Context-free Grammars for Permutations and Increasing Trees

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Dedicated to the Memory of Professor Dominique Dumont

Abstract. We introduce the notion of a grammatical labeling to describe a recursive process of generating combinatorial objects based on a context-free grammar. By labeling the ascents and descents of Stirling permutations, we obtain a grammar for the second-order Eulerian polynomials. Using the grammar for 0-1-2 increasing trees given by Dumont, we obtain a grammatical derivation of the generating function of the André polynomials obtained by Foata and Schützenberger. We also find a grammar for the number $T(n,k)$ of permutations of $[n] = \{1,2,\ldots,n\}$ with k exterior peaks. We demonstrate that Gessel's formula for the generating function of $T(n,k)$ can be deduced from this grammar. From a grammatical point of view, it is easily seen that the number of the permutations on $[n]$ with k exterior peaks equals the number of increasing trees on $[n]$ with $2k+1$ vertices of even degree. We present a combinatorial proof of this fact, which is in the spirit of the recursive construction of the correspondence between even increasing trees and up-down permutations, due to Kuznetsov, Pak and Postnikov.

Keywords: Context-free grammar, Eulerian grammar, grammatical labeling, increasing tree, exterior peak of a permutation, Stirling permutation

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1 Introduction

In this paper, a *context-free grammar* G over a set $V = \{x, y, z, \ldots,\}$ of variable is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V. For a context-free grammar G over V, the formal derivative D (introduced in [2]) with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the following relations,

$$D(u+v) = D(u) + D(v),$$

$$D(uv) = D(u)v + uD(v).$$

For a constant c, we have $D(c) = 0$. Clearly, the Leibniz formula is also valid,

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

Since $D(ww^{-1}) = 0$, we have

$$D(w^{-1}) = -\frac{D(w)}{w^2}.$$

A formal derivative D is also associated with an exponential generating function. For a Laurent polynomial w of variables in V, let

$$\mathrm{Gen}(w,t) = \sum_{n\geq 0} D^n(w) \frac{t^n}{n!}.$$

Then we have the following relations

$$\mathrm{Gen}'(w,t) = \mathrm{Gen}(D(w),t), \tag{1.1}$$

$$\mathrm{Gen}(u+v,t) = \mathrm{Gen}(u,t) + \mathrm{Gen}(v,t), \tag{1.2}$$

$$\mathrm{Gen}(uv,t) = \mathrm{Gen}(u,t)\mathrm{Gen}(v,t), \tag{1.3}$$

where u, v and w are Laurent polynomials of variables in V and $\mathrm{Gen}'(w,t)$ stands for the derivative of $\mathrm{Gen}(w,t)$ with respect to t.

To illustrate the connection between context-free grammars and combinatorial structures, we recall the following grammar introduced by Dumont [3],

$$G\colon \quad x \to xy, \quad y \to xy. \tag{1.4}$$

He showed that it generates the *Eulerian polynomials* $A_n(x)$. Let S_n denote the set of permutations on $[n] = \{1, 2, \ldots, n\}$. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, an index $i \in [n-1]$ is called an ascent of π if $\pi_i < \pi_{i+1}$; otherwise, i is called a descent. Let $\mathrm{asc}(\pi)$ be the number of ascents of π and let

$$A_n(x) = \sum_{\pi\in S_n} x^{\mathrm{asc}(\pi)+1}. \tag{1.5}$$

To give a grammatical interpretation of $A_n(x)$, Dumont defined the bivariate polynomials $A_n(x,y)$ based on cyclic permutations. Let C_n denote the set of cyclic permutations on $[n]$. For a cyclic permutation $\sigma \in C_n$, an index i ($1 \leq i \leq n$) is called an ascent if $i < \sigma(i)$ or a descent if $i > \sigma(i)$. Let $\mathrm{asc}_c(\sigma)$ be the number of ascents of σ, and let $\mathrm{des}_c(\sigma)$ be the number of descents of σ. For $n \geq 1$, Dumont [3] defined the polynomial $A_n(x,y)$ as follows,

$$A_n(x,y) = \sum_{\sigma\in C_{n+1}} x^{\mathrm{asc}_c(\sigma)} y^{\mathrm{des}_c(\sigma)}. \tag{1.6}$$

By using the formal derivative D with respect to the grammar in (1.4), Dumont showed that

$$D^n(x) = A_n(x, y),$$

where D is the formal derivative with respect to the grammar G. Setting $y = 1$ in (1.6), we see that for $n \geq 1$,

$$A_n(x, y)|_{y=1} = A_n(x). \tag{1.7}$$

It should be noted that Dumont [3] used the notation $A_{n+1}(x, y)$ instead of $A_n(x, y)$ for the polynomial (1.6). Our notation is chosen to be consistent with the notation used in this paper.

In this paper, we introduce the concept of a grammatical labeling to generate combinatorial structures. This idea is implicit in the partition argument with respect to the grammar $f_i \to f_{i+1}g_1$, $g_i \to g_{i+1}$ to generate partitions as given in [2]. We find grammars for Stirling permutations, partitions into lists, permutations with a given number of exterior peaks, 0-1-2 increasing trees, and increasing trees with parity constraints on degrees. As will be seen, such a grammatical approach is not only useful for the computation of generating functions, but also helpful for finding bijections.

This paper is organized as follows. In Section 2, we give an explanation of relation (1.7) by labeling ascents and descents of a permutation instead of a cyclic permutation. Similarly, we obtain a grammatical interpretation of the second-order Eulerian polynomials. As another example, we give a grammatical explanation of the Lah numbers. We also demonstrate how to use grammar of Dumont to derive an identity on the Eulerian polynomials.

Section 3 is devoted to the applications of the grammar $x \to xy$, $y \to x$ found by Dumont [3] for the André polynomials defined in terms of 0-1-2 increasing trees. We give a grammatical derivation of the generating function of the André polynomials obtained by Foata and Schützenberger [8].

In Section 4, we present a grammatical approach to the number $T(n, k)$ of permutations on $[n]$ with k exterior peaks. We find the following grammar to generate $T(n, k)$:

$$G\colon \quad x \to xy, \quad y \to x^2.$$

This grammar was announced at the International Conference on Designs, Matrices and Enumerative Combinatorics held at the National Taiwan University in 2011. Ma [12] studied the connection between the peak statistics and the relations $D_z(x) = xy$ and $D_z(y) = x^2$, where $x = \sec(z)$, $y = \tan(z)$ and D_z is the derivative with respect to z. We show that Gessel's formula for the generating function of $T(n, k)$ can be deduced from this grammar.

In Section 5, by specializing the following grammar of Dumont [3] for increasing trees,

$$G\colon \quad x_i \to x_0 x_{i+1},$$

we are led to a grammar to generate the number of increasing trees on $[n]$ with respect to the number of vertices with even degree. More precisely, the degree of a vertex in a rooted tree is meant to be the number of its children. As a consequence, we find that the number of permutations of $[n]$ with k exterior peaks equals the number of increasing trees on $[n]$ with $2k+1$ vertices of even degree, and we conclude this paper with a combinatorial proof of this fact, which can be considered as an extension of their bijection given by Kuznetsov, Pak and Postnikov [11] between up-down permutations and even trees.

2 Grammatical Labelings

A *grammatical labeling* of a combinatorial structure is an assignment of the elements of that structure with constants or variables in a grammar. For example, consider the grammar (1.4) given by Dumont [3]. We shall use a grammatical labeling on permutations to show that the Eulerian polynomial $A_n(x)$ can be expressed in terms of the formal derivative with respect to the grammar G.

