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ON BLOW-UP CRITERION FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. The blowup is studied for the nonlinear Schrödinger equation $iu_t + \Delta u + |u|^{p-1}u = 0$ with p is odd and $p \geq 1 + \frac{4}{N-2}$ (the energy-critical or energy-supercritical case). It is shown that the solution with negative energy $E(u_0) < 0$ blows up in finite or infinite time. A new proof is also presented for the previous result in [9], in which a similar result in a case of energy-subcritical was shown.

1. **Introduction.** The Schrödinger equation is the fundamental equation in quantum mechanics. Its general form is

$$iu_t + \Delta u - Vu = 0, \tag{1}$$

where V denote the potential and $|u|^2/\|u\|^2_{L^2(\mathbb{R}^N)}$ is the probability density that the particle appears at the point (x,t). The solution u is called wave function. In this paper, we study the following well-known focusing nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{2}$$

This equation received a great deal of attention from mathematicians, in particular because of its applications to nonlinear optics, see for examples, Bergé [1], Sulem and Sulem [21]. For (2), the potential $V = -|u|^{p-1}$. Notice that V depends on the wave function u. This give the term *nonlinear*. The potential V becomes negative very large when the probability density $|u|^2/\|u\|^2_{L^2(\mathbb{R}^N)}$ is very large. This property

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brings another term *focusing*. The equation (2) has very important scaling invariant symmetry:

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \tag{3}$$

in the sense that both the equation and the $\dot{H}^{s_c}$-norm are invariant under the scaling transformation, where

$$s_c = \frac{N}{2} - \frac{2}{p-1}. \tag{4}$$

This gives the notation *critical regularity* , the lowest regularity assumption that the equation (2) is well-posed. If the critical regularity of the problem (2) is higher/lower than s, we call the problem $\dot{H}^s$*-subcritical/supercritical*. In view of this, the Schrödinger equation (2) is called energy-subcritical when $p < 1 + \frac{4}{N-2}$, which is equivalent to $s_c < 1$ (in particular, it is called the mass-critical when $p = 1 + \frac{4}{N}$ or $s_c = 0$); it is called energy-critical when $p = 1 + \frac{4}{N-2}$, which is equivalent to $s_c = 1$; and it is called energy-supercritical when $p > 1 + \frac{4}{N-2}$, which is equivalent to $s_c > 1$.

The solution of equation (2) obeys the mass, momentum and energy conservation laws, which read as

$$\begin{aligned}
M(u(t)) &:= \int |u(x,t)|^2\,dx = M(u_0),\\
P(u(t)) &:= \mathrm{Im}\int \overline{u(x,t)}\nabla u(x,t)\,dx = P(u_0),\\
E(u(t)) &:= \int |\nabla u(x,t)|^2 - \frac{2}{p+1}\int |u(x,t)|^{p+1}\,dx = E(u_0).
\end{aligned} \tag{5}$$

The local well-posedness for the initial data problem (2) with $u_0 \in H^1(\mathbb{R}^N)$ was studied in Cazenave and Weissler [3] in the energy-subcritical/critical cases. It was also shown in [12] that the problem (2) in the energy-supercritical case is locally well-posed under some assumptions on the dimension N and the power p. A natural question is whether the local solution exists globally. In the mass-subcritical case, it follows easily from the Gagliardo-Nirenberg inequality that the global solution exists. From the global theory for small data, we know that if the Sobolev norm (H^{s_c}-norm) of the initial data is sufficiently small, then there exists a unique global solution to (1.1). However, for large initial data, under suitable smoothness and decay assumptions, the virial identity guarantees that finite time blowup may occur. In particular, Glassey [7] proved that if initial data satisfies $xu_0 \in L^2(\mathbb{R}^N)$ with negative energy, then the corresponding solution blows up in finite time.

