MACMAHON'S PARTITION ANALYSIS XV: PARITY

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Dedicated to the memory of our friend Marko Petkovšek.

Abstract. We apply the methods of partition analysis to partitions in which the parity of parts plays a role. We begin with an in-depth treatment of the generating function for the partitions from the first Göllnitz-Gordon identity. We then deduce a Schmidt-type theorem related to the false theta functions. We also consider: (1) position parity, (2) partitions with distinct even parts, (3) partitions with distinct odd parts. One of the corollaries of these last considerations is a new interpretation of Hei-Chi Chan's cubic partitions. A second part of our article is devoted to the algorithmic derivation of identities and arithmetic congruences related to the generating functions considered in part one, including cubic partitions. To this end, Smoot's implementation of Radu's Ramanujan-Kolberg algorithm is used. Finally, we give a short description which explains how to use the Omega package to derive special instances of the results of part one.

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1. Introduction

This paper is a natural successor to the last two in this series on MacMahon's partition analysis. These three papers collectively apply the Schmidt phenomenon to: first, ordinary partitions [9], second, partitions with n copies of n [10], and now to partitions in which parity plays a role.

Section 2 is devoted to the collection of some necessary background results. Section 3 is devoted to our prototypical example, the first Göllnitz-Gordon series. The partition analysis not only yields the familiar series, but also points naturally to the combinatorial construction of the series.

Section 4 is devoted to variations on the Göllnitz-Gordon theme by mixing it with parity conditions related to the indices of the parts.

Section 5 will be devoted to the simplest portion of this project. Here we begin with (starting from the largest part) each part with even index being even. For example, the six relevant partitions of 8 are $8, 6+2, 5+2+1, 4+4, 4+2+2, 2+2+2+2$. Naturally, we have three other cases depending on the parity of the part and the parity of the index. This also yields nice results in the case of Schmidt specializations.

Section 6 is devoted to partitions with distinct evens, and Section 7 to distinct odds.

In joint collaboration with the authors of this article, partition analysis has been implemented by Axel Riese in the form of the Omega package written in Mathematica. Section 8 will give a brief description of how to use Omega to compute special instances of the results presented in Sections 2 to 7.

As in the predecessor articles [9] and [10] we will use Smoot's implementation [27] of Radu's Ramanujan-Kolberg algorithm [24] to derive a variety of identities and arithmetic congruences related to the combinatorial generating functions of this paper. Section 9 is devoted to results related

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to a generating function which arose from a Schmidt process considered in Section 7. Section 10 continues on the Ramanujan-Kolberg theme and presents results related to cubic partitions.

Finally, in Section 11 the reader finds supplementing descriptions on the usage of Smoot's package to put her/him into the position to derive all the results presented in Sections 9 and 10 by herself/himself.

2. Background

Our main tool is MacMahon's partition analysis, [21, Sec VIII] and [22, Ch. 10]. The MacMahon operator $\Omega_{\geq}$ is given by

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \ldots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r}, \tag{2.1}$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\ldots,s_r}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of $A_{s_1,\ldots,s_r}$. In practice the $A_{s_1,\ldots,s_r}$ are usually monomials in r or fewer variables.

Also, we should mention what we mean by Schmidt type partitions. Schmidt's original observation was posed as a problem in the American Mathematical Monthly [26] which we state in the form of a theorem:

Let $p(n)$ denote the number of partitions of the integer n, and let $f(n)$ denote the number of partitions $a_1+a_2+a_3+\ldots$ satisfying $a_1 > a_2 > a_3 > \ldots$ and $n = a_1+a_3+a_5+\ldots$. For example, $p(5)$ counts the 7 partitions 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$, and $f(5)$ counts the 7 partitions 5, $5+1$, $5+2$, $5+3$, $5+4$, $4+3+1$, and $4+2+1$. Then

$$p(n) = f(n), \quad n \geq 1.$$

This concept of summing a subset of the summands of a partition will recur throughout this paper.

As technical ingredients we need a couple of elementary lemmas related to actions of the $\Omega_{\geq}$ operator.

For non-negative integer n let

$$\chi(n) := \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}. \tag{2.2}$$

Lemma 2.1. *We have*

$$\sum_{N\geq 1} x^N \lambda^{N-\chi(N)} = \frac{x\lambda(1+x)}{1-x^2\lambda^2}. \tag{2.3}$$

Proof.

$$\begin{aligned} \sum_{N\geq 1} x^N \lambda^{N-\chi(N)} &= \sum_{N\geq 1} x^{2N} \lambda^{2N-1} + \sum_{N\geq 1} x^{2N-1} \lambda^{2N-1} \\ &= (1+x) \sum_{N\geq 1} x^{2N-1} \lambda^{2N-1} = \frac{x\lambda(1+x)}{1-x^2\lambda^2}. \end{aligned}$$

□

From [21, p. 102, Entry 348] we know

$$\underset{\geq}{\Omega} \frac{1}{(1-x\lambda^2)\left(1-\frac{y}{\lambda}\right)} = \frac{1+xy}{(1-x)(1-xy^2)}. \tag{2.4}$$

We need a slight generalization of this in the following lemma. The above proof is typical so we omit subsequent details that only involve the summation of geometric series.

Lemma 2.2. *For an integer $A \geq 0$,*

$$\underset{\geq}{\Omega} \frac{\lambda^{-2A}}{(1-x\lambda^2)\left(1-\frac{y}{\lambda}\right)} = \frac{x^A(1+xy)}{(1-x)(1-xy^2)}. \tag{2.5}$$

Proof.

$$\underset{\geq}{\Omega} \frac{\lambda^{-2A}}{(1-x\lambda^2)\left(1-\frac{y}{\lambda}\right)} = \underset{\geq}{\Omega} \sum_{r,s\geq 0} x^r y^s \lambda^{2r-s-2A} = \sum_{r\geq A} \sum_{s=0}^{2r-2A} x^r y^s = \frac{x^A(1+xy)}{(1-x)(1-xy^2)}.$$

□

Lemma 2.3. *For an integer $A \geq 0$,*

$$\underset{\geq}{\Omega} \frac{\lambda^{-2A-1}}{(1-x\lambda^2)\left(1-\frac{y}{\lambda}\right)} = \frac{x^{A+1}(1+y)}{(1-x)(1-xy^2)}. \tag{2.6}$$

Proof.

$$\underset{\geq}{\Omega} \frac{\lambda^{-2A-1}}{(1-x\lambda^2)\left(1-\frac{y}{\lambda}\right)} = \underset{\geq}{\Omega} \sum_{r,s\geq 0} x^r y^s \lambda^{2r-s-2A-1} = \sum_{r\geq A+1} \sum_{s=0}^{2r-2A-1} x^r y^s = \frac{x^{A+1}(1+y)}{(1-x)(1-xy^2)}.$$

□

Lemma 2.4. *For an integer $A \geq 0$,*

$$\underset{\geq}{\Omega} \frac{\lambda^{-2A}}{(1-x\lambda^2)\left(1-\frac{y}{\lambda^2}\right)\left(1-\frac{z}{\lambda^2}\right)} = \frac{x^A}{(1-x)(1-xy)(1-xz)}. \tag{2.7}$$

Proof. As before. □

Lemma 2.5. *For an integer $A \geq -1$,*

$$\underset{\geq}{\Omega} \frac{\lambda^{-2A-1}}{(1-x\lambda^2)\left(1-\frac{y}{\lambda^2}\right)\left(1-\frac{z}{\lambda^2}\right)} = \frac{x^{A+1}}{(1-x)(1-xy)(1-xz)}. \tag{2.8}$$

Proof. As before. □

3. The Göllnitz-Gordon Identity

This result is stated classically as

$$\sum_{k\geq 0} \frac{q^{k^2}(-q;q^2)_k}{(q^2;q^2)_k} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n-1})(1-q^{8n-4})(1-q^{8n-7})} \tag{3.1}$$

where $(A;q)_0 := 1$ and

$$(A;q)_N := (1-A)(1-Aq)\dots(1-Aq^{N-1}).$$

The following theorem greatly refines the kth term of the left hand side of (3.1). Now the exponent of the variable x_i accounts for the ith part of the partition in question.

Theorem 1. *The generating function for partitions with k parts and difference 2 between parts (> 2 between evens) is*

$$\frac{x_1^{2k-1}x_2^{2k-3}\dots x_{k-1}^3 x_k(1+x_1)(1+x_1^2x_2)(1+x_1^2x_2^2x_3)\dots(1+x_1^2x_2^2\dots x_{k-1}^2x_k)}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)\dots(1-x_1^2x_2^2\dots x_k^2)}. \tag{3.2}$$

Proof. The $\Omega_{\geq}$ operator allows us to embed the various difference conditions in the exponents of the λ's. Hence, in terms of partition analysis our generating function is given by the following expression (MacMahon used the term "crude generating function" for such objects):

$$\begin{aligned}
&\underset{\geq}{\Omega} \sum_{\substack{n_1,\dots,n_{k-1}\geq 0\\ n_k\geq 1}} x_1^{n_1}x_2^{n_2}\dots x_k^{n_k}\lambda_1^{n_1-n_2-\chi(n_1)-2}\lambda_2^{n_2-n_3-\chi(n_2)-2}\dots\lambda_{k-1}^{n_{k-1}-n_k-\chi(n_{k-1})-2}\\
&\quad=\underset{\geq}{\Omega}\frac{x_1\lambda_1(1+x_1)\lambda_1^{-2}}{1-x_1^2\lambda_1^2}\cdot\frac{\frac{x_2}{\lambda_1}\lambda_2\big(1+\frac{x_2}{\lambda_1}\big)\lambda_2^{-2}}{1-\frac{x_2^2\lambda_2^2}{\lambda_1^2}}\cdot\frac{\frac{x_3}{\lambda_2}\lambda_3\big(1+\frac{x_3}{\lambda_2}\big)\lambda_3^{-2}}{1-\frac{x_3^2\lambda_3^2}{\lambda_2^2}}\cdots\frac{\frac{x_{k-1}}{\lambda_{k-2}}\lambda_{k-1}\big(1+\frac{x_{k-1}}{\lambda_{k-2}}\big)\lambda_{k-1}^{-2}}{1-\frac{x_{k-1}^2\lambda_{k-1}^2}{\lambda_{k-2}^2}}\\
&\qquad\times\frac{x_k\lambda_{k-1}^{-1}}{1-\frac{x_k}{\lambda_{k-1}}}\qquad\text{(by Lemma 2.1)}\\
&\quad=\underset{\geq}{\Omega}\frac{x_1x_2\dots x_{k-1}x_k(1+x_1)\big(1+\frac{x_2}{\lambda_1}\big)\big(1+\frac{x_3}{\lambda_2}\big)\dots\big(1+\frac{x_{k-1}}{\lambda_{k-2}}\big)\lambda_1^{-2}\lambda_2^{-2}\dots\lambda_{k-1}^{-2}}{(1-x_1^2\lambda_1^2)\big(1-\frac{x_2^2\lambda_2^2}{\lambda_1^2}\big)\big(1-\frac{x_3^2\lambda_3^2}{\lambda_2^2}\big)\dots\big(1-\frac{x_{k-1}^2\lambda_{k-1}^2}{\lambda_{k-2}^2}\big)\big(1-\frac{x_k}{\lambda_{k-1}}\big)}.
\end{aligned}\tag{3.3}$$

We now proceed to eliminate the λ's in the order $\lambda_1,\lambda_2,\dots,\lambda_{k-1}$. Each elimination uses Lemmas 2.4 and 2.5 for $\lambda_1,\dots,\lambda_{k-2}$. The final elimination uses Lemma 2.2.

