AN ALGORITHM TO COMPUTE ALGEBRAIC RELATIONS BETWEEN MODULAR FUNCTIONS

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Dedicated to our friend Krishnaswami Alladi: we pay highest tribute to all his efforts as Founding Editor of the Ramanujan Journal.

Abstract. The existence of an algebraic relation between two modular functions, in short: a modular equation, is implied by a classical fact from the theory of compact Riemann surfaces. In this article, we present a new, purely algebraic proof of the existence of modular equations. Our setting consists of an algorithmic framework which is based on a reduction procedure for tuples of formal Laurent series. The resulting algorithm MultiSamba ("sub-algebra module basis algorithm") is part of Hemmecke's computer algebra package QEta which has been implemented in FriCAS, a general purpose computer algebra system which is freely available as open source. QEta is a powerful tool-box for actual computations. For example, MultiSamba has been used for computer-assisted discovery and proofs of Ramanujan-Sato series. In this article, we describe the mathematics underlying the MultiSamba algorithm. Moreover, we explain in detail how MultiSamba works for the derivation of a well-known modular equation between the modular λ-function and the Klein j function. Other examples of the automatic discovery and proving of modular equations include identities by Alladi and others, which suggest relations of Ramanujan-Göllnitz-Gordon type as another promising area of MultiSamba application.

1. Introduction

In 1993, Krishna Alladi spent his first sabbatical at Penn State with George Andrews. One of the various beautiful fruits of this stay is Alladi's study [1] of the Göllnitz-Gordon functions,

$$G(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty} \tag{1}$$

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and

$$H(q) := \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n} q^{n^2+2n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}; \tag{2}$$

here we use standard notation: $(a;q)_0 := 1$,

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), n \geq 1, \text{ and } (a;q)_\infty = \prod_{j=1}^{\infty}(1-aq^{j-1}).$$

Alladi's second sabbatical at Penn State in 2013, again with Andrews, brought along a continuation and combinatorial completion of this work which produced, as a by-product of a more general result, a combinatorial proof [2, Sec. 4] of

$$G(-q^2) - qH(-q^2) = \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty}. \tag{3}$$

This relation can be viewed as an identity between modular functions for the congruence subgroup $\Gamma_1(32)$.

Such identities can be proven by means of Hemmecke's QEta [10] package.

In this paper we derive an algorithm that also been implemented in QEta. This algorithm allows computer-assisted *discovery and proving* of (3) and of many other such identities. For example, with QEta one can find and prove:

$$\frac{1}{H(-q^2)} - \frac{q}{G(-q^2)} = \frac{(q;q)_\infty (q^4;q^4)_\infty (q^8;q^8)_\infty}{(q^2;q^2)_\infty (q^{16};q^{16})_\infty^2} \tag{4}$$

or

$$\frac{(G(-q^2) - qH(-q^2))^2}{G(-q^2) + qH(-q^2)} = \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^3}. \tag{5}$$

The first relation (4) follows the pattern of the left side of (3) but with the inverse functions of G and H. In fact, using elementary q-product transformations (4) can be rewritten into the form (3). In contrast, relation (5) is essentially different from (3); using elementary q-product transformations it can be rewritten as,

$$G(-q^2) + qH(-q^2) = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty (q^4;q^4)_\infty^2}. \tag{6}$$

Similar to the case of the Rogers-Ramanujan functions [3] there are whole families of algebraic relations involving $G(\pm q^j)$ and $H(\pm q^j)$. For example, consider

$$R(q) := q^{\frac{1}{2}} \frac{G(q)}{H(q)}$$

which is a representation of the so-called Ramanujan-Göllnitz-Gordon continued fraction. The article [5] by H.H. Chan and S.-S. Huang presents many identities between $R(q)$ and $R(q^j)$; an example [5, Theorem 2.2(ii)] is

$$R(q)^2 = R(q^2)\frac{1-R(q^2)}{1+R(q^2)}, \tag{7}$$

which can be automatically derived (and proven) with the algorithm MultiSamba. Ralf Hemmecke has implemented this algorithm in FriCAS as a part of the program package QEta, freely available at `https://caa.risc.jku.at/software`.

The word "Multi" in MultiSamba stands for doing computations with tuples, in this way extending the algorithm Samba ("sub-algebra module basis algorithm"), originating from [13] and [9], which works with univariate Laurent series (in the Fourier variable q).

MultiSamba is the main algorithmic engine of QEta to compute algebraic relations between modular functions. It is a powerful computational tool which can be used for a broad spectrum of possible applications. In a recent prime application, MultiSamba (and related QEta tools) served as a main algorithmic instrument in the computer-assisted construction [11] of Ramanujan-Sato series for $1/\pi$. For example, as explained in [11], based on the theory of Ramanujan-Sato series, MultiSamba enables a computer-assisted discovery and proof of Ramanujan's celebrated series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\cdot\sum_{n=0}^{\infty}(1103n+26390)\frac{(4n)!}{n!^4}\left(\frac{1}{396^4}\right)^n. \tag{8}$$

The main objective of this article is to describe the mathematical ideas used in setting up the MultiSamba algorithm, and also to provide sufficient information to enable readers to use MultiSamba and QEta in their own research.

Before summarizing the contents of this article, we want to point to a fundamental ingredient of our algorithmic framework, Theorem 3.11. The setting of this theorem is a purely algebraic one. Nevertheless, it provides a new constructive proof of a classic fact in the theory of compact Riemann surfaces; moreover, in the form of MultiSamba it also finds an algorithmic version for actual computations. The following statement of this classic Riemann surfaces fact is taken from Milne's lecture notes [12, Prop. 1.16].

Proposition 1.1. *Let F be a nonconstant meromorphic function with n poles (counting multiplicities) on a compact Riemann surface X. Then every meromorphic function G on X is a root of a polynomial of degree $\leq n$ with coefficients in $\mathbb{C}(F)$.*

Proof. A short proof sketch is given in [12, Prop. 1.16]. A detailed treatment can be found, for instance, in Forster's monograph [7, §8.2 and §8.3]. □

The structure of our article is as follows. Section 2 introduces the basic notions used; a crucial ingredient of our setting is the strict partial order relation $>_F$ on tuples of formal Laurent series. In Section 3, we prove the main theorem of this article, Theorem 3.11; to this end, the notion of *modular algebras*, an algebraic abstraction of modular function properties, is introduced. In Section 4 we turn the proof of our main theorem, Theorem 3.11 into an algorithm; more concretely, we show how the choice of the elements of the set S in the proof of Theorem 3.11 can be done algorithmically. The resulting algorithm MultiSamba is a tool for computing a basis of the module representation (33) of the polynomial algebra $\mathbb{C}[F, G]$. In Section 5, we show how the algorithm MultiSamba can be used to compute modular equations between F and G. Subsection 5.1 takes the modular equation between the Klein j and the modular λ function as an example to present the algorithmic steps in full detail. In Subsection 5.2, we refer to further applications, all available as executable QEta examples; these examples range from closed form evaluations of modular functions at elements of imaginary quadratic fields, to identities such as the Ramanujan-Göllnitz-Gordon relation (3) by Alladi. Section 6 concludes this article with some historical remarks on the FriCAS system.

2. Basic notions

This section introduces basic notions used. A crucial ingredient of our setting is the strict partial order relation $>_F$ on $\mathbb{C}((q))^n$; see Definition 2.5.

