A Grammatical Calculus for the Ramanujan Polynomials

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Dedicated to Krishnaswami Alladi for his leadership and friendship

Abstract

The Ramanujan polynomials arise in three intertwined contexts. As remarked by Berndt-Evans-Wilson, no combinatorial perspective seems to be alluded to in the original definition of Ramanujan. On a different stage, Dumont-Ramamonjisoa uncovered a combinatorial structure underneath an equation also considered by Ramanujan. Around the same time, Shor came up with the same construction as a refinement of the classical formula of Cayley for trees. We present a labeling scheme for rooted trees by employing an extra label marking improper edges. Harnessed by this grammar, we develop a grammatical calculus for the Ramanujan polynomials heavily relying on the constant properties. Moreover, we provide a grammatical formulation of a correspondence that leads to the recurrence relation due to Berndt-Evans-Wilson and Shor.

Key words: Ramanujan polynomials, grammatical calculus, rooted trees, improper edges.

AMS subject classification: 05A05, 05A15.

1 Introduction

Speaking of the Ramanujan polynomials, which have emerged in various contexts, see [1, 2, 6–8, 10–18, 20, 21], one should be reminded of the three intertwining scenes. First, they refer to the polynomials $\psi_k(r,x)$ defined by Ramanujan via the following relation [1, 15]. Assume that r is a nonnegative integer and that

$$\sum_{k=0}^{\infty} \frac{(x+k)^{r+k} e^{-u(x+k)} u^k}{k!} = \sum_{k=1}^{r+1} \frac{\psi_k(r,x)}{(1-u)^{k+r}}, \tag{1.1}$$

from which the following recurrence relation follows:

$$\psi_k(r+1,x) = (x-1)\psi_k(r,x-1) + \psi_{k-1}(r+1,x) - \psi_{k-1}(r+1,x-1), \tag{1.2}$$

where $\psi_1(0,x) = 1$, $\psi_0(r,x) = 0$ and $\psi_k(r,x) = 0$ for $k > r+1$. The first few values of $\psi_k(r,x)$ are given in Table 1.

Table 1: $\psi_k(r,x)$ for $0 \le r \le 3$ and $1 \le k \le r+1$.

$r \backslash k$	1	2	3	4	$\sum_k$
0	1	0	0	0	1
1	$x-1$	1	0	0	x
2	x^2-3x+2	$3x-5$	3	0	x^2
3	$x^3-6x^2+11x-6$	$6x^2-26x+26$	$15x-35$	15	x^3

From the defining relation (1.1), Berndt-Evans-Wilson [1, 2] derived another recurrence relation

$$\psi_k(r,x) = (x-r-k+1)\psi_k(r-1,x) + (r+k-2)\psi_{k-1}(r-1,x). \tag{1.3}$$

For the background of $\psi_k(r,x)$, we refer the reader to Ramanujan [15] and Berndt-Evans-Wilson [2]. Assume that a is real with $|a| \ge e$. Then the solution of the equation

$$x = a \log x \tag{1.4}$$

admits the following expansion for any real number n,

$$\frac{x^n}{n} = \sum_{k=0}^{\infty} \frac{(n+k)^{k-1}}{a^k k!}.$$

Ramanujan extended this sum to

$$F_r(n) = \sum_{k=0}^{\infty} \frac{(n+k)^{r+k}}{a^k k!},$$

and considered the question of how to determine the function $\psi_r(n)$ such that

$$x^n \psi_r(n) = F_r(n). \tag{1.5}$$

When r is a nonnegative integer, by setting $x = e^u$, we see from (1.4) that $a = e^u/u$. Under this substitution, $\psi_r(n)$ is the left side of (1.1). Ramanujan showed that for any nonnegative integer r,

$$\sum_{k=1}^{r+1} \psi_k(r,x) = x^r. \tag{1.6}$$

As remarked by Berndt-Evans-Wilson [2], no combinatorial perspective seems to be alluded to in Ramanujan's original definition. Just on the contrary, the polynomials $\psi_k(r,x)$ are destined to be under the umbrella of combinatorial analysis.

On a different stage, Dumont-Ramamonjisoa [10], with the aid of a context-free grammar, uncovered a combinatorial structure underneath an equation also considered by Ramanujan,

$$x = ye^{-y} + \frac{a-1}{a}(e^{-y} - 1). \tag{1.7}$$

They obtained expansions of y and e^y as power series in x, where the coefficients are determined by the distribution of improper edges in labeled rooted trees. An associated insertion algorithm was also found. Around the same time, Shor [17] independently developed the same insertion algorithm. Subsequently, Zeng [21] noted that these polynomials in Shor [17] and Dumont-Ramamonjisoa [10] are linked to the Ramanujan polynomials $\psi_k(r,x)$, thereby providing a combinatorial interpretation.

