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Classification of conformal minimal immersions of constant curvature from S^2 to Q_3

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Abstract. In this paper, we study geometry of conformal minimal two-spheres immersed in complex hyperquadric Q_3. We firstly use Bahy-El-Dien and Wood's results to obtain some characterizations of the harmonic sequences generated by conformal minimal immersions from S^2 to $G(2,5;\mathbb{R})$. Then we give a classification theorem of linearly full totally unramified conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$, or equivalently, a complex hyperquadric Q_3.

1. Introduction.

The classification of minimal surfaces of constant curvature is an important topic of differential geometry. Bryant [**4**] gave a classification of minimal surfaces with constant curvature in $S^n(1)$. Kenmotsu and Masuda [**12**] classified all minimal surfaces of constant curvature in two-dimensional complex space forms. Bolton et al. [**3**] proved that a linearly full conformal minimal immersion of S^2 in $\mathbb{C}P^n$ with constant curvature belongs to the Veronese sequence, up to a holomorphic isometry of $\mathbb{C}P^n$. Generally, if the ambient space is not a real (or complex) space form, for example, complex Grassmannian $G(k,n;\mathbb{C})$, complex hyperquadric Q_n and quaternionic projective space HP^n and so on, the classification of minimal 2-spheres of constant curvature in them is not easy. It is well known that Hoffman and Osserman [**9**] gave some results about minimal surfaces in $\mathbb{R}^n$ whose Gaussian image in Q_{n-2} has constant curvature, and Chi and Zheng [**7**] classified all holomorphic curves from Riemann spheres into $G(2,4)$ whose curvature is equal to 2 into two families. Recently, J. Wang and the second author ([**10**], [**13**]) determined curvatures and Kähler angles of conformal minimal 2-spheres in Q_2 if their curvature is constant and all the totally real conformal minimal two-spheres of constant curvature in Q_n (only when $n = 2, 3, 4, 5$). Previously, in [**8**], the authors gave a classification theorem of linearly full totally unramified conformal minimal immersions of constant curvature from S^2 to HP^2. Here our interest is to study conformal minimal 2-spheres immersed in Q_n with constant curvature.

As is well known, $G(2,n;\mathbb{R})$ may be identified with complex hyperquadric Q_{n-2} in $\mathbb{C}P^{n-1}$ (for detailed descriptions see the Preliminaries below). In 1989 Bahy-El-Dien and Wood [**2**] gave the explicit construction of all harmonic two-spheres in $G(2,n;\mathbb{R})$,

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which is considered as totally geodesic submanifolds in complex Grassmann manifolds $G(2,n;\mathbb{C})$. In this paper we study classification of conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$ by theory of harmonic maps, and discuss the Kähler angle of conformal minimal immersions of S^2 in Q_n.

Our arrangement is as follows.

In the second section of this paper, firstly we identify Q_{n-2} and $G(2,n;\mathbb{R})$, then we give some fundamental results concerning $G(k,n;\mathbb{C})$ from the view of harmonic sequences, at last we give some brief descriptions of Veronese sequence and the rigidity theorem in $\mathbb{C}P^n$. In the third section, we use Bahy-El-Dien and Wood's results to study some properties of the harmonic sequence generated by a harmonic map from S^2 to $G(2,5;\mathbb{R})$ and obtain some characteristics of the corresponding harmonic map in $G(2,5;\mathbb{R})$. In the last section, we discuss geometric properties of conformal minimal 2-spheres immersed in $G(2,5;\mathbb{R})$ with constant curvature and give a classification theorem of linearly full totally unramified conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$ (see Theorem 4.9). In addition, we give a formula about Kähler angle of conformal minimal immersions from S^2 to Q_n.

2. Preliminaries.

(A) For $0 \leq k \leq n$, let $G(k,n;\mathbb{R})$ denote the Grassmannian of all real k-dimensional subspaces of $\mathbb{R}^n$ and

$$\sigma : G(k,n;\mathbb{C}) \to G(k,n;\mathbb{C})$$

denote the complex conjugation of $G(k,n;\mathbb{C})$. It is easy to see that σ is an isometry with the standard Riemannian metric of $G(k,n;\mathbb{C})$. Its fixed point set is $G(k,n;\mathbb{R})$, thus $G(k,n;\mathbb{R})$ lies totally geodesically in $G(k,n;\mathbb{C})$.

Map

$$Q_{n-2} \to G(2,n;\mathbb{R})$$

by

$$q \mapsto \frac{\sqrt{-1}}{2} Z \wedge \overline{Z},$$

where $q \in Q_{n-2}$ and Z is a homogeneous coordinate vector of q. It is clear that the map is well defined. We can easily check that the map is one-to-one and onto, and it is an isometry. Thus we can identify Q_{n-2} and $G(2,n;\mathbb{R})$ (for more details see [**14**]). Here we suppose that the metric on $G(2,n;\mathbb{R})$ is given by Section 2 of [**11**], then the metric is twice as much as the standard metric on Q_{n-2} induced by the inclusion $\tau : Q_{n-2} \to \mathbb{C}P^{n-1}$, where this latter space is given the Fubini-Study metric of constant holomorphic sectional curvature 4.

(B) In this section we simply introduce harmonic maps and harmonic sequences in $G(k,n;\mathbb{C})$ and calculate some corresponding geometric quantities.

Let M be an arbitrary Riemann surface and let $\varphi : M \to G(k,n;\mathbb{C})$ be a map. We shall frequently use one-to-one correspondence between maps $\varphi : M \to G(k,n;\mathbb{C})$ and rank k subbundles $\underline{\varphi}$ of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$ given by setting the fibre $\underline{\varphi}_x = \varphi(x)$ for all $x \in M$. Then $\underline{\varphi}$ is called (a) *harmonic* ((*sub-*) *bundle*) whenever φ is a harmonic map (cf. [**5**]).

Let $(z,\bar{z})$ be a complex coordinate on M. We take the metric $ds_M^2 = dzd\bar{z}$ on M. Denote

$$\partial = \frac{\partial}{\partial z}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}.$$

Let $\varphi : S^2 \to G(k,n;\mathbb{C})$ be a smooth harmonic map. Then from φ two harmonic sequences are derived as follows:

$$\underline{\varphi} = \underline{\varphi}_0 \xrightarrow{\partial'} \underline{\varphi}_1 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi}_\alpha \xrightarrow{\partial'} \cdots, \tag{2.1}$$

$$\underline{\varphi} = \underline{\varphi}_0 \xrightarrow{\partial''} \underline{\varphi}_{-1} \xrightarrow{\partial''} \cdots \xrightarrow{\partial''} \underline{\varphi}_{-\alpha} \xrightarrow{\partial''} \cdots, \tag{2.2}$$

where $\underline{\varphi}_\alpha = \partial' \underline{\varphi}_{\alpha-1}$ and $\underline{\varphi}_{-\alpha} = \partial'' \underline{\varphi}_{-\alpha+1}$ are Hermitian orthogonal projections from $S^2 \times \mathbb{C}^n$ onto $\underline{Im}\,(\varphi_{\alpha-1}^\perp \partial \varphi_{\alpha-1})$ and $\underline{Im}\,(\varphi_{-\alpha+1}^\perp \bar{\partial} \varphi_{-\alpha+1})$ respectively, $\alpha = 1,2,\ldots$.

As in [**2**] call a harmonic map $\varphi : S^2 \to G(k,n;\mathbb{C})$ (*strongly*) *isotropic* if $\varphi_\alpha \perp \varphi$ $\forall \alpha \in \mathbb{Z}$, $\alpha \neq 0$.

For an arbitrary harmonic map $\varphi : S^2 \to G(k,n;\mathbb{C})$, define its *isotropy order* (cf. [**5**]) to be the greatest integer r such that $\varphi_\alpha \perp \varphi$ for all α with $1 \le \alpha \le r$; if $\underline{\varphi}$ is isotropic, set $r = \infty$.

