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The Grammatical Calculus

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Abstract

We give an introduction to the grammatical calculus, which originated from the umbral calculus of Rota. Context-free grammars arise in computer science. Differential operators can be viewed as the production rules of context-free grammars. By the properties of differential operators, we may perform the grammatical calculus on a rigorous basis, for the purposes of deriving combinatorial identities and computing generating functions. A grammatical labeling serves as a bridge between a combinatorial structure and the grammar. A grammar may be instrumental in constructing a bijection between two combinatorial structures sharing the same grammar. Such a bijection is called a grammar assisted bijection. Many combinatorial structures admit recursive constructions that are context-free in some sense, such as those corresponding to the Eulerian polynomials, Bell polynomials, the Ramanujan polynomials and the Narayana polynomials.

Key words: Grammatical calculus, grammatical labelings, Eulerian polynomials, Bell polynomials, Ramanujan polynomials, Narayana polynomials.

MSC subject classification: 05A15, 05A19.

1 Introduction

The notion of context-free grammars is due to Chomsky, and it forms the foundation of programming languages, see, for example, [23, 24, 46]. Meanwhile, the linear differential operator D can be viewed as having a context-free property, namely,

$$D(fg) = D(f)\,g + f\,D(g). \tag{1.1}$$

In other words, when D is applied to f, the result is independent of g and vice versa. That is to say that we only need a differential algebra, that is, an algebra equipped with a derivation D. For example, consider bivariate polynomials in x and y, and define the derivative D by

$$D(x) = xy,\ D(y) = xy. \tag{1.2}$$

From now on, we call such derivatives D formal derivatives. In the terminology of formal languages, the operator D defined above can be represented by the following production rules

$$x \to xy,\ y \to xy, \tag{1.3}$$

which turns out to be the grammar for the Eulerian polynomials. Strictly speaking, the grammars we are concerned with are not devised to describe the recursive generation of a formal language, but rather, a recursive construction of a combinatorial structure. While the letters of a formal language are obviously noncommutative, we usually assume that the variables are commutative. The noncommutative case of the grammatical calculus will be discussed in the last section.

The idea of the grammatical calculus was presented in [10] along with grammatical derivations of some classical formulas, which is regarded as putting the traditional symbolic argument on a firm foundation by Johnson [49]. It can be seen that the Lagrange inversion formula is equivalent to Cayley's formula on labeled trees, in the sense that these two formulas correspond to essentially the same grammatical formulation, see [10]. It is worth mentioning that the umbral calculus provides a rigorous mechanism for the classical symbolic method, see [68]. In fact, the idea of the grammatical calculus was motivated by the umbral calculus of Rota.

The formal derivative (1.2) can be expressed as a differential operator

$$D = xy\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{1.4}$$

Dumont [31] pointed out that the production rules are more convenient in dealing with combinatorial structures. On the other hand, when it comes to computation, it is natural to employ the above differential operator representation while using a CAS (computer algebra system), such as Maple and Mathematica.

Consider the homogeneous bivariate Eulerian polynomials $A_n(x,y)$, see [63, 76]. Set $A_0(x,y) = y$. For $n \geq 1$, define

$$A_n(x,y) = D^n(y). \tag{1.5}$$

The first few values of $A_n(x,y)$ are as follows:

$$\begin{aligned}
A_1(x,y) &= xy, \\
A_2(x,y) &= xy^2 + x^2y, \\
A_3(x,y) &= xy^3 + 4x^2y^2 + x^3y, \\
A_4(x,y) &= xy^4 + 11x^2y^3 + 11x^3y^2 + x^4y, \\
A_5(x,y) &= xy^5 + 26x^2y^4 + 66x^3y^3 + 26x^4y^2 + x^5y.
\end{aligned}$$

As $A_n(x,y)$ can be generated by D, the operator D can be considered a creation operator, but it also possesses the differential property of an annihilation operator. Setting $y = 1$, $A_n(x,y)$ becomes the usual Eulerian polynomial $A_n(x)$.

For $n \geq 1$, let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. The set of descents and the number of descents of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ are defined by

$$\mathrm{Des}(\sigma) = \{i \mid 1 \leq i \leq n, \sigma_i > \sigma_{i+1}\}, \quad \mathrm{des}(\sigma) = |\mathrm{Des}(\sigma)|,$$

where we append a zero to the end of σ, that is, $\sigma_{n+1} = 0$. Likewise, we assume that $\sigma_0 = 0$ and define the ascent set and the number of ascents of σ by

$$\mathrm{Asc}(\sigma) = \{i \mid 0 \leq i \leq n-1, \sigma_i < \sigma_{i+1}\}, \quad \mathrm{asc}(\sigma) = |\mathrm{Asc}(\sigma)|.$$

An index in the ascent set or descent set is called an ascent or a descent, respectively.

Below is the combinatorial definition of the bivariate Eulerian polynomials. For $n \geq 1$,

$$A_n(x,y) = \sum_{\sigma \in S_n} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)}. \tag{1.6}$$

The equivalence of the operator definition and the combinatorial definition of $A_n(x,y)$ can be easily justified by induction. However, the idea of a grammatical labeling just serves this purpose, and it is indeed the reason it was introduced in [14]. Roughly speaking, a grammatical labeling is to associate a combinatorial structure with labels of a grammar such that the recursive construction of the combinatorial structure is consistent with the production rules of the grammar. In this way, the grammar replaces the usual recurrence relation in describing the involved combinatorial numbers, see also [44, 54, 55, 84, 85] for more examples of grammatical labelings.

Given the context-free property (1.1) of a grammar, the operator D satisfies the Leibniz formula

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v). \tag{1.7}$$

In particular,

$$D(u^{-1}) = -u^{-2} D(u). \tag{1.8}$$

For example, in the case $D(x) = x$, we have

$$D(x^{-1}) = -x^{-1}. \tag{1.9}$$

Via examples, we will give an outline of the grammatical calculus, involving set partitions, the Bell numbers, the Eulerian polynomials, the peak polynomials of permutations, up-down run polynomials, the Diaconis-Evans-Graham theorem, the operator expansion of $(g(x)D)^n f(x)$, the Ramanujan polynomials, plane trees, the ballot numbers and the Narayana numbers.

2 The Umbral Calculus and the Grammatical Calculus

The simplest example of the umbral calculus is the following argument of Rota for the binomial inversion: The relation

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \tag{2.1}$$

holds for all n if and only if

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \tag{2.2}$$

holds for all n. The idea of the classical symbolic method is to treat a_n as a^n. In doing so, (2.1) can be written as $a^n = (b+1)^n$. On the other hand, b_n can be written as $(b+1-1)^n$. Thus (2.2) follows from the binomial theorem. Rota [68] defined the linear functional $L(x^n) = b_n$, so that (2.1) takes the form $a_n = L((x+1)^n)$. Now we have $b_n = L(((x+1)-1)^n)$, and so the binomial expansion yields (2.2). This explains why one can get valid results while treating a_n as a^n.

Let us take the above example to illustrate the idea of the grammatical calculus. Define

$$x \to x, \quad b_i \to b_{i+1} \; (i \geq 0).$$

Let D be the formal derivative of the above grammar; i.e.,

$$D(x) = x, \; D^i(b_0) = b_i \; (i \geq 0).$$

By the Leibniz formula (1.7), (2.1) can be expressed as $a_n = x^{-1} D^n(b_0 x)$. Note that

$$b_n = D^n(b_0) = D^n(b_0 x x^{-1}).$$

Since $D(x^{-1}) = -x^{-1}$, applying the Leibniz formula gives (2.2).

The second example is concerned with the following identity

$$(2n)!! = \sum_{k=0}^{n} \binom{n}{k} (2k-1)!! \, (2(n-k)-1)!!, \tag{2.3}$$

where $(2k-1)!! = 1 \cdot 3 \cdot \cdots \cdot (2k-1)$, $(2n)!! = 2 \cdot 4 \cdot \cdots \cdot (2n)$, $(-1)!!$ and $0!!$ are defined to be 1, see Triana-Castro [77].

Let D be the formal derivative of the grammar $a \to a^3$. For $n \geq 0$, it is clear that

$$D^n(a) = (2n-1)!! \, a^{2n+1}, \quad D^n(a^2) = (2n)!! \, a^{2n+2}.$$

Expanding $D^n(a^2)$ using the Leibniz formula, we obtain (2.3), which is equivalent to the classical identity:

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n.$$

The third example is the formula for the higher order derivatives of a composite function $f(g(x))$, that is, the Faà di Bruno formula. Let $f_i = f^{(i)}(g(x))$ and $g_i = g^{(i)}(x)$. Then we have

$$(f(g(x)))^{(n)} = \sum_{k=0}^{n} f_k \sum_{k_1,k_2,\ldots,k_n \geq 0} \frac{n!}{k_1!k_2!\cdots k_n!\,1!^{k_1}2!^{k_2}\cdots n!^{k_n}} g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}, \tag{2.4}$$

where the second sum ranges over $k_1+k_2+\cdots+k_n = k$ and $k_1+2k_2+\cdots+nk_n = n$. The coefficients in the above expansion have a combinatorial interpretation. A partition of a set $[n]$ can be written as $\{B_1, B_2, \ldots, B_k\}$, where the B_i are disjoint nonempty subsets of $[n]$ such that $B_1 \cup \cdots \cup B_k = [n]$, and each B_i is called a block. The type of a partition π is denoted by $1^{k_1}2^{k_2}\cdots n^{k_n}$, where k_i is the number of blocks of π with i elements. The coefficient in the Faà di Bruno formula equals the number of partitions of $[n]$ having type $1^{k_1}2^{k_2}\cdots n^{k_n}$. The classical proof as given in [66] is a paradigm of the symbolic method. Below is an argument using the grammar

$$f_i \to f_{i+1}\, g_1 \ (i \geq 0), \quad g_i \to g_{i+1} \ (i \geq 1). \tag{2.5}$$

Here are some computations associated with this grammar:

$$\begin{aligned}
D(f_0) &= f_1 g_1, \\
D^2(f_0) &= D(f_1)\, g_1 + f_1 D(g_1) = f_2\, g_1^2 + f_1\, g_2, \\
D^3(f_0) &= f_3\, g_1^3 + 3 f_2\, g_1\, g_2 + f_1\, g_3.
\end{aligned}$$

In general, $D^n(f_0)$ agrees with the formula of Faà di Bruno. We now explain this formula by means of a grammatical labeling. Given a partition σ of $[n]$ with k blocks, we use f_k to label the whole partition and use g_i to label a block with i elements, so that the weight of σ equals $f_k g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$, where k_i denotes the number of blocks with i elements. The grammar (2.5) reflects the two cases of constructing a partition of $[n+1]$ from a partition of $[n]$: (1) Insert $n+1$ as a new block. This operation is captured by the production rule $f_k \to f_{k+1} g_1$. (2) Insert $n+1$ into a block containing i elements. In this case, we have the production rule $g_i \to g_{i+1}$.

Using a simplified version of the above grammar, that is,

$$G = \{a \to ax,\ x \to x\}, \tag{2.6}$$

we can generate the Stirling numbers of the second kind and the Bell numbers. To be more specific, for $1 \leq k \leq n$, the Stirling number of the second kind $S(n,k)$ refers to the number of partitions of $[n]$ with k blocks, and the Bell number B_n refers to the number of partitions of $[n]$.

Theorem 2.1 (Chen [10]). *Let D be the formal derivative of the grammar (2.6). For $n \geq 1$, we have*

$$D^n(a) = a \sum_{k=1}^{n} S(n,k) x^k, \tag{2.7}$$

$$B_n = a^{-1} D^n(a)\big|_{x=1}. \tag{2.8}$$

In the following two examples, we illustrate how the grammatical calculus can be employed to prove identities involving the Bell numbers. Spivey [75] obtained the following identity

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} j^{n-k} S(m,j) \binom{n}{k} B_k.$$

Using the grammar (2.6), we see that for $i \geq 0$, $D^i(x^j) = j^i x^j$. Thus,

$$D^{n+m}(a) = D^n(D^m(a)) = D^n\left(\sum_{j=0}^{m} S(m,j) a x^j\right) = \sum_{j=0}^{m} S(m,j) D^n(a x^j).$$

Applying the Leibniz formula to $D^n(ax^j)$ and setting $a = x = 1$, we arrive at Spivey's identity. A refinement of this identity goes back to Verde-Star [78], and a grammatical derivation can be found in [10]. For generalizations and applications of this relation, see [40, 72, 80].

Shattuck [71] obtained the following identity involving the Bell numbers. For $0 \leq j \leq n$,

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i} B_{n+1-i} = \sum_{k=0}^{n-j} \binom{n-j}{k} B_{n-k}. \tag{2.9}$$

Using the grammar (2.6), according to (1.9), the sum

$$\sum_{i=0}^{j} (-1)^i x^{-1} \binom{j}{i} D^{n+1-i}(a),$$

can be expressed as

$$\sum_{i=0}^{j} \binom{j}{i} D^i\left(x^{-1}\right) D^{j-i}\left(D^{n+1-j}(a)\right).$$

By the Leibniz formula (1.7), this equals

$$D^j\left(x^{-1} D^{n+1-j}(a)\right).$$

Since $D(a) = ax$ and $D(x) = x$, we further get

$$D^j\left(x^{-1}D^{n-j}(ax)\right) = D^j\left(x^{-1}\sum_{k=0}^{n-j}\binom{n-j}{k}D^k(x)\,D^{n-j-k}(a)\right)$$
$$= \sum_{k=0}^{n-j}\binom{n-j}{k}D^{n-k}(a).$$

Hence we obtain

$$\sum_{i=0}^{j}(-1)^i x^{-1}\binom{j}{i}D^{n+1-i}(a) = \sum_{k=0}^{n-j}\binom{n-j}{k}D^{n-k}(a). \tag{2.10}$$

Setting $a = x = 1$, we are led to (2.9).