After this result, many attempts have been made to remove/relax the finite variance assumption. Especially, in the energy-subcritical case, Ogawa and Tsutsumi [18] removed the finite variance assumption in the radial symmetry case (see [11] for the energy-critical case). The radiality condition was relaxed to some nonisotropic ones by Martel [15]. In the 1D mass-critical case ($p = 5$), Ogawa and Tsutsumi [19] completely removed the finite variance assumption. As a remark in the famous paper [14], Merle and Raphael showed that in the mass-critical case, if the mass of the initial data is close to the mass of the ground state, then the solution with negative energy blows up in finite time. The similar result was obtained by Raphael and Szeftel [20] for the radial quintic nonlinear Schrödinger equation in any dimension:

$$iu_t + \Delta u + |u|^4 u = 0, (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Besides the finite time blow-up criterion, the other interesting topic is to see what happens if one only assumes that the initial data has negative energy. In [5], Glangetas and Merle proved that in the mass-critical/mass-supercritical, energy subcritical cases with $E(u_0) < 0$, the solution blows up in finite or infinite time, in the sense of

$$\sup_{t\in(-T_-(u_0),T_+(u_0))} \|u(t)\|_{H^1} = +\infty,$$

where $(-T_-(u_0), T_+(u_0))$ is the maximal lifespan of the solution with the initial data u_0. The method is a geometrical approach. See also Nawa [17] in the mass-critical case. In particular, when $N = 3, p = 3$, a similar but more general result was established by Holmer and Roudenko [9] using the concentration-compactness argument, see also [2, 8] for some related results by using the argument in [9]. However, it's not clear how to generalize the argument to the energy-critical/energy-supercritical cases. In this paper, we give a new argument to prove it. Our argument is suitable for the energy-critical/energy-supercritical cases, and gives a similar result about it.

Here comes our theorem, which is about energy-critcal/energy-supercritical cases. For the sake of simplicity, we only focus our attention on the odd values of the power p.

Theorem 1.1. *Suppose that p is odd, $p \geq 1 + 4/(N-2)$, $N \geq 3$, and $s > s_c$. Let the initial data $u_0 \in H^s(\mathbb{R}^N)$ with $E(u_0) < 0$, and let u be the corresponding solution with the lifetime $[0, T_{max})$. Then one of the following two statements holds true,*

- *$T_{max} < \infty$, that is, the solution blows up in finite time. Moreover,*

$$\lim_{t\uparrow T_{max}} \|u(t)\|_{H^s} = \infty.$$

- *$T_{max} = \infty$, and there exists a time sequence $\{t_n\}$ such that $t_n \to \infty$, and for any $q > p+1$,*

$$\lim_{t_n\uparrow\infty} \|u(t_n)\|_{L^q} = \infty.$$

A similar result remains true for negative time.

Remark 1. Roughly speaking, Case 1 refers to the finite time blow-up, Case 2 refers to the infinite time blow-up (one may certainly substitute L^q-norm to H^s-norm in this case, by Sobolev's embedding). At this stage, it is not clear whether Case 2 could be ruled out, or it would indeed happen.

Thanks to the Galilei transformation, we may extend the negative energy condition to the following.

Corollary 1. *Theorem 1.1 still holds true when the condition $E(u_0) < 0$ is reduced to*

$$E(u_0) < P(u_0)^2/M(u_0). \tag{6}$$

Besides the energy-critical and energy-supercritical cases, our method also could be used in the energy-subcritical case, that is, $p < 1 + 4/(N-2)$. Let Q be the ground state of the nonlinear elliptic equation

$$-Q + \Delta Q + |Q|^{p-1}Q = 0, \qquad Q = Q(x), \qquad x \in \mathbb{R}^N. \tag{7}$$

As mentioned above, Holmer, Roudenko [9] and Guo [8] proved the following result.

Theorem 1.2. *Let $1+\frac{4}{N}<p<1+\frac{4}{N-2}$, u be the solution of (2) with the lifetime $[0,T_{max})$, and let the initial data $u_0\in H^1(\mathbb{R}^N)$. Then if*

$$M(u_0)^{1-s_c}E(u_0)^{s_c}<M(Q)^{1-s_c}E(Q)^{s_c},\quad \|u_0\|_{L^2}^{1-s_c}\|\nabla u_0\|_{L^2}^{s_c}>\|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}, \tag{8}$$

then one of the following two statements holds true,

- *$T_{max}<\infty$, and*
$$\lim_{t\uparrow T_{max}}\|\nabla u(t)\|_{L^2}=\infty.$$
- *$T_{max}=\infty$, and there exists a time sequence $\{t_n\}$ such that $t_n\to\infty$, and*
$$\lim_{t_n\uparrow\infty}\|\nabla u(t_n)\|_{L^2}=\infty.$$

Remark 2. Using energy conservation, it's easy to see that in Theorem 1.2 the blow-up norm $\|\nabla u(t)\|_{L^2}$ could be improved to $\|u(t)\|_{L^q}$ for any $q\geq p+1$. But this is not the case in the energy-supercritical.