Denote by $A(n,m)$ the number of permutations of $[n]$ with $m-1$ ascents. The generating function

$$A_n(x) = \sum_{m=1}^{n} A(n,m)x^m$$

is known as the Eulerian polynomial.

It is not difficult to see that $A_n(x,y)$ can also be expressed in terms of permutations in S_n. For a permutation π in S_n, we give a labeling of π as follows. An index i $(1 \leq i \leq n-1)$, is called an ascent if $\pi_i < \pi_{i+1}$, or a descent if $\pi_i > \pi_{i+1}$. Set $\pi_0 = \pi_{n+1} = 0$. For $0 \leq i \leq n$, if $\pi_i < \pi_{i+1}$, we label π_i by x; if $\pi_i > \pi_{i+1}$, we label π_i by y. The weight of π is defined as the product of the labels, that is,

$$w(\pi) = x^{\mathrm{asc}(\pi)+1} y^{\mathrm{des}(\pi)+1},$$

where $\mathrm{asc}(\pi)$ denotes the number of ascents in π and $\mathrm{des}(\pi)$ denotes the number of descents in π. For $n \geq 1$, it can be shown that the polynomial $A_n(x,y)$ defined in (1.6) possesses the following equivalent expression:

$$A_n(x,y) = \sum_{\pi \in S_n} x^{\mathrm{asc}(\pi)+1} y^{\mathrm{des}(\pi)+1}.$$

To illustrate the relation between the action of the formal derivative D and the insertion of the element $n+1$ into a permutation on $[n]$, we give the following example. Let $n=6$ and $\pi = 325641$. The grammatical labeling of π reads

$$\begin{array}{ccccccc} & 3 & 2 & 5 & 6 & 4 & 1 \\ x & y & x & x & y & y & y, \end{array}$$

where the element 0 is made invisible. If we insert 7 after 5, the resulting permutation and its grammatical labeling are given below,

$$\begin{array}{cccccccc} & 3 & 2 & 5 & 7 & 6 & 4 & 1 \\ x & y & x & x & y & y & y & y. \end{array}$$

Notice that the insertion of 7 after 5 corresponds to applying the rule $x \to xy$ to the label x associated with 5. We have a similar situation when the new element is inserted after an element labeled by y. Hence the action of the formal derivative D on the set of weights of permutations in S_n gives the set of weights of permutations in S_{n+1}. This yields a grammatical expression for $A_n(x,y)$.

Theorem 2.1 *Let D be the formal derivative with respect to the grammar* (1.4)*. For $n \geqslant 1$, we have*

$$D^n(x) = \sum_{m=1}^{n} A(n,m) x^m y^{n+1-m}.$$

From Theorem 2.1, it follows that $D^n(x)|_{y=1} = A_n(x)$. Here we give a grammatical proof of the following classical recurrence for the Eulerian polynomials $A_n(x)$.

Proposition 2.2 *For $n \geq 1$, we have*

$$A_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(x)(x-1)^{n-1-k}, \tag{2.1}$$

where $A_0(x) = 1$.

Proof. By the grammar (1.4), we have $D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y$. Hence

$$D(x^{-1}y) = x^{-1}D(y) + yD(x^{-1}) = x^{-1}y(x-y). \tag{2.2}$$

Since $(x-y)$ is a constant with respect to D, we see that

$$D^n(x^{-1}y) = x^{-1}y(x-y)^n. \tag{2.3}$$

By the Leibniz formula, we find that for $n \geq 1$,

$$D^n(x) = D^n(y) = D^n(xx^{-1}y) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(x^{-1}y). \tag{2.4}$$

Substituting (2.3) into (2.4), we get

$$(x-y)x^{-1}D^n(x) = \sum_{k=0}^{n-1} \binom{n}{k} x^{-1}yD^k(x)(x-y)^{n-k}.$$

Setting $y = 1$, we arrive at (2.1). ∎

Next, we introduce a grammar to generate Stirling permutations.

Let $[n]_r$ denote the multiset $\{1^r, 2^r, \ldots, n^r\}$, where i^r stands for r occurrences of i. An r-Stirling permutation is a permutation on $[n]_r$ such that the elements between two occurrences of i are not smaller than i. In particular, a 2-Stirling permutation is usually referred to as a Stirling permutation, see Gessel and Stanley [9]. For example, 123321455664 is a Stirling permutation on $[6]_2$.

For a Stirling permutation $\pi = \pi_1\pi_2\cdots\pi_{2n}$, an index i $(1 \leq i \leq 2n-1)$, is called an ascent if $\pi_i < \pi_{i+1}$, or a descent if $\pi_i > \pi_{i+1}$, or a plateaux if $\pi_i = \pi_{i+1}$. We shall show that the following grammar

$$G\colon \quad x \to xy^2, \quad y \to xy^2 \tag{2.5}$$

can be used to generate Stirling permutations. Let $\pi = \pi_1\pi_2\cdots\pi_{2n}$ be a Stirling permutation on $[n]_2$. We label an ascent of $\pi_0\pi_1\pi_2\cdots\pi_{2n}\pi_{2n+1}$ by x and label a descent or a plateau by y, where we set $\pi_0 = \pi_{2n+1} = 0$. For example, let $\pi = 244215566133$. The grammatical labeling of π is given below

$$\begin{array}{ccccccccccccc} & 2 & 4 & 4 & 2 & 1 & 5 & 5 & 6 & 6 & 1 & 3 & 3 \\ x & x & y & y & y & x & y & x & y & y & x & y & y. \end{array}$$

If we insert 77 after the first occurrence of 4, we get

$$\begin{array}{ccccccccccccccc} & 2 & 4 & 7 & 7 & 4 & 2 & 1 & 5 & 5 & 6 & 6 & 1 & 3 & 3 \\ x & x & x & y & y & y & y & x & y & x & y & y & x & y & y. \end{array}$$

Inserting 77 after the second occurrence of 1 gives

$$\begin{array}{ccccccccccccccc} & 2 & 4 & 4 & 2 & 1 & 5 & 5 & 6 & 6 & 1 & 7 & 7 & 3 & 3 \\ x & x & y & y & y & x & y & x & y & y & x & y & y & y & y. \end{array}$$

Clearly, each Stirling permutation on $[n]_2$ can be obtained by inserting nn into a Stirling permutation on $[n-1]_2$. This leads to a grammatical interpretation of the second-order Eulerian polynomials, namely, the generating functions of the Stirling permutations.

Theorem 2.3 *Let D be the formal derivative with respect to the grammar* (2.5)*. Then we have*

$$D^n(x) = \sum_{m=1}^{n} C(n,m)x^m y^{2n+1-m}, \tag{2.6}$$

where $C(n,m)$ denotes the number of Stirling permutations of $[n]_2$ with $m-1$ ascents.

We adopt the notation $C_n(x)$ as used in Bóna [1] for the second-order Eulerian polynomials

$$C_n(x) = \sum_{m=1}^{n} C(n,m)x^m.$$

From Theorem 2.3, we see that $C_n(x) = D^n(x)|_{y=1}$.

In general, the grammar

$$G\colon \quad x \to xy^r, \quad y \to xy^r \tag{2.7}$$

can be used to generate r-Stirling permutations.