DEFINITION 2.1. Let $\varphi : S^2 \to G(k,n;\mathbb{C})$ be a map. φ is *linearly full* if $\underline{\varphi}$ cannot be contained in any proper trivial subbundle $S^2 \times \mathbb{C}^m$ of $S^2 \times \mathbb{C}^n$ $(m < n)$.

In this paper, we always assume that φ is linearly full.

Suppose that $\varphi : S^2 \to G(2,n;\mathbb{C})$ is a linearly full harmonic map and belongs to the following harmonic sequence:

$$\underline{\varphi}_0 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi} = \underline{\varphi}_\alpha \xrightarrow{\partial'} \underline{\varphi}_{\alpha+1} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\varphi}_{\alpha_0} \xrightarrow{\partial'} 0 \tag{2.3}$$

for $\alpha = 0,\ldots,\alpha_0$. We choose the local unit orthogonal frame $e_1^{(\alpha)}, e_2^{(\alpha)}, \ldots, e_{k_\alpha}^{(\alpha)}$ such that they locally span subbundle $\underline{\varphi}_\alpha$ of $S^2 \times \mathbb{C}^n$, where $k_\alpha = \operatorname{rank} \underline{\varphi}_\alpha$.

Let $W_\alpha = \left(e_1^{(\alpha)}, e_2^{(\alpha)}, \ldots, e_{k_\alpha}^{(\alpha)}\right)$ be $(n \times k_\alpha)$-matrix. Then we have

$$\varphi_\alpha = W_\alpha W_\alpha^*,$$
$$W_\alpha^* W_\alpha = I_{k_\alpha \times k_\alpha}, \quad W_\alpha^* W_{\alpha+1} = 0, \quad W_\alpha^* W_{\alpha-1} = 0. \tag{2.4}$$

By (2.4), a straightforward computation shows that

$$\begin{cases} \partial W_\alpha = W_{\alpha+1}\Omega_\alpha + W_\alpha \Psi_\alpha, \\ \bar{\partial} W_\alpha = -W_{\alpha-1}\Omega^*_{\alpha-1} - W_\alpha \Psi^*_\alpha, \end{cases} \tag{2.5}$$

where Ω_α is a $(k_{\alpha+1} \times k_\alpha)$-matrix and Ψ_α is a $(k_\alpha \times k_\alpha)$-matrix.

Set $L_\alpha = \mathrm{tr}(\Omega_\alpha \Omega^*_\alpha)$. By a straightforward calculation, the metric induced by φ_α is given by

$$ds^2_\alpha = (L_{\alpha-1} + L_\alpha) dz d\bar{z}. \tag{2.6}$$

The Laplacian $\triangle_\alpha$ and the curvature K_α of ds^2_α are given by

$$\triangle_\alpha = \frac{4}{L_{\alpha-1} + L_\alpha} \partial\bar{\partial}, \qquad K_\alpha = -\frac{2}{L_{\alpha-1} + L_\alpha} \partial\bar{\partial} \log(L_{\alpha-1} + L_\alpha). \tag{2.7}$$

Set

$$\delta_\alpha = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} L_\alpha d\bar{z} \wedge dz. \tag{2.8}$$

In the following, we give a definition of the unramified harmonic map as follows:

DEFINITION 2.2 ([**11**]). If $\det(\Omega_\alpha \Omega^*_\alpha) dz^{k_{\alpha+1}} d\bar{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (2.3), we say that $\varphi_\alpha : S^2 \to G(k_\alpha, n; \mathbb{C})$ is *unramified.* If $\det(\Omega_\alpha \Omega^*_\alpha) dz^{k_{\alpha+1}} d\bar{z}^{k_{\alpha+1}} \neq 0$ everywhere on S^2 in (2.1) (resp. (2.2)) for each $\alpha = 0, 1, 2, \ldots$, we say that the harmonic sequence (2.1) (resp. (2.2)) is *totally unramified.* If (2.1) and (2.2) are both totally unramified, we say that φ is *totally unramified.*

Now recall ([**5**, Section 3A]) that a harmonic map $\varphi : S^2 \to G(k, n; \mathbb{C})$ in (2.1) (resp. (2.2)) is said to be *∂'-irreducible* (resp. *∂''-irreducible*) if rank $\underline{\varphi}$ = rank $\underline{\varphi}_1$ (resp. rank $\underline{\varphi}$ = rank $\underline{\varphi}_{-1}$) and *∂'-reducible* (resp. *∂''-reducible*) otherwise. In particular, let φ be a harmonic map from S^2 to $G(2, n; \mathbb{R})$, then φ is ∂'-irreducible (resp. ∂'-reducible) if and only if φ is ∂''-irreducible (resp. ∂''-reducible). In this case we simply call that φ is irreducible (resp. reducible). Assume that φ_α in (2.3) is ∂'-irreducible and unramified, then $|\det \Omega_\alpha|^2 dz^{k_\alpha} d\bar{z}^{k_\alpha}$ is a well-defined invariant and has no isolated zeros on S^2, then we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{S^2} \partial\bar{\partial} \log |\det \Omega_\alpha|^2 d\bar{z} \wedge dz = -2k_\alpha. \tag{2.9}$$

(C) In this section, we review the rigidity theorem of conformal minimal immersions with constant curvature from S^2 to $\mathbb{C}P^n$.

Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion, a harmonic sequence is derived as follows

$$0 \xrightarrow{\partial'} \underline{\psi}^{(n)}_0 \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\psi} = \underline{\psi}^{(n)}_p \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{\psi}^{(n)}_n \xrightarrow{\partial'} 0, \tag{2.10}$$

for some $p = 0, 1, \ldots, n$.

We define a sequence $f_0^{(n)}, \ldots, f_n^{(n)}$ of local sections of $\underline{\psi}_0^{(n)}, \ldots, \underline{\psi}_n^{(n)}$ inductively such that $f_0^{(n)}$ is a nowhere zero local section of $\underline{\psi}_0^{(n)}$ (without loss of generality, we assume that $\overline{\partial} f_0^{(n)} \equiv 0$) and $f_{p+1}^{(n)} = \psi_p^{(n)\perp}(\partial f_p^{(n)})$ for $p = 0, \ldots, n-1$. Then we have some formulae as follows

$$\partial f_p^{(n)} = f_{p+1}^{(n)} + \frac{\langle \partial f_p^{(n)}, f_p^{(n)} \rangle}{|f_p^{(n)}|^2} f_p^{(n)}, \quad p = 0, \ldots, n,$$

$$\overline{\partial} f_p^{(n)} = -\frac{|f_p^{(n)}|^2}{|f_{p-1}^{(n)}|^2} f_{p-1}^{(n)}, \quad p = 1, \ldots, n.$$

Let

$$l_p^{(n)} = |f_{p+1}^{(n)}|^2 / |f_p^{(n)}|^2, \quad p = 0, \ldots, n-1, \quad l_{-1}^{(n)} = l_n^{(n)} = 0. \tag{2.11}$$

Then Bolton et al ([**3**]) proved the following unintegrated Plücker formula

$$\partial\overline{\partial} \log l_p^{(n)} = l_{p+1}^{(n)} - 2 l_p^{(n)} + l_{p-1}^{(n)}, \quad p = 0, \ldots, n-1.$$

Let $F_p^{(n)} = f_0^{(n)} \wedge \cdots \wedge f_p^{(n)}$ be a local lift of the p-th osculating curve, where $p = 0, \ldots, n$. We write $F_p^{(n)} = g(z) \tilde{F}_p^{(n)}$, where $g(z)$ is the greatest common divisor of the $\binom{n+1}{p+1}$ components of $F_p^{(n)}$. Then $\tilde{F}_p^{(n)}$ is a nowhere zero holomorphic curve, and the degree $\delta_p^{(n)}$ of $F_p^{(n)}$ is given by $\delta_p^{(n)} = (1/2\pi\sqrt{-1}) \int_{S^2} \partial\overline{\partial} \log |F_p^{(n)}|^2 d\overline{z} \wedge dz$, which is equal to the degree of the polynomial function $\tilde{F}_p^{(n)}$. By a simple calculation we have

$$\delta_p^{(n)} = \frac{1}{2\pi\sqrt{-1}} \int_{S^2} l_p^{(n)} d\overline{z} \wedge dz, \tag{2.12}$$

which is consistent with (2.8) in the case $k = 1$.