In this paper, we give a simplified proof, which will be presented in Section 3.

To prove the main theorems, we adopt the idea of Glassey [7]. Because in our case, the initial data may not have finite variance, we shall deal with localized virial identities. There are some technical difficulties, which could be overcome by one observation and two techniques borrowed from scattering theory. The observation is that the gradient part in the localized virial identities could be controlled by the gradient part in the energy. The first technique is the small L^2-estimate in the exterior ball. It holds true in the relatively long time, which depends on the radius of the ball. The second is the following elementary estimate. Suppose $f\in L^1$, then

$$\int_{|x|<R}|x|^k|f|\,dx=o(R^k),\ \text{as } R\to\infty.$$

Note that one may not expect that the small L^2-estimate in the exterior ball keeps being right all the time. However, the time period, in which the small L^2-estimate holds true, is long enough to complete the proof.

This paper is organized as follows. In section 2, we give the proof of Theorem 1.1 and Corollary 1. Finally we prove Theorem 1.2 in Section 3.

2. Proof of Theorem 1.1.

The major part of this section is the following theorem: Theorem 2.1, one corollary of which is Theorem 1.1. Before stating this theorem, we introduce some quantities. Let the quantity

$$K(u):=\int|\nabla u(x)|^2\,dx-\frac{N(p-1)}{2(p+1)}\int|u(x)|^{p+1}\,dx, \tag{9}$$

then it is well-known as the virial identity that for the solution u of the equation (2),

$$\frac{d^2}{dt^2}\int|x|^2|u(t,x)|^2\,dx=8K(u(t)).$$

It implies by Glassey's argument (see [7]) that the solution u blows up in finite time if $xu_0\in L^2(\mathbb{R}^d)$ and there exists $\beta_0<0$ such that

$$\sup_{t\in[0,T_{max})}K(u(t))\leq\beta_0<0. \tag{10}$$

Theorem 2.1. *Let N, p, s be the same as in Theorem 1.1 or Theorem 1.2. Then if there exists $\beta_0 < 0$ such that (10) holds, there exists no global solution $u \in C(\mathbb{R}^+; H^s)$ with*

$$\sup_{t\in\mathbb{R}^+} \|u(t,\cdot)\|_{L^q_x} < \infty, \qquad \text{for some } q > p+1. \tag{11}$$

2.1. **The local theory.** In this subsection, we establish the following local result on the problem (2).

Proposition 1 (Local existence)**.** *Let $s \geq s_c$, and N, p, s_c be as in Theorem 1.1. Then for any $u_0 \in H^s(\mathbb{R}^N)$, there exists a unique local solution $u \in C([0,T); H^s(\mathbb{R}^N))$ of (2). Moreover, if $s > s_c$, the lifetime T is only dependent on $\|u_0\|_{H^s}$.*

Proof. Since the argument is standard, see c.f. [3], [12], we give the proof much briefly. More generally, we may consider $p > s$ or p is odd. Let $I = [0, \delta]$, for some small $\delta > 0$ decided later. According to the Duhamel formula, for $F(u) = |u|^{p-1}u$, we define

$$\Phi(u(t)) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}F(u(\tau))\,d\tau.$$

Let the Strichartz space

$$SN_s = \bigcap_{(\rho,\gamma,\sigma)\in\Lambda_s} L^\rho_t W^{\sigma,\gamma}_x(I\times\mathbb{R}^N), \quad \Lambda_s = \{(\rho,\gamma,\sigma) : \frac{2}{\rho} + \frac{d}{\gamma} - \sigma = \frac{d}{2} - s, 2 \leq \rho, \gamma, \leq \infty\}.$$

Making using of Strichartz estimates (see [6], [10]) and Sobolev inequality, we have

$$\|\Phi(u)\|_{SN_s} \leq \|e^{it\Delta}u_0\|_{SN_s} + \||\nabla|^s F(u)\|_{L^{q_0'}_t L^{p_0'}_x(I\times\mathbb{R}^N)},$$

where $\frac{2}{q_0} + \frac{N}{p_0} = \frac{N}{2}, 2 \leq q_0 \leq \infty, 2 \leq p_0 < \infty$. Then the proposition follows by the standard fixed point theory (in which for the sake of convenience one may choose the weaker norm $L^\rho_t L^\gamma_x(I\times\mathbb{R}^N)$, for some $(\rho,\gamma,0) \in \Lambda_s$ to be the distance, in order to avoiding differentiating), once we establish

$$\||\nabla|^s F(u)\|_{L^{q_0'}_t L^{p_0'}_x(I\times\mathbb{R}^N)} \leq C\|u\|_{SN_s}\|u\|^{p-1}_{SN_{s_c}}. \tag{12}$$