To conclude this section, we define the Lah grammar as follows:

$$G\colon \quad z \to xyz, \quad x \to xy, \quad y \to xy \tag{2.8}$$

and we show that it generates partitions into lists. A partition of $[n]$ into lists is a partition of $[n]$ for which the elements of each block are linearly ordered. For a partition into lists, label the partition itself by z. Express a list $\sigma_1\sigma_2\cdots\sigma_m$ by $0\sigma_1\sigma_2\cdots\sigma_m0$ and label an ascent by x and a descent by y. For example, the labeling $\pi = \{325, 614, 7\}$ is given by

$$\begin{array}{ccccccccccc} & & 3 & 2 & 5 & & 6 & 1 & 4 & & 7 \\ z & x & y & x & y & x & y & x & y & x & y. \end{array}$$

Notice that the elements 0 are omitted in the above expression. It can be easily checked that grammar (2.8) generates partitions into lists.

Theorem 2.4 *Let $C(n,k,m)$ be the number of partitions of $[n]$ into k lists with $m-k$ ascents. Then*

$$D^n(z) = \sum_{k=1}^{n}\sum_{m=k}^{n} C(n,k,m)x^m y^{k+n-m} z.$$

In particular, setting $y = x$ in grmmar (2.8), we get the grammar for the *Lah numbers*

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}.$$

Corollary 2.5 *Let D be the formal derivative with respect to the grammar*

$$G\colon \quad z \to x^2 z, \quad x \to x^2.$$

Then we have

$$D^n(z) = x^n z \sum_{k=1}^{n} L(n,k)x^k.$$

3 The André Polynomials

In this section, we use the grammar found by Dumont [3] to give a proof of the generating function formula for the André polynomials. This formula was first obtained by Foata and Schützenberger [8] using a differential equation. Later, Foata and Han [7] found a way to compute the generating function of $E_n(x,1)$ without solving a differential equation.

Recall that the André polynomials are defined in terms of 0-1-2 increasing trees. An increasing tree on $[n]$ is a rooted tree with vertex set $\{0,1,2,\ldots,n\}$ in which the labels of the vertices are increasing along any path from the root. Note that 0 is the root. A 0-1-2 increasing tree is an increasing tree in which the degree of any vertex is at most two. The degree of a vertex in a rooted tree is meant to be the number of its children. Given a 0-1-2 increasing tree T, let $l(T)$ denote the number of leaves of T, and let $u(T)$ denote the number of vertices of T with degree 1. The *André polynomial* $E_n(x,y)$ is defined by

$$E_n(x,y)=\sum_T x^{l(T)}y^{u(T)},$$

where the sum ranges over 0-1-2 increasing trees on $\{0,1,\ldots,n-1\}$.

Setting $x=y=1$, $E_n(x,y)$ reduces to the n-th Euler number E_n, which counts 0-1-2 increasing trees on $\{0,1,\ldots,n-1\}$ as well as alternating permutations on $[n]$, see [6,8,11].

Dumont [3] introduced the grammar

$$G\colon\quad x\to xy,\quad y\to x \tag{3.1}$$

and showed that it generates the André polynomials $E_n(x,y)$, namely,

$$D^n(y)=E_n(x,y), \tag{3.2}$$

where D is the formal derivative with respect to the grammar G in (3.1). This fact can be justified intuitively in terms of the following grammatical labeling. Given a 0-1-2 increasing tree T, a leaf is labeled by x, a vertex of degree 1 is labeled by y and a vertex of degree 2 is labeled by 1. The following figure illustrates the labeling of a 0-1-2 increasing tree on $\{0,1,2,3,4,5\}$.

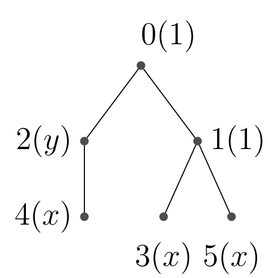
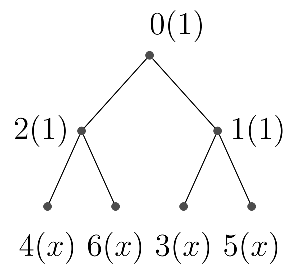


Figure 3.1: The labeling of a 0-1-2 increasing tree on $\{0,1,2,3,4,5\}$

If we add 6 as a child of 2, the resulting tree is



Once the vertex 6 is added, the label y of 2 becomes the label 1, and the vertex 6 gets a new label x. This corresponds to the substitution rule $y \to x$. Similarly, adding the vertex 6 to a leaf of the increasing tree in Figure 3.1 corresponds to the substitution rule $x \to xy$. So the above grammatical labeling leads to the relation (3.2).

The following classical relation

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \tag{3.3}$$

immediately follows from the above grammar. Since $2D^{n+1}(y) = 2D^n(x) = 2D^{n-1}(xy) = D^n(y^2)$, by the Leibniz formula, we get

$$2D^{n+1}(y) = \sum_{k=0}^{n} \binom{n}{k} D^k(y) D^{n-k}(y).$$

Replacing $D^n(y)$ by $E_n(x, y)$, we obtain that

$$2E_{n+1}(x, y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x, y) E_{n-k}(x, y). \tag{3.4}$$

Setting $x = y = 1$ yields (3.3).

Using the grammar (3.1), we also get equivalent formulations of (3.3) and (3.4). Since $D^n(y) = D^{n-1}(x) = D^{n-2}(xy)$ for $n \geq 2$, by the Leibniz formula, we are led to

$$D^n(y) = \sum_{k=0}^{n-2} \binom{n-2}{k} D^k(x) D^{n-2-k}(y).$$

Noting that for $k \geq 1$, $D^k(y) = D^{k-1}(x) = E_k(x, y)$, we see that

$$E_n(x, y) = \sum_{k=0}^{n-2} \binom{n-2}{k} E_{k+1}(x, y) E_{n-2-k}(x, y) \tag{3.5}$$

for $n \geq 2$. Setting $x = y = 1$ in (3.5) yields the known identity

$$E_n = \sum_{k=0}^{n-2} \binom{n-2}{k} E_{k+1} E_{n-2-k}. \tag{3.6}$$

A combinatorial interpretation of (3.6) was given by Donaghey [5].

Next we use the grammar G to derive the generating function of $E_n(x, y)$ without solving a differential equation.

Theorem 3.1 (Foata and Schützenberger) *We have*

$$\sum_{n=0}^{\infty} \frac{E_n(x,y)}{n!} t^n = \frac{x\sqrt{2x-y^2} + y(2x-y^2)\sin(t\sqrt{2x-y^2}) - (x-y^2)\sqrt{2x-y^2}\cos(t\sqrt{2x-y^2})}{(x-y^2)\sin(t\sqrt{2x-y^2}) + y\sqrt{2x-y^2}\cos(t\sqrt{2x-y^2})}. \tag{3.7}$$

Proof. By the Leibniz formula, we have

$$\mathrm{Gen}(x^{-1}y, t) = \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}(y, t). \tag{3.8}$$

Differentiating both sides of (3.8) with respect to t yields

$$\mathrm{Gen}'(x^{-1}y, t) = \mathrm{Gen}'(x^{-1}, t)\mathrm{Gen}(y, t) + \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}'(y, t). \tag{3.9}$$

Since $D(x^{-1}) = -x^{-1}y$, we have

$$\mathrm{Gen}'(x^{-1}, t) = \mathrm{Gen}(D(x^{-1}), t) = -\mathrm{Gen}(x^{-1}y, t). \tag{3.10}$$

