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A REFINEMENT OF THE DIACONIS-EVANS-GRAHAM THEOREM

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Abstract. The note is devoted to refining a theorem by Diaconis, Evans, and Graham concerning successions and fixed points of permutations. This refinement specifically addresses non-adjacent successions, predecessors, excedances, and drops of permutations.

1. **Introduction.** The main objective of this note is to give a refinement of a theorem of Diaconis-Evans-Graham [4] on successions and fixed points of permutations.

Let $\mathfrak{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. For a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index $1 \leq i \leq n-1$ is called a succession if $\sigma_i + 1 = \sigma_{i+1}$, whereas an index $1 \leq i \leq n$ is called a fixed point if $\sigma_i = i$. We write $\mathrm{Suc}(\sigma)$ for the set of successions of σ, i.e.,

$$\mathrm{Suc}(\sigma) = \{1 \leq i \leq n-1 : \, \sigma_i + 1 = \sigma_{i+1}\}. \tag{1}$$

We further define $\overline{\mathrm{Fix}}(\sigma)$ as the set of fixed points of σ excluding the index n:

$$\overline{\mathrm{Fix}}(\sigma) = \{1 \leq i \leq n-1 : \, \sigma_i = i\}. \tag{2}$$

For a subset $I \subseteq [n-1]$, let $\mathrm{Suc}_n(I)$ denote the set of permutations σ of $[n]$ satisfying $\mathrm{Suc}(\sigma) = I$ and let $\overline{\mathrm{Fix}}_n(I)$ denote the set of permutations $\sigma \in \mathfrak{S}_n$ satisfying $\overline{\mathrm{Fix}}(\sigma) = I$. Diaconis, Evans and Graham [4] discovered the following beautiful result.

Theorem 1.1. (Diaconis-Evans-Graham) *For all integers $n \geq 2$ and subsets $I \subseteq [n-1]$, there exists a bijection between $\mathrm{Suc}_n(I)$ and $\overline{\mathrm{Fix}}_n(I)$.*

Note that Diaconis, Evans, and Graham [4] refer to a succession of $\sigma = \sigma_1 \cdots \sigma_n$ as an unseparated pair $(k, k+1)$ of σ provided that $\sigma_k + 1 = \sigma_{k+1}$. This terminology and the motivation for studying this concept stem from regarding a permutation as the outcome of shuffling a deck of n cards. The succession has also been extensively studied in the literature, see, for example, [1, 3, 5, 6, 7, 8, 9], and the references therein.

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Notable extensions and alternative proofs of Theorem 1.1 have appeared in the literature. Chen [2] gave a bijective proof of the case $I = \emptyset$ via the first fundamental transformation. Brenti and Marietti [1] generalized this theorem to colored permutations in the complex reflection groups $G(r, p, n)$. More recently, Chen and Fu [3] established a left succession analogue of Theorem 1.1 via grammar-assisted bijection. Separately, Ma, Qi, Yeh and Yeh [5] employed grammatical labeling techniques to show that two triple set-valued statistics on permutations are equidistributed over $\mathfrak{S}_n$. This implies that the number of permutations in $\mathfrak{S}_n$ with a given set I of fixed points (excluding 1) equals the number of permutations in $\mathfrak{S}_n$ having I as a set of σ_{i+1} such that $\sigma_i + 1 = \sigma_{i+1}$ for $1 \le i \le n-1$.

2. **A Refinement and its Proof.** Inspired by the work of Chen and Fu [3], we discover a refinement of Theorem 1.1 that incorporates non-adjacent successions and predecessors. To state our main result, we first formalize these two new statistics, along with restricted versions of classic excedance and drop statistics on permutations.