Algebraic domains are denoted as usual: $C[X_1, \ldots, X_n]$ is the polynomial ring in the variables $X_1, \ldots, X_n$ with coefficients from C. Often $C = \mathbb{C}$, the field of complex numbers; but C can also stand for a (polynomial) ring. By $\mathbb{C}[[q]]$ we refer to the ring of formal power series in q with complex coefficients; $\mathbb{C}((q))$ denotes its quotient field consisting of formal Laurent series. By a $\mathbb{C}$-algebra A we understand a ring A which in addition is also a vector space over $\mathbb{C}$. For example, $\mathbb{C}[[q]]$ and $\mathbb{C}((q))$ are $\mathbb{C}$-algebras.

The usual notion of an order of a series is extended to tuples of Laurent series.

Definition 2.1. *Let* $F = \sum_{k=m}^{\infty} a(k)q^k \in \mathbb{C}((q))$ *be such that* $a(m) \neq 0$. *Then we define* $\operatorname{ord}(F) := m$. *For* $F = (F_1, \ldots, F_n) \in \mathbb{C}((q))^n$, *we define*

$$\operatorname{ord}(F) := (\operatorname{ord}(F_1), \ldots, \operatorname{ord}(F_n)).$$

Order relations "$\geq$" and "$>$" on integer tuples are defined componentwise.

Definition 2.2. *For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ we write $x \geq y$, resp. $x > y$, if $x_i \geq y_i$, resp. $x_i > y_i$, for all $i \in \{1, \ldots, n\}$.*

Next, for fixed $F \in \mathbb{C}((q))^n$ with $\mathrm{ord}(F) < 0$, and a tuple of functions $G = (G_1, \ldots, G_n)$, we define the functions F-degree, $\deg_F(G)$, and F-index, $\mathrm{ind}_F(G)$, applied to elements $G \in \mathbb{C}((q))^n$.

Definition 2.3. *Let $F \in \mathbb{C}((q))^n$ be such that*

$$\mathrm{ord}(F) = (f_1, \ldots, f_n) < 0. \tag{9}$$

For $G = (G_1, \ldots, G_n) \in \mathbb{C}((q))^n$ with $\mathrm{ord}(G) = (g_1, \ldots, g_n)$ we define

$$\deg_F(G) := \max\left\{\frac{g_i}{f_i} : i \in \{1, \ldots, n\}\right\} \in \mathbb{Q} \cup \{-\infty\}, \tag{10}$$

and

$$\mathrm{ind}_F(G) := \max\left\{i \in \{1, \ldots, n\} : \frac{g_i}{f_i} = \deg_F(G)\right\} \in \mathbb{Z}_{>0}. \tag{11}$$

Notice that for $0 = (0, \ldots, 0) \in \mathbb{C}((q))^n$,

$$\deg_F(0) = -\infty, \tag{12}$$

owing to $\mathrm{ord}(0) = \infty$ for $0 \in \mathbb{C}((q))^n$.

Remark 2.4. The condition (9) has the following reason. Basically our reduction argument in Section 3 will proceed by reducing the pole-orders of the series involved. In the setting of modular algebras, this relates to the positivity of the $\deg_F$-function which is induced by (9); see Lemma 3.4.

The F-degree and F-index functions give rise to a strict partial order $>_F$ on $\mathbb{C}((q))^n$; i.e., a transitive relation where no element is related to itself (never $G >_F G$), and where either $G >_F H$ or $H >_F G$ but not both.

Definition 2.5. *Let $F \in \mathbb{C}((q))^n$ be such that $\mathrm{ord}(F) < 0$. We introduce the strict partial order relation $>_F$ on $\mathbb{C}((q))^n$ as follows. Let $G, H \in \mathbb{C}((q))^n$, then we define*

$$\begin{aligned} G >_F H :\Leftrightarrow\ & (i)\ \deg_F(G) > \deg_F(H),\ \textit{or} \\ & (ii)\ \textit{if}\ \deg_F(G) = \deg_F(H),\ \textit{then}\ \mathrm{ind}_F(G) > \mathrm{ind}_F(H). \end{aligned}$$

Invoking the notion of lexicographical order on $(\mathbb{Q} \cup \{-\infty\}) \times \mathbb{Z}_{>0}$, the relation $G >_F H$ can be written in more compact fashion.

Definition 2.6. *For (a, b) and (c, d) from $(\mathbb{Q} \cup \{-\infty\}) \times \mathbb{Z}_{>0}$ we define,*

$$\begin{aligned} (a, b) >_{\mathrm{lex}} (c, d) :\Leftrightarrow\ & (i)\ a > c,\ \textit{or} \\ & (ii)\ \textit{if}\ a = c\ \textit{then}\ b > d. \end{aligned}$$

As a consequence we have,

$$G >_F H \Leftrightarrow (\deg_F(G), \mathrm{ind}_F(G)) >_{\mathrm{lex}} (\deg_F(H), \mathrm{ind}_F(H)). \tag{13}$$

3. Modular Subalgebras and the Main Theorem

In this section, we prove the main theorem of this article, Theorem 3.11. To prepare for this we need to introduce the notion of *modular algebras* and to prove some related lemmas.

From now on, we view $\mathbb{C}((q))^n = (\mathbb{C}((q))^n; +, \cdot)$ as a ring with componentwise addition and multiplication. Obviously, it is a ring with unity $1 := (1, \ldots, 1)$, and also a $\mathbb{C}$-algebra. For multiplication the dot "$\cdot$" usually is omitted; i.e.,

$$(F_1, \ldots, F_n)(G_1, \ldots, G_n) = (F_1, \ldots, F_n) \cdot (G_1, \ldots, G_n) = (F_1G_1, \ldots, F_nG_n).$$

In particular, for multiplication with a scalar $c \in \mathbb{C}$ one has,

$$c(G_1, \ldots, G_n) = (cG_1, \ldots, cG_n) = (c, \ldots, c)(G_1, \ldots, G_n).$$

A fundamental role is played by subrings called *modular algebras*.

Definition 3.1. *Let $A \subseteq \mathbb{C}$ be a subring of $\mathbb{C}$. A subring $R \subseteq \mathbb{C}((q))^n$ containing the unity element $1 = (1, \ldots, 1)$ is called a modular A-algebra or, if $A = \mathbb{C}$, a modular algebra, if for all $G \in R$,*
(i) $\mathrm{ord}(G) = 0$ *implies* $G = (c, \ldots, c)$ *for some* $c \in \mathbb{C}$,
(ii) if $\mathrm{ord}(G) > 0$ *then* $G = (0, \ldots, 0)$, *and*
(iii) for all $a \in A$ *one has* $aR \subseteq R$.

The notion of modular algebras is nothing but a formal Laurent series abstraction of classical modular function properties. The following definition and lemma makes this more precise.

Definition 3.2. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with n cusps in $\mathbb{Q} \cup \{\infty\}$. By $M^!(\Gamma; n)$ we denote the algebra of modular functions for the group Γ which are holomorphic on $\mathbb{H}$; i.e., poles are supported at the n cusps of Γ only.*

Lemma 3.3. *Let $F \in M^!(\Gamma; n)$ with $\{c_1, \ldots, c_n\} \subseteq \mathbb{Q} \cup \{\infty\}$ being a complete set of representatives of the cusps. For $1 \le j \le n$ let w_j be the width of c_j; moreover, let $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_j \infty = c_j$. Let $(F_1(X), \ldots, F_n(X)) \in \mathbb{C}((X))^n$ be such that for $1 \le j \le n$ in a suitable neighborhood of c_j,*

$$F(\tau) = F_j(q_j) \text{ where } q_j = q_j(\tau) = \exp(2\pi i \gamma_j^{-1} \tau / w_j).$$

Then the mapping

$$\phi : M^!(\Gamma; n) \to \mathbb{C}((X))^n, \quad F \mapsto \phi(F) := (F_1(X), \ldots, F_n(X)),$$

is an algebra isomorphism.

