

# Super and Weak Poincaré Inequalities for Sticky-Reflected Diffusion Processes\*

Feng-Yu Wang

Center for Applied Mathematics and KL-AAGDM, Tianjin University, Tianjin 300072, China

wangfy@tju.edu.cn

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## Abstract

As a continuation to [3] where the Poincaré and log-Sobolev inequalities were studied for the sticky-reflected Brownian motion on Riemannian manifolds with boundary, this paper establishes the super and weak Poincaré inequalities for more general sticky-reflected diffusion processes. As applications, the convergence rate and uniform integrability of the associated diffusion semigroups are characterized. The main results are illustrated by concrete examples.

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## 1 Introduction

Let  $(M, \langle \cdot, \cdot \rangle)$  be a  $d$ -dimensional open Riemannian manifold with smooth boundary  $(\partial M, \langle \cdot, \cdot \rangle_{\partial})$ . We consider a Markov process on  $\bar{M}$  as follows:

- (1) Starting from a point in  $M$  it moves as a diffusion process in  $M$  until hits the boundary  $\partial M$ ; if the starting point is on the boundary, the hitting time is 0.

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- (2) From the hitting time to  $\partial M$ , it stays at the hitting point (i.e. without boundary diffusion), or it moves as another diffusion process on  $\partial M$  (i.e. with boundary diffusion), until a random time determined by the strength of reflection.
- (3) At the random time, it is reflected into  $M$  and moves as the diffusion in  $M$  again until hits the boundary, and repeatedly.

This process is called a sticky-reflected diffusion process, or diffusion process with Wentzell's boundary condition since the study goes back to Wentzell [29], and has been used to describe interacting particle systems with singular boundary or zero-range pair interaction, see [1, 7, 11, 19] and references therein. Rigorous constructions of sticky-reflected diffusion processes were presented in [16, 17, 23, 28] for  $M$  being a special domain (e.g. ball) in  $\mathbb{R}^d$ , and in [4, 22] for more general domains. See [8, 15] for the study by using Dirichlet forms.

Recently, optimal constants in the Poincaré and log-Sobolev inequalities have been estimated in [2, 3] for the sticky-reflected (weighted) Brownian motions on  $\bar{M}$ , which extend the corresponding results derived in [18] for strictly convex manifolds with positive curvature. In this paper, we study the super and weak Poincaré inequalities, which were introduced in [25] and [20] respectively, for the sticky-reflected diffusion processes.

Let  $\Lambda$  and  $\Lambda_\partial$  be the volume measures on  $\bar{M}$  and  $\partial M$  respectively. Let  $V \in C^1(M)$  and  $W \in C^1(\partial M)$  be such that

$$Z_V := \int_M e^V d\Lambda < \infty, \quad Z_W^\partial := \int_{\partial M} e^W d\Lambda_\partial < \infty.$$

Then

$$\mu_V(dx) := \frac{1}{Z_V} e^{V(x)} \Lambda(dx), \quad \mu_W^\partial(dx) := \frac{1_{\partial M}(x)}{Z_W^\partial} e^{W(x)} \Lambda_\partial(dx)$$

are probability measures on  $\bar{M}$ , where  $\mu_W^\partial$  is fully supported on the boundary  $\partial M$ .

For two constants  $\delta \in [0, \infty)$ ,  $\gamma \in (0, \infty)$ , we consider the operator

$$\mathcal{L} := 1_M(\Delta + \nabla V) + 1_{\partial M}(\delta[\Delta^\partial + \nabla^\partial W] + \gamma e^{V-W} N),$$

where  $\Delta$  and  $\nabla$  are the Laplacian and gradient operators in  $M$ ,  $\Delta^\partial$  and  $\nabla^\partial$  are the corresponding ones on  $\partial M$ , and  $N$  is the unit inward normal vector field on  $\partial M$ . The diffusion process generated by  $\mathcal{L}$  is called sticky-reflected diffusions with inside diffusion generated by  $\Delta + \nabla V$  and boundary diffusion generated by  $\delta(\Delta^\partial + \nabla^\partial W)$ . The constant  $\gamma > 0$  measures the strength of reflection, and the model converges to the reflected diffusion process as  $\gamma \rightarrow \infty$ . When  $\delta = 0$ , the process is called sticky-reflected diffusion process without boundary diffusion, and if moreover  $\gamma = 0$  it becomes the diffusion with absorbing (i.e. killed) boundary.

To formulate the associated Dirichlet form, let

$$(1.1) \quad \theta := \frac{\gamma Z_W^\partial}{\gamma Z_W^\partial + Z_V} \in (0, 1).$$

Then the associate invariant probability measure for the sticky-reflected diffusion process is the following convex combination of  $\mu_V$  and  $\mu_W^\partial$ :

$$\mu := \theta\mu_V + (1 - \theta)\mu_W^\partial.$$

Indeed, by the integration by parts formula, for any  $f, g \in C_0^2(\bar{M})$ , the class of  $C^2$ -functions on  $\bar{M}$  with compact support, we have

$$-\int_{\bar{M}} (f\mathcal{L}g)d\mu = \mathcal{E}_\delta(f, g) := \int_{\bar{M}} \{\langle \nabla f, \nabla g \rangle + \delta \langle \nabla^\partial f, \nabla^\partial g \rangle_\partial\} d\mu.$$

Then it is standard that the form  $(\mathcal{E}_\delta, C_0^2(\bar{M}))$  is closable and its closure is a regular symmetric local Dirichlet form in  $L^2(\mu)$ , so that it associates with a unique diffusion process on  $\bar{M}$  with generator  $\mathcal{L}$ , see [10]. In particular, when  $M$  is a smooth domain in  $\mathbb{R}^d$ , the sticky-reflected diffusion process can be constructed by solving the SDE

$$\begin{aligned} dX_t = & 1_M(X_t) \left\{ \sqrt{2}dB_t + \nabla V(X_t)dt \right\} \\ & + 1_{\partial M}(X_t) \left\{ \sqrt{2\delta}P(X_t) \circ dB_t + \delta \nabla^\partial W(X_t)dt + \gamma e^{V-W}N(X_t)dt \right\}, \quad t \geq 0, \end{aligned}$$

where  $B_t$  is the  $d$ -dimensional Brownian motion,  $dB_t$  and  $\circ dB_t$  are Itô's and Stratonovich's differentials respectively,  $N$  is the inward unit normal vector field on  $\partial M$ , and for each  $x \in \partial M$ ,

$$P(x) : \mathbb{R}^d \rightarrow T_x \partial M$$

is the orthogonal projection operator. According to [15],  $\theta$  in (1.1) is the average time for  $X_t$  staying in  $M$ :