Moreover, if (2.10) is a totally unramified harmonic sequence (i.e. $\psi_p^{(n)}$ is unramified, $p = 0, \ldots, n$), then (cf. [**3**])

$$\delta_p^{(n)} = (p+1)(n-p). \tag{2.13}$$

Let

$$0 \longrightarrow \underline{V}_0^{(n)} \xrightarrow{\partial'} \underline{V}_1^{(n)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{V}_n^{(n)} \xrightarrow{\partial'} 0,$$

which is called the Veronese sequence, defined by $V_p^{(n)} = (v_{p,0}, \ldots, v_{p,n})^T$, where, for $z \in S^2$,

$$v_{p,r}(z) = \frac{p!}{(1+z\overline{z})^p}\sqrt{\binom{n}{r}}z^{r-p}\sum_k(-1)^k\binom{r}{p-k}\binom{n-r}{k}(z\overline{z})^k,$$

$\max\{0, p-r\} \leq k \leq \min\{p, n-r\}$, and $|V_p^{(n)}|^2 = (n!p!/(n-p)!)(1+z\overline{z})^{n-2p}$. Each map $\underline{V}_p^{(n)}$ has induced metric

$$ds_p^2 = \frac{n+2p(n-p)}{(1+z\overline{z})^2}dzd\overline{z}, \tag{2.14}$$

the corresponding constant curvature K_p and constant Kähler angle θ_p are given by

$$K_p = \frac{4}{n+2p(n-p)}, \quad \left(\tan\frac{1}{2}\theta_p\right)^2 = \frac{p(n-p+1)}{(p+1)(n-p)}. \tag{2.15}$$

By Calabi's rigidity theorem, Bolton et al proved the following rigidity result (cf. [**3**]).

LEMMA 2.3 ([**3**]). *Let $\psi : S^2 \to \mathbb{C}P^n$ be a linearly full conformal minimal immersion of constant curvature. Then, up to a holomorphic isometry of $\mathbb{C}P^n$, the harmonic sequence determined by ψ is the Veronese sequence.*

3. Characterization of harmonic maps from S^2 to $G(2,5;\mathbb{R})$.

We analyze harmonic maps from S^2 to $G(2,5;\mathbb{R})$ by reducible and irreducible case respectively. It follows from [**2**] that all reducible harmonic maps from S^2 to $G(2,5;\mathbb{R})$ with finite isotropy order have been characterized by harmonic maps from S^2 to $\mathbb{C}P^4$, and for the strongly isotropic ones we will discuss in detail in Subsection 4.1 below.

Now we only consider irreducible harmonic maps $\varphi : S^2 \to G(2,5;\mathbb{R})$ of isotropy order r. If φ has finite isotropy order, then $r = 1$ by ([**2**, Proposition 2.8 and Lemma 2.15]); if φ is strongly isotropic, then $r = \infty$. But for any irreducible harmonic map from S^2 to $G(2,n;\mathbb{R})$, if it is strongly isotropic, then we have $n \geq 6$. Therefore the isotropy order r of φ must be finite and $r = 1$.

Here we state one of Bahy-El-Dien and Wood' results ([**2**, Theorem 4.7]) as follows:

LEMMA 3.1 ([**2**]). *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be an irreducible harmonic map of isotropy order r. We know that $r = 1$. Then there is a unique sequence of harmonic maps $\varphi^i : S^2 \to G(2,5;\mathbb{C})$, $(i = 0,1,2)$ such that*

- (i) *$\underline{\varphi}^0$ is a real mixed pair, in fact $\underline{\varphi}^0 = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$, where $f_0^{(4)} \in H_5^1$;*
- (ii) *$\underline{\varphi} = \underline{\varphi}^2$;*
- (iii) *$\underline{\varphi}^1$ is obtained from $\underline{\varphi}^0$ by forward replacement of $\underline{f}_0^{(4)}$;*
- (iv) *$\underline{\varphi}^2$ is obtained from $\underline{\varphi}^1$ by backward replacement of $\underline{V}^\perp \cap \underline{\varphi}^1$, where $\underline{V}$ is a holomorphic line subbundle of $\underline{\varphi}^1$ not equal to the image of the first ∂'-return map of $\underline{\varphi}^1$.*

Firstly we recall ([**2**, Section 4]) that H_n^s denote the set of all holomorphic maps $f_0^{(m)} : S^2 \to \mathbb{C}P^m \subset \mathbb{C}P^{n-1}$, $m < n$ satisfying

$$\begin{cases} \langle f_i^{(m)}, \overline{f}_0^{(m)} \rangle = 0 \quad (0 \leq i \leq 2s+1), \\ \langle f_{2s+2}^{(m)}, \overline{f}_0^{(m)} \rangle \neq 0 \end{cases}$$

for any integers $n \geq 3$, $s \geq 0$, where $0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_{2s+1}^{(m)} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0$ is a harmonic sequence in $\mathbb{C}P^m \subset \mathbb{C}P^{n-1}$.

Let $\varphi : S^2 \to G(2, 5; \mathbb{R})$ be a linearly full irreducible harmonic map of isotropy order 1. In the following we characterize φ explicitly by Lemma 3.1.

In (i) of Lemma 3.1, φ^0 with isotropy order 3 belongs to the harmonic sequence as follows:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_4^{(4)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \underline{\overline{f}}_1^{(4)} \xleftarrow{\partial''} \underline{\varphi}^0 \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{3.1}$$

where $\underline{\varphi}^0 = \underline{\overline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ and

$$0 \xrightarrow{\partial'} \underline{f}_0^{(4)} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \underline{f}_2^{(4)} \xrightarrow{\partial'} \underline{f}_3^{(4)} \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0 \tag{3.2}$$

is a harmonic sequence in $\mathbb{C}P^4$. Since $f_0^{(4)} \in H_5^1$, then we have

$$\begin{cases} \langle \overline{f}_0^{(4)}, f_i^{(4)} \rangle = 0 \quad \text{for } 0 \leq i \leq 3, \\ \langle \overline{f}_0^{(4)}, f_4^{(4)} \rangle \neq 0. \end{cases} \tag{3.3}$$

Thus we get

$$\underline{\overline{f}}_0^{(4)} = \underline{f}_4^{(4)}, \quad \underline{\overline{f}}_1^{(4)} = \underline{f}_3^{(4)}, \quad \underline{\overline{f}}_2^{(4)} = \underline{f}_2^{(4)},$$

and

$$l_0^{(4)} = l_3^{(4)}, \quad l_1^{(4)} = l_2^{(4)}.$$

By (iii) of Lemma 3.1, $\underline{\varphi}^1$ is obtained from $\underline{\varphi}^0$ by forward replacement of $\underline{f}_0^{(4)}$, using (3.1) we have

$$\underline{\varphi}^1 = \underline{\overline{f}}_0^{(4)} \oplus \underline{f}_1^{(4)}.$$

The isotropy order of φ^1 is 2, and a harmonic sequence is derived as follows:

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_4^{(4)} \xleftarrow{\partial''} \underline{\overline{f}}_3^{(4)} \xleftarrow{\partial''} \underline{\overline{f}}_2^{(4)} \xleftarrow{\partial''} \underline{\varphi}_{-1}^1 \xleftarrow{\partial''} \underline{\varphi}^1 \xrightarrow{\partial'} \underline{f}_2^{(4)} \xrightarrow{\partial'} \underline{f}_3^{(4)} \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{3.4}$$

where $\underline{\overline{\varphi}}^1_{-1} = \underline{\varphi}^1$.

