Rigidity of the Clifford torus as a Lagrangian self-shrinker in $\mathbb{C}^2$

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Abstract. In this paper, we prove several rigidity results of the Clifford torus as a Lagrangian self-shrinker in $\mathbb{C}^2$ under some restrictions of the Kähler angle with respect to the twisted complex structure $\bar{J} = (J, -J)$. Besides, we discuss the rotational symmetry case.

Keywords and Phrases. Lagrangian self-shrinker, moduli space, Kähler angle, Clifford torus, generalized rotational symmetry.

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1 Introduction

Let $\Omega : M^n \to \mathbb{R}^{n+p}$ be a n-dimensional submanifold in the $(n+p)$-dimensional Euclidean space. Its mean curvature flow is the solution of the following system of parabolic equations:

$$\begin{cases} \dfrac{\partial}{\partial t}\Omega(p,t) = \vec{H}(p,t), \quad (p,t) \in M^n \times [0,T) \\ \Omega(\cdot, 0) = \Omega_0, \end{cases}$$

where $\vec{H}$ is the mean curvature vector of M_t at $\Omega(p,t)$ in M^n. An immersed submainfold is called a *self-shrinker* if it satisfies

$$\vec{H} = -\Omega^{\perp}, \tag{1.1}$$

where $\Omega^{\perp}$ denotes the normal projection of the position vector Ω. One of the most important research questions in mean curvature flow is to understand how its singularities generate, and the self-shrinkers play a crucial role in singularity analysis since they describe the various possibilities of singularity blowing up.

Additionally, some scholars have studied the related theories of $\xi-$submanifold [19], which means that there is a parallel normal vector field ξ such that the mean curvature vector field $\vec{H}$ satisfies

$$\vec{H} + \Omega^{\perp} = \xi.$$

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Here, we mainly focus on self-shrinker in the usual sense.

For one dimension, Abresch-Langer [1] completely classified all self-similar curves in $\mathbb{R}^2$ and these curves are called Abresch-Langer curves.

For one codimension, Huisken [11, 12] showed that smooth self-shrinking hypersurfaces in $\mathbb{R}^{n+1}$ with nonnegative mean curvature, bounded $|A|$ and polynomial volume growth are isometric to $\Gamma \times \mathbb{R}^{n-1}$ or $\mathbb{S}^{n-m}(\sqrt{m}) \times \mathbb{R}^m (1 \leq m \leq n-1)$, where $|A|$ denotes the norm of the second fundamental form, Γ is a Abresch-Langer curve, and $\mathbb{S}^n(\sqrt{n})$ is the sphere of radius $\sqrt{n}$. Colding-Minicozzi [9] further proved that Huisken's classification holds without the assumption that $|A|$ is bounded.

The complete classifications get more complicated in higher codimensions. For example, Smoczyk [20] showed that let M^n be a compact self-shrinker in $\mathbb{R}^{n+p}$, then M^n is spherical if and only if $\vec{H} \neq 0$ and $\nabla^\perp v = 0$, where $v = \vec{H}/|\vec{H}|$ is principal normal vector.

It has been an important subject in differential geometry to apply Atiyah-Singer index theory to study the geometry and topology of the moduli space of immersed surfaces in a real four-dimensional Riemannian manifold (see [3]). In this paper, we consider the Langrangian self-shrinker solutions in $\mathbb{C}^2$. We know that the Clifford torus is very important in such solutions space. Firstly we recall some rigidity results about the Clifford torus. Castro-Lerma [6] provided the following result.

Theorem 1.1. *(see [6, Theorem 1.2]) Let $M^2 \to \mathbb{C}^2$ be a compact orientable Lagrangian self-shrinker. If $|A|^2 \leq 2$, then $|A|^2 = 2$ and M^2 is a topological torus. If, in addition, the Gauss curvature K of M^2 is nonnegative or nonpositive, then M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

They conjectured that the condition "the Gauss curvature K of M^2 is nonnegative or nonpositive" is unnecessary. Li-Wang [17] verified their conjecture and showed that the Clifford torus is the unique compact orientable Lagrangian self-shrinker in $\mathbb{C}^2$ when $|A|^2$ is constant. Recently, Cheng-Hori-Wei [7] proved that if $\Omega : M^2 \to \mathbb{C}^2$ is a complete Lagrangian self-shrinker with $|A|^2$ being constant, then $\Omega : M^2 \to \mathbb{C}^2$ is isometric to one of $\mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}^1$ or $\mathbb{S}^1 \times \mathbb{S}^1$.

From the viewpoint of minimal surface, Lawson conjectured that the Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$ and Brendle [4] gave an affirmative answer to this conjecture. Lately, analogous to Lawson's conjecture, Lee [15] asked the following question.

Question. *Is the Clifford torus unique as an embedded closed Lagrangian self-shrinker in $\mathbb{C}^2$?*

Lee in [15] obtained a new geometric characterization of the Clifford torus as the unique embedded closed Lagrangian self-shrinker symmetric with respect to a hyperplane in $\mathbb{C}^2$. Recently, Evans-Lotay-Schulze [10] showed that the Clifford torus is locally unique as a self-shrinker in $\mathbb{C}^2$, despite having infinitesimal deformations which do not arise from rigid motions.

Given an immersed surface in a Kähler manifold, Chern-Wolfson [8] defined Kähler angle, which is a very important geometric quantity. Li-Li [18] discussed the Kähler angle of immersed surfaces in $\mathbb{C}^2$ and obtained two pinching results for the Kähler angle which implies rigidity theorems of the self-shrinker with Kähler angle under the condition that $\int_{M^2}|A|^2 e^{-\frac{x^2}{2}}dV_{M^2}<\infty$, where x denotes the position vector of the surface. Taking inspiration from the two complex structures defined on product of the spheres in [14], we also consider the canonical complex structure $\tilde{J}=(J,J)$ and the twisted one $\bar{J}=(J,-J)$ on $\mathbb{C}^2\cong\mathbb{R}^2\times\mathbb{R}^2$, where J is the standard complex structure defined on $\mathbb{R}^2$. For a Lagrangian self-shrinker M^2 in $\mathbb{C}^2$ with respect to $\tilde{J}$, we know that the Kähler angle with respect to $\tilde{J}$ is $\frac{\pi}{2}$, while the Kähler angle α with respect to $\bar{J}$ is not trivial. Here we continue to develop Chern-Wolfson's theory on the moduli space of immersed surfaces in $\mathbb{C}^2$ and study the geometry and topology of its compact Lagrangian self-shrinker solutions. We prove that the space of such solutions satisfying three conditions that $|\nabla\alpha|^2$ is a constant, $\Delta\alpha$ (or $\mathcal{L}\alpha$) vanishes at the point that $|\Omega|^2$ attains minimum, and curvature pinching $4|A|^2-|\nabla\alpha|^2\geq 8$ holds, is composed of the products of two closed Abresch-Langer curves in $\mathbb{R}^2$ (see Sec.3.2). Since the circle is the only embedded one of them (see the note after [1, Theorem A]), then we obtain the rigidity results of Clifford torus as a compact embedded Lagrangian self-shrinker in $\mathbb{C}^2$ under these conditions. Notice that $|\nabla\alpha|^2$ is a part of $|A|^2$. It means that the condition of $|\nabla\alpha|^2$ being constant is weaker than the one that $|A|^2$ is constant.

Theorem 1.2. *Let $\Omega: M^2\to\mathbb{C}^2$ be a compact orientable embedded Lagrangian self-shrinker. If the Kähler angle α with respect to the twisted complex structure $\bar{J}=(J,-J)$ is constant, then M^2 is the Clifford torus $\mathbb{S}^1\times\mathbb{S}^1$.*

Theorem 1.3. *Let $\Omega: M^2\to\mathbb{C}^2$ be a compact orientable embedded Lagrangian self-shrinker and α be the Kähler angle with respect to the twisted complex structure $\bar{J}=(J,-J)$. If $|\nabla\alpha|^2$ is constant, $4|A|^2-|\nabla\alpha|^2\geq 8$ and $\Delta\alpha$ vanishes at the point that $|\Omega|^2$ attains minimum, then $|\nabla\alpha|^2=0$ and M^2 is the Clifford torus $\mathbb{S}^1\times\mathbb{S}^1$.*

Theorem 1.4. *Let $\Omega: M^2\to\mathbb{C}^2$ be a compact orientable embedded Lagrangian self-shrinker and α be the Kähler angle with respect to the twisted complex structure $\bar{J}=(J,-J)$. If $|\nabla\alpha|^2$ is constant, $4|A|^2-|\nabla\alpha|^2\geq 8$ and $\mathcal{L}\alpha$ vanishes at the point that $|\Omega|^2$ attains minimum, then $|\nabla\alpha|^2=0$ and M^2 is the Clifford torus $\mathbb{S}^1\times\mathbb{S}^1$.*