Definition 2.1 (Non-adjacent succession)**.** Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index i $(1 \le i \le n-2)$ is called a *non-adjacent succession* of σ if there exists an integer j with $i+2 \le j \le n$ such that $\sigma_j = \sigma_i + 1$. The set of non-adjacent successions of σ is denoted by $\mathrm{najSuc}(\sigma)$, that is,

$$\mathrm{najSuc}(\sigma) = \{1 \le i \le n-2 : \ \sigma_i + 1 = \sigma_j \ \text{ for some } \ i+2 \le j \le n\}. \tag{3}$$

Definition 2.2 (Predecessor)**.** Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index i $(2 \le i \le n)$ is called a *predecessor* of σ if there exists an integer j with $1 \le j < i$ such that $\sigma_j = \sigma_i + 1$. The set of predecessors of σ is denoted by $\mathrm{Pred}(\sigma)$, that is,

$$\mathrm{Pred}(\sigma) = \{2 \le i \le n : \ \sigma_i + 1 = \sigma_j \ \text{ for some } \ 1 \le j < i\}. \tag{4}$$

For the permutation $\sigma = 4\,1\,2\,6\,7\,5\,3$, a direct computation gives

$$\mathrm{Suc}(\sigma) = \{2, 4\}, \quad \mathrm{najSuc}(\sigma) = \{1, 3\}, \quad \text{and} \quad \mathrm{Pred}(\sigma) = \{6, 7\}.$$

Recall that for a permutation $\sigma \in \mathfrak{S}_n$, an index $1 \le i \le n$ is called an excedance if $\sigma_i > i$ and an index $1 \le i \le n$ is called a drop if $\sigma_i < i$. We define their restricted versions (excluding the index n) as follows:

$$\overline{\mathrm{Drop}}(\sigma) = \{\sigma_i : \ 1 \le i \le n-1, \, \sigma_i < i\} \tag{5}$$

and

$$\overline{\mathrm{Exc}}(\sigma) = \{\sigma_i : \ 1 \le i \le n-1, \, \sigma_i > i\}. \tag{6}$$

As with $\overline{\mathrm{Fix}}(\sigma)$, both $\overline{\mathrm{Drop}}(\sigma)$ and $\overline{\mathrm{Exc}}(\sigma)$ exclude the index n in their definitions. We now state the main result of this paper.

Theorem 2.3. *For $n \ge 1$, there exists a bijection ϕ between $\mathfrak{S}_n$ and $\mathfrak{S}_n$ such that for $\sigma \in \mathfrak{S}_n$ and $\tau = \phi(\sigma)$, we have*

$$\overline{\mathrm{Fix}}(\sigma) = \mathrm{Suc}(\tau), \quad \overline{\mathrm{Drop}}(\sigma) = \mathrm{najSuc}(\tau) \quad \textit{and} \quad \overline{\mathrm{Exc}}(\sigma) = \mathrm{Pred}(\tau). \tag{7}$$

Proof. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, we define $\tau = \phi(\sigma)$ via three steps:

Step 1 (Reverse Complement). Define $\overline{\sigma} = \overline{\sigma}_1\overline{\sigma}_2 \cdots \overline{\sigma}_n$, where for $1 \le i \le n$, $\overline{\sigma}_i = n + 1 - \sigma_{n-i+1}$.

Step 2 (Cyclic Shift and Non-Canonical Cycle Form). Let $\hat{\sigma} = \hat{\sigma}_1\hat{\sigma}_2 \cdots \hat{\sigma}_n$ denote the cyclic right shift of $\overline{\sigma}$, so $\hat{\sigma}_i = \overline{\sigma}_{i+1}$, for $1 \le i \le n-1$ and $\hat{\sigma}_n = \overline{\sigma}_1$.

We write $\hat{\sigma}$ in cycle form as $(a_1, a_2, \ldots, a_r)(b_1, b_2, \ldots, b_s) \cdots (c_1, c_2, \ldots, c_t)$, subject to the following two rules:

- The first cycle contains the element n, with n placed as its final entry;
- All other cycles are written with their minimal element first, and the cycles are ordered in decreasing order of their minimal elements.

Note that this cycle form differs slightly from the canonical cycle form, and we thus refer to it as a non-canonical cycle form.