After connecting modular algebras to algebras of modular functions, we continue with our preparations for the statement and proof of Theorem 3.11.

Lemma 3.4. *Let $G \in R$ where $R \subseteq \mathbb{C}((q))^n$ is a modular algebra. Then for any $F \in \mathbb{C}((q))^n$ with $\operatorname{ord}(F) < 0$,*

$$\deg_F(G) = 0 \quad \Rightarrow \quad G = (c, \ldots, c) \text{ for some } c \in \mathbb{C}, \tag{14}$$

and

$$\deg_F(G) < 0 \quad \Rightarrow \quad G = (0, \ldots, 0). \tag{15}$$

Proof. Let $\operatorname{ord}(F) = (f_1, \ldots, f_n)$ and $\operatorname{ord}(G) = (g_1, \ldots, g_n)$. Then for each $j \in \{1, \ldots, n\}$,

$$\frac{g_j}{f_j} \leq \max\left\{\frac{g_i}{f_i} : i \in \{1, \ldots, n\}\right\} = \deg_F(G) \leq 0.$$

This means, if $\deg_F(G) = 0$ then each $g_j = 0$, and property (i) of Definition 3.1 implies the statement (14); if $\deg_F(G) < 0$ then each $g_j > 0$, and property (ii) of Definition 3.1 implies the statement (15). □

The next property for the partial order $>_F$ is simple to prove but crucial for making things constructive.

Lemma 3.5. *Let $R \subseteq \mathbb{C}((q))^n$ be a modular algebra and let $F \in R$ be such that $\operatorname{ord}(F) = (f_1, \ldots, f_n) < 0$. Then each decreasing chain in R,*

$$G_1 >_F G_2 >_F G_3 >_F \cdots \text{ where } G_1, G_2, G_3, \ldots \in R, \tag{16}$$

is finite.

Proof. Suppose that (16) is an infinite chain. Then for $\ell := |f_1 \cdots f_n|$ also

$$G_1^\ell >_F G_2^\ell >_F G_3^\ell >_F \cdots \tag{17}$$

is an infinite chain in R. Recalling (13) one has that if $G >_F H$ with $\deg_F(G) = \deg_F(H)$, then $\operatorname{ind}_F(G) > \operatorname{ind}_F(H)$ is required to hold. Hence in a chain as (17) maximally $n - 1$ successive equalities in $\deg_F$ can occur. As a consequence, without loss of generality (by choosing a suitable subchain and relabelling) we can assume that our chain is such that

$$\deg_F(G_1^\ell) > \deg_F(G_2^\ell) > \deg_F(G_3^\ell) > \cdots, \tag{18}$$

which is an infinite chain of *integers* owing to the definition of ℓ and $\deg_F(G_j^\ell) = \ell \deg_F(G_j)$. From some index on, say j, these values must be negative which by property (15) implies $G_j = (0, \ldots, 0)$. In other words, the chain must terminate at index j. This completes the proof of the lemma. □

The next lemma describes a situation where an element $G \in \mathbb{C}((q))^n$ can be reduced with respect to $>_F$.

Lemma 3.6. *Let* $F, G, H \in \mathbb{C}((q))^n$ *such that* $\operatorname{ord}(F) = (f_1, \ldots, f_n) < 0$,

$$k := \deg_F(G) - \deg_F(H) \in \mathbb{Z}_{\geq 0}, \text{ and } \operatorname{ind}_F(G) = \operatorname{ind}_F(H).$$

Then there exists unique $a \in \mathbb{C}$ *such that*

$$G >_F (G - aF^kH).$$

Proof. First note that for $\operatorname{ord}(H) = (h_1, \ldots, h_n)$,

$$\deg_F(F^kH) = \max\left\{\frac{kf_i + h_i}{f_i} : i \in \{1, \ldots, n\}\right\} = k + \deg_F(H) = \deg_F(G), \tag{19}$$

and

$$\operatorname{ind}_F(F^kH) = \operatorname{ind}_F(H). \tag{20}$$

Equation (19) implies

$$\deg_F(G) \geq \deg_F(G - aF^kH) \text{ for any } a \in \mathbb{C}. \tag{21}$$

To do a refined analysis, let

$$F = (F_1, \ldots, F_n), G = (G_1, \ldots, G_n), \text{ and } H = (H_1, \ldots, H_n),$$

set $s := \operatorname{ind}_F(G)$, and consider,

$$G - aF^kH = (G_1 - aF_1^kH_1, \ldots, G_s - aF_s^kH_s, \ldots, G_n - aF_n^kH_n).$$

By (19) we have

$$\operatorname{ord}(G_s) = \operatorname{ord}(F_s^kH_s),$$

hence there exists a uniquely determined $a \in \mathbb{C}$ such that

$$\operatorname{ord}(G_s) > \operatorname{ord}(G_s - aF_s^kH_s).$$

Consequently, for this constant $a \in \mathbb{C}$, in view of (21), we either have

$$\begin{aligned}\deg_F(G) &= \max\left\{\frac{\operatorname{ord}(G_i)}{f_i} : i \in \{1, \ldots, n\}\right\} \\ &> \frac{\operatorname{ord}(G_j - F_j^kH_j)}{f_j} \text{ for all } j \in \{1, \ldots, n\},\end{aligned}$$

or there is an index $j < s$ (this inequality is by (20)) such that

$$\deg_F(G) = \frac{\operatorname{ord}(G_j - F_j^kH_j)}{f_j}.$$

In the first case, we have

$$\deg_F(G) > \deg_F(G - aF^kH),$$

in the second one,

$$\deg_F(G) = \deg_F(G - aF^kH) \text{ and } \operatorname{ind}_F(G) > \operatorname{ind}_F(G - aF^kH).$$

According to (13) this means,

$$G >_F (G - aF^k H),$$

which completes the proof of the lemma. □

The following final preparations for Theorem 3.11 will provide convenient notions for further reasoning.

Definition 3.7. *Let $R \subseteq \mathbb{C}((q))^n$ be a modular algebra and let $F \in R$ be such that $\operatorname{ord}(F) = (f_1, \ldots, f_n) < 0$. For any $G \in R$ we define a modular subalgebra of R by*

$$R_F(G) := \mathbb{C}[F, G] = \Big\{ \sum_{\substack{0 \le a,b \\ \text{finite}}} c_{a,b} F^a G^b : c_{a,b} \in \mathbb{C} \Big\} \subseteq R. \tag{22}$$

Moreover,

$$R_F(G)^\star := \mathbb{C}[F, G] \setminus \{0\}. \tag{23}$$

In addition to being a modular subalgebra of R, $R_F(G)$ can be also viewed as a $\mathbb{C}[F]$-module. We introduce an equivalence relation on $R_F(G)^\star$.

Definition 3.8. *Let $R_F(G) = \mathbb{C}[F, G]$ with F and G as in Definition 3.7. For $t_1, t_2 \in R_F(G)^\star$ define a relation $\sim$ as follows,*

$$t_1 \sim t_2 \quad :\Leftrightarrow \quad \deg_F(t_1) - \deg_F(t_2) \in \mathbb{Z} \text{ and } \operatorname{ind}_F(t_1) = \operatorname{ind}_F(t_2).$$

Lemma 3.9. *Let $R_F(G) = \mathbb{C}[F, G]$ with F as in Definition 3.7. The relation $\sim$ specified in Definition 3.8 is an equivalence relation on $R_F(G)^\star$ with finitely many equivalence classes $T_{j,d}$ such that*

$$R_F(G)^\star = \bigcup_{\substack{1 \le j \le n \\ 1 \le d \le f_j}} T_{j,d},$$

where

$$T_{j,d} := \{t \in R_F(G) : f_j \deg_F(t) \equiv d \pmod{f_j} \quad \text{and} \quad \operatorname{ind}_F(t) = j\}, \tag{24}$$

Moreover,

for each $t \in R$ there exist $1 \le j \le n$ and $1 \le d \le f_j$ such that

$$f_j \deg_F(t) \equiv d \pmod{f_j} \quad \text{and} \quad \operatorname{ind}_F(t) = j. \tag{25}$$

Proof. The proofs of all these statements are straightforward verifications. □

We want to remark explicitly that for certain choices of j and d the set $T_{j,d}$ in (24) could be empty.