In the first step we eliminate λ_1 from (3.3):

$$\begin{aligned}
&x_1x_2\dots x_{k-1}x_k(1+x_1)\\
&\quad\times\underset{\geq}{\Omega}\frac{\big(1+\frac{x_2}{\lambda_1}\big)\lambda_1^{-2}}{(1-x_1^2\lambda_1^2)\big(1-\frac{x_2^2\lambda_2^2}{\lambda_1^2}\big)}\cdot\frac{\big(1+\frac{x_3}{\lambda_2}\big)\dots\big(1+\frac{x_{k-1}}{\lambda_{k-2}}\big)\lambda_2^{-2}\dots\lambda_{k-1}^{-2}}{\big(1-\frac{x_3^2\lambda_3^2}{\lambda_2^2}\big)\dots\big(1-\frac{x_{k-1}^2\lambda_{k-1}^2}{\lambda_{k-2}^2}\big)\big(1-\frac{x_k}{\lambda_{k-1}}\big)}\\
&=x_1x_2\dots x_{k-1}x_k(1+x_1)\\
&\quad\times\underset{\geq}{\Omega}\Big(\frac{x_1^2}{(1-x_1^2)(1-x_1^2x_2^2\lambda_2^2)}+x_2\frac{x_1^4}{(1-x_1^2)(1-x_1^2x_2^2\lambda_2^2)}\Big)\\
&\qquad\times\frac{\big(1+\frac{x_3}{\lambda_2}\big)\dots\big(1+\frac{x_{k-1}}{\lambda_{k-2}}\big)\lambda_2^{-2}\dots\lambda_{k-1}^{-2}}{\big(1-\frac{x_3^2\lambda_3^2}{\lambda_2^2}\big)\dots\big(1-\frac{x_{k-1}^2\lambda_{k-1}^2}{\lambda_{k-2}^2}\big)\big(1-\frac{x_k}{\lambda_{k-1}}\big)}\qquad\text{(by Lemmas 2.4 and 2.5)}\\
&=\frac{x_1^3x_2\dots x_{k-1}x_k(1+x_1)(1+x_1^2x_2)}{1-x_1^2}\\
&\qquad\times\underset{\geq}{\Omega}\frac{\big(1+\frac{x_3}{\lambda_2}\big)\lambda_2^{-2}}{(1-x_1^2x_2^2\lambda_2^2)\big(1-\frac{x_3^2\lambda_3^2}{\lambda_2^2}\big)}\cdot\frac{\big(1+\frac{x_4}{\lambda_3}\big)\dots\big(1+\frac{x_{k-1}}{\lambda_{k-2}}\big)\lambda_3^{-2}\dots\lambda_{k-1}^{-2}}{\big(1-\frac{x_4^2\lambda_4^2}{\lambda_3^2}\big)\dots\big(1-\frac{x_{k-1}^2\lambda_{k-1}^2}{\lambda_{k-2}^2}\big)\big(1-\frac{x_k}{\lambda_{k-1}}\big)}.
\end{aligned}\tag{3.4}$$

To eliminate λ_2 from (3.4) use again Lemmas 2.4 and 2.5 to obtain

$$\begin{aligned}
&\frac{x_1^5x_2^3x_3\dots x_k(1+x_1)(1+x_1^2x_2)(1+x_1^2x_2^2x_3)}{(1-x_1^2)(1-x_1^2x_2^2)}\\
&\qquad\times\underset{\geq}{\Omega}\frac{\big(1+\frac{x_4}{\lambda_3}\big)\dots\big(1+\frac{x_{k-1}}{\lambda_{k-2}}\big)\lambda_3^{-2}\dots\lambda_{k-1}^{-2}}{(1-x_1^2x_2^2x_3^2\lambda_3^2)\big(1-\frac{x_4^2\lambda_4^2}{\lambda_3^2}\big)\dots\big(1-\frac{x_{k-1}^2\lambda_{k-1}^2}{\lambda_{k-2}^2}\big)\big(1-\frac{x_k}{\lambda_{k-1}}\big)}.
\end{aligned}$$

The pattern up through λ_{k-2} is now clear and easily proved via mathematical induction. Here is the final stage where we treat the elimination of λ_{k-1}:

$$\frac{x_1^{2k-3}x_2^{2k-5}\dots x_{k-2}^3x_{k-1}x_k(1+x_1)(1+x_1^2x_2)\dots(1+x_1^2x_2^2\dots x_{k-2}^2x_{k-1})}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)\dots(1-x_1^2x_2^2x_3^2\dots x_{k-2}^2)}$$
$$\times\,\Omega_{\geq}\,\frac{\lambda_{k-1}^{-2}}{(1-x_1^2x_2^2\dots x_{k-1}^2\lambda_{k-1}^2)\big(1-\frac{x_k}{\lambda_{k-1}}\big)}$$
$$=\frac{x_1^{2k-3}x_2^{2k-5}\dots x_{k-2}^3x_{k-1}x_k(1+x_1)(1+x_1^2x_2)\dots(1+x_1^2x_2^2\dots x_{k-2}^2x_{k-1})}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)\dots(1-x_1^2x_2^2x_3^2\dots x_{k-2}^2)}$$
$$\times\,\frac{x_1^2x_2^2\dots x_{k-1}^2(1+x_1^2x_2^2\dots x_{k-1}^2x_k)}{(1-x_1^2x_2^2\dots x_{k-1}^2)(1-x_1^2x_2^2\dots x_{k-1}^2x_k^2)}.$$

□

Remark. Note how beautifully Lemma 2.2 with $A = 1$ completes the proof. Note also that Lemma 2.3 was unnecessary but was included for completeness.

Finally setting all the $x_i = q$ in (3.2), we obtain the kth term in the Göllnitz-Gordon series

$$\frac{q^{k^2}(1+q)(1+q^3)\dots(1+q^{2k-1})}{1-q^2)(1-q^4)\dots(1-q^{2k})}.$$

In addition, one may read off from Theorem 1 the combinatorial interpretation of the series. Indeed, the result is precisely the construction given in [2].

4. Partitions With Distinct Even Parts

The generating function for partitions with distinct even parts is clearly given by

$$\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^{2n-1}}. \tag{4.1}$$

So, we really do not need partition analysis to reveal the obvious. However, by using partition analysis we shall obtain some appealing results relating Schmidt-type partitions with overpartitions. As in the previous section, the variable x_i keeps track of the ith part of the partition. The only difference between Theorem 2 and Theorem 1 is that 0 has replaced 2. Now parts may repeat, but even parts differ from other parts by at least 1.

Theorem 2. *The generating function for partitions with k parts and distinct evens is*

$$\frac{x_1x_2\dots x_k(1+x_1)(1+x_1^2x_2)(1+x_1^2x_2^2x_3)\dots(1+x_1^2x_2^2\dots x_{k-1}^2x_k)}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)\dots(1-x_1^2x_2^2\dots x_k^2)}. \tag{4.2}$$

Proof. The proof is almost identical with the proof of Theorem 1. The only change is the disappearance of the 2'a in the lambda exponents. Thus the monomial $x_1^{2k-1}x_2^{2k-3}\dots x_k$ is replaced by $x_1x_2\dots x_k$. □

Now, if we set each $x_i = q$, we easily verify our initial observation in this section. Namely, the generating function for partitions with distinct even parts can be represented as

$$\sum_{k\geq 0}\frac{q^k(-q;q^2)_k}{(q^2;q^2)_k}=\prod_{n=1}^{\infty}\frac{1+q^{2n}}{1-q^{2n-1}} \tag{4.3}$$

by [5, p. 17, eq. (2.2.1)].

Theorem 3. *Consider Schmidt-type partitions in which: (1) there is an odd number of parts; (2) only the parts of odd index (starting from the largest part) are summed; and (3) no even part is repeated. The generating function for these partitions is*

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\binom{n+1}{2}}. \tag{4.4}$$

Proof. In (4.2) we replace k by $2n+1$ and set $x_{2i+1}=q$, $x_{2i}=1$ for each i. Summing over all n, the result is the desired generating function,

$$\sum_{n\geq 0}\frac{q^{n+1}(-q;q)_{2n+1}}{(q^2;q^2)_n^2(1-q^{2n+2})}=\frac{q(1+q)}{1-q^2}\sum_{n\geq 0}\frac{(-q^2;q^2)_n(-q^3;q^2)_n q^n}{(q^2;q^2)_n(q^4;q^2)_n}$$
$$=\frac{q(1+q)}{1-q^2}\frac{(-q^2;q^2)_\infty(-q^3;q^2)_\infty}{(q^4;q^2)_\infty(q;q^2)_\infty}\sum_{n\geq 0}\frac{(-q^2;q^2)_n(-q^2)^n}{(-q^3;q^2)_n}\quad\text{(by [5, p. 38, last line])}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{\binom{n+1}{2}}\quad\text{(by [6, p. 227, Entry 9.3.1, } a=q\text{])}.$$

□

Corollary 1. *The number of overpartitions of n in which the first missing non-overlined part is even equals the number of Schmidt-type partitions of n listed in Theorem 3.*

Proof. The generating function for the partitions in question is given by

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{\binom{n+1}{2}}=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}\left(q^{\binom{2n+2}{2}}-q^{\binom{2n+3}{2}}\right)$$
$$=\sum_{n\geq 0}q^{1+2+3+\cdots+2n+1}\frac{\prod_{m\geq 1}(1+q^m)}{\prod_{\substack{m\geq 1\\ m\neq 2n+2}}(1-q^m)},$$

and the latter expression is the generating function for the overpartitions in question. □

Example. The six Schmidt-type partitions of 4 are

$$4,3+3+1,3+2+1,3+1+1+1,2+1+1+1+1,\ \text{and}\ 1+1+1+1+1+1+1+1.$$

The relevant overpartitions are

$$3+1,\overline{3}+1,\overline{2}+1+1,\overline{2}+\overline{1}+1,1+1+1+1,\overline{1}+1+1+1.$$

Theorem 4. *Consider Schmidt-type partitions in which: (1) there is an even number of parts; (2) only the parts of odd index (starting from the largest part) are summed; and (3) no even part is repeated. The generating function for these partitions is*

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}.$$

Proof. In (4.2) we replace k by $2n$ and set $x_{2i+1}=q$, $x_{2i}=1$ for each i. Summing over all n, the result is the desired generating function,

$$\sum_{n\geq 0}\frac{q^n(-q;q)_{2n}}{(q^2;q^2)_n^2}=\sum_{n\geq 0}\frac{(-q;q^2)_n(-q^2;q^2)_n q^n}{(q^2;q^2)_n(q^2;q^2)_n}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}\frac{(-q;q^2)_n(-q)^n}{(-q^2;q^2)_n}\quad\text{(by [5, p. 38, last line])}$$
$$=\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}\quad\text{(by [6, p. 227, Entry 9.3.1, } a=1\text{])}.$$

□

Corollary 2. *The number of overpartitions of n in which the first missing non-overlined part is odd equals the number of Schmidt-type partitions of n listed in Theorem 4.*

Proof. The generating function for the partitions in question is given by

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}} = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 0}\left(q^{\binom{2n+1}{2}} - q^{\binom{2n+2}{2}}\right)$$
$$= \sum_{n\geq 0} q^{1+2+3+\cdots+2n}\frac{\prod_{m\geq 1}(1+q^m)}{\prod_{\substack{m\geq 1\\ m\neq 2n+1}}(1-q^m)},$$

and the latter expression is the generating function for the overpartitions in question. □

Example. The eight Schmidt-type partitions of 4 are

$$4{+}3, 4{+}2, 4{+}1, 3{+}3{+}1{+}1, 3{+}2{+}1{+}1, 3{+}1{+}1{+}1, 2{+}1{+}1{+}1{+}1, \text{ and } 1{+}1{+}1{+}1{+}1{+}1{+}1{+}1.$$

The eight relevant overpartitions are

$$4, \overline{4}, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, 2+\overline{2}, 2+1+1, 2+1+\overline{1}.$$

5. Partitions With Distinct Even Parts

One of the valuable aspects of partition analysis is that it allows us to discover truly novel generating functions. For example, let $W(n)$ denote the number of partitions of n in which if a part with odd index is odd then it is greater than the next part. Then

$$\sum_{n\geq 0} W(n)q^n = (1+q)\prod_{n=1}^{\infty}\frac{1+q^{4n-1}+q^{4n}+q^{4n+1}}{1-q^{2n}} \tag{5.1}$$
$$= 1+q+q^2+2q^3+4q^4+5q^5+6q^6+8q^7+12q^8+\dots .$$

Thus $W(7)=8$ with the eight relevant partitions being

$$7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 3+2+2, 2+2+2+1.$$

As in all the previous cases, we use partition analysis to prove a more refined result in which the variable x_i keeps track of the ith part. Before stating the theorem we introduce a variant of $\chi(n)$ where the roles of even and odd are swapped:

For non-negative integer n let

$$\psi(n) := 1-\chi(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}. \tag{5.2}$$

Lemma 5.1. *We have*

$$\sum_{N\geq 0} x^N \lambda^{N-\psi(N)} = \frac{1+x}{1-x^2\lambda^2}. \tag{5.3}$$

Proof. Analogous to the proof of Lemma 2.1. □

Theorem 5. *The generating function for partitions into at most N parts in which odd indexed parts that are odd are larger than the next part (with the exponent on x_i being the ith part of the partition under consideration) is: if $N=2\nu$,*

$$(1+x_1)(1+x_1^2x_2^2\dots x_{2\nu-1}^2x_{2\nu})\frac{\prod_{j=1}^{\nu-1}\left(1+x_1^2x_2^2\dots x_{2j-1}^2(x_{2j}+x_{2j}x_{2j+1}+x_{2j}^2x_{2j+1})\right)}{\prod_{j=1}^{2\nu}(1-x_1^2x_2^2\dots x_j^2)}, \tag{5.4}$$

and if $N=2\nu+1$,

$$(1+x_1)\frac{\prod_{j=1}^{\nu}\left(1+x_1^2x_2^2\dots x_{2j-1}^2(x_{2j}+x_{2j}x_{2j+1}+x_{2j}^2x_{2j+1})\right)}{\prod_{j=1}^{2\nu+1}(1-x_1^2x_2^2\dots x_j^2)}. \tag{5.5}$$

Proof. First note that the case $N = 2\nu$ follows from the case $N = 2\nu + 1$ by setting $x_{2\nu+1} = 0$.

We now proceed by mathematical induction on N odd. If $N = 1$, then clearly the generating function is

$$\frac{1}{1 - x_1}$$

which is what the theorem asserts.

Next we rewrite the case $N = 2\nu + 1$ in terms of the "crude generating function"; i.e., as the result of the action of the $\Omega_{\geq}$ operator:

$$\underset{\geq}{\Omega} \sum_{n_1,\ldots,n_{2\nu+1}\geq 0} x_1^{n_1} x_2^{n_2} \ldots x_{2\nu+1}^{n_{2\nu+1}} \lambda_1^{n_1-n_2-\psi(n_1)} \lambda_2^{n_2-n_3} \ldots \lambda_{2\nu-1}^{n_{2\nu-1}-n_{2\nu}-\psi(n_{2\nu-1})} \lambda_{2\nu}^{n_{2\nu}-n_{2\nu+1}}. \tag{5.6}$$

Observe that the n_{2i} series are simple geometric series, while the n_{2i-1} series are of the form as in Lemma 5.1.