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t 1_M(X_s)ds = \theta.$$

On the other hand, functional inequalities are power tools in the study of Markov processes, for instances, the Sobolev/Nash type inequality characterizes heat kernel estimates (see e.g. [5]), Gross' log-Sobolev inequality [12, 13] describes the hypercontractivity and exponential ergodicity in entropy, the Poincaré (spectral gap) inequality is equivalent to the exponential ergodicity in  $L^2$ , the super Poincaré inequality introduced in [25] is equivalent to the empty of essential spectrum and uniform integrability of semigroup, and the weak Poincaré inequality introduced in [20] describes general ergodicity rate which is slower than exponential. Recently, by using log-Sobolev inequalities, L. Gorss [14] showed the invariance of intrinsic ultracontractivity for Schrödinger operators under a class of unbounded perturbations, so that the corresponding result in [6] under bounded perturbations is improved.

In the following, we simply denote  $\nu(f) = \int_{\bar{M}} f d\mu$  for a measure  $\nu$  on  $\bar{M}$  and a function  $f \in L^1(\nu)$ . Let

$$|\nabla f|^2 := \langle \nabla f, \nabla f \rangle, \quad |\nabla^\partial f|_\partial^2 := \langle \nabla^\partial f, \nabla^\partial f \rangle_\partial.$$

We have

$$\mathcal{E}_\delta(f, f) = \theta \mu_V(|\nabla f|^2) + (1 - \theta) \delta \mu_W^\partial(|\nabla^\partial f|_\partial^2).$$

In this paper, we investigate the super Poincaré inequality

$$(1.2) \quad \mu(f^2) \leq r \mathcal{E}_\delta(f, f) + \beta(r) \mu(|f|)^2, \quad r > 0, \quad f \in C_b^1(\bar{M})$$

and the weak Poincaré inequality

$$(1.3) \quad \mu(f^2) \leq \alpha(r) \mathcal{E}_\delta(f, f) + r \|f\|_\infty^2, \quad r > 0, \quad f \in C_b^1(\bar{M}), \quad \mu(f) = 0,$$

where  $\beta : (0, \infty) \rightarrow (0, \infty)$  is crucial to estimate the associated diffusion semigroup and higher order eigenvalues of the generator, see [25]; and  $\alpha : (0, \infty) \rightarrow (0, \infty)$  corresponds to the convergence rate of the associated Markov semigroup, see [20].

We will estimate the smallest rate functions  $\alpha$  and  $\beta$  for the above introduced sticky-reflected diffusion process:

$$\begin{aligned} \beta(r) &:= \sup \{ \mu(f^2) - r \mathcal{E}_\delta(f, f) : f \in C_b^1(\bar{M}), \mu(|f|) = 1 \}, \\ \alpha(r) &:= \sup \{ (\mu(f^2) - r \|f\|_\infty^2)^+ : f \in C_b^1(\bar{M}), \mu(f) = 0, \mathcal{E}_\delta(f, f) = 1 \}, \quad r > 0. \end{aligned}$$

It is easy to see that  $\alpha(r) = 0$  for  $r \geq 1$ , and  $\beta(r) \geq 1$  with  $\beta(r) = 1$  for  $r \geq C(P)$ , where  $C(p)$  is the smallest positive constant such that the Poincaré inequality holds:

$$\mu(f^2) \leq C(p) \mathcal{E}_\delta(f, f) + \mu(f)^2, \quad f \in C_b^1(\bar{M}).$$

So, it suffices to estimate  $\alpha(r)$  and  $\beta(r)$  for small  $r > 0$ , in particular, to characterize the rates of  $\alpha(r) \uparrow \infty$  and  $\beta(r) \uparrow \infty$  as  $r \downarrow 0$ .

Since the super/weak Poincaré inequalities have been well studied for elliptic diffusions on manifolds with Neumann boundary or without boundary, see [26], we will estimate  $\alpha(r)$  and  $\beta(r)$  using the rate functions  $\beta_V, \beta_W^\partial, \alpha_V$  and  $\alpha_W^\partial$  in the following functional inequalities:

$$(1.4) \quad \mu_V(f^2) \leq r \mu_V(|\nabla f|^2) + \beta_V(r) \mu_V(|f|)^2, \quad r > 0, \quad f \in C_b^1(\bar{M}),$$

$$(1.5) \quad \mu_W^\partial(f^2) \leq r \mu_W^\partial(|\nabla^\partial f|_\partial^2) + \beta_W^\partial(r) \mu_W^\partial(|f|)^2, \quad r > 0, \quad f \in C_b^1(\partial M),$$

$$(1.6) \quad \mu_V(f^2) \leq \alpha_V(r) \mu_V(|\nabla f|^2) + r \|f\|_\infty^2, \quad r > 0, \quad f \in C_b^1(\bar{M}), \quad \mu_V(f) = 0,$$

$$(1.7) \quad \mu_W^\partial(f^2) \leq \alpha_W(r) \mu_W^\partial(|\nabla^\partial f|_\partial^2) + r \|f\|_\infty^2, \quad r > 0, \quad f \in C_b^1(\partial M), \quad \mu_W^\partial(f) = 0.$$

In Section 2, we recall some known results on super and weak Poincaré inequalities, which will be applied to the sticky-reflected diffusions. In Section 3 and Section 4 we establish these inequalities for  $\mathcal{E}_\delta$  with  $\delta > 0$  (the case with boundary diffusion), and  $\delta = 0$  (the case without boundary diffusion), respectively. Some examples are presented to illustrate our main results.

## 2 A review on super and weak Poincaré inequalities

Let  $(E, \mathcal{F}, \mu)$  be a separable probability space, let  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  be a Dirichlet form on  $L^2(\mu)$ , let  $(L, \mathcal{D}(L))$  and  $P_t := e^{tL}$  be the associated generator and (sub-) Markov semigroup. For any  $p, q \in [1, \infty]$ , let  $\|\cdot\|_{p \rightarrow q}$  denote the operator norm from  $L^p(\mu)$  to  $L^q(\mu)$ . In this section, we summarize some results on super and weak Poincaré inequalities, where detailed proofs can be found in the book [26] or [25, 20].