We need one more lemma; it concerns a fact from polynomial algebra which is straightforward to prove.

Lemma 3.10. *Let C be a field and F an element from an extension field being transcendental over C. Let $p(X) \in C[F][X]$ be monic such that*

$p(G) = 0$ *for an element G from some ring extension of $C[F]$.*

Let $q(X) \in C[F][X]$ be of minimial degree such that

$q(G) = 0$ *and the polynomial* gcd *of all the coefficients of $q(X)$ is equal to* 1.

Then $q(X)$ is also monic.

Now we are ready to state and prove the main theorem of this article.

Theorem 3.11. *Let $R \subseteq \mathbb{C}((q))^n$ be a modular algebra and let $F \in R$ be such that* $\operatorname{ord}(F) = (-f_1, \ldots, -f_n) < 0$. *Then for any $G \in R$ there exists a monic polynomial $p(X) \in \mathbb{C}[F][X]$ such that*

$$p(G) = 0$$

and

$$\deg p(X) \leq \sum_{i=1}^{n} f_i.$$

In fact, from Theorem 3.11 follows the existence of a polynomial $p(X, Y) \in \mathbb{C}[X, Y]$ with $p(F, G) = 0$ just as in Proposition 1.1, but with the restriction on $\operatorname{ord}(F)$. The importance of this Theorem actually lies in the fact that from its proof, we can extract an algorithm to actually compute such a polynomial explicitly.

Proof. Let $R_F(G)^\star$ be defined as in Definition 3.7 with $T_{j,d}$ as in Lemma 3.9 such that

$$R_F(G)^\star = \bigcup_{\substack{1 \leq j \leq n \\ 1 \leq d \leq f_j}} T_{j,d}$$

where

$$T_{j,d} := \{t \in R_F(G) : f_j \deg_F(t) \equiv d \pmod{f_j} \quad \text{and} \quad \operatorname{ind}_F(t) = j\}.$$

Out of each $T_{j,d}$ we choose a representative $t_{j,d}$ of minimal F-degree; i.e., $t_{j,d} \in T_{j,d}$ such that

$$\deg_F(t_{j,d}) = \min\{\deg_F(t) : t \in T_{j,d}\};$$

if $T_{j,d} = \emptyset$ we set $t_{j,d} := 0$. Let

$$S := \{t_{1,1}, \ldots, t_{1,f_1}, t_{2,1}, \ldots, t_{2,f_2}, \ldots, t_{n,1}, \ldots, t_{n,f_n}\} - \{0\}$$

be this subset of $R_F(G)$.

By (25) for each $t \in R_F(G)^\star$ there exists a pair (j,d) with $1 \leq j \leq n$ and $1 \leq d \leq f_j$ such that

$$f_j \deg_F(t) \equiv d \pmod{f_j} \quad \text{and} \quad j = \mathrm{ind}_F(t); \tag{26}$$

for the corresponding equivalence class representative $t_{j,d} \in S$ we also have that

$$f_j \deg_F(t_{j,d}) \equiv d \pmod{f_j} \quad \text{and} \quad j = \mathrm{ind}_F(t_{j,d}). \tag{27}$$

The congruence conditions combine into

$$k := \deg_F(t) - \deg_F(t_{j,d}) \in \mathbb{Z}_{\geq 0}, \tag{28}$$

using also the F-degree minimality of $t_{j,d}$. In addition to (28), we have

$$j = \mathrm{ind}_F(t) = \mathrm{ind}_F(t_{j,d}). \tag{29}$$

According to Lemma 3.6, setting $t_0 := t$, $t_{j_0,d_0} := t_{j,d}$, and $k_0 := k$, there exists a unique $a_0 \in \mathbb{C}$ such that

$$t_0 >_F (t_0 - a_0 F^{k_0} t_{j_0,d_0}).$$

By the same argument and setting

$$t_1 := t_0 - a_0 F^{k_0} t_{j_0,d_0},$$

there exist a unique $a_1 \in \mathbb{C}$ and $t_{j_1,d_1} \in S$ such that

$$t_1 >_F (t_1 - a_1 F^{k_1} t_{j_1,d_1}),$$

where $k_1 := \deg_F(t_1) - \deg_F(t_{j_1,d_1})) \in \mathbb{Z}_{\geq 0}$ and $j_1 = \mathrm{ind}_F(t_1) = \mathrm{ind}_F(t_{j_1,d_1})$.

This way, one can continue to construct a chain

$$t_0 >_F t_1 >_F t_2 := t_1 - a_1 F^{k_1} t_{j_1,d_1} >_F \cdots >_F t_{J+1}$$

where $t_{J+1} = (0, \ldots, 0)$. The reason for termination at the 0-tuple t_{J+1} is by the same argument as in the proof of Lemma 3.5.

Finally, back substitution gives,

$$\begin{aligned} t_J &= t_{J+1} + a_J F^{k_J} t_{j_J,d_J} = (0, \ldots, 0) + a_J F^{k_J} t_{j_J,d_J}, \\ t_{J-1} &= t_J + a_{J-1} F^{k_{J-1}} t_{j_{J-1},d_{J-1}} \\ &= (0, \ldots, 0) + a_{J-1} F^{k_{J-1}} t_{j_{J-1},d_{J-1}} + a_J F^{k_J} t_{j_J,d_J} \\ &\cdots \\ t = t_0 &= \sum_{j=0}^{J} a_j F^{k_j} t_{i_j,d_j}. \end{aligned} \tag{30}$$

Because of $S \subseteq R_F(G)$ and the Definition 3.7 of $R_F(G)$, every $t_{i,d} \in S$ can be represented as $p_{i,d}(G)$ for some polynomial $p_{i,d}(X) \in \mathbb{C}[F][X]$. Let m be the maximal degree with respect to the variable X among all the polynomials $p_{i,d}(X)$, $1 \leq i \leq n$ and $1 \leq d \leq f_i$.

In the consideration above, we are free to choose the element $t \in R_F(G)^\star$. Hence we can choose $t = G^{m+1}$, and we see that the corresponding polynomial

$$X^{m+1} - \sum_{j=0}^{J} a_j F^{k_j} p_{i_j,d_j}(X)$$

is monic and has $X = G$ as a root.

Because of Lemma 3.10 the minimal polynomial $p(X) \in \mathbb{C}[F][X]$ of G must also be monic. Let m_G be the degree of $p(X)$ with respect to the variable X. Owing to the fact that $p(X) \in \mathbb{C}[F][X]$ is monic, every element in $t \in R_F(G)$ can be written as a linear combination

$$t = c_0(F) + c_1(F)G + \cdots + c_{m_G-1}(F)G^{m_G-1} \text{ with polynomials } c_j(Y) \in \mathbb{C}[Y].$$

In other words,

$$R_F(G) \text{ is a free } \mathbb{C}[F]\text{-module with basis } \{1, G, \ldots, G^{m_G-1}\}. \tag{31}$$

The elements of S span $R_F(G)$ viewed as a $\mathbb{C}$-vector space. Hence the number of the elements of S provide a bound for the degree m_G, the rank of $R_F(G)$ as a $\mathbb{C}[F]$-module, and we have,

$$m_G \leq |S| \leq \sum_{i=1}^{n} f_i,$$

which completes the proof of our theorem. □

We want to point explicitly to an important mathematical ingredient of the proof which connects to the representation of polynomial algebras in the form of modules.