Thus the previous $\Omega_{\geq}$ expression equals

$$(1 + x_1)\, \underset{\geq}{\Omega} \frac{\big(1 + \frac{x_3}{\lambda_2}\big)\big(1 + \frac{x_5}{\lambda_4}\big) \ldots \big(1 + \frac{x_{2\nu-1}}{\lambda_{2\nu-2}}\big)}{(1 - \lambda_1^2 x_1^2)\big(1 - \frac{\lambda_2 x_2}{\lambda_1}\big)\big(1 - \frac{\lambda_3^2 x_3^2}{\lambda_2^2}\big) \ldots \big(1 - \frac{\lambda_{2\nu} x_{2\nu}}{\lambda_{2\nu-1}}\big)\big(1 - \frac{x_{2\nu+1}}{\lambda_{2\nu}}\big)}. \tag{5.7}$$

To reduce the $2\nu + 1$ case to the $2\nu - 1$ case, we must eliminate both λ_1 and λ_2.

To eliminate λ_1, we apply Lemma 2.2 to obtain

$$\begin{aligned} &\frac{1 + x_1}{1 - x_1^2}\, \underset{\geq}{\Omega} \frac{(1 + x_1^2 x_2 \lambda_2)\big(1 + \frac{x_3}{\lambda_2}\big)\big(1 + \frac{x_5}{\lambda_4}\big) \ldots \big(1 + \frac{x_{2\nu-1}}{\lambda_{2\nu-2}}\big)}{(1 + x_1^2 x_2^2 \lambda_2^2)\big(1 - \frac{\lambda_3^2 x_3^2}{\lambda_2^2}\big) \ldots \big(1 - \frac{\lambda_{2\nu} x_{2\nu}}{\lambda_{2\nu-1}}\big)\big(1 - \frac{x_{2\nu+1}}{\lambda_{2\nu}}\big)} \\ &\quad = \frac{1 + x_1}{1 - x_1^2}\, \underset{\geq}{\Omega} \frac{\big(1 + x_1^2 x_2 \lambda_2 + x_1^2 x_2 x_3 + \frac{x_3}{\lambda_2}\big)\big(1 + \frac{x_5}{\lambda_4}\big) \ldots \big(1 + \frac{x_{2\nu-1}}{\lambda_{2\nu-2}}\big)}{(1 + x_1^2 x_2^2 \lambda_2^2)\big(1 - \frac{\lambda_3^2 x_3^2}{\lambda_2^2}\big) \ldots \big(1 - \frac{\lambda_{2\nu} x_{2\nu}}{\lambda_{2\nu-1}}\big)\big(1 - \frac{x_{2\nu+1}}{\lambda_{2\nu}}\big)}. \end{aligned}$$

The above expression new separates into 4 terms (according to the 4 term expression in the numerator). In each of these we eliminate λ_2 according to the relevant instance of Lemmas 2.4 and 2.5.

Hence the piecewise expression is equal to

$$\frac{1 + x_1}{1 - x_1^2} \cdot \frac{1 + x_1^2 x_2 + x_1^2 x_2 x_3 + x_1^2 x_2^2 x_3}{1 - x_1^2 x_2^2}\, \underset{\geq}{\Omega} \frac{\big(1 + \frac{x_5}{\lambda_4}\big) \ldots \big(1 + \frac{x_{2\nu-1}}{\lambda_{2\nu-2}}\big)}{(1 + x_1^2 x_2^2 x_3^2 \lambda_3^2)\big(1 - \frac{\lambda_4 x_4}{\lambda_3}\big) \ldots \big(1 - \frac{\lambda_{2\nu} x_{2\nu}}{\lambda_{2\nu-1}}\big)\big(1 - \frac{x_{2\nu+1}}{\lambda_{2\nu}}\big)}. \tag{5.8}$$

Now this expression involving $\Omega_{\geq}$ is precisely of the same form as the original $\Omega_{\geq}$ expression in (5.7) except that each index is increased by 2, ν is replaced by $\nu - 1$ and x_1 is replaced by $x_1 x_2 x_3$. I.e., we have a first order recurrence between the $N = 2\nu + 1$ and the $N = 2\nu - 1$ case. Now we note that the product on the right in (5.5) satisfies the same recurrence. Thus (5.5) is proved by mathematical induction. As noted previously, (5.4) follows from (5.5). □

6. Partitions With Distinct Odd Parts

Also in the next theorem the exponent of x_i is devoted to the ith part of the partition in question.

Theorem 6. *The generating function for partitions with at most k parts and distinct odds is*

$$\frac{(1 + x_1)(1 + x_1^2 x_2)(1 + x_1^2 x_2^2 x_3) \ldots (1 + x_1^2 x_2^2 \ldots x_{k-1}^2 x_k)}{(1 - x_1^2)(1 - x_1^2 x_2^2)(1 - x_1^2 x_2^2 x_3^2) \ldots (1 - x_1^2 x_2^2 \ldots x_k^2)}. \tag{6.1}$$

Proof. Exactly as in the proof of Theorem 2 with the change being that $\chi(N)$ is replaced by $\psi(N) = 1 - \chi(N)$. □

Corollary 3. *If we now invoke the Schmidt process where only odd subscripted summands are summed, then the single variable generating function is*

$$\frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Remark. This is the generating function for the "cubic" partitions considered by Hei-Chi Chan [12].

Proof. We let $k \to \infty$ in Theorem 6 and then set $x_{2j+1} = q, x_{2j} = 1$. This yields

$$\frac{(-q;q)_\infty}{(q^2;q^2)_\infty^2} = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

□

As a final comment in this section we note that we may do partition analysis to partitions where all parts differ by at least 2 and odd parts differ from other parts by at least 3. This will produce n-variable generating functions for the partitions in the so-called little Göllnitz theorems [3, Thms. 2.9 and 2.10] and [3, pp. 166–167].

7. Position Parity

This final topic is less intricate than the previous topics. However, it possesses a few surprises.

Theorem 7. *Let $A(n)$ denote the number of partitions of n in which the even indexed parts (ordered from largest to smallest) are even. Let $B(n)$ denote the number of partitions of n with no repeated odds and no part $\equiv 3 \pmod 4$. Then for $n \geq 0$,*

$$A(n) = B(n).$$

For the proof we need the following elimination rule [21, p. 102, Entry 348],

$$\underset{\geq}{\Omega} \frac{1}{(1-x\lambda)\big(1-\frac{y}{\lambda^2}\big)} = \frac{1}{(1-x)(1-x^2y)}. \tag{7.1}$$

Proof of Theorem 7. As in previous results, we consider the multivariable case in which the exponent of x_i is the ith part of the partition. The generating function for the $A(n)$-partitions is

$$\begin{aligned}
&\underset{\geq}{\Omega} \sum_{n_1,n_2,n_3,\cdots\geq 0} x_1^{n_1}x_2^{2n_2}x_3^{n_3}x_4^{2n_4}\cdots \lambda_1^{n_1-2n_2}\lambda_2^{2n_2-n_3}\lambda_3^{n_3-2n_4}\lambda_4^{2n_4-n_5}\cdots \\
&\quad = \underset{\geq}{\Omega} \frac{1}{(1-x_1\lambda_1)\big(1-\frac{x_2^2\lambda_2^2}{\lambda_1^2}\big)\big(1-\frac{x_3\lambda_3}{\lambda_2}\big)\big(1-\frac{x_4^2\lambda_4^2}{\lambda_3^2}\big)\cdots} \\
&\quad = \frac{1}{1-x_1}\underset{\geq}{\Omega} \frac{1}{(1-x_1^2x_2^2\lambda_2^2)\big(1-\frac{x_3\lambda_3}{\lambda_2}\big)\big(1-\frac{x_4^2\lambda_4^2}{\lambda_3^2}\big)\big(1-\frac{x_5\lambda_5}{\lambda_4}\big)\cdots} \quad \text{(by (7.1))} \\
&\quad = \frac{1}{(1-x_1)(1-x_1^2x_2^2)}\underset{\geq}{\Omega} \frac{1+x_1^2x_2^2x_3\lambda_3}{(1-x_1^2x_2^2x_3^2\lambda_3^2)\big(1-\frac{x_4^2\lambda_4^2}{\lambda_3^2}\big)\big(1-\frac{x_5\lambda_5}{\lambda_4}\big)\cdots} \quad \text{(by (2.4))} \\
&\quad = \frac{(1+x_1)(1+x_1^2x_2^2x_3)}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)}\underset{\geq}{\Omega} \frac{1}{(1-x_1^2x_2^2x_3^2x_4^2\lambda_4^2)\big(1-\frac{x_5\lambda_5}{\lambda_4}\big)\cdots} \quad \text{(by (2.7) and (2.8))} \\
&\quad \vdots \\
&\quad = \frac{(1+x_1)(1+x_1^2x_2^2x_3)(1+x_1^2x_2^2x_3^2x_4^2x_5)\cdots}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)\cdots}.
\end{aligned} \tag{7.2}$$

Putting each $x_i = q$, we find the resulting generating function the $A(n)$ to be

$$\prod_{n=0}^{\infty} \frac{1+q^{4n+1}}{1-q^{2n+2}}, \tag{7.3}$$

which is clearly also the generating function for the $B(n)$. □

If we perform the Schmidt process of setting $x_{2i+1} = q$ and $x_{2i} = 1$, we obtain a generating function

$$\begin{aligned} \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty^2} &= \frac{\left(\sum_{r=0}^{\infty}(-q)^{\binom{r+1}{2}}\right)^{-1}}{(q^2;q^2)_\infty} \qquad \text{(by [5, (2.2.13)])} \\ &= \frac{1}{(q;q)_\infty (q^4;q^4)_\infty}. \end{aligned} \tag{7.4}$$

Remark. This generating function continues the theme of cubic partitions from Corollary 3. Namely, as special instances $k = 2$ and $k = 4$ of generating functions of the form

$$\frac{1}{(q;q)_\infty (q^k;q^k)_\infty} = \sum_{n=0}^{\infty} p_k(n) q^n. \tag{7.5}$$

The $p_k(n)$ can be interpreted as the number of 2-color partitions where one of the colors appears only in parts that are multiples of k. These numbers satisfy a variety of arithmetic congruences. For example, for $n \geq 0$,

$$p_2(25n+22) \equiv 0 \pmod 5, \tag{7.6}$$

as proven in [13, Thm. 1.3], and

$$p_4(25n+20) \equiv 0 \pmod 5, \tag{7.7}$$

as proven in [1, eq. (1.11)]. There is quite some literature concerning congruences for various k, and there are also infinite families of congruences for powers of 3 and 5. In addition to the two references mentioned, here is an incomplete selection of further articles: [11], [12], [14], [15], [18], [28], and [29]. It was Byungchan Kim [19, p. 1] who coined the name "cubic" partitions. Kim [19, before Thm. 1] also points to the work of Eggan [17] who independently, and as special instances of a more general context, also considered infinite families of cubic partitions modulo powers of 3 and 5.

In Section 10 we return to the theme of cubic partitions. In particular, we will show how (7.6) and (7.7) can be proven algorithmically with an implementation of Radu's Ramanujan-Kolberg algorithm [24].

Continuing with "Schmidtization" of Theorem 2 and (7.2), one can similarly treat the case where the odd indexed parts are even. The resulting generating function is

$$\frac{(1+x_1^2x_2)(1+x_1^2x_2^2x_3^2x_4)(1+x_1^2x_2^2x_3^2x_4^2x_5^2x_6)\dots}{(1-x_1^2)(1-x_1^2x_2^2)(1-x_1^2x_2^2x_3^2)\dots}, \tag{7.8}$$

the perfect complement to (7.2), and the single generating function is

$$\frac{(-q^3;q^4)_\infty}{(q^2;q^2)_\infty^2} \tag{7.9}$$

being the complement to (7.3).

It is plausible to consider next partitions where the even indexed or odd indexed parts are odd. The results do not apparently lead to any striking partition identities or other results of combinatorial or arithmetic interest. So, we will only record one such result.

The generating function for partitions with $2n-1$ parts, where the odd indexed parts are odd, is given by

$$\frac{x_1x_2\dots x_k(1+x_1^2x_2)(1+x_1^2x_2^2x_3^3x_4)\dots(1+x_1^2\dots x_{2n-3}^2x_{2n-2})}{(1-x_1^2)(1-x_1^2x_2^2)\dots(1-x_1^2x_2^2\dots x_{2n-1}^2)}. \tag{7.10}$$

Perhaps it is worth noting that the case $n=3$ relates to triangles with integer sides.

Proposition 1. *The number of partitions of n into three parts wherein the first and third are odd equals the number of triangles with integer sides and perimeter n.*

Proof. Setting $n=2$ and each $x_i=q$ in (7.10), we find that the generating function for partitions in which the first and third part is odd is

$$\frac{q^3(1+q^3)}{(1-q^2)(1-q^4)(1-q^6)}=\frac{q^3}{(1-q^2)(1-q^3)(1-q^4)},$$

and this is also the generating function for the number of triangles with perimeter n and integer sides [16]. □

8. How to Use the Omega Package

In joint collaboration with the authors of this article, partition analysis has been implemented by Axel Riese in the form of the Omega package, [7] and [8], written in Mathematica. In this section we briefly describe how to use Omega to compute special instances of results presented in the Sections 2 to 7.