From (3.4), the image of the first ∂'-return map of $\underline{\varphi}^1$ is $\underline{\overline{f}}_0^{(4)}$. By (iv) of Lemma 3.1, let $V = f_1^{(4)} + x_0\overline{f}_0^{(4)}$, where x_0 is a smooth function on S^2 expect some isolated points. Moreover, let $X = -|f_0^{(4)}|^2\overline{x}_0 f_1^{(4)} + |f_1^{(4)}|^2\overline{f}_0^{(4)}$, it satisfies $\underline{X} = \underline{V}^\perp \cap \underline{\varphi}^1$. Since $\underline{\varphi}^2$ is obtained from $\underline{\varphi}^1$ by backward replacement of $\underline{X}$, then we have $\underline{\varphi}^2 = \underline{V} \oplus \underline{W}$ where $W = \varphi^{1\perp}\overline{\partial}X$. Moreover, $\underline{\varphi}^2$ with isotropy order 1 belongs to the harmonic sequence as follows:

$$0 \overset{\partial''}{\longleftarrow} \underline{\overline{f}}_4^{(4)} \overset{\partial''}{\longleftarrow} \underline{\overline{f}}_3^{(4)} \overset{\partial''}{\longleftarrow} \underline{Y} \oplus \underline{\overline{f}}_2^{(4)} \overset{\partial''}{\longleftarrow} \underline{\varphi}^2 \overset{\partial'}{\longrightarrow} \underline{\overline{Y}} \oplus \underline{f}_2^{(4)} \overset{\partial'}{\longrightarrow} \underline{f}_3^{(4)} \overset{\partial'}{\longrightarrow} \underline{f}_4^{(4)} \overset{\partial'}{\longrightarrow} 0, \tag{3.5}$$

where $\underline{Y} = \underline{W}^\perp \cap \underline{\overline{\varphi}}^1$. Applying the equation $W = \varphi^{1\perp}\overline{\partial}X$ we obtain

$$W = |f_1^{(4)}|^2\overline{V}, \tag{3.6}$$

which implies that

$$\underline{W} = \underline{\overline{V}},\ \underline{Y} = \underline{\overline{X}}.$$

Obviously, $\underline{\overline{X}}, \underline{X}, \underline{\overline{V}}$ and $\underline{V}$ are mutually orthogonal. Then we have $\underline{\varphi} = \underline{\overline{V}} \oplus \underline{V}$ and (3.5) becomes

$$0 \overset{\partial''}{\longleftarrow} \underline{\overline{f}}_4^{(4)} \overset{\partial''}{\longleftarrow} \underline{\overline{f}}_3^{(4)} \overset{\partial''}{\longleftarrow} \underline{\overline{X}} \oplus \underline{\overline{f}}_2^{(4)} \overset{\partial''}{\longleftarrow} \underline{\varphi} \overset{\partial'}{\longrightarrow} \underline{X} \oplus \underline{f}_2^{(4)} \overset{\partial'}{\longrightarrow} \underline{f}_3^{(4)} \overset{\partial'}{\longrightarrow} \underline{f}_4^{(4)} \overset{\partial'}{\longrightarrow} 0.$$

Since $\underline{V}$ is a holomorphic line subbundle of $\underline{\varphi}^1$, we get

$$\varphi^1(\overline{\partial}V) \in \underline{V}. \tag{3.7}$$

Through a direct computation, condition (3.7) is equivalent to the following equation

$$\partial\overline{x}_0 + \overline{x}_0\partial\log|f_0^{(4)}|^2 = 0. \tag{3.8}$$

Then we have

PROPOSITION 3.2. *The map $\varphi : S^2 \to G(2,5;\mathbb{R})$ is a linearly full irreducible harmonic map if and only if $\underline{\varphi} = \underline{\overline{V}} \oplus \underline{V}$, where $V = f_1^{(4)} + x_0\overline{f}_0^{(4)}$, $f_0^{(4)} \in H_5^1$, and the corresponding coefficient x_0 satisfies the equation* (3.8).

PROOF. Through the construction of φ as shown above, the necessity is obvious. Since $f_0^{(4)} \in H_5^1$, using (3.8), this is a straightforward computation $\varphi^\perp\partial\overline{\partial}\varphi\varphi = 0$, which implies that $\underline{\varphi}$ is harmonic. Thus we get the sufficiency. □

4. Conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$.

In this section, we regard harmonic maps from S^2 to $G(2,5;\mathbb{R})$ as conformal minimal immersions of S^2 in $G(2,5;\mathbb{R})$. Then we consider the harmonic maps of constant curvature from S^2 to $G(2,5;\mathbb{R})$ by reducible case and irreducible case. So we divide these two cases into the following two subsections.

4.1. Reducible harmonic maps of constant curvature from S^2 to $G(2,5;\mathbb{R})$.

Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full reducible harmonic map, then by ([**2**, Proposition 2.12]) we know that φ is a real mixed pair with finite isotropy order 1 or 3, or φ is strongly isotropic. In the following we discuss these three cases with φ of constant curvature respectively.

(I) If φ is a linearly full real mixed pair with isotropy order 1, then

$$\underline{\varphi} = \underline{\overline{f}}_0^{(m)} \oplus \underline{f}_0^{(m)}$$

for $2 \le m \le 4$. By using φ, a harmonic sequence is derived as follows

$$0 \xleftarrow{\partial''} \underline{\overline{f}}_m^{(m)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \underline{\overline{f}}_1^{(m)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0,$$

where

$$0 \xrightarrow{\partial'} \underline{f}_0^{(m)} \xrightarrow{\partial'} \underline{f}_1^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0$$

is a harmonic sequence in $\mathbb{C}P^m \subset \mathbb{C}P^4$ satisfies

$$\langle \overline{f}_0^{(m)}, f_0^{(m)} \rangle = 0, \quad \langle \overline{f}_0^{(m)}, f_1^{(m)} \rangle = 0, \quad \langle \overline{f}_0^{(m)}, f_2^{(m)} \rangle \neq 0. \tag{4.1}$$

The induced metric of φ is given by

$$ds^2 = 2l_0^{(m)} dz d\bar{z}, \tag{4.2}$$

where $l_0^{(m)} dz d\bar{z}$ is the induced metric of $f_0^{(m)}$.

Then we prove

LEMMA 4.1. *There does not exist linearly full real mixed pair of constant curvature from S^2 to $G(2,5;\mathbb{R})$ with isotropy order* 1.

PROOF. Since $\underline{\varphi}$ is of constant curvature, using (4.2) we get that the constant curvature K of φ satisfies $K = 2/m$. By Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(m)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that $f_0^{(m)} = UV_0^{(m)}$, where $U \in U(5)$ and $V_0^{(m)}$ has the standard expression given in part (C) of Section 2 (adding zeros to $V_0^{(m)}$ such that $V_0^{(m)} \in \mathbb{C}^5$). Then from (4.1) we have

$$\begin{cases} \langle UV_0^{(m)}, \overline{UV_0^{(m)}} \rangle = 0, \\ \langle UV_1^{(m)}, \overline{UV_0^{(m)}} \rangle = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \mathrm{tr} W V_0^{(m)} V_0^{(m)T} = 0, \\ \mathrm{tr} W V_1^{(m)} V_0^{(m)T} = 0, \end{cases} \tag{4.3}$$

where $W = U^T U$, it satisfies $W \in U(5)$ and $W^T = W$.

Define a set

$$G_W \triangleq \{U \in U(5) | U^T U = W\}. \tag{4.4}$$

For a given W, the following can be easily checked

(1) $\forall\ U \in G_W,\ A \in SO(5)$, we have $AU \in G_W$;
(2) $\forall\ U, V \in G_W, \exists\ A \in SO(5),\ s.t.\ U = AV$.

In the following we discuss W in cases $m = 2, 3, 4$ respectively.

(Ia) $m = 4,\ K = 1/2$.