Using $D(y) = x$, we obtain

$$\mathrm{Gen}(x^{-1}, t)\mathrm{Gen}'(y, t) = \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}(D(y), t) = \mathrm{Gen}(x^{-1}, t)\mathrm{Gen}(x, t) = 1. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), we deduce that

$$\mathrm{Gen}'(x^{-1}y, t) = 1 - \mathrm{Gen}(x^{-1}y, t)\mathrm{Gen}(y, t),$$

and hence

$$\mathrm{Gen}(y, t) = \frac{1 - \mathrm{Gen}'(x^{-1}y, t)}{\mathrm{Gen}(x^{-1}y, t)}. \tag{3.12}$$

We now compute the generating function $\mathrm{Gen}(x^{-1}y, t)$. It is easily verified that for $m \geq 0$,

$$D^{2m}(x^{-1}y) = x^{-1}y(y^2 - 2x)^m \tag{3.13}$$

and

$$D^{2m+1}(x^{-1}y) = (1 - x^{-1}y^2)(y^2 - 2x)^m. \tag{3.14}$$

Using (3.13) and (3.14), we have

$$\begin{aligned}
\mathrm{Gen}(x^{-1}y,t) &= \sum_{n=0}^{\infty} \frac{D^n(x^{-1}y)}{n!} t^n \\
&= x^{-1}y \sum_{n=0}^{\infty} \frac{(y^2-2x)^n}{(2n)!} t^{2n} + (1-x^{-1}y^2) \sum_{n=0}^{\infty} \frac{(y^2-2x)^n}{(2n+1)!} t^{2n+1} \\
&= x^{-1}y \sum_{n=0}^{\infty} \frac{(-1)^n (t\sqrt{2x-y^2})^{2n}}{(2n)!} + \frac{1-x^{-1}y^2}{\sqrt{2x-y^2}} \sum_{n=0}^{\infty} \frac{(-1)^n (t\sqrt{2x-y^2})^{2n+1}}{(2n+1)!} \\
&= x^{-1}y \cos(t\sqrt{2x-y^2}) + \frac{1-x^{-1}y^2}{\sqrt{2x-y^2}} \sin(t\sqrt{2x-y^2}). \qquad (3.15)
\end{aligned}$$

Plugging (3.15) into (3.12), we arrive at (3.7), and hence the proof is complete. ∎

Setting $x = y = 1$, (3.7) reduces to the generating function of the Euler numbers:

$$\sum_{n=0}^{\infty} \frac{E_n}{n!} t^n = \sec t + \tan t.$$

4 Permutations of $[n]$ with k Exterior Peaks

In this section, we use the grammar

$$G: \quad x \rightarrow xy, \quad y \rightarrow x^2 \qquad (4.1)$$

to show that G generates the number $T(n,k)$ of permutations of $[n]$ with k exterior peaks. Let

$$T_n(x) = \sum_{k \geq 0} T(n,k) x^k.$$

We give a grammatical proof of the formula for the generating function of $T_n(x)$ due to Gessel, see [13]. We also obtain a recurrence relation of $T_n(x)$.

Recall that for a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, the index i is called an exterior peak if $1 < i < n$ and $\pi_{i-1} < \pi_i > \pi_{i+1}$, or $i = 1$ and $\pi_1 > \pi_2$. We shall use the following grammatical labeling of a permutation to show that the above grammar G generates the polynomials $T_n(x)$. For a permutation π of $[n]$, we first add an element 0 at the end of π. If i is an exterior peak, then we label π_i and π_{i+1} by x. In addition, the element 0 is labeled by x, and all other elements are labeled by y. The weight w of π is defined to be the product of all the labels. If π has k exterior peaks, then its weight is given by

$$w(\pi) = x^{2k+1} y^{n-2k}.$$

For example, let $\pi = 325641$. The labeling of π is as follows

$$\begin{array}{ccccccc} 3 & 2 & 5 & 6 & 4 & 1 & \\ x & x & y & x & x & y & x, \end{array}$$

and the weight of π is x^5y^2. If we insert 7 before 3, then the labeling of the resulting permutation is

$$\begin{array}{cccccccc} 7 & 3 & 2 & 5 & 6 & 4 & 1 & \\ x & x & y & y & x & x & y & x. \end{array}$$

We see that the label of 2 becomes y and the label of 7 becomes x. So this insertion corresponds to the rule $x \to xy$. If we insert 7 before 0, then we have

$$\begin{array}{cccccccc} 3 & 2 & 5 & 6 & 4 & 1 & 7 & \\ x & x & y & x & x & y & y & x, \end{array}$$

where the label of 0 remains the same and the label of 7 is y. In this case, the insertion corresponds to the rule $x \to xy$. If we insert 7 before 5, then we obtain

$$\begin{array}{cccccccc} 3 & 2 & 7 & 5 & 6 & 4 & 1 & \\ x & x & x & x & x & x & y & x, \end{array}$$

where the label of 5 becomes x and the label of 7 is x. Indeed, the above labeling leads to the following theorem.

Theorem 4.1 *Let D be the formal derivative with respect to the grammar* (4.1)*. For $n \geq 1$, we have*

$$D^n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)x^{2k+1}y^{n-2k}. \tag{4.2}$$

Proof. We proceed by induction on n. For $n = 1$, the statement is obvious. Assume that the theorem holds for n. To show that it is valid for $n+1$, we represent a permutation in S_n by adding a zero at the end. Let $\pi = \pi_1\pi_2\cdots\pi_n 0$ be a permutation of S_n. Recall that if i $(1 \leq i \leq n-1)$ is an exterior peak of π, then we label π_i and π_{i+1} by x. Moreover, the element 0 is labeled by x and all other elements are labeled by y. We consider the following two cases according to where $n+1$ is inserted into π to generate a permutation in S_{n+1}.

Case 1: $n+1$ is inserted after π_n. No matter π_n is labeled by x or y, the inserted element $n+1$ always gets a label y:

$$\begin{array}{ccc} \cdots & n+1 & \\ \cdots & y & x. \end{array}$$

Therefore, in either case, the insertion corresponds to the rule $x \to xy$.

Case 2: $n+1$ is inserted before π_i $(1 \leq i \leq n)$.

If π_i is labeled by y, that is, $i-1$ and i are not exterior peaks, we obtain

$$\begin{array}{ccc} \cdots & \pi_i & \cdots \\ \cdots & y & \cdots \end{array} \quad \Longrightarrow \quad \begin{array}{cccc} \cdots & n+1 & \pi_i & \cdots \\ \cdots & x & x & \cdots \end{array}.$$

So this insertion corresponds to the rule $y \to x^2$.

If π_i is labeled by x and i is an exterior peak, we have

$$\begin{array}{cccc} \cdots & \pi_i & \pi_{i+1} & \cdots \\ \cdots & x & x & \cdots \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} \cdots & n+1 & \pi_i & \pi_{i+1} & \cdots \\ \cdots & x & x & y & \cdots \end{array}.$$

Now, the insertion corresponds to the rule $x \to xy$.

If π_i is labeled by x and $i-1$ is an exterior peak, we obtain

$$\begin{array}{cccc} \cdots & \pi_{i-1} & \pi_i & \cdots \\ \cdots & x & x & \cdots \end{array} \quad \Longrightarrow \quad \begin{array}{ccccc} \cdots & \pi_{i-1} & n+1 & \pi_i & \cdots \\ \cdots & y & x & x & \cdots \end{array}.$$

In this case, the insertion also corresponds to the rule $x \to xy$.

Thus we have shown that the theorem is valid for $n+1$. This completes the proof. ∎

By Theorem 4.1, we obtain the following recurrence relation.

Proposition 4.2 *For $n \geq 1$, we have*

$$T_n(x) = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (1-x)^{\lfloor j/2 \rfloor} T_{n-j}(x), \tag{4.3}$$

where $T_0(x) = 1$.

