(\varepsilon)$ and $t \in [0, R - 2R_0(\varepsilon)]$, then we can combine Lemma 6.1 with the tail estimate (36) and obtain

$$y_R''(t) = -\frac{4d}{d-2} E_\phi(u, u_t) + \frac{4(d-1)}{d-2} \int_{\mathbb{R}^d} |u_t|^2 \varphi_R dx + \frac{4}{d-2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + O(\varepsilon^2).$$

We have already known $E_\phi(u,u_t) = E_\phi(u_0,u_1) < E_1(W,0)$. In addition, we claim that the inequality $\|\nabla u(\cdot,t)\|_{L^2} \geq \|\nabla W\|_{L^2}$ holds for all $t \geq 0$. Otherwise we would have $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ by Lemma 2.18. Using these inequalities and the identity $\|\nabla W\|_{L^2}^2 = d \cdot E_1(W,0)$ we can find a lower bound of the second derivative

$$\begin{aligned} y_R''(t) \geq & \frac{4d}{d-2}\left(E_1(W,0) - E_\phi(u_0,u_1)\right) + \frac{4(d-1)}{d-2}\int_{\mathbb{R}^d} |u_t|^2 \varphi_R dx \\ & + \frac{4}{d-2}\left(\int_{\mathbb{R}^d} |\nabla W|^2 dx - d\cdot E_1(W,0)\right) + O(\varepsilon^2) \\ \geq & \delta + \frac{4(d-1)}{d-2}\int_{\mathbb{R}^d} |u_t|^2 \varphi_R dx - C_1\varepsilon^2. \end{aligned}$$

Here the constant C_1 is determined solely by the dimension d while δ can be arbitrarily chosen in the interval $\left(0, \frac{4d}{d-2}\left(E_1(W,0) - E_\phi(u_0,u_1)\right)\right)$. Let us fix $\delta \ll 1$ and $\varepsilon = \delta^2$ so that

$$\delta - C_1\varepsilon^2 = \delta - C_1\delta^4 > \delta/2, \qquad \delta < \min\left\{C, \frac{1}{100C}, \frac{4d}{d-2}\left(E_1(W,0) - E_\phi(u_0,u_1)\right)\right\} \tag{37}$$

Here C is the constant in the inequality (36). As a result we have if $R > 2R_0(\delta^2)$ and $t \in [0, R - 2R_0(\delta^2)]$, then

$$y_R''(t) \geq \frac{\delta}{2} + \frac{4(d-1)}{d-2}\int_{\mathbb{R}^d} |u_t|^2\varphi_R dx; \quad \Longrightarrow \quad y_R''(t)y_R(t) \geq \frac{d-1}{d-2}[y_R'(t)]^2. \tag{38}$$

In addition, we have the following estimates on $y_R(0)$ and $|y_R'(0)|$. In the integrals below Ω represents the region $\{x : 2R_0(\delta^2) < |x| < 2R\}$.

$$\begin{aligned} y_R(0) \leq & \int_{|x|\leq 2R_0(\delta^2)} |u_0|^2 dx + 4R^2\int_\Omega \frac{|u_0|^2}{|x|^2}dx \leq \int_{|x|\leq 2R_0(\delta^2)} |u_0|^2 dx + 4C\delta^4R^2; \\ |y_R'(0)| \leq & 2\int_{|x|\leq 2R_0(\delta^2)} |u_0|\cdot|u_1|\,dx + 2\left(\int_\Omega |u_1|^2 dx\right)^{1/2}\left(4R^2\int_\Omega \frac{|u_0|^2}{|x|^2}dx\right)^{1/2} \\ \leq & 2\int_{|x|\leq 2R_0(\delta^2)} |u_0|\cdot|u_1|\,dx + 4C\delta^4 R. \end{aligned}$$

As a result, we always have the following estimates for sufficiently large $R > R_1$:

$$y_R(0) < 5C\delta^4R^2; \qquad |y_R'(0)| < 5C\delta^4 R; \qquad R > 100R_0(\delta^2). \tag{39}$$