### 2.1 Super Poincaré inequality

We say that  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  satisfies the super Poincaré inequality, if there exists a (decreasing) function  $\beta : (0, \infty) \rightarrow (0, \infty)$  such that

$$(2.1) \quad \mu(f^2) \leq r\mathcal{E}(f, f) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}).$$

This inequality was introduced in [25] to study the essential spectrum of the generator  $L$ , and has been further used to estimate the semigroup  $P_t$ .

We first introduce the link between (2.1) and the uniform integrability of  $P_t$ .

**Theorem 2.1** ([26], Lemma 3.3.5, Theorem 3.3.6). *The following assertions hold.*

(1) *The inequality (2.1) holds if and only if*

$$\mu(|P_t f|^2) \leq e^{-2rt}\mu(f^2) + \beta(r^{-1})(1 - e^{-2rt})\mu(|f|)^2, \quad r > 0, t \geq 0, f \in L^2(\mu).$$

(2) *Let*

$$\Gamma_t(s) = \inf\{r \geq 0 : \beta(1/r)(e^{2rt} - 1) \geq s^2\}, \quad s \geq 0.$$

*If (2.1) holds, then*

$$\sup_{\mu(f^2)=1} \mu((P_t f)^2 1_{\{|P_t f|>r\}}) \leq \exp[-2t\Gamma_t(\varepsilon r)] / (1 - \varepsilon)^2, \quad r > 0, \varepsilon \in (0, 1).$$

(3) *If  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is symmetric and there exists  $t > 0$  such that*

$$\phi_t(s) := \sup\{\mu((P_t f)^2 1_{\{|P_t f|>s\}}) : \mu(f^2) = 1\} \rightarrow 0$$

*as  $s \rightarrow \infty$ , then (2.1) holds with*

$$\beta(r) = \frac{r[\phi_t^{-1}(e^{-2t/r}/2)]^2 e^{2t/r}}{4t}, \quad r > 0,$$

*where  $\phi_t^{-1}(r) = \inf\{s > 0 : \phi_t(s) \leq r\}$ .*

**Corollary 2.2** ([26], Corollary 3.3.10). *Let  $\delta \in (0, 1]$ . If (2.1) holds with*

$$\beta(r) = \exp[c(1 + r^{-1/\delta})], \quad r > 0$$

*for some  $c > 0$ , then there exists decreasing  $C : (0, \infty) \rightarrow (0, \infty)$  such that*

$$(2.2) \quad \sup_{\mu(f^2)=1} \int_E (P_t f)^2 \exp \left\{ C_t [\log(1 + (P_t f)^2)]^\delta \right\} d\mu < \infty, \quad t \in (0, \infty).$$

*On the other hand, if  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is symmetric and (2.2) holds for some  $t > 0$  and  $C_t > 0$ , then (2.1) holds with  $\beta(r) = \exp[c(1 + r^{-1/\delta})]$  for some  $c > 0$ .*

Next, we consider three boundedness properties of  $P_t$  by using (2.1).

**Definition 2.1.** Let  $(E, \mathcal{F}, \mu)$  be a measure space and  $P_t$  a semigroup on  $L^2(\mu)$  which is bounded on  $L^p(\mu)$  for all  $p \in [1, \infty]$ .  $P_t$  is called *hyperbounded* if  $\|P_t\|_{2 \rightarrow 4} < \infty$  for some  $t > 0$ ; *superbounded* if  $\|P_t\|_{2 \rightarrow 4} < \infty$  for all  $t > 0$ ; and *ultrabounded* if  $\|P_t\|_{1 \rightarrow \infty} < \infty$  for all  $t > 0$ .

By Riesz-Thorin interpolation theorem,  $P_t$  is hyperbounded if and only if  $\|P_t\|_{p \rightarrow q} < \infty$  holds for some constants  $t > 0$  and  $1 < p < q < \infty$ . It is clear that the superboundedness implies the hyperboundedness, and they are implied by the ultraboundedness.

**Theorem 2.3** ([26], Theorems 3.3.13, 3.3.14, 3.3.15). *We have the following assertions on the hyper/super/ultraboundedness of  $P_t$ .*

- (1) *If (2.1) holds with  $\beta(r) = \exp[c(1 + r^{-1})]$  for some  $c > 0$ , then  $P_t$  is hyperbounded, and the converse result holds if  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is symmetric.*
- (2) *If (2.1) holds for some  $\beta$  with  $\lim_{r \rightarrow 0} r \log \beta(r) = 0$ , then  $P_t$  is superbounded. Conversely, if  $P_t$  is superbounded and  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is symmetric, then (2.1) holds for*

$$\beta(r) := \inf_{s \leq r} \left( \frac{s}{3e} \wedge 2 \right) \inf_{t > 0} \frac{(1 + \|P_t\|_{2 \rightarrow 4}^2)^2}{t} \exp[6t/s], \quad r > 0,$$

*which satisfies  $\lim_{r \rightarrow 0} r \log \beta(r) = 0$ .*

- (3) *If (2.1) holds with  $\beta$  satisfying*

$$\Psi(t) := \int_t^\infty \frac{\beta^{-1}(r)}{r} dr < \infty, \quad t > \inf \beta,$$

*then  $P_t$  is ultrabounded with*

$$\|P_t\|_{1 \rightarrow \infty} \leq \inf_{\varepsilon \in (0, 1)} \max \{ \varepsilon^{-1} \inf \beta, \Psi^{-1}((1 - \varepsilon)t) \}, \quad t > 0,$$

*where  $\Psi^{-1}(t) := \inf \{ r \geq \inf \beta : \Psi(r) \leq t \}$ . On the other hand, if  $P_t$  is ultra-bounded, then (2.1) holds for*

$$\beta(r) = \inf_{s \leq r, t > 0} \frac{s \|P_t\|_{1 \rightarrow \infty}}{t} \exp [t/s - 1], \quad r > 0.$$

The following is a direct consequence of Theorem 2.3(3).