Indeed, it easily follows from the chain rule and Hölder's inequality for the regular case, thus we only consider the case when $0 < p - [s] < 1$, where we also need additional tool of the fractional chain rule (see [12, Lemma 2.6] for example). In this case,

$$\begin{aligned}\||\nabla|^s F(u)\|_{L^{q_0'}_t L^{p_0'}_x(I\times\mathbb{R}^N)} &\leq C\left\||\nabla|^{s-[s]}\big(\nabla^{[s]}F(u)\big)\right\|_{L^{q_0'}_t L^{p_0'}_x(I\times\mathbb{R}^N)}\\ &\leq C\left\||\nabla|^{s-[s]}\big(F_1(u)\,F_2(u)\big)\right\|_{L^{q_0'}_t L^{p_0'}_x(I\times\mathbb{R}^N)},\end{aligned} \tag{13}$$

where $F_1(u)$ is a combination of the terms typing as

$$\partial^{\alpha_1}u\cdots\partial^{\alpha_J}u\cdot\partial^{\beta_1}\bar{u}\cdots\partial^{\beta_K}\bar{u},$$

for $\alpha_1 + \cdots + \alpha_J + \beta_1 + \cdots + \beta_K = [s], 0 \leq |\alpha_j|, |\beta_k| \leq [s]$ for $1 \leq j \leq J, 1 \leq k \leq K$; and $F_2(u)$ is a Hölder continuous function of order $p - 2[\frac{p}{2}] > s - [s]$. Then by Hölder's inequality and the fractional chain rule,

$$\begin{aligned}(13) \leq & C\||\nabla|^{s-[s]}F_1(u)\|_{L^{q_1}_t L^{r_1}_x}\|F_2(u)\|_{L^{q_2}_t L^{r_2}_x} + C\|F_1(u)\|_{L^{q_3}_t L^{r_3}_x}\||\nabla|^{s-[s]}F_2(u)\|_{L^{q_4}_t L^{r_4}_x}\\ \leq & C\|u\|_{SN_s}\|u\|^{2[\frac{p}{2}]-1}_{SN_{s_c}}\cdot\|u\|^{p-2[\frac{p}{2}]}_{SN_{s_c}} + C\|u\|^{2[\frac{p}{2}]}_{SN_{s_c}}\cdot\||\nabla|^\alpha u\|^{\frac{s-[s]}{\alpha}}_{L^{q_5}_t L^{r_5}_x}\|u\|^{p-2[\frac{p}{2}]-\frac{s-[s]}{\alpha}}_{L^{q_6}_t L^{r_6}_x}\\ \leq & C\|u\|_{SN_s}\|u\|^{p-1}_{SN_{s_c}},\end{aligned}$$

where

$$\frac{1}{q_0'}=\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q_3}+\frac{1}{q_4},\quad \frac{1}{p_0'}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4},\quad \big((p-2[\frac{p}{2}])q_2,(p-2[\frac{p}{2}])r_2,0\big)\in\Lambda_{s_c};$$

and

$$\Big(2[\frac{p}{2}]q_3,2[\frac{p}{2}]r_3,0\Big)\in\Lambda_{s_c},\quad \frac{s-[s]}{p-2[\frac{p}{2}]}<\alpha<1,\quad (q_5,r_5,\alpha)\in\Lambda_s,\quad (q_6,r_6,0)\in\Lambda_{s_c}.$$

This proves (12) and thus finishes the proof of the proposition. □

2.2. **The proof of Theorem 2.1.** Roughly speaking, Theorem 2.1 says that there exist no global solutions whose L^q norms are uniformly bounded in time. We prove the Theorem 2.1 by contradiction argument. Assume the contrary, then we have

$$C_0:=\sup_{t\in\mathbb{R}^+}\|u(t)\|_{L^q}<+\infty.$$

Then we can show that there exists $0<\overline{C_0}=\overline{C_0}(C_0,M(u_0),E(u_0))<\infty$, such that

$$\overline{C_0}=\sup_{t\in\mathbb{R}^+}\|\nabla u(t)\|_{L^2}.$$

Indeed, it is first bounded for L^{p+1}-norm by interpolation between L^q and L^2. Then the boundedness of $\dot H^1$-norm follows from the energy conservation law.