For $n \geq 1$, the exponential polynomials are defined by

$$\phi_n(x) = \sum_{k=1}^{n} S(n,k)\,x^k,$$

which permits a grammatical representation

$$\phi_n(x) = a^{-1}D^n(a).$$

Therefore, (2.10) can be written as

$$\sum_{i=0}^{j}(-1)^i\binom{j}{i}\phi_{n+1-i}(x) = x\sum_{k=0}^{n-j}\binom{n-j}{k}\phi_{n-k}(x). \tag{2.11}$$

The involution for (2.9) given in [71] also applies to the relation (2.11) for the exponential polynomials.

3 Grammatical Labelings

In this section, we give three examples to demonstrate the notion of a grammatical labeling. In fact, we have already encountered a grammatical labeling of partitions related to the Faà di Bruno formula. Roughly speaking, a grammatical labeling is introduced to facilitate the conversion of a recursive construction of a combinatorial structure into a context-free grammar, rather than using recurrence relations. Then this grammar itself is sufficient to perform the grammatical calculus for the purpose of enumeration.

The first example is a grammatical labeling for the cycle representation of a permutation. Let $n \geq 1$ and σ be a permutation of $[n]$. For $n \geq 2$, it is known that the number of permutations of $[n]$ such that 1 and 2 belong to the same cycle equals the number of permutations such that 1 and 2 occur in different cycles, see [64]. In the context of a grammatical labeling, this fact becomes obvious. Assume that we always place the minimum element of a cycle at the beginning, and label the position after each element by y, which indicates this is a valid position to insert a new element in the recursive construction of a permutation of $[n+1]$ (in the cycle notation) from a permutation of $[n]$ (in the cycle notation). In addition, we place a label a at the end outside the cycles to signify a position to insert $n+1$ as a new cycle. For instance, here is the labeling of the cyclic representation of a permutation:

$$(1\,y\,7\,y\,3)\ (2\,y\,4\,y)\ (5\,y\,6\,y)\ a.$$

Then the recursive construction of permutations yields the following grammar:

$$a \to ay, \quad y \to y^2. \tag{3.1}$$

When $n \geq 2$, the total weight of permutations such that 1 and 2 belong to the same cycle is given by $D^{n-2}(ay^2)$. Likewise, the total weight of permutations such that 1 and 2 occur in different cycles is also given by $D^{n-2}(ay^2)$, as expected.

If we take the number of cycles into account, we may use x to label a cycle. Consequently, we come to the grammar

$$a \to axy, \quad y \to y^2. \tag{3.2}$$

Let $c(n,k)$ denote the signless Stirling number of the first kind, i.e., the number of permutations of $[n]$ with k cycles, and let D be the formal derivative of the above grammar (3.2). Then for $n \geq 1$,

$$D^n(a) = ay^n \sum_{k=1}^{n} c(n,k)x^k. \tag{3.3}$$

Ma [55] found the following grammar for $c(n,k)$,

$$a \to ab, \quad b \to bc, \quad c \to c^2. \tag{3.4}$$

It has been shown that for $n \geq 1$,

$$D^n(a) = a \sum_{k=1}^{n} c(n,k)b^k c^{n-k}.$$

Setting $b = xy$, $c = y$, the grammar (3.4) gives rise to the grammar (3.2).

There is a simple operation of breaking cycles which explains the equality between the number of permutations with 1 and 2 in the same cycle and the number of permutations with 1 and 2 in different cycles. This operation has been employed by Chen [12] to give a combinatorial interpretation of the fact that for $n \geq 1$, the number of permutations of $[2n]$ all of whose cycles have odd length equals the number of permutations of $[2n]$ all of whose cycles have even length. Using the same grammatical reasoning, we may deduce that for $n \geq r \geq 2$, the number of permutations of $[n]$ with $1, 2, \ldots, r$ in the same cycle equals $(r-1)!$ times the number of permutations of $[n]$ with $1, 2, \ldots, r$ in different cycles. These two numbers are $n!/r$ and $n!/r!$, respectively. This number can be obtained via a grammatical argument. As for the grammar (3.1), we see that

$$D(ay^i) = (i+1)ay^{i+1},$$

and so

$$D^{n-r}(ay^r) \mid_{a=y=1} = (r+1)(r+2)\cdots n.$$

In view of the grammar (3.2), we may treat x as a constant. Thus, for $i \geq 0$, we have

$$D(ay^i) = (x+i)ay^{i+1}.$$

It follows that for $n \geq 1$,

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1).$$

The second example is a grammatical labeling for the bivariate Eulerian polynomials (1.5), which can be defined by a differential operator or a grammar. Assume that $n \geq 1$. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$, we adopt the convention that $\sigma_0 = \sigma_{n+1} = 0$. For $0 \leq i \leq n-1$, if i is an ascent, that is, $\sigma_i < \sigma_{i+1}$, then place a label x between σ_i and σ_{i+1}; otherwise, place a label y instead. When the element $n+1$ is inserted at the position between σ_i and σ_{i+1}, the change of labels is captured by the production rules $x \to xy$ and $y \to xy$.

For example, let $n = 9$ and $\sigma = 8\,4\,9\,6\,1\,2\,5\,3\,7$. The labeling of σ is shown below:

$$x\,8\,y\,4\,x\,9\,y\,6\,y\,1\,x\,2\,x\,5\,y\,3\,x\,7\,y. \tag{3.5}$$

The bivariate Eulerian polynomials can also be defined on increasing binary trees. For the correspondence between permutations and increasing binary trees, see Stanley [76].

To assign labels to increasing binary trees, we need to adopt the version of complete increasing binary trees. To be more specific, a complete increasing binary tree is generated by attaching leaves to make the increasing binary tree complete. The internal vertices form an increasing binary tree, and a left leaf is labeled by x and a right leaf is labeled by y. The labels are assigned only to the leaves, see Figure 1.

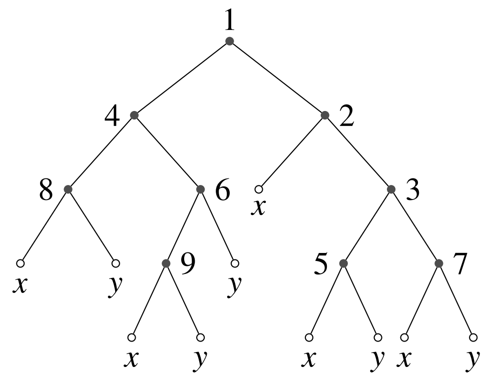

Figure 1: A complete increasing binary tree with (x,y)-labels.

The third example is connected with m-regular partitions. A partition of $[n]$ is said to be m-regular if any two elements in the same block differ by at least m, that is, the absolute difference of any two elements in the same block is at least m. This notion arises in computational biology. In particular, the number of 2-regular partitions of $[n]$ is the Bell number B_{n-1}, where B_n is the number of partitions of $[n]$. Yang [81] showed that the number of m-regular partitions of $[n]$ equals the number of $(m-1)$-regular partitions of $[n-1]$. Chen-Deng-Du [13] presented a bijective algorithm to reduce an m-regular partition to an $(m-1)$-regular partition and showed that this reduction preserves the noncrossing property.

From the perspective of grammatical calculus, the generation of m-regular partitions and partitions without restrictions are governed by the same grammar. First, notice that in an m-regular partition of $[n]$, where $n \geq m$, the largest $m-1$ elements $n-m+2, n-m+3, \ldots, n$ are scattered in different blocks. These $m-1$ blocks are not labeled, whereas the remaining blocks are labeled by x. Now, we cannot insert $n+1$ into any unlabeled block, but we can insert $n+1$ into any block labeled by x. In addition, we label the whole partition by a. Let us examine how the labels change after the insertion of the element $n+1$.

There are two cases. Case 1. When $n+1$ forms a new block, the block containing $n-m+2$ gets a label x, since the insertion of $n+2$ into this block still yields an m-regular partition. Consequently, the new block consisting of $n+1$ has no label. Case 2. When $n+1$ is inserted into a block labeled by x, the label x is removed from the resulting block, but the block containing $n-m+2$ will get a label x. This labeling scheme gives a recursive construction of m-regular partitions, along with a grammar

$$a \to ax, \; x \to x. \tag{3.6}$$

Indeed, this is just the grammar to generate ordinary partitions.

Theorem 3.1. *Assume that $m \geq 1$ and $n \geq m$. Let D be the formal derivative of the grammar (3.6). Then the total weight of m-regular partitions of $[n]$ equals $D^{n-m}(ax)$.*

Comparing the procedures to generate m-regular partitions and $(m-1)$-regular partitions, we are led to a bijection, called a grammar assisted bijection, which turns out to be equivalent to the reduction algorithm of Chen-Deng-Du [13]. It appears that the grammar setting of the bijection is easier to justify.

4 Eulerian Polynomials and the Grammatical Calculus

The definition of the bivariate Eulerian polynomials was given in (1.6) in the Introduction. Setting $y=1$, $A_n(x,y)$ becomes the Eulerian polynomial $A_n(x)$. Eulerian polynomials are the generating functions of permutations with respect to the numbers of ascents (or descents); see the labeling in the example (3.5). It should be noted that they differ from the Euler numbers and the Euler polynomials.

There is a classical theorem of Foata-Schützenberger [37] on the Eulerian polynomials: For $n \geq 1$, $A_n(x)$ can be uniquely expanded as

$$A_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{n,k} x^k (1+x)^{n-2k+1}, \tag{4.1}$$

where $\gamma_{n,k}$ are nonnegative integers. The expression (4.1) is called the γ-expansion of $A_n(x)$.

It can be reformulated in the bivariate form:

$$A_n(x,y) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \gamma_{n,k}\,(xy)^k (x+y)^{n-2k+1}. \tag{4.2}$$

The coefficients $\gamma_{n,k}$ are called the γ-coefficients. The nonnegativity of these coefficients is called the γ-positivity.

Ma-Ma-Yeh [56] gave a grammatical derivation of the γ-positivity of $A_n(x,y)$. Given the grammar (1.3), namely, $G = \{x \to xy,\ y \to xy\}$, make the following change of variables

$$u = xy, \quad v = x+y.$$

Clearly,

$$D(u) = D(xy) = xy^2 + x^2y = uv,$$
$$D(v) = D(x+y) = 2xy = 2u.$$

Thus G takes the following form in the variables u and v:

$$H = \{u \to uv, v \to 2u\}.$$

For $n \geq 1$, we have

$$A_n(x,y) = D_G^n(y) = D_H^{n-1}(u)\ |_{u=xy,\, v=x+y},$$

where D_G and D_H are the formal derivatives of G and H, respectively. Now the γ-positivity of $A_n(x,y)$ becomes transparent. Moreover, the known combinatorial interpretations of the γ-coefficients in terms of permutations and binary trees can also be deduced from the grammar.

Euler obtained the following identity for $A_n(x)$, see [47]. Here we give a derivation using the grammatical calculus, see Chen-Fu [14].

Theorem 4.1. *For $n \geq 1$,*

$$A_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(x)(x-1)^{n-1-k}, \tag{4.3}$$

where $A_0(x) = x$.

Proof. For the grammar (1.3), we have $D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y$, and

$$D(x^{-1}y) = x^{-1}D(y) + yD(x^{-1}) = x^{-1}y(x-y). \tag{4.4}$$

Since $D(x-y)=0$, we find that

$$D^n(x^{-1}y) = x^{-1}y(x-y)^n. \tag{4.5}$$

By the Leibniz formula, for $n \geq 1$,

$$\begin{aligned} D^n(y) = D^n(xx^{-1}y) &= \sum_{k=0}^{n} \binom{n}{k} D^k(x)\, D^{n-k}(x^{-1}y) \\ &= \sum_{k=0}^{n-1} \binom{n}{k} D^k(x)\, D^{n-k}(x^{-1}y) + x^{-1}yD^n(x). \end{aligned} \tag{4.6}$$

Plugging (4.5) into (4.6), we get

$$(x-y)x^{-1}\, D^n(y) = \sum_{k=0}^{n-1} \binom{n}{k} x^{-1}yD^k(x)\,(x-y)^{n-k}.$$

Here we have used the fact that for $n \geq 1$, $D^n(x) = D^n(y)$. Setting $y=1$ yields (4.3).

As remarked by Carlitz-Scoville [9], it is not so easy to recover the generating function of the Eulerian polynomials from the recurrence relation of the Eulerian numbers. The grammatical calculus for this purpose seems to be a shortcut, see Chen-Fu [15].