**Corollary 2.4** ([26], Theorem 3.3.15). *We have the following correspondence between  $\beta$  and  $\|P_t\|_{1 \rightarrow \infty}$ .*

(1) *Let  $\delta > 1$ . (2.1) with  $\beta(r) = \exp[c(1 + r^{-1/\delta})]$  for some  $c > 0$  is equivalent to*

$$\|P_t\|_{1 \rightarrow \infty} \leq \exp[\lambda(1 + t^{-1/(\delta-1)})], \quad t > 0,$$

*for some  $\lambda > 0$ .*

(2) *Let  $p > 0$ . (2.1) with  $\beta(r) = c(1 + r^{-p/2})$  for some  $c > 0$  is equivalent to*

$$(2.3) \quad \|P_t\|_{1 \rightarrow \infty} \leq \lambda(1 + t^{-p/2})$$

*for some  $\lambda > 0$  and all  $t > 0$ .*

## 2.2 Weak Poincaré inequality

In this part, we assume that the Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is irreducible and conservative, i.e.  $1 \in \mathcal{D}(\mathcal{E})$  with  $\mathcal{E}(1, 1) = 0$ , and  $f \in \mathcal{D}(\mathcal{E})$  with  $\mathcal{E}(f, f) = 0$  implies that  $f$  is constant. In this case, we have

$$\lim_{t \rightarrow \infty} \mu(|P_t f - \mu(f)|^2) = 0, \quad f \in L^2(\mu).$$

In the following we introduce the link between the convergence rate for  $\|P_t - \mu\|_{\infty \rightarrow 2} \rightarrow 0$  as  $t \rightarrow \infty$ , and the function  $\alpha : (0, \infty) \rightarrow (0, \infty)$  in the weak Poincaré inequality

$$(2.4) \quad \mu(f^2) \leq \alpha(r)\mathcal{E}(f, f) + r\|f\|_\infty^2, \quad f \in \mathcal{D}(\mathcal{E}), \quad \mu(f) = 0.$$

**Theorem 2.5** ([20], Theorems 2.1 and 2.3). *If (2.4) holds, then*

$$\|P_t - \mu\|_{\infty \rightarrow 2}^2 \leq \xi(t), \quad t \geq 0$$

*holds for*

$$\xi(t) := \inf \left\{ 2r : r > 0, -\frac{1}{2}\alpha(r) \log r \leq t \right\}$$

*which goes to 0 as  $t \uparrow \infty$ .*

*On the other hand, if  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is symmetric and*

$$\xi(t) := \|P_t - \mu\|_{\infty \rightarrow 2}^2 \rightarrow 0 \text{ as } t \rightarrow \infty,$$

*then (2.4) holds with*

$$\alpha(r) = 2r \inf_{s>0} \frac{1}{s} \xi^{-1}(s \exp[1 - s/r]), \quad r > 0,$$

*where  $\xi^{-1}(t) := \inf\{r > 0 : \xi(r) \leq t\}$ .*

When  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is symmetric, we have  $\|P_t - \mu\|_{\infty \rightarrow 1} = \|P_t - \mu\|_{\infty \rightarrow 2}^2$ , so that the following is a consequence of Theorem 2.5.

**Corollary 2.6** ([20], Corollary 2.4). *Let  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  be symmetric, then the following assertions hold.*

- (1) *Let  $\varepsilon \in (0, 1)$ . Then (2.4) holds with*

$$\alpha(r) = c_1 + c_2 [\log(1 + r^{-1})]^{(1-\varepsilon)/\varepsilon}$$

*for some constants  $c_1, c_2 \in (0, \infty)$  if and only if*

$$\|P_t - \mu\|_{\infty \rightarrow 1} \leq \exp[c'_1 - c'_2 t^\varepsilon]$$

*holds for some constants  $c'_1, c'_2 \in (0, \infty)$ .*

- (2) *Let  $p, q \in (1, \infty)$  with  $p^{-1} + q^{-1} = 1$ . Then (2.4) holds with*

$$\alpha(r) = cr^{1-p}$$

*for some constant  $c \in (0, \infty)$  if and only if*

$$\|P_t - \mu\|_{\infty \rightarrow 1} \leq c't^{1-q}$$

*for some constant  $c' \in (0, \infty)$ .*

- (3) *Let  $p > 0$ . Then (2.4) holds with*

$$\alpha(r) = \exp[c(1 + r^{-1/p})]$$

*for some  $c \in (0, \infty)$  if and only if*

$$\|P_t - \mu\|_{\infty \rightarrow 1} \leq c'[\log(1 + t)]^{-p}$$

*holds for some  $c' \in (0, \infty)$ .*

### 3 The case with boundary diffusion: $\delta > 0$

When  $\delta > 0$ , it is easy to estimate  $\beta(r)$  by using  $\beta_V(r)$  and  $\beta_W^\partial(r)$ . However, to estimate  $\alpha(r)$  using  $\alpha_V(r)$  and  $\alpha_W^\partial(r)$ , we need the following assumption, where the function  $h$  can be constructed by using the distance function to the boundary, see Example 3.1 and Example 4.1 for details.

- (A) There exists a function  $h \in C^2(\bar{M})$  such that

$$Nh|_{\partial M} = 1, \quad \|\nabla h\|_{L^2(\mu)} + \|1_{\partial M}(W - V)^+\|_\infty + \|(L_V h)^-\|_{L^2(\mu_V)} < \infty,$$

where  $L_V := \Delta + \nabla V$  on  $M$ .