After placing the package in a directory where we open a Mathematica session, we read it in as follows[1]:

```
In[1]:= << RISC`Omega`
```

> Omega Package version 2.49 written by Axel Riese (in cooperation with George E. Andrews and Peter Paule) © Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz

Example 1. As a first example, we derive special instances of Lemma 2.2. The case $A=0$ is the left side of (2.4). MacMahon called an expression of this kind a crude generating function; we input it as follows:

```
In[2]:= crude0 = 1/((1 - λ^2 x) (1 - y/λ));
```

Next we eliminate λ:

```
In[3]:= OR[crude0, {λ}]
Out[3]=   1 + x y
        ---------------
        (1 - x) (1 - x y^2)
```

Let us do each of the cases $A=1$ and $A=2$ of (2.5) "in one stroke":

```
In[4]:= OR[λ^-2/((1 - λ^2 x) (1 - y/λ)), {λ}]
Out[4]=   x y + x
        ---------------
        (1 - x) (1 - x y^2)
```

and

```
In[5]:= OR[λ^-4/((1 - λ^2 x) (1 - y/λ)), {λ}]
Out[5]=   x^3 y + x^2
        ---------------
        (1 - x) (1 - x y^2)
```

We remark that the package can handle quite complicated (crude) generating functions; however, it cannot do this in generic form. For instance, the integer variable A in Lemmas 2.2 to 2.5 needs to be specified to a concrete integer.

[1]The package is freely available at `https://caa.risc.jku.at/software` upon password request via email to the second named author.

Example 2. As another example, let us see how Omega deals with the special instance $k = 4$ of Theorem 1. From (3.3), the elimination problem in this case is,

$$\underset{\geq}{\Omega}\, \frac{x_1x_2x_3x_4(1+x_1)\big(1+\frac{x_2}{\lambda_1}\big)\big(1+\frac{x_3}{\lambda_2}\big)\lambda_1^{-2}\lambda_2^{-2}\lambda_3^{-2}}{(1-x_1^2\lambda_1^2)\big(1-\frac{x_2^2\lambda_2^2}{\lambda_1^2}\big)\big(1-\frac{x_3^2\lambda_3^2}{\lambda_2^2}\big)\big(1-\frac{x_4}{\lambda_3}\big)}.$$

Omega eliminates all the λ_j variables "in one stroke" as follows:

```
In[6]:= OR[(x1 x2 x3 x4 (1 + x1) (1 + x2/λ1) (1 + x3/λ2) λ1^-2 λ2^-2 λ3^-2)/
          ((1 - x1^2 λ1^2) (1 - x2^2 λ2^2/λ1^2) (1 - x3^2 λ3^2/λ2^2) (1 - x4/λ3)), {λ1, λ2, λ3}]

Eliminating  λ3 ...
Eliminating  λ2 ...
Eliminating  λ1 ...

Out[6]= (x1^7 (x1^6 x3^6 x4^2 x2^10 + x3^3 x4 x2^5 + x1^2 (x3^5 x4^2 x2^7 + x3^4 x4 x2^7 + x3^3 x4 x2^6) + x1^4 (x3^6 x4^2 x2^9 + x3^5 x4^2 x2^8 + x3^4 x4 x2^8)))/
          ((1 - x1) (1 - x1^2 x2^2) (1 - x1^2 x2^2 x3^2) (1 - x1^2 x2^2 x3^2 x4^2))

In[7]:= Factor[Numerator[%]]

Out[7]= x1^7 x2^5 (1 + x1^2 x2) x3^3 (1 + x1^2 x2^2 x3) x4 (1 + x1^2 x2^2 x3^2 x4)
```

We note that Omega follows built-in criteria for how to choose the order in which the λ_j variables are eliminated; here λ_3 first, then λ_2, and finally λ_1, as Omega indicates in the output. The user can execute these steps also individually, for instance, to eliminate in the first step only λ_3 is done as follows:

```
In[8]:= OR[(x1 x2 x3 x4 (1 + x1) (1 + x2/λ1) (1 + x3/λ2) λ1^-2 λ2^-2 λ3^-2)/
          ((1 - x1^2 λ1^2) (1 - x2^2 λ2^2/λ1^2) (1 - x3^2 λ3^2/λ2^2) (1 - x4/λ3)), {λ3}]

Eliminating  λ3 ...

Out[8]= (x1 x2 x3^3 x4 (λ1 λ3^2 + λ2^3 x1 x2 + λ2^3 x2 + λ1 λ2^3 x1 + x3^2 (λ2 x1 x2 x4 + λ2 x2 x4 + λ1 λ2 x1 x4 + λ1 λ2 x4)))/
          (λ1^3 λ2^7 (1 - λ1^2 x1^2) (1 - x3^2 x4^2/λ2^2) (1 - x3/λ2) (1 - λ2^2 x2^2/λ1^2))
```

The criteria used by Omega for the elimination of the λ_js originate from considerations of generic situations. This means, if the user in specific situations chooses a different elimination order could lead to more compact representations or to more transparent patterns better suitable for mathematical induction. We restrict to show what happens if, as in the proof of Theorem 1, in the first step λ_1 instead of λ_3 is eliminated:

```
In[9]:= OR[(x1 x2 x3 x4 (1 + x1) (1 + x2/λ1) (1 + x3/λ2) λ1^-2 λ2^-2 λ3^-2)/
          ((1 - x1^2 λ1^2) (1 - x2^2 λ2^2/λ1^2) (1 - x3^2 λ3^2/λ2^2) (1 - x4/λ3)), {λ1}]

Eliminating  λ1 ...

Out[9]= (x1^3 (x1^2 ((x3^2 x4 x2^2)/λ2 + x3 x4 x2^2) + (x2 x3^2 x4)/λ2 + x2 x3 x4))/
          (λ2^2 λ3^2 (1 - x1) (1 - λ2^2 x1^2 x2^2) (1 - x4/λ3) (1 - λ3^2 x3^2/λ2^2))

In[10]:= Factor[%]

Out[10]= (x1^3 x2 (x2 x1^2 + 1) x3 x4 (λ2 + x3))/
          (λ2 λ3 (x1 - 1) (λ2 x1 x2 - 1) (λ2 x1 x2 + 1) (x4 - λ3) (λ3 x3 - λ2) (λ2 + λ3 x3))
```

The factorization `Out[10]` of expression `Out[9]` indeed gives the first elimination step used in our proof of Theorem 1:

$$\frac{x_1^3x_2x_3x_4\left(1+x_1^2x_2\right)\left(1+\frac{x_3}{\lambda_2}\right)}{\lambda_2^2\lambda_3^2\,(1-x_1)\,(1-x_1^2x_2^2\lambda_2^2)\left(1-\frac{\lambda_3^2x_3^2}{\lambda_2^2}\right)\left(1-\frac{x_4}{\lambda_3}\right)}.$$

Example 3. As another example we revisit the proof of Theorem 5 for odd N; i.e., $N = 2\nu + 1$.

The first non-trivial case is $\nu = 1$. According to (5.7) the expression for the crude generating function is

```
In[11]:= crude1 = (1 + x1)/((1 - λ1^2 x1^2) (1 - λ2 x2/λ1) (1 - x3/λ2))
```

In contrast to the order used in the proof of Theorem 5, we now eliminate λ_2 first:

```
In[12]:= crude1a = OR[crude1, {λ2}]
```

```
                Eliminating  λ2 ...
                          1 + x1
Out[12]= ------------------------------------
         (1 - λ1^2 x1^2) (1 - x2 x3/λ1) (1 - x2/λ1)
```

For this step, we note that Omega executes one of MacMahon's most elementary rules [21, bottom of p. 92].

Next we let Omega eliminate λ_1,

```
In[13]:= crude1a = OR[crude1a, {λ1}]
                Eliminating  λ1 ...
                          1 + x1^2 (x2 + x3 x2 + x3 x2^2)
Out[13]= ------------------------------------------------
         (1 - x1) (1 - x1^2 x2^2) (1 - x1^2 x2^2 x3^2)
```

which confirms the statement of Theorem 5 for $N = 3$.

Remark. We want to note that in this step Omega executes another elimination rule of MacMahon [21, p. 103, Entry 348], a different generalization of (2.4).

The next case is $\nu = 2$; i.e., $N = 5$. According to (5.7) the expression for the crude generating function is

```
In[14]:= crude2 = (1 + x1) (1 + x3/λ2) / ((1 - λ1^2 x1^2) (1 - λ2 x2/λ1) (1 - λ3^2 x3^2/λ2^2) (1 - λ4 x4/λ3) (1 - x5/λ4))
```

Now one sees the advantage of proceeding as in our proof. Namely, in order to eliminate again λ_2 first, one would need to introduce another elimination rule. And, we can see that also Omega prefers the elimination order as used in our proof:

```
In[15]:= OR[crude2, {λ1, λ2}]
                Eliminating  λ1 ...
                Eliminating  λ2 ...
                          1 + x1^2 (x2 + x2 x3 + x2^2 x3)
Out[15]= ------------------------------------------------------------
         (1 - x1) (1 - x1^2 x2^2) (1 - λ3 x1^2 x2^2 x3^2) (1 - λ4 x4/λ3) (1 - x5/λ4)
```

In the output expression `Out[15]` one can see that the pattern for mathematical induction takes ground: compare the λ-pattern to the expression `Out[11]`.

Example 4. The final example for this section concerns the proof of Theorem 7. Consider the instance

$$\underset{\geq}{\Omega} \sum_{n_1,n_2,n_3,n_4\geq 0} x_1^{n_1} x_2^{2n_2} x_3^{n_3} x_4^{2n_4} \lambda_1^{n_1-2n_2} \lambda_2^{2n_2-n_3} \lambda_3^{n_3-2n_4} \lambda_4^{2n_4}.$$

Omega supports to rewrite this crude generating as a rational function:

```
In[16]:= crude4 = OSum[x1^n1 x2^(2 n2) x3^n3 x4^(2 n4) λ1^(n1 - 2 n2) λ2^(2 n2 - n3) λ3^(n3 - 2 n4) λ4^(2 n4), {n1 ≥ 0, n2 ≥ 0, n3 ≥ 0, n4 ≥ 0}, λ]
Out[16]= Ω≥ 1 / ((1 - λ1 x1) (1 - λ2^2 x2^2/λ1^2) (1 - λ3 x3/λ2) (1 - λ4^2 x4^2/λ3^2))
```

Then one eliminates the λ_j as follows:

```
In[17]:= crude4a = crude4[[1]]
                          1
Out[17]= ------------------------------------------
         (1 - λ1 x1) (1 - λ2^2 x2^2/λ1^2) (1 - λ3 x3/λ2) (1 - λ4^2 x4^2/λ3^2)
In[18]:= OR[crude4a, {λ1, λ2, λ3, λ4}]
                Eliminating  λ4 ...
                Eliminating  λ3 ...
                Eliminating  λ2 ...
                Eliminating  λ1 ...
                          1 + x1^2 x3 x2^2
Out[18]= ------------------------------------------------------------------
         (1 - x1) (1 - x1^2 x2^2) (1 - x1^2 x2^2 x3^2) (1 - x1^2 x2^2 x3^2 x4^2)
```

The instance for 6 variables $x_1, \ldots, x_6$ works as follows:

```
In[19]:= crude6 = OSum[x1^n1 x2^(2 n2) x3^n3 x4^(2 n4) x5^n5 x6^(2 n6) λ1^(n1 - 2 n2) λ2^(2 n2 - n3) λ3^(n3 - 2 n4) λ4^(2 n4) λ5^(n5 - 2 n6) λ6^(2 n6),
                       {n1 ≥ 0, n2 ≥ 0, n3 ≥ 0, n4, n5 ≥ 0, n6 ≥ 0}, λ]
```

```
Out[19]=  Ω  ≥  1 / ((1 - λ1 x1) (1 - λ2^2 x2^2 / λ1^2) (1 - λ3 x3 / λ2) (1 - λ4^2 x4^2 / λ3^2) (1 - λ5 x5 / λ4) (1 - λ6^2 x6^2 / λ5^2))

In[20]:= crude6a = OR[crude6[[1]], {λ1, λ2, λ3, λ4, λ5, λ6}]
Eliminating  λ6 . . .
Eliminating  λ5 . . .
Eliminating  λ4 . . .
Eliminating  λ2 . . .
Eliminating  λ1 . . .
Eliminating  λ3 . . .

Out[20]=  (1 + x1^2 x2^2 x3 + x1^2 x2^2 x3^2 (x1^2 x2^2 x3 x5 x4^2 + x5 x4^2)) / ((1 - x1) (1 - x1^2 x2^2) (1 - x1^2 x2^2 x3^2) (1 - x1^2 x2^2 x3^2 x4^2) (1 - x1^2 x2^2 x3^2 x4^2 x5^2) (1 - x1^2 x2^2 x3^2 x4^2 x5^2 x6^2))

In[21]:= Factor[Numerator[crude6a]

Out[21]=  (1 + x1^2 x2^2 x3) (1 + x1^2 x2^2 x3^2 x4^2 x5)
```

This output confirms the form of the infinite product on the right side of (7.2).

9. Some Identities derived with Radu's Ramanujan-Kolberg Algorithm

As in [9] and [10] we include a couple of identities and arithmetical theorems related to the new generating functions considered; all these results were derived using Smoot's implementation [27] of Radu's Ramanujan-Kolberg algorithm [24]. More details on the usage of Smoot's package are given in Section 11.