By the standard expression of $V_0^{(4)}$ and $V_1^{(4)}$, we get $V_1^{(4)} V_0^{(4)T}$ is a polynomial matrix in z and $\bar{z}$. But W is a constant matrix. Using the method of indeterminate coefficients by (4.3), assume $W = (a_{ij}),\ 1 \le i, j \le 5$, we get

$$W = \begin{pmatrix} 0 & 0 & a_{13} & -\sqrt{6}a_{23} & a_{15} \\ 0 & (-\sqrt{6}/2)a_{13} & a_{23} & a_{24} & -\sqrt{6}a_{34} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ -\sqrt{6}a_{23} & a_{24} & a_{34} & (-\sqrt{6}/2)a_{35} & 0 \\ a_{15} & -\sqrt{6}a_{34} & a_{35} & 0 & 0 \end{pmatrix},$$

where

$$a_{15} + 3a_{33} + 4a_{24} = 0.$$

Applying the equation $a_{15} + 3a_{33} + 4a_{24} = 0$, using the property of the unitary matrix, this is a straightforward computation

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & -a_{24} \\ 0 & 0 & 0 & a_{24} & 0 \\ 0 & 0 & -a_{24} & 0 & 0 \\ 0 & a_{24} & 0 & 0 & 0 \\ -a_{24} & 0 & 0 & 0 & 0 \end{pmatrix} \in U(5).$$

With a simple test we have

$$\mathrm{tr} W V_2^{(4)} V_0^{(4)T} = 0,$$

i.e. $\langle \overline{f}_0^{(4)}, f_2^{(4)} \rangle = 0$, which contradicts $r = 1$. Thus this case does not exist.

(Ib) $m = 3,\ K = 2/3$.
Similar to (Ia), we have

$$W = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & (-2/\sqrt{3})a_{13} & (-1/3)a_{14} & a_{24} & a_{25} \\ a_{13} & (-1/3)a_{14} & (-2/\sqrt{3})a_{24} & 0 & a_{35} \\ a_{14} & a_{24} & 0 & 0 & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{pmatrix}.$$

Moreover, using the property of the unitary matrix, we have

$$W = \begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 \\ 0 & (-2/\sqrt{3})a_{13} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & a_{45} & 0 \end{pmatrix},$$

which contradicts $W \in U(5)$, thus this case does not exist.

(Ic) $m = 2,\ K = 1$.
From (4.3), this is a straightforward computation

$$W = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & -a_{13} & 0 & a_{24} & a_{25} \\ a_{13} & 0 & 0 & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{pmatrix}.$$

Moreover, using the property of the unitary matrix, we have

$$W = \begin{pmatrix} 0 & 0 & a_{13} & 0 & 0 \\ 0 & -a_{13} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{45} & a_{55} \end{pmatrix}, \tag{4.5}$$

where $|a_{13}| = 1$ and $\begin{pmatrix} a_{44} & a_{45} \\ a_{45} & a_{55} \end{pmatrix} \in U(2)$. Obviously φ is not linearly full in this condition.
In summary we get the conclusion. □

(II) If φ is a linearly full real mixed pair with isotropy order 3, then we have $\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ belongs to the following harmonic sequence:

$$0 \xleftarrow{\partial''} \overline{\underline{f}}_4^{(4)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \overline{\underline{f}}_1^{(4)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0, \tag{4.6}$$

where

$$0 \xrightarrow{\partial'} \underline{f}_0^{(4)} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0$$

is a harmonic sequence in $\mathbb{C}P^4$ satisfies

$$\overline{\underline{f}}_0^{(4)} = \underline{f}_4^{(4)}, \quad \overline{\underline{f}}_1^{(4)} = \underline{f}_3^{(4)}, \quad \overline{\underline{f}}_2^{(4)} = \underline{f}_2^{(4)}. \tag{4.7}$$

The induced metric of φ is given by $ds^2 = 2l_0^{(4)} dz d\overline{z}$. Since $\underline{\varphi}$ is of constant curvature, then the constant curvature K of φ is $1/2$. By Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(4)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that $f_0^{(4)} = UV_0^{(4)}$, where $U \in U(5)$ and $V_0^{(4)}$ has the standard expression given in part (C) of Section 2.

Then we have

LEMMA 4.2. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full real mixed pair with isotropy order 3. If the curvature K of φ is constant, then up to an isometry of $G(2,5;\mathbb{R})$, $\underline{\varphi} = \overline{\underline{UV}_0^{(4)}} \oplus \underline{UV}_0^{(4)}$ with $K = 1/2$ for some $U \in G \triangleq \{U \in U(5) | \overline{U} = UW_0\}$, where $W_0 = antidiag\{1, -1, 1, -1, 1\}$.*

PROOF. Equation $\overline{\underline{f}}_0^{(4)} = \underline{f}_4^{(4)}$ is equivalent to

$$\overline{UV_0^{(4)}} = \lambda UV_4^{(4)}, \tag{4.8}$$

where λ is a parameter.

Set $W_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. From part (C) of Section 2, we get

$$V_0^{(4)} = (1,\ 2z,\ \sqrt{6}z^2,\ 2z^3,\ z^4)^T$$

and

$$V_4^{(4)} = \frac{4!}{(1+z\overline{z})^4}(\overline{z}^4,\ -2\overline{z}^3,\ \sqrt{6}\overline{z}^2,\ -2\overline{z},\ 1)^T,$$

which implies $V_4^{(4)} = (4!/(1+z\overline{z})^4) W_0 \overline{V}_0^{(4)}$. Then condition (4.8) becomes

$$\overline{U} = UW_0,$$

Define a set

$$G \triangleq \{U \in U(5) | \overline{U} = UW_0\},$$

then the following can be easily checked

(1) $\forall\ U \in G,\ A \in SO(5)$, we have $AU \in G$;
(2) $\forall\ U, V \in G, \exists\ A \in SO(5),\ s.t.\ U = AV$.

So we get the conclusion. □

REMARK 4.3. $G \neq \emptyset$. Simply choose $U_0 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ \sqrt{-1}/\sqrt{2} & 0 & 0 & 0 & -\sqrt{-1}/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & \sqrt{-1}/\sqrt{2} & 0 & \sqrt{-1}/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$, we have $U_0 \in G$ and $\forall\ U \in G$ can be obtained from U_0 by an $SO(5)$-motion. Then up to an isometry of $G(2,5;\mathbb{R})$,

$$\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$$

with

$$f_0^{(4)} = \left(1 + z^4,\ \sqrt{-1}(1 - z^4),\ 2(z - z^3),\ 2\sqrt{-1}(z + z^3),\ 2\sqrt{3}z^2\right)^T. \tag{4.9}$$

(III) If φ is a linearly full reducible harmonic map with isotropy order ∞. By using φ, a harmonic sequence is derived as follows

$$0 \xleftarrow{\partial''} \overline{\underline{f}}_m^{(m)} \xleftarrow{\partial''} \cdots \xleftarrow{\partial''} \overline{\underline{f}}_{p+1}^{(m)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{f}_{p+1}^{(m)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_m^{(m)} \xrightarrow{\partial'} 0, \tag{4.10}$$

where $m \leq 4$ and $\overline{\underline{f}}_m^{(m)}, \ldots, \overline{\underline{f}}_{p+1}^{(m)}, \underline{\varphi}, \underline{f}_{p+1}^{(m)}, \ldots, \underline{f}_m^{(m)}$ are mutually orthogonal. Since φ is a map from S^2 to $G(2,5;\mathbb{R})$, then $m - p \leq 1$.

Then we have

LEMMA 4.4. *There does not exist linearly full harmonic map of constant curvature from S^2 to $G(2,5;\mathbb{R})$ with isotropy order ∞.*

PROOF. From (4.10) we know that $f_p^{(m)}$ and $\overline{f}_p^{(m)}$ are two local sections of $\underline{\varphi}$.

If $\overline{\underline{f}}_p^{(m)} = \underline{f}_p^{(m)}$, applying the inequality $m - p \leq 1$, we have $p = 1, m = 2$. Then (4.10) becomes

$$0 \xleftarrow{\partial''} \underline{f}_0^{(2)} \xleftarrow{\partial''} \underline{\varphi} \xrightarrow{\partial'} \underline{f}_2^{(2)} \xrightarrow{\partial'} 0. \tag{4.11}$$

From (4.11), by a straightforward calculation, we have

$$\underline{\varphi} = \underline{f}_1^{(2)} \oplus \underline{g},$$

where $\underline{g}$ is a constant vector in $\mathbb{C}^5$ and $\overline{\underline{f}}_1^{(2)} = \underline{f}_1^{(2)}$. Obviously φ is included in $G(2,4;\mathbb{R})$, so it is not linearly full.