At last, we focus on examples of Lagrangian self-shrinkers in $\mathbb{C}^2\cong\mathbb{R}^4$. Most of the known examples [2][16](see also [7]) are rotationally symmetric surfaces (see [13])

$$M:\Omega(x,y)=(f(x)\cos(cy),f(x)\sin(cy),g(x)\cos(dy),g(x)\sin(dy)),\tag{1.2}$$

where $c>0,d>0,x\in I_1,y\in I_2$ and $f(x),g(x)$ are arbitrary smooth functions. We introduce different complex structures to rotationally symmetric surfaces and investigate when the compact Lagrangian self-shrinker with rotational symmetry is the Clifford torus. Using three different corresponding methods of $\mathbb{R}^4\cong\mathbb{R}^2\times\mathbb{R}^2$ given by $(x_1,x_2,x_3,x_4)\leftrightarrow((x_1,x_2),(x_3,x_4))$ or $((x_1,x_3),(x_2,x_4))$ or $((x_1,x_4),(x_2,x_3))$, we define three different complex structures of type $\tilde{J}_i=(J,J)$ $(i=1,2,3)$. The first complex structure is as follows:

$$\Omega_x=(f'e^{icy},g'e^{idy}),\tag{1.3a}$$

$$\tilde{J}_1\Omega_x=(if'e^{icy},idg'e^{idy}),\tag{1.3b}$$

the second complex structure is

$$\varOmega_x = (f'\cos(cy) + ig'\cos(dy), f'\sin(cy) + ig'\sin(dy)), \tag{1.4a}$$

$$\tilde{J}_2\varOmega_x = (-g'\cos(dy) + if'\cos(cy), -g'\sin(dy) + if'\sin(cy)), \tag{1.4b}$$

and the third complex structure is

$$\varOmega_x = (f'\cos(cy) + ig'\sin(dy), f'\sin(cy) + ig'\cos(dy)), \tag{1.5a}$$

$$\tilde{J}_3\varOmega_x = (-g'\sin(dy) + if'\cos(cy), -g'\cos(dy) + if'\sin(cy)). \tag{1.5b}$$

This leads us to the following proposition.

Proposition 1.5. *(a)Let $\varOmega : M^2 \to \mathbb{C}^2$ be a compact Lagrangian self-shinker with rotational symmetry with respect to the first complex structure and α be the Kähler angle with respect to the complex structure $\bar{J} = (J, -J)$. If $|\nabla\alpha|^2$ is constant, then M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

(b)Let $\varOmega : M^2 \to \mathbb{C}^2$ be a compact Lagrangian self-shinker with rotational symmetry with respect to the second complex structure and α be the Kähler angle with respect to the complex structure $\bar{J} = (J, -J)$. If $|\nabla\alpha|^2$ is constant, then M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.

(c)There does not exist compact Lagrangian self-shinker with rotational symmetry with respect to the third complex structure in $\mathbb{C}^2$ satisfying that $|\nabla\alpha|^2$ is constant, where α is the Kähler angle with respect to the complex structure $\bar{J} = (J, -J)$.

Thus we arrive at the following theorem:

Theorem 1.6. *Let $\varOmega : M^2 \to \mathbb{C}^2$ be a compact Lagrangian self-shinker with rotational symmetry and α be the Kähler angle with respect to the twisted complex structure $\bar{J} = (J, -J)$. If $|\nabla\alpha|^2$ is a constant, then M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

Remark 1.7. *We believe that it is worthwhile to further consider whether the Clifford torus is the unique compact orientable embedded Lagrangian self-shrinker in $\mathbb{C}^2$ when $|\nabla\alpha|^2$ is constant.*

This paper is organized as follows. In section 2, we provide some basic definitions and compute the Laplace equation of the Kähler angle α for the Lagrangian surface in $\mathbb{C}^2$. In section 3, we derive some fundamental equations about the Lagrangian self-shrinkers in $\mathbb{C}^2$, discuss this solutions space and give the proofs of the Theorems 1.2, 1.3 and 1.4. In section 4, we discuss the situation of the rotationally symmetric surface and give the proof of the Theorem 1.6.

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2 Preliminaries

Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ via $(x_1, x_2, x_3, x_4) \leftrightarrow (x_1 + ix_2, x_3 + ix_4)$, where $i = \sqrt{-1}$. Let $\langle , \rangle_{\mathbb{C}}$ be the standard Hermitian product in $\mathbb{C}^2$. Then the Euclidean inner product in $\mathbb{R}^4$ is given by the real part of $\langle , \rangle_{\mathbb{C}}$ under the identification.

Let $\mathbb{R}^2$ be a 1-dimensional Kähler manifold and its standard complex structure J is given as follows. For every $(u,v)\in T\mathbb{R}^2$

$$J(u,v)=(-v,u).$$

Analogous to Kimura-Suizu's [14] study on Lagrangian surfaces in $\mathbb{S}^2\times\mathbb{S}^2$, we investigate Lagrangian surfaces in $\mathbb{R}^2\times\mathbb{R}^2$. We define a complex structure $\tilde{J}$ and product structure $\tilde{P}$ on $\mathbb{R}^2\times\mathbb{R}^2$

$$\tilde{J}(X_1,X_2)=(JX_1,JX_2), \tag{2.1}$$

$$\tilde{P}(X_1,X_2)=(X_1,-X_2) \tag{2.2}$$

for any tangent vectors (X_1,X_2) on $\mathbb{R}^2\times\mathbb{R}^2$. The product metric $\langle,\rangle$ on $\mathbb{R}^2\times\mathbb{R}^2$ is defined by

$$\langle (X_1,X_2),(Y_1,Y_2)\rangle = X_1\cdot Y_1+X_2\cdot Y_2,$$

where $\cdot$ denotes the Euclidean inner product in $\mathbb{R}^2$, then $\mathbb{R}^2\times\mathbb{R}^2$ is a Kähler manifold with respect to the complex structure $\tilde{J}$. From (2.1) and (2.2), we deduce

$$\tilde{P}\tilde{J}=\tilde{J}\tilde{P}. \tag{2.3}$$

Let $\mathit{\Omega}: M^2\to\mathbb{R}^2\times\mathbb{R}^2$ be a Lagrangian immersion with respect to the complex structure $\tilde{J}$. Then for any tangent vector X of M^2, $\tilde{J}X$ is contained in the normal space of M^2. We denote the Levi-Civita connection of M^2 by ∇ and A is the second fundamental form of $M^2\to\mathbb{R}^2\times\mathbb{R}^2$. The Lagrangian condition implies that we have the following

$$\langle A(X,Y),\tilde{J}Z\rangle=\langle A(Y,Z),\tilde{J}X\rangle=\langle A(Z,X),\tilde{J}Y\rangle, \tag{2.4}$$

$$\nabla^\perp_X(\tilde{J}Y)=\tilde{J}\nabla_XY \tag{2.5}$$

for tangent vectors X,Y and Z of M^2, where $\nabla^\perp$ is the connection of the normal bundle.

For a Lagrangian immersion $\mathit{\Omega}:M^2\to\mathbb{R}^2\times\mathbb{R}^2$, we choose $\{e_1,e_2\}$ as an orthonormal basis for the tangent space T_pM^2 at $p\in M^2$, which is compatible with the orientation of M^2, then $\left\{e_1,e_2,\tilde{J}e_1,\tilde{J}e_2\right\}$ is an orthonormal frame in $T_{\mathit{\Omega}(p)}(\mathbb{R}^2\times\mathbb{R}^2)$. Set

$$e_{1^*}=\tilde{J}e_1, e_{2^*}=\tilde{J}e_2.$$

Then $\{e_1,e_2,e_{1^*},e_{2^*}\}$ is called an adapted Lagrangian frame field. The dual frame fields of e_1,e_2 are ω_1,ω_2. The Levi-Civita connection forms and normal connection forms are ω_{ij} and $\omega_{i^*j^*}$, respectively. Writting $A(e_i,e_j)=\sum\limits_k h^{k^*}_{ij}e_{k^*}$, from (2.4), we have

$$h^{k^*}_{ij}=h^{i^*}_{jk}=h^{j^*}_{ki}. \tag{2.6}$$

By virtue of (2.5), we obtain

$$\omega_{12}=\omega_{1^*2^*}. \tag{2.7}$$

The structure equations are

$$d\mathit{\Omega}=\sum_i\omega_ie_i, \tag{2.8}$$

$$de_i=\sum_j\omega_{ij}e_j+\sum_{j,k}h^{k^*}_{ij}\omega_je_{k^*}, \tag{2.9}$$

$$de_{k^*}=-\sum_{i,j}h^{k^*}_{ij}\omega_je_i+\sum_l\omega_{k^*l^*}e_{l^*}. \tag{2.10}$$