The first object we look at is the result when we perform the Schmidt process of setting $x_{2i+1} = q$ and $x_{2i} = 1$ in (7.8); namely, the generating function

$$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty^2} = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^3}. \tag{9.1}$$

This leads us to consider the replacement $q^2 \to q$ and to define

$$S(q) = \sum_{k=0}^{\infty} s(k)q^k := \frac{(q^2;q^2)_\infty}{(q;q)_\infty^3} = 1 + 3q + 8q^2 + 19q^3 + 41q^4 + 83q^5 + 161q^6 + \dots . \tag{9.2}$$

We note that $S(q)$ is a relative to

$$D(q) = \sum_{k=0}^{\infty} d(k)q^k := \frac{(q^2;q^2)_\infty}{(q;q)_\infty^4},$$

the generating function for the Schmidt-type partitions obtained by summing the linking nodes in the plane partition diamonds of unrestricted length [9, Thm. 4]. Another relative to $S(q)$ is

$$\sum_{k=0}^{\infty} \mathrm{PDN1}(k)q^k := \frac{(q^2;q^2)_\infty}{(q;q)_\infty^5},$$

the generating function for the Schmidt-type partitions obtained by summing the linking nodes of modified partition diamonds of unrestricted length which was studied extensively in [10, Thm. 4].

In contrast to $d(k)$ and $\mathrm{PDN1}(k)$, the $s(k)$ do not seem to satisfy simple congruences involving small primes only. Nevertheless, with the help of Smoot's implementation [27] of Radu's Ramanujan-Kolberg algorithm [24], we found various identities we think are worth being listed. Details about how to derive and prove some of these identities are given in Section 11.

Theorem 8. *We have*

$$\sum_{n=0}^{\infty} s(2n)q^n \cdot \sum_{n=0}^{\infty} s(2n+1)q^n \equiv (q;q)_\infty^{11} \pmod 2. \tag{9.3}$$

Theorem 9. *We have*

$$\sum_{n=0}^{\infty} s(3n)q^n \cdot \sum_{n=0}^{\infty} s(3n+1)q^n \cdot \sum_{n=0}^{\infty} s(3n+2)q^n \equiv q\frac{(q^3;q^3)_\infty (q^9;q^9)_\infty^6}{(q;q)_\infty^{10}} \pmod 2. \tag{9.4}$$

Theorem 10. *We have*

$$\sum_{n=0}^{\infty} s(3n)q^n \cdot \sum_{n=0}^{\infty} s(3n+1)q^n \cdot \sum_{n=0}^{\infty} s(3n+2)q^n \equiv q\frac{(q^2;q^2)_\infty^{25}}{(q;q)_\infty^3} \pmod 3. \tag{9.5}$$

Theorem 11. *We have*

$$\sum_{n=0}^{\infty} s(5n+4)q^n \equiv \frac{(q^2;q^2)_\infty^6 (q^5;q^5)_\infty^{15}}{(q;q)_\infty^{18}(q^{10};q^{10})_\infty^5} \pmod 4. \tag{9.6}$$

Notice that

$$\frac{(q^2;q^2)_\infty^6 (q^5;q^5)_\infty^{15}}{(q;q)_\infty^{18}(q^{10};q^{10})_\infty^5} = S(q)^6 \cdot \frac{1}{S(q^5)^5}.$$

In view of this relation we want to conclude this list with a couple of identities for the inverse

$$T(q) = \sum_{k=0}^{\infty} t(k)q^k := \frac{1}{S(q)} = \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty} = 1 - 3q + q^2 + 2q^3 + 2q^4 - q^5 - 4q^6 + \dots \tag{9.7}$$

which seems to have a somewhat richer arithmetic structure than $S(q)$.

Before stating our identities beginning with Theorem 12, we provide some combinatorial facts on $T(q)$ and the $t(k)$.

Proposition 2. *For $k \geq 0$,*

$$t(k) \equiv Q(k) \pmod 4,$$

where $Q(n)$ is the number of partitions of n into distinct parts,

$$\sum_{n\geq 0} Q(n)q^n = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + 4q^6 + \dots .$$

Proof. By the fact $(1-q)^2 \equiv (1+q)^2 \pmod 4$,

$$\begin{aligned} T(q) = \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty} &\equiv \frac{(-q;q)^2(q;q)_\infty}{(q^2;q^2)_\infty} \pmod 4 \\ &= (-q;q)_\infty = \sum_{n\geq 0} Q(n)q^n. \end{aligned}$$

□

Proposition 3. *For $k \geq 0$ we have that $t(k)$ is even if k is not a pentagonal number. Otherwise $t(k)$ is odd.*

Proof.

$$\begin{aligned} (-q;q)_\infty &\equiv (q;q)_\infty \pmod 2 \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2}. \end{aligned}$$

□

Corollary 4. *For $k \geq 0$,*

$$t(5n+3) \equiv t(5n+4) \equiv 0 \pmod 2. \tag{9.8}$$

Proof. Pentagonal numbers are all $\equiv 0, 1, 2 \pmod 5$. □

Now, as announced above, we are ready to list further identities.

Theorem 12. *We have*

$$f \cdot \sum_{n=0}^{\infty} t(5n+1)q^n = -4 - 3\,t, \tag{9.9}$$

where

$$f = \frac{\left(q^2;q^2\right)_\infty \left(q^5;q^5\right)_\infty^2}{q(q;q)_\infty (q^{10};q^{10})_\infty^4} \quad \textit{and} \quad t = \frac{\left(q^2;q^2\right)_\infty \left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5}.$$

Theorem 13. *We have*

$$\sum_{n=0}^{\infty} t(5n+3)q^n \cdot \sum_{n=0}^{\infty} t(5n+4)q^n = 4\,\frac{(q;q)_\infty (q^5;q^5)_\infty (q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty}. \tag{9.10}$$

We note that the factor 4 distributes over the series on the left which gives an alternative proof of Corollary 4.

Consider the series expansion of the product on the right side of (9.10),

$$\frac{(q;q)_\infty (q^5;q^5)_\infty (q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty} c(n)q^n.$$

Theorem 14. *For this generating function we have*

$$\sum_{n=0}^{\infty} c(5n+4)q^n = \frac{(q;q)_\infty^3 (q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty (q^5;q^5)_\infty}. \tag{9.11}$$

Theorem 15. *We have*

$$\sum_{n=0}^{\infty} t(5n)q^n \cdot \sum_{n=0}^{\infty} t(5n+2)q^n = \frac{(q^5;q^5)_\infty^6}{(q^{10};q^{10})_\infty^2} = T(q^5)^2. \tag{9.12}$$

Note that we obtained the square of a dilated version of $T(q)$. The dilation $q \to q^5$ implies for $j \in \{1,2,3,4\}$ and $n \geq 0$,

$$\sum_{k=0}^{5n+j} t(25n+5j-5k)t(5k+2) = 0, \tag{9.13}$$

and the square implies for odd $n \geq 0$,

$$\sum_{k=0}^{5n} t(25n-5k)t(5k+2) \equiv 0 \pmod 2. \tag{9.14}$$

Theorem 16. *We have*

$$\sum_{n=0}^{\infty} t(7n+2)q^n = \frac{(q^7;q^7)_\infty^3}{(q^{14};q^{14})_\infty} = T(q^7). \tag{9.15}$$

The dilation $q \to q^7$ implies for $j \in \{1,2,\ldots,6\}$ and $n \geq 0$,

$$t(49n+7j+2) = 0. \tag{9.16}$$

Concerning $T(q)$ we restrict to list one more relation.

Theorem 17. *We have*

$$\sum_{n=0}^{\infty} t(7n)q^n \cdot \sum_{n=0}^{\infty} t(7n+1)q^n \cdot \sum_{n=0}^{\infty} t(7n+5)q^n \equiv -\frac{(q^7;q^7)_\infty^9}{(q^{14};q^{14})_\infty^3} \pmod 4. \tag{9.17}$$

Notice that

$$\frac{(q^7;q^7)_\infty^9}{(q^{14};q^{14})_\infty^3} = T(q^7)^3.$$

10. Cubic Partitions and the Ramanujan-Kolberg Algorithm

In this section we return to the theme of cubic partitions (7.5) which we are now reconsidering being equipped with Smoot's implementation of Radu's Ramanujan-Kolberg algorithm. For convenience, we recall the defining generating function for the case $k = 2$ where we write $a(n) = p_2(n)$ for short; i.e.,

$$\frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \sum_{n=0}^{\infty} a(n) q^n. \tag{10.1}$$

Again we present a couple of results which seem to be new and which we obtained algorithmically. With regard to how this was done, the reader finds supplementing descriptions in Section 11 which should put her/him into the position to derive these and also all our other entries herself/himself.

The cases $a(2n)$ and $a(2n+1)$.

Theorem 18. *We have*

$$\sum_{n=0}^{\infty} a(2n) q^n \cdot \sum_{n=0}^{\infty} a(2n+1) q^n = \frac{\left(q^2;q^2\right)_\infty^2 \left(q^8;q^8\right)_\infty^2}{\left(q^4;q^4\right)_\infty (q;q)_\infty^7}. \tag{10.2}$$

Notice that this implies

$$\sum_{n=0}^{\infty} a(2n) q^n \cdot \sum_{n=0}^{\infty} a(2n+1) q^n \equiv (q;q)_\infty^{11} \pmod 2; \tag{10.3}$$

i.e., modulo 2 we see the same pattern as in (9.3).

The case $a(3n+2)$.

Theorem 19. *We have*

$$f \cdot \sum_{n=0}^{\infty} a(3n+2) q^n = 3(8+t), \tag{10.4}$$

where

$$f = \frac{(q;q)_\infty \left(q^2;q^2\right)_\infty^7 \left(q^3;q^3\right)_\infty^6}{q(q^6;q^6)_\infty^{12}} \quad \textit{and} \quad t = \frac{(q;q)_\infty^5 \left(q^3;q^3\right)_\infty}{q(q^2;q^2)_\infty (q^6;q^6)_\infty^5} \tag{10.5}$$

are produced by Smoot's package.

Applying the algorithm described in [4, Thm 10.3], which converts power series into product form, to the term

$$q(8+t) = 1 + 3q + 6q^2 + 4q^3 - 3q^4 - 12q^5 - 8q^6 + 12q^7 + 30q^8 + \dots,$$

one obtains

$$q(8+t) = (q^3;q^6)_\infty^6 \cdot \frac{(q^3;q^6)_\infty^3}{(q;q^2)_\infty^3} = \frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^9}{(q;q)_\infty^3 (q^6;q^6)_\infty^9}. \tag{10.6}$$

Hence, using the product on the right side,

$$\begin{aligned} \sum_{n=0}^{\infty} a(3n+2) q^n = 3\frac{1}{f}(8+t) &= 3\,\frac{(q^6;q^6)_\infty^{12}}{(q;q)_\infty \left(q^2;q^2\right)_\infty^7 \left(q^3;q^3\right)_\infty^6} \cdot q(8+t) \\ &= 3\,\frac{\left(q^3;q^3\right)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 \left(q^2;q^2\right)_\infty^4}, \end{aligned} \tag{10.7}$$

which is Chan's identity [11, Thm. 1].

We remark that also the proof of the first equality in (10.6) can be done algorithmically with another symbolic computation method by Radu [25]. Together with the algorithmic production of the relation (10.4) this gives a purely algorithmic derivation and proof of Chan's identity (10.7).

The case $a(9n+8)$.

Another aspect of the method concerns the possibility of iterating it. Consider the power series expansion of Chan's product,

$$\frac{\left(q^3;q^3\right)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 \left(q^2;q^2\right)_\infty^4} = \sum_{n=0}^{\infty} c(n) q^n.$$

Theorem 20. *Smoot's package computes*

$$f \cdot \sum_{n=0}^{\infty} c(3n+2) q^n = 9(8+t)\, p(t), \tag{10.8}$$

where

$$f = \frac{(q;q)_\infty^4 \left(q^2;q^2\right)_\infty^{22} \left(q^3;q^3\right)_\infty^{15}}{q^5 \left(q^6;q^6\right)_\infty^{39}} \quad \textit{and} \quad t = \frac{(q;q)_\infty^5 \left(q^3;q^3\right)_\infty}{q \left(q^2;q^2\right)_\infty \left(q^6;q^6\right)_\infty^5}, \tag{10.9}$$

and

$$p(t) = 15552 + 6480t + 1017t^2 + 72t^3 + 2t^4. \tag{10.10}$$

The factor 9 on the right of (10.8) implies that $9 \mid c(3n+2)$, which by (10.7) implies that for $n \geq 0$,

$$27 \mid a(3(3n+2)+2) = a(9n+8);$$

which is in correspondence to the special case $k=2$ of Chan's infinite family [12] of congruences modulo powers of 5.

The case $a(25n+22)$. We recall (7.6); i.e., for $n \geq 0$,

$$a(25n+22) \equiv 0 \pmod 5. \tag{10.11}$$

In this case the Ramanujan-Kolberg proof is a bit more tricky, but up to a human observation can be done also algorithmically as follows.

To prove (10.11) we start with the arithmetic subsequence $a(5n+2)$.