Let us consider $t_0(R) \doteq \min\{t : 0 \leq t \leq 12C\delta R,\ y_R'(t) \geq C\delta^2 R\}$ for such a radius R. This is well-defined since we know $12C\delta R < 12R/100 < R - 2R_0(\delta^2)$ and

$$y_R'(12C\delta R) \geq y_R'(0) + 12C\delta R \cdot \inf_{0\leq t\leq 12C\delta R} y_R''(t) \geq -5C\delta^4 R + 12C\delta R\cdot\frac{\delta}{2} \geq C\delta^2 R.$$

In addition we have

$$y_R(t_0(R)) \leq y_R(0) + t_0(R)\cdot \max_{0\leq t\leq t_0(R)} y_R'(t) \leq 5C\delta^4R^2 + 12C\delta R\cdot C\delta^2 R \leq 17C^2\delta^3R^2. \tag{40}$$

Now let us define $z_R(t) = \dfrac{y_R'(t)}{y_R(t)}$ for $t \in [t_0(R), R - 2R_0(\delta^2)]$. The function $z_R(t)$ is always positive. By the estimate (40) and the definition of $t_0(R)$, we have $z(t_0(R)) \geq \dfrac{C\delta^2 R}{17C^2\delta^3R^2} = \dfrac{1}{17C\delta R}$. Combining basic differentiation and the estimate (38), we have

$$z_R'(t) = \frac{y_R''(t)y_R(t) - [y_R'(t)]^2}{y_R^2(t)} \geq \frac{1}{d-2}\left[\frac{y_R'(t)}{y_R(t)}\right]^2 = \frac{1}{d-2}z_R^2(t).$$

for any $t \in [t_0(R), R - 2R_0(\delta^2)]$. Dividing both sides by $z_R^2(t)$ and integrating in t, we obtain

$$\frac{1}{z(t_0(R))} - \frac{1}{z(R - 2R_0(\delta^2))} \geq \frac{1}{d-2}\left[(R - 2R_0(\delta^2)) - t_0(R)\right]$$

Using the upper bound of $t_0(R)$, the lower bound of $z(t_0(R))$ and the choice of R, we have

$$17C\delta R \geq \frac{1}{z(t_0(R))} > \frac{1}{d-2}\left[R - \frac{R}{50} - 12C\delta R\right] \quad \Longrightarrow \quad [17(d-2) + 12]C\delta R > \frac{49}{50}R.$$

This contradicts our choice of δ, please see (37).

7 Application on a Shifted Wave Equation on $\mathbb{H}^3$

In this section we consider the radial solutions to an energy-critical, focusing, semilinear shifted wave equation on the hyperbolic space $\mathbb{H}^3$

$$\begin{cases} \partial_t^2 v - (\Delta_{\mathbb{H}^3} + 1)v = |v|^4 v, & (y,t) \in \mathbb{H}^3 \times \mathbb{R}; \\ v(\cdot, 0) = u_0 \in H^{0,1}(\mathbb{H}^3); \\ \partial_t v(\cdot, 0) = u_1 \in L^2(\mathbb{H}^3); \end{cases} \tag{41}$$

as one application of our main theorem.

7.1 Background and the Space of Functions

Model of hyperbolic space There are various models for the Hyperbolic space $\mathbb{H}^3$. We select the hyperboloid model. Let us consider the Minkowswi space $\mathbb{R}^{1+3}$ equipped with the standard Minkowswi metric $-(dx^0)^2 + (dx^1)^2 + \cdots + (dx^3)^2$ and the bilinear form $[x, y] = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$. The hyperbolic space $\mathbb{H}^3$ can be defined as the hyperboloid $x_0^2 - x_1^2 - x_2^2 - x_3^2 = 1$ whose metric, covariant derivatives and measure μ are induced by the Minkowswi metric.