We denote the components of curvature tensors of ∇ and $\nabla^{\perp}$ by R_{ijkl} and $R_{i^*j^*kl}$, respectively, then the equations of Gauss, Codazzi and Ricci are as follows

$$R_{ijkl} = \sum_{p}(h_{ik}^{p^*}h_{jl}^{p^*} - h_{il}^{p^*}h_{jk}^{p^*}), \tag{2.11}$$

$$R_{jk} = \sum_{p} H^{p*}h_{jk}^{p^*} - \sum_{i,p} h_{ij}^{p^*}h_{ik}^{p^*}, \tag{2.12}$$

$$h_{ijk}^{p^*} = h_{ikj}^{p^*},\ 1 \leq i,j,k,p \leq 2, \tag{2.13}$$

$$R_{i^*j^*kl} = \sum_{m}(h_{mk}^{i^*}h_{ml}^{j^*} - h_{ml}^{i^*}h_{mk}^{j^*}), \tag{2.14}$$

$$R = H^2 - |A|^2, \tag{2.15}$$

where

$$\sum_{k} h_{ijk}^{p^*}\omega_k = dh_{ij}^{p^*} + \sum_{k} h_{ik}^{p^*}\omega_{kj} + \sum_{k} h_{kj}^{p^*}\omega_{ki} + \sum_{q} h_{ij}^{q^*}\omega_{q^*p^*}$$

and R_{jk} and R are the Ricci curvature and the scalar curvature of M^2, respectively. $|A|^2 = \sum\limits_{i,j,p}(h_{ij}^{p^*})^2$ is the squard norm of the second fundamental form, $\vec{H} = \sum\limits_{p} H^{p*}e_{p*} = \sum\limits_{i,p} h_{ii}^{p*}e_{p*}$ is the mean curvature vector field, $H = |\vec{H}|$ is the mean curvature of M^2. From (2.6) and (2.13), we obtain that $h_{ijk}^{q^*}$ is totally symmetric, i.e.,

$$h_{ijk}^{p^*} = h_{ikj}^{p^*} = h_{pjk}^{i^*}. \tag{2.16}$$

Regarding the product structure $\tilde{P}$ given in (2.2), we have the following decomposition

$$\tilde{P}X = PX + \tilde{J}QX \tag{2.17}$$

for $X \in T_pM^2$, where P and Q are linear endomorphisms in T_pM^2. We choose the unit eigenvector e_1 of P and its orthogonal unit vector e_2 such that $\{e_1, e_2\}$ forms an orthonormal basis of T_pM^2 compatible with the orientation of M^2. Using $\mathrm{tr}\tilde{P} = 0$ and $\left\langle \tilde{P}X, Y \right\rangle = \left\langle X, \tilde{P}Y \right\rangle$ yields $\mathrm{tr}P = 0$ and $\langle PX, Y\rangle = \langle X, PY\rangle$. Then we set

$$Pe_1 = \lambda e_1,\ Pe_2 = -\lambda e_2,$$

where $\lambda \in \mathbb{R}$. It follows from $\tilde{P}^2 = \mathrm{id}$ and $\left\langle \tilde{P}X, \tilde{J}Y \right\rangle = \left\langle X, \tilde{P}\tilde{J}Y \right\rangle$ that $P^2 - Q^2 = \mathrm{id}$, $PQ + QP = 0$ and $\langle QX, Y\rangle = -\langle X, QY\rangle$, which implies

$$Qe_1 = \mu e_2,\ Qe_2 = -\mu e_1,$$

where $\mu \in \mathbb{R}$ and $\lambda^2 + \mu^2 = 1$. So that we get

$$\tilde{P}e_1 = \lambda e_1 + \mu\tilde{J}e_2,\ \tilde{P}e_2 = -\lambda e_2 - \mu\tilde{J}e_1.$$

By (2.2), we have that for $i = 1, 2$,

$$e_i = \left(\frac{e_i + \tilde{P}e_i}{2}, \frac{e_i - \tilde{P}e_i}{2}\right).$$

Let $\alpha : M^2 \to [0, \pi]$ be the Kähler angle of Ω with respect to another complex structure $\bar{J} = (J, -J)$ on $\mathbb{R}^2 \times \mathbb{R}^2$ (cf.[8]), then

$$\langle \bar{J}e_1, e_2 \rangle = \cos\alpha. \tag{2.18}$$

A straightforward calculation shows $\langle \bar{J}e_1, e_2 \rangle = -\mu$, which yields $\mu = -\cos\alpha$. Thus we obtain $\lambda = \sin\alpha$. So

$$\begin{cases} Pe_1 = \sin\alpha e_1, \\ Pe_2 = -\sin\alpha e_2, \\ Qe_1 = -\cos\alpha e_2, \\ Qe_2 = \cos\alpha e_1. \end{cases} \tag{2.19}$$

Similar to the method of the Lagrangian surfaces in the product of two unit spheres, we calculate the gradient and the Laplace equation of α for the Lagrangian surfaces in $\mathbb{R}^2 \times \mathbb{R}^2$. Let $\bar{\nabla}$ be the Levi-Civita connection of $\mathbb{C}^2$. Using $\bar{\nabla}\tilde{P} = 0$ yields

$$\langle (\nabla_X P)e_1, e_1 \rangle = \langle A(X, Qe_1)e_1, \tilde{J}e_1 \rangle + \langle A(X, e_1), \tilde{J}Qe_1 \rangle$$

and

$$\langle (\nabla_X Q)e_1, e_2 \rangle = \langle A(X, e_1), \tilde{J}Pe_2 \rangle - \langle A(X, Pe_1), \tilde{J}e_2 \rangle.$$

The former gives the differentiability of α at points where α attains the boundary values 0 or π. Then we obtain that $\Big\{X\alpha - 2\langle \tilde{J}A(e_1, e_2), X\rangle\Big\} = 0$ holds for any tangent vector $X \in T_pM^2$ (see [14, (2.23)]), i.e.,

$$\frac{\nabla\alpha}{2} = \tilde{J}A(e_1, e_2), \tag{2.20}$$

which implies

$$\nabla\alpha = -2(h_{12}^{1^*}e_1 + h_{12}^{2^*}e_2). \tag{2.21}$$

Since

$$\nabla_X e_1 = \omega_{12}(X)e_2, \nabla_X e_2 = -\omega_{12}(X)e_1,$$

we have (see [14, (2.27)])

$$\begin{cases} \omega_{12}(e_1) = \frac{1}{2}(h_{11}^{1^*} - h_{12}^{2^*})\cot\alpha, \\ \omega_{12}(e_2) = \frac{1}{2}(h_{12}^{1^*} - h_{22}^{2^*})\cot\alpha, \end{cases} \tag{2.22}$$

where $\alpha \in (0, \pi)$. If α attains the boundary values 0 or π, then we have $h_{11}^{1^*} - h_{12}^{2^*} = 0 = h_{12}^{1^*} - h_{22}^{2^*}$. It follows from (2.21) that

$$\alpha_{,1} = -2h_{12}^{1^*}, \ \alpha_{,2} = -2h_{12}^{2^*}. \tag{2.23}$$

Applying (2.23) and (2.22), a straightforward computation shows

$$\begin{aligned} \Delta\alpha &= \alpha_{,11} + \alpha_{,22} \\ &= d\alpha_{,1}(e_1) + \alpha_{,2}\omega_{21}(e_1) + d\alpha_{,2}(e_2) + \alpha_{,1}\omega_{12}(e_2) \\ &= -2dh_{12}^{1^*}(e_1) - 2dh_{12}^{2^*}(e_2) + 2\omega_{12}(e_1)h_{12}^{2^*} - 2\omega_{12}(e_2)h_{12}^{1^*} \\ &= -2e_1h_{12}^{1^*} - 2e_2h_{12}^{2^*} + \Big\{h_{12}^{2^*}(h_{11}^{1^*} - h_{12}^{2^*}) - h_{12}^{1^*}(h_{12}^{1^*} - h_{22}^{2^*})\Big\}\cot\alpha. \end{aligned} \tag{2.24}$$

3 Lagrangian self-shrinkers in $\mathbb{C}^2$

In this section, we derive some fundamental equations about the Lagrangian self-shrinkers in $\mathbb{C}^2$ and give the proofs of the Theorems 1.2, 1.3 and 1.4.