Proof. Note that

$$D(x^{-1}) = -x^{-1}y, \; D(-x^{-1}y) = x^{-1}(y^2 - x^2), \; D(y^2 - x^2) = 0.$$

Hence for $m \geq 0$, we have

$$D^{2m}(x^{-1}) = x^{-1}(y^2 - x^2)^m, \tag{4.4}$$

and

$$D^{2m+1}(x^{-1}) = -x^{-1}y(y^2 - x^2)^m. \tag{4.5}$$

Setting $y = 1$ in (4.4) and (4.5), we obtain

$$D^j(x^{-1})|_{y=1} = (-1)^j x^{-1} (1 - x^2)^{\lfloor j/2 \rfloor}.$$

By the Leibniz formula, we find that

$$\begin{aligned} D^n(x^{-1}x)|_{y=1} = 0 &= \sum_{j=0}^{n} \binom{n}{j} D^j(x^{-1})|_{y=1} D^{n-j}(x)|_{y=1} \\ &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} x^{-1} (1-x^2)^{\lfloor j/2 \rfloor} D^{n-j}(x)|_{y=1}. \end{aligned} \tag{4.6}$$

By Theorem 4.1, we see that

$$D^n(x)|_{y=1} = xT_n(x^2).$$

Hence (4.3) follows from (4.6). ∎

Using the grammar (4.1), we give a derivation of the following generating function of $T_n(x)$ due to Gessel, see [13].

Theorem 4.3 (Gessel) *We have*

$$\sum_{n=0}^{\infty} \frac{T_n(x)}{n!} t^n = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(t\sqrt{1-x}) - \sinh(t\sqrt{1-x})}. \tag{4.7}$$

To prove Theorem 4.3, we need the following lemma.

Lemma 4.4 *For the the following grammar*

$$G\colon \quad u \rightarrow v^2, \quad v \rightarrow v, \tag{4.8}$$

we have

$$\mathrm{Gen}(u^{-1}v, t) = \frac{v}{u\cosh(t) + (v^2 - u)\sinh(t)}. \tag{4.9}$$

Proof. Let D be the formal derivative with respect to G in (4.8). Since $D(v) = v$, we have

$$\mathrm{Gen}(v, t) = ve^t.$$

By (1.3), we find that

$$\mathrm{Gen}(u^{-1}v^2, t) = \mathrm{Gen}(v, t)\mathrm{Gen}(u^{-1}v, t) = ve^t\mathrm{Gen}(u^{-1}v, t). \tag{4.10}$$

We proceed to compute $(\mathrm{Gen}(u^{-1}v^2, t))'$ in two ways. It is easily checked that

$$D(u^{-1}v^2) = -(u^{-1}v)^2(v^2 - 2u).$$

Thus, from (1.1) and (1.3) we deduce that

$$(\mathrm{Gen}(u^{-1}v^2, t))' = \mathrm{Gen}\left(D(u^{-1}v^2), t\right) = -\mathrm{Gen}^2(u^{-1}v, t)\mathrm{Gen}(v^2 - 2u, t). \tag{4.11}$$

On the other hand, since

$$D(u^{-1}v) = u^{-1}v(1 - u^{-1}v^2),$$

from (4.10) we find that

$$\begin{aligned}
(\mathrm{Gen}(u^{-1}v^2, t))' &= (ve^t\mathrm{Gen}(u^{-1}v, t))' \\
&= ve^t\mathrm{Gen}(u^{-1}v, t) + ve^t\mathrm{Gen}(D(u^{-1}v), t) \\
&= ve^t\mathrm{Gen}(u^{-1}v, t) + ve^t\mathrm{Gen}(u^{-1}v, t)\mathrm{Gen}(1 - u^{-1}v^2, t).
\end{aligned} \tag{4.12}$$

Comparing (4.11) with (4.12) yields

$$\begin{aligned}&-\operatorname{Gen}^2(u^{-1}v,t)\operatorname{Gen}(v^2-2u,t)\\&\quad=ve^t\operatorname{Gen}(u^{-1}v,t)+ve^t\operatorname{Gen}(u^{-1}v,t)\operatorname{Gen}(1-u^{-1}v^2,t),\end{aligned}$$

or, equivalently,

$$-\operatorname{Gen}(u^{-1}v,t)\operatorname{Gen}(v^2-2u,t)=ve^t+ve^t\operatorname{Gen}(1-u^{-1}v^2,t). \tag{4.13}$$

Since $D(v^2-2u)=0$, we obtain that

$$\operatorname{Gen}(v^2-2u,t)=v^2-2u.$$

Clearly, $\operatorname{Gen}(1-u^{-1}v^2,t)=1-\operatorname{Gen}(u^{-1}v^2,t)$. Thus (4.13) can be simplified to

$$-(v^2-2u)\operatorname{Gen}(u^{-1}v,t)=2ve^t-ve^t\operatorname{Gen}(u^{-1}v^2,t). \tag{4.14}$$

Plugging (4.10) into (4.14), we arrive at

$$\operatorname{Gen}(u^{-1}v,t)=\frac{2v}{v^2e^t-(v^2-2u)e^{-t}},$$

which can be rewritten in the form of (4.9), and so the proof is complete. ∎

To prove Theorem 4.3, we introduce a parameter w in the grammar G for permutations on $[n]$ with k peaks. More precisely, consider the grammar

$$G'\colon\quad x\to xy,\quad y\to wx^2, \tag{4.15}$$

where w is a constant. For a permutation π on $[n]$, we give a labeling which is essentially the same as the labeling used in the proof of Theorem 4.1. First, add a zero at the end of π. If i is an exterior peak, then we label π_i by wx and π_{i+1} by x. Moreover, the element 0 is labeled by x, and all other elements are labeled by y. For example, let $\pi=325641$. Then π has the following labeling

$$\begin{array}{ccccccc}3&2&5&6&4&1&\\ wx&x&y&wx&x&y&x.\end{array}$$

Clearly, w records the number of exterior peaks. It follows that

$$D^n(x)=\sum_{k=0}^{\lfloor n/2\rfloor}T(n,k)x^{2k+1}y^{n-2k}w^k. \tag{4.16}$$

Proof of Theorem 4.3. For the grammar (4.8) in Lemma 4.4, the following relations hold:

$$\begin{aligned}D(u^{-1}v)&=u^{-1}v(1-u^{-1}v^2),\\D(1-u^{-1}v^2)&=(u^{-1}v)^2(v^2-2u),\\D(v^2-2u)&=0.\end{aligned}$$

Comparing the above relations with the rules of the grammar in (4.15) and making the substitutions $x = u^{-1}v,\ y = 1 - u^{-1}v^2,\ w = v^2 - 2u$, we get the rules as in grammar (4.15), namely, $D(x) = xy$, $D(y) = wx^2$ and $D(w) = 0$. Hence (4.16) implies that

$$D^n(u^{-1}v) = \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(u^{-1}v)^{2k+1}(1 - u^{-1}v^2)^{n-2k}(v^2 - 2u)^k,$$

that is,

$$\mathrm{Gen}(u^{-1}v, t) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(u^{-1}v)^{2k+1}(1 - u^{-1}v^2)^{n-2k}(v^2 - 2u)^k. \tag{4.17}$$

Comparing (4.9) with (4.17), we get

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(u^{-1}v)^{2k}(1 - u^{-1}v^2)^{n-2k}(v^2 - 2u)^k = \frac{u}{u\cosh(t) + (v^2 - u)\sinh(t)}.$$

Since the above relation is valid for indeterminates u and v, we can set $v = \sqrt{u-1}$ to deduce the following relation

$$\sum_{n\geq 0} \frac{t^n u^{-n}}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(1 - u^2)^k = \frac{u}{u\cosh(t) - \sinh(t)}. \tag{4.18}$$

Substituting t by ut in (4.18), we find that

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k)(1 - u^2)^k = \frac{u}{u\cosh(ut) - \sinh(ut)}. \tag{4.19}$$

Finally, by setting $x = 1 - u^2$ in (4.19), we arrive at (4.7). This completes the proof. ∎

5 Increasing trees and peaks in permutations

In this section, we use a grammatical approach to establish a connection between permutations with a given number of exterior peaks and increasing trees with a given number of vertices of even degree. We also give a combinatorial interpretation of this fact.