To make this more explicit it is convenient to introduce some notation.

Definition 3.12. *Let C be a field and $F, G_1, \ldots, G_r$ elements from a C-algebra R. Then*

$$\begin{aligned}\langle\{G_1, \ldots, G_r\}\rangle_{C[F]} &:= \langle G_1, \ldots, G_r\rangle_{C[F]} := C[F]G_1 + \cdots + C[F]G_n \\ &:= \left\{ \sum_{j=1}^{r} c_j(F)G_j : c_j(F) \in C[F] \right\}.\end{aligned}$$

This means, $\langle G_1, \ldots, G_r\rangle_{C[F]}$ is the $C[F]$-module generated by $G_1, \ldots, G_r$.

Now we are ready to make the module connection of Theorem 3.11 precise; namely, relation (30) gives rise to the following crucial fact.

Corollary 3.13. *Let $R \subseteq \mathbb{C}((q))^n$ be a modular algebra and let $F, G \in R$ be such that* $\operatorname{ord}(F) = (-f_1, \ldots, -f_n) < 0$. *Let*

$$R_F(G) := \mathbb{C}[F, G] \subseteq R$$

and

$$S := \{t_{1,1}, \ldots, t_{1,f_1}, t_{2,1}, \ldots, t_{2,f_2}, \ldots, t_{n,1}, \ldots, t_{n,f_n}\} - \{0\}$$

be the subset of $R_F(G)$ defined as described in the proof of Theorem 3.11. Then any element $t \in \mathbb{C}[F, G]$ can be written as a linear combination

$$t = \sum_{\substack{1 \le j \le n \\ 1 \le d \le f_n}} c_{j,d}(F) t_{j,d} \quad \text{with polynomial coefficients } c_{j,d}(F) \in \mathbb{C}[F]; \tag{32}$$

in other words,

$$\mathbb{C}[F, G] = \langle S \rangle_{\mathbb{C}[F]}. \tag{33}$$

In the next section, we outline how the choice of the set S can be made constructive. In this way, the procedure described in the proof of Theorem 3.11 is turned into the algorithm MultiSamba.

We conclude this section with the remark that there is an extension of MultiSamba which generalizes (33) to

$$\mathbb{C}[F, G_1, \ldots, G_r] = \langle S \rangle_{\mathbb{C}[F]}. \tag{34}$$

4. The computation of minimal elements of the $T_{j,d}$

In this section, we turn the proof of our main theorem, Theorem 3.11 into an algorithm.

Concretely, recall $R_F(G) = \mathbb{C}[F, G]$ from Definition 3.7. In this section, we show how to compute $>_F$-minimal representatives of non-empty equivalence classes for $\sim$ from Definition 3.8 mentioned in (24) of Lemma 3.9, i.e., we compute a set S as mentioned in the proof of Theorem 3.11.

The process starts with a certain finite basis set $B \subseteq R_F(G)$ and will update it until B is "completed to S". Lemma 3.6 says that any non-zero element from $R_F(G)$ can be reduced, if we have chosen a minimal element of each of the equivalence classes $T_{j,d} \neq \emptyset$, i.e., such a set S. We can turn this statement into a completion process of B. The basis B will be complete if any non-zero element of $R_F(G)$ can be reduced (to zero) by F and B.

In fact, the process we are going to describe below, is similar to Buchberger's algorithm [4] for the comptation of Gröbner bases in the sense that it is a critical pair/completion algorithm, but where the critical pairs are products instead of S-polynomials and the reduction function is essentially given by Lemma 3.6.

For $H \in R_F(G)$ let us denote by $\mathrm{reduce}_{F,B}(H)$ the result of successively applying reduction steps with elements of B according to Lemma 3.6 until no reduction with an element of B can be done anymore.

To facilitated understanding of the crucial idea, we restrict to describe here a simplified version of the MultiSamba algorithm. The algorithm works as follows. Start with setting $B = \{1, G\}$ and let $U = \{\mathrm{reduce}_{F,B}(b_1 b_2) : b_1, b_2 \in B\}$. For each non-zero element $u \in U$, we remove from B a possibly existing element b with $b \sim u$ and add u to B. Note that if $b \sim u$, we must have $b >_F u$. This process is repeated with the new B until $U = \{0\}$. Let S be the set B at the end of this process.

Note that when elements of B are updated, for any $j \in \{1, \ldots, n\}$ and $d \in \{1, \ldots, f_j\}$ it always holds that $B \cap T_{j,d}$ is either empty or contains exactly one element. Additionally, the size of B never shrinks. Since there are only finitely many equivalence classes and there are no infinitely decreasing $>_F$-chains by Lemma 3.5, the above process must eventually stop with $U = \{0\}$, i.e., each product of the basis elements is reducible (to zero). Clearly, any F-multiple of such a product is also reducible, because for any non-zero $H \in R_F(G)$ we have $H \sim FH$. Note that $B \cup \{F\}$ generates $R_F(G)$, i.e., $R_F(G) = \mathbb{C}[F, B]$ and this relation holds in each step of the algorithm. Suppose that $S = \{b_0, \ldots, b_s\}$. Since each product of basis elements is reducible to zero, there are $c_k^{(i,j)}(X) \in \mathbb{C}[\mathrm{X}]$ such that $b_i b_j = \sum_{k=0}^{s} c_k^{(i,j)}(F)\, b_k$. By construction, there are $c_k(X) \in \mathbb{C}[X]$ such that $G = \sum_{k=0}^{s} c_k(F)\, b_k$. Then, clearly, such a relation also exists for G^2, and, therefore, for any power of G. Eventually, it follows that any elements $H \in R_F(G)$ is representable as

$$H = \sum_{k=0}^{s} c_k^H(F)\, b_k \tag{35}$$

for some $c_k^H(X) \in \mathbb{C}[X]$, i.e., $R_F(G) = \langle S \rangle_{\mathbb{C}[F]}$ as a $\mathbb{C}[F]$-modules.

It remains to show that for any non-zero H not only there is a representation (35), but also that $\mathrm{reduce}_{F,S}(H) = 0$ holds, i.e., that the elements of S are indeed a complete set of representatives for the non-empty equivalence classes $T_{j,d}$ and have the $>_F$-minimality property. To this end, it is sufficient to show that any non-zero $H \in R_F(G)$ is reducible.

Obviously, (35) can be rewritten as a sum of terms of the form $a_{m,k} F^m b_k$ for $a_{m,k} \in \mathbb{C}$. Suppose $t_1 = a_{m_1,k_1} F^{m_1} b_{k_1}$ an $t_2 = a_{m_2,k_2} F^{m_2} b_{k_2}$ are two such terms

from (35). Then $t_1 \not\sim t_2$, if $k_1 \neq k_2$ and $\deg_F(t_1) \neq \deg_F(t_2)$, if $m_1 \neq m_2$. In other words, there is exactly one term $aF^m b_k$ with $d = \deg(c_k^H(X))$ such that $\deg_F(aF^m b_k) = \deg_F(H)$ and $\mathrm{ind}_F(aF^m b_k) = \mathrm{ind}_F(H)$, i.e., $H \sim b_k$. Since there is equality in (35), it follows

$$H >_F H - aF^m b_k, \tag{36}$$

i.e., H is reducible.