Theorem 4.2. *Let $A_0(x,y)=y$. We have*

$$\sum_{n=0}^{\infty} A_n(x,y)\frac{t^n}{n!} = \frac{y-x}{1-xy^{-1}e^{(y-x)t}}. \tag{4.7}$$

Below is a grammatical proof of (4.7). Let f be a Laurent polynomial in x and y, and D be the formal derivative of the grammar (1.3). Define the exponential generating function of f with respect to D by

$$\mathrm{Gen}(f,t) = \sum_{n\geq 0} D^n(f)\frac{t^n}{n!}. \tag{4.8}$$

Let g also be a Laurent polynomial in x and y. By the linearity of D and the Leibniz formula, the following multiplicative property holds:

$$\mathrm{Gen}(fg,t) = \mathrm{Gen}(f,t)\,\mathrm{Gen}(g,t). \tag{4.9}$$

Proof. Under the convention $A_0(x,y)=y$, for $n \geq 0$, we have $A_n(x,y) = D^n(y)$, where D is the formal derivative of the grammar (1.3). We aim to compute $\mathrm{Gen}(y,t)$. By the multiplicative property (4.9), we have

$$\mathrm{Gen}(y,t) = \frac{1}{\mathrm{Gen}(y^{-1},t)}.$$

To compute $\mathrm{Gen}(y^{-1},t)$, we find that

$$D(y^{-1}) = -y^{-2}D(y) = -xy^{-1}. \tag{4.10}$$

Thus, for $n \geq 0$,

$$D^n(xy^{-1}) = xy^{-1}(y-x)^n. \tag{4.11}$$

By (4.10), we get

$$\mathrm{Gen}(y^{-1},t) = \sum_{n\geq 0} D^n(y^{-1})\frac{t^n}{n!} = y^{-1} - \sum_{n\geq 1} D^{n-1}(xy^{-1})\frac{t^n}{n!}.$$

Employing (4.11) and the constant property of $y-x$, we obtain

$$\begin{aligned}\mathrm{Gen}(y^{-1},t) &= y^{-1} - \sum_{n\geq 1} xy^{-1}(y-x)^{n-1}\frac{t^n}{n!} \\ &= y^{-1} - \frac{xy^{-1}}{y-x}\left(e^{(y-x)t} - 1\right) \\ &= \frac{1-xy^{-1}e^{(y-x)t}}{y-x},\end{aligned}$$

and thus,

$$\mathrm{Gen}(y,t) = \frac{y-x}{1-xy^{-1}e^{(y-x)t}}, \tag{4.12}$$

completing the proof. ∎

Next we consider refinements of Eulerian polynomials. A permutation statistic is called Eulerian if it is equidistributed with respect to the excedance number exc over S_n for every n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ be a permutation of $[n]$, an index $1 \leq i \leq n$ such that $\sigma_i > i$ is called an excedance of σ and the number of excedances of σ is denoted by $\mathrm{exc}(\sigma)$. A Stirling statistic of a permutation refers to a statistic that is equidistributed with respect to the number of cycles.

Examples of Stirling statistics include the number of left-to-right maxima, the number of right-to-left maxima. Let $\mathrm{cyc}(\sigma)$ denote the number of cycles of σ. A left-to-right maximum (minimum) refers to an element σ_i such that for all $j < i$, $\sigma_j < \sigma_i$ ($\sigma_i < \sigma_j$). Likewise, a right-to-left maximum (minimum) of σ refers to an element σ_i such that for all $j > i$ we have $\sigma_j < \sigma_i$ ($\sigma_j > \sigma_i$). Let $\mathrm{lrmax}(\sigma)$, $\mathrm{lrmin}(\sigma)$, $\mathrm{rlmax}(\sigma)$ and $\mathrm{rlmin}(\sigma)$ denote the numbers of left-to-right maxima, left-to-right minima, right-to-left maxima, and right-to-left minima, respectively.

Foata-Schützenberger [37] introduced the following polynomials, involving the Eulerian-Stirling statistics, which have been called the q-Eulerian polynomials by Brenti [6]:

$$A_n(x;q) = \sum_{\sigma\in S_n} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}. \tag{4.13}$$

Brenti [6] obtained the generating function of $A_n(x;q)$, and showed that $A_n(x;q)$ is log-concave. A grammar to generate $A_n(x;q)$ has been given by Ma-Ma-Yeh-Zhu [57].

Carlitz-Scoville [9] introduced the polynomials $A_n(x,y \mid \alpha,\beta)$, which have been called the (α,β)-Eulerian polynomials by Ji [48]. They are defined by the numbers of descents, ascents, left-to-right maxima, right-to-left maxima. More precisely, $A_0(x,y \mid \alpha,\beta) = 1$ and for $n \geq 1$,

$$A_n(x,y \mid \alpha,\beta) = \sum_{\sigma\in S_{n+1}} x^{\mathrm{asc}(\sigma)-1} y^{\mathrm{des}(\sigma)-1} \alpha^{\mathrm{lrmax}(\sigma)-1} \beta^{\mathrm{rlmax}(\sigma)-1}. \tag{4.14}$$

By the first fundamental transformation of Foata-Schützenberger [37], it can be seen that the q-Eulerian polynomials (4.13) are a specialization of $A_n(x,y \mid \alpha,\beta)$, as given by

$$A_n(x;q) = A_n(1,x \mid q,0).$$

Setting $\alpha = 0$ and $\beta = 1$ in (4.14), we recover the bivariate Eulerian polynomials, that is, for $n \geq 1$,

$$A_n(x,y \mid 0,1) = \sum_{\sigma\in S_n} x^{\mathrm{asc}(\sigma)-1} y^{\mathrm{des}(\sigma)} = x^{-1} A_n(x,y). \tag{4.15}$$

Below are the first few values of $A_n(x,y \mid \alpha,\beta)$:

$$\begin{aligned}
A_1(x,y \mid \alpha,\beta) &= \alpha x + \beta y,\\
A_2(x,y \mid \alpha,\beta) &= \alpha^2x^2 + \alpha xy + \beta xy + 2\alpha\beta xy + \beta^2y^2\\
A_3(x,y \mid \alpha,\beta) &= \alpha^3x^3 + \alpha x^2y + \beta x^2y + 3\alpha\beta x^2y + 3\alpha^2x^2y + 3\alpha^2\beta x^2y\\
&\quad + \alpha xy^2 + \beta xy^2 + 3\alpha\beta xy^2 + 3\alpha\beta^2xy^2 + 3\beta^2xy^2 + \beta^3y^3.
\end{aligned}$$

Carlitz-Scoville [9] obtained the generating function of $A_n(x,y \mid \alpha,\beta)$.

Theorem 4.3 (Carlitz-Scoville [9])**.** *We have*

$$\sum_{n\geq 0} A_n(x,y \mid \alpha,\beta)\frac{t^n}{n!} = \big(1 + xF(x,y;t)\big)^{\alpha}\big(1 + yF(x,y;t)\big)^{\beta}, \tag{4.16}$$

where

$$F(x,y;t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}. \tag{4.17}$$

The complement of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is defined as

$$\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n). \tag{4.18}$$

Taking the complementation of permutations and exchanging the roles of x and y, Ji [48] obtained an equivalent definition

$$A_n(x,y \mid \alpha,\beta) = \sum_{\sigma\in S_{n+1}} x^{\mathrm{des}(\sigma)-1} y^{\mathrm{asc}(\sigma)-1} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1}. \tag{4.19}$$

A grammar to generate $A_n(x,y \mid \alpha,\beta)$ was found by Ji [48].

Theorem 4.4 (Ji [48])**.** *Define a grammar E by*

$$E = \{a \to \alpha ax,\ b \to \beta by,\ x \to xy,\ y \to xy\}, \tag{4.20}$$

and let D_E be the formal derivative of E. For $n \geq 0$, we have

$$D_E^n(ab) = abA_n(x,y \mid \alpha,\beta). \tag{4.21}$$

A grammatical calculus for the generating function of $A_n(x,y \mid \alpha,\beta)$ was given by Ji [48].

The following proof presents an interplay between two formal derivatives.

Grammatical Proof of Theorem 4.3. Let

$$\mathrm{Gen}_E(ab,t) = \sum_{n=0}^{\infty} D_E^n(ab)\frac{t^n}{n!}.$$

Theorem 4.4 asserts that

$$\mathrm{Gen}_E(ab,t) = ab\sum_{n\geq 0} A_n(x,y \mid \alpha,\beta)\frac{t^n}{n!}. \tag{4.22}$$

Let D be the formal derivative of the grammar (1.3), where α and β are regarded as constants. Thus,

$$D(y^\alpha) = \alpha y^{\alpha-1}D(y) = \alpha y^\alpha x, \qquad D(x^\beta) = \beta x^{\beta-1}D(x) = \beta x^\beta y. \tag{4.23}$$

Let $a = y^\alpha$ and $b = x^\beta$. By (4.23), we find that

$$D(y^\alpha) = D_E(a), \quad D(x^\beta) = D_E(b).$$

Evidently,

$$D(x) = D_E(x), \quad D(y) = D_E(y).$$

By induction, we see that for $n \geq 0$,

$$D^n(y^\alpha) = D_E^n(a), \quad D^n(x^\beta) = D_E^n(b),$$

where $a = y^\alpha$ and $b = x^\beta$. It follows that for $n \geq 0$,

$$D_E^n(ab) = D^n(x^\beta y^\alpha), \tag{4.24}$$

which yields

$$\mathrm{Gen}_E(ab,t) = \mathrm{Gen}(x^\beta y^\alpha, t). \tag{4.25}$$

By the multiplicative property, the following relation

$$\mathrm{Gen}(x^\beta y^\alpha, t) = \big(\mathrm{Gen}(x,t)\big)^\beta \big(\mathrm{Gen}(y,t)\big)^\alpha \tag{4.26}$$

is valid when α and β are integers. Moreover, expressing a power series $f(x)^\alpha$ as $e^{\alpha \log f(x)}$, we see that (4.26) holds when α and β are regarded as parameters.

Along the lines of the proof of Theorem 4.2, we deduce that

$$\mathrm{Gen}(y,t) = \sum_{n\geq 0} D^n(y)\frac{t^n}{n!} = y\big(1 + xF(x,y;t)\big), \tag{4.27}$$

and

$$\mathrm{Gen}(x,t) = \sum_{n\geq 0} D^n(x)\frac{t^n}{n!} = x\big(1 + yF(x,y;t)\big), \tag{4.28}$$

where $F(x,y;t)$ is given by (4.17). Substituting (4.27) and (4.28) into (4.25) gives

$$\mathrm{Gen}_E(ab,t) = y^\alpha x^\beta \big(1 + xF(x,y;t)\big)^\alpha \big(1 + yF(x,y;t)\big)^\beta.$$

Comparing with (4.22) yields (4.16). This completes the proof. ∎

Next we present a grammatical calculus for the enumeration of permutations with respect to the three kinds of peaks. Let $n \geq 1$ and let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of $[n]$. Assume that $\sigma_0 = \sigma_{n+1} = 0$.

An index i is called

- a left peak if $1 \le i \le n-1$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- an interior peak if $2 \le i \le n-1$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- an exterior peak, or outer peak, if $1 \le i \le n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$;
- a valley if $2 \le i \le n-1$ and $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$.

Let $L(\sigma)$, $M(\sigma)$, $W(\sigma)$ and $V(\sigma)$ denote the numbers of left peaks, interior peaks, exterior peaks and valleys of σ, respectively. Such pictographic symbols $L(\sigma)$, $M(\sigma)$ and $W(\sigma)$ were proposed in [18]. Here the letter L resembles a peak on the left, the letter M resembles peaks in the middle, and the letter W indicates that we allow peaks at the beginning and at the end. For example, let $\sigma = 7\,1\,3\,8\,5\,9\,6\,2\,4$, then we have

$$L(\sigma) = 3,\ M(\sigma) = 2,\ W(\sigma) = 4,\ V(\sigma) = 3.$$

It is clear that

$$W(\sigma) - 1 = V(\sigma) = M(\sigma^c),$$

where σ^c is the complement of σ, as defined by (4.18).

For $n \ge 1$ and $0 \le k \le \lfloor n/2 \rfloor$, let $L(n,k)$ be the number of permutations of $[n]$ with k left peaks. Similarly, we may define $M(n,k)$ and $W(n,k)$, with appropriate ranges of k. For $n = 0$, set $L(0,0) = M(0,0) = W(0,0) = 1$.

Define the polynomials

$$L_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} L(n,k)\,x^k,$$

$$M_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} M(n,k)\,x^k,$$

$$W_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} W(n,k)\,x^k.$$

Let $L(x,t)$, $M(x,t)$ and $W(x,t)$ denote the exponential generating functions of $L_n(x)$, $M_n(x)$ and $W_n(x)$, respectively. The generating function of the interior peak polynomials $M_n(x)$ was derived by David-Barton [27] by establishing a partial differential equation. Equivalent formulas can be found in Entringer [36], Kitaev [50] and Zhuang [86]. It is easy to deduce $W(x,t)$ from $M(x,t)$. Gessel [62] found an explicit formula for $L(x,t)$.