Let us recall the notion of an improper edge of a labeled rooted tree. Let $n \geq 1$ and write $[n] = \{1, 2, \ldots, n\}$. An edge of a rooted tree T on $[n]$ is represented by a pair (i, j) of vertices with j being a child of i. We say that (i, j) is improper if there exists a descendant of j that is smaller than i, bearing in mind that any vertex of T is considered as a descendant of itself; otherwise, (i,j) is called a proper edge. We use $\mathscr{R}_{n,k}$ to denote the set of rooted trees on $[n]$ with k improper edges, and let $R(n,k)$ denote the number of rooted trees in $\mathscr{R}_{n,k}$. Meanwhile, we use $\mathscr{T}_{n+1,k}$ to denote the set of rooted trees on $[n+1]$ with root 1 and with k improper edges, and let $T(n,k)$ denote the cardinality of $\mathscr{T}_{n+1,k}$. The numbers $R(n,k)$ and $T(n,k)$ are listed as sequences A054589 and A217922 in OEIS.

The following polynomials are also referred to as the Ramanujan polynomials,

$$R_n(u) = \sum_{k=0}^{n-1} R(n,k)\,u^k,$$

$$T_n(u) = \sum_{k=0}^{n-1} T(n,k)\,u^k.$$

Their generating functions are defined by

$$R(u,t) = \sum_{n\geq 1} R_n(u)\frac{t^n}{n!},$$

$$T(u,t) = 1 + \sum_{n\geq 1} T_n(u)\frac{t^n}{n!}.$$

As shown in [10], (1.7) gives rise to the following expansions,

$$y = \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} R(n,k)a^{n+k}\right)\frac{x^n}{n!}, \tag{1.8}$$

$$e^y = 1 + \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} T(n,k)a^{n+k}\right)\frac{x^n}{n!}. \tag{1.9}$$

Around the same time, Shor [17] considered polynomials $Q_{n,k}(x)$ when x is treated as a positive integer, which we also call the Ramanujan polynomials. Zeng [21] gave an interpretation of $Q_{n,k}(x)$, where x is regarded as an indeterminate. The Ramanujan polynomials $Q_{n,k}(x)$ are given by

$$Q_{n,k}(x) = \sum_{T\in\mathscr{T}_{n+1,k}} x^{\deg_T(1)-1}, \tag{1.10}$$

or equivalently,

$$Q_{n,k}(x)=\sum_{T\in\mathscr{R}_{n,k}}(x+1)^{\deg_T(1)}, \tag{1.11}$$

where $\deg_T(i)$ denotes the degree of i in T. The degree of a vertex i in a rooted tree T is defined to be the number of children of i. Zeng [21] also observed that

$$Q_{n,k}(x)=\psi_{k+1}(n-1,x+n). \tag{1.12}$$

Consequently, Ramanujan's identity (1.6) aligns with Shor's identity [17]:

$$\sum_{k=0}^{n-1}Q_{n,k}(x)=(x+n)^{n-1}.$$

In this context, the work of Dumont-Ramamonjisoa [10] can be viewed as an investigation into the special values of $Q_{n,k}(x)$. In particular, they found that $Q_{n,k}(0)=R(n,k)$, $Q_{n,k}(1)=T(n,k)$, and $Q_{n,k}(-1)$ equals the number of rooted trees on $[n]$ with k improper edges for which the vertex 1 is a leaf. It is worth mentioning that Wang-Zhou [20] showed that $Q_{n,k}(-1)$ is related to refined orbifold Euler characteristics of the moduli space of stable curves of genus 0 with n marked points.

This paper is organized as follows. In Section 2, we present a grammar for the Ramanujan polynomials with an extra label marking improper edges. We call this grammar literal because each label signifies exactly one of the cases in the recursive construction of Shor and Dumont-Ramamonjisoa. Section 3 is devoted to a grammatical calculus for the generating functions for the polynomials $R_n(u)$, $T_n(u)$ and $Q_{n,k}(x)$, where constant properties play a central role. In Section 4, we provide a grammatical formulation of a bijection in connection with the recurrence relation due to Berndt-Evans-Wilson and Shor.

2 A literal labeling scheme for rooted trees

To provide a solution of the functional equation (1.7), Dumont-Ramamonjisoa [10] introduced the following grammar

$$A\rightarrow A^3S,\ S\rightarrow AS^2, \tag{2.1}$$

and gave it a combinatorial interpretation. Let D be the formal derivative associated with this grammar. For $n \geq 1$, Dumont-Ramamonjisoa obtained

$$D^{n-1}(AS) = R_n(A),$$

$$D^n(S) = A^n S^{n+1} T_n(A).$$

In this section, we present a labeling scheme for rooted trees (or planted rooted trees) based on the insertion algorithm due to Shor [17] and Dumont-Ramamonjisoa [10]; see also [16]. We call this labeling scheme literal in the sense that each label corresponds to exactly one of the cases in the recursive construction. In principle, such an understanding of a labeling scheme may be instrumental in converting a recursive construction into a grammar. Moreover, the grammar generated this way might be more convenient for carrying out the grammatical calculus.