If $\overline{\underline{f}}_p^{(m)} \neq \underline{f}_p^{(m)}$, this is a straightforward computation

$$\mathrm{tr}\partial\varphi\overline{\partial}\varphi = 2l_p^{(m)},$$

i.e. the induced metric of φ is given by $ds^2 = 2l_p^{(m)}dzd\overline{z}$. Since $\underline{\varphi}$ is of constant curvature, then the constant curvature K of φ satisfies $K = 2/(m-p)(p+1)$. By Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(m)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2\backslash\{pt\}$ so that $f_0^{(m)} = UV_0^{(m)}$, where $U \in U(5)$ and $V_0^{(m)}$ has the standard expression given in part (C) of Section 2 (adding zeros to $V_0^{(m)}$ such that $V_0^{(m)} \in \mathbb{C}^5$). Here $m = 3$ or 4.

For $m = 4$, $p = 3$, we can easily check that for any $U \in U(5)$ satisfies $\mathrm{tr}U^TUV_4^{(4)}V_4^{(4)T} = 0$, we also have $\mathrm{tr}U^TUV_3^{(4)}V_3^{(4)T} = 0$. Thus we have $\underline{\varphi} = \overline{\underline{f}}_3^{(4)} \oplus \underline{f}_3^{(4)}$, which implies that φ is irreducible. For $m = 3$, $p = 2$, a straightforward calculation shows that U^TU does not exist.

In summary we get the conclusion. □

REMARK 4.5. In the case $\overline{\underline{f}}_p^{(m)} = \underline{f}_p^{(m)}$ in Lemma 4.4, we have $\underline{\varphi} = \underline{f}_1^{(2)} \oplus \underline{g}$, where $\underline{g}$ is a constant vector in $\mathbb{C}^5$ and $\overline{\underline{f}}_1^{(2)} = \underline{f}_1^{(2)}$. Since $\underline{\varphi}$ is of constant curvature, then the curvature of $\underline{f}_1^{(2)}$ is also a constant. By Lemma 2.3, there exists some $U \in U(5)$ so that

$$\underline{f}_1^{(2)} = \underline{UV}_1^{(2)}, \quad \overline{\underline{UV}}_1^{(2)} = \underline{UV}_1^{(2)}.$$

By a straightforward calculation, we have, up to an isometry of $G(2,5;\mathbb{R})$,

$$f_1^{(2)} = \left(\sqrt{-1}(z-\overline{z}),\ z\overline{z}-1,\ z+\overline{z}\right)^T,$$

and the curvature of $\underline{\varphi}$ is 1. Here $\underline{\varphi} = \underline{f}_1^{(2)} \oplus \underline{g}$ is a linearly full harmonic map from S^2 into $G(2,4;\mathbb{R})$. Moreover we can check that φ is totally geodesic.

From Lemma 4.1, 4.2 and 4.4 we have

PROPOSITION 4.6. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full reducible harmonic map with constant curvature K. Then, up to an isometry of $G(2,5;\mathbb{R})$, $\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ with $K = 1/2$, where $f_0^{(4)}$ satisfies* (4.9).

4.2. Irreducible harmonic maps of constant curvature from S^2 to $G(2,5;\mathbb{R})$.

In this section, we discuss linearly full irreducible harmonic maps from S^2 to $G(2,5;\mathbb{R})$ with constant curvature in Section 3.

Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full irreducible harmonic map of isotropy order r. From the discussion of Section 3, we know that $r = 1$. By Proposition 3.2, we choose local frame

$$e_1 = \frac{\overline{V}}{|V|},\ e_2 = \frac{V}{|V|},\ e_3 = \frac{X}{|X|},\ e_4 = \frac{f_2^{(4)}}{|f_2^{(4)}|},\ e_5 = \frac{\overline{X}}{|X|},\ e_6 = \frac{\overline{f}_2^{(4)}}{|f_2^{(4)}|},\ e_7 = \frac{f_3^{(4)}}{|f_3^{(4)}|},$$

where $V = f_1^{(4)} + x_0\overline{f}_0^{(4)}$ and x_0 is a smooth function on S^2 except some isolate points. Since the isotropy order of φ is 1, the local frame we choose here is not unitary frame.

Set $W_0 = (e_1, e_2)$, $W_1 = (e_3, e_4)$, $W_{-1} = (e_5, e_6)$, and $W_2 = (e_7)$, then by (2.5), we obtain

$$\Omega_0 = \begin{pmatrix} -\frac{|f_1^{(4)}|}{|f_0^{(4)}|} & \frac{\langle \partial V, X\rangle}{|X||V|} \\ 0 & \frac{|f_2^{(4)}|}{|V|} \end{pmatrix},\quad \Omega_{-1} = -\begin{pmatrix} \frac{\langle \partial V, X\rangle}{|X||V|} & \frac{|f_2^{(4)}|}{|V|} \\ -\frac{|f_1^{(4)}|}{|f_0^{(4)}|} & 0 \end{pmatrix},\quad \Omega_1 = \left(0,\ \frac{|f_3^{(4)}|}{|f_2^{(4)}|}\right). \tag{4.12}$$

From (4.12), applying the equation $L_\alpha = \operatorname{tr}(\Omega_\alpha\Omega_\alpha^*)$, a straightforward computation shows

$$L_0 = L_{-1} = \frac{\langle \partial V, X\rangle\langle X, \partial V\rangle}{|X|^2|V|^2} + \frac{|f_2^{(4)}|^2}{|V|^2} + l_0^{(4)}, \tag{4.13}$$

$$L_1 = l_2^{(4)}, \tag{4.14}$$

$$|\det\Omega_0|^2 dz^2 d\overline{z}^2 = \frac{|f_0^{(4)}|^2}{|V|^2}(l_0^{(4)})^2 l_1^{(4)} dz^2 d\overline{z}^2, \tag{4.15}$$

$$\det\Omega_1\Omega_1^* dz d\overline{z} = l_2^{(4)} dz d\overline{z}. \tag{4.16}$$

Since $\varphi_{-1}, \varphi_0, \varphi_1$ are not mutually orthogonal, we can't use the unintegrated Plücker formula directly. But using (4.13) and (4.14), by a straightforward calculation, we also have

$$\partial\overline{\partial}\log|\det\Omega_0|^2 = L_{-1} - 2L_0 + L_1. \tag{4.17}$$

If φ is totally unramified, then $|\det\Omega_0|^2 dz^2 d\overline{z}^2 \neq 0$ and $\det\Omega_1\Omega_1^* dz d\overline{z} \neq 0$ everywhere on S^2. It follows from (4.15) and (4.16) that $l_p^{(4)} dz d\overline{z} \neq 0$ $(p = 0, 1, 2)$ everywhere on S^2 and $(|f_0^{(4)}|^2/|V|^2)l_0^{(4)}$ is well-defined on S^2. In Section 3 we have $l_0^{(4)} = l_3^{(4)}$ and $l_1^{(4)} = l_2^{(4)}$. So $l_p^{(4)} dz d\overline{z} \neq 0$ $(p = 0, 1, 2, 3)$ everywhere on S^2. Then the harmonic sequence

$$0 \xrightarrow{\partial'} \underline{f}_0^{(4)} \xrightarrow{\partial'} \underline{f}_1^{(4)} \xrightarrow{\partial'} \cdots \xrightarrow{\partial'} \underline{f}_4^{(4)} \xrightarrow{\partial'} 0$$

is also totally unramified.