Theorem 4.5 (Gessel [62]). *We have*

$$\sum_{n\geq 0} L_n(x)\,\frac{t^n}{n!} = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(t\sqrt{1-x})-\sinh(t\sqrt{1-x})}. \tag{4.29}$$

To set up the grammatical calculus for peak polynomials, we need the bivariate forms. For $n=0$, define $L_0(x,y)=x$, $M_0(x,y)=1$ and $W_0(x,y)=y$. For $n\geq 1$, define

$$L_n(x,y) = \sum_{k=0}^{\lfloor n/2\rfloor} L(n,k)\,x^{2k+1}y^{n-2k}, \tag{4.30}$$

$$M_n(x,y) = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} M(n,k)\,x^{2k+2}y^{n-2k-1}, \tag{4.31}$$

$$W_n(x,y) = \sum_{k=1}^{\lfloor (n+1)/2\rfloor} W(n,k)\,x^{2k}y^{n-2k+1}. \tag{4.32}$$

Consider the grammar

$$G = \{x \to xy,\ y \to x^2\}. \tag{4.33}$$

Let D be the formal derivative of the grammar (4.33). We have the following grammatical expressions for the three kinds of peak polynomials. For $n\geq 0$, we have

$$D^n(x) = L_n(x,y), \quad D^n(y) = W_n(x,y).$$

For $n\geq 1$, we have $D^n(y) = M_n(x,y)$. The grammar (4.33) together with the relation $D^n(x) = L_n(x,y)$ was independently obtained by Ma [53] and Chen-Fu [11, 14]. A grammatical labeling was given in [14].

The generating function of $L_n(x,y)$ can be derived using the grammatical calculus, see Chen-Fu [16].

Theorem 4.6 (Chen-Fu [16]). *Let $L_0(x,y)=x$. We have*

$$\sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!} = \frac{x\sqrt{y^2-x^2}}{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2})-y\sinh(t\sqrt{y^2-x^2})}. \tag{4.34}$$

Proof. Let D be the formal derivative of the grammar (4.33). For $n\geq 0$, we have $D^n(x) = L_n(x,y)$. We proceed to compute $\mathrm{Gen}(x,t)$. Notice that

$$D(x^{-1}) = -x^{-1}y, \quad D^2(x^{-1}) = x^{-1}(y^2-x^2).$$

It follows that for $n \geq 0$,

$$D^{2n}(x^{-1}) = x^{-1}(y^2 - x^2)^n, \tag{4.35}$$

$$D^{2n+1}(x^{-1}) = -x^{-1}y(y^2 - x^2)^n. \tag{4.36}$$

Now we have

$$\sum_{n=0}^{\infty} D^{2n}(x^{-1}) \frac{t^{2n}}{(2n)!} = x^{-1}\cosh(t\sqrt{y^2 - x^2}), \tag{4.37}$$

$$\sum_{n=0}^{\infty} D^{2n+1}(x^{-1}) \frac{t^{2n+1}}{(2n+1)!} = -\frac{x^{-1}y}{\sqrt{y^2 - x^2}}\sinh(t\sqrt{y^2 - x^2}). \tag{4.38}$$

Combining (4.37) and (4.38), we deduce that

$$\mathrm{Gen}(x^{-1}, t) = \frac{\sqrt{y^2 - x^2}\cosh(t\sqrt{y^2 - x^2}) - y\sinh(t\sqrt{y^2 - x^2})}{x\sqrt{y^2 - x^2}}, \tag{4.39}$$

or equivalently,

$$\mathrm{Gen}(x, t) = \frac{1}{\mathrm{Gen}(x^{-1}, t)} = \frac{x\sqrt{y^2 - x^2}}{\sqrt{y^2 - x^2}\cosh(t\sqrt{y^2 - x^2}) - y\sinh(t\sqrt{y^2 - x^2})},$$

as required. ∎

Dividing both sides of (4.34) by x, replacing x by x^2 and setting $y = 1$, we come to Gessel's formula (4.29).

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, assume that $\sigma_0 = 0$ and $\sigma_{n+1} = +\infty$. We now define the following terms.

- For $1 \leq i < n$, if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, then i is called a left double ascent of σ.
- For $1 < i < n$, if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, then i is called a double ascent of σ.
- For $1 < i \leq n$, if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, then i is called a right double ascent of σ.
- For $1 \leq i \leq n$, if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, then i is called a left-right double ascent of σ.

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, the following patterns are defined under the convention that $\sigma_0 = +\infty$ and $\sigma_{n+1} = 0$,

- For $1 \leq i < n$, if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, then i is called a left double descent of σ.

- For $1 < i < n$, if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, then i is called a double descent of σ.
- For $1 < i \leq n$, if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, then i is called a right double descent of σ.
- For $1 \leq i \leq n$, if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, then i is called a left-right double descent of σ.

The number of indices in the above definitions will be denoted by concatenation of the initials of the terms. For example, $\mathrm{lrdd}(\sigma)$ stands for the number of left-right double descents of σ. Denote by $U(n,k)$ the number of permutations of $[n]$ with k double descents. Define

$$U_n(y) = \sum_{k\geq 0} U(n,k)\, y^k, \tag{4.40}$$

that is, $U_n(y)$ is the generating function of $\mathrm{dd}(\sigma)$ over S_n. Elizalde-Noy [35] derived the generating function of $U_n(y)$ via an ordinary differential equation.

Theorem 4.7 (Elizalde-Noy [35])**.** *We have*

$$\sum_{n=0}^{\infty} U_n(y)\frac{t^n}{n!} = \frac{2\sqrt{(y-1)(y+3)}e^{t/2\cdot(1-y+\sqrt{(y-1)(y+3)})}}{1+y+\sqrt{(y-1)(y+3)}-(1+y-\sqrt{(y-1)(y+3)})e^{t\sqrt{(y-1)(y+3)}}}. \tag{4.41}$$

For the special case $y = 0$, Barry [2] and Basset [3] independently obtained the following formula.

Theorem 4.8 (Elizalde-Noy [35], Barry [2], Basset [3])**.** *We have*

$$\sum_{n=0}^{\infty} U(n,0)\frac{t^n}{n!} = \frac{\sqrt{3}}{2}\frac{e^{t/2}}{\cos(\sqrt{3}t/2+\pi/6)}. \tag{4.42}$$

Fu [38] presented a grammatical calculus for the joint distribution of the number of left peaks and the number of double descents over permutations of $[n]$. Consider the grammar

$$G = \{x \to xy,\ y \to xz,\ z \to zw,\ w \to xz\}. \tag{4.43}$$

It can be seen that this grammar is a unification of the grammar (1.3) for the Eulerian polynomials and the grammar (4.33) for the number of left peaks. Substituting w with x and z with y, G becomes (1.3). It should be noted that such substitutions are consistent with the original grammar. On the other hand, substituting z with x and w with y, G becomes the grammar (4.33). Let $V_n(i,j)$ denote the number of permutations of $[n]$ with i left peaks and j double descents. Let $V_0(x,y,z,w)=z$. For $n\geq 1$, define

$$V_n(x,y,z,w)=\sum_{i,j}V_n(i,j)x^iy^jz^{i+1}w^{n-2i-j}, \tag{4.44}$$

where the sum ranges over i and j subject to $2i+j\leq n$.

Fu [38] gave a grammatical labeling of permutations and obtained the generating function of $V_n(x,y,z,w)$ by the grammatical calculus.

Theorem 4.9 (Fu [38])**.** *We have*

$$\begin{aligned}\sum_{n\geq 0}&V_n(x,y,z,w)\frac{t^n}{n!}\\ &=\frac{2z\sqrt{(w+y)^2-4xz}e^{t/2\cdot(w-y+\sqrt{(w+y)^2-4xz})}}{w+y+\sqrt{(w+y)^2-4xz}-(w+y-\sqrt{(w+y)^2-4xz})e^{t\sqrt{(w+y)^2-4xz}}}.\end{aligned}$$

Theorem 4.9 has several specializations. Setting $y=z=w=1$, it reduces to Theorem 4.5. Setting $z=x$ and $w=y$, we are led to Theorem 4.6. Setting $x=z=w=1$ yields Theorem 4.7. Setting $x=z=w=1$ and $y=0$, we come to Theorem 4.8.

Carlitz-Scoville [9] investigated the joint distribution of several statistics of permutations, and obtained the following generating function, as restated by Stanley [76].

$$\sum_{n\geq 1}\left(\sum_{\sigma\in S_n}u^{V(\sigma)}v^{W(\sigma)-1}w^{\mathrm{lda}(\sigma)}z^{\mathrm{rdd}(\sigma)}\right)\frac{t^n}{n!}=F(x,y;t), \tag{4.45}$$

where $x+y=w+z$, $xy=uv$, and $F(x,y;t)$ is given by (4.17). Fu [38] presented a grammatical labeling along with the grammatical calculus for the above formula. The original formulation of Carlitz-Scoville can be recovered by differentiating (4.45) with respect to t. For $n\geq 1$, define

$$P_n(u,v,w,z)=\sum_{\sigma\in S_{n+1}}u^{V(\sigma)}v^{W(\sigma)-1}w^{\mathrm{rdd}(\sigma)}z^{\mathrm{lda}(\sigma)}. \tag{4.46}$$

For $n=0$, set $P_0(u,v,w,z)=1$. Notice that the above definition (4.46) aligns with the definition (4.44).

Theorem 4.10 (Carlitz-Scoville [9]). *We have*

$$\sum_{n\geq 0} P_n(u,v,w,z)\frac{t^n}{n!} = (1+yF(x,y;t))(1+xF(x,y;t)), \tag{4.47}$$

where $x+y=w+z$, $xy=uv$, and $F(x,y;t)$ is given by (4.17).

Ji [48] introduced the (α,β)-extension of the above polynomials $P_n(u,v,w,z)$, that is, for $n\geq 1$, define

$$P_n(u,v,w,z\mid \alpha,\beta) = \sum_{\sigma\in S_{n+1}} u^{\mathrm{V}(\sigma)}v^{W(\sigma)-1}w^{\mathrm{rdd}(\sigma)}z^{\mathrm{lda}(\sigma)}\alpha^{\mathrm{lrmax}(\sigma)-1}\beta^{\mathrm{rlmax}(\sigma)-1}. \tag{4.48}$$

By convention, set $P_0(u,v,w,z\mid\alpha,\beta)=1$. The first few values of $P_n(u,v,w,z\mid\alpha,\beta)$ are as follows:

$$P_1(u,v,w,z\mid\alpha,\beta) = \alpha z+\beta w;$$

$$P_2(u,v,w,z\mid\alpha,\beta) = \alpha^2z^2+2\alpha\beta wz+\beta^2w^2+\alpha vu+\beta vu;$$

$$\begin{aligned}P_3(u,v,w,z\mid\alpha,\beta) &= \alpha^3z^3+3\alpha^2\beta wz^2+3\alpha\beta^2w^2z+3\alpha\beta uvz+3\alpha^2uvz+\alpha uvz\\ &\quad+\beta uvz+\beta^3w^3+3\beta^2uvw+\alpha uvw+\beta uvw+3\alpha\beta uvw.\end{aligned}$$

By taking complementation of permutations, we have the following equivalent definition:

$$P_n(u,v,w,z\mid\alpha,\beta) = \sum_{\sigma\in S_{n+1}} (uv)^{M(\sigma)}w^{\mathrm{rda}(\sigma)}z^{\mathrm{ldd}(\sigma)}\alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1}. \tag{4.49}$$

The following grammar for $P_n(u,v,w,z\mid\alpha,\beta)$ was given by Ji [48]:

$$H=\{a\to\alpha az,\ b\to\beta bw,\ z\to uv,\ w\to uv,\ u\to uw,\ v\to vz\}$$

Theorem 4.11 (Ji [48]). *Let D_H be the formal derivative of the above grammar H. For $n\geq 0$, we have*

$$D_H^n(ab) = abP_n(u,v,w,z\mid\alpha,\beta).$$

Employing Theorem 4.11, Ji [48] obtained the following generating function.

Theorem 4.12 (Ji [48]). *We have*

$$\sum_{n\geq 0} P_n(u,v,w,z\mid\alpha,\beta)\frac{t^n}{n!} = \big(1+yF(x,y;t)\big)^{\frac{\alpha+\beta}{2}}\big(1+xF(x,y;t)\big)^{\frac{\alpha+\beta}{2}}e^{\frac{1}{2}(\beta-\alpha)(w-z)t},$$

where $x+y=w+z$, $xy=uv$, and $F(x,y;t)$ is given by (4.17).

There are several consequences of Theorem 4.12. First, we get an explicit expression for the generating function of $P_n(u,v,w,z \mid \alpha,\beta)$.

Theorem 4.13 (Ji [48])**.** *We have*

$$\sum_{n\geq 0} P_n(u,v,w,z \mid \alpha,\beta)\frac{t^n}{n!} = e^{\frac{1}{2}(\beta-\alpha)(w-z)t} \times \left(\cosh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right.$$
$$\left. -\frac{w+z}{\sqrt{(w+z)^2-4uv}}\sinh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right)^{-(\alpha+\beta)}.$$

When $\alpha=\beta=1$, Theorem 4.12 reduces to Theorem 4.10. Setting $\alpha=0$ in (4.49), we get

$$P_n(u,v,w,z \mid 0,\beta) = \sum_{\sigma\in S_n} (uv)^{L(\sigma)} w^{\mathrm{lrda}(\sigma)} z^{\mathrm{dd}(\sigma)} \beta^{\mathrm{rlmin}(\sigma)}. \tag{4.50}$$

In view of (4.50), a β-extension of Theorem 4.9 can be deduced from Theorem 4.13 by setting $\alpha=0$.

Gessel-Stanley [39] introduced the notion of Stirling permutations; see Section 9. The Eulerian polynomials (with respect to the number of descents) over Stirling permutations are referred to as the second-order Eulerian polynomials, which can be generated by the following grammar

$$x \to xy^2, \quad y \to xy^2,$$

see [14]. The higher order Eulerian polynomials over r-Stirling permutations can be generated by the grammar

$$x \to xy^r, \quad y \to xy^r.$$

5 Grammar Assisted Bijections

In this section, we use two examples to illustrate how grammars can be instrumental in constructing bijections. The first example is a bijection between permutations and increasing trees that maps descents of a permutation to the leaves of an increasing tree.