A rooted tree T on $[n]$ can be regarded as a planted tree on $\{0,1,\ldots,n\}$ for which 0 is the root and the root has exactly one child. Let T be a rooted tree on $[n-1]$ with $n \geq 1$. We describe the labeling scheme along with the algorithm to insert the element n into T. For $n = 0$, there is only one planted tree with the root 0.

1. For any vertex i, n may be added as a child of i. See Figure 1 for an illustration. A new edge is formed, and a new vertex is formed. If we use z to label a vertex, and v to label a new edge. Then we get a rule:

   $$z \to z(vz).$$

   The labels v and z in parentheses are meant to be the labels for the new vertex and the new edge (which is proper). The operation in this case is called a z-insertion.

2. For any edge (either proper or improper), when n is added, see Figure 2, a new edge is created, which is improper. So we should consider v as a label for every edge. Moreover, we should use an additional label u to mark an improper edge. In this case, only edges labeled by v are selected for the operation of inserting n. Notice that a new vertex is created. We get the rule:

   $$v \to v(uvz).$$

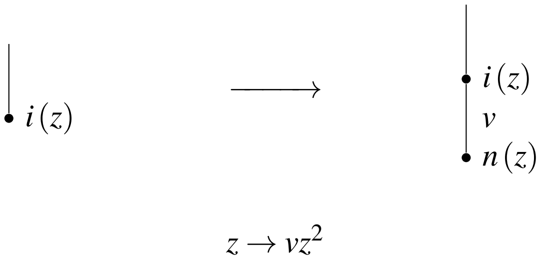


Figure 1: A z-insertion.

The operation in this case is called a v-insertion, which does not change the degree of the existing vertices.

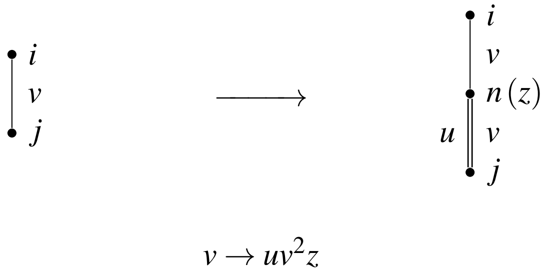


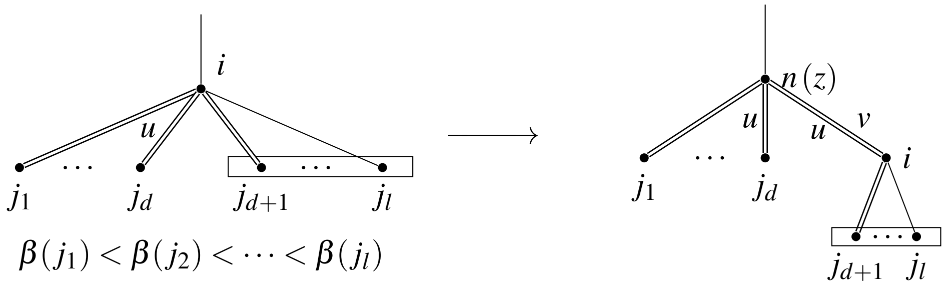
Figure 2: A v-insertion.

3. Each improper edge can be used to add n in another way. See Figure 3 for an illustration. In this case, an improper edge is created. Since a new vertex is created, we should use the rule:

   $$u \to u(uvz).$$

   The operation in this case is called a u-insertion and it does not change the degree of the existing vertices except the vertex i. Figure 3 depicts the insertion, where we adopt the notation $\beta(j)$ for the minimum vertex among the vertices in the subtree rooted at j.

It can be readily seen that the insertion algorithm is reversible. In accordance with the above procedure, it is natural to introduce the following labeling scheme for a planted tree T on $\{0,1,\ldots,n\}$ for any $n \geq 1$:



$$u \to u^2vz$$

Figure 3: A u-insertion.

- each vertex i other than 0 is labeled by z;
- each edge is labeled by v;
- each improper edge carries an additional label u.

In other words, a proper edge is labeled by v, whereas an improper edge is labeled by uv.

It should be noted that this labeling scheme has a special feature that the labels are in one-to-one correspondence with the insertions of the recursive construction. In this setting, an extra label is at our disposal. This updated labeling scheme seems to offer more flexibility in further exploring the Ramanujan polynomials subject to certain constraints.

The rules are summarized into the following grammar:

$$G = \{z \to vz^2,\ v \to uv^2z,\ u \to u^2vz\}. \tag{2.2}$$

Let D denote the formal derivative with respect to this grammar G.

Theorem 2.1. For $n \geq 1$, we have

$$D^{n-1}(vz) = v^n z^n R_n(u).$$

Likewise, the polynomials $T_n(u)$ can be generated by applying the operator D^{n-1} to z.

Theorem 2.2. For $n \geq 1$, we have

$$D^{n-1}(z) = v^{n-1} z^n T_{n-1}(u),$$

where $T_0(u)$ is defined to be 1.