3.1 Some fundamental equations

Let $\mathit{\Omega} : M^2 \to \mathbb{C}^2$ be a compact orientable Lagrangian self-shrinker. The self-shrinker equation (1.1) is equivalent to

$$H^{p*} = -\langle \mathit{\Omega}, e_{p*}\rangle, \quad 1 \le p \le 2. \tag{3.1}$$

From the above equation, we have the following

$$H^{p*}_{,i} = \nabla_i H^{p*} = -\nabla_i \langle \mathit{\Omega}, e_{p*}\rangle = \sum_j h^{p^*}_{ij}\langle \mathit{\Omega}, e_j\rangle, \tag{3.2}$$

which implies

$$-(e_1\langle \mathit{\Omega}, e_{2^*}\rangle + e_2\langle \mathit{\Omega}, e_{1^*}\rangle) = 2(h^{1^*}_{12}\langle \mathit{\Omega}, e_1\rangle + h^{2^*}_{12}\langle \mathit{\Omega}, e_2\rangle) - H^{1^*}\omega_{12}(e_1) + H^{2^*}\omega_{12}(e_2). \tag{3.3}$$

Since $\nabla\alpha = -2(h^{1^*}_{12}e_1 + h^{2^*}_{12}e_2)$, then

$$\langle \nabla\alpha, \mathit{\Omega}\rangle = -2\langle h^{1^*}_{12}e_1 + h^{2^*}_{12}e_2, \mathit{\Omega}\rangle = -2(h^{1^*}_{12}\langle \mathit{\Omega}, e_1\rangle + h^{2^*}_{12}\langle \mathit{\Omega}, e_2\rangle). \tag{3.4}$$

From (3.3) and (3.4), we know

$$e_1\langle \mathit{\Omega}, e_{2^*}\rangle + e_2\langle \mathit{\Omega}, e_{1^*}\rangle = \langle \nabla\alpha, \mathit{\Omega}\rangle + H^{1^*}\omega_{12}(e_1) - H^{2^*}\omega_{12}(e_2). \tag{3.5}$$

By virtue of (3.1), together with (2.6), we obtain

$$h^{1^*}_{11} + h^{2^*}_{12} = -\langle \mathit{\Omega}, e_{1*}\rangle, \;\; h^{1^*}_{12} + h^{2^*}_{22} = -\langle \mathit{\Omega}, e_{2*}\rangle,$$

which gives us by (3.5)

$$-(e_1h^{1^*}_{12} + e_2h^{2^*}_{12}) - (e_2h^{1^*}_{11} + e_1h^{2^*}_{22}) = \langle \nabla\alpha, \mathit{\Omega}\rangle + \frac{1}{2}\left[H^{1^*}(h^{1^*}_{11} - h^{2^*}_{12}) - H^{2^*}(h^{1^*}_{12} - h^{2^*}_{22})\right]\cot\alpha. \tag{3.6}$$

On the other hand, using $h^{1^*}_{121} = h^{1^*}_{112}$ and $h^{2^*}_{221} = h^{2^*}_{122}$, we have

$$\begin{cases} e_1h^{1^*}_{12} - e_2h^{1^*}_{11} + \omega_{12}(e_1)(h^{1^*}_{11} - 2h^{2^*}_{12}) + 3\omega_{12}(e_2)h^{1^*}_{12} = 0, \\ e_1h^{2^*}_{22} - e_2h^{2^*}_{12} + 3\omega_{12}(e_1)h^{2^*}_{12} + \omega_{12}(e_2)(h^{2^*}_{22} - 2h^{1^*}_{12}) = 0. \end{cases} \tag{3.7}$$

In terms of (3.7), together with (2.22), we obtain

$$-(e_1h^{1^*}_{12} + e_2h^{2^*}_{12}) + (e_2h^{1^*}_{11} + e_1h^{2^*}_{22}) = \frac{1}{2}\left\{(h^{1^*}_{11} - h^{2^*}_{12})(h^{1^*}_{11} - 5h^{2^*}_{12}) - (h^{1^*}_{12} - h^{2^*}_{22})(h^{2^*}_{22} - 5h^{1^*}_{12})\right\}\cot\alpha. \tag{3.8}$$

From (3.6) and (3.8), we have

$$-2(e_1h^{1^*}_{12} + e_2h^{2^*}_{12}) = \langle \nabla\alpha, \mathit{\Omega}\rangle + \left\{(h^{1^*}_{11} - h^{2^*}_{12})(h^{1^*}_{11} - 2h^{2^*}_{12}) - (h^{1^*}_{12} - h^{2^*}_{22})(h^{2^*}_{22} - 2h^{1^*}_{12})\right\}\cot\alpha. \tag{3.9}$$

Substituting (3.9) into (2.24) yields

$$\Delta\alpha - \langle \nabla\alpha, \varOmega \rangle = \left\{ (h^{1^*}_{11} - h^{2^*}_{12})^2 + (h^{1^*}_{12} - h^{2^*}_{22})^2 \right\} \cot\alpha. \tag{3.10}$$

Next we consider the following operator $\mathcal{L}$ which was introduced by Colding-Minicozzi [9]:

$$\mathcal{L}f = \Delta f - \langle \varOmega, \nabla f \rangle,$$

where f is a smooth function. So the equation (3.10) can be written as

$$\mathcal{L}\alpha = \left\{ (h^{1^*}_{11} - h^{2^*}_{12})^2 + (h^{1^*}_{12} - h^{2^*}_{22})^2 \right\} \cot\alpha. \tag{3.11}$$

Since M^2 is a Lagrangian surface in $\mathbb{C}^2$, we have

$$\begin{aligned} |A|^2 &= (h^{1^*}_{11})^2 + 3(h^{1^*}_{12})^2 + 3(h^{2^*}_{12})^2 + (h^{2^*}_{22})^2, \\ H^2 &= (h^{1^*}_{11} + h^{2^*}_{12})^2 + (h^{1^*}_{12} + h^{2^*}_{22})^2, \\ |\nabla\alpha|^2 &= 4(h^{1^*}_{12})^2 + 4(h^{2^*}_{12})^2, \end{aligned}$$

then equation (3.11) becomes

$$\mathcal{L}\alpha = (2|A|^2 - H^2 - |\nabla\alpha|^2)\cot\alpha. \tag{3.12}$$

Because $\varOmega : M^2 \to \mathbb{C}^2$ is a Lagrangian self-shrinker, according to Cheng-Hori-Wei's calculation (see [7, (2.23)])

$$\begin{aligned} \mathcal{L}h^{p^*}_{ij} &= \sum_k h^{p^*}_{ijkk} - \langle \varOmega, \sum_k \nabla_k h^{p^*}_{ij} e_k \rangle \\ &= (3K+1)h^{p^*}_{ij} - K(H^{p^*}\delta_{ij} + H^{i^*}\delta_{pj}) - \sum_{q,k} h^{p^*}_{ik} h^{q^*}_{jk} H^{q^*}, \end{aligned} \tag{3.13}$$

where $K = \dfrac{1}{2}(H^2 - |A|^2)$ is the Gauss curvature of M^2.

Since $\nabla\alpha = \sum\limits_{i=1}^{2} \alpha_{,i} e_i$, where $\alpha_{,i} = -2h^{i^*}_{12}$. Define

$$\sum_j \alpha_{,ij}\omega_j = d\alpha_{,i} + \alpha_{,k}\omega_{ki}$$

and

$$\sum_k \alpha_{,ijk}\omega_k = d\alpha_{,ij} + \alpha_{,kj}\omega_{ki} + \alpha_{,ik}\omega_{kj}.$$

A straightforward calculation shows

$$\alpha_{,ij} = -2h^{i^*}_{12j} + E^{i^*}E^{j^*}\cot\alpha \tag{3.14}$$

and

$$\begin{aligned} \alpha_{,ijk} =& -2h^{i^*}_{12jk} + (h^{i^*}_{11j} - h^{i^*}_{22j})E^{k^*}\cot\alpha \\ &+ (h^{i^*}_{11k} - h^{i^*}_{22k})E^{j^*}\cot\alpha + (h^{j^*}_{11k} - h^{j^*}_{22k})E^{i^*}\cot\alpha \\ &- \alpha_{,i}E^{j^*}E^{k^*}\cot^2\alpha - \alpha_{,j}E^{i^*}E^{k^*}\cot^2\alpha - \frac{1}{\sin^2\alpha}\alpha_{,k}E^{i^*}E^{j^*}, \end{aligned} \tag{3.15}$$

where $E^{i^*} = h_{11}^{i^*} - h_{22}^{i^*}$ $(i = 1, 2)$. Using (3.2),(3.13),(3.14) and (3.15), we obtain

$$\begin{aligned}\mathcal{L}\alpha_{,p} &= \sum_k \alpha_{,pkk} - \sum_k \alpha_{,pk}\langle \Omega, e_k\rangle \\ &= -2(3K+1)h_{12}^{p^*} + 2KH^{1^*}\delta_{p2} + 2\sum_{q,k} h_{1k}^{p^*}h_{2k}^{q^*}H^{q^*} + 2\sum_k (h_{11k}^{p^*} - h_{22k}^{p^*})E^{k^*}\cot\alpha \\ &- \alpha_{,p}\sum_k (E^{k^*})^2\cot^2\alpha - E^{p^*}\sum_k (\alpha_{,k}E^{k^*})(2\cot^2\alpha + 1).\end{aligned}$$