Dumont [31] found the following grammar for increasing trees:

$$x_i \to x_0 x_{i+1}, \quad i \geq 0, \tag{5.1}$$

where x_i is the label of a vertex with outdegree i (the number of its children). Let T be an increasing tree on $\{0,1,2,\ldots,n\}$. When the new vertex $n+1$ is attached to T, some labels will be updated. Assume that $n+1$ is inserted to T as a child of a vertex v. If v has outdegree i, then the label of v will be updated by x_{i+1} and the leaf $n+1$ will be labeled by x_0. This change of labels is captured by the rule $x_i \to x_0 x_{i+1}$. Let D be the formal derivative of the grammar (5.1). Dumont [31] showed that

$$D^n(x_0) = \sum_T x_{\deg_T(0)} x_{\deg_T(1)} \cdots x_{\deg_T(n)}, \tag{5.2}$$

where the sum ranges over increasing trees T on $\{0,1,2,\ldots,n\}$ and $\deg_T(i)$ denotes the outdegree of i in T.

If we use different labels to distinguish internal vertices and leaves of an increasing tree, and, more precisely, use x to label an internal vertex and use y to label a leaf, then we are led to the grammar

$$x \to xy, \quad y \to xy,$$

which is exactly the grammar for the Eulerian polynomials. This means that there is a combinatorial interpretation of the Eulerian polynomials in terms of increasing trees, see Dumont [31]. A generating function proof of this fact was given by Lin-Liu-Wang-Zang [52].

Theorem 5.1 (Dumont [31])**.** *For $n \geq 1$, the number of permutations of $[n]$ with k descents equals the number of increasing trees on $\{0,1,2,\ldots,n\}$ with k leaves.*

Guided by the grammar, it is easy to construct a bijection for the above correspondence.

Following this line of reasoning, Chen-Fu [14] obtained the following correspondence.

Theorem 5.2 (Chen-Fu [14])**.** *The number of permutations of $[n]$ with k left peaks equals the number of increasing trees on $\{0,1,2,\ldots,n\}$ with $2k+1$ vertices of even degree.*

The grammatical justification takes only one line. In the grammar (5.1), setting $x_{2i} = x$ and $x_{2i+1} = y$, we are led to the grammar (4.33) for the number of left peaks of a permutation. This gives a confirmation of Theorem 5.2. A grammar assisted bijection is provided by Chen-Fu [16].

As for exterior peaks, Chen-Fu [16] obtained the following correspondence and found a grammar assisted bijection.

Theorem 5.3 (Chen-Fu [16]). *For $n \geq 1$, there is a bijection from the set of permutations of $[n]$ to the set of increasing trees on $\{0,1,\ldots,n\}$ that maps the number k of exterior peaks of a permutation to the number j of nonroot vertices of even degree such that $k = \lfloor (j+1)/2 \rfloor$.*

There are two kinds of runs of a permutation. First, after patching a zero to the beginning of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, that is, setting $\sigma_0 = 0$, define an up-down run to be a maximal ascending segment or a maximal descending segment (of consecutive elements). If we drop the assumption that $\sigma_0 = 0$, a maximal ascending or descending segment is called an alternating run. For example, the permutation 21 has two up-down runs 0 2 and 2 1), but it has only one alternating run 2 1.

Using a recurrence relation, Ma [54] discovered a grammar for up-down runs, that is,

$$G = \{a \to ax,\ x \to xy,\ y \to x^2\}.$$

A grammatical labeling for up-down runs was found by Chen-Fu [16]. This labeling scheme demonstrates how an insertion of the element $n+1$ to a permutation of $[n]$ affects the number of up-down runs. This labeling gives a clue to establish a correspondence between permutations and increasing trees that maps the number of up-down runs to the number of vertices in an increasing tree subject to a certain degree constraint.

Theorem 5.4 (Chen-Fu [16]). *For $n \geq 1$, there is a bijection from the set of permutations of $[n]$ to the set of increasing trees on $\{0,1,\ldots,n\}$ that maps the number of up-down runs of a permutation to the number of nonroot vertices of even degree.*

The second example exhibits a connection between a grammar of Dumont and a theorem of Diaconis-Evans-Graham on permutation statistics. Let $\sigma = \sigma_1\cdots\sigma_n \in S_n$. An index $1 \leq i \leq n$ is called an excedance if $\sigma_i > i$, a drop if $\sigma_i < i$, and a fixed point if $\sigma_i = i$. Obviously, n cannot be an excedance and 1 cannot be a drop. The numbers of excedances, drops and fixed points of σ are denoted by $\mathrm{exc}(\sigma)$, $\mathrm{drop}(\sigma)$ and $\mathrm{fix}(\sigma)$, respectively. A drop is also called an anti-excedance.

Dumont [31] obtained the following grammar for the joint distribution $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})$ over permutations:

$$G = \{a \to az, z \to xy, x \to xy, y \to xy\}. \tag{5.3}$$

Define

$$F_n(x,y,z) = \sum_{\sigma \in S_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} z^{\mathrm{fix}(\sigma)},$$

and let $F_0(x,y,z) = 1$.

Theorem 5.5 (Dumont [31])**.** *Let D be the formal derivative of the grammar (5.3). For $n \geq 0$, we have*

$$D^n(a) = aF_n(x,y,z).$$

It should be noted that the work of Roselle [67] was not accurately cited by Dumont [31]. However, the pointer to the work of Roselle turned out to be unexpectedly informative. Namely, Roselle [67] in fact investigated the joint distribution of the number of ascents and the number of successions.

Assume that $n \geq 1$ and $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is a permutation of $[n]$. An index $1 \leq i \leq n-1$ is called a succession if $\sigma_{i+1} = \sigma_i + 1$ and the number of successions of σ is denoted by $\mathrm{suc}(\sigma)$. Roselle found the generating function for the joint distribution of $(\mathrm{asc}, \mathrm{suc})$ over permutations. A related statistic is called the big ascent by Ma-Qi-Yeh-Yeh [59]. If $1 \leq i \leq n-1$ and $\sigma_i + 2 \leq \sigma_{i+1}$, then i is called a big ascent of σ. The number of big ascents of σ is denoted by $\mathrm{basc}(\sigma)$. The generating function of the joint distribution of $(\mathrm{basc}, \mathrm{des}, \mathrm{suc})$ over permutations has been derived in [59].

While the grammar of Dumont does not seem to apply to the joint distribution of $(\mathrm{asc}, \mathrm{suc})$, Chen-Fu [18] observed that it instead applies to the joint distribution of $(\mathrm{exc}, \mathrm{fix})$, as well as to the joint distribution of $(\mathrm{jump}, \mathrm{lsuc})$, where the statistics jump and lsuc are defined as follows. For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, assume that $\sigma_0 = 0$. An index $1 \leq i \leq n$ is called a left succession if $\sigma_{i-1} + 1 = \sigma_i$, and a jump if $\sigma_{i-1} + 2 \leq \sigma_i$. The numbers of left successions and jumps of σ are denoted by $\mathrm{lsuc}(\sigma)$ and $\mathrm{jump}(\sigma)$, respectively.

For $n \geq 1$ and a permutation $\sigma \in S_n$, define

$$S(\sigma) = \{i \mid 1 \leq i \leq n-1, \sigma_i + 1 = \sigma_{i+1}\}, \tag{5.4}$$

$$G(\sigma) = \{i \mid 1 \leq i \leq n-1, \sigma_i = i\}, \tag{5.5}$$

$$F(\sigma) = \{i \mid 1 \leq i \leq n, \sigma_i = i\}. \tag{5.6}$$

It should be noted that the index n is excluded in the definition of $G(\sigma)$. Given a subset $I \subseteq [n-1]$, let $M_n(I)$ denote the set of permutations $\sigma \in S_n$ such that $S(\sigma) = I$, and let

$G_n(I)$ denote the set of permutations $\sigma \in S_n$ such that $G(\sigma) = I$, and let $F_n(I)$ denote the set of permutations $\sigma \in S_n$ such that $F(\sigma) = I$. Diaconis-Evans-Graham [29] discovered the following correspondence.

Theorem 5.6 (Diaconis-Evans-Graham [29])**.** *Let $n \geq 1$ and $I \subseteq [n-1]$. Then there exists a one-to-one correspondence between $M_n(I)$ and $G_n(I)$.*

For $n \geq 1$ and $\sigma \in S_n$, define

$$\overline{S}(\sigma) = \{\sigma_i \mid 1 \leq i \leq n-1, \sigma_i + 1 = \sigma_{i+1}\}.$$

It is easily seen that for any $\sigma \in S_n$,

$$\overline{S}(\sigma^{-1}) = S(\sigma), \tag{5.7}$$

$$G(\sigma^{-1}) = G(\sigma), \tag{5.8}$$

$$F(\sigma^{-1}) = F(\sigma), \tag{5.9}$$

where σ^{-1} stands for the inverse permutation σ. For $n \geq 1$ and a subset $I \subseteq [n-1]$, let $\overline{S}_n(I)$ denote the set of permutations $\sigma \in S_n$ such that $\overline{S}(\sigma) = I$. Then Theorem 5.6 can be restated as follows, see Chen-Fu [18].

Theorem 5.7. *Let $n \geq 1$ and $I \subseteq [n-1]$. There exists a bijection between $\overline{S}_n(I)$ and $G_n(I)$.*

Chen-Fu [18] observed the connection between the grammar (5.3) of Dumont [31] and Theorem 5.6 of Diaconis-Evans-Graham, and obtained the left succession analogue along with a grammatical labeling and a grammar assisted bijection.

Analogous to $\overline{S}(\sigma)$, for $n \geq 1$ and a permutation σ of $[n]$, define

$$\overline{L}(\sigma) = \{\sigma_i \mid 1 \leq i \leq n, \sigma_{i-1} + 1 = \sigma_i\}.$$

Moreover, for a subset I of $[n]$, define $\overline{L}_n(I)$ to be the set of permutations σ of $[n]$ such that $\overline{L}(\sigma) = I$.

Theorem 5.8 (Chen-Fu [18])**.** *Let $n \geq 1$ and $I \subseteq [n]$. There exists a bijection ϕ from $\overline{L}_n(I)$ to $F_n(I)$ such that* $(\mathrm{jump}(\sigma), \mathrm{des}(\sigma) - 1) = (\mathrm{exc}(\phi(\sigma)), \mathrm{drop}(\phi(\sigma)))$ *for any permutation σ of $[n]$.*

6 The Expansion of $(g(x)D)^n f(x)$

In this section, we discuss the expansion of $(g(x)D)^n$ when applied to a function $f(x)$, where $D = d/dx$, and $g(x)$ can be viewed as an operator of multiplication by $g(x)$. Noting that

$$(Dg(x))(f(x)) = (g(x)D)(f(x)) + g'(x)(f(x)),$$

we have the relation

$$Dg = gD + g', \tag{6.1}$$

where the multiplication operator g does not commute with D. By (6.1), we obtain a grammar for the expansion of $(g(x)D)^n f$ and we may also look at the expansion of $(g(x)D)^n f$ from the perspective of normal ordering defined below; see (6.7).

The expansion of $(g(x)D)^n f(x)$ has been extensively studied, see Scherk [69], Comtet [25], Mansour-Schork [60] and Han-Ma [43]. In particular, Carlitz-Scoville [8] investigated the expansion of $((1+x^2)D)^n f(x)$. In fact, the involved polynomials, called the derivative polynomials, can be traced back to the work of Knuth-Buckholtz [51] on the Euler numbers. For the generating functions and the grammatical calculus of the derivative polynomials, see [17, 45].

Write

$$F_n = (g(x)D)^n f(x) = \sum_{k=0}^{n} A_{n,k}(g_0, g_1, \ldots, g_{n-k}) f_k, \tag{6.2}$$

where $f_i = D^i(f)$ and $g_i = D^i(g)$. The polynomials $A_{n,k}(g_0, g_1, \ldots, g_{n-k})$ have been called the A-polynomials. The coefficients of the A-polynomials are listed as A139605 in OEIS [62].

Let $f_0 = f$, $g_0 = g$, and $A_{0,0} = 1$. For $n \leq 4$, we have

$$\begin{aligned}
F_0 &= f_0, \\
F_1 &= g_0 f_1, \\
F_2 &= g_0 g_1 f_1 + g_0^2 f_2, \\
F_3 &= (g_0^2 g_2 + g_0 g_1^2) f_1 + 3 g_0^2 g_1 f_2 + g_0^3 f_3, \\
F_4 &= (g_0^3 g_3 + 4 g_0^2 g_1 g_2 + g_0 g_1^3) f_1 + (4 g_0^3 g_2 + 7 g_0^2 g_1^2) f_2 + 6 g_0^3 g_1 f_3 + g_0^4 f_4.
\end{aligned}$$

The A-polynomials first appeared in the thesis of Scherk in 1823 [69], where he computed F_n for $n \leq 5$, see Blasiak-Flajolet [1]. Murphy [61] in 1837 obtained explicit expressions for the coefficients of f_k in F_n for $k = n, n-1, n-2$. Comtet [25] obtained the following formula for general n.