In this case, we prove

PROPOSITION 4.7. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full irreducible totally unramified harmonic map with constant curvature K. Then, up to an isometry of $G(2,5;\mathbb{R})$, $\underline{\varphi} = \overline{\underline{UV}_1^{(4)}} \oplus \underline{UV}_1^{(4)}$ with $K = 1/5$ for some $U \in G$.*

PROOF. Since the harmonic sequence $\underline{f}_0^{(4)}, \ldots, \underline{f}_4^{(4)} : S^2 \to \mathbb{C}P^4$ is totally unramified, it follows from (2.13) that

$$\delta_0^{(4)} = \delta_3^{(4)} = 4, \quad \delta_1^{(4)} = \delta_2^{(4)} = 6. \tag{4.18}$$

From (2.8) and (2.9) we have

$$\delta_1 - 2\delta_0 + \delta_{-1} = -4, \tag{4.19}$$

where $\delta_\alpha = (1/2\pi\sqrt{-1}) \int_{S^2} L_\alpha d\bar{z} \wedge dz$, $\alpha = -1, 0, 1$. It follows from (4.13) and (4.14) that $\delta_0 = \delta_{-1}$ and $\delta_1 = \delta_2^{(4)} = 6$. So that

$$\delta_0 = 10. \tag{4.20}$$

Since φ is of constant curvature K, using (4.20) we know that $K = 1/5$, and we can choose a complex coordinate z on $\mathbb{C} = S^2 \backslash \{pt\}$ so that the induced metric $ds^2 = 2L_0 dz d\bar{z}$ of φ is given by

$$ds^2 = \frac{20}{(1+z\bar{z})^2} dz d\bar{z},$$

which implies

$$L_0 = \frac{10}{(1+z\bar{z})^2}. \tag{4.21}$$

Consider the local lift of the p-th osculating curve $F_p^{(4)} = f_0^{(4)} \wedge \cdots \wedge f_p^{(4)}$ ($p = 0, \ldots, 4$). We choose a nowhere zero holomorphic $\mathbb{C}^5$-valued function $f_0^{(4)}$, then $F_p^{(4)}$ is a nowhere zero holomorphic curve and is a polynomial function on $\mathbb{C}$ of degree $\delta_p^{(4)}$ satisfying $\partial\bar{\partial} \log |F_p^{(4)}|^2 = l_p^{(4)}$. So using (4.13) (4.14) (4.15) and (4.17), we obtain

$$\partial\bar{\partial} \log \frac{(1+z\bar{z})^{10} |f_0^{(4)}|^2}{|F_0^{(4)}|^6 |V|^2} = 0. \tag{4.22}$$

By (4.15) we know that $(|f_0^{(4)}|^2/|V|^2) l_0^{(4)}$ is a globally defined function without zeros on S^2. Then it follows from (4.18) that $(1+z\bar{z})^{10}|f_0^{(4)}|^2/|F_0^{(4)}|^6|V|^2$ is globally defined on $\mathbb{C}$ and has a positive constant limit $1/c$ as $z \to \infty$. Thus from (4.22) we obtain

$$\frac{(1+z\bar{z})^{10} |f_0^{(4)}|^2}{|F_0^{(4)}|^6 |V|^2} = \frac{1}{c}.$$

Moreover we have

$$|V|^2 = \frac{c(1+z\overline{z})^{10}}{|f_0^{(4)}|^4}. \tag{4.23}$$

Applying the equation $V = f_1^{(4)} + x_0\overline{f}_0^{(4)}$, (4.23) becomes

$$|x_0|^2|f_0^{(4)}|^4 + |F_1^{(4)}|^2 = \frac{c(1+z\overline{z})^{10}}{|f_0^{(4)}|^2}. \tag{4.24}$$

By equation (3.8) we get $\overline{\partial}(x_0|f_0^{(4)}|^2) = 0$. Observing (4.24), we find that $x_0|f_0^{(4)}|^2$ is a holomorphic function on $\mathbb{C}$ at most with the pole $z = \infty$. So it is a polynomial function about z. Without loss of generality, we set

$$x_0|f_0^{(4)}|^2 = h(z), \tag{4.25}$$

then (4.24) becomes

$$|h|^2 + |F_1|^2 = \frac{c(1+z\overline{z})^{10}}{|f_0^{(4)}|^2}. \tag{4.26}$$

Since both sides of (4.26) are polynomial functions and $\delta_0^{(4)} = 4$, then we have

$$|f_0^{(4)}|^2 = \mu(1+z\overline{z})^4, \tag{4.27}$$

where μ is a real parameter.

If $h \neq 0$, then $1+z\overline{z}$ is a factor of it, which contradicts the fact that h is holomorphic. Thus we have $h = 0$, which implies that $x_0 = 0$. Then we get

$$V = f_1^{(4)}, \quad \underline{\varphi} = \overline{\underline{f}}_1^{(4)} \oplus \underline{f}_1^{(4)}.$$

From (4.27), by Lemma 2.3, up to a holomorphic isometry of $\mathbb{C}P^4$, $f_0^{(4)}$ is a Veronese surface. We can choose a complex coordinate z on $\mathbb{C} = S^2\backslash\{pt\}$ so that $f_0^{(4)} = UV_0^{(4)}$, where $U \in U(5)$ and $V_0^{(4)}$ has the standard expression given in part (C) of Section 2. Thus we have $\underline{\varphi} = \overline{\underline{UV}_1^{(4)}} \oplus \underline{UV}_1^{(4)}$. To determine φ, we just need to determine the matrix U. Since $\overline{\underline{f}}_0^{(4)} = \underline{f}_4^{(4)}$, using the standard expression of $V_0^{(4)}$, we have $\overline{U} = UW_0$. Similar to Lemma 4.2, we get the conclusion. □

REMARK 4.8. We choose the same U_0 as the one shown in Remark 4.3, then

$$\underline{\varphi} = \overline{\underline{f}}_1^{(4)} \oplus \underline{f}_1^{(4)} \in G(2,5;\mathbb{R})$$

with

$$f_1^{(4)} = \big((2(z^3 - \overline{z}),\ -2\sqrt{-1}(z^3 + \overline{z}),\ (1 - 3z\overline{z}) - z^2(3 - z\overline{z}),$$
$$\sqrt{-1}[(1 - 3z\overline{z}) + z^2(3 - z\overline{z})],\ 2\sqrt{3}z(1 - z\overline{z})\big)^T. \tag{4.28}$$

Moreover we can check that φ is totally geodesic.

By Proposition 4.6 and Proposition 4.7, we obtain a classification of conformal minimal immersions of constant curvature from S^2 to $G(2,5;\mathbb{R})$ as follows:

THEOREM 4.9. *Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full conformal minimal immersion of constant curvature. Then, up to an isometry of $G(2,5;\mathbb{R})$,*

(i) *If φ is reducible, $\underline{\varphi} = \overline{\underline{f}}_0^{(4)} \oplus \underline{f}_0^{(4)}$ with constant curvature $1/2$, where $f_0^{(4)}$ satisfies (4.9);*
(ii) *If φ is totally unramified irreducible, $\underline{\varphi} = \overline{\underline{f}}_1^{(4)} \oplus \underline{f}_1^{(4)}$ with constant curvature $1/5$, where $f_1^{(4)}$ satisfies (4.28).*

Theorem 4.9 shows that all linearly full totally unramified conformal minimal immersions of two-spheres in Q_3 with constant curvature are presented by the Veronese curves in $\mathbb{C}P^4$. We believe that these maps are homogeneous.
For the isotropy order r of φ, we have

REMARK 4.10. Let $\varphi : S^2 \to G(2,5;\mathbb{R})$ be a linearly full conformal minimal immersion with constant curvature. Suppose that the isotropy order of φ is r. We then have

(i) If φ is reducible, then $r = 3$;
(ii) If φ is irreducible, then $r = 1$.