Consider the local Virial identity and let

$$I(t)=\int\phi(x)|u(t,x)|^2\,dx, \tag{14}$$

then by direct computations (see for examples [7], [11]), one has

Lemma 2.2. *For any $\phi\in C^4(\mathbb{R}^N)$,*

$$I'(t)=2Im\int\nabla\phi\cdot\nabla u\bar u\,dx; \tag{15}$$

$$I''(t)=4Re\sum_{j,k}^N\int\partial_j\partial_k\phi\cdot\partial_j u\partial_k\bar u\,dx-2\frac{p-1}{p+1}\int\Delta\phi|u|^{p+1}\,dx-\int\Delta^2\phi|u|^2\,dx. \tag{16}$$

If ϕ is radial, then one may find that

$$I'(t)=2\mathrm{Im}\int\phi'\,\frac{x\cdot\nabla u}{r}\bar u\,dx, \tag{17}$$

$$\begin{aligned}I''(t)=&4\int\frac{\phi'}{r}|\nabla u|^2\,dx+4\int\Big(\frac{\phi''}{r^2}-\frac{\phi'}{r^3}\Big)|x\cdot\nabla u|^2\,dx\\&-2\frac{p-1}{p+1}\int\Big(\phi''+(N-1)\frac{\phi'}{r}\Big)|u|^{p+1}\,dx-\int\Delta^2\phi|u|^2\,dx,\end{aligned} \tag{18}$$

here and in the sequel, r denotes $|x|$.

2.2.1. *Virial identity-I and L^2-estimate in the exterior ball.* Fix some large constant $R>0$, which will be decided later, and choose ϕ in (14) such that

$$\phi=\begin{cases}0, & 0\le r\le R/2,\\ 1, & r\ge R,\end{cases} \tag{19}$$

and

$$0\le\phi\le1,\quad 0\le\phi'\le\frac{4}{R}.$$

Let $\|u_0\|_{L^2} = m_0$, then by (17),

$$\begin{aligned} I(t) &= I(0) + \int_0^t I'(t')\,dt' \\ &\leq I(0) + t\|\phi'\|_{L^\infty} \sup_{t'\in[0,t]} \big(\|u\|_{L^2}\,\|\nabla u\|_{L^2}\big) \\ &\leq \int_{|x|\geq R/2} |u_0|^2\,dx + \frac{4m_0\overline{C_0}t}{R}. \end{aligned}$$

Observe that

$$\int_{|x|\geq R/2} |u_0(x)|^2\,dx = o_R(1),$$

and

$$\int_{|x|\geq R} |u(t,x)|^2\,dx \leq I(t).$$

To summarize, we obtain that

Lemma 2.3. *Fixing $\eta_0 > 0$, then for any $t \leq \eta_0 R/(4m_0\overline{C_0})$, we have*

$$\int_{|x|\geq R} |u(t,x)|^2\,dx \leq \eta_0 + o_R(1). \tag{20}$$

Remark 3. Roughly speaking, the lemma above means that the solution has the almost finite speed of propagation. To the authors' best knowledge, the property was first discovered by Lin and Strauss [13] for the defocusing equation, and widely used in the scattering theory since then. The readers may refer [4, 16] for detailed introduction.

2.2.2. *Virial identity-II.* We rewrite $I''(t)$ in (18) as

$$I''(t) = 8K(u(t)) + R_1 + R_2 + R_3, \tag{21}$$

and

$$\begin{aligned} R_1 =& 4\int \big(\frac{\phi'}{r} - 2\big)|\nabla u|^2\,dx + 4\int \Big(\frac{\phi''}{r^2} - \frac{\phi'}{r^3}\Big)|x\cdot\nabla u|^2\,dx, \\ R_2 =& -2\frac{p-1}{p+1}\int \Big(\phi'' + (N-1)\frac{\phi'}{r} - 2N\Big)|u|^{p+1}\,dx, \\ R_3 =& -\int \Delta^2\phi|u|^2\,dx. \end{aligned} \tag{22}$$