For $0 \leq n \leq 4$, the values of $D^n(vz)$ and $D^n(z)$ are given below,

$$\begin{aligned}
D^0(vz) &= vzR_1(u) = vz,\\
D^1(vz) &= v^2z^2R_2(u) = v^2z^2(1+u),\\
D^2(vz) &= v^3z^3R_3(u) = v^3z^3(2+4u+3u^2),\\
D^3(vz) &= v^4z^4R_4(u) = v^4z^4(6+18u+25u^2+15u^3),\\
D^4(vz) &= v^5z^5R_5(u) = v^5z^5(24+96u+190u^2+210u^3+105u^4)
\end{aligned}$$

and

$$\begin{aligned}
D^0(z) &= zT_0(u) = z,\\
D^1(z) &= vz^2T_1(u) = vz^2,\\
D^2(z) &= v^2z^3T_2(u) = v^2z^3(2+u),\\
D^3(z) &= v^3z^4T_3(u) = v^3z^4(6+7u+3u^2),\\
D^4(z) &= v^4z^5T_4(u) = v^4z^5(24+46u+40u^2+15u^3).
\end{aligned}$$

3 The grammatical calculus

In this section, we develop a grammatical calculus leading to functional equations for the generating functions of $R_n(u)$, $T_n(u)$ and $Q_{n,k}(x)$.

Recall that for a Laurent series f in the variables involved in the grammar G, the associated generating function is defined by

$$\mathrm{Gen}(f,t) = \sum_{n=0}^{\infty} D^n(f)\frac{t^n}{n!}.$$

For two Laurent series f and g, the Leibniz formula holds for $n \geq 0$,

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k(f) D^{n-k}(g),$$

which implies the multiplicative property

$$\mathrm{Gen}(fg,t) = \mathrm{Gen}(f,t)\,\mathrm{Gen}(g,t).$$

In particular, we have

$$\mathrm{Gen}(f,t)\mathrm{Gen}(f^{-1},t) = 1.$$

Constants and eigenfunctions are basic ingredients of the grammatical calculus, see, for example, [3–5, 9]. A function f is called a constant with respect to D if $D(f) = 0$, and a k-th order constant if $D^k(f) = 0$. An eigenfunction f is a function satisfying $D(f) = cf$, where c is a constant.

First, we consider the functional equation satisfied by the generating function of $T_n(u)$.

Theorem 3.1. Given the grammar

$$G = \{z \to vz^2,\ v \to uv^2z,\ u \to u^2vz\},$$

the generating function $\mathrm{Gen}(z,t)$ satisfies

$$\mathrm{Gen}(z,t) = ze^{(z^{-1}-u^{-1}z^{-1}+u^{-1}vt)\mathrm{Gen}(z,t)+u^{-1}-1}. \tag{3.1}$$

From Theorem 2.2, we have for $n \geq 0$,

$$D^n(z)\mid_{v=z=1} = T_n(u).$$

Thus we obtain the following functional equations for $T(u,t)$ by setting $v = 1$ and $z = 1$ in Theorem 3.1:

$$T(u,t) = e^{(1-u^{-1}+u^{-1}t)T(u,t)+u^{-1}-1}. \tag{3.2}$$

For a labeled tree T on $[n+1]$ rooted at 1, any edge incident to the root is proper. After relabeling, such a rooted tree can be viewed as a forest F on $\{1,2,\ldots,n\}$, and an edge of T is improper if and only if it is improper in a component of F. It follows that

$$T(u,t) = e^{R(u,t)}.$$

Combining this identity with (3.2), we obtain

$$e^{R(u,t)} = e^{(1-u^{-1}+u^{-1}t)e^{R(u,t)}+u^{-1}-1},$$

and hence

$$R(u,t) = (1-u^{-1}+u^{-1}t)e^{R(u,t)}+u^{-1}-1. \tag{3.3}$$

Equation (3.3) can be transformed into (1.7) of Ramanujan under the substitutions $u = a$, $t = ax$ and $y = R(a,ax)$.

The following properties are needed in the proof of Theorem 3.1. Since the formal derivative D with respect to grammar G can be viewed as a differential operator

$$D = u^2vz\frac{\partial}{\partial u} + uv^2z\frac{\partial}{\partial v} + vz^2\frac{\partial}{\partial z},$$

and so these properties can be derived by solving the corresponding partial differential equations.

Proposition 3.2. The following relations hold

$$D(u^{-1}v) = 0, \tag{3.4}$$

$$D(ze^{u^{-1}}) = 0, \tag{3.5}$$

$$D((u-1)v^{-1}z^{-1}) = 1, \tag{3.6}$$

$$D(z^{-1}-u^{-1}z^{-1}) = u^{-1}v, \tag{3.7}$$

$$D(e^{u^{-1}}(uv^{-1}-v^{-1})) = ze^{u^{-1}}. \tag{3.8}$$

These identities can be verified by direct computations. Especially, we note that (3.7) follows from (3.4) and (3.6) since

$$z^{-1}-u^{-1}z^{-1} = (u^{-1}v)\cdot\left((u-1)v^{-1}z^{-1}\right),$$

and similarly, (3.8) follows from (3.5) and (3.6), since

$$e^{u^{-1}}(uv^{-1}-v^{-1}) = (ze^{u^{-1}})\cdot\left((u-1)v^{-1}z^{-1}\right).$$

We are now ready to give a grammatical derivation of Theorem 3.1.