Define $\overline{\tau}$ to be the linearization of $\hat{\sigma}$ (i.e., the permutation obtained by erasing all parentheses from the non-canonical cycle form of $\hat{\sigma}$). This non-canonical cycle form ensures $\hat{\sigma}$ is uniquely recoverable from $\overline{\tau}$: insert a left parenthesis before $\overline{\tau}_1$ and a right parenthesis after n; insert a left parenthesis before every left-to-right minimum of $\overline{\tau}$ that appears after n; finally, place a right parenthesis before each inserted left parenthesis (except the first) and at the end of the linear string.

Step 3 (Inverse and Reverse Complement). Let $\overline{\tau}^{-1} = \overline{\tau}_1^{-1} \cdots \overline{\tau}_n^{-1}$ be the inversion of $\overline{\tau}$. Define $\tau = \phi(\sigma) = \tau_1 \cdots \tau_n$ as the reverse complement of τ^{-1}, that is, for $1 \le i \le n$, $\tau_i = n + 1 - \overline{\tau}_{n-i+1}^{-1}$.

We proceed to demonstrate that σ and $\tau = \phi(\sigma)$ satisfy the relations (7).

Let

$$k \in \overline{\mathrm{Fix}}(\sigma), \quad \sigma_r \in \overline{\mathrm{Drop}}(\sigma), \quad \text{and} \quad \sigma_s \in \overline{\mathrm{Exc}}(\sigma).$$

By definition, $\sigma_k = k$, $\sigma_r < r$ and $\sigma_s > s$. Moreover, $k, r, s \neq n$.

Set $K = n + 1 - k$, $R = n + 1 - r$ and $S = n + 1 - s$. Since $k, r, s \neq n$, we see that $K, R, S \neq 1$. From Step 1 (reverse complement),

$$\overline{\sigma}_K = K, \quad \overline{\sigma}_R = n + 1 - \sigma_r > R, \quad \text{and} \quad \overline{\sigma}_S = n + 1 - \sigma_s < S.$$

Step 2 (cyclic shift) further gives

$$\hat{\sigma}_{K-1} = \overline{\sigma}_K = K, \quad \hat{\sigma}_{R-1} = \overline{\sigma}_R > R, \quad \text{and} \quad \hat{\sigma}_{S-1} = \overline{\sigma}_S < S. \tag{8}$$

(a) $k \in \overline{\mathrm{Fix}}(\sigma) \implies k \in \mathrm{Suc}(\tau)$. The equality $\hat{\sigma}_{K-1} = K$ implies the non-canonical cycle form of $\hat{\sigma}$ contains a cycle of the form $(\ldots, K-1, K, \ldots)$. Linearization to $\overline{\tau}$ gives $\overline{\tau}_j = K - 1$ and $\overline{\tau}_{j+1} = K$ for some $1 \le j \le n-1$. Taking inverses, $\overline{\tau}_{K-1}^{-1} = j$ and $\overline{\tau}_K^{-1} = j + 1$. By Step 3,

$$\tau_k = n + 1 - \overline{\tau}_{n+1-k}^{-1} = n - j \quad \text{and} \quad \tau_{k+1} = n + 1 - \overline{\tau}_{n-k}^{-1} = n + 1 - j.$$

Thus $\tau_{k+1} = \tau_k + 1$, so $k \in \mathrm{Suc}(\tau)$.

For example, let $\sigma = 7\,2\,6\,4\,1\,3\,5$ (from Remark 2.4 (a)), which has a fixed point at $k = 2$ ($K = 6$). In $\overline{\tau}$, we have $\overline{\tau}_6 = 5$ and $\overline{\tau}_7 = 6$ ($j = 6$), so $\tau_2 = 1$ and $\tau_3 = 2$, confirming $k = 2 \in \mathrm{Suc}(\tau)$.