Summarizing, given F, G from a modular algebra R such that $\mathrm{ord}(F) < 0$, MultiSamba computes a finite set $S \subseteq R$ such that

$$R_F(G) = \langle S \rangle_{\mathbb{C}[F]}. \tag{37}$$

Additionally to the module representation (37), we have proved termination of the MultiSamba algorithm and the property,

$$\text{for any } H \in R_F(G): \quad \mathrm{reduce}_{F,S}(H) = 0. \tag{38}$$

Property (38) is useful for checking membership in $R_F(G)$. Namely, for membership testing we need for any $H \in R$,

$$\mathrm{reduce}_{F,S}(H) = 0 \quad \Longleftrightarrow \quad H \in R_F(G). \tag{39}$$

Notice that (38) is the non-trivial direction of (39).

The full MultiSamba algorithm actually works for any finite number $G_1, \ldots, G_r$ as input instead of just a single element G, i.e., in this case MultiSamba computes $S \subseteq R$ such that $\mathbb{C}[F, G_1, \ldots, G_r] = \langle S \rangle_{\mathbb{C}[F]}$. The essential difference is that it starts the process with $B = \{1, G_1, \ldots, G_r\}$. We have refrained from presenting the algorithm in its full generality, since in this article, we are only concerned with computing modular polynomials and in this case, we are actually not interested in the basis set S, but rather in a way of obtaining the modular polynomial.

In fact, we can modify the above process in the following way. We start with $B = \{b_0\}$ where $b_0 = 1$ and compute in the first step $b_1 := \mathrm{reduce}_{F,B}(Gb_0)$. Tracing the reduction steps, gives $Gb_0 = p_{1,0}(F)b_0 + b_1$ for some $p_{1,0}(X) \in \mathbb{C}[X]$. If $b_1 \neq 0$, it is added to B; otherwise, B is not changed and the process stops. In the second step, we compute $b_2 := \mathrm{reduce}_{F,B}(Gb_1)$. This now gives $Gb_1 = p_{2,0}(F)b_0 + p_{2,1}(F)b_1 + b_2$ for some $p_{2,0}(X), p_{2,1}(X) \in \mathbb{C}[X]$. Continuing this process successively finds representations $Gb_{m-1} = \sum_{k=0}^{m-1} p_{m-1,k}(F)b_k + b_m$ in the m-th step for $m > 2$ and because this process must terminate after $s+1$ steps for the same reason that MultiSamba terminates, we get

$$Gb_s = \sum_{k=0}^{s} p_{s,k}(F)b_k + b_{s+1} \tag{40}$$

with $b_{s+1} = 0$. By backsubstituting earlier relations, (40) gives a polynomial $p(X, Y) \in \mathbb{C}[X, Y]$ such that $p(F, G) = 0$. This modified algorithm is implemented in QEta under the name modularPolynomial.

5. Examples: Algebraic relations between modular functions

In this section, we show how the algorithm MultiSamba, which actually computes a module basis for the algebra $\mathbb{C}[F,G]$, can be used to compute modular equations between F and G. Subsection 5.1 takes the modular equation between the Klein j and the modular λ function as an example to present the algorithmic steps in full detail. In Subsection 5.2, we refer to further applications, all available as executable QEta examples; these examples range from singular value computations to identities such as the Ramanujan-Göllnitz-Gordon relation (3) by Alladi.

5.1. The modular λ function and the Klein j invariant.

In this subsection, we present the steps to compute the well-known relation,

$$j(\tau) = \frac{256(1-\lambda(\tau)+\lambda(\tau)^2)^3}{\lambda(\tau)^2(1-\lambda(\tau))^2}, \tag{41}$$

between the Klein j invariant and the modular λ function by using their q-expansions ($q = e^{\pi i \tau}$) at the cusps of $\Gamma(2)$ and applying the MultiSamba algorithm to them. We give a detailed presentation of the steps of this kind of MultiSamba application, namely, to compute a modular polynomial. A more informal description can be found in [11, Example 7.6].

These functions can be given in terms of the Dedekind η function,

$$\lambda(\tau) = \frac{16\eta\left(\frac{\tau}{2}\right)^8 \eta(2\tau)^{16}}{\eta(\tau)^{24}}, \qquad j(\tau) = \left(\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^8 + 2^8\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{16}\right)^3,$$

where

$$\eta(\tau) = \exp\left(\frac{\pi i \tau}{12}\right)\prod_{n=1}^{\infty}\left(1-\exp\left(2\pi i n \tau\right)\right).$$

The package QEta [10] is able to compute the q-expansions of modular eta-quotients at any cusp of $\Gamma_0(N)$ and also of generalized eta-quotients at the cusps of $\Gamma_1(N)$.

To apply MultiSamba it is, in fact, unimportant how the expansions of the respective functions are obtained, since MultiSamba only works with tuples of Laurent series.

Clearly, it is impossible to compute with arbitrary complex numbers on a computer. However, in the cases we deal with, it is enough to work over an algebraic extension of the rational numbers. Depending on the congruence subgroup and the modular functions involved, QEta computes a certain primitive k-th root of unity ξ and does all computations in the coefficient domain $\mathbb{Q}[\xi]$.

Expansions of the Klein j invariant and the modular λ function are implemented in QEta, see the example notebook that computes these expansions on the website below. Concerning the coefficient field of the expansions of j and λ at the cusps, we can work with the rational numbers $\mathbb{Q}$.

Remark 5.1. Note that QEta relies on the implementation of lazy Laurent series from the computer algebra system FriCAS [8]. They are implemented in such a way that any coefficient is computed on demand; i.e., QEta does not need to compute a truncation point for the involved series before MultiSamba starts. Nevertheless, given sufficiently many coefficients of the expansions of j and λ at the cusps, the operations we demonstrate below can be repeated in any computer algebra system.

Let us denote the expansions of $j(\tau)$ and $\lambda(\tau)$ at the cusps $\infty, 0, 1$ of $\Gamma(2)$, in terms of $q = e^{\pi i \tau}$, by (j_∞, j_0, j_1) and $(\lambda_\infty, \lambda_0, \lambda_1)$, respectively.

Then $j_\infty = j_0 = j_1$ are equal to the following series:

$$q^{-2} + 744 + 196884\,q^2 + 21493760\,q^4 + 864299970\,q^6 + 20245856256\,q^8 + O\left(q^9\right).$$

The series for $\lambda_\infty, \lambda_0, \lambda_1$ in this order are given by the following series

$$16\,q - 128\,q^2 + 704\,q^3 - 3072\,q^4 + 11488\,q^5 - 38400\,q^6 + 117632\,q^7 + O\left(q^8\right),$$
$$1 - 16\,q + 128\,q^2 - 704\,q^3 + 3072\,q^4 - 11488\,q^5 + 38400\,q^6 - 117632\,q^7 + O\left(q^8\right),$$
$$-\frac{1}{16}\,q^{-1} + \frac{1}{2} - \frac{5}{4}\,q + \frac{31}{8}\,q^3 - \frac{27}{2}\,q^5 + \frac{641}{16}\,q^7 - \frac{409}{4}\,q^9 + O\left(q^{10}\right),$$

respectively. Assuming that the above tuples (j_∞, j_0, j_1) and $(\lambda_\infty, \lambda_0, \lambda_1)$ of series are stored in the variables `j` and `l`, the modular polynomial in QEta is computed with one procedure call as follows:

```
modularPolynomial([j,l]) $ QEtaModularEquation(QQ, An QQ)
```

The system returns as output,

$$y^6 - 3\,y^5 + \left(-\frac{1}{256}\,x + 6\right) y^4 + \left(\frac{1}{128}\,x - 7\right) y^3 + \left(-\frac{1}{256}\,x + 6\right) y^2 - 3\,y + 1,$$

which after replacing x by $j(\tau)$ and y by $\lambda(\tau)$ is equivalent to (41).