Theorem 21. *Smoot's package gives*

$$f \cdot \sum_{n=0}^{\infty} a(5n+2) q^n = p(t), \tag{10.12}$$

where

$$f = \frac{(q;q)_\infty^3 \left(q^2;q^2\right)_\infty^9 \left(q^5;q^5\right)_\infty^{10}}{q^5 \left(q^{10};q^{10}\right)_\infty^{20}} \quad \textit{and} \quad t = \frac{\left(q^2;q^2\right)_\infty \left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty \left(q^{10};q^{10}\right)_\infty^5} \tag{10.13}$$

and

$$p(t) = t(48 - 28t + 53t^2 + 7t^3 + 3t^4). \tag{10.14}$$

Now the crucial (human) observation is this:

$$p(t) \equiv 3t(1+t)^4 \pmod 5.$$

In view of this nice factorization and our experience made with (10.6), we again apply the algorithm from [4, Thm 10.3], which converts power series into product form, to the expression

$$q(1+t) = 1 + 2q + q^2 + 2q^3 + 2q^4 - 2q^5 - q^6 - 4q^8 - 2q^9 + \dots .$$

As the result of this conversion we obtain

$$q(1+t) = \frac{(q^2;q^2)_\infty^4 (q^5;q^5)_\infty^2}{(q;q)_\infty^2 (q^{10};q^{10})_\infty^4}. \tag{10.15}$$

Hence, using the product on the right side,

$$\sum_{n=0}^{\infty} a(5n+2)q^n \equiv 3\frac{1}{f}\,t(1+t)^4 \equiv 3\frac{\left(q^{10};q^{10}\right)_\infty^{20}}{(q;q)_\infty^3\,(q^2;q^2)_\infty^9\,(q^5;q^5)_\infty^{10}}(qt)\big(q(1+t)\big)^4 \pmod 5$$

$$\equiv 3\,\frac{\left(q^{10};q^{10}\right)_\infty^{20}}{(q;q)_\infty^3\,(q^2;q^2)_\infty^9\,(q^5;q^5)_\infty^{10}}\cdot\frac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{(q;q)_\infty\,(q^{10};q^{10})_\infty^5}\cdot\frac{(q^2;q^2)_\infty^{16}(q^5;q^5)_\infty^8}{(q;q)_\infty^8(q^{10};q^{10})_\infty^{16}} \pmod 5$$

$$\equiv 3\frac{\left(q^2;q^2\right)_\infty^8\left(q^5;q^5\right)_\infty^3}{(q;q)_\infty^{12}\,(q^{10};q^{10})_\infty} \equiv 3\frac{\left(q^2;q^2\right)_\infty^8(q;q)_\infty^{15}}{(q;q)_\infty^{12}(q^2;q^2)_\infty^5} \equiv 3\,(q;q)_\infty^3(q^2;q^2)_\infty^3 \pmod 5. \tag{10.16}$$

Consider the power series expansion of the product on the right side of (10.16),

$$(q;q)_\infty^3(q^2;q^2)_\infty^3 = \sum_{n=0}^{\infty} A(n)q^n.$$

Theorem 22. *Smoot's package computes,*

$$\sum_{n=0}^{\infty} A(5n+4)q^n = 25\,q\left(q^5;q^5\right)_\infty^3\left(q^{10};q^{10}\right)_\infty^3. \tag{10.17}$$

The factor 25 on the right of (10.17) implies that $25 \mid A(5n+4)$. Hence for all $n \geq 0$,

$$A(5n+4) \equiv 0 \pmod 5,$$

which by (10.16) implies that for $n \geq 0$,

$$a(5(5n+4)+2) = a(25n+22) \equiv 0 \pmod 5, \tag{10.18}$$

which is (7.6) and also the special case $\alpha = 0$ of Xiong's infinite family of congruences [28, Thm. 1.3].

Remark. An alternative proof of(10.18) emerges when using the first product in (10.16),

$$\frac{\left(q^2;q^2\right)_\infty^8\left(q^5;q^5\right)_\infty^3}{(q;q)_\infty^{12}\,(q^{10};q^{10})_\infty} = \sum_{n=0}^{\infty} B(n)q^n.$$

For this generating function Smoot's package computes,

$$f\cdot\sum_{n=0}^{\infty} B(5n+4)q^n = 5\,t\,p(t) \tag{10.19}$$

with f, t being products involving only terms of the form $(1-q^\alpha)^\beta$, and where $p(t)$ is a polynomial in t of the form

$$p(t) = 872415232 + 7952400384t + \cdots + 367t^{12}.$$

We conclude this section with a few results on cubic partitions $b(n) := p_4(n)$; i.e., for the case $k = 4$ of (7.5) which is

$$\frac{1}{(q;q)_\infty(q^4;q^4)_\infty} = \sum_{n=0}^{\infty} b(n)q^n. \tag{10.20}$$

The cases $b(2n)$ and $b(2n+1)$.

Theorem 23. *We have*

$$f\cdot\sum_{n=0}^{\infty} b(2n)q^n\cdot\sum_{n=0}^{\infty} b(2n+1)q^n = -2+t, \tag{10.21}$$

where

$$f = \frac{(q;q)_\infty^5\left(q^8;q^8\right)_\infty^4}{q\left(q^4;q^4\right)_\infty\left(q^{16};q^{16}\right)_\infty^4} \quad \textit{and} \quad t = \frac{\left(q^8;q^8\right)_\infty\left(q^2;q^2\right)_\infty^5}{q(q;q)_\infty^2\,(q^4;q^4)_\infty^2\,(q^{16};q^{16})_\infty^2} \tag{10.22}$$

are produced by Smoot's package.

Applying the algorithm described in [4, Thm 10.3], which converts power series into product form, to the term

$$q(-2+t) = 1 + 2q^4 - q^8 - 2q^{12} + 3q^{16} + 2q^{20} - 4q^{24} - 4q^{28} + 5q^{32} + \dots,$$

one obtains

$$q(-2+t) = \frac{(q^8;q^8)_\infty^6}{(q^4;q^4)_\infty^2 (q^{16};q^{16})_\infty^4}. \tag{10.23}$$

Consequently, we found and proved the following identity.

Theorem 24. *We have*

$$\sum_{n=0}^{\infty} b(2n)q^n \cdot \sum_{n=0}^{\infty} b(2n+1)q^n = \frac{1}{qf} \cdot q(-2+t) = \frac{\left(q^8;q^8\right)_\infty^2}{\left(q^4;q^4\right)_\infty (q;q)_\infty^5}. \tag{10.24}$$

The case $b(25n+20)$. We recall (7.7); i.e., for $n \geq 0$,

$$b(25n+20) \equiv 0 \pmod 5. \tag{10.25}$$

Also in this case the Ramanujan-Kolberg proof is bit tricky, but again—up to some extra (human) insight—can be done algorithmically as follows.

To prove (10.11) we start with the arithmetic subsequence $b(5n)$.

Theorem 25. *Smoot's package computes*

$$f \cdot \sum_{n=0}^{\infty} b(5n)q^n = p_1(t) + p_2(t)(s-t), \tag{10.26}$$

where

$$f = \frac{(q;q)_\infty^5 \left(q^4;q^4\right)_\infty^9 \left(q^{10};q^{10}\right)_\infty^{10}}{q^{11} \left(q^2;q^2\right)_\infty^2 \left(q^{20};q^{20}\right)_\infty^{20}}, t = \frac{\left(q^4;q^4\right)_\infty^4 \left(q^{10};q^{10}\right)_\infty^2}{q^2 \left(q^2;q^2\right)_\infty^2 \left(q^{20};q^{20}\right)_\infty^4}, \text{ and } s = \frac{\left(q^4;q^4\right)_\infty \left(q^5;q^5\right)_\infty^5}{q^3 (q;q)_\infty \left(q^{20};q^{20}\right)_\infty^5};$$

in addition,

$$p_1(t) = 125 - 250t + 190t^2 - 79t^3 + 11t^4 + 3t^5 \text{ and } p_2(t) = -15t^2 + 14t^3 + t^4.$$

Now the extra insight is this:

$$p_1(t) \equiv 3t^3(-2+t)(-1+t) \pmod 5 \quad \text{and} \quad p_2(t) \equiv t^3(-1+t) \pmod 5.$$

This turns the right side of (10.26) into

$$p_1(t) + p_2(t)(s-t) \equiv t^3(-1+t)\big(3(-2+t) + s - t\big) \pmod 5.$$

As in our proof of $5|a(25n+22)$, in view of this factorization we again apply the algorithm from [4, Thm 10.3], which converts power series into product form, to

$$q^2(-1+t) = 1 + q^2 + q^4 + 2q^6 + 2q^8 - 2q^{10} - q^{12} - 4q^{16} - 2q^{18} + 5q^{20} + 2q^{22} + \dots,$$

and to the expression

$$q^3\big(3(-2+t) + (s-t)\big) = 1 + 8q + 42q^2 + 176q^3 + 635q^4 + 2057q^5 + 6134q^6 + \dots.$$

As the result of this conversions we obtain

$$\begin{aligned} q^2(-1+t) &= \frac{\left(q^4;q^4\right)_\infty \left(q^{10};q^{10}\right)_\infty^5}{\left(q^2;q^2\right)_\infty \left(q^{20};q^{20}\right)_\infty^5} \\ &\equiv \frac{\left(q^2;q^2\right)_\infty^{24}}{\left(q^4;q^4\right)_\infty^{24}} \pmod 5 \end{aligned} \tag{10.27}$$

and

$$q^3(3(-2+t)+(s-t)) = \frac{\left(q^2;q^4\right)_\infty^4 (q^4;q^4)_\infty^2 \left(q^{10};q^{20}\right)_\infty^4}{(q;q^2)_\infty^3 (q^5;q^{10})_\infty \left(q^{20};q^{20}\right)_\infty^2} = \frac{(q^2;q^2)_\infty^7 (q^{10};q^{10})_\infty^5}{(q;q)_\infty^3 (q^4;q^4)_\infty^2 (q^5;q^5)_\infty (q^{20};q^{20})_\infty^6}$$

$$\equiv \frac{(q^2;q^2)_\infty^{32}}{(q;q)_\infty^8 (q^4;q^4)_\infty^{32}} \pmod 5. \tag{10.28}$$

Now, collecting things:

$$\begin{aligned}\sum_{n=0}^{\infty} b(5n)q^n &= \frac{1}{f}\big(p_1(t)+p_2(t)(s-t)\big)\\ &\equiv \frac{1}{q^{11}f}\cdot (q^2t)^3\cdot q^2(-1+t)\cdot q^3\big(3(-2+t)+s-t\big) \pmod 5\\ &\equiv \frac{\left(q^2;q^2\right)_\infty^2 \left(q^4;q^4\right)_\infty^{100}}{(q;q)_\infty^5 \left(q^4;q^4\right)_\infty^9 \left(q^2;q^2\right)_\infty^{50}} \cdot \frac{\left(q^2;q^2\right)_\infty^{24}}{\left(q^4;q^4\right)_\infty^{48}} \cdot \frac{\left(q^2;q^2\right)_\infty^{24}}{\left(q^4;q^4\right)_\infty^{24}} \cdot \frac{(q^2;q^2)_\infty^{32}}{(q;q)_\infty^8 (q^4;q^4)_\infty^{32}} \pmod 5\\ &\equiv \frac{\left(q^2;q^2\right)_\infty^{32}}{(q;q)_\infty^{13}\left(q^4;q^4\right)_\infty^{13}}. \qquad (10.29)\end{aligned}$$

Finally, consider the power series expansion of the product on the right side of (10.29),

$$\frac{\left(q^2;q^2\right)_\infty^{32}}{(q;q)_\infty^{13}\left(q^4;q^4\right)_\infty^{13}} = \sum_{n=0}^{\infty} C(n)q^n.$$

Theorem 26. *Smoot's package computes*

$$f\cdot\sum_{n=0}^{\infty} C(5n+4)q^n = p_1(t)+p_2(t)(s-t), \tag{10.30}$$

where

$$f = \frac{\left(q^5;q^5\right)_\infty (q;q)_\infty^4 \left(q^4;q^4\right)_\infty^8 \left(q^{10};q^{10}\right)_\infty^{10}}{q^{10}\left(q^2;q^2\right)_\infty^{10}\left(q^{20};q^{20}\right)_\infty^{19}}, t = \frac{\left(q^4;q^4\right)_\infty^4 \left(q^{10};q^{10}\right)_\infty^2}{q^2\left(q^2;q^2\right)_\infty^2\left(q^{20};q^{20}\right)_\infty^4} \text{ and}$$

$$s = \frac{\left(q^4;q^4\right)_\infty \left(q^5;q^5\right)_\infty^5}{q^3(q;q)_\infty\left(q^{20};q^{20}\right)_\infty^5};$$

in addition,

$$p_1(t) = 25(-1+t)^2(499-199t+143t^2+17t^3) \text{ and } p_2(t) = 50(-1+t)^2(63+31t).$$

Consequently, one can extract the common factor 25 from the expression on the right of (10.30) which implies that $25 \mid C(5n+4)$. Hence for all $n \geq 0$,

$$C(5n+4) \equiv 0 \pmod 5,$$

which by (10.30) implies that for $n \geq 0$,

$$b(5(5n+4)) = b(25n+20) \equiv 0 \pmod 5, \tag{10.31}$$

and which completes the algorithmic derivation and proof of (7.7).

11. How to Use Smoot's Ramanujan-Kolberg Package

In this section we briefly describe how to use Smoot's package RaduRK [27], an implementation of Radu's Ramanujan-Kolberg algorithm written in Mathematica.[2] In particular, we list all the procedure calls needed to derive, and thus to prove, all of the theorems stated in the Sections 9 and 10.