Roughly speaking, R_1, R_2, and R_3 are the error terms from the localization. We choose ϕ such that

$$0 \leq \phi \leq r^2, \quad \phi'' \leq 2, \quad \phi^{(4)} \leq \frac{4}{R^2}, \tag{23}$$

and

$$\phi = \begin{cases} r^2, & 0 \leq r \leq R, \\ 0, & r \geq 2R. \end{cases} \tag{24}$$

Then we have

Lemma 2.4. *There exist two constants $\widetilde{C}(s,p,N,m_0,C_0) > 0, \theta_q > 0$, such that*

$$I''(t) \leq 8K(u(t)) + \widetilde{C}\|u\|^{\theta_q}_{L^2(|x|>R)}.$$

Proof. We first claim that

$$R_1 \le 0. \tag{25}$$

To prove it, we divide the space $\mathbb{R}^N$ into two parts:

$$\Big\{\frac{\phi''}{r^2} - \frac{\phi'}{r^3} \le 0\Big\} \text{ and } \Big\{\frac{\phi''}{r^2} - \frac{\phi'}{r^3} > 0\Big\}.$$

If $\frac{\phi''}{r^2} - \frac{\phi'}{r^3} \le 0$ it is obviously true since $\phi' \le 2r$. If

$$\frac{\phi''}{r^2} - \frac{\phi'}{r^3} \ge 0,$$

then since $\phi'' \le 2$,

$$R_1 \le 4\int (\phi'' - 2)|\nabla u|^2\, dx \le 0.$$

So we have proved (25). Moreover, since

$$\operatorname{supp}(\phi'' + (N-1)\frac{\phi'}{r} - 2N) \subset [R, \infty),$$

by interpolation there exists $0 < \theta_q \le 1$, such that

$$R_2 \le C\|u\|_{L^q(|x|>R)}^{1-\theta_q}\|u\|_{L^2(|x|>R)}^{\theta_q} \le CC_0^{1-\theta_q}\|u\|_{L^2(|x|>R)}^{\theta_q}, \tag{26}$$

where $C > 0$, is only dependent on p, s, N. Furthermore,

$$R_3 \le CR^{-2}\|u\|_{L^2(|x|>R)}^2. \tag{27}$$

Thus, combining (21) with (25)–(27), one obtains that for $R > 1$,

$$I''(t) \le 8K(u(t)) + \widetilde{C}\|u\|_{L^2(|x|>R)}^{\theta_q},$$

where the constant $\widetilde{C} = \widetilde{C}(s, p, N, m_0, C_0) > 0$. The lemma is now proved. □

2.2.3. *The proof of Theorem 2.1.*

Proof of Theorem 2.1. Applying (20) and Lemma 2.4, one finds that for any $t \le T := \eta_0 R/(4m_0\overline{C_0})$,

$$I''(t) \le 8K(u(t)) + \widetilde{C}\big(\eta_0^{\theta_q} + o_R(1)\big).$$

Integrating from 0 to T, and using (10), one gets

$$\begin{aligned} I(T) \le & I(0) + I'(0)T + \int_0^T \int_0^t \Big(8K(u(t')) + \widetilde{C}\eta_0^{\theta_q} + o_R(1)\Big)\, dt'dt \\ \le & I(0) + I'(0)T + \Big(8\beta_0 + \widetilde{C}\eta_0^{\theta_q} + o_R(1)\Big)\cdot\frac{1}{2}T^2. \end{aligned}$$

Choosing η_0 such that

$$\widetilde{C}\eta_0^{\theta_q} = -\beta_0,$$

and taking R large enough, then for $T = \eta_0 R/(4m_0\overline{C_0})$ one has

$$I(T) \le I(0) + I'(0)\eta_0 R/(4m_0\overline{C_0}) + \alpha_0 R^2, \tag{28}$$

where the constant

$$\alpha_0 = \beta_0\eta_0^2/(4m_0\overline{C_0})^2 < 0.$$

We note here that α_0 is independent of R. Now we need the following two claims:

$$I(0) = o_R(1)R^2, \quad I'(0) = o_R(1)R. \tag{29}$$

Indeed,

$$\begin{aligned}I(0) \leq & \int_{|x|<\sqrt{R}} |x|^2|u_0(x)|^2\,dx + \int_{\sqrt{R}<|x|<2R} |x|^2|u_0(x)|^2\,dx\\ \leq & Rm_0^2 + R^2\int_{|x|>\sqrt{R}} |u_0(x)|^2\,dx\\ = & o_R(1)R^2.\end{aligned}$$

A similar argument can be used to obtain the second estimate and thus proves (29).