(b) $\sigma_r \in \overline{\mathrm{Drop}}(\sigma) \implies \sigma_r \in \mathrm{najSuc}(\tau)$. From $\hat{\sigma}_{R-1} = \overline{\sigma}_R > R$, the non-canonical cycle form of $\hat{\sigma}$ has a cycle $(\ldots, R-1, \overline{\sigma}_R, \ldots)$. Linearization gives $\overline{\tau}_i = R - 1$ and $\overline{\tau}_{i+1} = \overline{\sigma}_R > R$ for some $1 \le i \le n-1$. Taking inverses, $\overline{\tau}_{R-1}^{-1} = i$ and $\overline{\tau}_{\overline{\sigma}_R}^{-1} = i + 1$. By Step 3,

$$\tau_{r+1} = n + 1 - \overline{\tau}_{R-1}^{-1} = n + 1 - i \quad \text{and} \quad \tau_{n+1-\overline{\sigma}_R} = n + 1 - \overline{\tau}_{\overline{\sigma}_R}^{-1} = n - i.$$

Since $n+1-\overline{\sigma}_R = \sigma_r < r$, we have $\tau_{\sigma_r} + 1 = \tau_{r+1}$ with $\sigma_r \le r-1$, so $\sigma_r \in \mathrm{najSuc}(\tau)$.

For the example presented in Remark 2.4 (a), $\sigma = 7\,2\,6\,4\,1\,3\,5$ has a drop at $r = 5$ ($\sigma_5 = 1$, $R = 3$). In $\overline{\tau}$, $\overline{\tau}_3 = 2$ and $\overline{\tau}_4 = 7$ ($i = 3$), so $\tau_6 = 5$ and $\tau_1 = 4$, confirming $\sigma_5 = 1 \in \mathrm{najSuc}(\tau)$.

(c) $\sigma_s \in \overline{\mathrm{Exc}}(\sigma) \implies \sigma_s \in \mathrm{Pred}(\tau)$. From (8), $\hat{\sigma}_{S-1} \le S-1$. The non-canonical cycle form of $\hat{\sigma}$ has one of two forms for the cycle containing $S-1$:

(a) A cycle $(\ldots, S-1, \hat{\sigma}_{S-1}, \ldots)$;
(b) A cycle $(\hat{\sigma}_{S-1}, \ldots, S-1)$ (including the 1-cycle $(S-1)$ as a special case).

Subcase (a). A cycle $(\ldots, S-1, \hat{\sigma}_{S-1}, \ldots)$ implies $\overline{\sigma}_S = \hat{\sigma}_{S-1} \le S-2$, so $\sigma_s = n+1-\overline{\sigma}_S \ge s+2$. Linearization gives $\overline{\tau}_t = S-1$ and $\overline{\tau}_{t+1} = \hat{\sigma}_{S-1} \le S-2$ for some $1 \le t \le n-1$. Taking inverses, $\overline{\tau}^{-1}_{S-1} = t$ and $\overline{\tau}^{-1}_{\overline{\sigma}_S} = t+1$. By Step 3,

$$\tau_{s+1} = n+1-\overline{\tau}^{-1}_{S-1} = n+1-t \quad \text{and} \quad \tau_{n+1-\overline{\sigma}_S} = n+1-\overline{\tau}^{-1}_{\overline{\sigma}_S} = n-t.$$

Since $n+1-\overline{\sigma}_S = \sigma_s \ge s+2$, we have $\tau_{\sigma_s}+1 = \tau_{s+1}$ with $\sigma_s \ge s+2$, so $\sigma_s \in \mathrm{Pred}(\tau)$.

For the example in Remark 2.4 (a), $\sigma = 7\,2\,6\,4\,1\,3\,5$ has an excedance at $s = 3$ ($\sigma_3 = 6$, $S = 5$). In $\overline{\tau}$, $\overline{\tau}_2 = 4$ and $\overline{\tau}_3 = 2$ ($t = 2$), so $\tau_4 = 6$ and $\tau_6 = 5$, confirming $\sigma_3 = 6 \in \mathrm{Pred}(\tau)$.