Theorem 5.1 *The number of permutations on $[n]$ with m exterior peaks equals the number of increasing trees on $\{0, 1, 2, \ldots, n\}$ with $2m + 1$ vertices of even degree.*

We first present a proof of the theorem using the following grammar given by Dumont [3]:

$$G: \quad x_i \to x_0 x_{i+1}. \tag{5.1}$$

Let D be the formal derivative with respect to G. Dumont [3] showed that

$$D^n(x_0) = \sum_T x_0^{m_0(T)} x_1^{m_1(T)} x_2^{m_2(T)} \cdots, \tag{5.2}$$

where the sum ranges over increasing trees T on $\{0, 1, 2, \ldots, n\}$ and $m_i(T)$ denotes the number of vertices of degree i in T. Relation (5.2) can be justified by labeling a vertex of degree i with x_i. Here is an example.

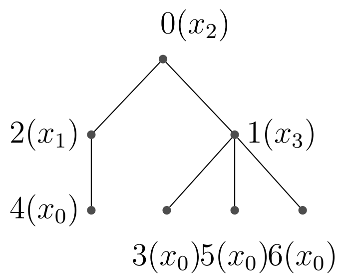


Figure 5.2: A labeling on an increasing tree

Let T be an increasing tree on $\{0, 1, 2, \ldots, n\}$ with the above labeling. When adding the vertex $n+1$ to T as the child of a vertex v of degree i, the label of v changes from x_i to x_{i+1} and $n+1$ gets a label x_0. This corresponds to the rule $x_i \to x_0 x_{i+1}$, which proves (5.2).

Setting $x_{2i} = x$ and $x_{2i+1} = y$, we see that the grammar (5.1) reduces to the grammar (4.1) that generates the polynomial $T_n(x)$ for permutations with a given number of exterior peaks. This gives a grammatical reasoning of Theorem 5.1.

To conclude this paper, we give a combinatorial proof of Theorem 5.1. More precisely, we provide a bijection Φ between permutations and increasing trees such that a permutation on $[n]$ with m exterior peaks corresponds to an increasing tree on $\{0, 1, 2, \ldots, n\}$ with $2m+1$ vertices of even degree. Recall that a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ on $[n]$ is called an up-down permutation if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$. Similarly, σ is called a down-up permutation if $\sigma_1 > \sigma_2 < \sigma_3 > \cdots$. An even increasing tree is meant to be an increasing tree such that each vertex possibly except for the root is of even degree. Kuznetsov, Pak and Postnikov [11] found a bijection between up-down permutations and even increasing trees.

Clearly, an up-down permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ can be transformed into a down-up permutation $\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_n$ by taking the complement $\sigma'_i = n+1-\sigma_i$ for $1 \le i \le n$.

Our bijection Φ from down-up permutations to increasing trees can be considered as an extension of the bijection given by Kuznetsov, Pak and Postnikov [11].

The increasing tree $\Phi(\sigma)$ is constructed in n steps. At each step, a vertex is added to a forest of increasing trees. At the k-th step for $1 \leq k \leq n$, we obtain a forest of increasing trees with k vertices, and finally obtain an increasing tree on $\{0, 1, 2, \ldots, n\}$.

The bijection Φ can be described as follows. Let $(c_1, c_2, \ldots, c_n)$ be the code of σ, that is, for $1 \leq i \leq n$, c_i is the number of elements σ_j such that $j > i$ and $\sigma_i > \sigma_j$. In the first step, we start with an increasing tree F_1 with a single vertex $i_1 = n - c_1$. In the k-th step for $k > 1$, we assume that an increasing forest F_{k-1} has been obtained at the $(k-1)$-th step. Denote by I_{k-1} and J_{k-1} the set of vertices and the set of roots of F_{k-1}, respectively. Let $\bar{I}_{k-1}$ be the complement of I_{k-1}, that is, $\bar{I}_{k-1} = [n] \setminus I_{k-1}$. The goal of the k-th step is to construct an increasing forest F_k by adding an element i_k from $\bar{I}_{k-1}$ to F_{k-1} such that i_k is the father of those roots larger than i_k in F_{k-1}.

Let $j_1, j_2, \ldots, j_l$ be the elements of J_{k-1} listed in decreasing order. For notational convenience, we assume that $j_0 = n+1$, $j_{l+1} = 0$ and $c_0 = 0$. Let

$$U_k = \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } s \geq 0\}, \tag{5.3}$$

$$V_k = \{m \in \bar{I}_{k-1} \mid j_{2s+1} < m < j_{2s} \text{ for some } s \geq 0\}. \tag{5.4}$$

It is clear that $U_k \cap V_k = \emptyset$ and $U_k \cup V_k = \bar{I}_{k-1}$.

Define

$$M_k = \begin{cases} U_k, & \text{if } c_{k-2} \leq c_{k-1} \leq c_k \text{ or } c_{k-2} > c_{k-1} > c_k; \\ V_k, & \text{otherwise.} \end{cases} \tag{5.5}$$

Let $m_1, m_2, \ldots$ be the elements of M_k listed in increasing order.

We claim that for $2 \leq k \leq n$, at the k-th step of the construction of Φ, we have

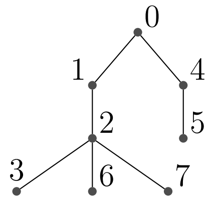
$$|M_k| = \begin{cases} c_{k-1}, & \text{if } c_{k-1} > c_k; \\ n-k+1-c_{k-1}, & \text{if } c_{k-1} \leq c_k. \end{cases} \tag{5.6}$$

It implies that there are at least $c_k + 1$ elements in M_k if $c_{k-1} > c_k$. Otherwise, there are at least $n-k+1-c_k$ elements in M_k. Thus it is feasible to set

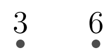
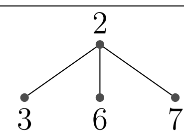
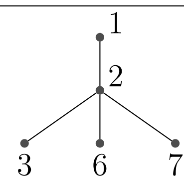
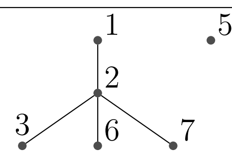
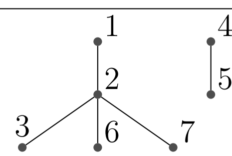
$$i_k = \begin{cases} m_{c_k+1}, & \text{if } c_{k-1} > c_k; \\ m_{n-k+1-c_k}, & \text{if } c_{k-1} \leq c_k. \end{cases}$$

Then we add vertex i_k to F_{k-1} as the father of each vertex $j_s \in J_{k-1}$ as long as $j_s > i_k$, and denote the resulting forest by F_k. When $k < n$, we may iterate the above process until we obtain an increasing forest F_n on $[n]$. Finally, we add vertex 0 to F_n as the father of each root, so that we obtain an increasing tree T.