Under **(A)**, we have

$$\begin{aligned} C_0 &:= \frac{Z_V e^{\|1_{\partial M}(W-V)^+\|_\infty}}{Z_W^\partial} < \infty, \\ C_1 &:= \|(L_V h)^-\|_{L^2(\mu_V)} < \infty, \\ C_2 &:= \|\nabla h\|_{L^2(\mu_V)} < \infty. \end{aligned}$$

**Theorem 3.1.** *Let  $\delta > 0$ .*

(1) *We have*

$$(3.1) \quad \beta(r) \leq \max \left\{ \frac{\beta_V(r)}{\theta}, \frac{\beta_W^\partial(\delta r)}{1-\theta} \right\}, \quad r > 0.$$

(2) *If **(A)** holds, then for any  $r > 0$ ,*

$$(3.2) \quad \alpha(r) \leq \inf \left\{ \max \left\{ (1 + C_0^2 C_1^2) \alpha_V(s_1) + \frac{1-\theta}{\theta} C_0^2 C_1^2, \frac{1}{\delta} \alpha_W^\partial(s_2) \right\} : \right. \\ \left. s_1, s_2 > 0, \theta(1 + C_0^2 C_1^2) s_1 + (1-\theta) s_2 \leq \frac{r}{4} \right\}.$$

*In particular, taking*

$$s_1 = s_2 := s(r) := \frac{r}{4 + 4\theta C_0^2 C_1^2}, \quad r > 0,$$

*we find a constant  $c > 1$  such that*

$$(3.3) \quad \begin{aligned} \alpha(r) &\leq \max \left\{ (1 + C_0^2 C_1^2) \alpha_V(s(r)) + \frac{1-\theta}{\theta} C_0^2 C_1^2, \frac{1}{\delta} \alpha_W^\partial(s(r)) \right\} \\ &\leq c + c \alpha_V(r/c) + c \alpha_W^\partial(r/c), \quad r > 0. \end{aligned}$$

*Proof.* (1) For any  $f \in C_b^1(\bar{M})$ , (1.4) and (1.5) imply

$$\begin{aligned} \mu(f^2) &= \theta \mu_V(f^2) + (1-\theta) \mu_W^\partial(f^2) \\ &\leq \theta r \mu_V(|\nabla f|^2) + \theta \beta_V(r) \mu_V(|f|)^2 + (1-\theta) \delta r \mu_W^\partial(|\nabla^\partial f|_\partial^2) + (1-\theta) \beta_W^\partial(\delta r) \mu_W^\partial(|f|)^2 \\ &\leq r [\theta \mu_V(|\nabla f|^2) + (1-\theta) \delta \mu_W^\partial(|\nabla^\partial f|_\partial^2)] \\ &\quad + \left( \frac{\beta_V(r)}{\theta} \vee \frac{\beta_W^\partial(\delta r)}{1-\theta} \right) [\theta \mu_V(|f|) + (1-\theta) \mu_W^\partial(|f|)]^2 \\ &= r \mathcal{E}_\delta(f, f) + \left( \frac{\beta_V(r)}{\theta} \vee \frac{\beta_W^\partial(\delta r)}{1-\theta} \right) \mu(|f|)^2, \quad r > 0. \end{aligned}$$

So, (3.1) holds.

(2) Let  $f \in C_b^1(\bar{M})$  such that

$$(3.4) \quad \mu(f) := \theta\mu_V(f) + (1 - \theta)\mu_W^\partial(f) = 0.$$

Then

$$\begin{aligned} & \theta\mu_V(f)^2 + (1 - \theta)\mu_W^\partial(f)^2 \\ &= \mu(f)^2 + \theta(1 - \theta)|\mu_V(f) - \mu_W^\partial(f)|^2 \\ &= \theta(1 - \theta)|\mu_V(f) - \mu_W^\partial(f)|^2. \end{aligned}$$

Combining this with (1.6) and (1.7), we derive

$$\begin{aligned} (3.5) \quad & \mu(f^2) = \theta\mu_V(f^2) + (1 - \theta)\mu_W^\partial(f^2) \\ &= \theta\mu_V(|f - \mu_V(f)|^2) + (1 - \theta)\mu_W^\partial(|f - \mu_W^\partial(f)|^2) + \theta\mu_V(f)^2 + (1 - \theta)\mu_W^\partial(f)^2 \\ &\leq \theta\alpha_V(s_1)\mu_V(|\nabla f|^2) + (1 - \theta)\alpha_W^\partial(s_2)\mu_W^\partial(|\nabla^\partial f|_\partial^2) \\ &\quad + (\theta s_1 + (1 - \theta)s_2)(\|f - \mu_V(f)\|_\infty^2 \vee \|1_{\partial M}(f - \mu_W^\partial(f))\|_\infty^2) \\ &\quad + \theta(1 - \theta)|\mu_V(f) - \mu_W^\partial(f)|^2. \end{aligned}$$

By  $Nh|_{\partial M} = 1$  and the integration by parts formula, we have

$$\begin{aligned} & |\mu_V(f) - \mu_W^\partial(f)| = |\mu_W^\partial(f - \mu_V(f))| \\ &\leq \frac{e^{\|1_{\partial M}(W-V)^+\|_\infty}}{Z_W^\partial} \int_{\partial M} |f - \mu_V(f)| e^V d\Lambda_\partial \\ &= \frac{e^{\|1_{\partial M}(W-V)^+\|_\infty}}{Z_W^\partial} \int_{\partial M} |f - \mu_V(f)| (Nh) e^V d\Lambda_\partial \\ &= -C_0 \int_M \left[ |f - \mu_V(f)| L_V h + \langle \nabla |f - \mu_V(f)|, \nabla h \rangle \right] d\mu_V \\ &\leq C_0 C_1 \|f - \mu_V(f)\|_{L^2(\mu_V)} + C_0 C_2 \|\nabla f\|_{L^2(\mu_V)}. \end{aligned}$$

Noting that

$$\theta(1 - \theta)(a + b)^2 \leq \theta a^2 + (1 - \theta)b^2, \quad a, b \geq 0,$$

this implies

$$(3.6) \quad \theta(1 - \theta)|\mu_V(f) - \mu_W^\partial(f)|^2 \leq \theta C_0^2 C_1^2 \mu_V(|f - \mu_V(f)|^2) + (1 - \theta) C_0^2 C_2^2 \mu_V(|\nabla f|^2).$$

Combining this with (3.5) and

$$\|f - \mu_V(f)\|_\infty \vee \|f - \mu_W^\partial(f)\|_\infty \leq 2\|f\|_\infty,$$

for any  $s_1, s_2 > 0$  and  $f \in C_b^1(\bar{M})$  with  $\mu(f) = 0$  we have

$$\begin{aligned} \mu(f)^2 &\leq \theta \alpha_V(s_1) \mu_V(|\nabla f|^2) + (1 - \theta) \alpha_W^\partial(s_2) \mu_W^\partial(|\nabla^\partial f|_\partial^2) \\ &\quad + 4(\theta s_1 + (1 - \theta)s_2) \|f\|_\infty^2 \\ &\quad + \theta C_0^2 C_1^2 \mu_V(|f - \mu_V(f)|^2) + (1 - \theta) C_0^2 C_2^2 \mu_V(|\nabla f|^2) \\ &\leq \max \left\{ (1 + C_0^2 C_1^2) \alpha_V(s_1) + \frac{1 - \theta}{\theta} C_0^2 C_1^2, \frac{1}{\delta} \alpha_W^\partial(s_2) \right\} \mathcal{E}_\delta(f, f) \\ &\quad + 4 \left[ \theta (1 + C_0^2 C_1^2) s_1 + (1 - \theta) s_2 \right] \|f\|_\infty^2. \end{aligned}$$