Proof of Theorem 3.1. In view of (3.5), we see that

$$\mathrm{Gen}(ze^{u^{-1}},t) = ze^{u^{-1}}. \tag{3.9}$$

Noting that

$$\mathrm{Gen}(e^{u^{-1}},t) = e^{\mathrm{Gen}(u^{-1},t)},$$

by (3.9), we obtain

$$\mathrm{Gen}(z,t)e^{\mathrm{Gen}(u^{-1},t)} = ze^{u^{-1}}, \tag{3.10}$$

and thus

$$e^{\mathrm{Gen}(u^{-1},t)} = ze^{u^{-1}}\mathrm{Gen}(z^{-1},t). \tag{3.11}$$

By (3.7) and (3.4), we find that

$$\mathrm{Gen}(z^{-1}-u^{-1}z^{-1},t) = z^{-1}-u^{-1}z^{-1}+u^{-1}vt. \tag{3.12}$$

Therefore,

$$\mathrm{Gen}(z^{-1},t)-\mathrm{Gen}(u^{-1},t)\mathrm{Gen}(z^{-1},t) = z^{-1}-u^{-1}z^{-1}+u^{-1}vt. \tag{3.13}$$

Multiplying both sides by $\mathrm{Gen}(z,t)$, we get

$$\mathrm{Gen}(u^{-1},t) = \mathrm{Gen}(z,t)(\mathrm{Gen}(z^{-1},t)-z^{-1}+u^{-1}z^{-1}-u^{-1}vt). \tag{3.14}$$

Substituting (3.14) into (3.11), we obtain

$$e^{\mathrm{Gen}(z^{-1},t)} = e^{z^{-1}-u^{-1}z^{-1}+u^{-1}vt}(ze^{u^{-1}}\mathrm{Gen}(z^{-1},t))^{\mathrm{Gen}(z^{-1},t)}, \tag{3.15}$$

which implies (3.1). This completes the proof. ∎

The functional equation for $\mathrm{Gen}(u,t)$ can be established by using (3.1) and (3.11). More precisely, we have

$$(1-\mathrm{Gen}(u^{-1},t))e^{\mathrm{Gen}(u^{-1},t)} = e^{u^{-1}}(1-u^{-1}+u^{-1}vzt). \tag{3.16}$$

As for $\mathrm{Gen}(v,t)$, by (3.4) we have

$$\mathrm{Gen}(u^{-1}v,\, t) = \mathrm{Gen}(u^{-1},t)\mathrm{Gen}(v,t) = u^{-1}v.$$

Thus,

$$\mathrm{Gen}(u^{-1},t)=u^{-1}v\,\mathrm{Gen}(v^{-1},t),$$

which, together with (3.16), yields

$$\left(1-u^{-1}v\,\mathrm{Gen}(v^{-1},t)\right)e^{u^{-1}v\,\mathrm{Gen}(v^{-1},t)}=e^{u^{-1}}(1-u^{-1}+u^{-1}vzt). \tag{3.17}$$

We next turn to the grammatical derivation of the functional equation of the generating function of $Q_{n,k}(x)$. In doing so, we introduce two additional labels a and x along with their corresponding substitution rules, on top of the grammar in Section 2:

$$G=\{z\rightarrow vz^2,\ v\rightarrow uv^2z,\ u\rightarrow u^2vz\}.$$

Given a rooted tree T on $[n]$ with root 1, label the root by a. The children of the root are labeled by x and the rest of the vertices are labeled by z. The edges are labeled by v, and the improper edges have an extra label u.

Starting with the root 1 with label a, $D(a)$ gives a labeled tree on [2] with 2 being a child of the root. This gives the rule:

$$a\rightarrow axv.$$

Inspecting the insertions involving the children of the root and other vertices, we get the rules

$$x\rightarrow xvz,\ z\rightarrow vz^2,\ v\rightarrow uv^2z,\ u\rightarrow u^2vz,$$

and so we have the grammar

$$G=\{a\rightarrow axv,\ x\rightarrow xvz,\ z\rightarrow vz^2,\ v\rightarrow uv^2z,\ u\rightarrow u^2vz\}. \tag{3.18}$$

By a slight abuse of notation, we still denote the updated grammar by G. This grammar has an extra label compared with the grammar in [7].