Subcase (b). A cycle $(\hat{\sigma}_{S-1}, \ldots, S-1)$ (not containing n, by non-canonical cycle form rules) appears after the first cycle in the linearization $\overline{\tau}$. Let $\overline{\tau}_{t+1} = \hat{\sigma}_{S-1} = \overline{\sigma}_S$ for some $1 \le t \le n-1$; the non-canonical cycle ordering implies $\overline{\tau}_t = T > \overline{\tau}_{t+1} = \overline{\sigma}_S$. Taking inverses, $\overline{\tau}^{-1}_T = t$ and $\overline{\tau}^{-1}_{\overline{\sigma}_S} = t+1$. By Step 3,

$$\tau_{n+1-T} = n+1-\overline{\tau}^{-1}_T = n+1-t \quad \text{and} \quad \tau_{n+1-\overline{\sigma}_S} = n+1-\overline{\tau}^{-1}_{\overline{\sigma}_S} = n-t.$$

Since $\sigma_s = n+1-\overline{\sigma}_S > n+1-T$, we have $\tau_{\sigma_s}+1 = \tau_{n+1-T}$, so $\sigma_s \in \mathrm{Pred}(\tau)$.

For the example in Remark 2.4 (a), $\sigma = 7\,2\,6\,4\,1\,3\,5$ has an excedance at $s = 1$ ($\sigma_1 = 7$, $S = 7$). In $\overline{\tau}$, $\overline{\tau}_4 = 7$ and $\overline{\tau}_5 = 1$ ($t = 4$, $T = 7$), so $\tau_1 = 4$ and $\tau_7 = 3$, confirming $\sigma_1 = 7 \in \mathrm{Pred}(\tau)$.

All three steps of the construction are reversible, and the reverse construction preserves the relations in (7). Thus ϕ is a bijection, and the proof is complete. $\square$

Remark 2.4. (a) We illustrate the bijection ϕ with the permutation $\sigma = 7\,2\,6\,4\,1\,3\,5$ (used by Diaconis, Evans, and Graham in [4, Remark 4.2]):

- **Step 1.** Reverse complement: $\overline{\sigma} = 3\,5\,7\,4\,2\,6\,1$;
- **Step 2.** Cyclic shift: $\hat{\sigma} = 5\,7\,4\,2\,6\,1\,3$, non-canonical cycle form: $(3\,4\,2\,7)(1\,5\,6)$, linearization: $\overline{\tau} = 3\,4\,2\,7\,1\,5\,6$;
- **Step 3.** Inverse of $\overline{\tau}$: $\overline{\tau}^{-1} = 5\,3\,1\,2\,6\,7\,4$, reverse complement: $\tau = \phi(\sigma) = 4\,1\,2\,6\,7\,5\,3$.

This permutation differs from the image $\widehat{\rho}(\sigma) = 7\,1\,2\,5\,6\,4\,3$ obtained via the Diaconis–Evans–Graham bijection [4]. Direct computation confirms

$$\overline{\mathrm{Fix}}(\sigma) = \mathrm{Suc}(\tau) = \{2,4\},\ \overline{\mathrm{Drop}}(\sigma) = \mathrm{najSuc}(\tau) = \{1,3\},\ \overline{\mathrm{Exc}}(\sigma) = \mathrm{Pred}(\tau) = \{6,7\}.$$

(b) Table 1 illustrates Theorem 2.3 for the case $n = 4$.

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Table 1. Corresponding statistics for permutations in $\mathfrak{S}_4$ via bijection ϕ in Theorem 2.3