Hence, (3.2) holds, which implies (3.3) for the given choice of  $s_1$  and  $s_2$ .  $\square$

To illustrate the above result, we present below an example to derive sharp functional inequalities for the sticky-reflected diffusion process, where the semigroup is ultrabounded if and only if  $\tau > 2$ , hypercontractive if and only if  $\tau = 2$ ,  $L^2$ -uniformly integrable if and only if  $\tau > 1$ ,  $L^2$ -exponential ergodic if and only if  $\tau = 1$ , and sub-exponential ergodic when  $\tau \in (0, 1)$ . For examples with weaker convergence rate, for instance the algebraic or logarithmic convergence, one may take  $V$  and  $W$  with slower growth as in [20, Example 1.4].

**Example 3.1.** Let  $M := (0, \infty) \times \mathbb{R}^d$  for some  $d \geq 1$ , and let  $W(x) = V(x) = -|x|^\tau$  for some constant  $\tau > 0$ . We have  $\partial M = \{0\} \times \mathbb{R}^d \equiv \mathbb{R}^d$ .

(1) It is known that (1.4) holds for some  $\beta_V$  if and only if  $\tau > 1$ , and in this case the exact order of  $\beta_V(r)$  and  $\beta_W^\partial(r)$  is  $e^{cr^{-\frac{\tau}{2(\tau-1)}}}$  for small  $r > 0$ , see [25, Corollary 2.5]. So, by Theorem 3.1(1), there exists a constant  $c_1 > 0$  such that the sticky-reflected diffusion process on  $\bar{M} := [0, \infty) \times \mathbb{R}^d$  with  $\partial M = \{0\} \times \mathbb{R}^d$  satisfies

$$\mu(f^2) \leq r \mathcal{E}_\delta(f, f) + \exp \left[ c_1 + c_1 r^{-\frac{\tau}{2(\tau-1)}} \right] \mu(|f|)^2, \quad r > 0, \quad f \in C_b^1(\bar{M}),$$

that is,

$$(3.7) \quad \beta(r) \leq \exp \left[ c_1 + c_1 r^{-\frac{\tau}{2(\tau-1)}} \right] \mu(|f|)^2, \quad r > 0.$$

This is sharp for small  $r > 0$ , since it implies

$$\mu_V(f^2) \leq r \mu_V(|\nabla f|^2) + \exp \left[ c_1 + c_1 r^{-\frac{\tau}{2(\tau-1)}} \right] \mu_V(|f|)^2, \quad f \in C_0^1(M),$$

and this inequality is sharp as shown in [25, Corollary 2.5]. Consequently:

- (a) If  $\tau \in (1, 2)$ , then by Corollary 2.2, the Markov semigroup  $P_t$  associated with  $\mathcal{E}_\delta$  is  $L^2$ -uniformly integrable with

$$\sup_{\mu(f^2)=1} \int_{\bar{M}} (P_t f)^2 \exp \left[ C_t \{ \log(1 + (P_t f)^2) \}^{\frac{2(\tau-1)}{\tau}} \right] d\mu < \infty, \quad t > 0$$

for some  $C : (0, \infty) \rightarrow (0, \infty)$ .

(b) If  $\tau = 2$ , then by [26, Theorem 3.3.13(1)] the defective log-Sobolev inequality holds, which together with [27, Corollary 1.2] implies the strict log-Sobolev inequality, since it is well known that the defective log-Sobolev inequality together with the Poincaré inequality implies the strict log-Sobolev inequality. So, according to [12] or [13], in this case  $P_t$  is hypercontractive, i.e.  $\|P_t\|_{L^2(\mu) \rightarrow L^4(\mu)} \leq 1$  holds for large enough  $t > 0$ .

(c) If  $\tau > 2$ , then by Corollary 2.4,  $P_t$  is ultrabounded with

$$\|P_t\|_{L^1(\mu) \rightarrow L^\infty(\mu)} \leq e^{c+ct^{-\frac{\tau}{\tau-2}}}, \quad t > 0$$

for some constant  $c > 0$ .

(2) Next,  $\mu_V$  and  $\mu_W^\partial$  satisfy the Poincaré inequality if and only if  $\tau = 1$ , see [24], so that in this case  $\mathcal{E}_\delta$  satisfies the Poincaré inequality as well, due to Theorem 3.1(2) with  $h(r, x) := r$  for  $(r, x) \in \bar{M}$  and bounded  $\alpha_V$  and  $\alpha_W^\partial$ . Consequently, when  $\tau = 1$ ,

$$\|P_t - \mu\|_{L^2(\mu)} \leq e^{-\lambda t}, \quad t \geq 0$$

holds for some constant  $\lambda > 0$ .

(3) Moreover, when  $\tau \in (0, 1)$ , by [20, Example 1.4(c)] which applies also to  $(0, \infty) \times \mathbb{R}^d$  in place of  $\mathbb{R}^{d+1}$ ,  $\alpha_V(r)$  and  $\alpha_W^\partial(r)$  behaves as  $[\log r^{-1}]^{\frac{4(1-\tau)}{\tau}}$  for small  $r > 0$ , so that by Theorem 3.1(2) for  $h(r, x) := r$  for  $(r, x) \in \bar{M}$ , there exists a constant  $c_2 > 0$  such that

$$\mu(f^2) \leq [\log(c_2 + r^{-1})]^{\frac{4(1-\tau)}{\tau}} \mathcal{E}_\delta(f, f) + r\|f\|_\infty^2, \quad r > 0, \quad f \in C_b^1(\bar{M}), \quad \mu(f) = 0.$$

Consequently, by Corollary 2.6, the Markov semigroup  $P_t$  associated with  $\mathcal{E}_\delta$  is sub-exponential ergodic

$$\|P_t - \mu\|_{L^\infty(\mu) \rightarrow L^2(\mu)} \leq e^{-ct^{\frac{\tau}{4-3\tau}}}, \quad t > 0$$

for some constant  $c > 0$ .