Theorem 3.3. For $n\geq 1$, we have

$$D^n(a)=av^n\sum_{k=0}^{n-1}\sum_{T\in\mathscr{T}_{n+1,k}}x^{\deg_T(1)}z^{n-\deg_T(1)}u^k, \tag{3.19}$$

or equivalently,

$$D^n(a) = axv^n z^{n-1} \sum_{k=0}^{n-1} Q_{n,k}(xz^{-1})u^k. \tag{3.20}$$

In the spirit of the grammatical calculus for the generating functions of $R_n(u)$ and $T_n(u)$, we can carry out a grammatical calculus for the generating function of $Q_{n,k}(x)$, leading to the functional equation (1.7) of Ramanujan.

First, we find it more convenient to make a change of variables by setting $b = xv$ and $c = vz$ for the grammar G in (3.18). Let D_G denote the formal derivative with respect to G. Since

$$D_G(xv) = xv^2z(1+u) = bc(1+u), \tag{3.21}$$

$$D_G(vz) = v^2z^2(1+u) = c^2(1+u), \tag{3.22}$$

the grammar G is transformed into

$$H = \{a \to ab, b \to bc(1+u), c \to c^2(1+u), u \to cu^2\}. \tag{3.23}$$

Let D_H denote the formal derivative with respect to H. Evidently,

$$D_H(a)\,|_{b=xv,c=vz} = D_G(a). \tag{3.24}$$

In the notation of the grammar H, the following properties hold.

Proposition 3.4. We have

$$D_H(uc^{-1}e^{-u^{-1}}) = 0, \tag{3.25}$$

$$D_H(bc^{-1}) = 0, \tag{3.26}$$

$$D_H(ae^{bc^{-1}u^{-1}}) = 0, \tag{3.27}$$

$$D_H((u-1)c^{-1}) = 1. \tag{3.28}$$

The following theorem gives a functional equation satisfied by the generating function $\mathrm{Gen}(a,t)$.

Theorem 3.5. Let $y \in \mathbb{C}[[t,u]]$ be the solution of the equation

$$(1-u^{-1}+y)ue^{-y} = u-1+vzt. \tag{3.29}$$

Then

$$\mathrm{Gen}(a,t) = ae^{xz^{-1}y}. \tag{3.30}$$

Proof. Let Gen_H denote the generating function with respect to grammar H in (3.23). It follows from (3.26) and (3.27) that

$$\mathrm{Gen}_H(bc^{-1},t) = bc^{-1}, \tag{3.31}$$

$$\mathrm{Gen}_H(a,t)e^{\mathrm{Gen}_H(bc^{-1},t)\mathrm{Gen}_H(u^{-1},t)} = ae^{bc^{-1}u^{-1}}. \tag{3.32}$$

Substituting (3.31) into (3.32), we get

$$\mathrm{Gen}_H(a,t) = ae^{bc^{-1}(u^{-1}-\mathrm{Gen}_H(u^{-1},t))}. \tag{3.33}$$

To compute $\mathrm{Gen}_H(u^{-1},t)$, using the constant property (3.25), we find that

$$\mathrm{Gen}_H(u,t)\mathrm{Gen}_H(c^{-1},t)e^{\mathrm{Gen}_H(-u^{-1},t)} = uc^{-1}e^{-u^{-1}}, \tag{3.34}$$

and thus

$$\mathrm{Gen}_H(c^{-1},t) = uc^{-1}e^{-u^{-1}}\mathrm{Gen}_H(u^{-1},t)e^{\mathrm{Gen}_H(u^{-1},t)}. \tag{3.35}$$

From (3.28), we obtain

$$\mathrm{Gen}_H(u-1,t)\mathrm{Gen}_H(c^{-1},t) = (u-1)c^{-1}+t. \tag{3.36}$$

Substituting (3.35) into (3.36) yields

$$ue^{-u^{-1}}e^{\mathrm{Gen}_H(u^{-1},t)}(1-\mathrm{Gen}_H(u^{-1},t)) = u-1+ct. \tag{3.37}$$

Writing $\mathrm{Gen}_H(u^{-1},t) = u^{-1}-y$, we have

$$ue^{-u^{-1}}e^{u^{-1}-y}(1-u^{-1}+y) = u-1+ct, \tag{3.38}$$

which simplifies to

$$(1-u^{-1}+y)ue^{-y} = u-1+ct. \tag{3.39}$$

Hence by (3.33), we get

$$\mathrm{Gen}_H(a,t) = ae^{bc^{-1}y}, \tag{3.40}$$

where y is the solution of (3.39).