Then we get

$$\begin{aligned}\frac{1}{2}\mathcal{L}|\nabla\alpha|^2 &= \sum_{p,k}(\alpha_{,pk})^2 + \sum_p \alpha_{,p}\mathcal{L}\alpha_{,p} \\ &= |\nabla^2\alpha|^2 + (H^2 - 2|A|^2 + \frac{1}{2}|\nabla\alpha|^2 + 1)|\nabla\alpha|^2 - 8h_{12}^{1^*}h_{12}^{2^*}H^{1^*}H^{2^*} \\ &- 4\sum_{p,k} h_{12}^{p^*}(h_{11k}^{p^*} - h_{22k}^{p^*})E^{k^*}\cot\alpha - |\nabla\alpha|^2\sum_k (E^{k^*})^2\cot^2\alpha \\ &- (\sum_k \alpha_{,k}E^{k^*})^2(2\cot^2\alpha + 1).\end{aligned} \tag{3.16}$$

3.2 Solutions space

Suppose that N is a real four-dimensional Riemannian manifold. Let M^2 be a Riemann surface and $\Omega : M^2 \to N$ be an isometric immersion. We denote the induced metric on M^2 by $<,>$. If $A : TM^2 \times TM^2 \to T^\perp M^2$ is the second fundamental form of Ω, the *ellipse of curvature* is the subset of the normal plane defined as

$$\{A(v,v) \in T_p^\perp M^2 :< v, v >= 1, v \in T_pM^2, p \in M^2\}.$$

To see that it is an ellipse, we consider an orthonormal tangent frame $\{e_1, e_2\}$, denote $h_{ij} = A(e_i, e_j), i, j = 1, 2$, and look at the following formula for $v = \cos\tau e_1 + \sin\tau e_2$:

$$A(v,v) = \frac{\vec{H}}{2} + \cos 2\tau \frac{h_{11} - h_{22}}{2} + \sin 2\tau h_{12}, \tag{3.17}$$

where $\vec{H} = \mathrm{trace}(A)$ is the mean curvature vector of Ω. As v goes once around the unit tangent circle, $A(v,v)$ goes twice around the ellipse, which could degenerate into a line segment or a point. The center of the ellipse is $\frac{\vec{H}}{2}$. And the following properties are equivalent at a point of the immersed surface: (i) the ellipse degenerates into a line segment or a point, (ii) $(h_{11} - h_{22})/2$ and h_{12} are linearly dependent, (iii) the normal bundle is flat, and (iv) if e_β $(\beta = 3, 4)$ is an orthonormal normal frame, the second fundamental forms A_{e_β} are simultaneously diagonalizable. Here, A_{e_β} is the symmetric endomorphism of TM^2 defined by $< A(X,Y), e_\beta >=< A_{e_\beta}X, Y >$, where $X, Y \in TM^2$.

Let $\mathcal{M}$ be the moduli space of immersed surfaces in $\mathbb{C}^2$. We study the geometry and topology of its Lagrangian self-shrinker solutions. Let M^2 be a compact Lagrangian self-shrinker in $\mathbb{C}^2$ with respect to $\tilde{J} = (J, J)$ and α be its Kähler angle with respect to

$\bar{J}=(J,-J)$, we take the same frame $\{e_1,e_2\}$ as (2.19) and set $e_3=e_{1^*},\ e_4=e_{2^*}$. Then the squared lengths of the two axes of the ellipse of curvature are respectively

$$|h_{11}-h_{22}|^2=2|A|^2-H^2-|\nabla\alpha|^2,\ 4|h_{12}|^2=|\nabla\alpha|^2. \tag{3.18}$$

We find that the product of two closed Abresch-Langer curves in $\mathbb{R}^2$ has vanishing $|\nabla\alpha|^2$, which means that, under frame (2.19), its ellipse of curvature degenerates into the first axis by (3.18). Conversely, if $|\nabla\alpha|^2$ is zero, then $K=0$, which implies that M^2 is the product of two closed Abresch-Langer curves in $\mathbb{R}^2$ by [17, Proposition 5.5]. So the space of compact solutions with vanishing $|\nabla\alpha|^2$ consists of the products of two closed Abresch-Langer curves in $\mathbb{R}^2$. Since the circle is the only embedded one of them (see the note after [1, Theorem A]), then we obtain the rigidity of the Clifford torus as a compact embedded Lagrangian self-shrinker with vanishing $|\nabla\alpha|^2$.

In order to get similar rigidity result, we hope to prove that the space of compact solutions with some geometric conditions is composed of the products of two closed Abresch-Langer curves in $\mathbb{R}^2$. It is natural to consider the condition that $|\nabla\alpha|^2$ is a nonzero constant. Combining it with (3.16), we get a complicated curvature formula. From (3.10) and (3.12), we surprisingly find that the condition that $\Delta\alpha$ (or $\mathcal{L}\alpha$) vanishes at the point that $|\Omega|^2$ attains minimum, which is equivalent to that $2|A|^2-H^2-|\nabla\alpha|^2=0$ or $\alpha=\frac{\pi}{2}$ at this point, is very helpful for simplifying the complicated curvature formula at this point. However, the first case of $2|A|^2-H^2-|\nabla\alpha|^2=0$ means that, under frame (2.19), the ellipse of curvature of the solution degenerates into the second axis, whose length is a nonzero constant by (3.18). Geometrically, this seems to cause the solution and the product of two closed Abresch-Langer curves to diverge increasingly. Fortunately, we have found a curvature pinching condition of $4|A|^2-|\nabla\alpha|^2\geq 8$ that prevents this divergence from happening. Whether the solution can be transformed into a product of two closed Abresch-Langer curves via $SO(4)$ matrices to eliminate this divergence merits further exploration. In the following, we will prove that the space of compact Lagrangian self-shrinker solutions with the above three conditions consists of the products of two closed Abresch-Langer curves. Then embedded condition leads to the rigidity result.

3.3 Proof of Theorem 1.2

Proof. Since α is constant, it follows from (2.21) that $h_{12}^{1^*}=h_{12}^{2^*}=0$, which implies $K\equiv 0$. According to [17, Proposition 5.5], M^2 is the product of two closed Abresch-Langer curves in $\mathbb{R}^2$. Since the circle is the only embedded one of them (see the note after [1, Theorem A]), then M^2 is the Clifford torus $\mathbb{S}^1\times\mathbb{S}^1$. □

3.4 Proofs of Theorems 1.3, 1.4

In this subsection, we give the proofs of Theorems 1.3 and 1.4. The idea of our proofs is similar to Li-Wang[17], except that we consider a different angle function α. Before giving our proofs, we introduce the following lemmas.

Lemma 3.1. *(cf.[5, 9]) Let $\Omega : M^2 \to \mathbb{C}^2$ be a compact orientable Lagrangian self-shrinker, we have*

$$
\begin{aligned}
0 &= \int_{M^2} \frac{1}{2}\Delta(|\Omega|)^2 dv = \int_{M^2} (2-|H|^2) dv, \\
0 &= \int_{M^2} \frac{1}{2}\mathcal{L}(|\Omega|)^2 e^{-\frac{|\Omega|^2}{2}} dv = \int_{M^2} (2-|\Omega|^2) e^{-\frac{|\Omega|^2}{2}} dv.
\end{aligned} \tag{3.19}
$$

Lemma 3.2. *(see [6, Theorem 1.1]) Let $\Omega : M^2 \to \mathbb{C}^2$ be a compact Lagrangian self-shrinker. If H^2 is constant or $H^2 \le 2$ or $H^2 \ge 2$, then M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

Proof of Theorem 1.3. Since M^2 is compact, there exists a point $p_1 \in M^2$ such that $|\Omega|^2$ attains minimum at p_1. Then

$$|\Omega|^2 \ge |\Omega|^2(p_1).$$

It's not difficult to get that $|\Omega|^2_{,i} = 0$ at p_1, which implies that $\langle \Omega, e_i\rangle = 0$ at p_1. It follows that $|\Omega|^2 = H^2$ at p_1 and

$$\left\{(E^{1^*})^2 + (E^{2^*})^2\right\} \cot\alpha = 0 \tag{3.20}$$

by (3.10) and that $\Delta\alpha = 0$ at p_1. And since $|\nabla\alpha|^2 = (\alpha_{,1})^2 + (\alpha_{,2})^2$ is constant, we have $(|\nabla\alpha|^2)_{,i} = 0,\ 1 \le i \le 2$. Hence,

$$
\begin{aligned}
\alpha_{,1}\alpha_{,11} + \alpha_{,2}\alpha_{,21} &= 0, \\
\alpha_{,1}\alpha_{,12} + \alpha_{,2}\alpha_{,22} &= 0.
\end{aligned} \tag{3.21}
$$

From that $\Delta\alpha = 0$ at p_1, we get

$$\alpha_{,11} + \alpha_{,22} = 0. \tag{3.22}$$

By substituting (3.22) into (3.21), we get

$$\begin{pmatrix} \alpha_{,1} & \alpha_{,2} \\ -\alpha_{,2} & \alpha_{,1} \end{pmatrix} \begin{pmatrix} \alpha_{,11} \\ \alpha_{,12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.23}$$

We know that the determinant of the matrix is

$$det = (\alpha_{,1})^2 + (\alpha_{,2})^2 = |\nabla\alpha|^2.$$

Hence, we have the following two cases:

(i)At p_1, $|\nabla\alpha|^2 = 0$. Since $|\nabla\alpha|^2$ is constant, $|\nabla\alpha|^2 \equiv 0$ in M^2, i.e., $h^{1^*}_{12} = h^{2^*}_{12} = 0$. Then $H^2 = |A|^2$ and $K = 0$. According to [17, Proposition 5.5], M^2 is the product of two closed Abresch-Langer curves in $\mathbb{R}^2$. Since the circle is the only embedded one of them (see the note after [1, Theorem A]), then we deduce that M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$.