## 4 The case without boundary diffusion: $\delta = 0$

When  $\delta = 0$ , the Dirichlet form for the sticky-reflected diffusion reduces to

$$\mathcal{E}_\delta(f, f) = \mathcal{E}_0(f, f) := \mu(|\nabla f|^2) = \theta \mu_V(|\nabla f|^2), \quad f \in C_b^1(\bar{M}).$$

So, to establish the weak and super Poincaré inequalities, we need to bound the  $L^2(\mu_W^\partial)$ -norm using the Neumann Dirichlet form  $\mu_V(|\nabla f|^2)$ . To this end, we need the following assumption, which is slightly stronger than **(A)**.

(B) There exists a function  $h \in C^2(\bar{M})$  such that

$$Nh|_{\partial M} = 1, \quad \|\nabla h\|_\infty + \|1_{\partial M}(W - V)^+\|_\infty + \|(L_V h)^-\|_\infty < \infty,$$

where  $L_V := \Delta + \nabla V$  on  $M$ .

Under (B), besides  $C_0 := \frac{Z_V e^{\|1_{\partial M}(W-V)^+\|_\infty}}{Z_W^\partial} < \infty$ , we have

$$\bar{C}_1 := \|(L_V h)^-\|_\infty < \infty, \quad \bar{C}_2 := \|\nabla h\|_\infty < \infty.$$

**Theorem 4.1.** Assume (B) and let  $\delta = 0$ . Then

$$(4.1) \quad \beta(r) \leq \left( \frac{2}{\theta r} C_0^2 \bar{C}_2^2 + C_0 \bar{C}_1 \right) \beta_V \left( \frac{\theta^2 r^2}{4C_0^2 \bar{C}_2^2 + 2\theta C_0 \bar{C}_1 r} \right), \quad r > 0,$$

$$(4.2) \quad \alpha(r) \leq \frac{A}{\theta} \alpha_V \left( \frac{r}{4A} \right) + \frac{B}{\theta},$$

where

$$\begin{aligned} A &:= \theta + (1 - \theta)(C_0^2 \bar{C}_2^2 + C_0 \bar{C}_1)((1 - \theta)C_0^2 C_1^2 + 1), \\ B &:= 1 - \theta + \frac{(1 - \theta)^2}{\theta} C_0^2 C_2^2. \end{aligned}$$

*Proof.* (a) By  $Nh|_{\partial M} = 1$ , and the integration by parts formula, we obtain

$$\begin{aligned} (4.3) \quad \mu_W^\partial(f^2) &= \mu_W^\partial(f^2 Nh) \leq \frac{e^{\|1_{\partial M}(W-V)^+\|_\infty}}{Z_W^\partial} \int_{\partial M} f^2(Nh) e^V d\Lambda_\partial \\ &= -C_0 \mu_V(f^2 L_V h + \langle \nabla f^2, \nabla h \rangle) \\ &\leq C_0 \bar{C}_1 \mu_V(f^2) + 2C_0 \bar{C}_2 \sqrt{\mu_V(f^2) \mu_V(|\nabla f|^2)} \\ &\leq s \mu_V(|\nabla f|^2) + (s^{-1} C_0^2 \bar{C}_2^2 + C_0 \bar{C}_1) \mu_V(f^2), \quad s > 0. \end{aligned}$$

Combining this with (1.4), we obtain

$$\begin{aligned} \mu_W^\partial(f^2) &\leq 2s \mu_V(|\nabla f|^2) + (s^{-1} C_0^2 \bar{C}_2^2 + C_0 \bar{C}_1) \beta_V \left( \frac{s}{s^{-1} C_0^2 \bar{C}_2^2 + C_0 \bar{C}_1} \right) \mu_V(|f|)^2 \\ &= \frac{2s}{\theta} \mathcal{E}_0(f, f) + (s^{-1} C_0^2 \bar{C}_2^2 + C_0 \bar{C}_1) \beta_V \left( \frac{s^2}{C_0^2 \bar{C}_2^2 + s C_0 \bar{C}_1} \right) \mu_V(|f|)^2. \end{aligned}$$

By taking  $s = \frac{\theta}{2} r$ , we derive the estimate (4.1) on  $\beta(r)$ .

(b) Let  $f \in C_b^1(\bar{M})$  with  $\mu(f) = \theta \mu_V(f) + (1 - \theta) \mu_W^\partial(f) = 0$ . Then by the triangle inequality,

$$|\mu_V(f)| \leq |\theta \mu_V(f) + (1 - \theta) \mu_W^\partial(f)| + |(1 - \theta) \mu_V(f) - (1 - \theta) \mu_W^\partial(f)|$$

$$= (1 - \theta)|\mu_V(f) - \mu_W^\partial(f)|.$$

Combining this with (3.6) we derive

$$\mu_V(f)^2 \leq (1 - \theta)C_0^2C_1^2\mu_V(|f - \mu_V(f)|^2) + \frac{(1 - \theta)^2}{\theta}C_0^2C_2^2\mu_V(|\nabla f|^2).$$

Combining this with (1.6) and

$$\mu_V(f^2) = \mu_V(|f - \mu_V(f)|^2) + \mu_V(f)^2,$$

we obtain

$$\begin{aligned} \mu_V(f^2) &\leq (1 + (1 - \theta)C_0^2C_1^2)\mu_V(|f - \mu_V(f)|^2) + \frac{(1 - \theta)^2}{\theta}C_0^2C_2^2\mu_V(|\nabla f|^2) \\ (4.4) \quad &\leq \left[ \frac{(1 - \theta)^2}{\theta}C_0^2C_2^2 + (1 + (1 - \theta)C_0^2C_1^2)\alpha_V(s) \right] \mu_V(|\nabla f|^2) \\ &\quad + (1 + (1 - \theta)C_0^2C_1^2)s\|f - \mu_V(f)\|_\infty^2, \quad s > 0. \end{aligned}$$