Radial Functions We can introduce polar coordinates (r, Θ) on the hyperbolic space $\mathbb{H}^3$. More precisely, we use the pair $(r, \Theta) \in [0, \infty) \times \mathbb{S}^2$ to represent the point $(\cosh r, \Theta \sinh r) \in \mathbb{R}^{1+3}$ in the hyperboloid model above. One can check that the r coordinate of a point in $\mathbb{H}^3$ represents the distance from that point to the "origin" $\mathbf{0} \in \mathbb{H}^n$, which is the point $(1, 0, 0, 0)$ in the Minkiwski space. In terms of the polar coordinate, the measure and Laplace operator can be given by

$$d\mu = \sinh^2 r \, dr d\Theta; \qquad \Delta_{\mathbb{H}^3} = \partial_r^2 + 2\coth r \cdot \partial_r + \sinh^{-2} r \cdot \Delta_{\mathbb{S}^2}$$

Here $d\Theta$ corresponds the usual unnormalized measure on the sphere $\mathbb{S}^2$. As in Euclidean spaces, for any $y \in \mathbb{H}^3$ we also use the notation $|y|$ for the distance from y to $\mathbf{0}$. Namely

$$r = |y| = d(y, \mathbf{0}), \qquad y \in \mathbb{H}^n.$$

A function f defined on $\mathbb{H}^3$ is radial if it is independent of Θ. By convention we can use the notation $f(r)$ to mention a radial function f.

Function Space The homogenous Sobolev space $H^{0,1}(\mathbb{H}^3)$, which is the counterpart of $\dot{H}^1(\mathbb{R}^d)$ in the hyperbolic space $\mathbb{H}^3$, is defined by

$$H^{0,1}(\mathbb{H}^3) = (-\Delta_{\mathbb{H}^3} - 1)^{-1/2} L^2(\mathbb{H}^3) \qquad \|u\|_{H^{0,1}(\mathbb{R}^3)} = \|(-\Delta_{\mathbb{H}^3} - 1)^{1/2} u\|_{L^2(\mathbb{H}^3)}$$

If $f \in C_0^\infty(\mathbb{H}^3)$, then its $H^{0,1}(\mathbb{H}^3)$ norm can also be given by $(|\nabla f| = (\mathbf{D}_\alpha f \mathbf{D}^\alpha f)^{1/2})$

$$\|f\|_{H^{0,1}(\mathbb{R}^3)} = \int_{\mathbb{H}^3} \left(|\nabla f(y)|^2 - |f(y)|^2\right) d\mu(y). \tag{42}$$

Please pay attention that the spectrum of the Laplace operator $-\Delta_{\mathbb{H}^3}$ is $[1, \infty)$, which is much different from that of the Laplace operator on $\mathbb{R}^d$. As a result, the integral above is always nonnegative.

Sobolev Embedding As in Euclidean Spaces, we have the Sobolev embedding $H^{0,1}(\mathbb{H}^3) \hookrightarrow L^6(\mathbb{H}^3)$. (Please see [29] for more details.) This implies that the energy

$$E(v, \partial_t v) = \frac{1}{2}\|v\|^2_{\dot{H}^{0,1}(\mathbb{H}^3)} + \frac{1}{2}\|\partial_t v\|^2_{L^2(\mathbb{H}^3)} - \frac{1}{6}\|v\|^6_{L^6(\mathbb{H}^3)} = E(v_0, v_1) \tag{43}$$

is a finite constant as long as $(v_0, v_1) \in \dot{H}^{0,1} \times L^2(\mathbb{H}^3)$.

Local Theory Both the Strichartz estimates and local theory have been discussed in the author's recent work [30]. Generally speaking, the local theory is similar to that of a wave equation on the Euclidean space. Given any initial data $(v_0, v_1) \in \dot{H}^{0,1} \times L^2(\mathbb{H}^3)$, there is a unique solution v defined on a maximal interval of time I, such that $(v, \partial_t v) \in C(I; H^{0,1} \times L^2(\mathbb{H}^3))$ and the inequality $\|v\|_{L^5 L^{10}(J \times \mathbb{H}^3)} < \infty$ holds for any bounded closed subinterval J of I.