(ii)At p_1, $|\nabla\alpha|^2 \ne 0$, i.e., $\alpha_{,11} = \alpha_{,12} = 0$. Then from (3.22), we get $\alpha_{,22} = 0$. Under this condition, $|\nabla^2\alpha|^2 = 0$ at p_1. If $\alpha(p_1) \ne \frac{\pi}{2}$, then $E^{i^*}(p_1) = 0$ by (3.20), which implies that $|A|^2(p_1) = H^2(p_1) = |\nabla\alpha|^2$. Here we can get the same conclusions when $\alpha(p_1)$ attains the boundary values 0 or π. Since $4|A|^2 - |\nabla\alpha|^2 \ge 8$, we obtain that

$$|\nabla\alpha|^2 \ge \frac{8}{3}.$$

So we deduce that

$$|\Omega|^2(p_1) = H^2(p_1) \geq \frac{8}{3},$$

then

$$|\Omega|^2 \geq \frac{8}{3} > 2 \quad \text{on} \quad M^2.$$

It is a contradiction with Lemma 3.1. Therefore, this situation does not exist.

If $\alpha(p_1) = \frac{\pi}{2}$, then at p_1, using (3.16) we have

$$(H^2 - 2|A|^2 + \frac{1}{2}|\nabla\alpha|^2 + 1)|\nabla\alpha|^2 - 8h^{1^*}_{12}h^{2^*}_{12}H^{1^*}H^{2^*} - (\sum_k \alpha_{,k}E^{k^*})^2 = 0. \tag{3.24}$$

From (3.24) we deduce that

$$(H^2 - 2|A|^2 + \frac{1}{2}|\nabla\alpha|^2 + 1)|\nabla\alpha|^2 \geq -\frac{1}{2}|\nabla\alpha|^2 H^2.$$

Since $|\nabla\alpha|^2 \neq 0$ at p_1, then the above inequality at p_1 also can be written as:

$$H^2 \geq \frac{1}{3}(4|A|^2 - |\nabla\alpha|^2 - 2), \tag{3.25}$$

which means that $H^2(p_1) \geq \frac{1}{3}(4|A|^2 - |\nabla\alpha|^2 - 2) \geq 2$. It follows that $|\Omega|^2 \geq |\Omega|^2(p_1) = H^2(p_1) \geq 2$. Using Lemma 3.1, we obtain that $|\Omega|^2 = 2$, which implies that $H^2 = 2$. Applying Lemma 3.2, we know that M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$. It follows that on M^2

$$|A|^2 = 2, H^2 = 2, |\nabla\alpha|^2 = 0,$$

which contradicts $|\nabla\alpha|^2 \neq 0$ at p_1. Therefore, this situation does not exist too. Thus we complete the proof of Theorem 1.3. □

Combining the above theorem with the relationship between $|A|^2$ and $|\nabla\alpha|^2$, we get the following result:

Corollary 3.3. *Let $\Omega : M^2 \to \mathbb{C}^2$ be a compact orientable embedded Lagrangian self-shrinker and α be the Kähler angle with respect to the twisted complex structure $\bar{J} = (J, -J)$. If $|\nabla\alpha|^2$ is constant and $\Delta\alpha$ vanishes at the point that $|\Omega|^2$ attains minimum, then $|\nabla\alpha|^2 < 4$.*

Proof . By way of contradiction, suppose that $|\nabla\alpha|^2 \geq 4$. Since M^2 is compact, there exists a point $p_2 \in M^2$ such that $|\Omega|^2$ attains minimum at p_2. Then

$$|\Omega|^2 \geq |\Omega|^2(p_2).$$

Since $|\nabla\alpha|^2$ is constant and $\Delta\alpha = 0$ at p_2, then (3.20)-(3.23) all hold. If $\alpha(p_2) \neq \frac{\pi}{2}$, then we obtain that $|A|^2(p_2) = H^2(p_2) = |\nabla\alpha|^2$. Since $|\nabla\alpha|^2 \geq 4$, then $|\Omega|^2 \geq |\Omega|^2(p_2) = H^2(p_2) \geq 4$. It contradicts Lemma 3.1.

If $\alpha(p_2) = \frac{\pi}{2}$, then at p_2, (3.25) is true, which means that $H^2(p_2) \geq \frac{1}{3}(4|A|^2 - |\nabla\alpha|^2 - 2)$. Since

$$|A|^2 \geq \frac{3}{4}|\nabla\alpha|^2 \quad \text{on} \quad M^2,$$

we have that

$$H^2(p_2) \geq \frac{2}{3}(|\nabla\alpha|^2 - 1) \geq 2.$$

Hence we have

$$|\Omega|^2 \geq |\Omega|^2(p_2) = H^2(p_2) \geq 2.$$

Applying Lemma 3.1 and Lemma 3.2, we know that M^2 is the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$. It follows that on M^2

$$|A|^2 = 2, H^2 = 2, |\nabla\alpha|^2 = 0,$$

which contradicts $|\nabla\alpha|^2 \geq 4$. Thus we complete the proof. □

Proof of Theorem 1.4. Since $\mathcal{L}\alpha = \Delta\alpha - \langle\Omega, \nabla\alpha\rangle$, then the condition of $\mathcal{L}\alpha$ vanishing at the point that $|\Omega|^2$ attains minimum deduces the one that $\Delta\alpha$ vanishes at the point that $|\Omega|^2$ attains minimum. The result follows from proof of Theorem 1.3. □

4 Rotational symmetry case (proof of Theorem 1.6)

For the generalized rotational surface

$$\Omega(x,y) = (f(x)\cos(cy), f(x)\sin(cy), g(x)\cos(dy), g(x)\sin(dy)),$$

the orthonormal tangent frame field T_1, T_2 and orthonormal normal frame field N_1, N_2 of M can be considered as follows:

$$T_1 = \frac{1}{\sqrt{(f')^2 + (g')^2}}(f'\cos(cy), f'\sin(cy), g'\cos(dy), g'\sin(dy)), \tag{4.1a}$$

$$T_2 = \frac{1}{\sqrt{c^2f^2 + d^2g^2}}(-cf\sin(cy), cf\cos(cy), -dg\sin(dy), dg\cos(dy)), \tag{4.1b}$$

and

$$N_1 = \frac{1}{\sqrt{(f')^2 + (g')^2}}(g'\cos(cy), g'\sin(cy), -f'\cos(dy), -f'\sin(dy)), \tag{4.2a}$$

$$N_2 = \frac{1}{\sqrt{c^2f^2 + d^2g^2}}(-dg\sin(cy), dg\cos(cy), cf\sin(dy), -cf\cos(dy)) \tag{4.2b}$$

respectively. Also, the mean curvature vector and Gaussian curvature of the generalized rotation surface M are given by

$$\vec{H} = \frac{(c^2f^2 + d^2g^2)(g'f'' - f'g'') + (d^2gf' - c^2fg')(f'^2 + g'^2)}{(f'^2 + g'^2)^{\frac{3}{2}}(c^2f^2 + d^2g^2)}N_1, \tag{4.3}$$

and

$$K = \frac{(c^2f^2 + d^2g^2)(g'f'' - f'g'')(d^2gf' - c^2fg') - c^2d^2(gf' - fg')^2(f'^2 + g'^2)}{(f'^2 + g'^2)^2(c^2f^2 + d^2g^2)^2} \tag{4.4}$$

respectively.