In the following, we discuss the Kähler angle of a curve from S^2 to Q_n. Throughout this section, we agree on the following ranges of indices

$$1 \le \alpha, \beta, \gamma, \ldots \le n, \quad 1 \le A, B, C, \ldots \le n+1.$$

Let $f : S^2 \to Q_n$ be a map, and $\tau : Q_n \to \mathbb{C}P^{n+1}$ denote the inclusion. The algebraic variety is given by

$$(w_0)^2 + (w_1)^2 + \cdots + (w_{n+1})^2 = 0,$$

where $(w_0, w_1, \ldots, w_{n+1})$ are homogeneous coordinate system on $\mathbb{C}P^{n+1}$. If $w^0 \neq 0$, let $z_1 = w_1/w_0, \ldots, z_{n+1} = w_{n+1}/w_0$, then we have

$$1 + z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = 0, \tag{4.29}$$

where $(z_1, \ldots, z_{n+1})$ are inhomogeneous coordinate system on $\mathbb{C}P^{n+1}$. A natural complex structure J on $\mathbb{C}P^{n+1}$ is defined by $J(\partial/\partial z_A) = \sqrt{-1}(\partial/\partial z_A)$. Suppose $z_{n+1} \neq 0$, then we have complex coordinate system $(\widetilde{z}_1, \ldots, \widetilde{z}_n)$ of Q_n such that $\widetilde{z}_1 = z_1, \ldots, \widetilde{z}_n = z_n$.

Therefore a natural complex structure $\widetilde{J}$ on Q_n is given by $\widetilde{J}(\partial/\partial\widetilde{z}_\alpha) = \sqrt{-1}(\partial/\partial\widetilde{z}_\alpha)$.

By differentiating (4.29) we obtain

$$\frac{\partial z_{n+1}}{\partial \widetilde{z}_\alpha} = -\frac{z_\alpha}{z_{n+1}},$$

which implies that

$$\tau_*\left(\frac{\partial}{\partial \widetilde{z}_\alpha}\right) = \frac{\partial}{\partial z_\alpha} - \frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}}.$$

Then we have

$$(\tau\circ f)_*\left(\frac{\partial}{\partial z}\right) = \sum_\alpha \left(\frac{\partial f^\alpha}{\partial z}\frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial z}\frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}} + \frac{\partial \overline{f}^\alpha}{\partial z}\frac{\partial}{\partial \overline{z}_\alpha} - \frac{\partial \overline{f}^\alpha}{\partial z}\frac{\overline{z}_\alpha}{\overline{z}_{n+1}}\frac{\partial}{\partial \overline{z}_{n+1}}\right), \quad (4.30)$$

and

$$(\tau\circ f)_*\left(\frac{\partial}{\partial \overline{z}}\right) = \sum_\alpha \left(\frac{\partial f^\alpha}{\partial \overline{z}}\frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial \overline{z}}\frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}} + \frac{\partial \overline{f}^\alpha}{\partial \overline{z}}\frac{\partial}{\partial \overline{z}_\alpha} - \frac{\partial \overline{f}^\alpha}{\partial \overline{z}}\frac{\overline{z}_\alpha}{\overline{z}_{n+1}}\frac{\partial}{\partial \overline{z}_{n+1}}\right), \quad (4.31)$$

where $f(z) = (f^1(z), \ldots, f^n(z))$.

For a conformal immersion $f : S^2 \to Q_n$, we define the *Kähler angle* of f to be the function $\theta : S^2 \to [0, \pi]$ given in terms of a complex coordinate $z = x + \sqrt{-1}y$ on S^2, where θ is the angle between $\widetilde{J}f_*(\partial/\partial x)$ and $f_*(\partial/\partial y)$. Since τ is a holomorphic isometry, by a simple calculation, θ is also the angle between $J(\tau\circ f)_*(\partial/\partial x)$ and $(\tau \circ f)_*(\partial/\partial y)$. It is clear that θ is globally defined. Thus we have

$$\left(\tan\frac{\theta}{2}\right)^2 = \frac{\left|(\tau\circ f)_*\left(\frac{\partial}{\partial y}\right) - J(\tau\circ f)_*\left(\frac{\partial}{\partial x}\right)\right|^2}{\left|(\tau\circ f)_*\left(\frac{\partial}{\partial y}\right) + J(\tau\circ f)_*\left(\frac{\partial}{\partial x}\right)\right|^2}.$$

Let $L = 1 + |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2$, from the metric $ds^2 = \sum_{A,B}((L\delta_{AB} - \overline{z}^A z^B)/L^2)dz^A d\overline{z}^B$ of $\mathbb{C}P^{n+1}$, using (4.30) and (4.31), we directly compute to obtain

$$\left(\tan\frac{\theta}{2}\right)^2 = \frac{\sum_\alpha\left|\frac{\partial \overline{f}^\alpha}{\partial z}\frac{\partial}{\partial \overline{z}_\alpha} - \frac{\partial \overline{f}^\alpha}{\partial z}\frac{\overline{z}_\alpha}{\overline{z}_{n+1}}\frac{\partial}{\partial \overline{z}_{n+1}}\right|^2}{\sum_\alpha\left|\frac{\partial f^\alpha}{\partial z}\frac{\partial}{\partial z_\alpha} - \frac{\partial f^\alpha}{\partial z}\frac{z_\alpha}{z_{n+1}}\frac{\partial}{\partial z_{n+1}}\right|^2}. \quad (4.32)$$

Since

$$\sum_{\alpha}\left|\frac{\partial f^{\alpha}}{\partial z}\frac{\partial}{\partial z_{\alpha}}-\frac{\partial f^{\alpha}}{\partial z}\frac{z_{\alpha}}{z_{n+1}}\frac{\partial}{\partial z_{n+1}}\right|^{2}=-\frac{1}{L^{2}}\left|\sum_{A}\partial f_{A}\overline{f}_{A}\right|^{2}+\frac{1}{L}\Big(\sum_{A}|\partial f_{A}|^{2}\Big),$$

$$\sum_{\alpha}\left|\frac{\partial \overline{f}^{\alpha}}{\partial z}\frac{\partial}{\partial \overline{z}_{\alpha}}-\frac{\partial \overline{f}^{\alpha}}{\partial z}\frac{\overline{z}_{\alpha}}{\overline{z}_{n+1}}\frac{\partial}{\partial \overline{z}_{n+1}}\right|^{2}=-\frac{1}{L^{2}}\left|\sum_{A}\overline{\partial} f_{A}\overline{f}_{A}\right|^{2}+\frac{1}{L}\Big(\sum_{A}|\overline{\partial} f_{A}|^{2}\Big),$$

then (4.32) becomes

$$\Big(\tan\frac{\theta}{2}\Big)^{2}=\frac{L\Big(\sum_{A}|\overline{\partial} f_{A}|^{2}\Big)-\left|\sum_{A}\overline{\partial} f_{A}\overline{f}_{A}\right|^{2}}{L\Big(\sum_{A}|\partial f_{A}|^{2}\Big)-\left|\sum_{A}\partial f_{A}\overline{f}_{A}\right|^{2}}. \tag{4.33}$$

Take $\underline{\psi}=\underline{f}_1^{(4)}=\underline{U}_0\underline{V}_1^{(4)}$ as an example, where U_0 is the one in Remark 4.3 and $f_1^{(4)}$ satisfies (4.28). We can easily checked that $f_1^{(4)}$ is an immersion of S^2 in Q_3. A straightforward calculation shows that

$$L\Big(\sum_{A}|\partial f_{A}|^{2}\Big)-\left|\sum_{A}\partial f_{A}\overline{f}_{A}\right|^{2}=\frac{3(1+z\overline{z})^{6}}{2|z^{3}-\overline{z}|^{4}},$$

$$L\Big(\sum_{A}|\overline{\partial} f_{A}|^{2}\Big)-\left|\sum_{A}\overline{\partial} f_{A}\overline{f}_{A}\right|^{2}=\frac{(1+z\overline{z})^{6}}{|z^{3}-\overline{z}|^{4}}.$$

Using (4.33), the Kähler angle θ of $\underline{\psi}=\underline{f}_1^{(4)}$ is given by

$$\tan^{2}\frac{\theta}{2}=\frac{2}{3}. \tag{4.34}$$

REMARK 4.11. For the example above, we can check that the Kähler angle in (4.34) satisfies (2.15). In fact, the conformal immersion from S^2 into Q_n is also a conformal immersion from S^2 into $\mathbb{C}P^{n+1}$, it is not difficult to check that their Kähler angles are equal.

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