σ	$\overline{\mathrm{Fix}}(\sigma)$	$\overline{\mathrm{Drop}}(\sigma)$	$\overline{\mathrm{Exc}}(\sigma)$	$\tau=\phi(\sigma)$	$\mathrm{Suc}(\tau)$	$\mathrm{najSuc}(\tau)$	$\mathrm{Pred}(\tau)$
1 2 3 4	$\{1,2,3\}$	$\emptyset$	$\emptyset$	1 2 3 4	$\{1,2,3\}$	$\emptyset$	$\emptyset$
1 2 4 3	$\{1,2\}$	$\emptyset$	$\{4\}$	2 3 4 1	$\{1,2\}$	$\emptyset$	$\{4\}$
1 3 2 4	$\{1\}$	$\{2\}$	$\{3\}$	2 3 1 4	$\{1\}$	$\{2\}$	$\{3\}$
1 3 4 2	$\{1\}$	$\emptyset$	$\{3,4\}$	3 4 2 1	$\{1\}$	$\emptyset$	$\{3,4\}$
1 4 2 3	$\{1\}$	$\{2\}$	$\{4\}$	1 2 4 3	$\{1\}$	$\{2\}$	$\{4\}$
1 4 3 2	$\{1,3\}$	$\emptyset$	$\{4\}$	3 4 1 2	$\{1,3\}$	$\emptyset$	$\{4\}$
2 1 3 4	$\{3\}$	$\{1\}$	$\{2\}$	2 1 3 4	$\{3\}$	$\{1\}$	$\{2\}$
2 1 4 3	$\emptyset$	$\{1\}$	$\{2,4\}$	3 2 4 1	$\emptyset$	$\{1\}$	$\{2,4\}$
2 3 1 4	$\emptyset$	$\{1\}$	$\{2,3\}$	3 2 1 4	$\emptyset$	$\{1\}$	$\{2,3\}$
2 3 4 1	$\emptyset$	$\emptyset$	$\{2,3,4\}$	4 3 2 1	$\emptyset$	$\emptyset$	$\{2,3,4\}$
2 4 1 3	$\emptyset$	$\{1\}$	$\{2,4\}$	2 1 4 3	$\emptyset$	$\{1\}$	$\{2,4\}$
2 4 3 1	$\{3\}$	$\emptyset$	$\{2,4\}$	4 3 1 2	$\{3\}$	$\emptyset$	$\{2,4\}$
3 1 2 4	$\emptyset$	$\{1,2\}$	$\{3\}$	1 3 2 4	$\emptyset$	$\{1,2\}$	$\{3\}$
3 1 4 2	$\emptyset$	$\{1\}$	$\{3,4\}$	2 4 3 1	$\emptyset$	$\{1\}$	$\{3,4\}$
3 2 1 4	$\{2\}$	$\{1\}$	$\{3\}$	3 1 2 4	$\{2\}$	$\{1\}$	$\{3\}$
3 2 4 1	$\{2\}$	$\emptyset$	$\{3,4\}$	4 2 3 1	$\{2\}$	$\emptyset$	$\{3,4\}$
3 4 1 2	$\emptyset$	$\{1\}$	$\{3,4\}$	1 4 3 2	$\emptyset$	$\{1\}$	$\{3,4\}$
3 4 2 1	$\emptyset$	$\{2\}$	$\{3,4\}$	4 2 1 3	$\emptyset$	$\{2\}$	$\{3,4\}$
4 1 2 3	$\emptyset$	$\{1,2\}$	$\{4\}$	3 1 4 2	$\emptyset$	$\{1,2\}$	$\{4\}$
4 1 3 2	$\{3\}$	$\{1\}$	$\{4\}$	1 4 2 3	$\{3\}$	$\{1\}$	$\{4\}$
4 2 1 3	$\{2\}$	$\{1\}$	$\{4\}$	1 3 4 2	$\{2\}$	$\{1\}$	$\{4\}$
4 2 3 1	$\{2,3\}$	$\emptyset$	$\{4\}$	4 1 2 3	$\{2,3\}$	$\emptyset$	$\{4\}$
4 3 1 2	$\emptyset$	$\{1\}$	$\{3,4\}$	2 4 1 3	$\emptyset$	$\{1\}$	$\{3,4\}$
4 3 2 1	$\emptyset$	$\{2\}$	$\{3,4\}$	4 1 3 2	$\emptyset$	$\{2\}$	$\{3,4\}$

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