Theorem 6.1 (Comtet [25]). *For $1 \leq k \leq n$, we have*

$$A_{n,k} = \frac{g_0}{k!} \sum (2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1}) \frac{g_{k_1}}{k_1!} \cdots \frac{g_{k_{n-1}}}{k_{n-1}!}, \tag{6.3}$$

where the sum ranges over sequences of nonnegative integers $(k_1, k_2, \ldots, k_{n-1})$ such that $k_1 + k_2 + \cdots + k_{n-1} = n-k$ and $k_1 + \cdots + k_j \leq j$ for any $1 \leq j \leq n-1$.

The idea of using labeled trees to describe the action of partial differential operators on multivariate functions goes back to Cayley. Following this direction, Copeland [26] and Blasiak-Flajolet [1] independently obtained an interpretation of $A_{n,k}$ in terms of increasing trees.

Let T be an increasing tree on $\{0, 1, \ldots, n\}$. As before, let $\deg_T(i)$ denote the outdegree of i in T. Define the weight of T by

$$g_{\deg_T(1)} \cdots g_{\deg_T(n)} f_{\deg_T(0)}.$$

Theorem 6.2 (Copeland [26], Blasiak-Flajolet [1]). *For $n \geq 1$, F_n equals the total weight of increasing trees on $\{0, 1, \ldots, n\}$.*

Chen-Fu-Wang [1] obtained a grammar that generates F_n.

Theorem 6.3. *Let D be the formal derivative of the following grammar*

$$G = \{f_i \rightarrow g_0 f_{i+1},\ \ g_i \rightarrow g_0 g_{i+1} \mid i = 0, 1, 2, \ldots\}. \tag{6.4}$$

Then $F_n = D^n(f_0)$.

Note that the above increasing tree interpretation of the A-polynomials immediately follows from the grammar (6.4). The increasing tree interpretation can be reformulated in terms of inversion sequences, because the Prüfer codes of increasing trees are precisely inversion sequences.

[1] W.Y.C. Chen, A.M. Fu and E.L. Wang, The grammatical calculus and normal ordering. Preprint, 2026

To be more specific, a sequence $e = (e_1, e_2, \ldots, e_n)$ of nonnegative integers is called an inversion sequence if for $1 \le i \le n$, we have $0 \le e_i < i$. Han-Ma [43] obtained an expansion of $(gD)^n f$ in terms of inversion sequences and gave a derivation of Comtet's formula in Theorem 6.1.

Next we show how the Prüfer codes or the inversion codes lead to the formula of Comtet. Let T be an increasing tree on $\{0, 1, 2, \ldots, n\}$. Notice that the maximum element n must be a leaf. Let e_n denote the parent of n. After this leaf is removed, we get an increasing tree on $\{0, 1, 2, \ldots, n-1\}$ in which $n-1$ must be a leaf. Denote the parent of $n-1$ by e_{n-1}. Repeating this procedure yields a sequence $(e_n, e_{n-1}, \ldots, e_1)$. Reversing this sequence, we get an inversion code $(e_1, \ldots, e_n)$. Observe that the number of occurrences of i equals the outdegree of the vertex i in the increasing tree. Thus, we obtain

$$\sum_T t_0^{\deg_T(0)} t_1^{\deg_T(1)} \cdots t_n^{\deg_T(n)} = t_0 (t_0 + t_1) \cdots (t_0 + t_1 + \cdots + t_{n-1}), \tag{6.5}$$

where the sum ranges over increasing trees T on $\{0, 1, 2, \ldots, n\}$. To expand the product on the right side of (6.5) we may consider the power of t_{n-1} first, then the power of t_{n-2}, and so on. By the Prüfer correspondence, a power t_i^j in the expansion indicates that the vertex i has outdegree j, contributing the factor g_j in the A-polynomial. Thus, we arrive at the formula of Comtet, see Chen-Fu-Wang [1].

For the case $g(x) = x$, the expansion of $(xD)^n$ can be viewed as a special case of the normal ordering problem. As noted by Schork [70], the idea of normal ordering goes back to Scherk [69]. Consider the operators X and D satisfying the relation

$$DX - XD = I, \tag{6.6}$$

where I is the identity operator. The associative algebra generated by X and D, subject to (6.6), is called the first Weyl algebra.

Any monomial in X and D can be written as

$$\omega = X^{r_1} D^{s_1} X^{r_2} D^{s_2} \cdots X^{r_n} D^{s_n},$$

where $r_k, s_k \ge 0$. The word ω is said to be in normal ordered form if it is written as

$$\omega = \sum_{r,s \in \mathbb{N}} S_{r,s}(\omega) X^r D^s, \tag{6.7}$$

where the coefficients $S_{r,s}(\omega)$ are uniquely determined and are called the normal ordering coefficients.

Ma-Mansour-Yeh-Yeh [58] considered the normally ordered expansion of $(w(x,y,\ldots)D_G)^n$, where D_G is the formal derivative of a grammar G. For example, given a grammar

$$G = \{x \to u(x,y),\, y \to v(x,y)\}$$

and a function $w(x,y)$, one may expand $(w(x,y)D_G)^n$ as

$$(w(x,y)D_G)^n = \sum_{k=0}^{n} \xi_{n,k}(x,y)w^k(x,y)D_G^k.$$

The coefficients $\xi_{n,k}(x,y)$ are determined by the grammar G and the weight function $w(x,y)$.

Let $\mathrm{cdes}(\pi)$ denote the number of cycle descents of a permutation π. A cycle descent of a permutation is defined as follows. In the following, we always write a permutation in standard cycle form, in which each cycle has its smallest element first and the cycles are written in increasing order of their first elements. If j immediately follows i in this linear representation and $i > j$, then we call (i,j) a cycle descent. For example, the permutation $\pi = (1,4,2)(3,5,7)(6,9,8)$ has two cycle descents $(4,2)$ and $(9,8)$. Ma-Mansour-Yeh-Yeh [58] obtained the following result.

Theorem 6.4 (Ma-Mansour-Yeh-Yeh [58]). *Let*

$$G = \{x \to y,\, y \to py\}.$$

For $n \geq 1$, we have

$$(xD_G)^n = \sum_{\pi \in S_n} x^{n-\mathrm{exc}(\pi)} y^{\mathrm{exc}(\pi)} p^{\mathrm{cdes}(\pi)} D_G^{\mathrm{cyc}(\pi)}. \tag{6.8}$$

If we use a variable q to replace the operator D_G in the above sum, we get multivariate polynomials called $(\mathrm{cdes},\mathrm{cyc})(p,q)$-Eulerian polynomials. Ma-Mansour-Yeh-Yeh [58] also showed, by means of grammars, that the Eulerian polynomials, the second-order Eulerian polynomials, the type B Eulerian polynomials and the up-down run polynomials can all be represented by normally ordered expansions.

7 The Ramanujan Polynomials

As remarked by Berndt [4, 5], no combinatorial perspective seems to be alluded to in the original definition of Ramanujan polynomials. As time passes, we have come to see that they legitimately belong to the subject of enumerative combinatorics.

The Ramanujan polynomials arise in three seemingly unrelated contexts. First, Ramanujan [4, 65] defined the polynomials $\psi_k(r,x)$ which we call the Ramanujan polynomials, via the following relation,

$$\sum_{k=0}^{\infty} \frac{(x+k)^{r+k}\, e^{-u(x+k)}\, u^k}{k!} = \sum_{k=1}^{r+1} \frac{\psi_k(r,x)}{(1-u)^{r+k}}. \tag{7.1}$$

The second definition of the Ramanujan polynomials has a different story. As a refinement of Cayley's formula for labeled trees, Shor [73] came up with an insertion algorithm for labeled rooted trees based on the notion of improper edges. Let T be a rooted tree on $[n]$, and (i,j) be an edge of T with j being a child of i. If i is smaller than any descendant of j (including j), then (i,j) is called a proper edge; otherwise (i,j) is called an improper edge. The algorithm leads to a recurrence relation for the number of rooted trees on $[n]$ with k improper edges. Shor then introduced the polynomials $Q_{n,k}(x)$, where x is treated as a positive integer.

Zeng [83] gave an interpretation of $Q_{n,k}(x)$ by treating x as an indeterminate and observed a connection between Shor's polynomial and the Ramanujan polynomials $\psi_k(r,x)$, namely

$$Q_{n,k}(x) = \psi_{k+1}(n-1, x+n). \tag{7.2}$$

Let $\mathscr{T}_{n+1,k}$ be the set of rooted trees on $[n+1]$ with root 1 and with k improper edges. Here is the interpretation of Zeng:

$$Q_{n,k}(x) = \sum_{T \in \mathscr{T}_{n+1,k}} x^{\deg_T(1)-1}, \tag{7.3}$$

where $\deg_T(i)$ denotes the outdegree of the vertex i in T.

The third story about the Ramanujan is also surprising. Around the same time of Shor's algorithm, Dumont-Ramamonjisoa [32] looked at a functional equation of Ramanujan:

$$x = ye^{-y} + \frac{a-1}{a}(e^{-y}-1), \tag{7.4}$$

where y is a series in x without a constant term and a is a parameter. To solve this equation in combinatorial terms, Dumont-Ramamonjisoa independently developed the insertion algorithm as provided by Shor. Moreover, as noted by Chen-Fu-Wang [19], the functional equation satisfied by the generating function of the Ramanujan polynomials turns out to be the functional equation (7.4) also due to Ramanujan. This leaves one to wonder whether Ramanujan realized that the polynomials $\psi_k(r,x)$ are somehow related to the functional equation (7.4).

In order to derive combinatorial expansions of y and e^y, Dumont-Ramamonjisoa started from the functional equation (7.4) and derived the grammar

$$A \to A^3S,\ S \to AS^2. \tag{7.5}$$

Then they devised an insertion algorithm for rooted trees and showed that the recurrence relation for the enumeration of rooted tree with respect to the number of improper edges is consistent with the recurrence relation governed by the grammar, and so the functional equation was solved combinatorially.

There are two versions of the enumeration of rooted trees with respect to the number of improper edges. Let $\mathscr{R}_{n,k}$ denote the set of rooted trees on $[n]$ with k improper edges, and let $R(n,k)$ denote the number of rooted trees in $\mathscr{R}_{n,k}$. On the other hand, let $T(n,k)$ denote the number of trees in $\mathscr{T}_{n+1,k}$, that is, the number of rooted trees on $[n+1]$ with root 1 and with k improper edges. The numbers $R(n,k)$ and $T(n,k)$ are listed as sequences A054589 and A217922 in OEIS [62].

Dumont-Ramamonjisoa found the following solution to the functional equation (7.4), along with the expansion of e^y,

$$y = \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} R(n,k)\, a^{n+k}\right)\frac{x^n}{n!},$$

$$e^y = 1 + \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} T(n,k)\, a^{n+k}\right)\frac{x^n}{n!}.$$

They also gave combinatorial interpretations of some special cases of $Q_{n,k}(x)$, such as $Q_{n,k}(0) = R(n,k)$, $Q_{n,k}(1) = T(n,k)$. In addition, they showed that $Q_{n,k}(-1)$ equals the number of trees on $[n]$ with k improper edges in which vertex 1 is a leaf. It is worth mentioning that Wang-Zhou [79] showed that $Q_{n,k}(-1)$ is related to refined orbifold Euler characteristics of the moduli space of stable curves of genus 0 with n marked points.

Based on the grammar of Dumont-Ramamonjisoa, Chen-Yang [21] gave a grammar for the Ramanujan polynomials $Q_{n,k}(x)$. By utilizing an extra label, Chen-Fu-Wang [19] presented a grammar for $Q_{n,k}(x)$ called a literal grammar, in the sense that the grammar is a direct translation of the recursive construction. The idea of a literal grammar not only provides a clue to build a grammar, it may also be informative for specializations. Moreover, a literal grammar might be more convenient to perform the grammatical calculus.

The insertion algorithm of Shor and Dumont-Ramamonjisoa leads to the following literal grammar:

$$a \to axv,\ x \to xvz,\ z \to vz^2,\ v \to uv^2z,\ u \to u^2vz. \tag{7.6}$$

Assume that $n \geq 1$ and T is a tree on $[n-1]$ with root 1. Consider the insertion of n into T to generate a rooted tree T' on $[n]$ with root 1. Now, the root 1 is labeled by a, the children of the root are labeled by x, the rest of the vertices are labeled by z. All edges are labeled by v, and an improper edge is endowed with an extra label u. The reason to label the children of the root is for the sake of recording the outdegree of the root, as required for the computation of $Q_{n,k}(x)$.

Depending on the outdegree of n in T', we encounter three cases, namely, the outdegree is 0, or 1, or greater than 1. The three cases can be identified with three labels z, v and u. Accordingly, the insertions in the three cases are called the leaf-insertion, the v-insertion and the u-insertion.

1. The leaf-insertion. First, if n is inserted as child of the root, we get the rule

   $$a \to axv.$$

   Second, if n is inserted as a child of a vertex labeled by x, we get the rule

   $$x \to xvz.$$

   Third, if n is inserted as a child of a vertex i labeled by z, that is, i is neither the root nor a child of the root, we get the rule

   $$z \to zvz,$$

   as illustrated in Figure 2.

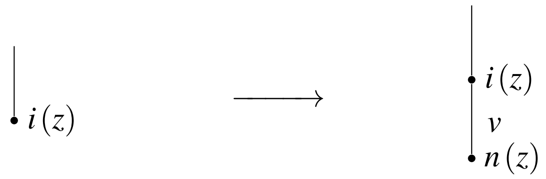

Figure 2: A leaf-insertion.