Together (28) with (29), and choosing R large enough, one obtains that for $T = \eta_0 R/(4m_0\overline{C_0})$,

$$\begin{aligned}I(T) &\leq o_R(1)R^2 + \alpha_0 R^2\\ &\leq \frac{1}{2}\alpha_0 R^2.\end{aligned}$$

Since $\alpha_0 < 0$, one finally gets

$$I(T) < 0.$$

But this is a contradiction with the definition, the proof of Theorem 2.1 is now completed. □

2.3. **The proof of Theorem 1.1.**

Proof of Theorem 1.1. From the local theory Proposition 1, we could define the maximal lifespan T_{max}. There are two cases,

(i) $T_{max} < \infty$. This yields

$$\lim_{t\to T_{max}} \|u(t)\|_{H^s} = \infty.$$

Otherwise, there exists a sequence $\{t_n\}_n$ such that $t_n \to T_{max}$, such that

$$\sup_{t_n} \|u(t_n)\|_{H^s} < \infty.$$

Using Proposition 1 with the initial data of t_n, we get a contradiction with T_{max} for large n.

(ii) $T_{max} = \infty$. We first observe that

$$K(u(t)) \leq E(u_0) < 0, \text{ for any } t \in \mathbb{R}.$$

Thus (10) always holds with $\beta_0 = E(u_0)$ under the assumption in this theorem. Now using Theorem 2.1, we prove that there exists a time sequence $\{t_n\}$ such that $t_n \to \infty$, and for any $q > p+1$,

$$\lim_{t_n\uparrow\infty} \|u(t_n)\|_{L^q} = \infty.$$

This concludes Theorem 1.1. □

At the end of this section, we give the proof of Corollary 1.

Proof of Corollary 1. From the Galilean transformation,

$$\tilde{u}(t,x) = e^{ix\cdot\xi_0}e^{-it|\xi_0|^2}\,u(t, x - 2\xi_0 t). \tag{30}$$

If u is the solution of (2), then so is $\tilde{u}$. Moreover, taking $\xi_0 = -\frac{P(u_0)}{M(u_0)}$, then

$$E(u_0) - P(u_0)^2/M(u_0) = E(\tilde{u}_0).$$

Therefore, the conclusion follows by considering $\tilde{u}$ instead. □

3. **The proof of Theorem 1.2.** To this end, we shall firstly check that (8) implies (10), that is, there exists some strictly negative constant β_0, such that

$$\sup_{t\in[0,T_{max})} K(u(t)) \le \beta_0 < 0.$$

This was essentially obtained in [9], however we also give the proof here for completeness (with a different argument).

First, we claim that the hypothesis (8) implies that for any $t \in [0, T_{max})$,

$$\|u(t)\|_{L^2}^{1-s_c}\|\nabla u(t)\|_{L^2}^{s_c} > \|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}. \tag{31}$$

Indeed, suppose not, then by continuity, there exists $\tilde{t} \in (0, T_{max})$, such that

$$\|u(\tilde{t})\|_{L^2}^{1-s_c}\|\nabla u(\tilde{t})\|_{L^2}^{s_c} = \|Q\|_{L^2}^{1-s_c}\|\nabla Q\|_{L^2}^{s_c}. \tag{32}$$

Then by (32) and the sharp Gagliardo-Nirenberg inequality (see [22]),

$$\|u\|_{L^{p+1}}^{p+1} \le C_{\rm GN}\,\|\nabla u\|_{L^2}^{\frac{N(p-1)}{2}}\,\|u\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}}, \tag{33}$$

where

$$C_{\rm GN} = \|Q\|_{L^{p+1}}^{p+1}\Big/\|\nabla Q\|_{L^2}^{\frac{N(p-1)}{2}}\,\|Q\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}},$$