Back to the grammar G, we conclude that

$$\mathrm{Gen}(a,t) = ae^{xz^{-1}y}, \tag{3.41}$$

where $y \in \mathbb{C}[[t,u]]$ is the solution of the equation

$$(1-u^{-1}+y)ue^{-y} = u-1+vzt. \tag{3.42}$$

This completes the proof. ∎

In light of Theorem 3.3, we see that

$$\mathrm{Gen}(a,t) = a + axz^{-1}\sum_{n\geq 1}\sum_{k=0}^{n-1} Q_{n,k}(xz^{-1})u^k \frac{(vzt)^n}{n!}. \tag{3.43}$$

Note that Zeng [21] defined the generating function of $Q_{n,k}(x)$ in the following form

$$Y(u,t) = \sum_{n\geq 1}\sum_{k=0}^{n-1} \frac{Q_{n,k}(x)}{(1-u)^k}\frac{t^n}{n!} \tag{3.44}$$

and showed that

$$Y(u,t) = \frac{e^{xy}-1}{x}, \tag{3.45}$$

where $y \in \mathbb{C}[[t,u]]$ is the solution of the equation

$$(1-u)t = ye^{-y} + u(e^{-y}-1). \tag{3.46}$$

Notice that by setting $a = v = z = 1$ and replacing u with $1/(1-u)$, the functional equation of $Y(u,t)$ is equivalent to that of $\mathrm{Gen}(a,t)$.

To conclude this section, we notice that equation (3.46) is equivalent to Ramanujan's equation (1.7) by replacing $(1-u)t$ with x and u with $(a-1)/a$. In fact, from the expansion of y given by Dumont-Ramamonjisoa (1.8), resorting to the exponential formula for labeled structures, see Stanley [19], we are led to the following expansion

$$e^{xy} = 1 + x\sum_{n\geq 1}\left(\sum_{k=0}^{n-1} Q_{n,k}(x)a^{n+k}\right)\frac{x^n}{n!}, \tag{3.47}$$

where $Q_{n,k}(x)$ are endowed with the combinatorial interpretation as in (1.10).

4 The grammatical calculus behind a bijection

This section is concerned with a grammatical formulation of a recurrence relation of the Ramanujan polynomials. Recall that Berndt-Evans-Wilson [2] derived the recurrence relation (1.3) for $\psi_k(r,x)$. On the other hand, Shor [17] asked for a combinatorial proof of a recurrence relation for $Q_{n,k}(x)$, that is,

$$Q_{n,k}(x) = (x-k+1)Q_{n-1,k}(x+1) + (n+k-2)Q_{n-1,k-1}(x+1). \tag{4.1}$$

It turns out that these two recurrences are equivalent. Chen-Guo [6] presented a rather involved bijection for this recurrence, and later Guo [11] found a simpler construction. Based on the grammar mentioned in Section 3, Chen-Yang [7] gave a grammatical proof using generating functions. Albeit these efforts, a better combinatorial understanding is still in demand. Perhaps an alternative insertion algorithm is needed for this purpose.

Chen-Guo [6] showed that the recurrence (4.1) can be deduced from the following correspondence.

Theorem 4.1. For $n \geq 2$, there is a bijection

$$\mathscr{T}_{n,k}[\deg_T(2) > 0, \deg_T(1) = r] \longleftrightarrow \mathscr{T}_{n,k+1}[\deg_T(n) > 0, \deg_T(1) = r],$$

where the conditions in brackets specify degree constraints.

Let's translate the above correspondence into a grammatical statement, where the grammar G is given in (3.18), that is,

$$G = \{a \rightarrow axv,\ x \rightarrow xvz,\ z \rightarrow vz^2,\ v \rightarrow uv^2z,\ u \rightarrow u^2vz\}.$$

The insertion process of generating rooted trees on $[n]$ with root 1 yields

$$D^{n-1}(a) = av^{n-1} \sum_{k=0}^{n-2} \sum_{T \in \mathscr{T}_{n,k}} x^{\deg_T(1)} z^{n-\deg_T(1)-1} u^k.$$

Start with the edge $(1,2)$, where the vertex 1 is labeled by a, the vertex 2 is labeled by x and the edge is labeled by v. Observe that once a vertex becomes an internal vertex, it remains an internal vertex. The condition $\deg(2) > 0$ indicates that the vertex 2 is not allowed to be a leaf. Concerning the trees on $[n]$ with $\deg(2) = 0$, since the vertex 2 never gets involved in an x-insertion, the generating polynomial of such trees is $xD^{n-2}(av)$.

On the other hand, the condition $\deg(n) = 0$ says that the vertex n is a leaf. Assume that T is obtained from T' with $n-1$ vertices by the insertion of a leaf n. There are three cases. If n is a child of the root 1, that is, an a-insertion gets involved, the generating polynomial will be $xvD^{n-2}(a)$. For an x-insertion or a z-insertion, the generating polynomials of both cases shall be $(n-2)vzD^{n-2}(a)$.

To give a grammatical restatement of the bijection in Theorem 4.1 the number of improper edges should be taken into consideration. This is reflected by a factor u on the left side in the following identity,

$$u(D^{n-1}(a) - xD^{n-2}(av)) = D^{n-1}(a) - xvD^{n-2}(a) - (n-2)vzD^{n-2}(a).$$

So the grammatical identity we wish to establish can be formulated as follows, which is not hard to justify by induction.