To set up the stage for the above `modularPolynomial` call, there is some code needed. We do not include it here, but rather refer to RISC-JKU website

(42) `https://risc.jku.at/people/hemmecke/papers/modrings`

where additional information is given, including the computational execution of the iterations presented below.

In the following, we go through the actual steps that are done while the function `modularPolynomial` is executed; i.e., we demonstrate the steps of the variant of MultiSamba that computes a modular polynomial.

For MultiSamba to work, corresponding to $F \in \mathbb{C}((q))^n$ with $\operatorname{ord}(F) < 0$ above, we need a special element with negative orders in all components. Here we choose $F = (j_\infty, j_0, j_1)$, which corresponds to listing `j` as the first element in the procedure call `modularPolynomial` above.

Since its orders are equal to -2 in each component, it is easy to determine the component in which an element can be reduced. We must just look for its highest pole-order, i.e., the negation of the order of the respective Laurent series. In QEta, the function `qetaGrades` is the name for the function that returns the tuple of pole-orders.

For the following computation, we pair the tuple of the q-series expansions with an additional entry which can be thought of as the "representation part". All arithmetic operations are not only performed on the q-expansions, but simultaneously also on the representation part. In this fashion, we automatically record in which way the q-expansions in such a pair relate to the original elements. In our concrete example, we start with the pairs $((j_\infty, j_0, j_1), x)$ and $((\lambda_\infty, \lambda_0, \lambda_1), y)$ where x and y are indeterminates representing $j(\tau)$ and $\lambda(\tau)$, respectively. By slight abuse of notation, let us below denote by j and λ the above mentioned pairs.

As already described at the end of the previous section, the variant of the MultiSamba algorithm, adapted to the computation of a modular polynomial, works as follows. Let $b_0 = ((1,1,1),1)$. We start with the set $B = \{b_0\}$ as an initial basis and then roughly do the following steps.

(1) Reduce $\lambda \cdot b_0$ w.r.t. j and B to the element b_1 (which is then put into B).
(2) Reduce $\lambda \cdot b_1$ w.r.t. j and B to the element b_2 (which is then put into B).
(⋮) ...
(s) Reduce $\lambda \cdot b_{s-1}$ w.r.t. j and B to the element b_3 (which is then put into B).
(s+1) Stop the above process if $\operatorname{ord}(b_{s+1}) > 0$ and read off the modular polynomial p from the representation part of the pair $b_{s+1} = (0, p(x,y))$.

Iteration 1. Clearly λ is not reducible by j, because the highest pole-order of λ is in the last component λ_1; it is smaller in absolute value than the pole-order of j_1. Therefore, we simply add $b_1 = \lambda$ to the basis.

Now we have $B = \{b_0, b_1\}$ and the corresponding pole-order vectors are $\boxed{(0,0,0)}$ and $\boxed{(-1,0,1)}$. Since the pole-order vectors of the basis B are important to figure

out in which component the reduction can be done, we decided to frame them, so they can more easily be recognized in the text.

Iteration 2. We consider $z_2 = \lambda \cdot b_1$. It has pole-orders $(-2, 0, 2)$, representation y^2, and is given by the expansions
$\left(256\,q^2 - 4096\,q^3 + O(q^5), 1 - 32\,q + 512\,q^2 + O(q^3), \frac{1}{256}\,q^{-2} - \frac{1}{16}\,q^{-1} + \frac{13}{32} + O(q^1)\right)$.

Since $\mathrm{ind}_j(z_2) = 3$, it can be reduced by j. We get as $z_{2a} = z_2 - j/256$ an element with pole-orders $\boxed{(2,2,1)}$, representation $y^2 - \frac{1}{256}\,x$, and expansions
$\left(-\frac{1}{256}\,q^{-2} - \frac{93}{32} + O(q^1), -\frac{1}{256}\,q^{-2} - \frac{61}{32} + O(q^1), -\frac{1}{16}\,q^{-1} - \frac{5}{2} - \frac{5}{4}\,q + O(q^2)\right)$.

Although that reduction step looks like creating an element with a worse pole-order vector than that of z_2, it nevertheless holds that $z_2 >_j z_{2a}$.

We must find a reducer for the second component. However, none of the basis elements has their highest pole-order in the second component; i.e., z_{2a} is irreducible and enters the basis B as b_2.

Iteration 3. We consider $z_3 = \lambda \cdot b_2$. It has pole-orders $(1, 2, 2)$, representation $y^3 - \frac{1}{256}\,x\,y$, and expansions
$\left(-\frac{1}{16}\,q^{-1} + \frac{1}{2} - \frac{197}{4}\,q + O(q^2), -\frac{1}{256}\,q^{-2} + \frac{1}{16}\,q^{-1} - \frac{77}{32} + O(q^1), \frac{1}{256}\,q^{-2} + \frac{1}{8}\,q^{-1} - \frac{35}{32} + O(q^1)\right)$.

From $\mathrm{ind}_j(z_2) = 3$ it follows that z_3 must be reduced in the third component by subtracting an appropriate multiple of j from it. We get $z_{3a} = z_3 - j/256$ with pole-orders $(2, 2, 1)$, representation $y^3 - \frac{1}{256}\,x\,y - \frac{1}{256}\,x$, and expansions
$\left(-\frac{1}{256}\,q^{-2} - \frac{1}{16}\,q^{-1} - \frac{77}{32} + O(q^1), -\frac{1}{128}\,q^{-2} + \frac{1}{16}\,q^{-1} - \frac{85}{16} + O(q^1), \frac{1}{8}\,q^{-1} - 4 + \frac{101}{2}\,q + O(q^2).\right)$.

Since $\mathrm{ind}_j(z_{3a}) = 2$, we can use b_2 to reduce z_{3a} in the second component and get $z_{3b} = z_{3a} - 2b_2$ with pole-orders $\boxed{(2,1,1)}$, representation $y^3 - 2\,y^2 - \frac{1}{256}\,x\,y + \frac{1}{256}\,x$, and expansions
$\left(\frac{1}{256}\,q^{-2} - \frac{1}{16}\,q^{-1} + \frac{109}{32} + O(q^1), \frac{1}{16}\,q^{-1} - \frac{3}{2} + \frac{261}{4}\,q + O(q^2), \frac{1}{4}\,q^{-1} + 1 + 53\,q + O(q^2)\right)$.

We must find a reducer for the first component, since there is the highest pole-order. However, none of the basis elements have their highest pole-order in the first component, i.e. z_{3b} is irreducible and enters the basis B as b_3.

Iteration 4. We consider $z_4 = \lambda \cdot b_3$. It has pole-orders $(1, 1, 2)$, representation $y^4 - 2\,y^3 - \frac{1}{256}\,x\,y^2 + \frac{1}{256}\,x\,y$, and expansions
$\left(\frac{1}{16}\,q^{-1} - \frac{3}{2} + \frac{261}{4}\,q + O(q^2), \frac{1}{16}\,q^{-1} - \frac{5}{2} + \frac{389}{4}\,q + O(q^2), -\frac{1}{64}\,q^{-2} + \frac{1}{16}\,q^{-1} - \frac{25}{8} + O(q^1)\right)$.

Since $\mathrm{ind}_j(z_4) = 3$, z_4 must be reduced in the third component. This can be done by subtracting a suitable multiple of j. We get $z_{4a} = z_4 + j/64$ with pole-orders $(2, 2, 1)$, representation $y^4 - 2\,y^3 - \frac{1}{256}\,x\,y^2 + \frac{1}{256}\,x\,y + \frac{1}{64}\,x$, and expansions
$\left(\frac{1}{64}\,q^{-2} + \frac{1}{16}\,q^{-1} + \frac{81}{8} + O(q^1), \frac{1}{64}\,q^{-2} + \frac{1}{16}\,q^{-1} + \frac{73}{8} + O(q^1), \frac{1}{16}\,q^{-1} + \frac{17}{2} + \frac{5}{4}\,q + O(q^2)\right)$.