2. A v-insertion is illustrated in Figure 3. We get the rule

$$v \to v(uvz).$$

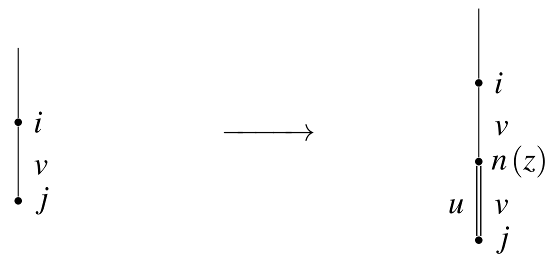

Figure 3: A v-insertion.

3. A u-insertion is associated with an improper edge with a label u. In this case, a new improper edge is created, see Figure 4. We get the rule

$$u \to u(uvz).$$

where we use $\beta(j)$ to denote the minimum vertex among all the descendants of j, and the vertices $j_1, j_2, \ldots, j_l$ are arranged subject to the condition $\beta(j_1) < \beta(j_2) < \cdots < \beta(j_l)$. Note that $\{j_{d+1}, \ldots, j_l\}$ is allowed to be empty.

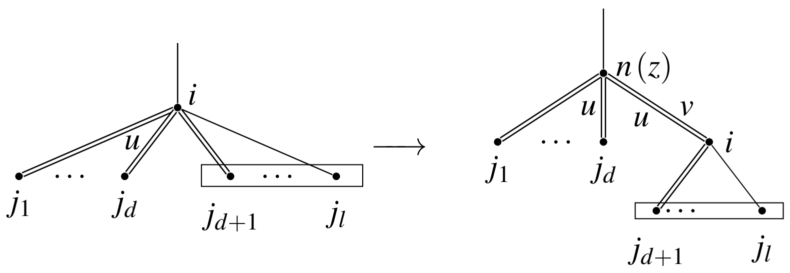

Figure 4: A u-insertion.

Let D be the formal derivative of the grammar (7.6). We have the following grammatical representation of $Q_{n,k}(x)$

$$D^n(a) = axv^n z^{n-1} \sum_{k=0}^{n-1} Q_{n,k}(xz^{-1})\, u^k. \tag{7.7}$$

The grammatical calculus can be employed to derive the functional equation for the generating function of the Ramanujan polynomials. Moreover, it can be used to justify a grammatical identity bearing a combinatorial interpretation.

Let

$$R_n(u) = \sum_{k=0}^{n-1} R(n,k)\, u^k.$$

The first few values of $R_n(u)$ are given below:

$$\begin{aligned}
R_1(u) &= 1, \\
R_2(u) &= 1+u, \\
R_3(u) &= 2+4u+3u^2, \\
R_4(u) &= 6+18u+25u^2+15u^3, \\
R_5(u) &= 24+96u+190u^2+210u^3+105u^4.
\end{aligned}$$

In the study of the monotone properties of polynomials, we often use q as the variable. Let $\{f_n(q)\}_{n\geq 0}$ be a sequence of polynomials in q. By writing $f(q) \geq_q g(q)$ we mean $f(q)-g(q)$ is a polynomial in q with nonnegative coefficients. If

$$f_{m+1}(q)\, f_{m-1}(q) \geq_q f_m(q)^2,$$

for all $m \geq 1$, then we say that $\{f_n(q)\}_{n\geq 0}$ is q-log-convex. If

$$f_{m-1}(q)\, f_{n+1}(q) \geq_q f_m(q)\, f_n(q),$$

for all $n \geq m \geq 1$, then we say that $\{f_n(q)\}_{n\geq 0}$ is strongly q-log-convex.

Chen-Wang-Yang [22] showed that $\{R_{n+1}(q)\}_{n\geq 0}$ is strongly q-log-convex. Sokal [74] obtained a more general result, namely, every minor of the Hankel matrix $(R_{i+j+1}(q))_{i,j\geq 0}$ is a polynomial with nonnegative coefficients.

8 The Insertion Algorithm for Plane Trees

We observe that the insertion algorithm of Shor and Dumont-Ramamonjisoa applies to labeled plane trees without any additional effort. The key point is to treat every edge as an improper edge. In other words, we may ignore the notion of improper edges while recursively constructing plane trees by insertions.

Let $n \geq 1$, and let $\mathscr{P}_n$ denote the set of labeled plane trees on $[n+1]$. It is clear that $|\mathscr{P}_n| = (n+1)!C_n$, where C_n is the n-th Catalan number, that is, the number of unlabeled plane trees with n edges, see sequence A000108 of OEIS [62]. Let $\mathscr{O}_n$ be the set of plane trees on $\{0,1,2,\ldots,n\}$ with root 0. It is clear that $|\mathscr{O}_n| = n!C_n$.

The ballot numbers $T(n,k)$ can be viewed as a refinement of the Catalan numbers, see sequence A009766 in OEIS [62]. For $n \geq 0$ and $0 \leq k \leq n$, the ballot numbers $T(n,k)$ are defined by

$$T(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{k},$$

which satisfy the recurrence relation

$$(n+2)\,T(n+1,k) = (n-k+2)\,T(n,k) + (2n+2k)\,T(n,k-1). \tag{8.1}$$

The number $T(n,k)$ can be interpreted as the number of unlabeled plane trees with $n+1$ edges such that the outdegree of the root equals $n-k+1$. Making use of the recurrence relation (8.1), Ma [55] found a grammar to generate $T(n,k)$.

Theorem 8.1 (Ma [55])**.** *Let D be the formal derivative of the grammar*

$$G = \{a \to a^2b^2,\ b \to b^3c^2,\ c \to b^2c^3\}. \tag{8.2}$$

For $n \geq 0$, we have

$$D^n(a^2b^2) = (n+1)!a^2b^{2n+2}\sum_{k=0}^{n} T(n,k)a^{n-k}c^{2k}. \tag{8.3}$$

The Narayana numbers can also be generated by a grammar. For $n \geq 1, 1 \leq k \leq n$, the Narayana numbers, see sequence A001263 in OEIS [62], are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

They are also a refinement of the Catalan numbers and satisfy the following recurrence relation

$$(n+2)N(n+1,k) = (n+2k)N(n,k) + (3n+4-2k)N(n,k-1). \tag{8.4}$$

It is well known that $N(n,k)$ equals the number of unlabeled rooted plane trees with $n+1$ vertices and k leaves. Employing the recurrence relation (8.4), Ma-Ma-Yeh [56] found a grammar for the Narayana polynomial $N_n(x)$ and showed that they are γ-positive, where

$$N_n(x) = \sum_{k=1}^{n} N(n,k)x^k.$$

Theorem 8.2 (Ma-Ma-Yeh [56])**.** *Let D be the formal derivative of the grammar*

$$G = \{u \to u^2v^3,\ v \to u^3v^2\}. \tag{8.5}$$

For $n \geq 1$, we have

$$D^n(u^2) = (n+1)! \sum_{k=1}^{n} N(n,k)\, u^{3n-2k+2}\, v^{n+2k}. \tag{8.6}$$

Moreover, $N_n(x)$ is γ-positive for all n.

As noted by Ma-Ma-Yeh [56], the γ-positivity of $N_n(x)$ can be deduced from the change of variables: $x = uv, y = u^2+v^2$ and $z = u^2$. Ma raised the question of finding a grammatical labeling to justify the grammar (8.5) for Theorem 8.2, see [52]. Lin-Liu-Wang-Zang [52] and Yang-Zhang [82] independently obtain grammatical labelings, thereby answering Ma's question. The grammar given by Yang-Zhang is as follows,

$$G = \{t \to t^2(x+y),\ x \to 2txy,\ y \to 2txy\}. \tag{8.7}$$

Let D be the formal derivative of the grammar (8.7). For $n \geq 1$, we have

$$D^n(y) = (n+1)!t^n \sum_{k=1}^{n} N(n,k)x^k y^{n-k+1}. \tag{8.8}$$

Set $t = uv$, $x = u^2$ and $y = v^2$, the grammar (8.5) is transformed into (8.7). Setting $y = x$, the grammar (8.7) reduces to the grammar for the Catalan numbers

$$G = \{t \to 2t^2x,\ x \to 2tx^2\}. \tag{8.9}$$

For $n \geq 0$, we have

$$D^n(x) = (n+1)!\,t^n x^{n+1} C_n. \tag{8.10}$$

Deutsch [28] obtained a formula for Dyck paths, which can be viewed as a unification of the formulas for the Narayana numbers and the ballot numbers. We call the involved numbers the Deutsch numbers, denoted $D_r(n,k)$. Combinatorially, $D_r(n,k)$ is the number of unlabeled plane trees with $n+1$ vertices with root outdegree r and k leaves.

Theorem 8.3 (Deutsch [28]). *For $n \geq 1$ and $1 \leq r,k \leq n$,*

$$D_r(n,k) = \frac{r}{k}\binom{n-1}{k-1}\binom{n-1-r}{k-r}. \tag{8.11}$$

Next we proceed to adapt the insertion algorithm of Shor and Dumont-Ramamonjisoa to derive a grammar for the Deutsch numbers, which specializes to grammars for the Catalan numbers and the Narayana numbers. For a plane tree in $\mathcal{O}_n$, define a labeling scheme as follows.

- A label r signifies a position where one can insert a new vertex as a child of the root.
- A label s signifies a position where one can insert a new vertex as a child of a vertex that is neither the root nor a leaf.
- A label t signifies the position where one can insert a new vertex as a child of a leaf.
- Assign a label v to every edge.
- In addition, for any nonroot internal vertex i, label the edge from i to its last child by w. Label the edge from i to its other children by u.

Figure 5 gives an illustration. The weight of a tree in $\mathcal{O}_n$ is defined to be the product of the labels. Let $G_n(r,s,t;u,v,w)$ denote the total weight of the trees in $\mathcal{O}_n$.

To read off the insertion algorithm in accordance with the above labeling scheme, it is straightforward to compose a grammar, with special attention only to the insertion associated with an edge labeled by w, which we call a w-insertion, as shown in Figure 6. Notice that a new leaf is created by a w-insertion. So we get the rule

$$w \to stuvw.$$

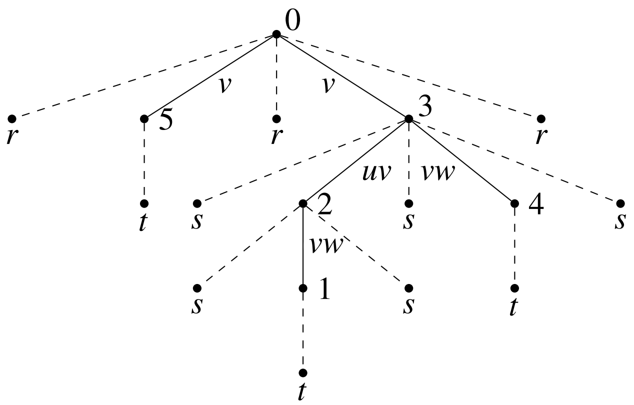

Figure 5: The labeling of a plane tree in $\mathcal{O}_5$

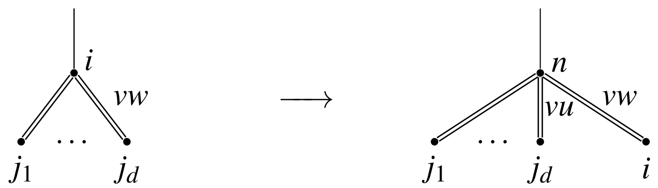

Figure 6: A w-insertion.

Theorem 8.4. *Let D be the formal derivative of the grammar*

$$G = \{r \to r^2tv,\ s \to s^2tuv,\ t \to s^2tvw,\ u \to s^2uvw,\ v \to s^2v^2w,\ w \to stuvw\}. \tag{8.12}$$

For $n \geq 0$, we have

$$D^n(r) = G_n(r,s,t;u,v,w). \tag{8.13}$$

One may compute the generating function of $D^n(r)$ by the grammatical calculus. Examples of using the grammatical calculus to compute generating functions can be found in [14, 16, 17, 19, 30].

Setting $r = a$, $s = u = c$ and $t = v = w = b$, the grammar (8.12) reduces to the grammar (8.2). If we replace u,t,v,w by s in (8.12), then we get the grammar

$$G = \{r \to r^2s^2,\ s \to s^5\}, \tag{8.14}$$

which is a grammar for the ballot numbers. Let D be the formal derivative of the above grammar. Indeed, for $n \geq 0$, we have

$$D^{n+1}(r)\big|_{s=1} = (n+1)!\sum_{k=0}^{n} T(n,k)r^{n-k+2}.$$

9 Grammars and Stable Polynomials

In this section, we demonstrate how grammars along with labeling schemes can play a role in constructing stable multivariate polynomials associated with certain combinatorial structures.

A multivariate polynomial $f(z_1,z_2,\ldots,z_n) \in \mathbb{C}[z_1,z_2,\ldots,z_n]$ is called stable if

$$f(z_1,z_2,\ldots,z_n) \neq 0$$

as long as $\mathrm{Im}(z_i) > 0$ for $(1 \le i \le n)$. A stable multivariate polynomial with real coefficients is said to be real stable. A univariate polynomial $f(z) \in \mathbb{R}[z]$ with real coefficients is stable if and only if it is real-rooted.