We first derive the equations characterizing the Lagrangian condition with respect to distinct complex structures. Let $|\varOmega_x|^2 = (f')^2 + (g')^2 = m^2, |\varOmega_y|^2 = c^2f^2 + d^2g^2 = L^2$. In the first complex structure, by the Lagrangian condition, we have

$$< \tilde{J}_1\varOmega_x, \varOmega_y >= Re[if'e^{icy}(-icfe^{-icy}) + ig'e^{idy}(-idge^{-idy})] = cff' + dgg',$$

which implies that

$$cff' + dgg' = 0. \tag{4.5}$$

In the second complex structure, the Lagrangian condition $< \tilde{J}_2\varOmega_x, \varOmega_y >= 0$ requires

$$\begin{aligned} < \tilde{J}_2\varOmega_x, \varOmega_y > &= Re[(-g'\cos(dy) + if'\cos(cy))(-cf\sin(cy) + idg\sin(dy)) \\ &\quad + (-g'\sin(dy) + if'\sin(cy))(cf\cos(cy) - idg\cos(dy))] \\ &= (cfg' + df'g)\sin[(c-d)y] \\ &= 0, \end{aligned}$$

that is,

$$c = d \tag{4.6}$$

or

$$cfg' + df'g = 0. \tag{4.7}$$

In the third complex structure, the Lagrangian condition $< \tilde{J}_3\varOmega_x, \varOmega_y >= 0$ requires

$$\begin{aligned} < \tilde{J}_3\varOmega_x, \varOmega_y > &= (df'g - cfg')\cos[(c+d)y] \\ &= 0. \end{aligned}$$

Since $c > 0, d > 0$, we have

$$cfg' - df'g = 0. \tag{4.8}$$

4.1 Proof of Prop 1.5(a)

Proof . From (1.1), a self-shinker of the mean curvature flow satisfies the quasilinear elliptic system

$$\vec{H} = -\varOmega^{\perp}. \tag{4.9}$$

Through calculation, we get

$$\begin{aligned} < \varOmega, N_1 > &= \frac{1}{m} Re[(f\cos(cy) + ig\cos(dy))(g'\cos(cy) + if'\cos(dy)) \\ &\quad + (f\sin(cy) + ig\sin(dy))(g'\sin(cy) + if'\sin(dy))] \\ &= \frac{fg' - gf'}{m}, \end{aligned}$$

and

$$\begin{aligned} < \varOmega, N_2 > = \frac{1}{L} & Re[(f\cos(cy) + ig\cos(dy))(-dg\sin(cy) - icf\sin(dy)) \\ &+ (f\sin(cy) + ig\sin(dy))(dg\cos(cy) + icf\cos(dy))] \\ &= 0. \end{aligned}$$

Moreover, by the definition of the self-shinker, we obtain

$$L^2(g'f'' - f'g'') + m^2(d^2gf' - c^2fg' + L^2fg' - L^2gf') = 0. \tag{4.10}$$

By the identity (4.5), we get

$$(cf^2 + dg^2)' = 0.$$

Hence,

$$cf^2 + dg^2 = \nu, \tag{4.11}$$

here, ν is a constant. Hence,

$$f = \pm\sqrt{\frac{\nu - dg^2}{c}}. \tag{4.12}$$

Next, we investigate the case where $f = \sqrt{\frac{\nu - dg^2}{c}}$. The discussion for $f = -\sqrt{\frac{\nu - dg^2}{c}}$ is similar. By this identity, (4.10) can be written in the following form

$$(c + d - \nu)[d(c - d)g^2 - c\nu] = 0. \tag{4.13}$$

We will now consider the equation under several different scenarios.

Case a1: $c + d - \nu \neq 0$

If $c = d$, we have $c\nu = 0$. This creates a contradiction with $c > 0, \nu > 0$, hence $c \neq d$.

So we have $g^2 = \frac{c\nu}{d(c-d)}$, implying that $c - d > 0$. From $f^2 = \frac{\nu - dg^2}{c} = \frac{\nu d}{c(d-c)}$, we deduce that $d - c > 0$. Creating a contradiction, so this situation is not valid.

Case a2: $c + d - \nu = 0$

When $c + d - \nu = 0$, we have $\frac{c}{c+d}f^2 + \frac{d}{c+d}g^2 = 1$. If we parametrise the curve

$$\gamma = re^{i\varphi} = (\sqrt{\frac{c}{c+d}}f, \sqrt{\frac{d}{c+d}}g) \tag{4.14}$$

by arclength and denote by θ the angle made by the tangent with any fixed axis like [2], we will obtain

$$\begin{cases} \dot{r} = \cos(\theta - \varphi), \\ \dot{\varphi} = \frac{1}{r}\sin(\theta - \varphi), \\ \dot{\theta} = (\lambda r - \frac{n-1}{r})\sin(\theta - \varphi), \end{cases} \tag{4.15}$$

where, $n = 2, \lambda = 1$. It follows that $r^2 = \frac{c}{c+d}f^2 + \frac{d}{c+d}g^2 = 1$, that is $r = 1$. Moreover, $\dot{r} = \cos(\theta - \varphi) = 0$ because r is a constant. Hence, $\theta - \varphi = \frac{\pi}{2}$ and $\dot{\varphi} = \frac{1}{r}sin\frac{\pi}{2} = 1$, which means that $\varphi = s$. So we have

$$f = \sqrt{\frac{c+d}{c}}\cos s, g = \sqrt{\frac{c+d}{d}}\sin s. \tag{4.16}$$

Thus the surface is

$$\varOmega(s,t) = \begin{bmatrix} \sqrt{\frac{c+d}{c}}\cos s\cos ct \\ \sqrt{\frac{c+d}{d}}\sin s\cos dt \\ \sqrt{\frac{c+d}{c}}\cos s\sin ct \\ \sqrt{\frac{c+d}{d}}\sin s\sin dt \end{bmatrix}. \tag{4.17}$$

Let's first calculate $|\nabla\alpha|^2$ in this scenario. Similar to the previous calculation process, we have

$$\cos\alpha = \frac{1}{mL}(cff' - dgg') = \frac{2c}{mL}ff', \tag{4.18}$$

where we used the condition of Lagrangian in the last equality. Moreover,

$$(\cos\alpha)_s = -2cff'[\frac{f'f'' + g'g''}{m^3L} + \frac{c^2ff' + d^2gg'}{mL^3}] + \frac{2c}{mn}(f'^2 + ff''),$$

$$(\cos\alpha)_t = 0,$$

and

$$|\nabla\cos\alpha|^2 = \frac{1}{m^2}|(\cos\alpha)_s|^2 = \frac{4c^2Q^2}{m^6L^6}, \tag{4.19}$$

where,

$$Q = (f'^2 + ff'')m^2L^2 - (c^2ff' + d^2gg')ff'm^2 - (f'f'' + g'g'')ff'L^2, \tag{4.20}$$

and so,

$$|\nabla\alpha|^2 = \frac{4c^2Q^2}{m^6L^4(m^2L^2 - 4c^2f^2f'^2)} = k, \tag{4.21}$$

where k is a constant.

By substituting (4.16) into (4.21), we obtain

$$|\nabla\alpha|^2 = \frac{4c^2Q^2}{\frac{(c+d)^7}{c^3d^4}(1 + \frac{d-c}{c}\sin^2 s)^4(1 + \frac{c-d}{d}\sin^2 s)^3 - 4(c+d)^2\sin^2 s(1 - \sin^2 s)} = k, \tag{4.22}$$

where,

$$\begin{aligned} Q = &\frac{(c+d)^3}{dc^2}(2\sin^2 s - 1)(1 + \frac{d-c}{c}\sin^2 s)(1 + \frac{c-d}{d}\sin^2 s) \\ &+ \frac{(c+d)^2(d^2 - c^2)}{4cd}\sin^2 s(1 + \frac{d-c}{c}\sin^2 s) \\ &+ \frac{(c+d)^3(c-d)}{4c^3d}\sin^2 s(1 + \frac{c-d}{d}\sin^2 s). \end{aligned}$$

By the equation (4.22), the coefficient of the highest degree term is $(\frac{c+d}{cd})^7(c-d)^7 = 0$, hence we have $c = d$. The surface equation (4.17) becomes

$$\varOmega(s,t) = \begin{bmatrix} \sqrt{2}\cos s\cos ct \\ \sqrt{2}\sin s\cos ct \\ \sqrt{2}\cos s\sin ct \\ \sqrt{2}\sin s\sin ct \end{bmatrix}. \tag{4.23}$$

Hence the surface is the Clifford torus.