Here is an example. Let $n = 7$ and $\sigma = 5346721$. The code of σ is $\mathrm{code}(\sigma) = (4, 2, 2, 2, 2, 1, 0)$. The corresponding increasing tree $\Phi(\sigma)$ is given below.



The values of I_k, J_k, M_k, i_k and the forests F_k are listed in the following table.

k	M_k	i_k	F_k	$I_k,\ J_k$
1	—	$i_1 = 3$		$I_1 = \{3\}$ $J_1 = \{3\}$
2	$M_2 = \{4, 5, 6, 7\}$	$i_2 = 6$		$I_2 = \{3, 6\}$ $J_2 = \{3, 6\}$
3	$M_3 = \{1, 2, 7\}$	$i_3 = 7$		$I_3 = \{3, 6, 7\}$ $J_3 = \{3, 6, 7\}$
4	$M_4 = \{1, 2\}$	$i_4 = 2$		$I_4 = \{2, 3, 6, 7\}$ $J_4 = \{2\}$
5	$M_5 = \{1\}$	$i_5 = 1$		$I_5 = \{1, 2, 3, 6, 7\}$ $J_5 = \{1\}$
6	$M_6 = \{4, 5\}$	$i_6 = 5$		$I_6 = \{1, 2, 3, 5, 6, 7\}$ $J_6 = \{1, 5\}$
7	$M_7 = \{4\}$	$i_7 = 4$		—

Next we give a proof of claim (5.6) which ensures that the map Φ is well defined.

Proof of (5.6). It is clear that $|M_2| = c_1$ if $c_1 > c_2$, and $|M_2| = n - 1 - c_1$ if $c_1 \le c_2$. In other words, (5.6) holds for $k = 2$. Assume that (5.6) holds for k. To compute $|M_{k+1}|$, we consider the following four cases:

Case 1: $c_{k-2} > c_{k-1} > c_k$. Let $j_1, j_2, \ldots, j_l$ be the elements of J_{k-1} listed in decreasing order, and let $j_0 = n + 1$ and $j_{l+1} = 0$. By the assumption $c_{k-2} > c_{k-1} > c_k$ and the definition of M_k, we get

$$M_k = U_k = \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } s \ge 0\}.$$

Since $i_k \in M_k$, there exists $t \geq 0$ such that $j_{2t+2} < i_k < j_{2t+1}$. So the set of roots of F_k is given by

$$J_k = \{i_k, j_{2t+2}, \ldots, j_l\}.$$

It follows that

$$\begin{aligned} U_{k+1} &= \{m \in \bar{I}_k \mid m < i_k,\ j_{2s+2} < m < j_{2s+1} \text{ for some } s \geq t\} \\ &= \{m \in M_k \mid m < i_k\}. \end{aligned}$$

Since $c_{k-1} > c_k$, i_k is the (c_k+1)-th smallest element in M_k. Hence

$$|U_{k+1}| = |\{m \in M_k \mid m < i_k\}| = c_k.$$

If $c_k > c_{k+1}$, by the assumption $c_{k-1} > c_k$, we have

$$|M_{k+1}| = |U_{k+1}| = c_k.$$

If $c_k \leq c_{k+1}$, by (5.5), we obtain that

$$M_{k+1} = V_{k+1} = \bar{I}_k \setminus U_{k+1},$$

which implies that

$$|M_{k+1}| = n - k - c_k.$$

So we have verified that in this case (5.6) also holds.

Case 2: $c_{k-2} > c_{k-1} \leq c_k$. In this case, we see that

$$M_k = V_k = \{m \in \bar{I}_{k-1} \mid j_{2s+1} < m < j_{2s} \text{ for some } s \geq 0\}. \tag{5.7}$$

Since $i_k \in M_k$, there exists $t \geq 0$ such that $j_{2t+1} < i_k < j_{2t}$. It follows that

$$J_k = \{i_k, j_{2t+1}, \ldots, j_l\}. \tag{5.8}$$

Since $I_k = I_{k-1} \cup \{i_k\}$ and $\bar{I}_k = \bar{I}_{k-1} \setminus \{i_k\}$, by (5.4) and (5.8), we find that

$$\begin{aligned} V_{k+1} &= \{m \in \bar{I}_k \mid i_k < m < j_0 \text{ or } j_{2s+2} < m < j_{2s+1} \text{ for some } s \geq t\} \\ &= \{m \in \bar{I}_k \mid i_k < m < j_0\} \cup \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } s \geq t\}. \end{aligned}$$

Since $j_{2t+1} < i_k < j_{2t}$, we see that $i_k < m$ for any $j_{2s+1} < m < j_{2s}$ or $j_{2s+2} < m < j_{2s+1}$, where $0 \leq s < t$. Noting that $\bar{I}_k = \bar{I}_{k-1} \setminus \{i_k\}$, we have

$$\begin{aligned} \{m \in \bar{I}_k \mid i_k < m < j_0\} &= \{m \in \bar{I}_{k-1} \mid i_k < m < j_0\} \\ &= \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } 0 \leq s < t\} \\ &\quad \cup \{m \in \bar{I}_{k-1} \mid m > i_k, j_{2s+1} < m < j_{2s} \text{ for some } 0 \leq s \leq t\}. \end{aligned}$$

Thus V_{k+1} can be rewritten as

$$\begin{aligned}V_{k+1} = \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } 0 \le s < t\} \\ \cup \{m \in \bar{I}_{k-1} \mid m > i_k, j_{2s+1} < m < j_{2s} \text{ for some } 0 \le s \le t\} \\ \cup \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } s \ge t\}.\end{aligned}$$

By the definition of U_k, we find that

$$\begin{aligned}U_k = \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } 0 \le s < t\} \\ \cup \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } s \ge t\}.\end{aligned}$$

Let $W_k = \{m \in \bar{I}_{k-1} \mid m > i_k,\ j_{2s+1} < m < j_{2s} \text{ for some } 0 \le s \le t\}$, so that V_{k+1} can be written as $V_{k+1} = U_k \cup W_k$. It is easy to check that $U_k \cap W_k = \emptyset$. Recalling that $j_{2t+1} < i_k < j_{2t}$, by the definition of M_k in (5.7), we deduce that

$$\begin{aligned}W_k &= \{m \in \bar{I}_{k-1} \mid m > i_k,\ j_{2s+1} < m < j_{2s} \text{ for some } 0 \le s \le t\} \\ &= \{m \in \bar{I}_{k-1} \mid m > i_k,\ j_{2s+1} < m < j_{2s} \text{ for some } s \ge 0\} \\ &= \{m \in M_k \mid m > i_k\}.\end{aligned}$$

Hence

$$|V_{k+1}| = |U_k| + |\{m \in M_k \mid m > i_k\}|. \tag{5.9}$$

Using the induction hypothesis and (5.7), we find that

$$|V_k| = |M_k| = n - k + 1 - c_{k-1}, \tag{5.10}$$

so that

$$|U_k| = |\bar{I}_{k-1} \setminus V_k| = c_{k-1}. \tag{5.11}$$

Since $c_{k-1} \le c_k$, it can be seen that i_k is the $(n-k+1-c_k)$-th smallest element in M_k. In other words,

$$|\{m \in M_k \mid m \le i_k\}| = n - k + 1 - c_k. \tag{5.12}$$

In view of (5.10) and (5.12), we obtain that

$$\begin{aligned}&|\{m \in M_k \mid m > i_k\}| \\ &\quad = |M_k| - |\{m \in M_k \mid m \le i_k\}| \\ &\quad = (n - k + 1 - c_{k-1}) - (n - k + 1 - c_k) \\ &\quad = c_k - c_{k-1}.\end{aligned} \tag{5.13}$$

Substituting (5.11) and (5.13) into (5.9) gives $|V_{k+1}| = c_k$, and hence $|U_{k+1}| = n-k-c_k$.