Many univariate combinatorial polynomials are real-rooted, while their multivariate refinements are stable. We now consider a family of such multivariate combinatorial polynomials. In answer to the two questions of Haglund-Visontai [41] on stable multivariate refinements of the second-order Eulerian polynomials, Chen-Hao-Yang [20] obtained grammars to generate the Legendre-Stirling permutations and marked Stirling permutations.

For $n \ge 1$, let $[n]_2$ denote the multiset $\{1,1,2,2,\ldots,n,n\}$. Let $\pi = \pi_1\pi_2\cdots\pi_{2n-1}\pi_{2n}$ be a permutation of $[n]_2$. We say that π is a Stirling permutation if for any $1 \le i \le n$, the elements between the two occurrences of i, if any, are greater than i. Moreover, we assume that $\pi_0 = \pi_{2n+1} = 0$. For $n \ge 1$, let Q_n denote the set of Stirling permutations of $[n]_2$.

For the purpose of producing stable multivariate polynomials related to Stirling permutations, we introduce the notion of marked Stirling permutations, see Chen-Hao-Yang [20]. Let $\pi = \pi_1\pi_2\cdots\pi_{2n}$ be a Stirling permutation of $[n]_2$. If π_i is the second occurrence of an element k and $\pi_i < \pi_{i+1}$, then we may mark the element π_i with a bar. Note that it is also eligible not to mark this element. A marked Stirling permutation is derived from a Stirling permutation by adding some bars. Let $\bar{Q}_n$ denote the set of marked Stirling permutations of $[n]_2$. For example, for $n = 2$, there are four marked Stirling permutations:

$$2211, \quad 1221, \quad 1122, \quad 1\bar{1}22.$$

Egge [33] introduced the notion of Legendre-Stirling permutations. For $n \ge 1$, let M_n denote the multiset $\{1,1,\bar{1},2,2,\bar{2},\ldots,n,n,\bar{n}\}$. Let $\pi = \pi_1\pi_2\ldots\pi_{3n}$ be a permutation of M_n.

We say that π is a Legendre-Stirling permutation if for any $1 \leq i \leq n$, the elements between the two occurrences of the unbarred i, if any, are greater than i. Note that the bars are ignored as far as the comparison is concerned. We also assume that $\pi_0 = \pi_{3n+1} = 0$.

We first define the multivariate polynomial associated with Legendre-Stirling permutations. Let $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, $Z = (z_1, z_2, \ldots, z_n)$, $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$. Define

$$B_n(X,Y,Z,U,V) = \sum_{\pi} \prod_{i \in X(\pi)} x_{\pi_i} \prod_{i \in Y(\pi)} y_{\pi_i} \prod_{i \in Z(\pi)} z_{\pi_i} \prod_{i \in U(\pi)} u_{\pi_i} \prod_{i \in V(\pi)} v_{\pi_i},$$

where π ranges over Legendre-Stirling permutations of M_n, $X(\pi), Y(\pi), Z(\pi), U(\pi)$ and $V(\pi)$ are defined by

$$\begin{aligned}
X(\pi) &= \{i \mid \pi_{i-1} \leq \pi_i,\ \pi_i \text{ is unbarred and is the first occurrence}\},\\
Y(\pi) &= \{i \mid \pi_i > \pi_{i+1},\ \pi_i \text{ is unbarred}\},\\
Z(\pi) &= \{i \mid \pi_{i-1} \leq \pi_i,\ \pi_i \text{ is unbarred and is the second occurrence}\},\\
U(\pi) &= \{i \mid \pi_{i-1} \leq \pi_i,\ \pi_i \text{ is barred}\},\\
V(\pi) &= \{i \mid \pi_i > \pi_{i+1},\ \pi_i \text{ is barred}\}.
\end{aligned}$$

The polynomials $B_n(X,Y,Z,U,V)$ can be generated by a grammar along with a labeling scheme. Using the grammar representation, the stability of the multivariate polynomials follows from the differential operator with respect to the grammar preserves the stability. Here we need a theorem of Borcea-Brändén [7] on linear operators that preserve the stability of multiaffine polynomials.

A polynomial $f(z_1, z_2, \ldots, z_n)$ is called multiaffine if the degree of any variable z_i is at most one. Let $\mathbb{C}^{\mathrm{ma}}[z_1, z_2, \ldots, z_n]$ denote the vector space of multiaffine polynomials in $z_1, z_2, \ldots, z_n$ over complex numbers. A linear operator T is called stability preserving if for any stable polynomial $f \in \mathbb{C}^{\mathrm{ma}}[z_1, z_2, \ldots, z_n]$, $T(f)$ is either stable or identically zero.

Theorem 9.1 (Borcea-Brändén [7]). *Let T be a linear operator from $\mathbb{C}^{\mathrm{ma}}[z_1, z_2, \ldots, z_n]$ to $\mathbb{C}[z_1, z_2, \ldots, z_n]$. If the polynomial*

$$T\left(\prod_{i=1}^{n}(z_i + w_i)\right) \in \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n]$$

is stable, then T is stability preserving on multiaffine polynomials.

Using the above criterion and the grammatical representation, the stability of $B_n(X,Y,Z,U,V)$ can be deduced.

Theorem 9.2 (Chen-Hao-Yang [20]). *For $n \geq 1$, the polynomial $B_n(X,Y,Z,U,V)$ is stable.*

Let $\pi = \pi_1\pi_2\cdots\pi_{2n}$ be a Stirling permutation of $[n]_2$. Assume that $\pi_0 = \pi_{2n+1} = 0$. Define

$$A(\pi) = \{i \mid \pi_{i-1} < \pi_i,\ 1 \leq i \leq 2n\},$$

$$D(\pi) = \{i \mid \pi_i > \pi_{i+1},\ 1 \leq i \leq 2n\},$$

$$P(\pi) = \{i \mid \pi_{i-1} = \pi_i,\ 1 \leq i \leq 2n\}.$$

We now come to the multivariate polynomials in connection with marked Stirling permutations. For a marked Stirling permutation $\bar{\pi}$, we define the sets $D(\bar{\pi})$, $A(\bar{\pi})$ and $P(\bar{\pi})$ as if the bars are invisible, and we still assume that $\pi_0 = \pi_{2n+1} = 0$. For example, for $\bar{\pi} = 1\,\bar{1}\,2\,2$, we have

$$A(\bar{\pi}) = \{1,3\}, \quad D(\bar{\pi}) = \{4\}, \quad P(\bar{\pi}) = \{2,4\}.$$

We next define the multivariate polynomial associated with marked Stirling permutations. Define the multivariate polynomials

$$T_n(X,Y,Z) = \sum_{\bar{\pi}} \prod_{i \in A(\bar{\pi})} x_{\pi_i} \prod_{i \in D(\bar{\pi})} y_{\pi_i} \prod_{i \in P(\bar{\pi})} z_{\pi_i},$$

where $\bar{\pi}$ ranges over marked Stirling permutations of $[n]_2$.

Theorem 9.3 (Chen-Hao-Yang [20]). *For $n \geq 1$, $T_n(X,Y,Z)$ is stable.*

Recently, using the grammatical approach Yang-Zhang [82] obtained a multivariate version of the Narayana polynomials and proved their stability.

10 Noncommutative Grammars and q-Grammars

In this final section, we discuss two recent developments.

First, the Leibniz formula holds for commutative variables. But the context-free property (1.1) is concerned with the characterization of a formal language in terms of production rules regardless of their contexts. In theory, we do not have to require that the variables be commutative. So the problem is how to employ appropriate operators in a noncommutative setting to carry out an effective grammatical calculus.

Ehrenborg [34] took the grammar of Dumont for the Eulerian polynomials to the noncommutative platform. Assume that a and b are noncommutative variables. Consider the substitution rules

$$a \to ba,\ b \to ab.$$

This noncommutative grammar is closely related to the ab-index and the cd-index of Eulerian posets. By studying the behavior of powers of the pyramid and prism operators on a product, Ehrenborg derived exponential generating functions for the cd-indices of the n-dimensional simplices and the n-dimensional cubes, thereby providing a new perspective on the theory of cd-indices in polyhedral combinatorics.

The second development is the q-grammatical calculus. Enumeration problems involving Mahonian statistics often lead to q-binomial coefficients, also called the Gaussian polynomials. A fundamental question is how to build a grammatical calculus with a suitable version of the q-Leibniz formula. Han-Ji-Xiong [42] introduced a q-analogue of the grammatical calculus, called the q-derivative grammar, or q-grammar for short. Specifically, for a special class of q-grammars (termed q-linear grammars), the associated q-derivative satisfies the q-Leibniz formula:

$$D^n(fg) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D^k(f) \uparrow^k \left(D^{n-k}(g)\right),$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$, or simply $\begin{bmatrix} n \\ k \end{bmatrix}$, are the q-binomial coefficients. As shall be seen, the q-grammars are associated with sequences of noncommutative variables $(x_0, x_1, x_2, \ldots)$, $(y_0, y_1, y_2, \ldots)$, ... They defined the shift operator $\uparrow$ on the sequences of noncommutative variables $(x_0, x_1, x_2, \ldots)$, $(y_0, y_1, y_2, \ldots)$, ... by

$$\uparrow(x_i) = x_{i+1},\ \uparrow(y_i) = y_{i+1},\ \ldots$$

for $i \geq 0$, which can be extended linearly and multiplicatively. Assume that for $c \in \mathbb{K}[q]$, $\uparrow(c) = c$. For example,

$$\uparrow\left(y_2^{-1}x_0x_0y_1\right) = y_3^{-1}x_1x_1y_2 \quad \text{and} \quad \uparrow^3\left(y_2^{-1}x_0x_0y_1\right) = y_5^{-1}x_3x_3y_4.$$

Consider the q-Eulerian polynomials $A_n^{\mathrm{inv}}(q;x,y)$ defined by

$$A_n^{\mathrm{inv}}(q;x,y) = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)},$$

where $\mathrm{inv}(\sigma)$ denotes the number of inversions of σ, that is,

$$\mathrm{inv}(\sigma) = |\{(i,j) : 1 \le i < j \le n,\ \sigma_i > \sigma_j\}|.$$

The first few values of $A_n^{\mathrm{inv}}(q;x,y)$ are:

$$\begin{aligned} A_1^{\mathrm{inv}}(q;x,y) &= xy, \\ A_2^{\mathrm{inv}}(q;x,y) &= x^2y + qxy^2, \\ A_3^{\mathrm{inv}}(q;x,y) &= x^3y + (2q+2q^2)x^2y^2 + q^3xy^3. \end{aligned}$$

The grammar for the Eulerian polynomials involves two variables x and y. For the q-grammar, we need two sequences of variables: $(x_0, x_1, x_2, \ldots)$ and $(y_0, y_1, y_2, \ldots)$. We further impose the following order on the variables:

$$x_0 < y_0 < x_1 < y_1 < x_2 < y_2 < \cdots. \tag{10.1}$$

The q-grammar is given by

$$G_{\mathrm{inv}} = \{x_j \to q^j y_j x_{j+1},\ \ y_j \to q^j y_j x_{j+1} \mid j = 0,1,2,\ldots\}. \tag{10.2}$$

The q-derivative operator D associated with G_{inv} is defined by

$$D(w_1 w_2 \cdots w_n) = \sum_{i=1}^{n} \rho\left(w_1 \cdots w_{i-1} R(w_i) \uparrow (w_{i+1} \cdots w_n)\right),$$

where we assume that $w_1 w_2 \cdots w_n$ is written in the normal form, $R(w_i)$ means to apply the substitution rules to w_i, and ρ denotes the normal form with respect to the order (10.1). Set $D^0(f) = f$, and for $k \ge 1$, define

$$D^k(f) = D(D^{k-1}(f)).$$

Define the evaluation map

$$\phi(x_j) = x, \qquad \phi(y_j) = y.$$

Let D denote the q-derivative with respect to the q-grammar (10.2). For $n \geq 1$, we have

$$\phi\left(D^n(x_0)\right) = A_n^{\mathrm{inv}}(q;x,y).$$

For example,

$$\begin{aligned}
D(x_0) &= R(x_0) = y_0x_1, \\
D^2(x_0) &= D(y_0x_1) = R(y_0)x_2 + y_0R(x_1) = y_0x_1x_2 + qy_0y_1x_2, \\
D^3(x_0) &= D(y_0x_1x_2 + qy_0y_1x_2) \\
&= R(y_0)x_2x_3 + y_0R(x_1)x_3 + y_0x_1R(x_2) + qR(y_0)y_2x_3 + qy_0R(y_1)x_3 + qy_0y_1R(x_2) \\
&= y_0x_1x_2x_3 + qy_0y_1x_2x_3 + q^2y_0x_1y_2x_3 + qy_0x_1y_2x_3 + q^2y_0y_1x_2x_3 + q^3y_0y_1y_2x_3.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\phi(D(x_0)) &= xy, \\
\phi(D^2(x_0)) &= x^2y + qxy^2, \\
\phi(D^3(x_0)) &= yx^3 + (2q + 2q^2)y^2x^2 + q^3y^3x.
\end{aligned}$$

Using a grammatical labeling for G_{inv}, Han-Ji-Xiong [42] showed that the q-Eulerian polynomials $A_n^{\mathrm{inv}}(q;x,y)$ can be generated by the above q-grammar. Furthermore, they derived its q-exponential generating function by the q-grammatical calculus. They also found a q-grammar for the q-Roselle polynomials and derived their q-exponential generating function. The q-grammars for two kinds of the q-André polynomials are also given.

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