[2]The package is freely available at `https://caa.risc.jku.at/software` upon password request via email to the second named author.

To prepare for the usage of RaduRK, follow the installation instructions given in [27], and invoke the package within a Mathematica session as follows:

```
In[22]:= << RaduRK`
```

```
math4ti2: Mathematica interface to 4ti2
(http://www.4ti2.de)
© 2017, Ralf Hemmecke <ralf@hemmecke.org>
© 2017, Silviu Radu <sradu@risc.jku.at>
```

```
RaduRK: Ramanujan–Kolberg Program Version 3.4
2021 written by Nicolas Smoot
<nicolas.smoot@risc.jku.at>
© Research Institute for Symbolic Computation (RISC), Johannes
Kepler University Linz
```

Before running the program, one needs to set the two global key variables q and t:

```
In[23]:= {SetVar1[q], SetVar2[t]}
Out[23]= {q, t}
```

11.1. **Algorithmic proofs of Theorems 22, 23, 25, and 26.** Our first example to illustrate the usage of the RaduRK package is the derivation and proof of Theorem 22 which we restate for convenience. Consider the third power of the inverse of the generating function of cubic partitions (10.1),

$$\alpha(q) := (q;q)_\infty^3 (q^2;q^2)_\infty^3 = \sum_{n=0}^{\infty} A(n) q^n,$$

then

$$\sum_{n=0}^{\infty} A(5n+4) q^n = 25\, q \left(q^5;q^5\right)_\infty^3 \left(q^{10};q^{10}\right)_\infty^3 = 25\, q\, \alpha(q^5). \tag{11.1}$$

Proof of Theorem 22. The input for the algorithmic derivation and proof of (11.1) is done with the procedure call

```
In[24]:= RK[10, 2, {3, 3}, 5, 4]
```

After a few seconds, Smoot's package returns

Out[3] =

N:	10
$\{M, (r_\delta)_{\delta\mid M}\}$:	$\{2, (3,3)\}$
m:	5
$P_{m,r}(j)$:	$\{4\}$
$f_1(q)$:	$\frac{\left(q^2;q^2\right)_\infty^2 \left(q^5;q^5\right)_\infty^7}{q^3 (q;q)_\infty^2 \left(q^{10};q^{10}\right)_\infty^{13}}$
t:	$\frac{\left(q^2;q^2\right)_\infty \left(q^5;q^5\right)_\infty^5}{q (q;q)_\infty \left(q^{10};q^{10}\right)_\infty^5}$
AB:	$\{1\}$
$\{p_g(t):\ g \in \mathrm{AB}\}$	$\{25t^2\}$
Common Factor:	25

The interpretation of the output is as follows:

• The assignment $\{M, (r_\delta)_{\delta\mid M}\} = \{2, (3,3)\}$ comes from the second and third entry of the procedure call `RK[10,2,{3,3},5,4]`; this corresponds to specifying $M = 2$ and $(r_\delta)_{\delta\mid 2} = (r_1, r_2) = (3,3)$ to define the generating function of the $A(n)$ by the respective product,

$$\sum_{n=0}^{\infty} A(n) q^n = \prod_{\delta\mid M} (q^\delta;q^\delta)_\infty^{r_\delta} = (q,q)_\infty^3 (q^2,q^2)_\infty^3.$$

• The last two entries in the procedure call `RK[10,2,{3,3},5,4]` correspond to the assignment $m = 5$ and $j = 4$, which means that we are interested in the generating function

$$\sum_{n=0}^{\infty} A(mn+j)q^n = \sum_{n=0}^{\infty} A(5n+4)q^n.$$

In the output expression $P_{m,r}(j)$ these parameters m and j are used; i.e., here $P_{m,r}(j) = P_{2,r}(4)$ with $r = (r_\delta)_{\delta|2} = (3,3)$.

• The first entry in the procedure call `RK[10,2,{3,3},5,4]` corresponds to specifying $N = 10$, which fixes the space of modular functions the program will work with:

$$M(\Gamma_0(N)) := \text{ the algebra of modular functions for } \Gamma_0(N).$$

Remark. For the definition of notions such as $\Gamma_0(N)$ or $M(\Gamma_0(N))$, together with a general introduction to Radu's Ramanujan-Kolberg algorithm, see [23]. For the proof of correctness and further details of the algorithm, resp. of the implementation, see [24], resp. [27].

• The output $P_{m,r}(j) = P_{5,r}(4) = \{4\}$, where $r = (3,3)$, means that there exists a q-product

$$f_1(q) = \frac{\left(q^2;q^2\right)_\infty^2 \left(q^5;q^5\right)_\infty^7}{q^3(q;q)_\infty^2 \left(q^{10};q^{10}\right)_\infty^{13}},$$

which is also delivered by RaduRK as an output, such that

$$f_1(q) \prod_{k\in P_{5,r}(4)} \sum_{n=0}^{\infty} A(5n+k)q^n = f_1(q) \sum_{n=0}^{\infty} A(5n+4)q^n \in M(\Gamma_0(N)) \text{ with } N = 10.$$

Note. In general, the set $P_{m,r}(j)$ need not be a singleton. For example, $P_{m,r}(j) = \{0,1\}$ in the proof of Theorem 23 as discussed in the next example. If $P_{m,r}(j)$ is a singleton one speaks of an identity of Ramanujan-type and of Kolberg-type, otherwise.

• The output

$$t = \frac{\left(q^2;q^2\right)_\infty \left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty \left(q^{10};q^{10}\right)_\infty^5}, \quad \text{AB} = \{1\}, \quad \text{and} \quad \{p_g(t) : g \in \text{AB}\} = \{25t^2\} \tag{11.2}$$

presents a solution to the following task: find a modular function $t \in M(\Gamma_0(N))$ and polynomials $p_g(t)$ such that

$$f_1(q) \sum_{n=0}^{\infty} A(5n+4)q^n = \sum_{g\in \text{AB}} p_g(t) \cdot g. \tag{11.3}$$

In general, the elements of the finite set AB constitute a $\mathbb{C}[t]$-module basis of $M(\Gamma_0(N))$, resp. of a large subspace of $M(\Gamma_0(N))$. The elements g of AB are $\mathbb{C}$-linear combinations of modular functions in $M(\Gamma_0(N))$ which are representable in q-product form such as $f_1(q)$ and t. In the specific case under consideration, the program delivers (11.2), which means, $p_1(t) = p(t) = 25t^2$ and

$$f_1(q) \sum_{n=0}^{\infty} A(5n+4)q^n = 25t^2 \cdot 1.$$

Multiplying both sides by $1/f_1(q)$ completes the algorithmic derivation and proof of (11.1). □

Next we revisit Theorem 23 for the case $k = 4$ of cubic partitions (7.5),

$$\frac{1}{(q;q)_\infty (q^4;q^4)_\infty} = \sum_{n=0}^{\infty} b(n)q^n,$$

which is a statement of Kolberg-type,

$$f \cdot \sum_{n=0}^{\infty} b(2n)q^n \cdot \sum_{n=0}^{\infty} b(2n+1)q^n = -2 + t, \tag{11.4}$$

where f and t are as in (10.22). Recall, Kolberg-type simply means that running the RaduRK package leads to a set $P_{m,r}(j)$ containing more than one element. This is seen in the output of the corresponding procedure call where we choose $m = 2$ and $j = 0$,

```
In[25]:= RK[16, 4,{−1, 0, −1}, 2, 0]
```

Proof of Theorem 23. The procedure call as in `In[25]` produces the Ramanujan-Kolberg type identity (11.4) in the following format:

Out[25] =

N:	16
$\{M, (r_\delta)_{\delta\|M}\}$:	$\{4, (-1, 0, -1)\}$
m:	2
$P_{m,r}(j)$:	$\{0, 1\}$
$f_1(q)$:	f as in (10.22)
t:	t as in (10.22)
AB:	$\{1\}$
$\{p_g(t)$: $g \in$AB$\}$	$\{-2 + t\}$
Common Factor:	None

□

Remark. This time the output relation involves modular functions in $M(\Gamma_0(N))$ with $N = 16$. But now, according to the output $P_{m,r}(j) = \{0, 1\}$, the witness identity involves a product of generating functions,

$$f_1(q) \prod_{k \in P_{m,r}(j)} \sum_{n=0}^{\infty} \mathrm{PDN1}(2n + k)q^n = f_1(q) \sum_{n=0}^{\infty} \mathrm{PDN1}(2n)q^n \sum_{n=0}^{\infty} \mathrm{PDN1}(2n + 1)q^n = -2 + t,$$

Identities involving products in this form were first studied in systematic manner by Kolberg [20]. The entry "Common Factor" in the last output line refers to a possible common factor of all the integer coefficients of $-2 + t$. Here this common factor is trivial ($= 1$), which is indicated by "None."

As a last explicit Ramanujan-Kolberg example we present the RaduRK derivation and proof of Theorem 25 which is obtained with the procedure call

```
In[26]:= RK[20, 4,{−1, 0, −1}, 5, 0]
```

Proof of Theorem 25. The procedure call as in `In[26]` produces:

Out[26] =

N:	20
$\{M, (r_\delta)_{\delta\|M}\}$:	$\{4, (-1, 0, -1)\}$
m:	5
$P_{m,r}(j)$:	$\{0\}$
$f_1(q)$:	f as in Thm. 25
t:	t as in Thm. 25
AB:	$\{1, s - t\}$ with s, t as in Thm. 25
$\{p_g(t)$: $g \in$AB$\}$	$\{p_1(t), p_2(t)\}$ as in Thm. 25
Common Factor:	None

According to (11.3) this output means that

$$f_1(q) \sum_{n=0}^{\infty} b(5n)q^n = \sum_{g \in \mathrm{AB}} p_g(t) \cdot g = p_1(t) \cdot 1 + p_2(t) \cdot (s - t),$$

which is (10.26). □

Proof of Theorem 26. The algorithmic derivation and proof of (10.30) works exactly as in the proof of (10.26); it is done with the procedure call

```
In[27]:= RK[20, 4, {−13, 32, −13}, 5, 4]
```

□

11.2. **Algorithmic proofs of Theorems 8, 9, 10, 11, 12, 13, 14, 15, 16, and 17.** In this subsection we provide short descriptions of algorithmic derivations and proofs of our results related to the partition generating functions $S(q)$ and $T(q)$ which were defined in (9.2) and (9.7) as

$$S(q)=\sum_{k=0}^{\infty}s(k)q^k:=\frac{(q^2;q^2)_\infty}{(q;q)_\infty^3}\ \text{ and }\ T(q)=\sum_{k=0}^{\infty}t(k)q^k:=\frac{1}{S(q)}.$$

Proof of Theorem 8. The procedure call

```
In[28]:= RK[16, 2, {-3, 1}, 2, 0]
```

produces the following output:

Out[28] =

N:	16
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{2,(-3,1)\}$
m:	2
$P_{m,r}(j)$:	$\{0,1\}$
$f_1(q)$:	$\frac{(q;q)_\infty^{13}\left(q^8;q^8\right)_\infty^6}{q^3(q^4;q^4)_\infty(q^2;q^2)_\infty^6(q^{16};q^{16})_\infty^8}$
t:	$\frac{\left(q^8;q^8\right)_\infty\left(q^2;q^2\right)_\infty^5}{q(q;q)_\infty^2(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}$
AB:	$\{1\}$
$\{p_g(t):\ g\in\mathrm{AB}\}$	$\{-32+32t-14t^2+3t^3\}$
Common Factor:	None

This means that

$$f_1(q)\sum_{n=0}^{\infty}s(2n)q^n\cdot\sum_{n=0}^{\infty}s(2n+1)q^n=\sum_{g\in\mathrm{AB}}p_g(t)\cdot g=(-32+32t-14t^2+3t^3)\cdot 1.$$

Observing that

$$-32+32t-14t^2+3t^3\equiv t^3\pmod 2,$$

Theorem 8 follows:

$$\begin{aligned}\frac{1}{f_1(q)}t^3&=\frac{q^3\left(q^4;q^4\right)_\infty\left(q^2;q^2\right)_\infty^6\left(q^{16};q^{16}\right)_\infty^8}{(q;q)_\infty^{13}\left(q^8;q^8\right)_\infty^6}\frac{\left(q^8;q^8\right)_\infty^3\left(q^2;q^2\right)_\infty^{15}}{q^3(q;q)_\infty^6\left(q^4;q^4\right)_\infty^6\left(q^{16};q^{16}\right)_\infty^6}\\&\equiv\frac{(q;q)_\infty^4(q;q)_\infty^{12}(q;q)_\infty^{128}}{(q;q)_\infty^{13}(q;q)_\infty^{48}}\frac{(q;q)_\infty^{24}(q;q)_\infty^{30}}{(q;q)_\infty^6(q;q)_\infty^{24}(q;q)_\infty^{96}}\pmod 2\\&\equiv(q;q)_\infty^{11}\pmod 2.\end{aligned}$$

□

Proof of Theorem 9. The proof of Theorem 9 works the same as the proof of Theorem 8. Hence we restrict to state the corresponding input line whose output is explicitly stated in the proof of Theorem 10,

```
In[29]:= RK[18, 2, -3, 1, 3, 0]
```

□

Proof of Theorem 10. We start with the procedure call which is the same as for the proof of Theorem 9,

```
In[30]:= RK[18, 2, -3, 1, 3, 0]
```

We obtain as output:

Out[30] =

N:	18
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{2,(-3,1)\}$
m:	3
$P_{m,r}(j)$:	$\{0,2,1\}$
$f_1(q)$:	$\frac{(q;q)_\infty^{30}\left(q^6;q^6\right)_\infty^{11}\left(q^9;q^9\right)_\infty^{12}}{q^{11}(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^{13}(q^{18};q^{18})_\infty^{24}}$
t:	$\frac{\left(q^6;q^6\right)_\infty\left(q^9;q^9\right)_\infty^3}{q(q^3;q^3)_\infty(q^{18};q^{18})_\infty^3}$
AB:	$\{1\}$
$\{p_g(t)\colon\ g\in\mathrm{AB}\}$	$\{p_1(t)\}$
Common Factor:	None

where

$$\begin{aligned} p_1(t) &= 1024t + 1536t^2 + 1152t^3 + 4224t^4 + 2880t^5 + 3168t^6 + 2784t^7 + 2016t^8 \\ &\quad + 1206t^9 + 313t^{10} + 24t^{11} \\ &\equiv t(1+t)^9 \pmod 3. \end{aligned}$$

Applying the algorithm described in [4, Thm 10.3], which converts power series into product form, to the term

$$q(1+t) = 1 + q + q^3 + q^6 - q^9 - q^{12} + q^{18} + 2q^{21} - 2q^{27} - 3q^{30} - q^{33} + \dots,$$

gives

$$q(1+t) = \frac{\left(q^9;q^9\right)_\infty\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty\left(q^{18};q^{18}\right)_\infty^2}. \tag{11.5}$$

Hence we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} s(3n)q^n \cdot \sum_{n=0}^{\infty} s(3n+1)q^n \cdot \sum_{n=0}^{\infty} s(3n+2)q^n &= \frac{1}{f_1(q)} \sum_{g\in\mathrm{AB}} p_g(t)\cdot g = \frac{1}{f_1(q)}\cdot p_1(t) \\ &\equiv \frac{q^{11}\left(q^2;q^2\right)_\infty^{10}\left(q^3;q^3\right)_\infty^{13}\left(q^{18};q^{18}\right)_\infty^{24}}{(q;q)_\infty^{30}\left(q^6;q^6\right)_\infty^{11}\left(q^9;q^9\right)_\infty^{12}} \cdot \frac{t}{q^9}\cdot\left(q(1+t)\right)^9 \pmod 3 \\ &\equiv \frac{q\left(q^2;q^2\right)_\infty^{10}(q;q)_\infty^{39}\left(q^2;q^2\right)_\infty^{216}}{(q;q)_\infty^{30}(q^2;q^2)_\infty^{33}(q;q)_\infty^{108}} \cdot \frac{\left(q^2;q^2\right)_\infty^3(q;q)_\infty^{27}}{(q;q)_\infty^3(q^2;q^2)_\infty^{27}} \cdot \frac{(q;q)_\infty^{81}\left(q^2;q^2\right)_\infty^{18}}{(q;q)_\infty^9(q^2;q^2)_\infty^{162}} \pmod 3 \\ &\equiv q\frac{(q^2;q^2)_\infty^{25}}{(q;q)_\infty^3} \pmod 3. \end{aligned}$$

This completes the proof of Theorem 10. □

Proof of Theorem 11. The proof of Theorem 11 works as the proof of Theorem 8. Hence we restrict to state the corresponding procedure call,

In[31]:= **RK[10, 2, {−3, 1}, 5, 4]**

□

Proof of Theorem 12. All the constituents of Theorem 12 are computed with the procedure call,

In[32]:= **RK[10, 2, {3, −1}, 5, 1]**

□

Proof of Theorem 13. The procedure call,

In[33]:= **RK[10, 2, {3, −1}, 5, 3]**

delivers as output:

Out[33] =

N:	10
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{2,(3,-1)\}$
m:	5
$P_{m,r}(j)$:	$\{3,4\}$
$f_1(q)$:	$\frac{\left(q^2;q^2\right)_\infty^2\left(q^5;q^5\right)_\infty^4}{q(q;q)_\infty^2(q^{10};q^{10})_\infty^8}$
t:	$\frac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$
AB:	$\{1\}$
$\{p_g(t)\colon\ g\in\mathrm{AB}\}$	$\{4t\}$
Common Factor:	4

These data constitute all the ingredients of Theorem 13. □

Proof of Theorem 14. The procedure call,

In[34]:= **RK[10, 10, {1, −1, 1, 3}, 5, 4]**

delivers as output:

Out[34] =

N:	10
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{10,(1,-1,1,3)\}$
m:	5
$P_{m,r}(j)$:	$\{4\}$
$f_1(q)$:	$\frac{\left(q^2;q^2\right)_\infty^2\left(q^5;q^5\right)_\infty^6}{q(q;q)_\infty^4(q^{10};q^{10})_\infty^8}$
t:	$\frac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$
AB:	$\{1\}$
$\{p_g(t)\colon\ g\in\mathrm{AB}\}$	$\{t\}$
Common Factor:	None

These data constitute all the ingredients of Theorem 14. □

Proof of Theorem 15. The procedure call,

In[35]:= **RK[10, 2, {3, −1}, 5, 0]**

delivers as output:

Out[35] =

N:	10
$\{M,(r_\delta)_{\delta\mid M}\}$:	$\{2,(3,-1)\}$
m:	5
$P_{m,r}(j)$:	$\{0,2\}$
$f_1(q)$:	$\frac{\left(q^2;q^2\right)_\infty^2\left(q^5;q^5\right)_\infty^4}{q^2(q;q)_\infty^2(q^{10};q^{10})_\infty^8}$
t:	$\frac{\left(q^2;q^2\right)_\infty\left(q^5;q^5\right)_\infty^5}{q(q;q)_\infty(q^{10};q^{10})_\infty^5}$
AB:	$\{1\}$
$\{p_g(t)\colon\ g\in\mathrm{AB}\}$	$\{t^2\}$
Common Factor:	None

These data constitute all the ingredients of Theorem 15. □

Proof of Theorem 16. The procedure call,

In[36]:= **RK[14, 2, {3, −1}, 7, 2]**

delivers as output:

Out[36] =

N:	14
$\{M, (r_\delta)_{\delta\|M}\}$:	$\{2, (3, -1)\}$
m:	7
$P_{m,r}(j)$:	$\{2\}$
$f_1(q)$:	$\frac{\left(q^2;q^2\right)_\infty \left(q^7;q^7\right)_\infty^4}{q^2(q;q)_\infty (q^{14};q^{14})_\infty^6}$
t:	$\frac{\left(q^2;q^2\right)_\infty \left(q^7;q^7\right)_\infty^7}{q^2(q;q)_\infty (q^{14};q^{14})_\infty^7}$
AB:	$\left\{1, \frac{\left(q^2;q^2\right)_\infty^8 \left(q^7;q^7\right)_\infty^4}{q^3(q;q)_\infty^4 (q^{14};q^{14})_\infty^8} - 4t\right\}$
$\{p_g(t)$: $g \in$AB$\}$	$\{t, 0\}$
Common Factor:	None

This means that

$$f_1(q) \sum_{n=0}^{\infty} t(7n+2)q^n = \sum_{g \in \mathrm{AB}} p_g(t) \cdot g = t \cdot 1 + 0 \cdot \Big(\frac{\left(q^2;q^2\right)_\infty^8 \left(q^7;q^7\right)_\infty^4}{q^3(q;q)_\infty^4 \left(q^{14};q^{14}\right)_\infty^8} - 4t \Big) = t,$$

and dividing by $1/f_1(q)$ completes the proof of Theorem 16. □

Proof of Theorem 17. The procedure call,

In[37]:= **RK[14, 2, {3, −1}, 7, 0]**

delivers as output:

Out[37] =

N:	14
$\{M, (r_\delta)_{\delta\|M}\}$:	$\{2, (3, -1)\}$
m:	7
$P_{m,r}(j)$:	$\{0, 1, 5\}$
$f_1(q)$:	$\frac{\left(q^2;q^2\right)_\infty^3 \left(q^7;q^7\right)_\infty^{12}}{q^6(q;q)_\infty^3 (q^{14};q^{14})_\infty^{18}}$
t:	$\frac{\left(q^2;q^2\right)_\infty \left(q^7;q^7\right)_\infty^7}{q^2(q;q)_\infty (q^{14};q^{14})_\infty^7}$
AB:	$\left\{1, \frac{\left(q^2;q^2\right)_\infty^8 \left(q^7;q^7\right)_\infty^4}{q^3(q;q)_\infty^4 (q^{14};q^{14})_\infty^8} - 4t\right\}$
$\{p_g(t)$: $g \in$AB$\}$	$\{-8t + 24t^2 + 3t^3, 8t\}$
Common Factor:	None

This means that

$$\begin{aligned} f_1(q) \sum_{n=0}^{\infty} t(7n+2)q^n &= \sum_{g \in \mathrm{AB}} p_g(t) \cdot g \\ &= (-8t + 24t^2 + 3t^3) \cdot 1 + 8t \cdot \Big(\frac{\left(q^2;q^2\right)_\infty^8 \left(q^7;q^7\right)_\infty^4}{q^3(q;q)_\infty^4 \left(q^{14};q^{14}\right)_\infty^8} - 4t \Big) \\ &\equiv -t^3 \pmod 4 \end{aligned}$$

and dividing by $1/f_1(q)$ completes the proof of Theorem 17. □

11.3. **Algorithmic proofs of Theorems 18, 19, 20, and 21.** In this subsection we provide short descriptions of algorithmic derivations and proofs of our results related to the partition generating function of cubic partitions (10.1),

$$\frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \sum_{n=0}^{\infty} a(n)q^n.$$

Proof of Theorem 18. The procedure call,

```
In[38]:= RK[16, 2, {-1, -1}, 2, 0]
```

delivers as output:

Out[38] =

N:	16
$\{M, (r_\delta)_{\delta\mid M}\}$:	$\{2, (-1,-1)\}$
m:	2
$P_{m,r}(j)$:	$\{0,1\}$
$f_1(q)$:	$\frac{(q;q)_\infty^7 \left(q^8;q^8\right)_\infty^4}{q\left(q^4;q^4\right)_\infty (q^2;q^2)_\infty^2 \left(q^{16};q^{16}\right)_\infty^4}$
t:	$\frac{\left(q^8;q^8\right)_\infty \left(q^2;q^2\right)_\infty^5}{q(q;q)_\infty^2 \left(q^4;q^4\right)_\infty^2 \left(q^{16};q^{16}\right)_\infty^2}$
AB:	$\{1\}$
$\{p_g(t):\ g \in \mathrm{AB}\}$	$\{-2+t\}$
Common Factor:	None

Applying the algorithm described in [4, Thm 10.3], which converts power series into product form, to the term

$$q(-2+t) = 1 + 2q^4 - q^8 - 2q^{12} + 3q^{16} + 2q^{20} - 4q^{24} - 4q^{28} + 5q^{32} + \dots,$$

gives

$$q(-2+t) = \frac{\left(q^8;q^8\right)_\infty^6}{\left(q^4;q^4\right)_\infty^2 \left(q^{16};q^{16}\right)_\infty^4}.$$

Hence we obtain

$$f_1(q) \sum_{n=0}^{\infty} a(2n) q^n \cdot \sum_{n=0}^{\infty} a(2n+1) q^n = \sum_{g \in \mathrm{AB}} p_g(t) \cdot g = \frac{\left(q^8;q^8\right)_\infty^6}{q\left(q^4;q^4\right)_\infty^2 \left(q^{16};q^{16}\right)_\infty^4},$$

and dividing by $1/f_1(q)$ completes the proof of Theorem 18. □

Proof of Theorem 19. The derivation and proof of Theorem 19 is obtained with the procedure call

```
In[39]:= RK[6, 2, {-1, -1}, 3, 2]]
```

which produces f and t as in (10.5). □

Proof of Theorem 20. The derivation and proof of Theorem 20 is obtained with the procedure call

```
In[40]:= RK[6, 6, {-4, -4, 3, 3}, 3, 2]
```

which produces f and t as in (10.9) and p as in (10.10). □

Proof of Theorem 21. The derivation and proof of Theorem 21 is obtained with the procedure call

```
In[41]:= RK[10, 2, {-1, -1}, 5, 2]
```

which produces f and t as in (10.13) and p as in (10.14). □

12. Conclusion

Apart from applying the Schmidt process in various ways, the major key idea in our article is to combine MacMahon's partition analysis with parity coded by the functions $\chi(n)$ and $\psi(n)$; see (2.2) and (5.2), respectively. To exploit this idea further, we are planning to integrate such kind of functions into the Omega package. For instance, it would be convenient to deal with expressions such as (5.6) in the same way as the Omega procedure call `OSum` does in Example 4; see `In[16]` and `Out[16]`.

Another major objective of our article is to show the huge potential of Radu's Ramanujan-Kolberg algorithm. Equipped with Smoot's implementation in the form of the Mathematica

package RaduRK, without trying to be exhaustive we derived a variety of results presented in the Sections 9 and 10. All of these identities are related to the combinatorics and to the partition analysis presented in the first part of our article. On this and similar combinatorial ground we expect that many more identities and arithmetic congruences are waiting for being discovered.

On the other hand, as already mentioned, the $s(k)$ defined by (9.2) at least at the first glance do not seem to satisfy simple congruences involving small primes, in contrast to the related $d(k)$ and PDN1(k).

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