Thus the Prop 1.5(a) has been proved. $\square$

Remark 4.1. *In [7], a special case of the above surface is given as follows. The relevant theories about this type of surface is first mentioned in [16].*

For positive integers m, n with $(m,n)=1$, defining $\Omega^{m,n}:\mathbb{R}^2\to\mathbb{R}^4$ by

$$\Omega^{m,n}(s,t)=\sqrt{m+n}\begin{bmatrix}\frac{\cos s}{\sqrt{n}}\cos\sqrt{\frac{n}{m}}t\\ \frac{\sin s}{\sqrt{m}}\cos\sqrt{\frac{m}{n}}t\\ \frac{\cos s}{\sqrt{n}}\sin\sqrt{\frac{n}{m}}t\\ \frac{\sin s}{\sqrt{m}}\sin\sqrt{\frac{m}{n}}t\end{bmatrix}. \tag{4.24}$$

$\Omega^{m,n}:\mathbb{R}^2\to\mathbb{R}^4$ is a rotational Lagrangian self-shinker in $\mathbb{R}^4$ with respect to the second complex structure. We compute

$$\cos\alpha=-\frac{\sqrt{mn}\sin(2s)}{n+(m-n)\sin^2 s},$$

thus,

$$(\cos\alpha)_t=0,$$

$$(\cos\alpha)_s=\sqrt{mn}\frac{(m-n)\sin^2(2s)-2\cos(2s)(n+(m-n)\sin^2 s)}{[n+(m-n)\sin^2 s]^2}.$$

Moreover,

$$|\nabla\alpha|^2=\frac{\frac{m+n}{n}+\frac{n^2-m^2}{mn}\sin^2 s}{(\frac{m+n}{m}+\frac{m^2-n^2}{mn}\sin^2 s)^2}\frac{mn[(m-n)\sin^2(2s)-2\cos(2s)(n+(m-n)\sin^2 s)]^2}{[n+(m-n)\sin^2 s]^4-mn\sin^2(2s)[n+(m-n)\sin^2 s]^2}, \tag{4.25}$$

it is easy to be proved that $|\nabla\alpha|^2$ is a constant if and only if $m=n$. Thus M^2 is the Clifford torus because of (4.24).

4.2 Proof of Prop 1.5(b)

Proof . In the situation of $c=d$, (4.10) can be simplified as follows:

$$L^2(g'f''-f'g'')+m^2(L^2-c^2)(fg'-gf')=0. \tag{4.26}$$

Hence, by

$$L^2(g'f''-f'g'')=-m^2(L^2-c^2)(fg'-gf'),$$

(4.4) can be simplified as follows:

$$K=\frac{c^2(L^2-2c^2)(fg'-gf')^2}{m^2L^4}. \tag{4.27}$$

We introduce the condition on the Kähler angle α and further investigate the above problem. From (2.18), Kähler angle α satisfies the following equation

$$\cos\alpha=<\bar{J}(\frac{\Omega_x}{|\Omega_x|}),\frac{\Omega_y}{|\Omega_y|}>. \tag{4.28}$$

The following discussion is based on the condition that $|\nabla\alpha|^2$ is a constant.

Case b1

When $c = d$, by (4.28), we have

$$\Omega(x, y) = (f(x)\cos(cy), f(x)\sin(cy), g(x)\cos(cy), g(x)\sin(cy)), \tag{4.29}$$

and

$$\cos\alpha = \frac{g'f - f'g}{m\sqrt{f^2+g^2}}\sin(2cy). \tag{4.30}$$

We notice that the induced metric is $\Omega^*(ds^2_{\mathbb{R}^4}) = m^2dx^2 + L^2dy^2$. Then,

$$(\cos\alpha)_x = \frac{\sin(2cy)N}{m^3(f^2+g^2)^{\frac{3}{2}}},$$

$$(\cos\alpha)_y = 2c\cos(2cy)\frac{fg' - f'g}{m\sqrt{f^2+g^2}},$$

and

$$|\nabla\cos\alpha|^2 = \frac{|(\cos\alpha)_x|^2}{m^2} + \frac{|(\cos\alpha)_y|^2}{L^2} = \frac{N^2\sin^2(2cy) + 4m^6\cos^2(2cy)(fg'-f'g)^2(f^2+g^2)}{(f^2+g^2)^3m^8}, \tag{4.31}$$

where,

$$N = m^2(fg'' - f''g)(f^2+g^2) - m^2(fg'-f'g)(ff'+gg') - (fg'-f'g)(f^2+g^2)(ff'+gg'). \tag{4.32}$$

Thus,

$$\begin{aligned} |\nabla\alpha|^2 &= \frac{|\nabla\cos\alpha|^2}{\sin^2\alpha} \\ &= \frac{4m^6(f^2+g^2)(fg'-f'g)^2 + [N^2 - 4m^6(f^2+g^2)(fg'-f'g)^2]\sin^2(2cy)}{m^8(f^2+g^2)^3 - m^6(f^2+g^2)^2(fg'-gf')^2\sin^2(2cy)} \\ &= k, \end{aligned} \tag{4.33}$$

where k is a constant. From this, we obtain

$$\begin{cases} \frac{(fg'-f'g)^2}{(f^2+g^2)^2} = \frac{km^2}{4}, \\ -km^6(f^2+g^2)^2(fg'-gf')^2 = N^2 - 4m^6(f^2+g^2)(fg'-f'g)^2. \end{cases} \tag{4.34}$$

If we parametrise the curve $\gamma = re^{i\varphi} = (f, g)$ as in (4.14), we will also obtain (4.15). We compute that

$$\frac{df}{ds} = \frac{df}{dr}\cdot\frac{dr}{ds} + \frac{df}{d\varphi}\cdot\frac{d\varphi}{ds} = \cos\varphi\cos(\theta-\varphi) - \sin\varphi\sin(\theta-\varphi) = \cos\theta,$$

$$\frac{dg}{ds} = \frac{dg}{dr}\cdot\frac{dr}{ds} + \frac{dg}{d\varphi}\cdot\frac{d\varphi}{ds} = \sin\varphi\cos(\theta-\varphi) + \cos\varphi\sin(\theta-\varphi) = \sin\theta.$$

Moreover, $\frac{(fg'-f'g)^2}{(f^2+g^2)^2}=\frac{km^2}{4}$ can be writen as follows:

$$\frac{\sin^2(\theta-\varphi)}{r^2}=\frac{k}{4}. \tag{4.35}$$

Let $\alpha=\theta-\varphi$, we have $\frac{\sin\alpha}{r}$ is a constant. Thus,

$$\frac{-\dot{r}\sin\alpha+r\cos\alpha\dot{\alpha}}{r^2}=0.$$

Namely,

$$\sin 2\alpha(\frac{1}{2}\lambda r^2-n)=0. \tag{4.36}$$

Therefore, we conclude that $r=2$ and the surface is Clifford torus.

Case b2

When $cfg'+df'g=0$, we have

$$(c\log g+d\log f)'=0,$$

then,

$$g=\beta f^{-\frac{d}{c}}. \tag{4.37}$$

Here, β is a constant.

Hence, (4.10) can be simplified as follows:

$$-\frac{c+d}{c}\beta f^{-\frac{d}{c}}(c^2f^2+d^2\beta^2f^{-\frac{2d}{c}})=0,$$

that is,

$$c^2f^2+d^2\beta^2f^{-\frac{2d}{c}}=0. \tag{4.38}$$

By $c>0, d>0$ we have

$$f=(-\frac{d^2\beta^2}{c^2})^{-(\frac{2d}{c}+2)}, \tag{4.39}$$

which means that f is a constant and Ω is a curve in R^4. This leads to a contradiction.

Thus Prop 1.5(b) has been proved. □

Remark 4.2. *Anciaux torus [7] is a special case of (4.29) when $c=1$,*

$$\Omega(x,y)=\begin{bmatrix} f(x)\cos y\\ f(x)\sin y\\ g(x)\cos y\\ g(x)\sin y\end{bmatrix}, \tag{4.40}$$

where $\gamma(x)=(f(x),g(x)), x\in I$ is a closed curve and its self-shinker condition (4.26) becomes

$$L^2(g'f''-f'g'')+m^2(L^2-1)(fg'-gf')=0, \tag{4.41}$$

where $m^2=(f')^2+(g')^2, L^2=f^2+g^2$. Hence, $\Omega(x,y)$ is a compact rotational Lagrangian self-shinker in $\mathbb{R}^4$ with respect to the first complex structure. At this time, (4.33) becomes

$$|\nabla\alpha|^2=\frac{4m^6(f^2+g^2)(fg'-f'g)^2+[N^2-4m^6(f^2+g^2)(fg'-f'g)^2]\sin^2(2y)}{m^8(f^2+g^2)^3-m^6(f^2+g^2)^2(fg'-gf')^2\sin^2(2y)},$$

where

$$N = m^2(fg'' - f''g)(f^2+g^2) - m^2(fg' - f'g)(ff' + gg') - (fg' - f'g)(f^2+g^2)(ff' + gg').$$

We can see that, in general, $|\nabla\alpha|^2$ is not a constant.

4.3 Proof of Prop 1.5(c)

Proof . When $cfg' - df'g = 0$, we have

$$(c\log g - d\log f)' = 0,$$

then,

$$g = \beta f^{\frac{d}{c}}, \tag{4.42}$$

where β is a constant.

Similar to Case b2, we can obtain that f is a constant and Ω is a curve in R^4. This leads to a contradiction.

Thus Prop 1.5(c) has been proved. □

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