Theorem 4.2. For $n \geq 2$, we have

$$D^{n-2}(av) = \frac{u-1}{xu} D^{n-1}(a) + \frac{v}{u}\left(1 + (n-2)\frac{z}{x}\right) D^{n-2}(a). \tag{4.2}$$

Proof. We proceed by induction on n. First, we observe the following constant properties

$$D\left(\frac{v}{u}\right) = 0, \quad D\left(\frac{z}{x}\right) = 0,$$

and

$$D\left(\frac{u-1}{xu}\right) = \frac{vz}{xu}.$$

Applying the operator D to both sides of (4.2), by the Leibniz rule we find that

$$\begin{aligned} D^{n-1}(av) &= D\left(\frac{u-1}{xu} D^{n-1}(a)\right) + D\left(\frac{v}{u}\left(1 + (n-2)\frac{z}{x}\right) D^{n-2}(a)\right) \\ &= \frac{vz}{xu} D^{n-1}(a) + \frac{u-1}{xu} D^{n}(a) + \frac{v}{u}\left(1 + (n-2)\frac{z}{x}\right) D^{n-1}(a). \end{aligned}$$

Thus,

$$D^{n-1}(av) = \frac{u-1}{xu} D^n(a) + \frac{v}{u}\left(1 + (n-1)\frac{z}{x}\right) D^{n-1}(a),$$

and so the proof is complete by induction. ∎

Acknowledgment. We wish to thank the referees for their insightful comments.

References

[1] Berndt, B.C.: Ramanujan's Notebooks, Part I, Chap. 3. Springer, New York (1985) 1, 1

[2] Berndt, B.C., Evans, R.J., Wilson, B.M.: Chapter 3 of Ramanujan's second notebook. Adv. Math. 49, 123–169 (1983) 1, 1, 1, 1, 4

[3] Chen, W.Y.C., Fu, A.M.: Context-free grammars for permutations and increasing trees. Adv. Appl. Math. 82, 58–82 (2017) 3

[4] Chen, W.Y.C., Fu, A.M.: A grammatical calculus for peaks and runs of permutations. J. Algebraic Combin. 57, 1139–1162 (2023)

[5] Chen, W.Y.C., Fu, A.M.: The Dumont ansatz for the Eulerian polynomials, peak polynomials and derivative polynomials. Ann. Combin. 27, 707–735 (2023) 3

[6] Chen, W.Y.C., Guo, V.J.W.: Bijections behind the Ramanujan polynomials. Adv. Appl. Math. 27, 336–356 (2001) 1, 4

[7] Chen, W.Y.C., Yang, H.R.L.: A context-free grammar for the Ramanujan-Shor polynomials. Adv. Appl. Math. 126, 101908 (2021) 3, 4

[8] Chen, X., Sokal, A.D.: Total positivity of some polynomial matrices that enumerate labeled trees and forests II. Rooted labeled trees and partial functional digraphs. Adv. Appl. Math. 157, 102703 (2024) 1

[9] Dong, J.J.W., Du, L.R., Ji, K.Q., Zhang, D.T.X.: New refinements of Narayana polynomials and Motzkin polynomials. Adv. Appl. Math. 166, 102855 (2025) 3

[10] Dumont, D., Ramamonjisoa, A.: Grammaire de Ramanujan et Arbres de Cayley. Electron. J. Combin. 3(2), #R17 (1996) 1, 1, 1, 1, 2, 2

[11] Guo, V.J.W.: A bijective proof of the Shor recurrence. European J. Combin. 70, 92–98 (2018) 4

[12] Guo, V.J.W., Zeng, J.: A generalization of the Ramanujan polynomials and plane trees. Adv. Appl. Math. 39, 96–115 (2007)

[13] Josuat-Vergès, M.: Derivatives of the tree function. Ramanujan J. 38, 1–15 (2015)

[14] Lin, Z., Zeng, J.: Positivity properties of Jacobi–Stirling numbers and generalized Ramanujan polynomials. Adv. Appl. Math. 53, 12–27 (2014)

[15] Ramanujan, S.: Notebooks, Vol. 1. Tata Institute of Fundamental Research, Bombay (1957) 1, 1

[16] Randazzo, L.: Arboretum for a generalisation of Ramanujan polynomials. Ramanujan J. 54, 591–604 (2021) 2

[17] Shor, P.W.: A new proof of Cayley's formula for counting labeled trees. J. Combin. Theory Ser. A 71, 154–158 (1995) 1, 1, 1, 2, 4

[18] Sokal, A.D.: Total positivity of some polynomial matrices that enumerate labeled trees and forests I: forests of rooted labeled trees. Monatsh. Math. 200(2), 389–452 (2023) 1

[19] Stanley, R.P.: Enumerative Combinatorics, Vol. 2. Cambridge University Press, Cambridge (1999) 3

[20] Wang, Z., Zhou, J.: Orbifold Euler characteristics of $\overline{\mathscr{M}}_{g,n}$. Peking Math. J., to appear. 1, 1

[21] Zeng, J.: A Ramanujan sequence that refines the Cayley formula for trees. Ramanujan J. 3, 45–54 (1999) 1, 1, 1, 1, 3