one obtains that

$$\begin{aligned}
M(Q)^{\frac{1-s_c}{s_c}}E(Q) &> M(u(\tilde{t}))^{\frac{1-s_c}{s_c}}E(u(\tilde{t}))\\
&= \|u(\tilde{t})\|_{L^2}^{\frac{2(1-s_c)}{s_c}}\|\nabla u(\tilde{t})\|_{L^2}^2 - \frac{2}{p+1}\|u(\tilde{t})\|_{L^2}^{\frac{2(1-s_c)}{s_c}}\|u(\tilde{t})\|_{L^{p+1}}^{p+1}\\
&\ge \|u(\tilde{t})\|_{L^2}^{\frac{2(1-s_c)}{s_c}}\|\nabla u(\tilde{t})\|_{L^2}^2\\
&\quad - \frac{2}{p+1}C_{\rm GN}\cdot\|u(\tilde{t})\|_{L^2}^{\frac{2(1-s_c)}{s_c}+2-\frac{(N-2)(p-1)}{2}}\|\nabla u(\tilde{t})\|_{L^2}^{\frac{N(p-1)}{2}}\\
&= \|u(\tilde{t})\|_{L^2}^{\frac{2(1-s_c)}{s_c}}\|\nabla u(\tilde{t})\|_{L^2}^2 - \frac{2}{p+1}C_{\rm GN}\cdot\Big[\|u(\tilde{t})\|_{L^2}^{\frac{1-s_c}{s_c}}\|\nabla u(\tilde{t})\|_{L^2}\Big]^{\frac{N(p-1)}{2}}\\
&= \|Q\|_{L^2}^{\frac{2(1-s_c)}{s_c}}\|\nabla Q\|_{L^2}^2 - \frac{2}{p+1}C_{\rm GN}\Big[\|Q\|_{L^2}^{\frac{1-s_c}{s_c}}\|\nabla Q\|_{L^2}\Big]^{\frac{N(p-1)}{2}}\\
&= M(Q)^{\frac{1-s_c}{s_c}}E(Q).
\end{aligned} \tag{34}$$

This gives a contradiction and thus proves (31).

By the definition (5) and (9), one has

$$K(u(t)) = \frac{N(p-1)}{4}E(u(t)) - \Big(\frac{N(p-1)}{4} - 1\Big)\|\nabla u(t)\|_{L^2}^2, \tag{35}$$

thus, by (31) and (8), one gives that

$$K(u(t)) < 0, \text{ for any } t \in [0, T_{max}).$$

This together with (33), and noting that $\frac{N(p-1)}{2} > 2$, yields that there exists some small $\epsilon_0 > 0$ such that

$$\|\nabla u(t)\|_{L^2} > \epsilon_0. \tag{36}$$

Now we further claim that there exists $\delta_0 > 0$ such that for any $t \in [0, T_{max})$,

$$K(u(t)) < -\delta_0\|\nabla u(t)\|_{L^2}^2. \tag{37}$$

Indeed, suppose not, there exists a time sequence $\{t_n\} \subset (0, T_{max})$ such that

$$-\delta_n\Big(\frac{N(p-1)}{4}-1\Big)\|\nabla u(t_n)\|_{L^2}^2 < K(u(t_n)) < 0,$$

where $\delta_n \to 0$ as $n \to \infty$. Then by (35), one has

$$E(u(t_n)) > (1-\delta_n)\Big(1-\frac{4}{N(p-1)}\Big)\|\nabla u(t_n)\|_{L^2}^2.$$

Therefore, by (31), one finds that

$$M(u(t_n))^{1-s_c}E(u(t_n))^{s_c} > (1-\delta_n)^{s_c}\Big(1-\frac{4}{N(p-1)}\Big)^{s_c} M(Q)^{1-s_c}\|\nabla Q\|_{L^2}^{2s_c}.$$

Recall that from the Pohozaev identities, $K(Q) = 0$. That is,

$$E(Q) = \Big(1-\frac{4}{N(p-1)}\Big)\|\nabla Q\|_{L^2}^2.$$

Hence we obtain

$$M(u(t_n))^{1-s_c}E(u(t_n))^{s_c} > (1-\delta_n)^{s_c} M(Q)^{1-s_c}E(Q)^{s_c}.$$

Thus taking $n \to \infty$ and making use of the mass and energy conservation laws, we prove that

$$M(u_0)^{1-s_c}E(u_0)^{s_c} \geq M(Q)^{1-s_c}E(Q)^{s_c}.$$

But this is contradicted with the hypothesis (8) and thus proves (37). Combining with (36), we obtain (10).

Now by Theorem 2.1, there exists no global solution $u \in C(\mathbb{R}^+; H^1)$ with (11). Then using Sobolev's embedding, one may replace L^q-norm by H^1-norm in (11), and thus proves Theorem 1.2. □

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