On the other hand, taking  $s = 1$  in (4.3), we obtain

$$\begin{aligned} \mu(f^2) &= \theta\mu_V(f^2) + (1 - \theta)\mu_W^\partial(f^2) \\ &\leq (1 - \theta)\mu_V(|\nabla f|^2) + [\theta + (1 - \theta)(C_0\bar{C}_2^2 + C_0\bar{C}_1)]\mu_V(f^2). \end{aligned}$$

Combining this with (4.4),  $\|f - \mu_V(f)\|_\infty \leq 2\|f\|_\infty$ , and the definitions of  $A$  and  $B$ , we arrive at

$$\begin{aligned} \mu(f^2) &\leq (B + A\alpha_V(s))\mu_V(|\nabla f|^2) + As\|f - \mu_V(f)\|_\infty^2 \\ &\leq \left( \frac{B}{\theta} + \frac{A}{\theta}\alpha_V(s) \right) \mathcal{E}_0(f, f) + 4As\|f\|_\infty^2, \quad s > 0. \end{aligned}$$

Taking  $s = \frac{\tau}{4A}$  we obtain (4.2). □

By Theorem 4.1, for the model in Example 3.1 with  $\delta = 0$ , the same assertion for  $\tau \in (0, 1]$  holds as in Example 3.1, and when  $\tau > 1$  we have

$$\beta(r) \leq \exp \left[ c_1 + c_1 r^{-\frac{\tau}{\tau-1}} \right], \quad r > 0$$

for some constant  $c_1 > 0$ , which is weaker than the corresponding ones for  $\delta > 0$ .

Below we consider the sticky-reflected diffusion on a compact manifold.

**Example 4.1.** Let  $\bar{M}$  be a  $d$ -dimensional compact Riemannian manifold with smooth boundary  $\partial M$ , and let  $\delta = 0$ . Then (1.2) holds for

$$(4.5) \quad \beta(r) = c(1 \wedge r)^{-d}, \quad r > 0$$

for some constant  $c > 0$ . When  $d = 1$  (4.5) can be improved as

$$(4.6) \quad \beta(r) = c(1 \wedge r)^{-\frac{1}{2}} = \beta(r) = c(1 \wedge r)^{-\frac{d}{2}}.$$

*Proof.* (a) Since  $W$  and  $V$  are bounded as  $\bar{M}$  is compact, it suffices to consider  $W = V = 0$ . So,

$$\mu_V(dx) = \frac{\Lambda(dx)}{\Lambda(M)} =: \mu_0(dx), \quad \mu_W^\partial(dx) = \frac{\Lambda_\partial(dx)}{\Lambda_\partial(\partial M)} =: \mu_0^\partial(dx).$$

Let  $\rho_\partial$  be the Riemannian distance to the boundary  $\partial M$ . Then there exists a constant  $s_0 > 0$  such that  $\rho_\partial \in C_b^2(\partial_{s_0} M)$ , where

$$\partial_{s_0} M := \{x \in \bar{M} : \rho_\partial(x) \leq s_0\}.$$

Consider the polar coordinates on  $\partial_{s_0} M$

$$[0, s_0] \times \partial M \ni (r, z) \mapsto \exp_z[rN(z)] \in \partial_{s_0} M.$$

Then there exists a constant  $c_0 > 1$  such that the volume measure  $\Lambda$  on  $\partial_{s_0} M$  satisfies

$$(4.7) \quad c_0^{-1} dr \Lambda_\partial(dz) \leq \Lambda(dr, dz) \leq c_0 dr \Lambda_\partial(dz).$$

Choose  $\xi \in C^\infty([0, \infty); [0, 1])$  such that  $\xi' \geq 0$ ,  $\xi(0) = 1$  and  $\xi(s) = 0$  for  $s \geq s_0$ , and let

$$h(x) := \begin{cases} r\xi(r), & \text{if } x = \exp_z(rN(z)) \in \partial_{s_0} M, \\ 0, & \text{if } x \in M \setminus \partial_{s_0} M. \end{cases}$$

Then  $h \in C_b^2(\bar{M})$  with  $h|_{\partial M} = 0$  and  $Nh|_{\partial M} = 1$ . So, the assumption **(B)** holds.

Now, since  $M$  is a  $d$ -dimensional compact manifold, the classical Nash inequality with dimension  $d$  holds for the Dirichlet form  $\mu_V(|\nabla f|^2)$ , so by [25, Corollary 3.3], there exists a constant  $c_1 > 0$  such that

$$\beta_V(r) \leq c_1(1 \wedge r^{-\frac{d}{2}}), \quad r > 0.$$

Then the desired assertion follows from Theorem 4.1.

(b) When  $d = 1$ , we may simply consider  $M = (0, 1)$  and  $\gamma = \frac{1}{2}$  so that  $\theta = \frac{1}{2}$ , hence

$$\mu(f) = \frac{1}{2} \int_0^1 f(s) ds + \frac{1}{4} f(0) + \frac{1}{4} f(1)$$

and

$$\mathcal{E}_0(f, f) = \frac{1}{2} \int_0^1 f'(s)^2 ds.$$

By the classical Nash inequality, there exists a constant  $c_1 > 0$  such that

$$\int_0^1 f(s)^2 ds \leq r \int_0^1 f'(s)^2 ds + c_1 (1 \wedge r)^{-\frac{1}{2}} \left( \int_0^1 |f(s)| ds \right)^2, \quad s > 0, \quad f \in C_b^1([0,1]).$$

Then there exists a constant  $c > 0$  such that

$$\begin{aligned} \mu(f^2) &= \frac{1}{2} \int_0^1 f(s)^2 ds + \frac{1}{4} f(0)^2 + \frac{1}{4} f(1)^2 \\ &\leq \frac{r}{2} \int_0^1 f'(s)^2 ds + \frac{1}{2} c_1 (1 \wedge r)^{-\frac{1}{2}} + \frac{1}{4} f(0)^2 + \frac{1}{4} f(1)^2 \\ &\leq r \mathcal{E}_0(f, f) + c (1 \wedge r)^{-\frac{1}{2}} \mu(|f|)^2, \quad r > 0, \quad f \in C_b^1([0,1]). \end{aligned}$$

Then (1.2) holds for  $\delta = 0$  and  $\beta(r)$  in (4.6).  $\square$

**Problem 4.1.** We hope that in Example 4.1, even for  $d \geq 2$  we still have (1.2) for  $\beta$  in (4.6). To this end, the general estimate (4.1) should be improved by more refined calculus.

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