7.2 A transformation

Let us consider the transformation $\mathbf{T} : L^2(\mathbb{H}^3) \to L^2(\mathbb{R}^3)$ defined by

$$(\mathbf{T}f)(r, \Theta) = \frac{\sinh r}{r} f(r, \Theta).$$

Here $(r, \Theta) \in [0, \infty) \times \mathbb{S}^2$ represents the polar coordinates, in either the hyperbolic space $\mathbb{H}^3$ or the Euclidean space $\mathbb{R}^3$. This transformation has been known for many years. (See, for instance, V. Pierfelice's work [24] in 2008.) It is trivial to check that the transformation $\mathbf{T}$ is an isometry from $L^2(\mathbb{H}^3)$ to $L^2(\mathbb{R}^3)$. In addition, the transformation $\mathbf{T}$ is also an isometry from $H^{0,1}_{rad}(\mathbb{H}^3)$ to $\dot{H}^1_{rad}(\mathbb{R}^3)$. Here the spaces $H^{0,1}_{rad}(\mathbb{H}^3)$ and $\dot{H}^1_{rad}(\mathbb{R}^3)$ consist of all radial functions in the corresponding Sobolev spaces:

$$H^{0,1}_{rad}(\mathbb{H}^3) = \{f \in H^{0,1}(\mathbb{H}^3) : f \text{ is radial}\} \qquad \dot{H}^1_{rad}(\mathbb{R}^3) = \{f \in \dot{H}^1(\mathbb{R}^3) : f \text{ is radial}\}.$$

This can be observed by using the identity (42) and conducting basic calculations. Furthermore, if f is a radial and smooth function defined on $\mathbb{H}^3$, then one can also verify

$$-\Delta_{\mathbb{R}^3} (\mathbf{T}f) = \mathbf{T}\left[(-\Delta_{\mathbb{H}^3} - 1)f\right].$$

Combining all the facts above, we have

Lemma 7.1. *If $v(y, t)$ is a solution to the equation (41) with initial data $(v_0, v_1) \in H^{0,1}_{rad} \times L^2_{rad}(\mathbb{H}^3)$ and a maximal lifespan I, then $u(\cdot, t) = \mathbf{T}v(\cdot, t)$ is a solution to the equation*

$$\partial_t^2 u - \Delta_{\mathbb{R}^3} u = \phi(x)|u|^4 u, \qquad (x, t) \in \mathbb{R}^3 \times \mathbb{R} \tag{44}$$

with the initial data $(\mathbf{T}v_0, \mathbf{T}v_1)$ and the same maximal lifespan I. Here the coefficient function ϕ is defined by $\phi(x) = \frac{|x|^4}{\sinh^4 |x|}$. In particular, the energy is preserved under this transformation. Namely, the energy of the solution u defined by

$$E_\phi(u, \partial_t u) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla u| + \frac{1}{2}|\partial_t u|^2 - \frac{\phi}{6}|u|^6\right] dx$$

remains the same as the energy $E(v, \partial_t v)$ defined in (43).

7.3 Conclusion

According to Remark 1.4, our main theorem may be applied to the equation (44). Combing our main theorem and Lemma 7.1, we immediately obtain

Theorem 7.2. *Given a pair of initial data $(v_0, v_1) \in H^{0,1}_{rad} \times L^2_{rad}(\mathbb{H}^3)$ with an energy $E(v_0, v_1) < E_1(W, 0)$, the global behaviour, and in particular, the maximal lifespan $I = (-T_-(v_0, v_1), T_+(v_0, v_1))$ of the corresponding solution v to the Cauchy problem (41) can be determined by:*

(I) If $\|v_0\|_{H^{0,1}(\mathbb{H}^3)} < \|\nabla W\|_{L^2(\mathbb{R}^3)}$, then $I = \mathbb{R}$ and v scatters in both time directions.

(II) If $\|v_0\|_{H^{0,1}(\mathbb{H}^3)} > \|\nabla W\|_{L^2(\mathbb{R}^3)}$, then v blows up within finite time in both two directions, namely

$$T_-(v_0, v_1) < +\infty; \qquad T_+(v_0, v_1) < +\infty.$$

Remark 7.3. *A similar argument works in the defocusing case as well and gives the following theorem. Please note that this theorem has already been proved in the author's previous work [30] by an application of a Morawetz-type inequality.*

Theorem 7.4. *Given any initial data $(v_0, v_1) \in H^{0,1}_{rad} \times L^2_{rad}(\mathbb{H}^3)$, the corresponding solution v to the defocusing shifted wave equation*

$$\partial_t^2 v - (\Delta_{\mathbb{H}^3} + 1)v = -|v|^4 v$$

must exist for all time $t \in \mathbb{R}$ and scatter in both two time directions.

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