If $c_k > c_{k+1}$, by the assumption $c_{k-1} \leq c_k$ and (5.5), we have $|M_{k+1}| = |V_{k+1}| = c_k$. If $c_k \leq c_{k+1}$, by (5.5), we get $|M_{k+1}| = |U_{k+1}| = n - k - c_k$. This shows that (5.6) holds for M_{k+1} in the case $c_{k-2} > c_{k-1} \leq c_k$.

For the other two cases, $c_{k-2} \leq c_{k-1} \leq c_k$ and $c_{k-2} \leq c_{k-1} > c_k$, $|M_{k+1}|$ can be determined by the same argument. The details are omitted. Thus we have shown that (5.6) holds for $k+1$. Hence (5.6) holds for $2 \leq k \leq n$. ∎

Combinatorial Proof of Theorem 5.1. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation with m exterior peaks on $[n]$. Let $T = \Phi(\sigma)$ and let $i_1, i_2, \ldots, i_n$ be the vertices successively generated in the construction of the increasing tree T. First, we show that a vertex of even degree, except for i_1, in T is an exterior peak or a valley in σ. For the vertex i_1, it is clear that it remains to be a leaf during the construction of T. However, as will be seen, i_1 is neither an exterior peak nor a valley in σ.

For $2 \leq k \leq n$, since $j_{2t+1} < i_k < j_{2t}$ for some $t \geq 0$, it can be checked that i_k is a vertex of even degree in T if and only if $i_k \in V_k$. Moreover, $i_k \in V_k$ if and only if $c_{k-2} \leq c_{k-1} > c_k$ or $c_{k-2} > c_{k-1} \leq c_k$. Consequently, $i_k \in V_k$ if and only if $\sigma_{k-2} < \sigma_{k-1} > \sigma_k$ or $\sigma_{k-2} > \sigma_{k-1} < \sigma_k$. Hence, for $2 \leq k \leq n$, i_k is a vertex of even degree if and only if $k-1$ is either an exterior peak or a valley. By the same argument, we find that i_1 is neither an exterior peak nor a valley.

To compute the number of vertices of even degree in T, we consider the total number of exterior peaks and valleys in σ. Setting $\sigma_0 = 0$, then σ begins with an exterior peak, alternately followed valleys and exterior peaks. If $\sigma_{n-1} < \sigma_n$, then σ ends up with a valley. Since there are m exterior peaks in σ, there are also m valleys in σ. It follows that there are a total number of $2m$ exterior peaks or valleys in σ, and hence there are $2m$ non-rooted vertices of even degree in T assuming that i_i is not taken into account. Given that i_1 is a leaf in T, there are $2m+1$ non-rooted vertices of even degree in T. Noting that there are an odd number of vertices of even degree for any rooted tree, the degree of the root 0 in T must be odd. Therefore, T has $2m+1$ vertices of even degree.

If $\sigma_{n-1} > \sigma_n$, then σ ends up with an exterior peak. By the above argument, we see that there are m exterior peaks and $m-1$ valleys in σ. Thus T has $2m$ non-rooted vertices of even degree. Moreover, the degree of the root 0 is also even. Thus T has $2m+1$ vertices of even degree. Based on the two cases discussed above, it follows that T is an increasing tree with $2m+1$ vertices of even degree. Therefore, Φ is well-defined.

It remains to prove that Φ is a bijection. To this end, we construct the inverse map Ψ of Φ. Let T be an increasing tree on $\{0, 1, \ldots, n\}$. Let F_n be the increasing forest obtained from T by deleting the root 0. From F_n, we construct a sequence $F_{n-1}, \ldots, F_1$ of increasing forests. For $k = n, n-1, \ldots, 2$, F_{k-1} is obtained from F_k as follows. Let i_k be the largest root of the increasing forest F_k, and let F_{k-1} be the increasing forest obtained from F_k by deleting the root i_k. For $k = 1$, set i_1 to be the largest root of F_1.

For $1 \le k \le n$, we let I_k denote the set of vertices in F_k and let J_k denote the set of roots in F_k. As before, let $\bar{I}_k$ denote the complement of I_k with respect to $[n]$. For $2 \le k \le n$, assume that U_k and V_k are defined as in (5.3) and (5.4), namely,

$$U_k = \{m \in \bar{I}_{k-1} \mid j_{2s+2} < m < j_{2s+1} \text{ for some } s \ge 0\},$$

$$V_k = \{m \in \bar{I}_{k-1} \mid j_{2s+1} < m < j_{2s} \text{ for some } s \ge 0\},$$

where $j_1, j_2, \ldots, j_l$ are the elements of J_{k-1} listed in decreasing order and $j_0 = n+1$, $j_{l+1} = 0$. Note that $i_k \in \bar{I}_{k-1}$ and $\bar{I}_{k-1}$ is the disjoint union of U_k and V_k. If $i_k \in U_k$, we set $M_k = U_k$. If $i_k \in V_k$, we set $M_k = V_k$.

Given $M_2, \ldots, M_n$, we construct a sequence $(c_1, c_2, \ldots, c_n)$, which will be shown to be a permutation code. It is easily seen that $|M_n| = 1$ and we set $c_n = 0$. For $k = n-1$, set

$$c_{n-1} = \begin{cases} 1, & \text{if the degree of the root 0 in } T \text{ is even,} \\ 0, & \text{if the degree of the root 0 in } T \text{ is odd.} \end{cases}$$

Moreover, for $k = n-2, n-3, \ldots, 1$, set

$$c_k = \begin{cases} |M_{k+1}|, & \text{if } M_{k+2} = U_{k+2} \text{ and } c_{k+1} > c_{k+2}, \\ n-k-|M_{k+1}|, & \text{if } M_{k+2} = U_{k+2} \text{ and } c_{k+1} \le c_{k+2}, \\ n-k-|M_{k+1}|, & \text{if } M_{k+2} = V_{k+2} \text{ and } c_{k+1} > c_{k+2}, \\ |M_{k+1}|, & \text{if } M_{k+2} = V_{k+2} \text{ and } c_{k+1} \le c_{k+2}. \end{cases} \tag{5.14}$$

Now, we verify that for $1 \le k \le n$, $0 \le c_k \le n-k$. Recalling that for $2 \le k \le n$, $i_k \in M_k$ and $M_k \subseteq \bar{I}_{k-1}$, hence $1 \le |M_k| \le |\bar{I}_{k-1}|$. On the other hand, by the definition of I_{k-1}, we find that $|\bar{I}_{k-1}| = n-k+1$. It follows that for $2 \le k \le n$,

$$1 \le |M_k| \le n-k+1. \tag{5.15}$$

Clearly, for $1 \le k \le n-1$, by (5.14), c_k equals $|M_{k+1}|$ or $n-k-|M_{k+1}|$. In view of (5.15), we conclude that $0 \le c_k \le n-k$ for $1 \le k \le n-1$.

Finally, we set $\Psi(T)$ to be the permutation σ with code $(c_1, c_2, \ldots, c_n)$. It can be checked that every step of the construction of Ψ is the inverse of the corresponding step of Φ, and hence the proof is complete. ∎

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