Clearly, z_{4a} can be reduced in the second component by b_2. We get $z_{4b} = z_{4a} + 4b_2$ with pole-orders $(1,1,1)$, representation $y^4 - 2\,y^3 + \left(-\frac{1}{256}\,x + 4\right)\,y^2 + \frac{1}{256}\,x\,y$, and expansions
$\left(\frac{1}{16}\,q^{-1} - \frac{3}{2} + \frac{261}{4}\,q + O(q^2), \frac{1}{16}\,q^{-1} + \frac{3}{2} - \frac{123}{4}\,q + O(q^2), -\frac{3}{16}\,q^{-1} - \frac{3}{2} - \frac{15}{4}\,q + O(q^2)\right)$.

Since $\mathrm{ind}_j(z_{4b}) = 3$ we must reduce z_{4b} in the third component by subtracting a constant multiple of $b_1 = \lambda$ and get $z_{4c} = z_{4b} - 3b_1$ with pole-orders $\boxed{(1,1,0)}$, representation $y^4 - 2\,y^3 + \left(-\frac{1}{256}\,x + 4\right)\,y^2 + \left(\frac{1}{256}\,x - 3\right)\,y$, and expansions
$\left(\frac{1}{16}\,q^{-1} - \frac{3}{2} + \frac{69}{4}\,q + O(q^2), \frac{1}{16}\,q^{-1} - \frac{3}{2} + \frac{69}{4}\,q + O(q^2), -3 - 256\,q^2 + O(q^3)\right)$.

The element z_{4c} would need to be reduced in the second component, but that can obviously not be done by any of the current basis elements. Therefore z_{4c} is added to the basis B as b_4.

Iteration 5. We consider $z_3 = \lambda \cdot b_4$. It has pole-orders $(0,1,1)$, representation $y^5 - 2\,y^4 + \left(-\frac{1}{256}\,x + 4\right)\,y^3 + \left(\frac{1}{256}\,x - 3\right)\,y^2$, and expansions
$\left(1 - 32\,q + 512\,q^2 + O(q^3), \frac{1}{16}\,q^{-1} - \frac{5}{2} + \frac{197}{4}\,q + O(q^2), \frac{3}{16}\,q^{-1} - \frac{3}{2} + \frac{79}{4}\,q + O(q^2)\right)$.

This element can be reduced in the third component by subtracting a constant multiple of b_1. We get $z_{5a} = z_5 + 3b_1$ with pole-orders $\boxed{(0,1,-1)}$, representation $y^5 - 2\,y^4 + \left(-\frac{1}{256}\,x + 4\right)\,y^3 + \left(\frac{1}{256}\,x - 3\right)\,y^2 + 3\,y$, and expansions
$\left(1 + 16\,q + 128\,q^2 + O(q^3), \frac{1}{16}\,q^{-1} + \frac{1}{2} + \frac{5}{4}\,q + O(q^2), 16\,q - 128\,q^2 + 704\,q^3 + O(q^4)\right)$.

The element z_{5a} can be reduced in the second component by subtracting b_4. We get $z_{5b} = z_{5a} - b_4$ with pole-orders $\boxed{(1,0,0)}$, representation $y^5 - 3\,y^4 + \left(-\frac{1}{256}\,x + 6\right)\,y^3 + \left(\frac{1}{128}\,x - 7\right)\,y^2 + \left(-\frac{1}{256}\,x + 6\right)\,y$, and expansions
$\left(-\frac{1}{16}\,q^{-1} + \frac{5}{2} - \frac{5}{4}\,q + O(q^2), 2 - 16\,q - 128\,q^2 + O(q^3), 3 + 16\,q + 128\,q^2 + O(q^3)\right)$.

We obtain an element that obviously cannot be reduced in the first component by any existing basis element. As a consequence, it enters the basis B as b_5.

If we look at the pole-orders of all the 6 basis elements $b_0, \ldots, b_5$, we see that relative to j (with pole-order 2 in each component), we can reduce any triple of q-expansions. In other words, the basis B is already complete and we expect a zero reduction in the next interation.

Iteration 6. We consider $z_3 = \lambda \cdot b_5$. It has pole-orders $(0,0,1)$, representation $y^6 - 3\,y^5 + \left(-\frac{1}{256}\,x + 6\right)\,y^4 + \left(\frac{1}{128}\,x - 7\right)\,y^3 + \left(-\frac{1}{256}\,x + 6\right)\,y^2$, and expansions
$\left(-1 + 48\,q - 384\,q^2 + O(q^3), 2 - 48\,q + 384\,q^2 + O(q^3), -\frac{3}{16}\,q^{-1} + \frac{1}{2} - \frac{15}{4}\,q + O(q^2)\right)$.

This element can be reduced in the third component by subtracting a constant multiple of b_1. We get $z_{6a} = z_6 - 3b_1$ with pole-orders $(0,0,0)$, representation $y^6 - 3\,y^5 + \left(-\frac{1}{256}\,x + 6\right)\,y^4 + \left(\frac{1}{128}\,x - 7\right)\,y^3 + \left(-\frac{1}{256}\,x + 6\right)\,y^2 - 3\,y$, and expansions

$\left(-1+O(q^3), -1+O(q^3), -1+O(q^3)\right)$.

Adding the vector for $b_0 = 1$ to z_{6a} gives an element with positive orders at all cusps and is, therefore, exactly zero.

The representation of this zero gives rise to the desired relation between j and λ that we have obtained as the result of calling `modularPolynomial`.

5.2. **Other examples: Alladi's identity, Ramanujan-Sato relations, etc.** Owing to the fact that algebraic relations between modular functions are ubiquitous "in nature", the MultiSamba algorithm has a huge application potential.

As already mentioned, MultiSamba (and related QEta tools) served as a main algorithmic instrument in the computer-assisted construction [11] of Ramanujan-Sato series for 1 over π. Here algebraic relations between modular functions were used to derive computer-assisted proofs of closed form evaluations of modular functions at elements of imaginary quadratic fields. For example, for

$$h(\tau) := \frac{\lambda(\tau)(1-\lambda(\tau))^2}{(1+\lambda(\tau))^4}, \tag{43}$$

with the help of MultiSamba one can find and prove [11, eq. (54)] that

$$h(i\sqrt{58}) = \frac{1}{396^4}.$$

The corresponding computational derivations are made fully explicit in example files coming with the distribution of the QEta package; i.e., all these applications can be even executed and modified.

QEta is freely available from its website

`https://risc.jku.at/sw/qeta`

at RISC-JKU.

Last but not least, we want to mention that in the QEta distribution a great variety of example sheets for of QEta and MultiSamba applications can be found, including the computer-assisted derivations and proofs of Alladi's identity (3) and of the other identities (4), (5), and (7) mentioned in the introduction.

6. Conclusion

We encourage the reader to do actual computations with MultiSamba and the QEta package written in the FriCAS system.

FriCAS has quite some history [6]. It began in 1965 with IBM's Scratchpad and was followed by Scratchpad II which around 1990 was renamed Axiom and withdrawn from the market in 2001 and re-released under the Modified BSD License. Tim Daly managed to bring the sources into shape to be able to start an open source project in 2003. In 2007, Axiom was forked as FriCAS by Waldek Hebisch. FriCAS is freely available for many common platforms such as Linux, macOS, Unix, and Microsoft Windows.

Links concerning FriCAS and QEta installation can be found at the website given in (42). QEta installation is rather straightforward; installation of FriCAS needs some prerequisites to compile, but should otherwise be relatively easy to install.

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